# Mathematical Finance

## Multi Period Market Model

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# Overview

- One-period models
  - Elementary market model
  - Single-period market model
- Multi-period market model
  - Role of Information
  - Can we 'split' into many single-period models?

# Elementary Market Model

- 2 states of nature
- 2 assets: riskless bank account (governemnt bond) and an 'underlying asset' - risky share
- The aim is to replicate particular contingent claim: form a portfolio of two assets which will give you exactly the same payoff as the contingent claim.
- Important notions are introduced
  - arbitrage
  - risk neutral probability measure (RNPM)
- We show that the replicating strategy (portfolio weights) is fully explained by RNPM
- In the elementary model there is a simple formula to compute RNPM. If there is no arbitrage you can *always* find RNPM and construct replicating strategy.
  - you can guess this is because there are two states of nature and two assets

# Single-Period Market Model

- $K \geq 2$ states of nature. We work with finite $K$.
- $N \geq 2$ risky assets and one riskless bank account (governemnt bond). So $N + 1$ securities in portfolio.
- The aim is the same: replicate particular contingent claim by forming a portfolio of assets which will give you exactly the same payoff as the contingent claim.
- Important notions are introduced
  - arbitrage
  - risk neutral probability measure (RNPM)
  - complete market model
  - attainable claim

# Single-Period Market Model

- FATP: Model $\mathcal{M}$ is arbitrage free $\Longleftrightarrow$ at least one RNPM exists.
  - Need a procedure to find RNPM. If we can construct a *RNPM* then there is *no arbitrage.* We discussed the procedure.
  - If we know *RNPM* we can use it to construct the *replicating strategy.*
  - If we have a *replicating strategy* then the contingent claim is *attainable.*
- Complete market: all contingent claims are attainable.
- Need to have at least as many (different) securities $(N+1)$ as (different) states of nature $(K)$.
- So, we are naturally interested in working with complete markets. There is a unique RNPM there. And we learned how to construct a replicating portfolio and so we learned how to find an arbitrage price of a contingent claim.
- However, when markets are incomplete, then certain claims can still be attainable.

# Multi-Period Market Model

- Many (possibly infinite number of) periods.
- $K \geq 2$ states of nature.
- $N \geq 2$ risky assets and one riskless bank account (governemnt bond). So $N + 1$ securities in portfolio.
- Unlike in single-period models, information at each period changes.
- We discussed ho to model the information set (partitions and filtrations). When computing expectations at period $t$ they are now conditional on 'information available at time $t$', not 'the whole information'.

# Outline

We will examine the following issues:

1. Trading Strategies and Arbitrage-Free Models
2. Risk-Neutral Probability Measures and Martingales
3. Fundamental Theorem of Asset Pricing
4. Arbitrage Pricing of Attainable Claims
5. Risk-Neutral Valuation of Non-Attainable Claims
6. Completeness of Multi-Period Market Models

# Primary Traded Assets

To specify a **multi-period market model** $\mathcal{M} = (B, S^1, \dots, S^n)$ we have to examine the concept of a dynamic trading strategy and the associated wealth process.
We first define **primary traded assets**

- Let $r$ be the interest rate. The **money market account** is denoted by $B_t$ for $t = 0, 1, \dots, T$ where

$$B_t = (1 + r)^t.$$

- There are $n$ risky assets, called **stocks**, with price processes denoted by $S_t^j$ for $t = 0, 1, \dots, T$.
- We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F}$ generated by price processes of stocks.

# Dynamic Trading Strategy and Wealth Process

A dynamic **trading strategy** in a multi-period market model is defined as a stochastic process $\phi_t = \left(\phi_t^0, \phi_t^1, \ldots, \phi_t^n\right)$ for $t = 0, 1, \ldots, T$ where:

- $\phi_t^0$ is the number of 'shares' of the money market account $B$ held at time $t$.
- $\phi_t^j$ is the number of shares of the $j$th stock held at time $t$.

## Definition (Value Process)

The **wealth process** (or **value process**) of a trading strategy $\phi = \left(\phi^0, \phi^1, \ldots, \phi^n\right)$ is the stochastic process $V(\phi)$ given by

$$V_t\left(\phi\right) = \phi_t^0 B_t + \sum_{j=1}^{n} \phi_t^j S_t^j.$$

# Self-Financing Trading Strategy

**Definition (Self-Financing Trading Strategy)**

A trading strategy $\phi$ is said to be **self-financing strategy** if for $t = 0, 1, \ldots, T-1$,

$$\phi_t^0 B_{t+1} + \sum_{j=1}^{n} \phi_t^j S_{t+1}^j = \phi_{t+1}^0 B_{t+1} + \sum_{j=1}^{n} \phi_{t+1}^j S_{t+1}^j. \tag{1}$$

- The LHS of (1) represents the value of the portfolio at time $t+1$ before its revision, whereas the RHS represents the value at time $t+1$ after the portfolio was revised.
- Condition (1) says that these two values must be equal and this means that no cash was withdrawn or added.
- For $t = T-1$, both sides of (1) represent the wealth at time $T$, that is, $V_T(\phi)$. We do not revise the portfolio at time $T$.

