1 Tutorial Week 5: Multi-Period Market Models

We will need the following definition (it will be discussed in lectures next week, but it is convenient to start working with this notion already in this tutorial). As we work with multi-period models, the portfolio can be revised at every time t. Consider mode with $t = 0, 1, \dots, T$.

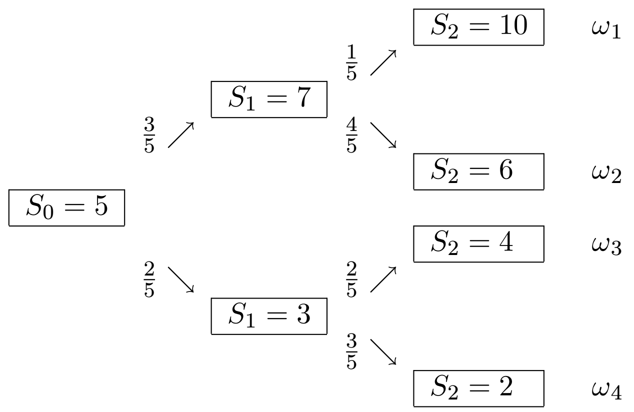
Definition 1 *A trading strategy ϕ is said to be* ***self-financing strategy*** *if for $t = 0, 1, \dots, T-1$,*

$$\phi_t^0 B_{t+1} + \sum_{j=1}^{n} \phi_t^j S_{t+1}^j = \phi_{t+1}^0 B_{t+1} + \sum_{j=1}^{n} \phi_{t+1}^j S_{t+1}^j \tag{1}$$

The LHS of (1) represents the value of the portfolio at time $t+1$ before its revision, whereas the RHS represents the value at time $t+1$ after the portfolio was revised.

- Condition (1) says that these two values must be equal and this means that no cash was withdrawn or added.
- For $t = T-1$, both sides of (1) represent the wealth at time T, that is, $V_T(\phi)$. We do not revise the portfolio at time T.

1. Consider the following two period market model, with money market account B evolving according to $B_0 = 1$, $B_1 = 1+r$, $B_2 = (1+r)^2$ with $r = 0.25$ as well as one stock S evolving according to the diagram:



1. (a) Compute the probabilities of the states $\omega_1, \omega_2, \omega_3, \omega_4$.
   (b) Decide whether the following strategies are adapted and self financing:

	$t=0$	$t=1$			
		ω_1	ω_2	ω_3	ω_4
ϕ^0	315	399	399	399	399
ϕ^1	-105	-120	-120	-140	-140
ϕ^0	315	567	567	483	483
ϕ^1	-210	-255	-255	-280	-280
ϕ^0	105	105	84	42	63
ϕ^1	210	-168	-168	-140	-140

2. Consider the two period market model from Exercise 1. Consider the Asian option

$$X = \left(\frac{1}{3}(S_0 + S_1 + S_2) - 4\right)^+ .$$

(a) What are its payoff's in each of the four states $\omega_1, \omega_2, \omega_3, \omega_4$?

(b) Compute the expectation $\mathbb{E}(X)$.

(c) Similar as in Example 5.5, compute all conditional expectations of S_2.

2 Tutorial Week 5: Solutions

1.

(a) For instance, $\mathbb{P}(\omega_1)$ is computed as follows

$$\begin{aligned}\mathbb{P}(\omega_1) &= \mathbb{P}(S_0 = 5, S_1 = 7, S_2 = 10) \\ &= \mathbb{P}(S_2 = 10 | S_1 = 7)\,\mathbb{P}(S_1 = 7 | S_0 = 5) = \frac{3}{5} \cdot \frac{1}{5} = \frac{3}{25}\end{aligned}$$

Therefore, the probabilities if the states $\omega_1, \omega_2, \omega_3, \omega_4$ are

$$\mathbb{P}(\omega_i) = \begin{cases} \frac{3}{5} \cdot \frac{1}{5} = \frac{3}{25}, & i = 1, \\ \frac{3}{5} \cdot \frac{4}{5} = \frac{12}{25}, & i = 2, \\ \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}, & i = 3, \\ \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}, & i = 4 \end{cases}$$

