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# Matroids and fair allocation

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In matroid theory, we explore  
A structure with properties galore  
From bases to circuits  
It never discourages  
Mathematicians always wanting more

The axioms it holds are so grand  
And its applications vast and unplanned  
From optimization to graphs  
It can solve so many tasks  
Matroid theory, truly a wonderland

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ChatGPT

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# Chapter 1

## Introduction

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# Chapter 2

## Background

### 2.1 Matroid theory

If a mathematical structure can be defined or axiomatized in multiple different, but not obviously equivalent, ways, the different definitions or axiomatizations of that structure make up a cryptomorphism. The many obtusely equivalent definitions of a matroid are a classic example of cryptomorphism, and belie the fact that the matroid is a generalization of concepts in many, seemingly disparate areas of mathematics.

First introduced by Hassler Whitney in 1935 [Whi35], in a seminal paper where he described two axioms for independence in the columns of a matrix, and defined any system obeying these axioms to be a “matroid”. Whitney’s key insight was that this abstraction of “independence” is applicable to both matrices and graphs. As a result of this, the terms used in matroid theory are borrowed from analogous concepts in both graph theory and linear algebra. Matroids have also received attention from researchers in fair allocation, as their properties make them useful for modeling user preferences; for instance, matroid rank functions are a natural way of formally describing course allocation for students [Ben21].

#### Independent sets

Perhaps the most common way to define a matroid is in terms of its *independent sets*. An independence system is a pair  $(E, \mathcal{I})$ , where  $E$  is the ground set of



elements,  $E \neq \emptyset$ , and  $\mathcal{I}$  is the set of independent sets,  $\mathcal{I} \subseteq 2^E$ . A matroid is an independence system with the following properties:

1. The empty set is independent:  $\emptyset \in \mathcal{I}$ .
2. The hereditary property: if  $A \subseteq B$  and  $B \in \mathcal{I}$ , then  $A \in \mathcal{I}$ .
3. The augmentation property: If  $A, B \in \mathcal{I}$  and  $|A| > |B|$ , then there exists  $e \in A$  such that  $B \cup \{e\} \in \mathcal{I}$ .

In practice, the ground set  $E$  represents the universe of elements in play, and the independent sets of typically represent the legal combinations of these items. In the context of fair allocation, the independent sets represent the legal (in the case of matroid constraints) or desired (in the case of matroid utilities) bundles of items.

## Rank

Given a matroid  $\mathfrak{M} = (E, \mathcal{I})$ , the *matroid rank function* (MRF) is a function  $rank : 2^E \rightarrow \mathbb{Z}^+$  that gives the *rank* of a set  $A \subseteq E$ , which is defined to be the size of the largest independent subset of  $A$ . Formally,

$$rank(A) = \max\{|X| : X \subseteq A \text{ and } X \in \mathcal{I}\}.$$

Matroid rank functions are *binary submodular*. Binary because they have binary marginals, that is,  $rank(A \cup \{e\}) - rank(A) \in \{0, 1\}$ , for all  $A \subseteq 2^E$  and  $e \in E$ . Submodularity refers to rank functions' natural diminishing returns property, namely that for any two sets  $X, Y \subseteq E$ , we have

$$rank(X \cup Y) + rank(X \cap Y) \leq rank(X) + rank(Y).$$

This diminishing returns property makes the rank function useful for modeling user preferences, and is a reason why matroids show up so often in economics and game theory (???).

## Closed sets

We also need to establish the concept of the *closed sets* of a matroid. A closed set is a set whose cardinality is maximal for its rank. Equivalently to the definition given above, we can define a matroid as  $\mathfrak{M} = (E, \mathcal{F})$ , where  $\mathcal{F}$  is the set of closed sets of  $\mathfrak{M}$ , satisfying the following properties:

1. The set of all elements is closed:  $E \in \mathcal{F}$
2. The intersection of two closed sets is a closed set: If  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$
3. If  $A \in \mathcal{F}$  and  $a, b \in E \setminus A$ , then  $b$  is a member of all sets in  $\mathcal{F}$  containing  $A \cup \{a\}$  if and only if  $a$  is a member of all sets in  $\mathcal{F}$  containing  $A \cup \{b\}$

## 2.2 Examples of matroids



### The free matroid



### The uniform matroid



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**The vector matroid**

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**The graphic matroid**

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## Chapter 3

# Random matroid generation

One goal for this project is to create the Julia library `Matroids.jl`, which will supply functionality for generating and interacting with random matroids. In the preparatory project delivered fall of 2022, I implemented Knuth’s 1974 algorithm for the random generation of arbitrary matroids via the erection of closed sets [Knu75]. With this, I was able to randomly generate matroids of universe sizes  $n \leq 12$ , but for larger values of  $n$  my implementation was unbearably slow. In this chapter, Knuth’s method for random matroid construction will be described, along with the steps I have taken to speed up my initial, naïve implementation. The random generation of other specific types of matroids is discussed as well.

### 3.1 Knuth’s matroid construction (KMC)

KNUTH-MATROID (given in Algorithm 1) accepts the ground set  $E$  and a list of enlargements  $X$ , and produces the matroid  $\mathfrak{M} = (E, \mathcal{F})$ , such that for each set  $X \in \mathcal{X}[r]$ , we have  $X \in \mathcal{F}$ ,  $\text{rank}(X) = r$ . The algorithm outputs the tuple  $(E, \mathbf{F})$ , where  $\mathbf{F} = [F_0, \dots, F_r]$ ,  $r$  being the final rank of  $\mathfrak{M}$  and  $F_i$  the family of closed sets of rank  $i$ . In the paper, Knuth shows that  $\bigcup_{i=0}^r F[i] = \mathcal{F}$ , and so the algorithm produces a valid matroid represented by its closed sets.

The *covers* of a closed set  $A$  of rank  $r$  are the sets obtained by adding one more element from  $E$  to  $A$ . The algorithm proceeds in a bottom-up manner, starting with the single closed set of rank 0 (the empty set) and for each rank  $r + 1$  adds the covers of the closed sets of rank  $r$ . The covers are generated with

the helper method `GENERATE-COVERS(F, r, E)`.

```

GENERATE-COVERS(F, r, E)
1  return {A ∪ {a} : A ∈ F[r], a ∈ E \ A}

```

Given no enlargements ( $X = []$ ), the resulting matroid is the free matroid over  $E$ . Arbitrary matroids can be generated by supplying different lists  $X$ . When enlarging, the sets in  $X[r + 1]$  are simply added to  $F[r + 1]$ .

