1 Exam

Answer ONE question from Part A and ONE from Part B

Part A

1. Consider a multi-period market model $M = (B, S)$ with two assets: the savings account B and the risky asset S. For a natural number T, consider the following model for the price of the risky asset S :

$$S_t = S_0 + Y_1 + Y_2 + ... + Y_t,$$

where the random variables $Y_1, \ . \ . \ . \ , Y_T$ are independent and identically distributed under the real-world probability measure $\mathbb{P}$, specifically,

$$P(Y_1 = +a) = .25, \ P(Y_1 = -a) = 0.75$$

where $a > 0$ is a strictly positive constant. We assume that the risk-free rate $r = 0$ so that the savings account equals $B_t = 1$ for every $t = 0, 1, ..., T$.

(a) Check whether the model $M = (B, S)$ is arbitrage-free and complete.

(b) Find the option price process $\pi_t(X), \ t = 0, 1, 2, 3$ and the replicating strategy $\varphi = (\varphi_0, \varphi_1)$ for the claim $X = (S_3 - S_1)^+$ maturing at time $T = 3$.

2. (a) If S_t follows geometric Brownian motion, what process does S_t^{-1} follow? Describe the process by an appropriate stochastic differential equation.

   (b) Show that Ae^{rt} is a solution of the Black-Scholes equation.

Part B

3. Discuss the connection between the Cox-Ross-Rubinstein pricing formula and the Black-Scholes pricing formula.

4. Discuss the relationship between martingales and arbitrage.

2 Solution

1. a.

$$\begin{aligned} q &= \frac{1 + r - d}{u - d} = \frac{1 + 0 - \left(1 - \frac{a}{S_k}\right)}{\left(1 + \frac{a}{S_k}\right) - \left(1 - \frac{a}{S_k}\right)} = \frac{1}{2} \\ 1 - q &= \frac{1}{2} \end{aligned}$$

$$X = (S_3 - S_1)^+$$

$$
\begin{array}{lllllllllll}
 & & & & & & S_0+3a & & \text{uuu} & 2a & \frac{2a}{8} \\
 & & & & & \frac{1}{2}\nearrow & \pi_3=2a & X=2a & \text{uud} & 0 & \\
 & & & & S_0+2a & & & & \text{udd} & 0 & \\
 & & & \frac{1}{2}\nearrow & \pi_2=a & \frac{1}{2}\searrow & \pi_3=0 & & \text{ddd} & 0 & \\
 & & S_0+a & & & & S_0+a & & \text{udu} & 0 & 0 \\
 & \frac{1}{2}\nearrow & \pi_1=a & \frac{1}{2}\searrow & & \frac{1}{2}\nearrow & \pi_3=0 & \pi_3=2a & \text{dud} & 0 & 0 \\
S_0 & & & & S_0 & & & & \text{duu} & 2\text{a} & \frac{2a}{8} \\
\pi_0=\frac{3}{4}a & \frac{1}{2}\searrow & & \frac{1}{2}\nearrow & \pi_2=a & \frac{1}{2}\searrow & \pi_3=0 & & \text{ddu} & 0 & \\
 & & S_0-a & & & & S_0-a & & & & \\
 & & \pi_1=\frac{a}{2} & \frac{1}{2}\searrow & & \frac{1}{2}\nearrow & \pi_3=0 & \pi_3=0 & & & \\
 & & & & S_0-2a & & & & & & \\
 & & & & \pi_2=0 & \frac{1}{2}\searrow & \pi_3=0 & & & & \\
 & & & & & & S_0-3a & & & &
\end{array}
$$

$$2\frac{2a}{8}=\frac{2}{4}a$$

2.

(a)

$$
\begin{aligned}
dS_t &= rS_tdt+\sigma S_tdW_t \\
\frac{dS_t}{S_t} &= rdt+\sigma dW_t \\
d\ln S_t &= rdt+\sigma dW_t \\
S_t &= S_0\exp\left(\sigma W_t+\left(r-\frac{\sigma^2}{2}\right)t\right)
\end{aligned}
$$

Ito Formula:

Suppose that $x(\cdot)$ solves the stochastic differential equation

$$
\begin{aligned}
dx(t) &= f(t,x(t))dt+\sigma(t,x(t))dW(t) \\
x(t) &= S_t \\
dS_t &= rS_tdt+\sigma S_tdW_t
\end{aligned}
$$

and that $G:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a function with continuous partial derivatives G_t,G_x and G_{xx}. Then the stochastic process $g(\cdot)$ defined by

$$
\begin{aligned}
g(t) &= G(t,x(t)) \\
g\left(t,S_t\right) &= S_t^{-1}=\frac{1}{S_0}\exp\left(-\left(\sigma W_t+\left(r-\frac{\sigma^2}{2}\right)t\right)\right)
\end{aligned}
$$

has the following dynamics

$$\begin{aligned}
dg(t) &= \{G_t(t,x(t)) + G_x(t,x(t))f(t,x(t)) \\
&\quad + \frac{1}{2}G_{xx}(t,x(t))\sigma^2(t,x(t))\Big\} dt \\
&\quad + G_x(t,x(t))\sigma(t,x(t))dW(t). \qquad (1)
\end{aligned}$$

