Computer Science 531
Spring 2023
Homework Set 2
Due Wednesday, March 1, at 10:00 PM on Gradescope

Problem 9.

For $A \subseteq \Sigma^*$ and $n \in \mathbb{N}$, we define the n^{th} *slice* of A to be the language

$$A_n = \{y \in \Sigma^* \mid <n, y> \in A\},$$

where $<n, y> = <s_n, y>$ and $s_0, s_1, \ldots$ is the standard enumeration of Σ^*.

Let $\mathcal{C}$ and $\mathcal{D}$ be classes of languages.

1. $\mathcal{C}$ *parametrizes* $\mathcal{D}$ (or $\mathcal{C}$ is *universal* for $\mathcal{D}$) if there exists $A \in \mathcal{C}$ such that $\mathcal{D} = \{A_n | n \in \mathbb{N}\}$.
2. $\mathcal{D}$ is $\mathcal{C}$-*countable* if there exists $A \in \mathcal{C}$ such that $\mathcal{D} \subseteq \{A_n | n \in \mathbb{N}\}$.

(a) Prove: A class $\mathcal{D}$ of languages is countable if and only if $\mathcal{D}$ is $\mathcal{P}(\Sigma^*)$-countable.

(b) Prove that DEC is not DEC-countable.

Problem 10.

(a) Assume that $\mathcal{C}$ and $\mathcal{D}$ are sets of languages and $g : \mathcal{C} \xrightarrow{\text{onto}} \mathcal{D}$. Prove: if $\mathcal{C}$ is countable, then $\mathcal{D}$ is countable.

(b) Prove: if $\mathcal{C}$ is a countable set of languages, then $\exists\mathcal{C}$ and $\forall\mathcal{C}$ are countable.

Problem 11. Prove that the class of countable classes of languages (defined as CTBL in class) is a σ-ideal on $\mathcal{P}(\Sigma^*)$.

Problem 12 Prove all the inclusions in the infinite diagram.

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{135}{$\subseteq$} & \\
\Sigma_3^0 & & & & \Pi_3^0 \\
 & \rotatebox{135}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & \Delta_3^0 & & \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{135}{$\subseteq$} & \\
\Sigma_2^0 & & & & \Pi_2^0 \\
 & \rotatebox{135}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & \Delta_2^0 & & \\
 & \rotatebox{45}{$\subseteq$} & & \rotatebox{135}{$\subseteq$} & \\
\Sigma_1^0 & & & & \Pi_1^0 \\
 & \rotatebox{135}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
 & & \Delta_1^0 & &
\end{array}
$$

Problem 13. Prove that there is a function $g : \mathbb{N} \to \mathbb{N}$ with the following properties.

(i) g is *nondecreasing*, i.e., $g(n) \leq g(n+1)$ holds for all $n \in \mathbb{N}$.

(ii) g is *unbounded*, i.e., for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $g(n) > m$.

(iii) For every computable, nondecreasing, unbounded function $f : \mathbb{N} \to \mathbb{N}$, $f(n) > g(n)$ holds for all but finitely many $n \in \mathbb{N}$.

Problem 14. Prove that a partial function $f :\subseteq \Sigma^* \to \Sigma^*$ is computable if and only if its *graph*

$$G_f = \{<x, f(x)> \mid x \in \text{dom } f\}$$

is c.e.

Problem 15. Let $A \subseteq \Sigma^*$ be c.e., and let B be an infinite decidable subset of A. Prove: If A is undecidable, then $A \setminus B$ is undecidable.

Problem 16. Let $A = L(U)$ be the universal c.e. language defined in class lectures, and let $B \subseteq \Sigma^*$. Prove: If $A \leq_m B$ and $\Sigma^* \setminus A \leq_m B$, then B is neither c.e. nor co-c.e.