Mathematical Finance

Single Period Market Model

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Semester 1

General Single-Period Market Model

- The main differences between the elementary and general single-period market models are:
  - The investor is allowed to invest in several risky securities instead of only one.
  - The sample set is bigger, that is, there are more possible states of the world at time $t = 1$.
- The sample space is $\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$ with $\mathcal{F} = 2^\Omega$.
- An investor's personal beliefs about the future behaviour of stock prices are represented by the probability measure $\mathbb{P}(\omega_i) = p_i > 0$ for $i = 1, 2, \dots, k$.
- The savings account B equals $B_0 = 1$ and $B_1 = 1 + r$ for some constant $r > -1$.
- The price of the jth stock at $t = 1$ is a random variable on Ω. It is denoted by S_t^j for $t = 0, 1$ and $j = 1, \dots, n$.
- A contingent claim $X = (X(\omega_1), \dots, X(\omega_k))$ is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Questions

1. Under which conditions a general single-period market model $\mathcal{M} = (B, S^1, \ldots, S^n)$ is arbitrage-free?
2. How to define a risk-neutral probability measure for a model?
3. How to use a risk-neutral probability measure to analyse a general single-period market model?
4. Under which conditions a general single-period market model is complete?
5. Is completeness of a market model related to risk-neutral probability measures?
6. How to define an arbitrage price of an attainable claim?
7. Can we still apply the risk-neutral valuation formula to compute the price of an attainable claim?
8. How to deal with contingent claims that are not attainable?
9. How to use the class of risk-neutral probability measures to value non-attainable claims?

Outline

We will examine the following issues:

1. Trading Strategies and Arbitrage-Free Models
2. Fundamental Theorem of Asset Pricing
3. Examples of Market Models
4. Risk-Neutral Valuation of Contingent Claims
5. Completeness of Market Models

Trading Strategy

Definition (Trading Strategy)

A **trading strategy** (or a **portfolio**) for an investor in a general single-period market model is defined as a vector

$$(x, \phi^1, \dots, \phi^n) \in \mathbb{R}^{n+1}$$

where x is the initial wealth of an investor and ϕ^j stands for the number of shares of the jth stock purchased at time $t = 0$.

If an investor adopts the trading strategy $(x, \phi^1, \dots, \phi^n)$ at time $t = 0$ then the cash value of his portfolio at time $t = 1$ equals

$$V_1(x, \phi^1, \dots, \phi^n) := \left(x - \sum_{j=1}^{n} \phi^j S_0^j \right) (1 + r) + \sum_{j=1}^{n} \phi^j S_1^j.$$

Wealth Process of a Trading Strategy

Definition (Wealth Process)

The **wealth process** (or the **value process**) of a trading strategy $(x, \phi^1, \ldots, \phi^n)$ is the pair

$$\big(V_0(x, \phi^1, \ldots, \phi^n), V_1(x, \phi^1, \ldots, \phi^n)\big).$$

The real number $V_0(x, \phi^1, \ldots, \phi^n)$ is the initial endowment

$$V_0(x, \phi^1, \ldots, \phi^n) := x$$

and the real-valued random variable $V_1(x, \phi^1, \ldots, \phi^n)$ represents the cash value of the portfolio at time $t = 1$

$$V_1(x, \phi^1, \ldots, \phi^n) := \left(x - \sum_{j=1}^{n} \phi^j S_0^j\right)(1 + r) + \sum_{j=1}^{n} \phi^j S_1^j.$$

Gains (Profits and Losses) Process

- Obviously, the profits or losses an investor obtains from the investment can be calculated by subtracting $V_0(\cdot)$ from $V_1(\cdot)$. This is called the (undiscounted) gains process.
- The 'gain' can be negative; hence it may also represent a loss.

Definition (Gains Process)

The **gains process** is defined as $G_0(x, \phi^1, \ldots, \phi^n) = 0$ and

$$\begin{aligned} G_1(x, \phi^1, \ldots, \phi^n) &:= V_1(x, \phi^1, \ldots, \phi^n) - V_0(x, \phi^1, \ldots, \phi^n) \\ &= \left(x - \sum_{j=1}^{n} \phi^j S_0^j\right) r + \sum_{j=1}^{n} \phi^j \Delta S_1^j \end{aligned}$$

where the random variable $\Delta S_1^j = S_1^j - S_0^j$ represents the nominal change in the price of the jth stock.

Discounted Stock Price and Wealth Process

- To understand whether the jth stock appreciates in real terms, we consider the **discounted stock prices** of the jth stock

$$\widehat{S}_0^j := S_0^j = \frac{S_0^j}{B_0}, \quad \widehat{S}_1^j := \frac{S_1^j}{1+r} = \frac{S_1^j}{B_1}.$$

- Similarly, we define the **discounted wealth process** as

$$\widehat{V}_0(x, \phi^1, \dots, \phi^n) := x, \quad \widehat{V}_1(x, \phi^1, \dots, \phi^n) := \frac{V_1(x, \phi^1, \dots, \phi^n)}{B_1}.$$

- It is easy to see that

$$\begin{aligned} \widehat{V}_1(x, \phi^1, \dots, \phi^n) &= \left(x - \sum_{j=1}^n \phi^j S_0^j \right) + \sum_{j=1}^n \phi^j \widehat{S}_1^j \\ &= x + \sum_{j=1}^n \phi^j (\widehat{S}_1^j - \widehat{S}_0^j). \end{aligned}$$

Discounted Gains Process

Definition (Discounted Gains Process)

The **discounted gains process** for the investor is defined as

$$\widehat{G}_0(x, \phi^1, \dots, \phi^n) = 0$$

and

$$\begin{aligned}\widehat{G}_1(x, \phi^1, \dots, \phi^n) &:= \widehat{V}_1(x, \phi^1, \dots, \phi^n) - \widehat{V}_0(x, \phi^1, \dots, \phi^n) \\ &= \sum_{j=1}^{n} \phi^j \Delta \widehat{S}_1^j\end{aligned}$$

where $\Delta \widehat{S}_1^j = \widehat{S}_1^j - \widehat{S}_0^j$ represents the change in the discounted price of the jth stock.

Arbitrage: Definition

The concept of an arbitrage in a general single-period market model is essentially the same as in the elementary market model. It is worth noting that $\mathbb{P}$ can be replaced here by any equivalent probability measure $\mathbb{Q}$.

Definition (Arbitrage)

A trading strategy $(x, \phi^1, \ldots, \phi^n)$ in a general single-period market model is called an **arbitrage opportunity** if

- A.1. $V_0(x, \phi^1, \ldots, \phi^n) = 0$,
- A.2. $V_1(x, \phi^1, \ldots, \phi^n)(\omega_i) \geq 0$ for $i = 1, 2, \ldots, k$,
- A.3. $\mathbb{E}_{\mathbb{P}} \left\{ V_1(x, \phi^1, \ldots, \phi^n) \right\} > 0$, that is,

$$\sum_{i=1}^{k} V_1(x, \phi^1, \ldots, \phi^n)(\omega_i) \, \mathbb{P}(\omega_i) > 0.$$

Arbitrage: Equivalent Conditions

The following condition is equivalent to A.3.

- A.3′. There exists $\omega \in \Omega$ such that $V_1(x, \phi^1, \dots, \phi^n)(\omega) > 0$.

The definition of arbitrage can be formulated using the discounted value and gains processes. This is sometimes very helpful.

Proposition (4.1)

A trading strategy $(x, \phi^1, \dots, \phi^n)$ in a general single-period market model is an arbitrage opportunity if and only if one of the following conditions holds:

1. *Assumptions A.1-A.3 in the definition of arbitrage hold with $\widehat{V}(x, \phi^1, \dots, \phi^n)$ instead of $V(x, \phi^1, \dots, \phi^n)$.*
2. *$x = 0$ and A.2-A.3 in the definition of arbitrage are satisfied with $\widehat{G}_1(x, \phi^1, \dots, \phi^n)$ instead of $G_1(x, \phi^1, \dots, \phi^n)$.*

Proof.