`SUPERPOSE!(F[r + 1], F[r])` ensures that the newly enlarged family of closed sets of rank  $r + 1$  is valid. If  $F_{r+1}$  contains two sets  $A, B$  whose intersection  $A \cap B \not\subseteq C$  for any  $C \in F_r$ , replace  $A, B$  with  $A \cup B$ . Repeat until no two sets exist in  $F_{r+1}$  whose intersection is not contained within some set  $C \in F_r$ .

```

SUPERPOSE!(Fr+1, Fr)
1  for A ∈ Fr+1
2  |   for B ∈ Fr+1
3  |   |   flag ← TRUE
4  |   |   for C ∈ Fr
5  |   |   |   if A ∩ B ⊆ C
6  |   |   |   |   flag ← FALSE
7  |   |
8  |   |   if flag = TRUE
9  |   |   |   Fr+1 ← Fr+1 \ {A, B}
10 |   |   |   Fr+1 ← Fr+1 ∪ {A ∪ B}

```

### 3.1.1 Randomized KMC

In the randomized version of `KNUTH-MATROID`, we generate matroids by applying a supplied number of random coarsening steps, instead of enlarging with supplied sets. This is done by applying `SUPERPOSE!` immediately after adding the covers, then choosing a random member  $A$  of  $F[r + 1]$  and a random element  $a \in E \setminus A$ , replacing  $A$  with  $A \cup \{a\}$  and finally reapplying `SUPERPOSE!`. The parameter  $p = (p_1, p_2, \dots)$  gives the number of such coarsening steps to be applied at each iteration of the algorithm.

The pseudocode given up to this point corresponds closely to the initial Julia implementation, which can be found in Appendix B.1. It should already

---

**Algorithm 1** KNUTH-MATROID( $E, X$ )

---

**Input:** The ground set of elements  $E$ , and a list of enlargements  $X$ .  
**Output:** The list of closed sets of the resulting matroid grouped by rank,  
 $F = [F_0, \dots, F_r]$ , where  $F_i$  is the set of closed sets of rank  $i$ .

```
1  $r = 0, F = [\{\emptyset\}]$ 
2 while TRUE
3    $\text{PUSH!}(F, \text{GENERATE-COVERS}(F, r, E))$ 
4    $F[r + 1] = F[r + 1] \cup X[r + 1]$ 
5    $\text{SUPERPOSE!}(F[r + 1], F[r])$ 
6   if  $E \notin F[r + 1]$ 
7      $r \leftarrow r + 1$ 
8   else
9     return  $(E, F)$ 
```

---

be clear that this brute force implementation leads to poor performance – for instance, the `SUPERPOSE!` method uses a triply nested for loop, which should be a candidate for significant improvement if possible. Section 3.1.2 describes the engineering work done to create a more performant implementation.

### 3.1.2 Improving performance

When recreating Knuth’s table of observed mean values for the randomly generated matroids, some of the latter configurations of  $n$  and  $(p_1, p_2, \dots)$  was unworkably slow, presumably due to my naïve implementation of the algorithm. Table 3.1 shows the performance of this first implementation.

The performance was measured using Julia’s `@timed`<sup>1</sup> macro, which returns the time it takes to execute a function call, how much of that time was spent in garbage collection and the number of bytes allocated. As is evident from the data, larger matroids are computationally quite demanding to compute with the current approach, and the time and space requirements scales exponentially with  $n$ . Can we do better? As it turns out, we can; after the improvements outlined in this section, we will be able to generate matroids over universes as

---

<sup>1</sup><https://docs.julialang.org/en/v1/base/base/#Base.@timed>





Table 3.1: Performance of `random_kmc_v1`.

$n$	$(p_1, p_2, \dots)$	Trials	Time	GC Time	Bytes allocated
10	(0, 6, 0)	100	0.0689663	0.0106786	147.237 MiB
10	(0, 5, 1)	100	0.1197194	0.0170734	251.144 MiB
10	(0, 5, 2)	100	0.0931822	0.0144022	203.831 MiB
10	(0, 6, 1)	100	0.0597314	0.0094902	132.460 MiB
10	(0, 4, 2)	100	0.1924601	0.0284532	406.131 MiB
10	(0, 3, 3)	100	0.3196838	0.0463972	678.206 MiB
10	(0, 0, 6)	100	1.1420602	0.1671325	2.356 GiB
10	(0, 1, 1, 1)	100	2.9283978	0.3569357	5.250 GiB
13	(0, 6, 0)	10	104.0171128	9.9214449	161.523 GiB
13	(0, 6, 2)	10	11.4881308	1.3777947	20.888 GiB
16	(6, 0, 0)	1	-	-	-


The idea is to define a family of closed sets of the same rank as `Set{UInt16}`. Using `UInt16` we can support ground sets of size up to 16. Each 16-bit number represents a set in the family. For example, the set  $\{2, 5, 7\}$  is represented by

$$164 = 0x00a4 = 0b0000000010100100 = 2^7 + 2^5 + 2^2.$$

At either end we have  $\emptyset \equiv 0x0000$  and  $E \equiv 0xffff$  (if  $n = 16$ ). The elementary set operations we will need have simple implementations using bitwise operations.

We can now describe the bitwise versions of the required methods. The bitwise implementation of `GENERATE-COVERS` finds all elements in  $E \setminus A$  by finding each value  $0 \leq i < n$  for which `A & 1 << i == 0`, meaning that the set represented by `1 << i` is not a subset of  $A$ . The bitwise implementation of `SUPERPOSE!` is unchanged apart from using the bitwise set operations described above.

Set operation	Bitwise operation
$A \cap B$	$A \text{ AND } B$
$A \cup B$	$A \text{ OR } B$
$A \setminus B$	$A \text{ AND NOT } B$
$A \subseteq B$	$A \text{ AND } B = A$

Table 3.2: Performance of random\_kmc\_v2.

$n$	$(p_1, p_2, \dots)$	Trials	Time	GC Time	Bytes allocated
10	[0, 6, 0]	100	0.0010723	0.0001252	1.998 MiB
10	[0, 5, 1]	100	0.0017543	0.0001431	3.074 MiB
10	[0, 5, 2]	100	0.0008836	0.0001075	2.072 MiB
10	[0, 6, 1]	100	0.0007294	6.73e-5	1.700 MiB
10	[0, 4, 2]	100	0.0020909	0.0001558	3.889 MiB
10	[0, 3, 3]	100	0.0024636	0.0002139	4.530 MiB
10	[0, 0, 6]	100	0.007082	0.0004801	9.314 MiB
10	[0, 1, 1, 1]	100	0.0132477	0.0008307	17.806 MiB
13	[0, 6, 0]	10	0.042543	0.0014988	31.964 MiB
13	[0, 6, 2]	10	0.0183313	0.0012176	21.062 MiB
16	[0, 6, 0]	10	1.2102877	0.0146129	450.052 MiB

The performance of random\_kmc\_v2 is shown in Table 3.2. It is clear that representing closed sets using binary numbers represents a substantial improvement – we are looking at performance increases of 100x-1000x across the board. Great stuff!

