Quadratic Voting Meets Groups

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Abstract

Quadratic Voting is argued to be robustly optimal by making the marginal cost of voting linear, making costs proportional to marginal benefit. However, the assumptions underlying Quadratic Voting's efficiency and robustness ignores the impact of voting in groups, a widely observed empirical phenomena. I set up a QV model with two polarized, homogeneous groups, and demonstrate that the equilibrium inefficiently privileges larger groups. This weakens QV's claims for efficiency, as in contrast to 1p1v democracy, it is intended to allow minorities with strong preferences to override the tyranny of the majority.

1 Introduction

Quadratic Voting is a voting mechanism proposed by Glenn Weyl [Posner and Weyl, 2017] The mechanism is intended to be a balance between one-person-one-vote democracy and a market, where those who value the good more receive more of it. It is designed such that collective decisions can incorporate intensity of preference, and to incentivize true reporting of preferences.

Weyl argues that not only is QV more efficient than a 1p1v democracy, it is also more robust to fraud and collusion than alternative social choice mechanisms such as VCG. However, Weyl's uses asymptotic efficiency to argue for robustness, which is only effective in large populations. This assumption may hold in the commonly envisioned setting for QV, which are large public elections. However, Weyl and Posner advocate for QV in far more intimate settings, as demonstrated by their paper *Quadratic Voting as Efficient Corporate Governance*. [Posner and Weyl, 2014] In an article in the New York Times, *How to Make Poison Pills Palatable* [Posner and Weyl, 2013], they argue that QV is a more efficient method to protect management from shareholder passivity during hostile takeovers by activist investors.

Two polarized groups breaks down many of assumptions underlying QV's robustness. I focus on one possible violation - coordination within groups. The model I build assumes two competing, homogeneous groups which can perfectly coordinate voting internally. I demonstrate that the equilibrium in which both groups vote inefficiently privileges larger groups. This leads to a situation similar to a tyranny of the majority, whereby even if it is more efficient for the minority, or the smaller group, to win the vote, the larger group may win due to their size advantage. This is an outcome that QV intends to avoid, as it incorporates preference intensity in order to allow minorities to express strong preferences and override the majority.

First, in sections 2 and 3, I explain Lalley and Weyl's core rationale for QV, and try to summarize, by Weyl's own admission what: "LW also show that proving the properties of the QV equilibria fully formally requires a rich statistical machinery; even in the simple environment they study the proof of their main theorem involves forty pages of dense mathematics." [Weyl, 2017]

However, in smaller settings, especially over divisive issues, the possibility of voting in dichotomous groups becomes non-negligible. In Section 4, I qualitatively detail how group voting violates the assumptions underpinning QV's robustness. Then in Section 5 I build a Full Cooperation Model, and show how the group coordination weakens QV's claimed efficiency.

2 Motivation for Quadratic Voting

Lalley and Weyl argue that collective decisions are essentially trading off between equality, embodied by the principle of rationing, and efficiency, embodied by the principle of trade:

> "Perhaps the basic problem underlying existing collective decision procedures is that they rely on the **principle of rationing** (viz., every individual is rationed a single vote on each political contest or issue) rather than on the market **principle of trade** (viz., individuals can exchange influence on issues less important to them for influence on those more important to them)." [Posner and Weyl, 2017]

Quadratic Voting is intended to provide an intermediate trade-off between equality and efficiency, as QV allows for expression of preference intensity, but also fixes the cost of voting.

> "As Hirschman (1982, p. 104) put it,
>
> > "the 'one man one vote' rule gives everyone a minimum share in public decision-making, but it also sets ... a maximum or ceiling; for example, it does not permit citizens to register the different intensities with which they hold their respective political convictions and opinions."
>
> QV, by allowing individuals truthfully to express this intensity, expands the freedom of individuals to participate in the political process. Even if this added freedom may be outweighed by the lesser equality allowed by QV, it seems to us a basic error to exclude it from the balance" [Posner and Weyl, 2017]

2.1 Corporate Governance and Quadratic Voting

In this system, shares do not provide shareholders the right to vote. Instead, anyone may buy votes on a corporate outcome. The price of the votes is a quadratic function of the votes purchased, i.e. you can buy 1 vote for \$1, 2 votes for \$4, 3 votes for \$9, etc. For voter i, within a δ band of the leave-one-out vote total, or the total vote without voter i's vote, each vote increases the probability that voter i's favored side wins. Overall, the vote is decided by majority rule.

