1 Tutorial Week 1: Probability Review

1. Assume that joint probability distribution of the two-dimensional random variable (X, Y), that is, the set of probabilities

$$\mathbb{P}(X = i, Y = j) = p_{i,j} \text{ for } i, j = 1, 2, 3,$$

is given by:

$$\begin{array}{lll} p_{1,1} = 1/9, & p_{1,2} = 1/9, & p_{1,3} = 0, \\ p_{2,1} = 1/3, & p_{2,2} = 0, & p_{2,3} = 1/6, \\ p_{3,1} = 1/9, & p_{3,2} = 1/18, & p_{3,3} = 1/9. \end{array}$$

   (a) Compute $\mathbb{E}_{\mathbb{P}}(X|Y)$, that is, $\mathbb{E}_{\mathbb{P}}(X|Y = j)$ for $j = 1, 2, 3$.

   (b) Check that the equality $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}(X|Y)]$ holds.

   (c) Are the random variables X and Y independent?

2. The joint probability density function $f_{(X,Y)}$ of random variables X and Y is given by

$$f_{(X,Y)}(x, y) = \frac{1}{y} e^{-\frac{x}{y}} e^{-y}, \quad \forall (x, y) \in \mathbb{R}^2_+$$

   and $f_{(X,Y)}(x, y) = 0$ otherwise

   (a) Check that $f_{(X,Y)}$ is a two-dimensional probability density function.

   (b) Show that $\mathbb{E}_{\mathbb{P}}(X|Y = y) = y$ for all $y \in \mathbb{R}_+$.

3. We assume that $\mathbb{P}(X = \pm 1) = \frac{1}{4}$, $\mathbb{P}(X = \pm 2) = \frac{1}{4}$ and we set $Y = X^2$. Check whether the random variables X and Y are correlated and/or dependent.

4. Let X be a random variable uniformly distributed over $(0, 1)$. Compute the conditional expectation $\mathbb{E}_{\mathbb{P}}(X|X < \frac{1}{2})$.

2 Tutorial Week 1: Solutions

1. (a) See Probability Review Slides (page 45)

$$p_{X|Y}(x_i \,|\, y_j) = \mathbb{P}(X = x_i \,|\, Y = y_j) := \frac{\mathbb{P}(X = x_i, Y = y_j)}{\mathbb{P}(Y = y_j)}$$

$$\begin{aligned} \mathbb{P}(X &= 1 \,|\, Y = 1) = \frac{\mathbb{P}(X = 1, Y = 1)}{\mathbb{P}(Y = 1)} = \frac{1/9}{1/9 + 1/3 + 1/9} = \frac{1}{5} \\ \mathbb{P}(X &= 2 \,|\, Y = 1) = \frac{\mathbb{P}(X = 2, Y = 1)}{\mathbb{P}(Y = 1)} = \frac{1/3}{1/9 + 1/3 + 1/9} = \frac{3}{5} \end{aligned}$$

and so on. We get the conditional distribution of X given $Y = j$ are

$$\begin{array}{ll} \text{for } j=1: & \left(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}\right) \\ \text{for } j=2: & \left(\frac{2}{3}, 0, \frac{1}{3}\right) \\ \text{for } j=3: & \left(0, \frac{3}{5}, \frac{2}{5}\right) \end{array}$$

and thus the conditional expectations

$$\mathbb{E}_{\mathbb{P}}(X \mid Y = y_j) := \sum_{i=1}^{\infty} x_i \, \mathbb{P}(X = x_i \mid Y = y_j)$$

gives

$$\begin{array}{r} \mathbb{E}_{\mathbb{P}}(X|Y=1) = 1 \times \frac{1}{5} + 2 \times \frac{3}{5} + 3 \times \frac{1}{5} = 2 \\ \mathbb{E}_{\mathbb{P}}(X|Y=2) = 1 \times \frac{2}{3} + 2 \times 0 + 3 \times \frac{1}{3} = \frac{5}{3} \\ \mathbb{E}_{\mathbb{P}}(X|Y=3) = 1 \times 0 + 2 \times \frac{3}{5} + 3 \times \frac{2}{5} = \frac{12}{5} \end{array}$$

(b) Marginal distributions:

$$\mathbb{P}(X = 1) = 1/9 + 1/9 + 0 = 2/9$$

and so on to get:

$$\begin{array}{ll} \mathbb{P}(X=i): & \left(\frac{2}{9}, \frac{3}{6}, \frac{5}{18}\right) \\ \mathbb{P}(Y=j): & \left(\frac{5}{9}, \frac{3}{18}, \frac{5}{18}\right) \end{array}$$

so, on the one hand, the unconditional expectation:

$$\mathbb{E}_{\mathbb{P}}(X) = \sum_{i=1}^{3} i\mathbb{P}(X = i) = 1 \times \frac{2}{9} + 2 \times \frac{3}{6} + 3 \times \frac{5}{18} = \frac{37}{18}$$

on the other hand

$$\mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(X|Y)\right) := \sum_{j=1}^{3} \mathbb{E}_{\mathbb{P}}(X|Y=j)\mathbb{P}(Y=j) = 2 \times \frac{5}{9} + \frac{5}{3} \times \frac{3}{18} + \frac{12}{5} \times \frac{5}{18} = \frac{37}{18}$$

and so that

$$\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(X|Y)\right)$$

(c) Random variables X and Y are not independent since

$$\mathbb{P}(X = i, Y = j) = \mathbb{P}(X = i)\mathbb{P}(Y = j) \quad \forall i, j = 1, 2, 3$$

is not satisfied. For example,

$$\mathbb{P}(X = 1, Y = 3) = 0 \neq \mathbb{P}(X = 1)\mathbb{P}(Y = 3) = \frac{4}{18} \times \frac{5}{18}$$