### Sorted superpose

Can we improve the running time of the algorithm further? It is clear that SUPERPOSE! takes up a large portion of the compute time. In the worst case, when no enlargements have been made,  $F_{r+1}$  is the set of all  $r+1$ -sized subsets of  $E$ ,  $|F_{r+1}| = \binom{n}{r+1}$ . Comparing each  $A, B \in F_{r+1}$  with each  $C \in F_r$  in a triply nested for loop requires  $\mathcal{O}(\binom{n}{r+1}^2 \binom{n}{r})$  operations. In the worst case, no

enlargements are made at all, and we build the free matroid in  $\mathcal{O}(2^{3n})$  time (considering only the superpose step).

After larger closed sets have been added to  $F[r+1]$ , SUPERPOSE! will cause sets to merge, so that only maximal dependent sets remain. Some sets will even simply disappear. In the case where  $X = \{1, 2\}$  was added by GENERATE-COVERS, and the  $Y = \{1, 2, 3\}$  was added manually as an enlargement, the smaller set will be fully subsumed in the bigger set, as  $\{1, 2\} \cap \{1, 2, 3\} = \{1, 2\}$  (which is not a subset of any set in  $F[r]$ ) and  $\{1, 2\} \cup \{1, 2, 3\} = \{1, 2, 3\}$ . In this situation,  $Y$  would “eat” the covers  $\{1, 3\}$  and  $\{2, 3\}$  as well. This fact is reflected in the performance data – notice how much more space is demanded by the 10-element matroid with  $p = [0, 0, 6]$  than the one with  $p = [0, 6, 0]$  in any of the performance tables in this section. Making enlargements at earlier ranks will result in smaller matroids as more sets will get absorbed.

```
function sorted_bitwise_superpose!(F, F_prev)
    As = sort!(collect(F), by = s -> length(bits_to_set(s)))
    while length(As) != 0
        A = popfirst!(As)

        for B in setdiff(F, A)
            if should_merge(A, B, F_prev)
                insert!(As, 1, A | B)
                setdiff!(F, [A, B])
                push!(F, A | B)
                break
            end
        end
    end
    return F
end
```

Since the larger sets will absorb so many of the smaller sets (around  $\binom{p}{r+1}$ , where  $p$  is the size of the larger set and  $r+1$  is the size of the smallest sets allowed to be added in a given iteration), might it be an idea to perform the superpose operation in descending order based on the size of the sets? This should result in fewer calls to SUPERPOSE!, as the bigger sets will remove the smaller sets that fully overlap with them in the early iterations, however, the repeated sorting of the sets might negate this performance gain. This is the idea behind `sorted_bitwise_superpose!`, which was used in `random_kmc_v3`. The full code can be found in Appendix B.2.

Unfortunately, as Table 3.3 shows, this implementation is a few times slower and more space demanding than the previous implementation. This is might be

due to the fact that an ordered list is more space inefficient than the hashmap-based `Set`.

Table 3.3: Performance of `random_kmc_v3`.

$n$	$(p_1, p_2, \dots)$	Trials	Time	GC Time	Bytes allocated
10	[0, 6, 0]	100	0.0023382	0.0001494	4.042 MiB
10	[0, 5, 1]	100	0.001853	0.0001433	4.383 MiB
10	[0, 5, 2]	100	0.0017845	0.0001341	4.043 MiB
10	[0, 6, 1]	100	0.0015145	0.0001117	3.397 MiB
10	[0, 4, 2]	100	0.0030704	0.0002125	6.385 MiB
10	[0, 3, 3]	100	0.0037838	0.0002514	7.018 MiB
10	[0, 0, 6]	100	0.008903	0.000557	14.159 MiB
10	[0, 1, 1, 1]	100	0.0142828	0.0008823	21.838 MiB
13	[0, 6, 0]	10	0.0627633	0.002094	51.492 MiB
13	[0, 6, 2]	10	0.0106478	0.0007704	20.774 MiB
16	[0, 6, 0]	10	0.6070136	0.0095656	310.183 MiB

### Iterative superpose

The worst-case  $\mathcal{O}(\binom{n}{r+1}^2 \binom{n}{r})$  runtime of SUPERPOSE! at step  $r$  is due to the fact that it takes in  $F$  after all covers and enlargements have been indiscriminately added to  $F[r+1]$  and then loops through to perform the superposition. Might there be something to gain by inserting new closed sets into the current family one at a time, and superposing on the fly?

```

# Superpose (random_kmc_v4)
push!(F, Set()) # Add F[r+1].
while length(to_insert) > 0
    A = pop!(to_insert)
    push!(F[r+1], A)

    for B in setdiff(F[r+1], A)
        if should_merge(A, B, F[r])
            push!(to_insert, A | B)
            setdiff!(F[r+1], [A, B])
            push!(F[r+1], A | B)
        end
    end
end
end
end

```

In `random_kmc_v4`, the full code of which can be found in Appendix B.3, the covers and enlargements are not added directly to  $F[r+1]$ , but to a temporary array `to_insert`. Each set  $A$  is then popped from `to_insert` one at a time, added to  $F[r+1]$  and compared with the other sets  $B \in F[r+1] \setminus \{A\}$  and  $C \in F[r]$  in the usual SUPERPOSE! manner. This results in fewer comparisons, as each set is only compared with the sets added before it; the first set is compared with no other sets, the second set with one other and the sets in  $F[r]$ , and so on. The number of such comparisons is therefore given by the triangular number  $T_{\binom{n}{r+1}}$ , and so we should have roughly halved the runtime at step  $r$ . It is worth noting that this implementation of SUPERPOSE! uses a subroutine `should_merge` that returns early when it finds one set  $C \in F[r]$  such that  $C \supseteq A \cap B$ , so in practice it usually does not require  $\binom{n}{r}$  comparisons in the innermost loop.

Table 3.4 shows that the iterative superpose was a meaningful improvement. For most input configurations, it is a few times faster and a few times less space demanding than `random_kmc_v2`.

## Rank table

While SUPERPOSE! is getting more efficient, it is still performing the same comparisons over and over again. Let's consider what we are really trying to achieve with this function, to see if we can't find a smarter way to go about it.