The full mechanism would include a refund system, whereby the money collected from votes would be transferred to the corporate treasury, and distributed back to shareholders, except that shareholders with more than 1% of stock would be capped at 1% of the money they personally spent to buy votes. Excess would be divided *pro rata* directly to the rest of the shareholders, as "This rule prevents the implicit price of votes from being (more than slightly) lower for a large voter compared to a small voter." [Posner and Weyl, 2014]

To simplify the modelling, I do not consider outsiders voting. I do not consider the refund system, as if we assume all players are shareholders with less than 1% holdings, the refund mechanism would be an non-substantive change to the cost function.

3 Lalley and Weyl's Original Model

All voters are voting on single proposal. They can either vote for or against said proposal. Without loss of generality, assume that a vote for the proposal is positive, and a vote against the proposal is negative. First, each voter i chooses $v_i \in \mathbb{R}$ number of votes to buy, paying v_i^2 dollars for these votes. Let u_i be the utility voter i receives if the vote is in her favor and 0 if the vote does not go in her favor. Thus the payoff for voter i is:

$$\Psi(V)u_i, \text{ where } V = \sum_{i=1}^{N} v_i \tag{1}$$

is the vote total. They specify a highly specific form for $\Psi(V)$, and describe it "as the payoff function, because it determines the quantity by which values u_i are multiplied to obtain the allocative component of each individuals utility."[Lalley and Weyl, 2014] Essentially this function is trying to smooth out what in practice step function payoff, i.e. if $V > 0$ the voter who supports the proposal receives all the utility, the voter who is against the proposal receives nothing, and vice-versa if $V < 0$. This function is attempting to smooth out this payoff into a likelihood or probability. The way Lalley and Weyl define it, within an intermediate region, or when the votal total is within absolute value δ around zero, then either outcome has a non-zero probability. Furthermore this function ensures that the likelihood for an outcome is increasing and differentiable as the vote aggregate increases.

Thus, in a *type-symmetric Bayes-Nash equilibrium*, a voter will be maximizing

$$E[u\Psi\left(V_{-1} + v\right)] - v^2 \tag{2}$$

$V_{-1} := \sum_{i \neq 1} v_i$ is termed the *one-out vote total*, or all the votes except for the votes of one individual.

Lalley and Weyl prove that a NE equilibrium exists and is asymptotically efficient, although the nature of the NE is radically different when the mean of the value distribution, $\mu \neq 0$. This will be further explained in the following section.

3.1 Core Rationale of QV

All their roughly 40 pages of math and statistics and proofs, however, is trying to justify and examine the complexities of what seems to be a simple maximization problem, where by differentiating by v we get the following first order condition:

$$uE[\psi(V_{-1} + v)] = 2v \Rightarrow v(u) = \underbrace{\frac{E[\psi(V_{-1} + v)]}{2}}_{\text{marginal pivotality}} u \tag{3}$$

To explain this conceptually, the Marginal Cost of an additional vote is thus the vote's marginal pivotality, or the amount her vote increases the probability of an alternative being adopted, multiplied by her value. The key benefit of Quadratic Voting here is that the Marginal Cost of voting, $2v$, is also linear, rather than fixed.

Intuitively speaking, as N grows large, most voters will realize that their vote will have an almost inconsequential impact on the vote total. If we extend this logic to each member of the population, and assume that there are a limited number of extreme voters, a large number of voters can reach a consensus marginal pivotality p that is constant throughout the population. Thus the number of votes a voter buys given a utility should satisfy the proportionality rule, where:

$$v(u) = pu \tag{4}$$

Moreover, they argue that this implies QV is efficient, as then the vote total would be $V = p\sum_i u_i$. Since the vote total is proportional to the sum of values, the vote would always result in an efficient outcome.

However, Lalley and Weyl caveat this rather intuitive assumption of a constant p, arguing that

> "Our main results will show, however, that the marginal pivotality is not constant; in fact, when the mean μ of the sampling distribution F is non-zero the marginal pivotality can have large jump discontinuities in the tail of the distribution. Thus, voters do not buy votes strictly in proportion to their values, and so in general the vote total will not be a scalar multiple of the aggregate value $\sum_i u_i$" [Posner and Weyl, 2014]

4 Voting Happens in Groups

A key assumption of the QV mechanism is an independent symmetric private values environment, whereby each voter i is characterized by a value, u_i. Each value is drawn from an *iid* continuous probability distribution F with C^∞, strictly positive density f supported by a finite closed interval $[u, \bar{u}]$. Each individual knows her own value, but not the values of any of the other $N-1$ voters. The sampling distribution F however is known to all members.