(b) We need to check that $\phi_t = (\phi_t^0, \phi_t^1)$ is $\mathcal{F}_t^S$-measurable for t=0,1, and

$$\phi_0^0 B_1 + \phi_0^1 S_1 = \phi_1^0 B_1 + \phi_1^1 S_1$$

Since $\mathcal{F}_1^S = \sigma(A_1, A_2)$ we need to check that

$$\begin{cases} \phi_0^0 B_1 + \phi_0^1 S_1(\omega) = \phi_1^0(\omega) B_1 + \phi_1^1(\omega) S_1(\omega), & \omega \in A_1, \\ \phi_0^0 B_1 + \phi_0^1 S_1(\omega) = \phi_1^0(\omega) B_1 + \phi_1^1(\omega) S_1(\omega), & \omega \in A_2, \end{cases}$$

In this model we obtain the following conditions

$$\begin{cases} 1.25\phi_0^0 + 7\phi_0^1 = 1.25\phi_1^0(\omega) + 7\phi_1^1(\omega), & \omega \in \{\omega_1, \omega_2\}, \\ 1.25\phi_0^0 + 3\phi_0^1 = 1.25\phi_1^0(\omega) + 3\phi_1^1(\omega), & \omega \in \{\omega_3, \omega_4\}, \end{cases}$$

- Consider process ϕ

	$t=0$	$t=1$			
		ω_1	ω_2	ω_3	ω_4
ϕ^0	105	105	84	42	63
ϕ^1	210	-168	-168	-140	-140

We observe that $\phi_1^0(\omega_1) \neq \phi_1^0(\omega_2)$ and thus the random variable ϕ_1^0 is not $\mathcal{F}_1^S$-measurable. In other words, the process ϕ is not $\mathbb{F}^S$ adapted and thus it does not define a valid trading strategy.

- Consider process ϕ

	$t=0$	$t=1$			
		ω_1	ω_2	ω_3	ω_4
ϕ^0	315	567	567	483	483
ϕ^1	-210	-255	-255	-280	-280

We observe that ϕ_1^0 and ϕ_1^1 are constant on events A_1 and A_2 and thus they are $\mathcal{F}_1^S$- measurable random variables. Hence, the process ϕ is $\mathbb{F}^S$ adapted and thus it is a valid trading strategy. We have

$$\begin{cases} 1.25 \cdot 315 + 7 \cdot (-210) = -1076.25 = 1.25 \cdot 567 + 7 \cdot (-255)\,, & \omega \in A_1, \\ 1.25 \cdot 315 + 3 \cdot (-210) = -236.25 = 1.25 \cdot 483 + 3 \cdot (-280)\,, & \omega \in A_2, \end{cases}$$

We conclude that ϕ is a trading strategy, and it is self-financing.

- Consider process ϕ

	$t=0$	$t=1$			
		ω_1	ω_2	ω_3	ω_4
ϕ^0	315	399	399	399	399
ϕ^1	-105	-120	-120	-140	-140

We observe that ϕ_1^0 and ϕ_1^1 are constant on events A_1 and A_2 and thus they are $\mathcal{F}_1^S-$measurable random variables. Hence, the process ϕ is $\mathbb{F}^S$ adapted and thus it is a valid trading strategy. We have

$$\begin{cases} 1.25 \cdot (-315) + 7 \cdot 105 = 341.25 = 1.25 \cdot (-399) + 7 \cdot 120, & \omega \in A_1, \\ 1.25 \cdot (-315) + 3 \cdot 105 = -78.75 = 1.25 \cdot (-399) + 3 \cdot 140, & \omega \in A_2, \end{cases}$$

We conclude that ϕ is self-financing.

2. Consider the Asian option

$$X = \left(\frac{1}{3}(S_0 + S_1 + S_2) - 4 \right)^+ .$$

1. (a) What are its payoff's in each of the four states $\omega_1, \omega_2, \omega_3, \omega_4$?