# Gains Process

## Lemma (6.1)

*For any self-financing trading strategy, the wealth process can be alternatively computed by, for* $t = 1, \ldots, T$*,*

$$V_t(\phi) = \phi^0_{t-1} B_t + \sum_{j=1}^{n} \phi^j_{t-1} S^j_t.$$

## Proof.

The statement is an immediate consequence of formula (1). $\square$

## Definition (Gains Process)

For a trading strategy $\phi$, the **gains process** $G(\phi) = (G_t(\phi))_{0 \leq t \leq T}$ is given by

$$G_t(\phi) := V_t(\phi) - V_0(\phi).$$

# Multi-Period Market Model

## Definition (Market Model)

A **multi-period market model** $\mathcal{M} = (B, S^1, \ldots, S^n)$ is given by the following data:

1. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$.
2. The money market account $B$ given by $B_t = (1+r)^t$.
3. A number of financial assets with prices $S^1, \ldots, S^n$, which are assumed to be $\mathbb{F}$-adapted stochastic processes.
4. The class $\Phi$ of all self-financing trading strategies.

# Increment Processes

- As in the single-period model, we define the discounted value process, gains process and discounted gains process.
- It will be convenient to use the increment processes for traded assets.

## Definition (Increment Processes)

- The **increment process** $\Delta S^j$ corresponding to the $j$th stock is defined by

$$\Delta S^j_{t+1} := S^j_{t+1} - S^j_t \quad \text{for} \quad t = 0, \dots, T-1.$$

- The **increment process** $\Delta B$ of the money market account are given by

$$\Delta B_{t+1} := B_{t+1} - B_t = (1+r)^t r = B_t r \quad \text{for} \quad t = 0, \dots, T-1.$$

# Discounted Processes

## Definition (Discounted Processes)

The **discounted stock prices** are given by

$$\hat{S}_t^j := \frac{S_t^j}{B_t}$$

and the increments of discounted prices are $\Delta \hat{S}_{t+1}^j := \hat{S}_{t+1}^j - \hat{S}_t^j$. The **discounted wealth process** $\hat{V}(\phi)$ of $\phi$ is given by

$$\hat{V}_t(\phi) := \frac{V_t(\phi)}{B_t}.$$

The **discounted gains process** $\hat{G}(\phi)$ of $\phi$ equals

$$\hat{G}_t(\phi) := \hat{V}_t(\phi) - \hat{V}_0(\phi).$$

# Discounted Processes

## Proposition (6.1)

*An $\mathbb{F}$-adapted trading strategy $\phi = (\phi_t)_{0 \le t \le T}$ is self-financing if and only if any of the two equivalent statements hold:*

1. *for every $t = 1, \ldots, T$*

$$G_t(\phi) = \sum_{u=0}^{t-1} \phi_u^0 \, \Delta B_{u+1} + \sum_{u=0}^{t-1} \sum_{j=1}^{n} \phi_u^j \, \Delta S_{u+1}^j.$$

2. *for every $t = 1, \ldots, T$*

$$\hat{G}_t(\phi) = \sum_{u=0}^{t-1} \sum_{j=1}^{n} \phi_u^j \, \Delta \hat{S}_{u+1}^j.$$

# Properties of Discounted Gains and Wealth

## Proof.

[Proof of Proposition 6.1] The proof is elementary and thus it is left as an exercise. □

- Note that the process $\hat{G}(\phi)$ given by condition 2) does not depend on the component $\phi^0$ of $\phi \in \Phi$.
- In view of Proposition 6.1, the discounted wealth process of any $\phi \in \Phi$ satisfies

$$\hat{V}_t(\phi) = \hat{V}_0(\phi) + \sum_{u=0}^{t-1} \sum_{j=1}^{n} \phi_u^j \, \Delta \hat{S}_{u+1}^j.$$

- Hence for every $t = 0, \dots, T-1$

$$\Delta \hat{V}_{t+1}(\phi) = \Delta \hat{G}_{t+1}(\phi) = \sum_{j=1}^{n} \phi_t^j \, \Delta \hat{S}_{t+1}^j.$$

# Arbitrage Opportunity

As usual, we work under the standing assumption that the sample space $\Omega$ is finite (or countable).

## Definition (Arbitrage Opportunity)

A trading strategy $\phi \in \Phi$ is an **arbitrage opportunity** if

1. $V_0(\phi) = 0$,
2. $V_T(\phi)(\omega) \geq 0$ for all $\omega \in \Omega$,
3. $V_T(\phi)(\omega) > 0$ for some $\omega \in \Omega$ or, equivalently, $\mathbb{E}_{\mathbb{P}}(V_T(\phi)) > 0$.

We say that a multi-period market model $\mathcal{M}$ is **arbitrage-free** if no arbitrage opportunities exist in $\Phi$.

# Arbitrage Conditions

- Note that in the arbitrage conditions one can use either the discounted wealth process $\widehat{V}$ or the discounted gains process $\widehat{G}$ (instead of the wealth $V$).
- It is also important to note that conditions 1)–3) hold under $\mathbb{P}$ whenever they are satisfied under some probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.
- The next step is to introduce the concept of a risk-neutral probability measure for a multi-period market model.
- Risk-neutral probability measures are very closely related to the question of arbitrage-free property and completeness of a multi-period market model.