[Proof of Proposition 4.1: First step] We will show that the following two statements are true:
The definition of arbitrage and condition 1 in Proposition 4.1 are equivalent.
In Proposition 4.1, condition 1 is equivalent to condition 2.
To prove the first statement, we use the relationships between $V(x, \phi^1, \ldots, \phi^n)$ and $\widehat{V}(x, \phi^1, \ldots, \phi^n)$, that is,

$$
\begin{aligned}
\widehat{V}_0(x, \phi^1, \ldots, \phi^n) &= V_0(x, \phi^1, \ldots, \phi^n) = x, \\
\widehat{V}_1(x, \phi^1, \ldots, \phi^n) &= \frac{1}{1+r} V_1(x, \phi^1, \ldots, \phi^n), \\
\mathbb{E}_{\mathbb{P}}\left\{\widehat{V}_1(x, \phi^1, \ldots, \phi^n)\right\} &= \frac{1}{1+r} \mathbb{E}_{\mathbb{P}}\left\{V_1(x, \phi^1, \ldots, \phi^n)\right\}.
\end{aligned}
$$

This shows that the first statement holds. $\square$

Proof.

[Proof of Proposition 4.1: Second step] To prove the second statement, we recall the relation between $\widehat{V}(x, \phi^1, \ldots, \phi^n)$ and $\widehat{G}_1(x, \phi^1, \ldots, \phi^n)$

$$\begin{aligned}\widehat{G}_1(x, \phi^1, \ldots, \phi^n) &= \widehat{V}_1(x, \phi^1, \ldots, \phi^n) - \widehat{V}_0(x, \phi^1, \ldots, \phi^n) \\ &= \widehat{V}_1(x, \phi^1, \ldots, \phi^n) - x.\end{aligned}$$

It is now clear that for $x = 0$ we have

$$\widehat{G}_1(x, \phi^1, \ldots, \phi^n) = \widehat{V}_1(x, \phi^1, \ldots, \phi^n).$$

Hence the second statement is true as well.
In fact, one can also observe that $\widehat{G}_1(x, \phi^1, \ldots, \phi^n)$ does not depend on x at all, since $\widehat{G}_1(x, \phi^1, \ldots, \phi^n) = \sum_{j=1}^{n} \phi^j \Delta \widehat{S}_1^j$. □

Verification of the Arbitrage-Free Property

- It can be sometimes hard to check directly whether arbitrage opportunities exist in a given market model, especially when dealing with several risky assets or in the multi-period setup.
- We have introduced the risk-neutral probability measure in the elementary market model and we noticed that it can be used to compute the arbitrage price of any contingent claim.
- We will show that the concept of a risk-neutral probability measure is also a convenient tool for checking whether a general single-period market model is arbitrage-free or not.
- In addition, we will argue that a risk-neutral probability measure can also be used for the purpose of valuation of a contingent claim (either attainable or not).

Risk-Neutral Probability Measure

Definition (Risk-Neutral Probability Measure)

A probability measure $\mathbb{Q}$ on Ω is called a **risk-neutral probability measure** for a general single-period market model $\mathcal{M}$ if:

R.1. $\mathbb{Q}(\omega_i) > 0$ for all $\omega_i \in \Omega$,

R.2. $\mathbb{E}_{\mathbb{Q}}\left(\Delta \widehat{S}_1^j\right) = 0$ for $j = 1, 2, \ldots, n$.

We denote by $\mathbb{M}$ the class of all risk-neutral probability measures for the market model $\mathcal{M}$.

- Condition R.1 means that $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures. A risk-neutral probability measure is also known as an **equivalent martingale measure**.
- Note that condition R.2 is equivalent to $\mathbb{E}_{\mathbb{Q}}(\widehat{S}_1^j) = \widehat{S}_0^j$ or, more explicitly,

$$\mathbb{E}_{\mathbb{Q}}\left(S_1^j\right) = (1+r)S_0^j$$

  for $j = 1, 2, \ldots, n$.

Example 4.1: Stock Prices

- We consider the following model featuring two stocks S^1 and S^2 on the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
- The interest rate $r = \frac{1}{10}$ so that $B_0 = 1$ and $B_1 = 1 + \frac{1}{10}$.
- We deal here with the market model $\mathcal{M} = (B, S^1, S^2)$.
- The stock prices at $t = 0$ are given by $S_0^1 = 2$ and $S_0^2 = 3$.
- The stock prices at $t = 1$ are represented in the table:

	ω_1	ω_2	ω_3
S_1^1	1	5	3
S_1^2	3	1	6

Example 4.1: Wealth Process

- For any trading strategy $(x, \phi^1, \phi^2) \in \mathbb{R}^3$, we have

$$V_1(x, \phi^1, \phi^2) = (x - 2\phi^1 - 3\phi^2)\left(1 + \frac{1}{10}\right) + \phi^1 S_1^1 + \phi^2 S_1^2.$$

- We set $\phi^0 := x - 2\phi^1 - 3\phi^2$. Then $V_1(x, \phi^1, \phi^2)$ equals

$$\begin{aligned}
V_1(x, \phi^1, \phi^2)(\omega_1) &= \phi^0\left(1 + \frac{1}{10}\right) + \phi^1 + 3\phi^2, \\
V_1(x, \phi^1, \phi^2)(\omega_2) &= \phi^0\left(1 + \frac{1}{10}\right) + 5\phi^1 + \phi^2, \\
V_1(x, \phi^1, \phi^2)(\omega_3) &= \phi^0\left(1 + \frac{1}{10}\right) + 3\phi^1 + 6\phi^2.
\end{aligned}$$

Example 4.1: Gains Process

- The increments ΔS_1^j are represented by the following table

	ω_1	ω_2	ω_3
ΔS_1^1	-1	3	1
ΔS_1^2	0	-2	3

- The gains $G_1(x, \phi^1, \phi^2)$ are thus given by

$$G_1(x, \phi^1, \phi^2)(\omega_1) = \frac{1}{10}\phi^0 - \phi^1 + 0\phi^2,$$
$$G_1(x, \phi^1, \phi^2)(\omega_2) = \frac{1}{10}\phi^0 + 3\phi^1 - 2\phi^2,$$
$$G_1(x, \phi^1, \phi^2)(\omega_3) = \frac{1}{10}\phi^0 + \phi^1 + 3\phi^2.$$

Example 4.1: Discounted Stock Prices

- Out next goal is to compute the discounted wealth process $\widehat{V}(x, \phi^1, \phi^2)$ and the discounted gains process $\widehat{G}_1(x, \phi^1, \phi^2)$.
- To this end, we first compute the discounted stock prices.
- Of course, $\widehat{S}_0^j = S_0^j$ for $j = 1, 2$.
- The following table represents the discounted stock prices $\widehat{S}_1^j$ for $j = 1, 2$ at time $t = 1$

	ω_1	ω_2	ω_3
$\widehat{S}_1^1$	$\frac{10}{11}$	$\frac{50}{11}$	$\frac{30}{11}$
$\widehat{S}_1^2$	$\frac{30}{11}$	$\frac{10}{11}$	$\frac{60}{11}$

Example 4.1: Discounted Wealth Process

The discounted wealth process $\widehat{V}(x, \phi^1, \phi^2)$ is thus given by

$$\widehat{V}_0(x, \phi^1, \phi^2) = V_0(x, \phi^1, \phi^2) = x$$

and

$$\begin{aligned}
\widehat{V}_1(x, \phi^1, \phi^2)(\omega_1) &= \phi^0 + \frac{10}{11}\phi^1 + \frac{30}{11}\phi^2, \\
\widehat{V}_1(x, \phi^1, \phi^2)(\omega_2) &= \phi^0 + \frac{50}{11}\phi^1 + \frac{10}{11}\phi^2, \\
\widehat{V}_1(x, \phi^1, \phi^2)(\omega_3) &= \phi^0 + \frac{30}{11}\phi^1 + \frac{60}{11}\phi^2,
\end{aligned}$$

where $\phi^0 = x - 2\phi^1 - 3\phi^2$ is the amount of cash invested in B at time 0 (as opposed to the initial wealth given by x).