$$\begin{aligned}
G_t(t,S(t)) &= 0 \\
G_S(t,S(t)) &= -S_t^{-2} \\
G_{SS}(t,S(t)) &= 2S_t^{-3} \\
f(t,S(t)) &= rS_t \\
\sigma(t,S(t)) &= \sigma S_t
\end{aligned}$$

$$\begin{aligned}
dg(t) &= \{G_t(t,x(t)) + G_x(t,x(t))f(t,x(t)) \\
&\quad + \frac{1}{2}G_{xx}(t,x(t))\sigma^2(t,x(t))\Big\} dt \\
&\quad + G_x(t,x(t))\sigma(t,x(t))dW(t) \qquad (2) \\
&= \left(0 - S_t^{-2}rS_t + \frac{1}{2}2S_t^{-3}\sigma^2 S_t^2\right) dt - S_t^{-2}\sigma S_t dW(t) \qquad (3)
\end{aligned}$$

$$\begin{aligned}
d\left(S_t^{-1}\right) &= -S_t^{-1}\left(r - \sigma^2\right) dt - S_t^{-1}\sigma dW(t) \\
d\left(S_t^{-1}\right) &= -S_t^{-1}\left(\left(r - \sigma^2\right) dt + \sigma dW(t)\right)
\end{aligned}$$

$$d\left(X_t\right) = -X_t\left(\left(r - \sigma^2\right) dt + \sigma dW(t)\right)$$

$$f\left(t, W_t\right) = \frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right)\right)$$

Ito Formula:

Suppose that $x(\cdot)$ solves the stochastic differential equation

$$\begin{aligned}
dx(t) &= f(t,x(t))dt + \sigma(t,x(t))dW(t) \\
x(t) &= W_t \\
dW_t &= 0dt + 1dW_t
\end{aligned}$$

and that $G : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a function with continuous partial derivatives G_t,G_x and G_{xx}. Then the stochastic process $g(\cdot)$ defined by

$$\begin{aligned}
g(t) &= G(t,x(t)) \\
G\left(t, W_t\right) &= \frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right)\right)
\end{aligned}$$

has the following dynamics

$$
\begin{aligned}
dg(t) &= \{G_t(t,W(t)) + G_x(t,W(t))f(t,x(t)) \\
&\quad + \frac{1}{2}G_{xx}(t,x(t))\sigma^2(t,x(t))\Big\}\, dt \\
&\quad + G_x(t,x(t))\sigma(t,x(t))dW(t). \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
G_t(t,x(t)) &= -\left(r-\frac{\sigma^2}{2}\right)\frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r-\frac{\sigma^2}{2}\right)t\right)\right) \\
G_W(t,W(t)) &= -\sigma\frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r-\frac{\sigma^2}{2}\right)t\right)\right) \\
G_{WW}(t,W(t)) &= \sigma^2\frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r-\frac{\sigma^2}{2}\right)t\right)\right) \\
f(t,x(t)) &= 0 \\
\sigma(t,x(t)) &= 1
\end{aligned}
$$

$$
\begin{aligned}
dg(t) &= \begin{pmatrix} -\left(r-\frac{\sigma^2}{2}\right)\frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r-\frac{\sigma^2}{2}\right)t\right)\right) \\ +\frac{1}{2}\sigma^2\frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r-\frac{\sigma^2}{2}\right)t\right)\right) \end{pmatrix} dt \qquad (5) \\
&\quad -\sigma\frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r-\frac{\sigma^2}{2}\right)t\right)\right)dW(t) \qquad (6) \\
dg(t) &= \frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r-\frac{\sigma^2}{2}\right)t\right)\right)\left(-r+\sigma^2\right)dt \qquad (7) \\
&\quad -\sigma\frac{1}{S_0}\exp\left(-\left(\sigma W_t + \left(r-\frac{\sigma^2}{2}\right)t\right)\right)dW(t) \qquad (8) \\
dg(t) &= S_t^{-1}\left(\left(-r+\sigma^2\right)dt - \sigma dW(t)\right) \qquad (9)
\end{aligned}
$$

$$
g(t,S_t) = S_t^{-1} = \frac{1}{S_0}\left(\sigma W_t + \left(r-\frac{\sigma^2}{2}\right)t\right)^{-1}
$$

(b)

$$
\begin{aligned}
V(t,x) &= Ae^{rt} \\
V_t(t,x) &= Are^{t} \\
V_x(t,x) &= 0 \\
V_{xx}(t,x) &= 0
\end{aligned}
$$

BS equation

$$
rV(t,x) - V_t(t,x) - rxV_x(t,x) - \frac{1}{2}\sigma^2x^2V_{xx}(t,x) = rAe^{rt} - Are^{t} = 0
$$

1. Hence $V(t,x)$ is the price
3. One is the limit of another
4. From lectures