# Risk-Neutral Probability Measure

## Definition (Risk-Neutral Probability Measure)

A probability measure $\mathbb{Q}$ on $\Omega$ is called a **risk-neutral probability measure** for a multi-period market model $\mathcal{M} = (B, S^1, \dots, S^n)$ whenever

1. $\mathbb{Q}(\omega) > 0$ for all $\omega \in \Omega$,
2. $\mathbb{E}_{\mathbb{Q}}(\Delta \hat{S}^j_{t+1} \,|\, \mathcal{F}_t) = 0$ for all $j = 1, \dots, n$ and $t = 0, \dots, T-1$.

We denote by $\mathbb{M}$ the class of all risk-neutral probability measures for the market model $\mathcal{M}$.

Observe that condition 2) is equivalent to the equality, for all $t = 0, \dots, T-1$,

$$\mathbb{E}_{\mathbb{Q}}\left(\hat{S}^j_{t+1} \,|\, \mathcal{F}_t\right) = \hat{S}^j_t.$$

The discounted stock price $\hat{S}^j$ is a martingale under any risk-neutral probability measure $\mathbb{Q} \in \mathbb{M}$.

# Discounted Wealth as a Martingale

## Proposition (6.2)

*Let $\phi \in \Phi$ be a trading strategy. Then the discounted wealth process $\hat{V}(\phi)$ and the discounted gains process $\hat{G}(\phi)$ are martingales under any risk-neutral probability measure $\mathbb{Q} \in \mathbb{M}$.*

## Proof.

- Recall that $\hat{V}_t(\phi) = \hat{V}_0(\phi) + \hat{G}_t(\phi)$ for every $t = 0, \ldots, T$,
- Since $\hat{V}_0(\phi)$ (the initial endowment) is a constant, it suffices to show that the process $\hat{G}(\phi)$ is a martingale under any $\mathbb{Q} \in \mathbb{M}$. From Proposition 6.1, we obtain

$$\hat{G}_{t+1}(\phi) = \hat{G}_t(\phi) + \sum_{j=1}^{n} \phi_t^j \, \Delta \hat{S}_{t+1}^j.$$

□

# Proof of Proposition 6.2

## Proof.

[Proof of Proposition 6.2 (Continued)]

- Hence

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\hat{G}_{t+1}(\phi)|\mathcal{F}_t) &= \hat{G}_t(\phi) + \sum_{j=1}^{n} \mathbb{E}_{\mathbb{Q}}\left(\phi_t^j \Delta \hat{S}_{t+1}^j \middle| \mathcal{F}_t\right) \\ &= \hat{G}_t(\phi) + \sum_{j=1}^{n} \phi_t^j \underbrace{\mathbb{E}_{\mathbb{Q}}\left(\Delta \hat{S}_{t+1}^j | \mathcal{F}_t\right)}_{=0} \\ &= \hat{G}_t(\phi)\end{aligned}$$

- We used the fact that $\phi_t^j$ is $\mathcal{F}_t$-measurable and the "take out what is known" property of the conditional expectation.
- We conclude that $\hat{G}(\phi)$ is a martingale under any $\mathbb{Q} \in \mathbb{M}$.

$\square$

# Fundamental Theorem of Asset Pricing

- We will show that the **F**undamental **T**heorem of **A**sset **P**ricing (FTAP) can be extended to a multi-period market model.
- Recall that the class of admissible trading strategies $\Phi$ in a multi-period market model is assumed to be the **full** set of all self-financing and $\mathbb{F}$-adapted trading strategies.
- It possible to show that in that case, the relationship between the existence of a risk-neutral probability measure $\mathbb{Q}$ and no arbitrage for the model $\mathcal{M}$ is "if and only if".
- We will only prove here the following implication:

$$\textbf{Existence of } \mathbb{Q} \in \mathbb{M} \;\Rightarrow\; \textbf{Model } \mathcal{M} \textbf{ is arbitrage-free}$$

# Fundamental Theorem of Asset Pricing

## Theorem (FTAP)

*Consider a multi-period market model* $\mathcal{M} = (B, S^1, \ldots, S^n)$*.*
*The following statements hold:*

1. *if the class* $\mathbb{M}$ *of risk-neutral probability measures for* $\mathcal{M}$ *is non-empty then there are no arbitrage opportunities in the class* $\Phi$ *of all self-financing trading strategies and thus the model* $\mathcal{M}$ *is arbitrage-free,*
2. *if there are no arbitrage opportunities in the class* $\Phi$ *of all self-financing trading strategies then there exists a risk-neutral probability measure for* $\mathcal{M}$*, so that the class* $\mathbb{M}$ *is non-empty.*

To sum up:

**Class** $\mathbb{M}$ **is non-empty** $\Leftrightarrow$ **Market model** $\mathcal{M}$ **is arbitrage-free**

# Proof of the FTAP ($\Rightarrow$)

## Proof.

[Proof of the FTAP ($\Rightarrow$)] Let us assume that a risk-neutral probability measure $\mathbb{Q}$ for $\mathcal{M}$ exists. Our goal it to show that the model $\mathcal{M}$ is arbitrage-free.

To this end, we argue by contradiction. Let us thus assume that there exists an arbitrage opportunity $\phi \in \Phi$. Such a strategy would satisfy the following conditions:

1. the initial endowment $\hat{V}_0(\phi) = 0$,
2. the discounted gains process $\hat{G}_T(\phi) \geq 0$,
3. there exists at least one $\omega \in \Omega$ such that $\hat{G}_T(\phi)(\omega) > 0$.