After adding the closed sets for a rank, SUPERPOSE! is run to maintain the closed set properties of the matroid (given in Section 2.1). These are maintained by ensuring that, for any two newly added sets  $A, B \in F[r+1]$ , there exists  $C \in F[r]$  such that  $A \cap B \subseteq C$ . Until this point, this has been done by checking if the intersection of each such  $A, B$  is contained in a set  $C$  of rank  $r$ . We

Table 3.4: Performance of random\_kmc\_v4.

$n$	$(p_1, p_2, \dots)$	Trials	Time	GC Time	Bytes allocated
10	[0, 6, 0]	100	0.0014585	3.94e-5	724.635 KiB
10	[0, 5, 1]	100	0.0007192	9.39e-5	659.729 KiB
10	[0, 5, 2]	100	0.0005943	3.53e-5	617.668 KiB
10	[0, 6, 1]	100	0.0003502	2.88e-5	408.666 KiB
10	[0, 4, 2]	100	0.001013	5.36e-5	887.618 KiB
10	[0, 3, 3]	100	0.0011847	5.03e-5	1.003 MiB
10	[0, 0, 6]	100	0.0015756	9.7e-5	1.066 MiB
10	[0, 1, 1, 1]	100	0.0046692	0.0001385	2.455 MiB
13	[0, 6, 0]	10	0.0118201	0.0005486	6.289 MiB
13	[0, 6, 2]	10	0.0075668	0.0002458	4.666 MiB
16	[0, 6, 0]	10	0.2819294	0.0040792	81.317 MiB
16	[0, 6, 1]	10	0.8268207	0.0070206	154.451 MiB
16	[0, 0, 6]	10	95.1959596	0.0290183	553.597 MiB

remember that one of the properties of the closed sets of a matroid is that the intersection of two closed sets is itself a closed set. Therefore, we do not need to find a closed set  $C$  that *contains*  $A \cap B$ , since if  $A$  and  $B$  are indeed closed sets, their intersection will be *equal* to some closed set  $C$  of lesser rank. This insight leads us to the next improvement: if we keep track of all added closed sets in a rank table, then we can memoize SUPERPOSE! and replace the innermost loop with a constant time dictionary lookup.

```
# The rank table maps from the representation of a set to its assigned rank.
rank = Dict{T, UInt8}()

[...]

# Superpose.
push!(F, Set()) # Add F[r+1].
while length(to_insert) > 0
    A = pop!(to_insert)
    push!(F[r+1], A)
    rank[A] = r
```

```

for B in setdiff(F[r+1], A)
  if !haskey(rank, A&B) || rank[A&B] >= r
    # Update insert queue.
    push!(to_insert, A | B)

    # Update F[r+1].
    setdiff!(F[r+1], [A, B])
    push!(F[r+1], A | B)

    # Update rank table.
    rank[A|B] = r
    break
  end
end
end
end

```

Table 3.5: Performance of random\_kmc\_v5.

$n$	$(p_1, p_2, \dots)$	Trials	Time	GC Time	Bytes allocated
10	[0, 6, 0]	100	0.0001335	0.0	138.966 KiB
10	[0, 5, 1]	100	0.0001436	0.0	158.691 KiB
10	[0, 5, 2]	100	0.0001928	0.0	167.487 KiB
10	[0, 6, 1]	100	0.0002204	0.0	148.812 KiB
10	[0, 4, 2]	100	0.0001578	0.0	173.455 KiB
10	[0, 3, 3]	100	0.0001743	0.0	202.566 KiB
10	[0, 0, 6]	100	0.0003433	0.0	431.089 KiB
10	[0, 1, 1, 1]	100	0.0004987	0.0	439.511 KiB
13	[0, 6, 0]	100	0.0004776	0.0	422.431 KiB
13	[0, 6, 2]	100	0.0003469	0.0	441.621 KiB
16	[0, 6, 0]	100	0.0009073	0.0	1010.452 KiB
16	[0, 6, 1]	100	0.0007939	0.0	997.022 KiB
16	[0, 0, 6]	100	0.0066951	0.0	8.564 MiB
20	[0, 6, 0]	100	0.0030797	0.0	4.042 MiB
20	[0, 6, 2]	10	0.0022849	0.0	4.547 MiB
32	[0, 6, 2, 1]	10	0.0269912	0.0	63.082 MiB

The full code for `random_kmc_v5` can be found in Appendix B.4. Table 3.5 shows that implementing a rank table was an extremely significant improvement. For

smaller matroids, it is around 5-10x faster, however it is for larger matroids that it truly outshines its predecessors – `random_kmc_v5` is a whopping 13 000 times faster than `random_kmc_v4` with  $n = 16, p = [0, 0, 6]$  as input.

### Non-redundant cover generation

Up to this point, our cover generation routine has not taken into account that any two sets of rank  $r$  will have at least one cover in common. To see this, consider a matroid-under-construction with  $n = 10$  where  $A = \{1, 2\}$  and  $B = \{1, 3\}$  are closed sets of rank 2. Currently, `GENERATE-COVERS` will happily generate the cover  $C = \{1, 2, 3\}$  twice, once as the cover of  $A$  and subsequently as the cover of  $B$ . Throughout this analysis, we will assume the worst case scenario of no enlargements, as any enlargements will strictly lower the number of sets in play at a given rank. In this case,  $|F[r]| = \binom{n}{r}$ , and for each closed set  $A$  of rank  $r$  we are generating  $|E \setminus A| = (n - r)$  covers, giving us a total of  $\binom{n}{r}(n - r)$  covers generated at each rank  $r$ , including the duplicates. With no enlargements, we know that there are  $\binom{n}{r+1}$  covers, and

$$\begin{aligned} (n - r) \binom{n}{r} &= \frac{n!(n - r)}{r!(n - r)!} \\ &= \frac{n!}{r!(n - r - 1)!} \\ &= (r + 1) \frac{n!}{(r + 1)!(n - r - 1)!} \\ &= (r + 1) \binom{n}{r + 1}. \end{aligned}$$

For each step  $r$ , we are generating  $r + 1$  times as many covers as we need to. Over the course of all steps  $0 \leq r \leq n$ , we are generating

$$\sum_{r=0}^n (r + 1) = \sum_{r=1}^{n+1} r = T_{n+1}$$

times the actual number of covers, where  $T_{n+1} = \frac{(n+1)(n+2)}{2}$  is the triangular number. In other words, if we find a way to generate each cover only once, we will have shaved off an  $n^2$  factor from the asymptotic complexity of our implementation.