Example 4.1: Discounted Gains Process

- The increments of the discounted stock prices equal

	ω_1	ω_2	ω_3
$\Delta\widehat{S}_1^1$	$-\frac{12}{11}$	$\frac{28}{11}$	$\frac{8}{11}$
$\Delta\widehat{S}_1^2$	$-\frac{3}{11}$	$-\frac{23}{11}$	$\frac{27}{11}$

- Hence the discounted gains $\widehat{G}_1(x, \phi^1, \phi^2)$ are given by

$$\widehat{G}_1(x, \phi^1, \phi^2)(\omega_1) = -\frac{12}{11}\phi^1 - \frac{3}{11}\phi^2,$$

$$\widehat{G}_1(x, \phi^1, \phi^2)(\omega_2) = \frac{28}{11}\phi^1 - \frac{23}{11}\phi^2,$$

$$\widehat{G}_1(x, \phi^1, \phi^2)(\omega_3) = \frac{8}{11}\phi^1 + \frac{27}{11}\phi^2.$$

Example 4.1: Arbitrage-Free Property

- The condition $\widehat{G}_1(x, \phi^1, \phi^2) \geq 0$ is equivalent to

$$\begin{aligned} -12\phi^1 - 3\phi^2 &\geq 0 \\ 28\phi^1 - 23\phi^2 &\geq 0 \\ 8\phi^1 + 27\phi^2 &\geq 0 \end{aligned}$$

- Can we find $(\phi^1, \phi^2) \in \mathbb{R}^2$ such that all inequalities are valid and at least one of them is strict?
- It appears that the answer is negative, since the unique vector satisfying all inequalities above is $(\phi^1, \phi^2) = (0, 0)$.
- Therefore, the single-period market model $\mathcal{M} = (B, S^1, S^2)$ is arbitrage-free.

Example 4.1: Risk-Neutral Probability Measure

- We will now show that this market model admits a unique risk-neutral probability measure on $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
- Let us denote $\mathbb{Q}(\omega_i) = q_i$ for $i = 1, 2, 3$. From the definition of a risk-neutral probability measure, we obtain the following linear system

$$
\begin{aligned}
-\frac{12}{11}q_1 + \frac{28}{11}q_2 + \frac{8}{11}q_3 &= 0 \\
-\frac{3}{11}q_1 - \frac{23}{11}q_2 + \frac{27}{11}q_3 &= 0 \\
q_1 + q_2 + q_3 &= 1
\end{aligned}
$$

- The unique solution equals $\mathbb{Q} = (q_1, q_2, q_3) = \left(\frac{47}{80}, \frac{15}{80}, \frac{18}{80}\right)$.

Fundamental Theorem of Asset Pricing (FTAP)

- In Example 4.1, we have checked directly that the market model $\mathcal{M} = (B, S^1, S^2)$ is arbitrage-free.
- In addition, we have shown that the unique risk-neutral probability measure exists in this model.
- Is there any relation between no arbitrage property of a market model and the existence of a risk-neutral probability measure?

Fundamental Theorem of Asset Pricing (FTAP)

- The following important result, known as the FTAP, gives a complete answer to this question within the present setup.
- The FTAP was first established by Harrison and Pliska (1981) and it was later extended to continuous-time market models.

Theorem (FTAP)

A general single-period market model $\mathcal{M} = (B, S^1, \ldots, S^n)$ is arbitrage-free if and only if there exists a risk-neutral probability measure for $\mathcal{M}$, that is, $\mathbb{M} \neq \emptyset$.

Proof.

The proof is optional and is given at the end of this set of slides. It is very easy to prove that if the set of all risk neutral probability measures $\mathbb{M}$ is not empty then there is no arbitrage. The proof of the opposite statement is much more difficult. $\square$

Example 4.1: Arbitrage-Free Market Model

- We consider the market model $\mathcal{M} = (B, S^1, S^2)$ introduced in Example 4.1.
- The interest rate $r = \frac{1}{10}$ so that $B_0 = 1$ and $B_1 = 1 + \frac{1}{10}$.
- The stock prices at $t = 0$ are given by $S_0^1 = 2$ and $S_0^2 = 3$.
- We have shown that the increments of the discounted stock prices $\widehat{S}^1$ and $\widehat{S}^2$ equal

	ω_1	ω_2	ω_3
$\Delta\widehat{S}_1^1$	$-\frac{12}{11}$	$\frac{28}{11}$	$\frac{8}{11}$
$\Delta\widehat{S}_1^2$	$-\frac{3}{11}$	$-\frac{23}{11}$	$\frac{27}{11}$

Example 4.1: Arbitrage-Free Market Model

- How to find RNP measure?
- If it exists then $\mathbb{E}_{\mathbb{Q}}\left(\Delta \widehat{S}_1^j\right) = 0$, in our case

$$
\begin{aligned}
-\frac{12}{11}q_1 + \frac{28}{11}q_2 + \frac{8}{11}q_3 &= 0 \\
-\frac{3}{11}q_1 - \frac{23}{11}q_2 + \frac{27}{11}q_3 &= 0
\end{aligned}
$$

- We have two vectors of discounted return $Z_1 = \begin{pmatrix} -12 \\ 28 \\ 8 \end{pmatrix}$ and $Z_2 = \begin{pmatrix} -3 \\ -23 \\ 27 \end{pmatrix}$ and equations above can be rewritten as inner products

$$
\begin{aligned}
\langle q, Z_1 \rangle &= 0 \\
\langle q, Z_2 \rangle &= 0
\end{aligned}
$$

Example 4.1: Arbitrage-Free Market Model

- The gain expressions

$$\widehat{G}_1(x,\phi^1,\phi^2)(\omega_1) = -\frac{12}{11}\phi^1 - \frac{3}{11}\phi^2$$
$$\widehat{G}_1(x,\phi^1,\phi^2)(\omega_2) = \frac{28}{11}\phi^1 - \frac{23}{11}\phi^2$$
$$\widehat{G}_1(x,\phi^1,\phi^2)(\omega_3) = \frac{8}{11}\phi^1 + \frac{27}{11}\phi^2$$

  determines a hyperplane in $\mathbb{R}^3$: $\phi^1 Z_1 + \phi^2 Z_2$, $\phi^1, \phi^2 \in \mathbb{R}$
- So, vector $q : \langle q, Z_j \rangle = 0$ must be orthogonal to the hyperplane.
- And $\sum_i^K q_i = 1$, $q_i > 0$
- If such vector exists then it defines RNP measure.
- For our example there is the unique risk-neutral probability measure.
- The FTAP confirms that the market model is arbitrage-free.

Example 4.2: Market Model with Arbitrage

- We consider the following model featuring two stocks S^1 and S^2 on the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$.
- The interest rate $r = \frac{1}{10}$ so that $B_0 = 1$ and $B_1 = 1 + \frac{1}{10}$.
- The stock prices at $t = 0$ are given by $S_0^1 = 1$ and $S_0^2 = 2$ and the stock prices at $t = 1$ are represented in the table:

	ω_1	ω_2	ω_3
S_1^1	1	$\frac{1}{2}$	3
S_1^2	$\frac{5}{2}$	4	$\frac{1}{10}$

- Does this market model admit an arbitrage opportunity?

Example 4.2: Market Model with Arbitrage

- The increments of discounted stock prices are represented in the following table

	ω_1	ω_2	ω_3
$\Delta\widehat{S}_1^1$	$-\frac{1}{11}$	$-\frac{6}{11}$	$\frac{19}{11}$
$\Delta\widehat{S}_1^2$	$\frac{3}{11}$	$\frac{18}{11}$	$-\frac{21}{11}$

- To tell whether a model is arbitrage-free it suffices to know the increments of discounted stock prices.

Example 4.2: Market Model with Arbitrage

- Recall that
$$\widehat{G}_1(x, \phi^1, \phi^2) = \phi^1 \Delta \widehat{S}_1^1 + \phi^2 \Delta \widehat{S}_1^2$$
Hence, the equation for hyperplane
$$\mathbb{W} = \left\{ \phi^1 \begin{pmatrix} -1 \\ -6 \\ 19 \end{pmatrix} + \phi^2 \begin{pmatrix} 3 \\ 18 \\ -21 \end{pmatrix} \mid \phi^1, \phi^2 \in \mathbb{R} \right\}.$$
- Let us take $\phi^1 = 3$ and $\phi^2 = 1$. Then we obtain the vector $(0, 0, 36)^T$, which means we have an arbitrage opportunity.

Example 4.2: Market Model with Arbitrage

- Still, we can solve the system for the 'orthogonal vector':

$$q = \left\{ \gamma \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} \mid \gamma \in \mathbb{R} \right\}.$$

- but must have $\mathbb{Q}(\omega) > 0$ for all $\omega \in \Omega$.
- Hence the FTAP confirms that the market model is not arbitrage-free.

Contingent Claims

- Since we now know how to check whether a given model is arbitrage-free, the following question arises:
- What should be the 'fair' price of a European call or put option in a general single-period market model?
- In a general single-period market model, the idea of pricing European options can be extended to any contingent claim.