On the one hand, from conditions 2. and 3. above, we deduce easily that

$$\mathbb{E}_{\mathbb{Q}}\left(\hat{G}_T(\phi)\right) > 0.$$

□

# Proof of the FTAP ($\Rightarrow$)

## Proof.

[Proof of the FTAP ($\Rightarrow$)] On the other hand, using Proposition 6.1, we obtain

$$\mathbb{E}_{\mathbb{Q}}\left(\hat{G}_T(\phi)\right) = \mathbb{E}_{\mathbb{Q}}\left(\sum_{j=1}^{n}\sum_{u=0}^{t-1}\phi_u^j \Delta \hat{S}_{u+1}^j\right) = \sum_{j=1}^{n}\sum_{u=0}^{t-1}\mathbb{E}_{\mathbb{Q}}\left(\phi_u^j \Delta \hat{S}_{u+1}^j\right)$$

$$= \sum_{j=1}^{n}\sum_{u=0}^{t-1}\mathbb{E}_{\mathbb{Q}}\left(\mathbb{E}_{\mathbb{Q}}(\phi_u^j \Delta \hat{S}_{u+1}^j \,|\, \mathcal{F}_u)\right)$$

$$= \sum_{j=1}^{n}\sum_{u=0}^{t-1}\mathbb{E}_{\mathbb{Q}}\left(\phi_u^j \underbrace{\mathbb{E}_{\mathbb{Q}}(\Delta \hat{S}_{u+1}^j \,|\, \mathcal{F}_u)}_{=0}\right) = 0.$$

This clearly contradicts the inequality obtained in the first step. Hence there cannot be an arbitrage in the market model $\mathcal{M}$. □

# Replicating Strategy

- Note that a contingent claim of European style can only be exercised at its maturity date $T$ (as opposed to contingent claims of American style).
- A **European contingent claim** in a multi-period market model is an $\mathcal{F}_T$-measurable random variable $X$ on $\Omega$ to be interpreted as the payoff at the terminal date $T$.
- For brevity, European contingent claims will also be referred to as **contingent claims** or simply **claims**.

### Definition (Replicating Strategy)

A **replicating strategy** (or a **hedging strategy**) for a contingent claim $X$ is a trading strategy $\phi \in \Phi$ such that $V_T(\phi) = X$, that is, the terminal wealth of the trading strategy matches the claim's payoff for all $\omega$.

# Principle of No-Arbitrage

## Definition (Principle of No-Arbitrage)

An $\mathbb{F}$-adapted stochastic process $(\pi_t(X))_{0 \leq t \leq T}$ is a price process for the contingent claim $X$ that **complies with the principle of no-arbitrage** if there is no $\mathbb{F}$-adapted and self-financing arbitrage strategy in the extended model $\widetilde{\mathcal{M}} = (B, S^1, \ldots, S^n, S^{n+1})$ with an additional asset $S^{n+1}$ given by $S_t^{n+1} = \pi_t(X)$ for $0 \leq t \leq T-1$ and $S_T^{n+1} = X$.

- The standard method to price a contingent claim is to employ the replication principle, if it can be applied.
- The price will now depend on time $t$ and thus one has to specify a whole price process $\pi(X)$, rather than just an initial price, as in the single-period market model.
- Obviously, $\pi_T(X) = X$ for any claim $X$.

# Arbitrage Pricing of Attainable Claims

- In the next result, we deal with an **attainable** claim, meaning that we assume a priori that a replicating strategy for $X$ exists.

## Proposition (6.3)

*Let $X$ be a contingent claim in an arbitrage-free multi-period market model $\mathcal{M}$ and let $\phi \in \Phi$ be any replicating strategy for $X$. Then the only price process of $X$ that complies with the principle of no-arbitrage is the wealth process $V(\phi)$.*

- The arbitrage price at time $t$ of an attainable claim $X$ is unique and it is also denoted as $\pi_t(X)$.
- Hence the equality $\pi_t(X) = V_t(\phi)$ holds for any replicating strategy $\phi \in \Phi$ for $X$.
- In particular, the price at time $t = 0$ is the initial endowment of any replicating strategy for $X$, that is, $\pi_0(X) = V_0(\phi)$ for any strategy $\phi \in \Phi$ such that $V_T(\phi) = X$.

# Example: Replication of a Digital Call Option

- The definition of hedging strategy is also used to price a contingent claim, which is called the **replication principle**.
- We will now examine replication of a contingent claim in a two-period market model.
- It is easy to check that this model is arbitrage-free.

## Example (6.1)

- Consider a two-period market model consisting of the savings account and one risky stock.
- The interest rate equals $r = \frac{1}{9}$ so that $B_t = (1+r)^t = \left(\frac{10}{9}\right)^t$.
- The price of the stock is represented in the following exhibit in which the real-world probability $\mathbb{P}$ is also specified.