When generating covers, `random_kmc_v6` improves upon the brute force cover generation described above by only adding the covers

$$\left\{ A \cup \{a\} : A \in F[r], a \in E \setminus A, a \notin \bigcup \{B : B \in F[r+1], A \subseteq B\} \right\}.$$

In other words, we find the covers of  $A$ , that is, the sets obtained by adding one more element  $a$  from  $E$  to  $A$ , but we do not include any  $a$  that is to be found in another, already added, cover  $B$  that contains  $A$ . This solves the problem described above; the cover  $\{1, 2, 3\} = B \cup \{2\}$  will not be generated, as  $2 \in C$  and  $B \subseteq C$ . This is implemented in the following manner:

```
# Generate minimal closed sets for rank r+1 (random_kmc_v6)
for y in F[r] # y is a closed set of rank r.
  t = E - y # The set of elements not in y.
  # Find all sets in F[r+1] that already contain y and remove excess elements
  from t.
  for x in F[r+1]
    if (x & y == y) t &= ~x end
    if t == 0 break end
  end
  # Insert y ∪ a for all a ∈ t.
  while t > 0
    x = y | (t & -t)
    add_set!(x, F, r, rank)
    t &= ~x
  end
end
end
```

We have extracted the iterative superpose logic described above into its own function to allow it to be performed on a cover-per-cover basis:

```
function add_set!(x, F, r, rank)
  if x in F[r+1] return end
  for y in F[r+1]
    if haskey(rank, x&y) && rank[x&y] < r
      continue
    end

    # x ∩ y has rank > r, replace with x ∪ y.
    setdiff!(F[r+1], y)
    return add_set!(x|y, F, r, rank)
  end

  push!(F[r+1], x)
  rank[x] = r
end
```

As such,  $F[r+1]$  is empty when the first cover  $y \in F$  is generated, and all covers  $\{y \cup \{a\} : a \in E \setminus y\}$  are added. For later sets  $y$ , we are comparing with the previously added covers, and dropping any element to be found in a cover  $x$  that fully includes  $y$ . This way, we avoid re-generating the cover  $x$ .

The full code for `random_kmc_v6` can be found in Appendix B.5.

Table 3.6: Performance of `random_kmc_v6`.

$n$	$(p_1, p_2, \dots)$	Trials	Time	GC Time	Bytes allocated
10	[0, 6, 0]	100	0.000157	0.0	11.306 KiB
10	[0, 5, 1]	100	0.0001427	0.0	12.257 KiB
10	[0, 5, 2]	100	0.000121	0.0	11.568 KiB
10	[0, 6, 1]	100	8.61e-5	0.0	10.447 KiB
10	[0, 4, 2]	100	0.0001237	0.0	13.597 KiB
10	[0, 3, 3]	100	0.0001233	0.0	14.029 KiB
10	[0, 0, 6]	100	0.0002856	0.0	15.414 KiB
10	[0, 1, 1, 1]	100	0.0001942	0.0	14.446 KiB
13	[0, 6, 0]	100	0.0004483	0.0	19.117 KiB
13	[0, 6, 2]	100	0.0004541	0.0	18.957 KiB
16	[0, 6, 0]	10	0.0014919	0.0	34.531 KiB
16	[0, 6, 1]	10	0.0014731	0.0	36.016 KiB
16	[0, 0, 6]	10	0.0168858	0.0	127.652 KiB
20	[0, 6, 0]	10	0.0061574	0.0	81.573 KiB
20	[0, 6, 2]	10	0.0059717	0.0	82.323 KiB
32	[0, 6, 2, 1]	10	0.1599507	0.0	279.531 KiB
63	[0, 6, 4, 2, 1]	1	11.138914	0.0	4.912 MiB
64	[0, 6, 4, 4, 2, 1]	1	12.508729	0.0	4.912 MiB
128	[0, 6, 6, 4, 4, 2, 1]	1	1232.8570	0.0114583	102.159 MiB




[REDACTED]

### 3.1.3 What do the generated matroids look like?

#### Observations

1. The average cardinality of the closed sets of a given rank is usually not very much higher than the rank. If the average cardinality were to stray much, all the sets merge instead and the sole closed set of that rank would become  $E$ . This might be a useful heuristic when finding the rank of a set.

[REDACTED]



### 3.1.4 Producing the rank function and independent sets

To build a general matroid library, we want to be able to access all properties of a generated matroid  $\mathfrak{M}$ . This would include:

1. the bases  $\mathcal{B}$  of  $\mathfrak{M}$ ,
2. the independent sets  $\mathcal{I}$  of  $\mathfrak{M}$ ,
3. the circuits  $\mathcal{C}$  of  $\mathfrak{M}$ ,
4. the closure function  $cl : 2^E \rightarrow \mathcal{F}$ , and
5. the rank function  $rank : 2^E \rightarrow \mathbb{Z}^*$  of  $\mathfrak{M}$ .

In this section, I will first describe an extension of KNUTH-MATROID that is also fully enumerates  $\mathcal{I}$  and  $\mathcal{C}$  for  $\mathfrak{M}$  when  $n$  is small enough. However, this approach does not scale well for larger values of  $n$ . For values of  $n$  up to 128, we will therefore restrict our attention to independent sets and the rank function, as these are the matroid properties that are relevant to our usecase of fair allocation.

#### Finding circuits and independent sets for smaller matroids

[Knu75] includes an ALGOL W [WH66] implementation that also generates the circuits and independent sets for the generated matroid. A later implementation in C called ERECTION.W can be found at [Knu03]. `random_erection` is an extension of `random_kmc_v6` that finds  $\mathcal{I}$  and  $\mathcal{C}$  by pre-populating the rank table with all subsets of  $E$ . The full source code for `random_erection` can be found in Appendix XXX.

```
# Populate rank table with 100+cardinality for all subsets of E.
k=1; rank[0]=100;
while (k<=mask)
  for i in 0:k-1 rank[k+i] = rank[i]+1 end
  k=k+k;
end
```

Covers are generated and sets inserted in the same manner as in `random_kmc_v6`. After all covers and enlargements have been inserted and superposed (meaning  $F[r+1]$  contains the closed sets of rank  $r+1$ ), a new operation, `mark_independent_subsets!` is called on each closed set.



