

Computer Science 531

Spring 2023

Homework Set 2

Due Wednesday, March 1, at 10:00 PM on Gradescope

Problem 9.

For $A \subseteq \Sigma^*$ and $n \in \mathbb{N}$, we define the n^{th} *slice* of A to be the language

$$A_n = \{y \in \Sigma^* \mid \langle n, y \rangle \in A\},$$

where $\langle n, y \rangle = \langle s_n, y \rangle$ and s_0, s_1, \dots is the standard enumeration of Σ^* .

Let \mathcal{C} and \mathcal{D} be classes of languages.

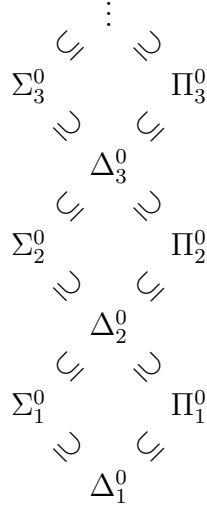
1. \mathcal{C} *parametrizes* \mathcal{D} (or \mathcal{C} is *universal* for \mathcal{D}) if there exists $A \in \mathcal{C}$ such that $\mathcal{D} = \{A_n \mid n \in \mathbb{N}\}$.
 2. \mathcal{D} is \mathcal{C} -*countable* if there exists $A \in \mathcal{C}$ such that $\mathcal{D} \subseteq \{A_n \mid n \in \mathbb{N}\}$.
- (a) Prove: A class \mathcal{D} of languages is countable if and only if \mathcal{D} is $\mathcal{P}(\Sigma^*)$ -countable.
- (b) Prove that DEC is not DEC-countable.

Problem 10.

- (a) Assume that \mathcal{C} and \mathcal{D} are sets of languages and $g : \mathcal{C} \xrightarrow{\text{onto}} \mathcal{D}$. Prove: if \mathcal{C} is countable, then \mathcal{D} is countable.
- (b) Prove: if \mathcal{C} is a countable set of languages, then $\exists \mathcal{C}$ and $\forall \mathcal{C}$ are countable.

Problem 11. Prove that the class of countable classes of languages (defined as CTBL in class) is a σ -ideal on $\mathcal{P}(\Sigma^*)$.

Problem 12 Prove all the inclusions in the infinite diagram.



Problem 13. Prove that there is a function $g : \mathbb{N} \rightarrow \mathbb{N}$ with the following properties.

- (i) g is *nondecreasing*, i.e., $g(n) \leq g(n+1)$ holds for all $n \in \mathbb{N}$.
- (ii) g is *unbounded*, i.e., for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $g(n) > m$.
- (iii) For every computable, nondecreasing, unbounded function $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(n) > g(n)$ holds for all but finitely many $n \in \mathbb{N}$.

Problem 14. Prove that a partial function $f : \subseteq \Sigma^* \rightarrow \Sigma^*$ is computable if and only if its *graph*

$$G_f = \{ \langle x, f(x) \rangle \mid x \in \text{dom } f \}$$

is c.e.

Problem 15. Let $A \subseteq \Sigma^*$ be c.e., and let B be an infinite decidable subset of A . Prove: If A is undecidable, then $A \setminus B$ is undecidable.

Problem 16. Let $A = L(U)$ be the universal c.e. language defined in class lectures, and let $B \subseteq \Sigma^*$. Prove: If $A \leq_m B$ and $\Sigma^* \setminus A \leq_m B$, then B is neither c.e. nor co-c.e.