I posit a key empirical observation, namely that people tend to vote in groups. Furthermore, I argue that in divisive decisions, voting groups are formed that are aligned in the direction of their interests, and include most of the voters. Finally, members within a group have incentive to share information and coordinate votes.[1]

There are many settings where the aforementioned properties may manifest. While one might initially think of applying QV to large, public elections, where Posner and Weyl's arguments for asymptotic efficiency best apply, they also advocate for applying QV to Corporate Governance. Specifically, in an article published in the New York Times titled *How to Make Poison Pills Palatable* they argue that QV could more efficiently resolve conflict in hostile corporate takeovers, in comparison to Poison Pills, which are designed to protect existing shareholders. They argue that because QV incorporates preference intensity, a takeover is only succesful if it maximizes shareholder value, whereas "By contrast, if a takeover is put to a regular vote, outsiders can exploit the passivity of most shareholders to obtain a controlling stake and thereby to transfer resources from remaining shareholders to themselves." [Posner and Weyl, 2013]

They explicitly split the conflict into two groups, managers and shareholder activists: "Managers disliked takeovers because they frequently lost their jobs in the resulting reorganization, and argued that takeovers disrupted a firms operations, caused layoffs, and destroyed shareholder value. Shareholder activists replied that managers used poison pills to entrench their positions at shareholders expense."[Posner and Weyl, 2013] It's thus easy to imagine that each groups's preferences for the outcome is largely the same, and that each camp would try to share information and coordinate their votes. Moreover, one could also hypothesize that since these are two groups are small, relatively homogenous, and elite, the utility each member receives in the group would be comparable. For example, the activists participating in the takeover could expect to receive a roughly equal proportion of the spoils from the takeover, or each high-level executive would want to save their bonuses or corporate jet.

Moreover, there are powerful mechanisms that exist already to coordinate and enforce group voting. For example, in a labor dispute, members of a union could share private values and coordinate such that each person buys an equal amount of votes. Unions could use funds from dues to subsidize laborers who do not perceive a major loss from a merger, and punish union workers who do not participate by excluding them from the benefits of a union.

Finally, pre-existing mechanisms of coordination may exert a strong influence on voters both values and voting. Lalley and Weyl use as an example voting on gay marriage to show how intensity of minority preference may have led gay marriage to be legalized. However, in many cases voters are ambivalent about their values and thus simply vote along group lines. For example, in the United States currently, I would argue a significant portion of local, state, and national races are determined not by the quality of the candidate but by the fact that the candidate is either Democrat or Republican. Thus, relatively homogeneous group voting

[1] "...Third, none of my analyses allowed for endogenous information acquisition or pre-election communication among voters. Such factors are crucial to how real-world elections operate." [Weyl, 2017]

is a significant empirical phenomenon that cannot be ignored, especially because it directly destabilizes the marginal cost of voting that Quadratic Voting argues is it's key contribution.

4.1 Weyl's Arguments for Robustness

This consideration of groups, and coordination within groups, is important as Lalley and Weyl not only argue that QV is more efficient than a 1p1v mechanisms, but also that because QV is 'simpler' it is more robust to collusion and fraud compared to other social allocation mechanisms, such as VCG:

> "...in the same environment [large population, iid values] LW consider, many of these mechanisms are efficient even in small populations or converge to efficiency more quickly than QV does. **LW argue that the principal advantage of QV over these alternatives is that it is is "simpler" than these other mechanisms and that its structure suggests that it may also be more robust to a variety of deviations from the baseline model.** " [Weyl, 2017]

However, Weyl's arguments for robustness depend on a large population. He focuses on 'extremists' who could lead to an inefficient outcome. Then, by finding bounds on possible inefficiency using order statistics on a sample drawn from a parameterized Pareto distribution, he approximates the size necessary for a colluding group to succeed. This asymptotic argument relies on large populations, a property he frequently assumes:

> "While collusion and fraud typically benefit their perpetrators, incentives for unilateral deviation and reactions from other participants force collusive groups to be quite large or fraud to be extremely ambitious in order to be efficacious **if the broader population is large.** Given that such large groups or fraudulent schemes plausibly could be detected by authorities, it seems unlikely that collusion and fraud would be severe threat to QVs efficiency **in large populations.**" [Weyl, 2017]

Weyl's derivations for robustness thus seem to have never left the large, public election setting, even though he also advocates for QV in different scenarios. For example, battles over a corporate takeover could easily violate the following conditions "... if coalitions are reasonably anticipated by the public, reasonably limited in their internal enforcement capacity and of a reasonable size, they are unlikely to cause significant inefficiency."[Weyl, 2017]

Thus, I believe QV's robustness in smaller settings with aligned groups is an overlooked area of research.