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A general

### 3.2 Other kinds of matroids

[Redacted text block]

#### 3.2.1 Uniform matroids

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[Redacted text block]

[Redacted text block]

3.2.2 Graphic matroids

[Redacted text block]

[Redacted text block]



[REDACTED]

**3.2.3 Vector matroids**

[REDACTED]

[REDACTED]

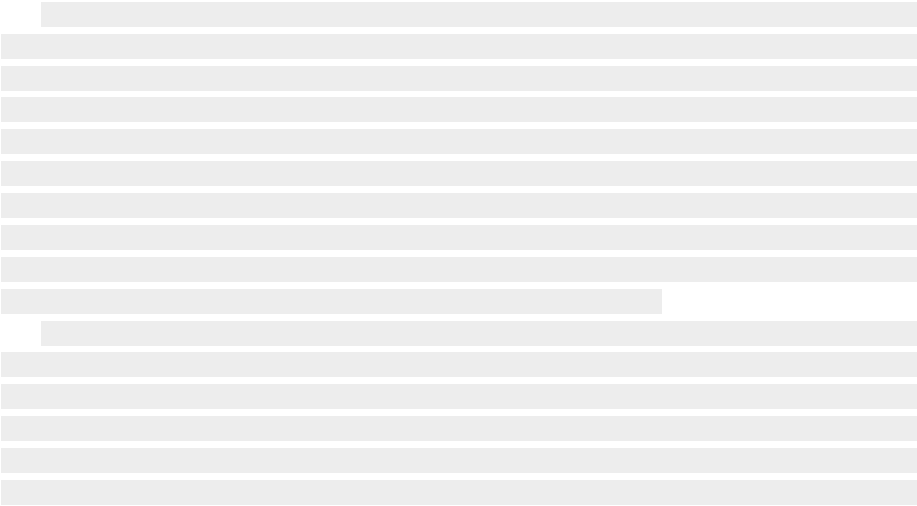
[REDACTED]



# Chapter 4

## A library for fair allocation with matroids

A goal for this project is to introduce Matroids.jl as a useful library for experimenting with fair allocation that require matroids. In the previous chapter, we explored how this library generates random matroids, however this is not very useful until we also have in place an API layer to allow fair allocation algorithms to interface with our matroids in a practical and efficient manner.





# 4.1 The matroid union algorithm

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## 4.2 Supporting universe sizes of $n > 128$

The larger the ground set, the closer we are to an instance of The cake-cutting problem. Typical fair allocation problems with indivisible items deal with less



3

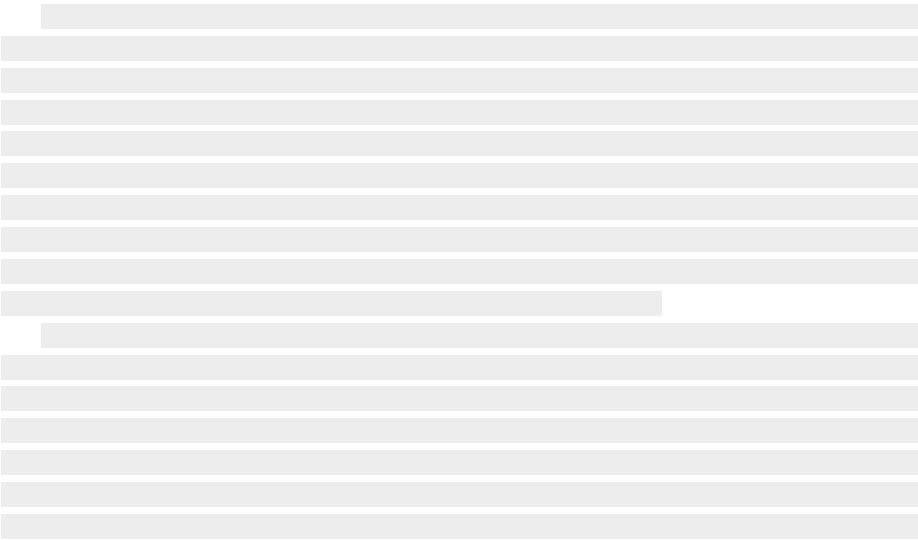
4

# Chapter 5

## Fair allocation with matroid rank utilities

We wish to study fair allocation in which the utility function of each agent is a matroid rank function. That is, the utility function for each agent  $i \in [n]$  is the rank function for a matroid  $\mathfrak{M}_i = ([m], \mathcal{I}_i)$ .

$$S_i \in \mathcal{I}_i \iff v_i(S_i) = |S_i|.$$



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5.1 Yankee Swap

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[Redacted text block]

**5.2 Barman and Verma’s MMS algorithm**

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# Chapter 6

## Fair allocation under matroid constraints



6.1 Gourvès and Monnot’s MMS approximation algorithm

[Redacted content]





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**6.2 Biswas and Barman’s algorithm for EF1 under cardinality constraints**

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### 6.3 Biswas and Barman’s algorithm for EF1 under matroid constraints

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[REDACTED]

# Chapter 7

## Results

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# Chapter 8

## Conclusions and future work

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# Notes

1. Skrive mer om hvordan  $\text{Set}\{\text{Set}\{\text{Integer}\}\}$  lagres i minnet og fordelene med å gå over til  $\text{Set}\{\text{Integer}\}$ .
2. @Benchmarking. Histogrammer. Beskrive variansen i matroide-størrelse ifht input.
3. Referer til Spliddit og vanlige størrelser på fordelingsproblemer
4. Beskriv åssen man kan oppgi valgfri Integer-type

# Bibliography

- [Ben21] et al. Benabbou, Nawal. Finding fair and efficient allocations for matroid rank valuations. *ACM Transactions on Economics and Computation*, 9:1–41, December 2021.
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- [WH66] Niklaus Wirth and C. A. R. Hoare. A contribution to the development of ALGOL. *Commun. ACM*, 9(6):413–432, jun 1966.
- [Whi35] Hassler Whitney. On the abstract properties of linear dependence. *American Journal of Mathematics*, 57:509–533, July 1935.

# Appendices

## Appendix A

# Some useful tricks when working with sets as numbers

Section 3.1.2 details a number of steps taken in order to build a performant Julia implementation of KNUTH-MATROID. Perhaps chief among these steps in terms of sheer performance gain compared to the initial, naïve implementation, was the transition from representing subsets of  $E$  as a `Set` of integers (or whatever type the elements of  $E$  might have), to representing them as a single integer, whose 1-bits denote which elements are in the set. This is possible as long as  $n$  is less than the widest numeric primitive (in out-of-the-box Julia, 128 bits). For simplicity, we reiterate the bitwise equivalents of the basic set operations: These bitwise

Set operation	Bitwise equivalent
$A \cap B$	$A \wedge B$
$A \cup B$	$A \vee B$
$A \setminus B$	$A \wedge \neg B$
$A \subseteq B$	$A \wedge B = A$

equivalents allow us to perform the set operations in constant time (right???), resulting in significant performance increases. In the code snippets included throughout Section 3.1.2 and Appendix B, a number of “tricks” are performed with bitwise operations whose workings and purpose might be a bit obtuse. This appendix came to be as I worked to get to grips with working with sets in this

manner.