Definition (Contingent Claim)

A **contingent claim** is a random variable X defined on Ω and representing a payoff at the maturity date.

- Derivatives nowadays are usually quite complicated and thus it makes sense to analyse valuation and hedging of a general contingent claim, and not only European call and put options.

No-Arbitrage Principle

Definition (Replication and Arbitrage Price)

A trading strategy $(x, \phi^1, \ldots, \phi^n)$ is called a **replicating strategy** (a **hedging strategy**) for a claim X when $V_1(x, \phi^1, \ldots, \phi^n) = X$. Then the initial wealth is also denoted as $\pi_0(X)$ and it is called the **arbitrage price** of X.

Proposition (No-Arbitrage Principle)

Assume that a contingent claim X can be replicated by means of a trading strategy $(x, \phi^1, \ldots, \phi^n)$. Then the unique price of X at 0 consistent with no-arbitrage principle equals $V_0(x, \phi^1, \ldots, \phi^n) = x$.

Proof.

If the price of X is higher (lower) than x, one can short sell (buy) X and buy (short sell) the replicating portfolio. This will yield an arbitrage opportunity in the extended market in which X is traded at time $t = 0$. □

Example 4.3: Stochastic Volatility Model

- In the elementary market model, a replicating strategy for any contingent claim always exists. However, in a general single-period market model, a replicating strategy may fail to exist for some claims.
- For instance, when there are more sources of randomness than there are stocks available for investment then replicating strategies do not exist for some claims.
- Consider a market model consisting of bond B, stock S, and a random variable v called the **volatility**.
- The volatility determines whether the stock price can make either a big or a small jump.
- This is a simple example of a **stochastic volatility model**.

Example 4.3: Stochastic Volatility Model

- The sample space is given by

$$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$$

and the volatility is defined as

$$v(\omega_i) = \begin{cases} h \text{ for } i = 1, 4, \\ l \text{ for } i = 2, 3. \end{cases}$$

- We furthermore assume that $0 < l < h < 1$. The stock price S_1 is given by

$$S_1(\omega_i) = \begin{cases} (1 + v(\omega_i))S_0 \text{ for } i = 1, 2, \\ (1 - v(\omega_i))S_0 \text{ for } i = 3, 4. \end{cases}$$

Example 4.3: Stochastic Volatility Model

- Unlike in examples we considered earlier, the amount by which the stock price in this market model jumps is random.
- It is easy to check that the model is arbitrage-free whenever $1 - h < 1 + r < 1 + h$.
- We claim that for some contingent claims a replicating strategy does not exist. In that case, we say that a claim is not **attainable**.
- To justify this claim, we consider the **digital call option** X with the payoff

$$X = \begin{cases} 1 \text{ if } S_1 > K, \\ 0 \text{ otherwise,} \end{cases}$$

where $K > 0$ is the strike price.

Example 4.3: Stochastic Volatility Model

- We assume that $(1+l)S_0 < K < (1+h)S_0$, so that

$$(1-h)S_0 < (1-l)S_0 < (1+l)S_0 < K < (1+h)S_0$$

and thus

$$X(\omega_i) = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that (x, ϕ) is a replicating strategy for X. Equality $V_1(x, \phi) = X$ becomes

$$(x - \phi S_0) \begin{pmatrix} 1+r \\ 1+r \\ 1+r \\ 1+r \end{pmatrix} + \phi \begin{pmatrix} (1+h)S_0 \\ (1+l)S_0 \\ (1-l)S_0 \\ (1-h)S_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 4.3: Stochastic Volatility Model

- two unknowns, four equations

$$(x - \phi S_0) \begin{pmatrix} 1+r \\ 1+r \\ 1+r \\ 1+r \end{pmatrix} + \phi \begin{pmatrix} (1+h)S_0 \\ (1+l)S_0 \\ (1-l)S_0 \\ (1-h)S_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- It is easy to see that the above system of equations has no solution and thus a digital call is *not an attainable* contingent claim within the framework of the stochastic volatility model.
- The heuristic explanation is that the randomness generated by the volatility cannot be replicated, we do not have anough traded assets to replicate volatility, since the volatility itself is not a traded asset in this model.

Valuation of Attainable Contingent Claim

We first recall the definition of attainability of a contingent claim.

Definition (Attainable Contingent Claim)

A contingent claim X is called to be **attainable** if there exists a replicating strategy for X.

Let us summarise the known properties of attainable claims:

- It is clear how to price attainable contingent claims by the replicating principle.
- There might be more than one replicating strategy, but no arbitrage principle leads the initial wealth x (and therefore, price) to be unique.
- In the two-state single-period market model, one can use the risk-neutral probability measure to price contingent claims.

Risk-Neutral Valuation Formula

Our next goal is to extend the **risk-neutral valuation formula** to any attainable contingent claim within the framework of a general single-period market model.

Proposition (4.2)

Let X be an attainable contingent claim and let $\mathbb{Q} \in \mathbb{M}$ be any risk-neutral probability measure. Then the arbitrage price of X at $t = 0$ equals

$$\pi_0(X) = \frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}\left(X\right).$$

Proof.

[Proof of Proposition 4.2] Recall that a trading strategy $(x, \phi^i, \ldots, \phi^n)$ is a replicating strategy for X whenever $V_1(x, \phi^1, \ldots, \phi^n) = X$ $\square$

Proof.

[Proof of Proposition 4.2] We divide both sides by $1+r$, to obtain

$$\frac{X}{1+r} = \frac{V_1(x,\phi^i,\dots,\phi^n)}{1+r} = \widehat{V}_1(x,\phi^1,\dots,\phi^n).$$

Hence

$$\begin{aligned}
\frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}(X) &= \mathbb{E}_{\mathbb{Q}}\left\{\widehat{V}_1(x,\phi^1,\dots,\phi^n)\right\} \\
&= \mathbb{E}_{\mathbb{Q}}\left\{x + \widehat{G}_1(x,\phi^1,\dots,\phi^n)\right\} \\
&= x + \mathbb{E}_{\mathbb{Q}}\left\{\sum_{j=1}^{n}\phi^j\Delta\widehat{S}_1^j\right\} \\
&= x + \sum_{j=1}^{n}\phi^j\,\mathbb{E}_{\mathbb{Q}}\left(\Delta\widehat{S}_1^j\right) = x. \quad \text{(from R.2.)}
\end{aligned}$$

$\square$

Example 4.3: Stochastic Volatility Model

- Proposition 4.2 shows that risk-neutral probability measures can be used to price attainable contingent claims.
- Consider the market model introduced in Example 4.3 with the interest rate $r = 0$.
- Recall that in this case the model is arbitrage-free since $1 - h < 1 + r = 1 < 1 + h$.
- The increments of the discounted stock price $\widehat{S}$ are represented in the following table

	ω_1	ω_2	ω_3	ω_4
$\Delta \widehat{S}_1$	hS_0	lS_0	$-lS_0$	$-hS_0$

Example 4.3: Stochastic Volatility Model

- The equation for 'hyperplane'

$$\mathbb{W} = \left\{ \varphi \begin{pmatrix} h \\ l \\ -l \\ -h \end{pmatrix} \,\middle|\, \varphi \in \mathbb{R} \right\}.$$

  so it is not a hyperplane which must be the plane of maximim dimension.