# Example: Replication of a Digital Call Option

## Example (6.1 Continued)

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | $\boxed{S_2 = 3}$ | $\omega_1$ | $\mathbb{P}(\omega_1) = 0.18$ |
| | | $\boxed{S_1 = 1.5}$ | $0.3 \nearrow$ | | | |
| | $\nearrow$ 0.6 | | $0.7 \searrow$ | $\boxed{S_2 = 1}$ | $\omega_2$ | $\mathbb{P}(\omega_2) = 0.42$ |
| $\boxed{S_0 = 1}$ | | | | | | |
| | 0.4 $\searrow$ | | $0.5 \nearrow$ | $\boxed{S_2 = 1.5}$ | $\omega_3$ | $\mathbb{P}(\omega_3) = 0.2$ |
| | | $\boxed{S_1 = 0.75}$ | | | | |
| | | | $0.5 \searrow$ | $\boxed{S_2 = 0.5}$ | $\omega_4$ | $\mathbb{P}(\omega_4) = 0.2$ |

# Example: Replication of a Digital Call Option

## Example (6.1 Continued)

- Consider a **digital call option** with the payoff function

$$X(\omega) = g\left(S_2(\omega)\right) = \begin{cases} 1 & \text{if } S_2(\omega) > 2 \\ 0 & \text{otherwise} \end{cases}$$

  Put another way

$$X = (X(\omega_1), X(\omega_2), X(\omega_3), X(\omega_4)) = (1, 0, 0, 0).$$

- By the definition of hedging strategy, we have $V_2(\phi) = X$ or, more explicitly,

$$V_2(\phi) = \phi_1^0 B_2 + \phi_1^1 S_2 = g(S_2).$$

- Observe that $\phi_1^i(\omega_1) = \phi_1^i(\omega_2)$ and $\phi_1^i(\omega_3) = \phi_1^i(\omega_4)$ for $i = 0, 1$ since $\phi$ is an $\mathbb{F}$-adapted process.

# Example: Replication of a Digital Call Option

## Example (6.1 Continued)

At time $t = 1$ we obtain two linear systems for $\phi_1 = (\phi_1^0, \phi_1^1)$

- For $\omega \in \{\omega_1, \omega_2\}$

$$\left(\frac{10}{9}\right)^2 \phi_1^0 + 3\phi_1^1 = 1$$
$$\left(\frac{10}{9}\right)^2 \phi_1^0 + \phi_1^1 = 0$$

- For $\omega \in \{\omega_3, \omega_4\}$

$$\left(\frac{10}{9}\right)^2 \phi_1^0 + \frac{3}{2}\phi_1^1 = 0$$
$$\left(\frac{10}{9}\right)^2 \phi_1^0 + \frac{1}{2}\phi_1^1 = 0$$

# Example: Replication of a Digital Call Option

## Example (6.1 Continued)

- By solving these linear systems, we obtain the replicating strategy at $t = 1$:

$$(\phi_1^0, \phi_1^1) = \left(-\frac{81}{200}, \frac{1}{2}\right) \quad \text{if } \omega \in \{\omega_1, \omega_2\}$$
$$(\phi_1^0, \phi_1^1) = (0, 0) \qquad \text{if } \omega \in \{\omega_3, \omega_4\}$$

- If $S_1 = 0.75$ then the price of the digital call at $t = 1$ equals 0 since $V_1(\phi) = 0 \cdot B_1 + 0 \cdot S_1 = 0$.
- If $S_1 = 1.5$ then the price of the digital call at $t = 1$ equals

$$V_1(\phi) = \phi_1^0 B_1 + \phi_1^1 S_1 = -\frac{81}{200} \cdot \frac{10}{9} + \frac{1}{2} \cdot \frac{3}{2} = 0.3$$

- The price of $X$ at time 1 equals $\pi_0(X) = 0.3\,\mathbb{1}_{\{S_1 = 1.5\}}$.

# Example: Replication of a Digital Call Option

## Example (6.1 Continued)

- We will now compute the price of the digital call at $t = 0$. The replicating strategy at time $t = 0$ satisfies

$$\frac{10}{9}\phi_0^0 + \frac{3}{2}\phi_0^1 = \frac{3}{10}$$
$$\frac{10}{9}\phi_0^0 + \frac{3}{4}\phi_0^1 = 0$$

- Then $\left(\phi_0^0, \phi_0^1\right) = (-0.27, 0.4)$. The price of the digital call at time $t = 0$ thus equals

$$V_0(\phi) = \phi_0^0 B_0 + \phi_0^1 S_0 = -0.27 + 0.4 = 0.13$$

- Hence the arbitrage price of $X$ at time 0 equals $\pi_0(X) = 0.13$.

# Example: Replication of a Digital Call Option

## Example (6.1 Continued)

Summary of pricing and hedging results for a digital call option.

- Recall that the price at time $t = 2$ equals $\pi_2(X) = X$.
- Replicating strategy $\phi$ satisfies $V_2(\phi) = X$.
- The arbitrage price process of $X$ equals $\pi(X) = V(\phi)$.

| | $\{\omega_1, \omega_2\}$ | $\{\omega_3, \omega_4\}$ |
|---|---|---|
| $t = 0$ | $(\phi_t^0, \phi_t^1) = (-0.27, 0.4)$ | $(\phi_t^0, \phi_t^1) = (-0.27, 0.4)$ |
| $t = 0$ | $\pi_0(X) = 0.13$ | $\pi_0(X) = 0.13$ |
| $t = 1$ | $(\phi_t^0, \phi_t^1) = (-\frac{81}{200}, \frac{1}{2})$ | $(\phi_t^0, \phi_t^1) = (0, 0)$ |
| $t = 1$ | $\pi_1(X) = 0.3$ | $\pi_1(X) = 0$ |

# Attainability of Contingent Claims and Completeness

**Definition (Attainable Contingent Claim)**

A contingent claim $X$ is called to be **attainable** if there exists a trading strategy $\phi \in \Phi$, which replicates $X$, i.e., $V_T(\phi) = X$.