2. (a)We need to show that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{(X,Y)}(x,y)dxdy = \int_0^{\infty}\int_0^{\infty}\frac{1}{y}e^{-\frac{x}{y}}e^{-y}dxdy \stackrel{?}{=} 1$$

We first compute

$$\begin{aligned} f_Y(y) &= \int_0^{\infty}\frac{1}{y}e^{-\frac{x}{y}}e^{-y}dx = \frac{1}{y}e^{-y}\int_0^{\infty}e^{-\frac{x}{y}}dx = \frac{1}{y}e^{-y}\left[-ye^{-\frac{x}{y}}\right]_0^{\infty} \\ &= \frac{1}{y}e^{-y}\left[-ye^{-\frac{\infty}{y}} + ye^{-\frac{0}{y}}\right] = \frac{1}{y}e^{-y}\left[0+y\right] = e^{-y} \end{aligned}$$

$$\begin{aligned} \int_0^{\infty}\int_0^{\infty}\frac{1}{y}e^{-\frac{x}{y}}e^{-y}dxdy &= \int_0^{\infty} f_Y(y)dy = \int_0^{\infty} e^{-y}dy \\ &= \left[-e^{-y}\right]_0^{\infty} = 0+1 = 1 \end{aligned}$$

so we can use this function as probability distribution function

(b) The conditional density of X given $Y=y$ equals for any fixed $y \geq 0$

$$f_{X|Y}(x|y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)} = \frac{1}{y}e^{-\frac{x}{y}}$$

Consequently, for all $y \geq 0$

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(X|Y &= y) = \int_0^{\infty} x f_{X|Y}(x|y)dx = \int_0^{\infty} x\frac{1}{y}e^{-\frac{x}{y}}dx \\ &= y\int_0^{\infty}\frac{x}{y}e^{-\frac{x}{y}}d\frac{x}{y} = y\int_0^{\infty} ze^{-z}dz \\ &= y\left(-\int_0^{\infty} zd\left(e^{-z}\right)\right) = -y\left(\left[ze^{-z}\right]_0^{\infty} - \int_0^{\infty}e^{-z}dz\right) \\ &= -y\left([0-0] + \left[e^{-z}\right]_0^{\infty}\right) = -y\left([0-0]+[0-1]\right) \\ &= y \end{aligned}$$

we integrated by parts

3. We assume that $\mathbb{P}(X=\pm 1) = \frac{1}{4}$, $\mathbb{P}(X=\pm 2) = \frac{1}{4}$ and we set $Y = X^2$. Check whether the random variables X and Y are correlated and/or dependent.

$$Cov\,(X,Y) = \mathbb{E}_{\mathbb{P}}\,(XY) - \mathbb{E}_{\mathbb{P}}(X)\mathbb{E}_{\mathbb{P}}(Y) = \mathbb{E}_{\mathbb{P}}\left(X^3\right) - \mathbb{E}_{\mathbb{P}}(X)\mathbb{E}_{\mathbb{P}}(X^2) = 0 - 0\times\mathbb{E}_{\mathbb{P}}(X^2) = 0$$

So the variables are not correlated. However, they are not independent as, for example

$$\mathbb{E}_{\mathbb{P}}(Y|X=1) = 1 \neq 4 = \mathbb{E}_{\mathbb{P}}(Y|X=2)$$

Recall that under independence of X and Y we have $\mathbb{E}_{\mathbb{P}}(Y|X) = \mathbb{E}_{\mathbb{P}}(Y)$ and $\mathbb{E}_{\mathbb{P}}(X|Y) = \mathbb{E}_{\mathbb{P}}(X)$

4. Let X be a random variable uniformly distributed over $(0,1)$. Compute the conditional expectation $\mathbb{E}_{\mathbb{P}}(X|X<\frac{1}{2})$.

   The conditional density given the event $X<\frac{1}{2}$

$$f_{X|X<\frac{1}{2}}(x)=\begin{cases} 0, x\leq 0 \\ 2, x\in(0,\frac{1}{2}) \\ 0, x\geq\frac{1}{2} \end{cases}$$

   Therefore

$$\mathbb{E}_{\mathbb{P}}(X|X<\frac{1}{2})=\int_{-\infty}^{+\infty} xf_{X|X<\frac{1}{2}}(x)dx=\int_0^{\frac{1}{2}} 2xdx=\frac{1}{4}$$