جامعة الإسكندرية
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Theory of Statistics (1)

Assignment

December 11th, 2023

Prove that:

$$X \sim Norm(\mu, \sigma^2) \longrightarrow M_x(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Since the PDF of the normal distribution is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\begin{aligned}
\therefore M_x(t) = E(e^{xt}) &= \int_{-\infty}^{\infty} e^{xt} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{xt-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx
\end{aligned}$$

Now take the substitution $z = \frac{x-\mu}{\sigma}$

$x = \sigma z + \mu \longrightarrow dx = \sigma z$

$$\begin{aligned}
\therefore M_x(t) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\sigma z t + \mu t - \frac{1}{2}z^2} \sigma dz \\
&= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{\sigma z t - \frac{1}{2}z^2} dz
\end{aligned}$$

By completing the square on the exponent $\sigma z t - \frac{1}{2}z^2$

$$\begin{aligned}
\therefore M_x(t) &= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2 + \frac{1}{2}\sigma^2 t^2} dz \\
&= e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz
\end{aligned}$$

Since $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\sigma t)^2}$ is just the PDF of the standard normal shifted by σt

$$\because \int PDF = 1 \longrightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\sigma t)^2}dz = 1$$

$$\begin{aligned} \therefore M_x(t) &= e^{\mu t}e^{\frac{1}{2}\sigma^2 t^2} \\ &= e^{\mu t+\frac{1}{2}\sigma^2 t^2} \end{aligned}$$