SHEET 1

1. If $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$, calculate the following:
   (a) $(5z_1 + 3z_2)^2$
   (b) $Re\left(\frac{1}{\bar{Z}_1}\right)$
   (c) $Re\left(z_2^2\right)\left(Re(z_1)\right)^2$

2. Find the modulus and the principal argument for the following complex numbers:
   (a) $\frac{i}{-2-2i}$
   (b) $\frac{-1+\sqrt{3}i}{\left(\sqrt{3}-i\right)(1+i)}$
   (c) $\frac{(1+i)\left(2-i2\sqrt{3}\right)}{\sqrt{2}i-\sqrt{6}}$

3. Solve the following equations:
   (a) $z^3 - 8i = 0$
   (b) $z^2 + 4z + 16 = 0$
   (c) $z^5 + 1 = 0$
   (d) $z^2 + (2i - 3)z + 5 - i = 0$

4. Assuming that $|z_1| \neq |z_2|$, prove that:
$$\left|\frac{z_1}{z_1 + z_2}\right| \leq \frac{z_1}{|\,|z_1| - |z_2|\,|}$$

5. Prove that:
$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$
   (Hint: use the fact that $|z|^2 = \bar{z}$)

6. Describe graphically the regions represented by each of the following:
   (a) $1 < |z - 2i| \leq 2$
   (b) $Re(z^2) > 1$
   (c) $|z + 3i| > 4$

7. Let S be the vertical strip $S = \{z = x + iy : 1 \leq x \leq 2\}$. Find the image of S under the mapping $f(z) = z^2$

8. Find the image of $|z - 2| < 1$ under the mapping $w = \dfrac{z}{z - 1}$

9. Describe the range of the function $f(z) = x^2 + 2i$ on the closed disk $|z| \leq 1$

10. A square S in the $\mathbb{Z}$-plane has vertices at $(0, 0), (1, 0), (0, 1), (1, 1)$. Determine the region in the $\mathbb{W}$-plane into which S is mapped under the transformation:
    (a) $w = z^2$
    (b) $w = \dfrac{1}{1 + z}$

11. Find the general solution of the following:
    (a) $\sinh z = \frac{i}{2}$
    (b) $\cosh z = \frac{1}{2}$
    (c) $\cos z = 2$
    (d) $\sin z = \cosh 4$

12. Evaluate the following expressions for $z_1 = 1 + i$ & $z_2 = 1 - i$ and write your answer in the form $a + ib$
    (a) e^{z_1}
    (b) $3ie^{z_2}$
    (c) $e^{z_1}e^{z_2}$
    (d) $Im(e^{z_1})$

13. Solve the following equations:
    (a) $e^z = 2 - 2i$
    (b) $e^{2z} = i$
    (c) $e^{-z+1} = -3 - 4i$

14. Compute $Log(z)$, when:
    (a) $z = -5$
    (b) $z = 1 - i$
    (c) $z = -5i$
    (d) $z = \sqrt{3} + i$

15. Compute $\log(z)$ for the following:
    (a) $z = 2i$
    (b) $z = -\frac{1}{2}$
    (c) $z = 4 + 4i$
    (d) $z = i$
    (e) $z = \dfrac{1}{(1+i)^4}$

16. Evaluate the principal value of:
    (a) 5^i
    (b) $(1+i)^{3+i}$
    (c) i^i
    (d) $\left(\dfrac{1+i}{1-i}\right)^2$