SHEET 2

1. Show that the function

$$f(z) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous at the origin $z = 0$

2. Show that the function defined by

$$f(z) = \begin{cases} \dfrac{4x^2 y}{\sqrt{x^4+y^2}} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is discontinuous at the origin $z = 0$

3. Show that the Cauchy-Riemann equations in polar coordinates are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \& \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

where $u(x, y)$ and $v(x, y)$ are the real and imaginary parts of the complex function $f(z)$

4. Show that $f'(z)$ and its derivative $f''(z)$ exist everywhere and use the formula

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y}$$

to find $f'(z)$ and $f''(z)$ when:

(a) $f(z) = e^{-x}(\cos y - i \sin y)$

(b) $f(z) = z^3$

(c) $f(z) = \cos x \cosh y - i \sin x \sinh y$

5. Show that $\bar{z}$ and $|z|$ are **not** differentiable anywhere

6. For the following complex functions, determine which of them is harmonic, and find the harmonic conjugate for each, and express $u + iv$ as an analytic function of the variable z

   (a) $u = 3x^2y + 2x^2 - y^3 - 2y^2$

   (b) $u = xe^x \cos y - ye^x \sin y$

   (c) $u = e^{-2xy} \sin\left(x^2 - y^2\right)$

   (d) $u = x^2 + y^2$

   (e) $u = 2x\left(1 - y\right)$

7. Is the following function entire? Explain your answer.

$$f(x + iy) = 3x + y + i(3y - x)$$

8. Verify that the Cauchy-Riemann equations are satisfied for the real and imaginary parts of the following functions:

   (a) $f(z) = \sin 2z$

   (b) $f(z) = ze^{-z}$

   (c) $f(z) = z^2 + 5iz + 3 - i$