## How do I...

### ...create a singleton set?

The left-shift operator ( $\ll$ ) can be used to set the  $i$ th bit to 1 and the others to 0. In general,  $\{a\} = 1 \ll a$ . This is used in an early version of GENERATE-COVERS:

```
function generate_covers_v2(F_r, n)
  Set([A | 1 << i for A ∈ F_r for i in 0:n-1 if A & 1 << i == 0])
end
```

### ...find the smallest element of a set?

Using its two's complement, denoted by  $-T = \neg T + 1$ , we can find the smallest element in a set  $T$  with  $T \wedge -T$ .

### ...enumerate all elements of a set one by one?

We can pop the smallest element from a set, using the previous trick to get the smallest element, in the following manner:



# Appendix B

## Code snippets

### B.1 random\_kmc\_v1

```
"""
Generate the set  $F_{\{r+1\}}$  of all "covers" of the sets in  $F_r$ , given the ground
set of elements  $E$ . Set-based.
"""
function generate_covers_v1(Fr, E)
    Set([A ∪ a for A ∈ Fr for a ∈ setdiff(E, A)])
end

"""
If  $F_{\{r+1\}}$  contains any two sets  $A, B$  whose intersection  $A \cap B$  is not
contained in  $C$  for any  $C \in F_r$ , replace  $A, B \in F_{\{r+1\}}$  by the single set
 $A \cup B$ . Repeat this operation until  $A \cap B \subseteq C$  for some  $C \in F_r$  whenever  $A$ 
and  $B$  are distinct members of  $F_{\{r+1\}}$ .

 $F$  and  $F_{\text{old}}$  should be Family: A Set of Sets of some type.

This is the first, Set-based implementation of this method.
"""
function superpose_v1!(F, F_old)
    for A ∈ F
        for B ∈ F
            should_merge = true
            for C ∈ F_old
                if  $A \cap B \subseteq C$ 
                    should_merge = false
                end
            end
            if should_merge
                A ∪ B
            end
        end
    end
end
```

```

        if should_merge
            setdiff!(F, [A, B])
            push!(F, A ∪ B)
        end
    end
end

return F
end

"""
First implementation of Knuth's random matroid construction through random "
coarsening".

n is the size of the universe.
p is a list (p_1, p_2, ...), where p_r is the number of coarsening steps to
apply at rank r in the construction. The first entry of p should usually be
0, since adding closed sets of size > 1 at rank 1 is equivalent to
shrinking E.

This uses the Set-based KMC methods.
"""
function random_kmc_v1(n, p, T)::KnuthMatroid{Set{Integer}}
    E = Set{Integer}([i for i in range(0, n-1)])

    # Step 1: Initialize.
    r = 1
    F = [family([])]
    pr = 0

    while true
        # Step 2: Generate covers.
        push!(F, generate_covers_v1(F[r], E))

        # Step 4: Superpose.
        superpose_v1!(F[r+1], F[r])

        # Step 5: Test for completion.
        if E ∈ F[r+1]
            return KnuthMatroid{Set{Integer}}(n, F, [], Set{Integer}(), Dict{Integer, Integer}())
        end

        if r ≤ length(p)
            pr = p[r]
        end

        while pr > 0
            # Random closed set in F_{r+1} and element in E \ A.
            A = rand(F[r+1])
            a = rand(setdiff(E, A))

            # Replace A with A ∪ {a}.
            F[r+1] = setdiff(F[r+1], A) ∪ Set{Integer}([A ∪ a])

            # Superpose again to account for coarsening step.
            superpose_v1!(F[r+1], F[r])
        end
    end
end

```

```

    # Step 5: Test for completion.
    if E ∈ F[r+1]
        return KnuthMatroid(Set{Integer})(n, F, [], Set(), Dict())
    end

    pr -= 1
end

r += 1
end
end

```

## B.2 random\_kmc\_v2 and random\_kmc\_v3

```

"""
Generate the set F_{r+1} of all "covers" of the sets in F_r, given the size of
the universe. Bit-based.
"""
function generate_covers_v2(F_r, n)
    Set([A | 1 << i for A ∈ F_r for i in 0:n-1 if A & 1 << i == 0])
end

"""
Returns whether the intersection of A and B is contained within
some C in F_prev.
"""
function should_merge(A, B, F_prev)
    for C in F_prev
        if subseq(A & B, C)
            return false
        end
    end
    return true
end

"""
If F contains any two sets A, B whose intersection A ∩ B is not contained in C
for any C ∈ F_prev, replace A, B ∈ F with the single set A ∪ B. Repeat this
operation until A ∩ B ⊆ C for some C ∈ F_prev whenever A and B are
distinct members of F.

This implementation represents the sets using bits.
"""
function bitwise_superpose!(F, F_prev)
    As = copy(F)
    while length(As) != 0
        A = pop!(As)
        for B in setdiff(F, A)

```

```

        if should_merge(A, B, F_prev)
            push!(As, A | B)
            setdiff!(F, [A, B])
            push!(F, A | B)
            break
        end
    end
end

return F
end

function sorted_bitwise_superpose!(F, F_prev)
    As = sort!(collect(F), by = s -> length(bits_to_set(s)))
    while length(As) != 0
        A = popfirst!(As)

        for B in setdiff(F, A)
            if should_merge(A, B, F_prev)
                insert!(As, 1, A | B)
                setdiff!(F, [A, B])
                push!(F, A | B)
                break
            end
        end
    end

    return F
end

"""
Bitwise implementation of Knuth's approach to random matroid generation through
a number of random "coarsening" steps. Supply the generate_covers and
superpose methods to study the effects of different implementations of
these.

n is the size of the universe.
p is a list (p_1, p_2, ...), where p_r is the number of coarsening steps to
apply at rank r in the construction. The first entry of p should usually be
0, since adding closed sets of size > 1 at rank 1 is equivalent to
shrinking E.
"""
function random_bitwise_kmc(generate_covers, superpose, n, p)::KnuthMatroid{Any}
    # Initialize.
    r = 1
    pr = 0
    F = [Set{0}]
    E = 2^n - 1 # The set of all elements in E.

    while true
        # Generate covers.
        push!(F, generate_covers(F[r], n))

        # Superpose.
        superpose(F[r+1], F[r])

        # Test for completion.
        if E ∈ F[r+1]