- The orthogonal complement of $\mathbb{W}$ is thus the three-dimensional subspace of $\mathbb{R}^4$ given by

$$\mathbb{W}^\perp = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, \left\langle \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}, \begin{pmatrix} h \\ l \\ -l \\ -h \end{pmatrix} \right\rangle = 0 \right\}.$$

Example 4.3: Stochastic Volatility Model

- Recall that a vector $(q_1, q_2, q_3, q_4)^\top$ must satisfy $\sum_{i=1}^4 q_i = 1$ holds and $q_i > 0$ for $i = 1, 2, 3, 4$.
- Specifically,

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \in \mathbb{M} \quad \Leftrightarrow \quad q_i > 0, \quad \sum_{i=1}^{4} q_i = 1$$

$$\text{and} \quad h(q_1 - q_4) + l(q_2 - q_3) = 0.$$

Example 4.3: Stochastic Volatility Model

- The class $\mathbb{M}$ of all risk-neutral probability measures in our stochastic volatility model is therefore given by

$$\mathbb{M} = \left\{ \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ 1 - q_1 - q_2 - q_3 \end{pmatrix} \middle| \begin{array}{c} q_1 > 0,\ q_2 > 0,\ q_3 > 0, \\ q_1 + q_2 + q_3 < 1, \\ l(q_2 - q_3) = h(1 - (2q_1 + q_2 + q_3)) \end{array} \right\}$$

- This set appears to be non-empty and thus we conclude that our stochastic volatility model is arbitrage-free.
- Recall that we have already shown that the digital call option is not attainable if

$$(1 + l)S_0 < K < (1 + h)S_0.$$

Example 4.3: Stochastic Volatility Model

- It is not difficult to check that for every $0 < q_1 < \frac{1}{2}$ there exists a probability measure $\mathbb{Q} \in \mathbb{M}$ such that $\mathbb{Q}(\omega_1) = q_1$.
- Indeed, it suffices to take $q_1 \in (0, \frac{1}{2})$ and to set

$$q_4 = q_1, \quad q_2 = q_3 = \frac{1}{2} - q_1.$$

- We apply the risk-neutral valuation formula to the digital call $X = (1, 0, 0, 0)^\top$. For $\mathbb{Q} = (q_1, q_2, q_3, q_4)^\top \in \mathbb{M}$, we obtain

$$\mathbb{E}_{\mathbb{Q}}(X) = q_1 \cdot 1 + q_2 \cdot 0 + q_3 \cdot 0 + q_4 \cdot 0 = q_1.$$

- Since q_1 is any number from $(0, \frac{1}{2})$, we see that every value from the open interval $(0, \frac{1}{2})$ can be achieved.

Extended Market Model and No-Arbitrage Principle

We no longer assume that a contingent claim X is attainable.

Definition (4.1)

A price $\pi_0(X)$ of a contingent claim X is said to be **consistent with the no-arbitrage principle** if the extended model, which consists of the bond B, original stocks $S^1, \ldots, S^n$, as well as an additional asset S^{n+1} satisfying $S_0^{n+1} = \pi_0(X)$ and $S_1^{n+1} = X$, is arbitrage-free.

The interpretation of Definition 4.1 is as follows:

- We assume that the market model $\mathcal{M} = (B, S^1, \ldots, S^n)$ is arbitrage-free.
- We regard the additional asset as a tradable risky asset in the extended market model $\widetilde{\mathcal{M}} = (B, S^1, \ldots, S^{n+1})$.
- We postulate its price at time 0 should be selected in such a way that the extended market model $\widetilde{\mathcal{M}}$ is still arbitrage-free.

Valuation of Non-Attainable Claims

- We already know that the risk-neutral valuation formula returns the arbitrage price for any attainable claim.
- The next result shows that it also yields a price consistent with the no-arbitrage principle when it is applied to any non-attainable claim.
- The price obtained in this way is not unique, however.

Proposition (4.3)

Let X be a possibly non-attainable contingent claim and $\mathbb{Q}$ is an arbitrary risk-neutral probability measure. Then $\pi_0(X)$ given by

$$\pi_0(X) := \frac{1}{1+r}\,\mathbb{E}_{\mathbb{Q}}\,(X) \qquad (1)$$

defines a price for the contingent claim at $t = 0$ consistent with the no-arbitrage principle.

Proof.

[Proof of Proposition 4.3] Let $\mathbb{Q} \in \mathbb{M}$ be an arbitrary risk-neutral probability measure
for the original market model $\mathcal{M}$.
We will show that $\mathbb{Q}$ is also a risk-neutral probability measure for the extended market model $\widetilde{\mathcal{M}} = (B, S^1, \ldots, S^{n+1})$ in which $S_0^{n+1} = \pi_0(X)$ and $S_1^{n+1} = X$.
For this purpose, we check that

$$\mathbb{E}_{\mathbb{Q}}\left(\Delta \widehat{S}_1^{n+1}\right) = \mathbb{E}_{\mathbb{Q}}\left\{\frac{X}{1+r} - \pi_0(X)\right\} = 0$$

and thus $\mathbb{Q} \in \widetilde{\mathbb{M}}$ is indeed a risk-neutral probability measure
in the extended market model.
By the Fundamental Theorem of Asset Pricing, the extended market model $\widetilde{\mathcal{M}}$ is arbitrage-free. Hence the price $\pi_0(X)$ given by (2) complies with the no-arbitrage principle. $\square$

Models of Complete and Incomplete Markets

- We categorise market models into two classes: **complete** and **incomplete** market models.

Definition

A financial market, described by a model, is called **complete** if for any contingent claim X there exists a replicating strategy $(x, \phi) \in \mathbb{R}^{n+1}$. A market is **incomplete** when there exists a claim X for which a replicating strategy does not exist.

- Given an arbitrage-free and complete markets, the issue of pricing contingent claims is completed solved.
- How can we recognize whether a given market is complete?

Algebraic Criterion for Market Completeness

Proposition (4.4)

Assume that a single-period market model $\mathcal{M} = (B, S^1, \dots, S^n)$ defined on the sample space $\Omega = \{\omega_1, \dots, \omega_k\}$ is arbitrage-free. Then $\mathcal{M}$ is complete if and only if the $k \times (n+1)$ matrix A

$$A = \begin{pmatrix} 1+r & S_1^1(\omega_1) & \cdot & \cdot & \cdot & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \cdot & \cdot & \cdot & S_1^n(\omega_2) \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ 1+r & S_1^1(\omega_k) & \cdot & \cdot & \cdot & S_1^n(\omega_k) \end{pmatrix} = (A_0, A_1, \dots, A_n)$$

has a full row rank, that is, $\mathrm{rank}(A) = k$. Equivalently, $\mathcal{M}$ is complete whenever the linear subspace spanned by the vectors $A_0, A_1, \dots, A_n$ coincides with the full space $\mathbb{R}^k$.

Proof.

[Proof of Proposition 4.4] By the linear algebra, A has a full row rank if and only if for every $X \in \mathbb{R}^k$ the equation $AZ = X$ has a solution $Z \in \mathbb{R}^{n+1}$.
If we set $\phi^0 = x - \sum_{j=1}^{n} \phi^j S_0^j$ then we have

$$\begin{pmatrix} 1+r & S_1^1(\omega_1) & \cdot & \cdot & \cdot & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \cdot & \cdot & \cdot & S_1^n(\omega_2) \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ 1+r & S_1^1(\omega_k) & \cdot & \cdot & \cdot & S_1^n(\omega_k) \end{pmatrix} \begin{pmatrix} \phi^0 \\ \phi^1 \\ \cdot \\ \cdot \\ \cdot \\ \phi^n \end{pmatrix} = \begin{pmatrix} V_1(\omega_1) \\ V_1(\omega_2) \\ \cdot \\ \cdot \\ \cdot \\ V_1(\omega_k) \end{pmatrix}$$

where $V_1(\omega_i) = V_1(x, \phi)(\omega_i)$. This shows that computing a replicating strategy for X is equivalent to solving the equation $AZ = X$. $\square$

Example 4.3: Incomplete Market

- Consider the stochastic volatility model from Example 4.3.
- We already know that this market is incomplete, since the digital call is not an attainable claim.
- The matrix A is given by

$$A = \begin{pmatrix} 1+r & S_1^1(\omega_1) \\ 1+r & S_1^1(\omega_2) \\ 1+r & S_1^1(\omega_3) \\ 1+r & S_1^1(\omega_4) \end{pmatrix}$$

- The rank of A is 2, and thus it is not equal to $k = 4$.
- In view of Proposition 4.4, this confirms that this market model is incomplete.

Probabilistic Criterion for Attainability

- Proposition 4.4 yields a method for determining whether a market model is complete.
- Given an incomplete model, how to recognize an attainable contingent claim?
- Recall that if a model $\mathcal{M}$ is arbitrage-free then the class $\mathbb{M}$ is non-empty.

Proposition (4.5)

Assume that a single-period market model $\mathcal{M} = (B, S^1, \dots, S^n)$ is arbitrage-free. Then a contingent claim X is attainable if and only if the expected value

$$\mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1}X\right)$$

has the same value for all risk-neutral probability measures $\mathbb{Q} \in \mathbb{M}$.

Proof.

Optional. See end of this set of slides. $\square$

Probabilistic Criterion for Market Completeness

Theorem (4.1)

Assume that a single-period market model $\mathcal{M} = (B, S^1, \dots, S^n)$ is arbitrage-free. Then the model $\mathcal{M}$ is complete if and only if the class $\mathbb{M}$ consists of a single element, that is, there exists a unique risk-neutral probability measure.