$\boxed{S_0 = 5}$ $\xrightarrow{\frac{3}{5}}$ $\boxed{S_1 = 7}$ $\xrightarrow{\frac{1}{5}}$ $\boxed{S_2 = 10}$ $\mathbb{P}(\omega_1) = \frac{3}{25}$ $X = \left(\frac{1}{3}(5+7+10) - 4\right)^+ = \frac{10}{3}$

$\boxed{S_0 = 5}$ $\xrightarrow{\frac{3}{5}}$ $\boxed{S_1 = 7}$ $\xrightarrow{\frac{4}{5}}$ $\boxed{S_2 = 6}$ $\mathbb{P}(\omega_2) = \frac{12}{25}$ $X = \left(\frac{1}{3}(5+7+6) - 4\right)^+ = \frac{6}{3}$

$\boxed{S_0 = 5}$ $\xrightarrow{\frac{2}{5}}$ $\boxed{S_1 = 3}$ $\xrightarrow{\frac{2}{5}}$ $\boxed{S_2 = 4}$ $\mathbb{P}(\omega_3) = \frac{4}{25}$ $X = \left(\frac{1}{3}(5+3+4) - 4\right)^+ = 0$

$\boxed{S_0 = 5}$ $\xrightarrow{\frac{2}{5}}$ $\boxed{S_1 = 3}$ $\xrightarrow{\frac{3}{5}}$ $\boxed{S_2 = 2}$ $\mathbb{P}(\omega_4) = \frac{6}{25}$ $X = \left(\frac{1}{3}(5+3+2) - 4\right)^+ = 0$

(b) Compute the expectation $\mathbb{E}(X)$.

$$\begin{aligned}
\mathbb{E}(X) &= \mathbb{P}(\omega_1)\cdot X(\omega_1)+\mathbb{P}(\omega_2)\cdot X(\omega_2)+\mathbb{P}(\omega_3)\cdot X(\omega_3)+\mathbb{P}(\omega_4)\cdot X(\omega_4)\\
&= \frac{3}{25}\cdot\frac{10}{3}+\frac{12}{25}\cdot\frac{6}{3}+\frac{4}{25}\cdot 0+\frac{6}{25}\cdot 0=\frac{10}{25}+\frac{24}{25}=\frac{34}{25}
\end{aligned}$$

(c) Similar as in Example 5.5, compute all conditional expectations of S_2.

$$\begin{aligned}
\mathbb{P}(S_2 &= 10\,|\,A_1)=\frac{\mathbb{P}(\{\omega\in A_1\}\cap\{S_2(\omega)=10\})}{\mathbb{P}(A_1)}=\frac{\mathbb{P}(\omega_1)}{\mathbb{P}(\{\omega_1,\omega_2\})}=\frac{\frac{3}{25}}{\frac{3}{5}}=\frac{1}{5}\\
\mathbb{P}(S_2 &= 6\,|\,A_1)=\frac{\mathbb{P}(\{\omega\in A_1\}\cap\{S_2(\omega)=6\})}{\mathbb{P}(A_1)}=\frac{\mathbb{P}(\omega_2)}{\mathbb{P}(\{\omega_1,\omega_2\})}=\frac{\frac{12}{25}}{\frac{3}{5}}=\frac{4}{5}\\
\mathbb{P}(S_2 &= 4\,|\,A_1)=\frac{\mathbb{P}(\{\omega\in A_1\}\cap\{S_2(\omega)=4\})}{\mathbb{P}(A_1)}=\frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_1,\omega_2\})}=0\\
\mathbb{P}(S_2 &= 2\,|\,A_1)=\frac{\mathbb{P}(\{\omega\in A_1\}\cap\{S_2(\omega)=2\})}{\mathbb{P}(A_1)}=\frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_1,\omega_2\})}=0\\
\mathbb{P}(S_2 &= 10\,|\,A_2)=\frac{\mathbb{P}(\{\omega\in A_2\}\cap\{S_2(\omega)=10\})}{\mathbb{P}(A_2)}=\frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_3,\omega_4\})}=0\\
\mathbb{P}(S_2 &= 6|\,A_2)=\frac{\mathbb{P}(\{\omega\in A_2\}\cap\{S_2(\omega)=6\})}{\mathbb{P}(A_2)}=\frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_3,\omega_4\})}=0\\
\mathbb{P}(S_2 &= 4\,|\,A_2)=\frac{\mathbb{P}(\{\omega\in A_2\}\cap\{S_2(\omega)=4\})}{\mathbb{P}(A_2)}=\frac{\mathbb{P}(\omega_3)}{\mathbb{P}(\{\omega_3,\omega_4\})}=\frac{\frac{4}{25}}{\frac{2}{5}}=\frac{2}{5}\\
\mathbb{P}(S_2 &= 2\,|\,A_2)=\frac{\mathbb{P}(\{\omega\in A_2\}\cap\{S_2(\omega)=2\})}{\mathbb{P}(A_2)}=\frac{\mathbb{P}(\omega_4)}{\mathbb{P}(\{\omega_3,\omega_4\})}=\frac{\frac{6}{25}}{\frac{2}{5}}=\frac{3}{5}
\end{aligned}$$