```

```

        return KnuthMatroid{Any}(n, F, [], Set(), Dict())
    end

    # Apply coarsening.
    if r <= length(p)
        pr = p[r]
    end

    while pr > 0
        # Get random closed set A in F_{r+1} and element a in E - A.
        A = rand(F[r+1])
        a = random_element(E ~ A)

        # Replace A with A ∪ {a}.
        F[r+1] = setdiff(F[r+1], A) ∪ Set([A | a])

        # Superpose again to account for coarsening step.
        superpose(F[r+1], F[r])

        # Step 5: Test for completion.
        if E ∈ F[r+1]
            return KnuthMatroid{Any}(n, F, [], Set(), Dict())
        end

        pr -= 1
    end

    r += 1
end

"""
Second implementation of random-KMC. This uses the bit-based KMC methods.
"""
function random_kmc_v2(n, p, T=UInt16)
    return random_bitwise_kmc(generate_covers_v2, bitwise_superpose!, n, p)
end

"""
Third implementation of random-KMC. This sorts the sets by size before
superposing.
"""
function random_kmc_v3(n, p, T=UInt16)
    return random_bitwise_kmc(generate_covers_v2, sorted_bitwise_superpose!, n, p)
end

```

## B.3 random\_kmc\_v4

```

"""
This is an attempt at a smarter implementation than directly following the setup
from Knuth's 1974 article. The superpose step is replaced by an insert

```

```

        operation that inserts new closed sets into the family of current rank one
        at a time, superposing on the fly.
"""
function randomized_knuth_matroid_construction_v4(n, p, T=UInt16)::KnuthMatroid{
    T}
    r = 1
    pr = 0
    F = [Set{0}]
    E = 2^n - 1 # The set of all elements in E.

    while true
        to_insert = generate_covers_v2(F[r], n)

        # Apply coarsening to covers.
        if r <= length(p) && E ∉ to_insert # No need to coarsen if E is added.
            pr = p[r]
            while pr > 0
                A = rand(to_insert)
                a = random_element(E - A)
                to_insert = setdiff(to_insert, A) ∪ [A | a]
                pr -= 1
            end
        end

        # Superpose.
        push!(F, Set{0}) # Add F[r+1].
        while length(to_insert) > 0
            A = pop!(to_insert)
            push!(F[r+1], A)

            for B in setdiff(F[r+1], A)
                if should_merge(A, B, F[r])
                    push!(to_insert, A | B)
                    setdiff!(F[r+1], [A, B])
                    push!(F[r+1], A | B)
                end
            end
        end

        if E ∈ F[r+1]
            return KnuthMatroid{T}(n, F, [], Set{0}, Dict{0})
        end

        r += 1
    end
end

```

## B.4 random\_kmc\_v5

```

"""

```

```

Fifth implementation of random-KMC. This one uses a dictionary to keep track of
previously seen sets.
"""
function random_kmc_v5(n, p, T=UInt16)::KnuthMatroid{T}
    r = 1
    pr = 0
    F::Vector{Set{T}} = [Set{T}(0)]
    E = 2^n-1
    rank = Dict{T, UInt8}(0=>0) # The rank table maps from the representation of a
                                # set to its assigned rank.

    while true
        to_insert = generate_covers_v2(F[r], n)

        # Apply coarsening to covers.
        if r <= length(p)
            pr = p[r]
            while length(to_insert) > 0 && pr > 0 && E ∉ to_insert # No need to
                coarsen if E is added.
                A = rand(to_insert)
                a = random_element(E - A)
                to_insert = setdiff(to_insert, A) ∪ [A | a]
                pr -= 1
            end
        end

        # Superpose.
        push!(F, Set{()}) # Add F[r+1].
        while length(to_insert) > 0
            A = pop!(to_insert)
            push!(F[r+1], A)
            rank[A] = r

            for B in setdiff(F[r+1], A)
                if !haskey(rank, A&B) || rank[A&B] >= r
                    # Update insert queue.
                    push!(to_insert, A | B)

                    # Update F[r+1].
                    setdiff!(F[r+1], [A, B])
                    push!(F[r+1], A | B)

                    # Update rank table.
                    rank[A|B] = r
                    break
                end
            end
        end

        if E ∈ F[r+1]
            return KnuthMatroid{T}(n, F, [], Set{()}, rank)
        end

        r += 1
    end
end
end

```

## B.5 random\_kmc\_v6

```
"""
Sixth implementation of random-KMC, in which a rank table is used to keep track
of set ranks, and the covers and enlargements are added one at a time,
ensuring the matroid properties at all times.
"""
function random_kmc_v6(n, p, T=UInt16)::KnuthMatroid{T}
    r = 1
    pr = 0
    F::Vector{Set{T}} = [Set{T(0)}]
    E::T = BigInt(2)^n-1
    rank = Dict{T, UInt8}(0==>0)

    while E ∉ F[r]
        # Create empty set.
        push!(F, Set{()})

        # Generate minimal closed sets for rank r+1.
        for y in F[r] # y is a closed set of rank r.
            t = E - y # The set of elements not in y.
            # Find all sets in F[r+1] that already contain y and remove excess
            # elements from t.
            for x in F[r+1]
                if (x & y == y) t &= ~x end
            end
            # Insert y ∪ a for all a ∈ t.
            while t > 0
                x = y | (t&-t)
                add_set!(x, F, r, rank)
                t &= ~x
            end
        end

        if E ∈ F[r+1]
            break
        end

        if r <= length(p)
            # Apply coarsening.
            pr = p[r]
            while pr > 0 && E ∉ F[r+1]
                A = rand(F[r+1])
                t = E-A
                one_element_added::Vector{T} = []
                while t > 0
                    x = A | (t&-t)
                    push!(one_element_added, x)
                    t &= ~x
                end
                Acupa = rand(one_element_added)
                setdiff!(F[r+1], A)
                add_set!(Acupa, F, r, rank)
                pr -= 1
            end
        end
    end
end
```



```

    end

    r += 1
end

return KnuthMatroid{T}(n, F, [], Set(), rank)
end

function add_set!(x, F, r, rank)
    if x in F[r+1] return end
    for y in F[r+1]
        if haskey(rank, x&y) && rank[x&y]<r
            continue
        end

        #  $x \cap y$  has rank > r, replace with  $x \cup y$ .
        setdiff!(F[r+1], y)
        return add_set!(x|y, F, r, rank)
    end

    push!(F[r+1], x)
    rank[x] = r
end

```