Proof.

[Proof of ($\Leftarrow$) in Theorem 4.1] Since $\mathcal{M}$ is assumed to be arbitrage-free, it follows from the FTAP that there exists at least one risk-neutral probability measure, that is, the class $\mathbb{M}$ is non-empty.

($\Leftarrow$) Assume first that a risk-neutral probability measure for $\mathcal{M}$ is unique. Then the condition of Proposition 4.5 is trivially satisfied for any claim X. Hence any claim X is attainable and thus the market model is complete. $\square$

Proof.

[Proof of ($\Rightarrow$) Theorem 4.1] Assume $\mathcal{M}$ is complete and consider any two risk-neutral probability measures $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ from $\mathbb{M}$. For a fixed, but arbitrary, $i = 1, \ldots, k$, let the contingent claim X^i be given by

$$X^i(\omega) = \begin{cases} 1 + r & \text{if } \omega = \omega_i, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{M}$ is now assumed to be complete, the contingent claim X^i is attainable. From Proposition 4.2, it thus follows that

$$\mathbb{Q}(\omega_i) = \mathbb{E}_{\mathbb{Q}}\left(\frac{X^i}{1+r}\right) = \pi_0(X^i) = \mathbb{E}_{\widehat{\mathbb{Q}}}\left(\frac{X^i}{1+r}\right) = \widehat{\mathbb{Q}}(\omega_i).$$

Since i was arbitrary, we see that the equality $\mathbb{Q} = \widehat{\mathbb{Q}}$ holds. $\square$

Summary

Let us summarise the properties of single-period market models:

- A single-period market model is arbitrage-free if and only if it admits at least one risk-neutral probability measure.
- An arbitrage-free single-period market model is complete if and only if the risk-neutral probability measure is unique.
- Under the assumption that the model is arbitrage-free:
  - An arbitrary attainable contingent claim X (that is, a claim that can be replicated by means of some trading strategy) has the unique arbitrage price $\pi_0(X)$.
  - The arbitrage price $\pi_0(X)$ of any attainable claim X can be computed from the risk-neutral valuation formula using any risk-neutral probability measure $\mathbb{Q}$.
  - If a claim X is not attainable then we may define a price of X consistent with the no-arbitrage principle. It can be computed using the risk-neutral valuation formula, but it depends on the choice of a risk-neutral probability measure $\mathbb{Q}$.

Proof of FTAP

Proof.

[Proof of ($\Leftarrow$) in FTAP] ($\Leftarrow$) We first prove the 'if' part.
We assume that $\mathbb{M} \neq \emptyset$, so that a risk-neutral probability measure $\mathbb{Q}$ exists.
Let $(0, \phi) = (0, \phi^1, \ldots, \phi^n)$ be any trading strategy with null initial endowment. Then for any $\mathbb{Q} \in \mathbb{M}$

$$\mathbb{E}_{\mathbb{Q}}\left(\widehat{V}_1(0,\phi)\right) = \mathbb{E}_{\mathbb{Q}}\left(\sum_{j=1}^{n} \phi^j \Delta \widehat{S}_1^j\right) = \sum_{j=1}^{n} \phi^j \underbrace{\mathbb{E}_{\mathbb{Q}}\left(\Delta \widehat{S}_1^j\right)}_{=0} = 0.$$

If we assume that $\widehat{V}_1(0,\phi) \geq 0$ then the last equation implies that the equality $\widehat{V}_1(0,\phi)(\omega) = 0$ must hold for all $\omega \in \Omega$.
Hence no trading strategy satisfying all conditions of an arbitrage opportunity may exist. $\square$

Geometric Interpretations

- The proof of the implication ($\Rightarrow$) in the FTAP needs some preparation, since it is based on geometric arguments.
- Any random variable on Ω can be identified with a vector in $\mathbb{R}^k$, specifically,

$$X = (X(\omega_1), \ldots, X(\omega_k))^T = (x_1, \ldots, x_k)^T \in \mathbb{R}^k.$$

- An arbitrary probability measure $\mathbb{Q}$ on Ω can also be interpreted as a vector in $\mathbb{R}^k$

$$\mathbb{Q} = (\mathbb{Q}(\omega_1), \ldots, \mathbb{Q}(\omega_k)) = (q_1, \ldots, q_k) \in \mathbb{R}^k.$$

- We note that

$$\mathbb{E}_{\mathbb{Q}}(X) = \sum_{i=1}^{k} X(\omega_i)\mathbb{Q}(\omega_i) = \sum_{i=1}^{k} x_i q_i = \langle X, \mathbb{Q} \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors in $\mathbb{R}^k$.

Axiliary Subsets

- We define the following classes:

$$\mathbb{W} = \left\{ X \in \mathbb{R}^k \;\middle|\; X = \widehat{V}_1(0, \phi^1, \dots, \phi^n) \text{ for some } \phi^1, \dots, \phi^n \right\}$$
$$\mathbb{W}^\perp = \left\{ Z \in \mathbb{R}^k \mid \langle X, Z \rangle = 0 \text{ for all } X \in \mathbb{W} \right\}$$

- The set $\mathbb{W}$ is the image of the map $\widehat{V}_1(0, \cdot, \dots, \cdot) : \mathbb{R}^n \to \mathbb{R}^k$.
- We note that $\mathbb{W}$ represents all discounted values at $t = 1$ of trading strategies with null initial endowment.
- The set $\mathbb{W}^\perp$ is the set of all vectors in $\mathbb{R}^k$ orthogonal to $\mathbb{W}$.
- We introduce the following sets of k-dimensional vectors:

$$\mathbb{A} = \left\{ X \in \mathbb{R}^k \mid X \neq 0,\ x_i \geq 0 \text{ for } i = 1, \dots, k \right\}$$
$$\mathcal{P}^+ = \left\{ \mathbb{Q} \in \mathbb{R}^k \mid \sum_{i=1}^k q_i = 1,\ q_i > 0 \right\}$$

Vector Spaces

Corollary

The sets $\mathbb{W}$ and $\mathbb{W}^{\perp}$ are vector (linear) subspaces of $\mathbb{R}^k$.

Proof.

It suffices to observe that the map $\widehat{V}_1(0, \cdot, \dots, \cdot) : \mathbb{R}^n \to \mathbb{R}^k$ is linear.
In other words, for any trading strategies $(0, \eta^1, \dots, \eta^n)$ and $(0, \kappa^1, \dots, \kappa^n)$ and arbitrary real numbers α, β

$$(0, \phi^1, \dots, \phi^n) = \alpha(0, \eta^1, \dots, \eta^n) + \beta(0, \kappa^1, \dots, \kappa^n)$$

is also a trading strategy. Hence $\mathbb{W}$ is a vector subspace of $\mathbb{R}^k$. In particular, the zero vector $(0, 0, \dots, 0)$ belongs to $\mathbb{W}$.
It can be checked directly that $\mathbb{W}^{\perp}$, that is, the orthogonal complement of $\mathbb{W}$ is a vector subspace as well. $\square$

Risk-Neutral Probability Measures

Lemma (4.1)

A single-period market model $\mathcal{M} = (B, S^1, \ldots, S^n)$ *is arbitrage free if and only if* $\mathbb{W} \cap \mathbb{A} = \emptyset$.

Proof.

The proof hinges on an application of Proposition 4.1. $\square$

Lemma (4.2)

A probability measure $\mathbb{Q}$ *is a risk-neutral probability measure for a single-period market model* $\mathcal{M} = (B, S^1, \ldots, S^n)$ *if and only if* $\mathbb{Q} \in \mathbb{W}^{\perp} \cap \mathcal{P}^{+}$.
Hence the set $\mathbb{M}$ *of all risk-neutral probability measures for the model* $\mathcal{M}$ *satisfies* $\mathbb{M} = \mathbb{W}^{\perp} \cap \mathcal{P}^{+}$ *and thus*

$$\mathbb{M} \neq \emptyset \quad \Leftrightarrow \quad \mathbb{W}^{\perp} \cap \mathcal{P}^{+} \neq \emptyset.$$

Proof.