- For attainable contingent claims, it is clear how to price them by the initial investment needed for a replicating strategy.
- As in single period market models, for some contingent claims a hedging strategy may fail to exist.

**Definition (Completeness)**

A multi period market model is said to be **complete** if and only if all contingent claims have replicating strategies. If a multi period market model is not complete, it is said to be **incomplete**.

# Risk-Neutral Valuation Formula

## Proposition (6.4)

*Let $X$ be a contingent claim (possibly non-attainable) and $\mathbb{Q}$ any risk-neutral probability measure for the multi-period market model $\mathcal{M}$. Then the* **risk-neutral valuation formula**

$$\pi_t(X) = B_t \, \mathbb{E}_{\mathbb{Q}}\left(\frac{X}{B_T} \mid \mathcal{F}_t\right)$$

*defines a price process $\pi(X) = (\pi_t(X))_{0 \leq t \leq T}$ for $X$ that complies with the principle of no-arbitrage.*

## Proof.

The proof hinges the same arguments as in the single-period case and thus it is left as an exercise. If $X$ is attainable then can also observe that $\widehat{V}(\phi)$ is a martingale under $\mathbb{Q}$ and apply the definition of a martingale. $\square$

# Example: Risk-Neutral Valuation

## Example (6.2)

$$\boxed{S_0 = 1} \quad \begin{cases} \nearrow 0.6 \quad \boxed{S_1 = 1.5} \quad \begin{cases} 0.3 \nearrow \quad \boxed{S_2 = 3} \\ 0.7 \searrow \quad \boxed{S_2 = 1} \end{cases} \\ \searrow 0.4 \quad \boxed{S_1 = 0.75} \quad \begin{cases} 0.5 \nearrow \quad \boxed{S_2 = 1.5} \\ 0.5 \searrow \quad \boxed{S_2 = 0.5} \end{cases} \end{cases}$$

| | | |
|---|---|---|
| $\boxed{S_2 = 3}$ | $\omega_1$ | $\mathbb{P}(\omega_1) = 0.18$ |
| $\boxed{S_2 = 1}$ | $\omega_2$ | $\mathbb{P}(\omega_2) = 0.42$ |
| $\boxed{S_2 = 1.5}$ | $\omega_3$ | $\mathbb{P}(\omega_3) = 0.2$ |
| $\boxed{S_2 = 0.5}$ | $\omega_4$ | $\mathbb{P}(\omega_4) = 0.2$ |

# Example: Risk-Neutral Valuation

## Example (6.2 Continued)

- Consider again the market model $\mathcal{M} = (B, S)$ introduced in Example 6.1. Recall that the conditional probabilities describe the movements under the real-world probability $\mathbb{P}$.
- Let $\mathbb{Q}$ be a risk-neutral probability measure from $\mathbb{M}$.
- We denote $q_i = \mathbb{Q}(\omega_i)$ for $i = 1, 2, 3, 4$.
- By the definition of the risk-neutral probability, we have

$$
\begin{aligned}
q_1 + q_2 + q_3 + q_4 &= 1 \\
\frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}\left(S_2 \,|\mathcal{F}_1\right) &= S_1 \\
\frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}\left(S_1 \,|\mathcal{F}_0\right) &= S_0
\end{aligned}
$$

# Example: Risk-Neutral Valuation

## Example (6.2 Continued)

- The conditional probabilities under $\mathbb{Q}$ at time $t = 1$ are

$$\mathbb{Q}\left(S_2 = 3 \,|\, \{\omega_1, \omega_2\}\right) = \frac{q_1}{q_1 + q_2}$$
$$\mathbb{Q}\left(S_2 = 1 \,|\, \{\omega_1, \omega_2\}\right) = \frac{q_2}{q_1 + q_2}$$
$$\mathbb{Q}\left(S_2 = 1.5 \,|\, \{\omega_3, \omega_4\}\right) = \frac{q_3}{q_3 + q_4}$$
$$\mathbb{Q}\left(S_2 = 0.5 \,|\, \{\omega_3, \omega_4\}\right) = \frac{q_4}{q_3 + q_4}$$

and

$$\mathbb{Q}\left(S_2 = 1.5 \,|\, \{\omega_1, \omega_2\}\right) = 0$$
$$\mathbb{Q}\left(S_2 = 0.5 \,|\, \{\omega_1, \omega_2\}\right) = 0$$
$$\mathbb{Q}\left(S_2 = 3 \,|\, \{\omega_3, \omega_4\}\right) = 0$$
$$\mathbb{Q}\left(S_2 = 1 \,|\, \{\omega_3, \omega_4\}\right) = 0$$

# Example: Risk-Neutral Probability Measure

## Example (6.2 Continued)

- Also, the probability distribution of $S_1$ reads

$$\mathbb{Q}\left(S_1 = 1.5\right) = q_1 + q_2$$
$$\mathbb{Q}\left(S_1 = 0.75\right) = q_3 + q_4$$

