Appendix

Proof of the bound of $T(m)$

Suppose $T(m)$ satisfies the following recurrence relations.

$$T(m) \leq \begin{cases} c_1 \text{ if } m = 0, \\ 2c_2|V|^2 + \sum_{i=0}^{p} T(m-1-i) + T(m-d-1) \text{ otherwise.} \end{cases} \tag{1}$$

where $p = k-1$ and $d = k$. In the worst-case, we replace p by $k-1$ and d by k, thus Recurrence Relation 1 is rewrote as

$$T(m) \leq \begin{cases} c_1 \text{ if } m = 0, \\ 2c_2|V|^2 + \sum_{i=0}^{k} T(m-1-i) \text{ otherwise.} \end{cases} \tag{2}$$

Lemma 1. *Let $H(m) = \alpha_k \gamma_k^m - \beta_k$ where γ_k is the largest real root of $x^{k+2} - 2x^{k+1} + 1 = 0$, $\beta_k = \frac{2c_2|V|^2}{k}$ and $\alpha_k = max_{i=1}^{k+1}\{\frac{(2^{i-1}k+1)\beta_k + 2^{i-1}c_1}{\gamma_k^i}\}$, $T(m)$ satisfies Inequality (1). Then, $T(m) \leq H(m)$ for $m \geq 0$.*

Proof. We first show that for $m = 1, \ldots, k+1$, $T(m) \leq 2^{m-1}k\beta_k + 2^{m-1}c_1$ by induction on m.

- Basis: for $m = 1$, $T(1) \leq 2c_2|V|^2 + T(0) \leq k\beta_k + c_1$
- Induction step: We assume that $T(m) \leq 2^{m-1}k\beta_k + 2^{m-1}c_1$ for all $1 \leq m \leq R$ where $1 \leq R \leq k$ and prove that it also holds for $m = R+1$.
  By Recurrence Relation 2, we have

$$\begin{aligned} T(m) &\leq 2c_2|V|^2 + T(m-1) + ... + T(0) \quad \because m \leq k+1 \\ &= 2c_2|V|^2 + c_1 + \sum_{i=1}^{m-1} T(i) \\ &\leq 2c_2|V|^2 + c_1 + \sum_{i=1}^{m-1} (2^{i-1}k\beta_k + 2^{i-1}c_1) \\ &= k\beta_k(1 + \sum_{i=1}^{m-1} 2^{i-1}) + c_1(\sum_{i=1}^{m-1} 2^{i-1} + 1) \\ &= 2^m k\beta_k + 2^m c_1 \end{aligned}$$

Then, we prove that $T(m) \leq H(m)$ for any $m \geq 0$, again, by induction on m.

- Basis: For $m = 1, \ldots, k+1$, we have

$$\begin{aligned} H(m) &= \alpha_k \gamma_k^m - \beta_k \\ &\geq \frac{(2^{m-1}k+1)\beta_k + 2^{m-1}c_1}{\gamma_k^m} \gamma_k^m - \beta_k \\ &= 2^{m-1}k\beta_k + 2^{m-1}c_1 \\ &\geq T(m) \end{aligned}$$

  where the first inequality follows the definitions of α_k
- Induction step: We assume that $T(m) \leq H(m)$ for all $1 \leq m \leq L$ where $L \geq k+1$ and prove that it also holds for $m = L+1$.

First, let us factorize $x^{k+2} - 2x^{k+1} + 1$ as $(x-1)(x^{k+1} - x^k - \cdots - 1)$. Denote $F_k(x) = (x^{k+1} - x^k - \cdots - 1)$. We have $F_k(1) < 0$ and $F_k(2) > 0$, indicating that $F_k(x) = 0$ has a root between $(1, 2)$. Combining the fact that γ_k is the largest root of $(x-1)F_k(x) = 0$, $gamma_k$ is also the root of $F_k(x) = 0$, thus i.e.,$\gamma_k^k + \cdots + 1 = \gamma_k k + 1$. Multiply both side an item γ_k^{L-k}, we have

$$\gamma_k^L + \cdots + \gamma_k^{L-k} = \gamma_k^{L+1}$$

Then, the following inequalities hold.

$$\begin{aligned} T(L+1) &\leq T(L) + T(L-1) + \cdots + T(L-k) + 2c|V|^2 \\ &\leq \alpha_k(\gamma_k^L + \cdots + \gamma_k^{L-k}) - (k+1)\beta_k + k\beta_k \\ &\leq \alpha_k(\gamma_k^L + \cdots + \gamma_k^{L-k}) - \beta_k \\ &= \alpha_k \gamma_k^{L+1} - \beta_k \\ &= H(L+1) \end{aligned}$$

□