[Proof of ($\Rightarrow$) in Lemma 4.2] ($\Rightarrow$) We assume that $\mathbb{Q}$ is a risk-neutral probability measure.
By the property R.1, it is obvious that $\mathbb{Q}$ belongs to $\mathcal{P}^+$.
Using the property R.2, we obtain for an arbitrary vector $X = \widehat{V}_1(0, \phi) \in \mathbb{W}$

$$\begin{aligned}\langle X, \mathbb{Q} \rangle &= \mathbb{E}_{\mathbb{Q}}\left(\widehat{V}_1(0, \phi)\right) = \mathbb{E}_{\mathbb{Q}}\left(\sum_{j=1}^{n} \phi^j \Delta \widehat{S}_1^j\right) \\ &= \sum_{j=1}^{n} \phi^j \underbrace{\mathbb{E}_{\mathbb{Q}}\left(\Delta \widehat{S}_1^j\right)}_{=0} = 0.\end{aligned}$$

We conclude that $\mathbb{Q}$ belongs to $\mathbb{W}^\perp$ as well.
Consequently, $\mathbb{Q} \in \mathbb{W}^\perp \cap \mathcal{P}^+$ as was required to show. □

Proof.

[Proof of ($\Leftarrow$) in Lemma 4.2] ($\Leftarrow$) We now assume that $\mathbb{Q}$ is an arbitrary vector in $\mathbb{W}^{\perp} \cap \mathcal{P}^{+}$.

Since $\mathbb{Q} \in \mathcal{P}^{+}$, we see that $\mathbb{Q}$ defines a probability measure satisfying condition R.1.

It remains to show that $\mathbb{Q}$ satisfies condition R.2 as well. To this end, for a fixed (but arbitrary) $j = 1, \dots, n$, we consider the trading strategy $(0, \phi^1, \dots, \phi^n)$ with

$$(\phi^1, \dots, \phi^n) = (0, \dots, 0, 1, 0, \dots, 0) = e_j.$$

This trading strategy only invests in the savings account and the jth asset. The discounted wealth of this strategy is $\widehat{V}_1(0, e_j) = \Delta \widehat{S}_1^j$. □

Proof.

[Proof of ($\Leftarrow$) in Lemma 4.2 (Continued)] Since

$$\widehat{V}_1(0, e_j) \in \mathbb{W} \quad \text{and} \quad \mathbb{Q} \in \mathbb{W}^\perp$$

we obtain

$$0 = \langle \widehat{V}_1(0, e_j), \mathbb{Q} \rangle = \langle \Delta \widehat{S}_1^j, \mathbb{Q} \rangle = \mathbb{E}_{\mathbb{Q}} \left(\Delta \widehat{S}_1^j \right).$$

Since j was arbitrary, we see that $\mathbb{Q}$ satisfies condition R.2. Hence $\mathbb{Q}$ is a risk-neutral probability measure. □

From Lemmas 4.1 and 4.2, we get the following purely geometric reformulation of the FTAP:

$$\mathbb{W} \cap \mathbb{A} = \emptyset \quad \Leftrightarrow \quad \mathbb{W}^\perp \cap \mathcal{P}^+ \neq \emptyset.$$

Separating Hyperplane Theorem: Statement

Theorem (Separating Hyperplane Theorem)

Let $B, C \subset \mathbb{R}^k$ *be nonempty, closed, convex sets such that that* $B \cap C = \emptyset$. *Assume, in addition, that at least one of these sets is compact (i.e., bounded and closed). Then there exist vectors* $a, y \in \mathbb{R}^k$ *such that*

$$\langle b - a, y \rangle < 0 \quad \text{for all} \quad b \in B$$

and

$$\langle c - a, y \rangle > 0 \quad \text{for all} \quad c \in C.$$

Proof.

[Proof of the Separating Hyperplane Theorem] The proof can be found in any textbook of convex analysis or functional analysis. It is sketched in the course notes. □

Separating Hyperplane Theorem: Interpretation

- Let the vectors $a, y \in \mathbb{R}^k$ be as in the statement of the Separating Hyperplane Theorem
- It is clear that $y \in \mathbb{R}^k$ is never a zero vector.
- We define the $(k-1)$-dimensional **hyperplane** $H \subset \mathbb{R}^k$ by setting

$$H = a + \left\{ x \in \mathbb{R}^k \mid \langle x, y \rangle = 0 \right\} = a + \{y\}^{\perp}.$$

- Then we say that the hyperplane H **strictly separates** the convex sets B and C.
- Intuitively, the sets B and C lie on different sides of the hyperplane H and thus they can be seen as geometrically separated by H.
- Note that the compactness of at least one of the sets is a necessary condition for the **strict** separation of B and C.

Separating Hyperplane Theorem: Corollary

- The following corollary is a consequence of the separating hyperplane theorem.
- It is more suitable for our purposes: it will be later applied to $B = \mathbb{W}$ and $C = \mathbb{A}^+ := \{X \in \mathbb{A} \,|\, \langle X, \mathbb{P} \rangle = 1\} \subset \mathbb{A}$.

Corollary (4.1)

Assume that $B \subset \mathbb{R}^k$ is a vector subspace and set C is a compact convex set such that $B \cap C = \emptyset$. Then there exists a vector $y \in \mathbb{R}^k$ such that

$$\langle b, y \rangle = 0 \quad \textit{for all} \quad b \in B$$

that is, $y \in B^\perp$, and

$$\langle c, y \rangle > 0 \quad \textit{for all} \quad c \in C.$$

Proof.

[Proof of Corollary 4.1: First step] We note that any vector subspace of $\mathbb{R}^k$ is a closed, convex set.
From the separating hyperplane theorem, there exist $a, y \in \mathbb{R}^k$ such that the inequality

$$\langle b, y\rangle < \langle a, y\rangle$$

is satisfied for all vectors $b \in B$. Since B is a vector subspace, the vector λb belongs to B for any $\lambda \in \mathbb{R}$. Hence for any $b \in B$ and $\lambda \in \mathbb{R}$ we have that

$$\langle \lambda b, y\rangle = \lambda\langle b, y\rangle < \langle a, y\rangle.$$

This in turn implies that $\langle b, y\rangle = 0$ for any vector $b \in B$, meaning that $y \in B^{\perp}$. Also, we have that $\langle a, y\rangle > 0$. $\square$

Proof.

[Proof of Corollary 4.1: Second step] To establish the second inequality, we observe that from the separating hyperplane theorem, we obtain

$$\langle c, y \rangle > \langle a, y \rangle \quad \text{for all} \quad c \in C.$$

Consequently, for any $c \in C$

$$\langle c, y \rangle > \langle a, y \rangle > 0.$$

We conclude that $\langle c, y \rangle > 0$ for all $c \in C$. □

We now are ready to establish the implication $(\Rightarrow)$ in the Fundamental Theorem of Asset Pricing, that is,

$$\mathbb{W} \cap \mathbb{A} = \emptyset \quad \Rightarrow \quad \mathbb{W}^{\perp} \cap \mathcal{P}^{+} \neq \emptyset.$$

Proof.

[Proof of ($\Rightarrow$) in FTAP: First step] We assume that the model is arbitrage-free. From Lemma 4.1, this is equivalent to the condition $\mathbb{W} \cap \mathbb{A} = \emptyset$.

Our goal is to show that the class $\mathbb{M}$ of risk-neutral probabilities is non-empty.

In view of Lemma 4.2, it thus suffices to show that

$$\mathbb{W} \cap \mathbb{A} = \emptyset \quad \Rightarrow \quad \mathbb{W}^{\perp} \cap \mathcal{P}^{+} \neq \emptyset.$$

We define an auxiliary set $\mathbb{A}^{+} = \{X \in \mathbb{A} \mid \langle X, \mathbb{P} \rangle = 1\}$.
Observe that $\mathbb{A}^{+}$ is a closed, bounded (hence compact) and convex subset of $\mathbb{R}^{k}$. Since $\mathbb{A}^{+} \subset \mathbb{A}$, it is clear that

$$\mathbb{W} \cap \mathbb{A} = \emptyset \;\Rightarrow\; \mathbb{W} \cap \mathbb{A}^{+} = \emptyset.$$

Hence in the next step we may assume that $\mathbb{W} \cap \mathbb{A}^{+} = \emptyset$. $\square$

Proof.