- Hence we obtain the following linear system:

$$\begin{aligned}
q_1 + q_2 + q_3 + q_4 &= 1 \\
\frac{9}{10}\left(\frac{3q_1}{q_1+q_2} + \frac{q_2}{q_1+q_2}\right) &= \frac{3}{2} = S_1\left(\omega_i\right) \text{ for } i = 1, 2 \\
\frac{9}{10}\left(\frac{3}{2}\frac{q_3}{q_3+q_4} + \frac{1}{2}\frac{q_4}{q_3+q_4}\right) &= \frac{3}{4} = S_1\left(\omega_i\right) \text{ for } i = 3, 4 \\
\frac{9}{10}\left[\frac{3}{2}\left(q_1+q_2\right) + \frac{3}{4}\left(q_3+q_4\right)\right] &= 1 = S_0
\end{aligned}$$

# Example: Risk-Neutral Valuation

## Example (6.2 Continued)

- Equivalently,

$$\begin{aligned} q_1 + q_2 + q_3 + q_4 &= 1 \\ 2q_1 - q_2 &= 0 \\ 2q_3 - q_4 &= 0 \\ \frac{3}{2}q_1 + \frac{3}{2}q_2 + \frac{3}{4}q_3 + \frac{3}{4}q_4 &= \frac{10}{9} \end{aligned}$$

- The unique solution reads

$$q_1 = \frac{13}{81},\ q_2 = \frac{26}{81},\ q_3 = \frac{14}{81},\ q_4 = \frac{28}{81}.$$

# Example: Risk-Neutral Valuation

## Example (6.2 Continued)

- The price of the digital call option considered in Example 6.1 can be computed as follows.
- For $\omega \in \{\omega_1, \omega_2\}$, we obtain

$$\begin{aligned}
\pi_1(X)(\omega) &= \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left(X \,|\{\omega_1, \omega_2\}\right) \\
&= \frac{9}{10}\left(1 \cdot \frac{\frac{13}{81}}{\frac{13}{81}+\frac{26}{81}} + 0 \cdot \frac{\frac{26}{81}}{\frac{13}{81}+\frac{26}{81}}\right) \\
&= 0.3
\end{aligned}$$

- For $\omega \in \{\omega_3, \omega_4\}$, we have $\pi_1(X)(\omega) = 0$ since the payoff is 0 at $t = 2$ if the stock price at time $t = 1$ equals $S_1 = 0.75$.

# Example: Risk-Neutral Valuation

## Example (6.2 Continued)

- The price of the digital call option at time 0 equals

$$\begin{aligned}
\pi_0(X) &= \frac{1}{(1+r)^2}\,\mathbb{E}_{\mathbb{Q}}\left(X \,|\, \mathcal{F}_0\right) = \frac{1}{(1+r)^2}\,\mathbb{E}_{\mathbb{Q}}\left(X\right) \\
&= \frac{81}{100}\left(1\cdot\frac{13}{81} + 0\cdot\frac{26}{81} + 0\cdot\frac{14}{81} + 0\cdot\frac{28}{81}\right) \\
&= 0.13
\end{aligned}$$

- These pricing results coincide with those obtained in Example 6.1, where we computed directly the wealth process $V(\phi)$ of the replicating strategy $\phi$ for $X$.
- As indicated earlier, a multi-period market model $\mathcal{M}$ can be decomposed into several single-period market models.

# Example: Backward Induction

## Example (6.3)

$$
\begin{array}{ccccccc}
 & & & & \boxed{S_2 = 3} & \omega_1 & \mathbb{P}(\omega_1) = 0.18 \\
 & & & 0.3 \nearrow & & & \\
 & & \boxed{S_1 = 1.5} & & & & \\
 & \nearrow & & 0.7 \searrow & & & \\
 & 0.6 & & & \boxed{S_2 = 1} & \omega_2 & \mathbb{P}(\omega_2) = 0.42 \\
\boxed{S_0 = 1} & & & & & & \\
 & 0.4 & & & \boxed{S_2 = 1.5} & \omega_3 & \mathbb{P}(\omega_3) = 0.2 \\
 & \searrow & & 0.5 \nearrow & & & \\
 & & \boxed{S_1 = 0.75} & & & & \\
 & & & 0.5 \searrow & & & \\
 & & & & \boxed{S_2 = 0.5} & \omega_4 & \mathbb{P}(\omega_4) = 0.2
\end{array}
$$

# Example: Backward Induction

## Example (6.3 Continued)

- We consider the market model in Example 6.1 once again.
- The two-period market model is composed of the following single-period market models:
  1. $S_1 = 1.5$, $S_2 = 3$ and $S_2 = 1$.
  2. $S_1 = 0.75$, $S_2 = 1.5$ and $S_2 = 0.5$.
  3. $S_0 = 1$, $S_1 = 1.5$ and $S_1 = 0.75$.
- Note that these models are elementary market models.
- Hence the unique risk-neutral probability measures can be computed using the formula

$$\tilde{p} = \frac{1 + r - d}{u - d}, \quad 1 - \tilde{p} = \frac{u - (1 + r)}{u - d}.$$

# Example: Backward Induction

## Example (6.3 Continued)

- Recall that if the elementary market model is arbitrage-free then it is also complete and all contingent claims can be priced using the risk-neutral probability.
- In the first model, the risk neutral probability measure is

$$\begin{aligned} \tilde{p}_1 &= \frac{1+r-d_1}{u_1-d_1} = \frac{1+\frac{1}{9}-\frac{2}{3}}{2-\frac{2}{3}} = \frac{1}{3} \\ 1-\tilde{p}_1 &= \frac{u_1-1-r}{u_1-d_1} = \frac{2-1-\frac{1}{9}}{2-\frac{2}{3}} = \frac{2}{3} \end{aligned}$$