5 The Full Cooperation Model

This model departs from Weyl's model by assuming that there are only two groups in the company, groups $i \in \{a, b\}$. Furthermore, assume full cooperation within each group, or that each group can collectively agree on a total number of votes to be purchased, and that each group can ensure each members purchases the same number of votes. Finally, assume that each member in group i receives the same u_i utility if their preferred outcome results.

Let v_i denote the number of votes an individual in group i purchases, and let $V_i = \sum_{n_i} v_{i_j}$ be the vote total for group i. Assume the minimum vote you can cast as a group is $V_i = 1$. Without loss of generality I analyze the model from the perspective of an individual, but focus on the total vote count of the group as that ultimately determines the payoff for each individual payer. Thus from an individual perspective each member of group i votes $\frac{V_i}{n_i}$.

Let the vote total of group $h \in \{a, b\}, h \neq i$. Define function $\Psi(V_i, V_h)$ such that:

$$\Psi_i(V_i, V_h) = \begin{cases} 1, & \text{if } V_i \geq V_h + \delta \\ \frac{V_i - V_h + \delta}{2\delta}, & \text{if } V_h - \delta < V_i < V_h + \delta \\ 0, & \text{if } V_i \leq V_h - \delta \end{cases}$$

The Ψ payoff function is essentially a linear version of Weyl's original specification.[2] To make Ψ more similar to a probability, I made the function non-symmetric, and changed the range to $[0,1]$. Within δ of the other group's vote V_h, there is a non-zero probability that group i can win. The probability of winning increases linearly over $[V_h-\delta, V_h+\delta]$.

Thus, the payoff for a member in group i is:

$$\Psi_i(V_i, V_h)u_i - \left(\frac{V_i}{n_i}\right)^2 \tag{5}$$

In the following sections I will first find the conditions necessary for an equilibrium where one group does not vote and the other group votes enough to win with probability one. This equilibrium I term the 'Tryannical Equilibrium.' I use these conditions to characterize the inverse of the Tryannical Equilibrium, when both groups vote, and an equilibrium I term the 'Probabilistic Equilibrium.'

5.1 Tyrannical Equilibrium

Here I will find the conditions necessary such that one group wins the vote with probability 1 and the other group has probability 0 of winning the election. Without loss of generality assume group a wins with probability 1.

If b buys no votes, then a has the following marginal pivotality function:

$$\Psi_i(V_a, V_b = 0) = \begin{cases} 1, & \text{if } V_a \geq \delta \\ \frac{V_a+\delta}{2\delta}, & \text{if } -\delta < V_a < \delta \\ 0, & \text{if } V_a \leq -\delta \end{cases}$$

To win with probability 1, it is optimal for group a to purchase δ votes since any extra vote would simply decrease utility. Thus a would receive payoff:

$$u_a - \left(\frac{\delta}{n_a}\right)^2$$

To show that δ is a best response, we need to show that

$$u_a - \left(\frac{\delta}{n_a}\right)^2 \geq \Psi(\widetilde{V_a}, 0)u_a - \left(\frac{\widetilde{V_a}}{n_a}\right)^2 \quad \forall \widetilde{V_a} \tag{6}$$

When $0 < \widetilde{V_a} < \delta$:

$$\Psi(\widetilde{V_a}, 0)u_a - \left(\frac{\widetilde{V_a}}{n_a}\right)^2 = \frac{\widetilde{V_a}+\delta}{2\delta}u_a - \left(\frac{\widetilde{V_a}}{n_a}\right)^2 < \frac{\delta+\delta}{2\delta}u_a - \left(\frac{\delta}{n_a}\right)^2 = u_a - \left(\frac{\delta}{n_a}\right)^2 \tag{7}$$

When $\widetilde{V_a} = 0$, to be a best reponse:

$$u_a - \left(\frac{\delta}{n_a}\right)^2 \geq \Psi(\widetilde{V_a}, 0)u_a - \left(\frac{\widetilde{V_a}}{n_a}\right)^2 = 0$$