[Proof of ($\Rightarrow$) in FTAP: Second step] By applying Corollary 4.1 to $B = \mathbb{W}$ and $C = \mathbb{A}^+$, we see that there exists a vector $Y \in \mathbb{W}^\perp$ such that

$$\langle X, Y \rangle > 0 \quad \text{for all} \quad X \in \mathbb{A}^+. \tag{2}$$

Our goal is to show that Y can be used to define a risk-neutral probability $\mathbb{Q}$. We need first to show that $y_i > 0$ for every i.
For this purpose, for any fixed $i = 1, \ldots, k$, we define

$$X_i = \frac{1}{\mathbb{P}(\omega_i)} (0, \ldots, 0, 1, 0 \ldots, 0) = \frac{1}{\mathbb{P}(\omega_i)} e_i$$

so that

$$\mathbb{E}_{\mathbb{P}}(X_i) = \langle X_i, \mathbb{P} \rangle = \frac{1}{\mathbb{P}(\omega_i)} \mathbb{P}(\omega_i) = 1$$

and thus $X_i \in \mathbb{A}^+$. □

Proof.

[Proof of ($\Rightarrow$) in FTAP: Third step] Let y_i be the ith component of Y. It follows from (1) that

$$0 < \langle X_i, Y \rangle = \frac{1}{\mathbb{P}(\omega_i)}\, y_i$$

and thus $y_i > 0$ for all $i = 1, \ldots, k$. We define

$$q_i := \frac{y_i}{y_1 + \cdots + y_k} = c y_i > 0$$

and we set $\mathbb{Q}(\omega_i) = q_i$ for $i = 1, \ldots, k$. It is clear that $\mathbb{Q}$ is a probability measure such that $\mathbb{Q} \in \mathcal{P}^+$.
Since $\mathbb{Q} = cY$ for some scalar c and $\mathbb{W}^\perp$ is a vector space, we have that $\mathbb{Q} \in \mathbb{W}^\perp$ (recall that $Y \in \mathbb{W}^\perp$). We conclude that $\mathbb{Q} \in \mathbb{W}^\perp \cap \mathcal{P}^+$ so that $\mathbb{W}^\perp \cap \mathcal{P}^+ \neq \emptyset$. In view of Lemma 4.2, $\mathbb{Q}$ is a risk-neutral probability measure, so that $\mathbb{M} \neq \emptyset$. □

Proof.

[Proof of Proposition 4.5 Step 1] ($\Rightarrow$) It is immediate from Proposition 4.2 that if a contingent claim X is attainable then the expected value

$$\mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1}X\right)$$

has the same value for all $\mathbb{Q} \in \mathbb{M}$.

($\Leftarrow$) We prove this implication by contrapositive. Let us thus assume that the contingent claim X is not attainable. Our goal is to find two risk-neutral probabilities, say $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$, for which

$$\mathbb{E}_{\mathbb{Q}}\left((1+r)^{-1}X\right) \neq \mathbb{E}_{\widehat{\mathbb{Q}}}\left((1+r)^{-1}X\right). \tag{3}$$

□

Proof.

[Proof of Proposition 4.5 Step 2] Consider the matrix A introduced in Proposition 4.4.
Since the claim X is not attainable, there is no solution $Z \in \mathbb{R}^{n+1}$ to the linear system

$$AZ = X.$$

We define the following subsets of $\mathbb{R}^k$

$$B = \text{image}\,(A) = \left\{AZ \,|\, Z \in \mathbb{R}^{n+1}\right\} \subset \mathbb{R}^k$$

and $C = \{X\}$.
Then B is a subspace of $\mathbb{R}^k$ and, obviously, the set C is convex and compact. Moreover, $B \cap C = \emptyset$. □

Proof.

[Proof of Proposition 4.5 Step 3] In view of Corollary 4.1, there exists a non-zero vector $Y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ such that

$$\begin{aligned} \langle b, Y \rangle &= 0 \ \text{ for all } \ b \in B, \\ \langle c, Y \rangle &> 0 \ \text{ for all } \ c \in C. \end{aligned}$$

In view of the definition of B and C, this means that for $j = 0, \ldots, n$

$$\langle A_j, Y \rangle = 0 \ \text{ and } \ \langle X, Y \rangle > 0 \tag{4}$$

where A_j is the jth column of the matrix A.
It is worth noting that the vector Y depends on X. $\square$

Proof.

[Proof of Proposition 4.5 Step 4] We assumed that the market model is arbitrage-free and thus,
by the FTAP, the class $\mathbb{M}$ is non-empty.
Let $\mathbb{Q} \in \mathbb{M}$ be an arbitrary risk-neutral probability measure.
We may choose a real number $\lambda > 0$ to be small enough in order to ensure that for every $i = 1, \ldots, k$

$$\widehat{\mathbb{Q}}(\omega_i) := \mathbb{Q}(\omega_i) + \lambda(1+r)y_i > 0. \tag{5}$$

In the next step, our next goal is to show that $\widehat{\mathbb{Q}}$ is also a risk-neutral probability measure and it is different from $\mathbb{Q}$.
In the last step, we will show that inequality (3) is valid. □

Proof.

[Proof of Proposition 4.5 Step 5] From the definition of A in Proposition 4.4 and the first equality in (4) with $j = 0$, we obtain

$$\sum_{i=1}^{k} \lambda(1+r)y_i = \lambda\langle A_0, Y\rangle = 0.$$

It then follows from (5) that

$$\sum_{i=1}^{k} \widehat{\mathbb{Q}}(\omega_i) = \sum_{i=1}^{k} \mathbb{Q}(\omega_i) + \sum_{i=1}^{k} \lambda(1+r)y_i = 1$$

and thus $\widehat{\mathbb{Q}}$ is a probability measure on the space Ω.
In view of (5), it is clear that $\widehat{\mathbb{Q}}$ satisfies condition R.1. $\square$

Proof.

[Proof of Proposition 4.5 Step 6] It remains to check that $\widehat{\mathbb{Q}}$ satisfies also condition R.2.
We examine the behaviour under $\widehat{\mathbb{Q}}$ of the discounted stock prices $\widehat{S}_1^j$. We have that, for every $j = 1, \ldots, n$,

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}\left(\widehat{S}_1^j\right) &= \sum_{i=1}^{k} \widehat{\mathbb{Q}}(\omega_i)\widehat{S}_1^j(\omega_i) \\
&= \sum_{i=1}^{k} \mathbb{Q}(\omega_i)\widehat{S}_1^j(\omega_i) + \lambda \sum_{i=1}^{k} \widehat{S}_1^j(\omega_i)(1+r)y_i \\
&= \mathbb{E}_{\mathbb{Q}}\left(\widehat{S}_1^j\right) + \lambda \underbrace{\langle A_j, Y\rangle}_{=0} \qquad \text{(in view of (4))} \\
&= \widehat{S}_0^j \qquad \text{(since } \mathbb{Q} \in \mathbb{M}\text{)}
\end{aligned}$$

□

Proof.

[Proof of Proposition 4.5 Step 7] We conclude that $\mathbb{E}_{\widehat{\mathbb{Q}}}\left(\Delta\widehat{S}_1^j\right) = 0$ and thus $\widehat{\mathbb{Q}} \in \mathbb{M}$, that is, $\widehat{\mathbb{Q}}$ is a risk-neutral probability measure for the market model $\mathcal{M}$.
From (5), it is clear that $\mathbb{Q} \neq \widehat{\mathbb{Q}}$. We have thus proven that if $\mathcal{M}$ is arbitrage-free and incomplete then there exists more than one risk-neutral probability measure.
To complete the proof, it remains to show that inequality (3) is satisfied for a contingent claim X.
Recall that X was a fixed non-attainable contingent claim and we constructed a risk-neutral probability measure $\widehat{\mathbb{Q}}$ corresponding to X. $\square$

Proof.

[Proof of Proposition 4.5 Step 8] We observe that

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbb{Q}}}\left(\frac{X}{1+r}\right) &= \sum_{i=1}^{k} \widehat{\mathbb{Q}}(\omega_i)\frac{X(\omega_i)}{1+r} \\
&= \sum_{i=1}^{k} \mathbb{Q}(\omega_i)\frac{X(\omega_i)}{1+r} + \underbrace{\lambda \sum_{i=1}^{k} y_i X(\omega_i)}_{>0} \\
&> \sum_{i=1}^{k} \mathbb{Q}(\omega_i)\frac{X(\omega_i)}{1+r} = \mathbb{E}_{\mathbb{Q}}\left(\frac{X}{1+r}\right)
\end{aligned}$$

since the second part of (4) and the inequality $\lambda > 0$ show that the braced expression is strictly positive. $\square$