- The price of the digital call option in the considered model equals

$$\pi_1^u(X) = \frac{1}{1+\frac{1}{9}} \left( \frac{1 \cdot 1}{3} + \frac{0 \cdot 2}{3} \right) = 0.3$$

# Example: Backward Induction

## Example (6.3 Continued)

- In the second model, the risk-neutral probability measure is

$$\begin{aligned}\tilde{p}_2 &= \frac{1+r-d_2}{u_2-d_2} = \frac{1+\frac{1}{9}-\frac{2}{3}}{2-\frac{2}{3}} = \frac{1}{3} \\ 1-\tilde{p}_2 &= \frac{u_2-1-r}{u_2-d_2} = \frac{2-1-\frac{1}{9}}{2-\frac{2}{3}} = \frac{2}{3}\end{aligned}$$

- The price of the digital call option in this model equals 0 since its payoff is 0. Formally,

$$\pi_1^d(X) = \frac{1}{1+\frac{1}{9}}\left(\frac{0\cdot 1}{3}+\frac{0\cdot 2}{3}\right) = 0$$

# Example: Backward Induction

## Example (6.3 Continued)

- We now move on to the last model. The risk neutral probability measure is

$$\begin{aligned}
\tilde{p}_3 &= \frac{1+r-d_3}{u_3-d_3} = \frac{1+\frac{1}{9}-\frac{3}{4}}{\frac{3}{2}-\frac{3}{4}} = \frac{13}{27} \\
1-\tilde{p}_3 &= \frac{u_3-1-r}{u_3-d_3} = \frac{\frac{3}{2}-1-\frac{1}{9}}{\frac{3}{2}-\frac{3}{4}} = \frac{14}{27}
\end{aligned}$$

We consider the contingent claim $\pi_1(X)$ with the payoff at time $t=1$ given by

$$\pi_1(X) = \begin{cases} \pi_1^u(X) = 0.3 & \text{if } S_1 = u_3 S_0 = 1.5 \\ \pi_1^d(X) = 0 & \text{if } S_1 = d_3 S_0 = 0.75 \end{cases}$$

# Example: Backward Induction

## Example (6.3 Continued)

- The price of $X$ at time 0 equals

$$\pi_0(X) = \frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}(\pi_1(X)) = \frac{1}{1+1/9}\left(\frac{0.3\cdot 13}{27} + \frac{0\cdot 14}{27}\right) = 0.13$$

- Hence $\pi_0(X) = 0.13$, $\pi_1(X)(\omega) = 0.3$ if $\omega \in \{\omega_1, \omega_2\}$ and $\pi_1(X)(\omega) = 0$ if $\omega \in \{\omega_3, \omega_4\}$.
- Note that the unique risk-neutral probability measure $\mathbb{Q}$ in two-period market model can be recomputed as follows:

$$\begin{aligned}
\mathbb{Q}(\omega_1) &= \tilde{p}_3\tilde{p}_1 = \frac{13}{27}\cdot\frac{1}{3} = \frac{13}{81}\\
\mathbb{Q}(\omega_2) &= \tilde{p}_3(1-\tilde{p}_1) = \frac{13}{27}\cdot\frac{2}{3} = \frac{26}{81}\\
\mathbb{Q}(\omega_3) &= (1-\tilde{p}_3)\,\tilde{p}_2 = \frac{14}{27}\cdot\frac{1}{3} = \frac{14}{81}\\
\mathbb{Q}(\omega_4) &= (1-\tilde{p}_3)(1-\tilde{p}_2) = \frac{14}{27}\cdot\frac{2}{3} = \frac{28}{81}
\end{aligned}$$

# Completeness

As a handy criterion for the market completeness, we have the theorem, which extends the known result for the single-period case.

**Theorem (6.1)**

*Assume that a multi-period market model* $\mathcal{M} = (B, S^1, \ldots, S^n)$ *is arbitrage-free. Then* $\mathcal{M}$ *is complete if and only if there is only one risk-neutral probability measure, that is,* $\mathbb{M} = \{\widetilde{\mathbb{P}}\}$ *is a singleton.*

In the context of a model decomposition, the following statements are known to hold:

- If all single-period models which compose a multi-period model are arbitrage-free then the multi-period model is also arbitrage-free.
- If they are also complete then the multi-period model is also complete.
- The converse of the above statement is also correct.

# Summary of Pricing and Hedging Approaches

We examined three pricing and hedging approaches:

1. The method based on the idea of **replication** of a contingent claim. It can only be applied to attainable contingent claim in a complete or incomplete model and it yields the hedging strategy and arbitrage price process.
2. The method relying on the concept of a **risk-neutral probability**, which can be applied in either a complete or incomplete model. It furnishes the unique arbitrage price process for any attainable claim and a possible arbitrage price process for a non-attainable contingent claim. In the latter case, the price process depends on the choice of a risk-neutral probability.
3. The **backward induction** approach in which a multi-period market model is decomposed into a family of single-period models. Pricing is performed in a recursive way starting from the date $T-1$ and moving step-by-step towards the initial date 0. Hedging strategy can also be computed provided that the claim is attainable.