[2] $\Psi :\to [-1,1]$ is an odd, nondecreasing, C^∞ function such that for some $\delta > 0$:

$$\begin{aligned} &\Psi(x) = \text{sgn}(x) \quad \text{for all } |x| \geq \delta \\ &\psi(x) := \Psi'(x) > 0 \quad \text{for all } x \in (-\delta, \delta) \\ &\psi'(x) > 0 \quad \text{for all } x \in (-\delta, 0); \quad \text{and} \\ &\psi \;\; \text{has a single inflection point in} \;\; x \in (-\delta, 0) \end{aligned}$$

This implies that:

$$\delta \leq n_a\sqrt{u_a} \tag{8}$$

Now we consider if buying 0 votes is optimal from the group b's perspective. If a buys δ votes, then b has the following marginal pivotality function:

$$\Psi_i(V_b, V_a = \delta) = \begin{cases} 1, & \text{if } V_b \geq 2\delta \\ \frac{V_b}{2\delta}, & \text{if } 0 \leq V_b \leq 2\delta \\ 0, & \text{if } V_b \leq 0 \end{cases}$$

Thus, to show that 0 is a best response, we need to show that:

$$0 \geq \Psi(\widetilde{V}_b, \delta)u_b - \left(\frac{\widetilde{V}_b}{n_b}\right)^2 \tag{9}$$

When $\widetilde{V}_b > 0$:

$$\frac{\widetilde{V}_b}{2\delta}u_b - \left(\frac{\widetilde{V}_b}{n_b}\right)^2 \leq 0 \tag{10}$$

$$\Rightarrow \delta \geq \frac{n_b^2 u_b}{2\widetilde{V}_b} \tag{11}$$

Assuming that the lowest number of votes a group can purchase is 1, then:

$$\delta \geq \frac{n_b^2 u_b}{2} \tag{12}$$

Thus, combining the conditions from (8) and (12) and finding the case where b dominates using symmetry, we derive the following conditions for a tyrannical equilibrium:

$$\frac{n_b^2 u_b}{2} \leq \delta \leq n_a\sqrt{u_a} \tag{13}$$

$$\frac{n_a^2 u_a}{2} \leq \delta \leq n_b\sqrt{u_b} \tag{14}$$

5.2 Probabilistic Equilibrium

This thus implies that for both parties to be vote, and achieve an equilibrium where both parties have some probability of winning, the following conditions must be met:

$$n_a\sqrt{u_a} < \delta < \frac{n_b^2 u_b}{2} \tag{15}$$

$$n_b\sqrt{u_b} < \delta < \frac{n_a^2 u_a}{2} \tag{16}$$

WLOG, finding a best response from the perspective of group a we find that:

$$\max_{V_a} \Psi(V_a, V_b)u_a - \left(\frac{V_a}{n_a}\right)^2 \tag{17}$$

$$= \left(\frac{V_a - V_b + \delta}{2\delta}\right)u_a - \left(\frac{V_a}{n_a}\right)^2 \tag{18}$$

Taking the FOC, we find:

$$\frac{u_a}{2\delta} - \frac{2V_a}{n_a^2} = 0 \tag{19}$$

$$V_a = \frac{n_a^2 u_a}{4\delta} \tag{20}$$

By symmetry $V_b = \frac{n_b^2 u_b}{4\delta}$.

Thus, wlog of generality, group a has a greater probability of winning if

$$V_a > V_b \Rightarrow n_a^2 u_a > n_b^2 u_b \tag{21}$$

However, the efficient outcome would be where group a wins only if

$$n_a u_a > n_b u_b \tag{22}$$

5.3 Numerical Example

This condition can result in an inefficient outcome. Assume $u_a = 5$, $n_a = 2$ and $u_b = 1.6$, $n_b = 5$. Let $\delta = 5$. It would be efficient for a to win with greater probability, as:

$$\text{Group a Total Utility} = 5 \times 2 > 1.6 \times 5 = \text{Group b Total Utility} \tag{23}$$

However, in the probabilistic equilibrium, both groups best respond by voting:

$$V_a = \frac{2^2 \times 5}{4 \times 5} = 1 \tag{24}$$

$$V_b = \frac{5^2 \times 1.6}{4 \times 5} = 2 \tag{25}$$

Thus since $V_b > V_a$ group b would win with greater probability

5.4 Implications

Since whether a group wins with higher probability depends on the square of the number of members, but efficiency depends on a linear relationship with the number of members, it is unsurprising that there are inefficient outcomes. Taking this equilibrium to the extreme, one could imagine as n grows to infinity, no matter the u, the larger side would also win. However, one must also consider that as n or u grows large in this model, δ must also grow large, as demonstrated in (15). This would provide a real world limitation, as for example it would be hard to imagine that the delta band could explode to infinity.

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