$$\mathbb{E}_{\mathbb{P}}(S_2\,|\,\mathcal{F}_1^S)(\omega)=10\cdot\frac{1}{5}+6\cdot\frac{4}{5}+4\cdot 0+2\cdot 0=\frac{34}{5}\quad\text{for }\ \omega\in A_1$$

$$\mathbb{E}_{\mathbb{P}}(S_2\,|\,\mathcal{F}_1^S)(\omega)=10\cdot 0+6\cdot 0+4\cdot\frac{2}{5}+2\cdot\frac{3}{5}=\frac{14}{5}\quad\text{for }\ \omega\in A_2$$

$$\mathbb{E}_{\mathbb{P}}(S_2|\,\mathcal{F}_1^S)=\begin{cases}\frac{34}{5} & \text{if }\ \omega\in\{\omega_1,\omega_2\}\\ \frac{14}{5} & \text{if }\ \omega\in\{\omega_3,\omega_4\}\end{cases}$$

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(S_2|\,\mathcal{F}_1^S)\right)=\frac{34}{5}\cdot\frac{3}{5}+\frac{14}{5}\cdot\frac{2}{5}=\frac{130}{25}$$

$$\mathbb{E}_{\mathbb{P}}(S_2)=10\cdot\frac{3}{25}+6\cdot\frac{12}{25}+4\cdot\frac{4}{25}+2\cdot\frac{6}{25}=\frac{130}{25}$$

$$\begin{array}{ccccccc}
 & & & & \boxed{S_2=10} & \mathbb{P}(\omega_1)=\frac{3}{25} & X=\left(\frac{1}{3}(5+7+10)-4\right)^+=\frac{10}{3}\\
 & & & \frac{1}{5}\nearrow & & & \\
 & & \boxed{S_1=7} & & & & \\
 & \frac{3}{5}\nearrow & & \frac{4}{5}\searrow & & & \\
 & & & & \boxed{S_2=6} & \mathbb{P}(\omega_2)=\frac{12}{25} & X=\left(\frac{1}{3}(5+7+6)-4\right)^+=\frac{6}{3}\\
\boxed{S_0=5} & & & & & & \\
 & & & & \boxed{S_2=4} & \mathbb{P}(\omega_3)=\frac{4}{25} & X=\left(\frac{1}{3}(5+3+4)-4\right)^+=0\\
 & \frac{2}{5}\searrow & & \frac{2}{5}\nearrow & & & \\
 & & \boxed{S_1=3} & & & & \\
 & & & \frac{3}{5}\searrow & & & \\
 & & & & \boxed{S_2=2} & \mathbb{P}(\omega_4)=\frac{6}{25} & X=\left(\frac{1}{3}(5+3+2)-4\right)^+=0
\end{array}$$

0: $\frac{1+r-d}{u-d} = \frac{1+\frac{1}{4}-\frac{3}{5}}{\frac{7}{5}-\frac{3}{5}} = \frac{13}{16}$

1:$\frac{1+r-d}{u-d} = \frac{1+\frac{1}{4}-\frac{6}{7}}{\frac{10}{7}-\frac{6}{7}} = \frac{11}{16}$

1:$\frac{1+r-d}{u-d} = \frac{1+\frac{1}{4}-\frac{2}{3}}{\frac{4}{3}-\frac{2}{3}} = \frac{7}{8}$