# 1 Tutorial Week 7: CRR Formula

1. We take for granted the CRR call option pricing formula

$$C_0 = S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \hat{p}^k (1-\hat{p})^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k}$$

   where

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \hat{p} = \frac{\tilde{p}u}{1+r}$$

   and $\hat{k}$ is the smallest integer $k$ such that

$$k \log\left(\frac{u}{d}\right) > \log\left(\frac{K}{S_0 d^T}\right).$$

   Use the CRR-option pricing formula from the lecture to compute the price (at time $t = 0$) of a European call option with strike price $K = 10$, maturity time $T = 5$ and initial stock price $S_0 = 9$. For the interest rate assume $r = 0.01$ and for the volatility $\sigma = 0.1$. Use the CRR-parametrization for $u$ and $d$, that is, set

$$u = e^{\sigma\sqrt{\Delta t}}, d = \frac{1}{u}$$

   with the time increment $\Delta t = 1$.

2. Consider the binomial tree model with the same specification, as in Exercise 1, compute the price process (i.e. prices at all times $t$) of the European call using the binomial tree method.

# 2 Tutorial Week 7: Solution

The CRR parameterisation for u and d with $\Delta t = 1$ gives

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} = e^{\sigma} = 1.105171 \\ d &= \frac{1}{u} = 0.904837 \end{aligned}$$

Consequently,

$$\begin{aligned} \tilde{p} &= \frac{1+r-d}{u-d} = 0.524938 \\ \hat{p} &= \frac{\tilde{p}u}{1+r} = 0.574402 \end{aligned}$$

1. We first compute the call option price at time 0. It is easy to check that $\hat{k} = 4$ and thus

$$\begin{aligned}
C_0 &= S_0 \sum_{k=4}^{5} \binom{5}{k} \hat{p}^k (1-\hat{p})^{5-k} - \frac{10}{(1.01)^5} \sum_{k=4}^{5} \binom{5}{k} \tilde{p}^k (1-\tilde{p})^{5-k} \\
&= S_0 \cdot 5 \cdot \hat{p}^4 (1-\hat{p}) + S_0 \cdot 1 \cdot \hat{p}^5 - \frac{10}{(1.01)^5} \cdot 5 \cdot \tilde{p}^4 (1-\tilde{p}) - \frac{10}{(1.01)^5} \cdot 1 \cdot \tilde{p}^5 \\
&= 9 \cdot 5 \cdot 0.574402^4 \cdot (1 - 0.574402) + 9 \cdot 1 \cdot 0.574402^5 \\
&\quad - \frac{10}{(1.01)^5} \cdot 5 \cdot 0.524938^4 \cdot (1 - 0.524938) - \frac{10}{(1.01)^5} \cdot 1 \cdot 0.524938^5 \\
&= 0.552247
\end{aligned}$$

so the price of option equals $C_0 = 0.552247$

2. Risk neutral probability, as before

$$\tilde{p} = \frac{1 + r - d}{u - d} = 0.524938$$

| $t=0$ | $t=1$ | $t=2$ | $t=3$ | $t=4$ | $t=5$ | |
|---|---|---|---|---|---|---|
| | | | | | $S_5 = u^5 S_0 = 14.8 > K$, $\pi_5 = 4.8$ | $\mathbb{Q}(\omega_1) = \tilde{p}^5 = 0.039\,86$ |
| | | | | $S_4 = u^4 S_0$, $\pi_4 = 3.48$ | | |
| | | | $S_3 = u^3 S_0$, $\pi_3 = 2.32$ | | $S_5 = u^4 d S_0 = 12.1 > K$, $\pi_5 = 2.1$ | $\mathbb{Q}(\omega_2) = 5\tilde{p}^4 (1-\tilde{p}) = 0.180\,37$ |
| | | $S_2 = u^2 S_0$, $\pi_2 = 1.47$ | | $S_4 = u^3 d S_0$, $\pi_4 = 1.09$ | | |
| | $S_1 = u S_0$, $\pi_1 = 0.9$ | | $S_3 = u^2 d S_0$, $\pi_3 = 0.57$ | | $S_5 = u^3 d^2 S_0 = 9.9 < K$, $\pi_5 = 0$ | $\mathbb{Q}(\omega_2) = 10\tilde{p}^3 (1-\tilde{p})^2$ |
| $S_0 = 9$, $\pi_0 = 0.55$ | | $S_2 = ud S_0$, $\pi_2 = 0.29$ | | $S_4 = u^2 d^2 S_0$, $\pi_4 = 0$ | | |
| | $S_1 = d S_0$, $\pi_1 = 0.15$ | | $S_3 = u d^2 S_0$, $\pi_3 = 0$ | | $S_5 = u^2 d^3 S_0$, $\pi_5 = 0$ | $\mathbb{Q}(\omega_2) = 10\tilde{p}^2 (1-\tilde{p})^3$ |
| | | $S_2 = d^2 S_0$, $\pi_3 = 0$ | | $S_4 = u d^3 S_0$, $\pi_4 = 0$ | | |
| | | | $S_3 = d^3 S_0$, $\pi_3 = 0$ | | $S_5 = u d^4 S_0$, $\pi_5 = 0$ | $\mathbb{Q}(\omega_2) = 5\tilde{p}(1-\tilde{p})^4$ |
| | | | | $S_4 = d^4 S_0$, $\pi_4 = 0$ | | |
| | | | | | $S_5 = d^5 S_0$, $\pi_5 = 0$ | $\mathbb{Q}(\omega_2) = (1-\tilde{p})^5$ |

As it is obvious that different techniques involve different number of operations and so the arithmetic rounding may lead to (slightly) different results even if no mistakes are made. Hence it is important that all operations are clearly written and not omitted.