Mathematical Finance

Filtrations and Conditioning

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Semester 1

New Features

- Two important new features of multi-period market models:
  - Investors can trade in assets at any specific time $t \in \{0, 1, 2, \ldots, T\}$, where T is the horizon date.
  - Investors can gather information over time, since the fluctuations of asset prices can be observed.
- We have to determine how the level of information evolves over time.
- The latter aspect leads to the probabilistic concepts of σ-fields and filtrations.

Outline

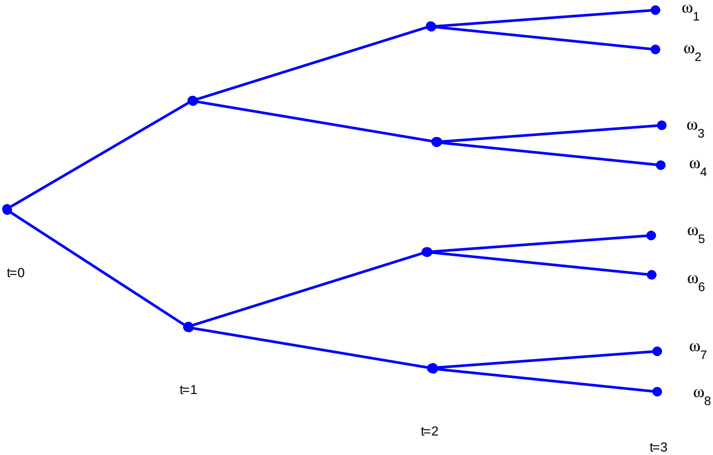
We will examine the following issues:

1. Partitions and σ-Fields
2. Filtrations and Adapted Stochastic Processes
3. Conditional Expectations
4. Change of a Probability Measure

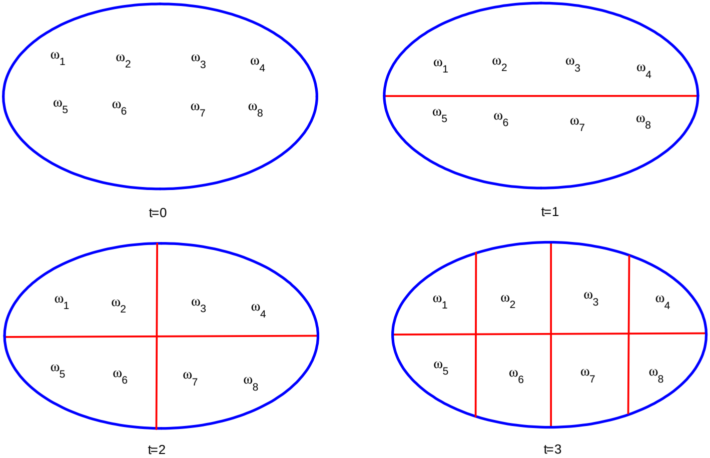
Example

- In one-period model we have the set of possible events (states of nature) $\Omega = \{\omega_1, \omega_2\}$ at time $t = 0$, and at time $t = 1$ we know which state of nature realised. $\{\omega_1, \omega_2\} = \{$'up', 'down'$\}$
- In multi-period model the structure of information is more comlex
  - We start with $\Omega = A_0 = \{\omega_1, ..., \omega_k, ...\}$ the set of all possible states of nature.
    ($A_0 = \{$'up at t=1', 'down at t=1', 'up at t=2',, 'down at t=2',...$\}$)
  - at each next period $t > 0$ we know that some events will never realise, $A_t \subseteq A_{t-1} \subseteq ... \subseteq A_1 \subseteq A_0$.
    ($A_1 = \{$'down at t=1', 'up at t=2',, 'down at t=2',...$\} \subset A_0$) So we may denote complement set $A_t^c = \Omega \backslash A_t$ which containes all states of nature which are ruled out at time t.

Example



Example


ω1 ω2 ω3 ω4 ω5 ω6 ω7 ω8
t=0
ω1 ω2 ω3 ω4 ω5 ω6 ω7 ω8
t=1
ω1 ω2 ω3 ω4 ω5 ω6 ω7 ω8
t=2
ω1 ω2 ω3 ω4 ω5 ω6 ω7 ω8
t=3

Example 5.4: Stock Price Model

$$
\begin{tikzcd}
 & & \boxed{S_2 = 1.8} & \omega_1 & \mathbb{P}(\omega_1) = 3/16 \\
 & \boxed{S_1 = 1.3} \arrow[ur] \arrow[dr] & & & \\
 & & \boxed{S_2 = 1.2} & \omega_2 & \mathbb{P}(\omega_2) = 5/16 \\
\boxed{S_0 = 1} \arrow[uur] \arrow[ddr] & & & & \\
 & & \boxed{S_2 = 1.1} & \omega_3 & \mathbb{P}(\omega_3) = 4/16 \\
 & \boxed{S_1 = 0.7} \arrow[ur] \arrow[dr] & & & \\
 & & \boxed{S_2 = 0.5} & \omega_4 & \mathbb{P}(\omega_4) = 4/16
\end{tikzcd}
$$

Example: Stock Price Model

Example (5.4 Continued)

- At time zero the investor observes $S_0 = 1$, and has no clue about the true future state, so $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- At $t = 2$ the investor observes S_2 and knows the true state ω. $\mathcal{F}_2 = 2^\Omega$ since the partition generating $\mathcal{F}_2$ is

$$\mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}.$$

- At $t = 1$, investor observes $S_1 = 1.3$ or $S_1 = 0.7$ so it is either $\{\omega_1, \omega_2\}$ or $\{\omega_3, \omega_4\}$. The relevant partition is $\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$
- Note that $\emptyset, \Omega$ must always be there as Ω is a union of all possible sets and $\emptyset$ is 'zero' and complement to Ω. In other words, $\mathcal{F}$ would not be an algebra/field.

Example: Stock Price Model

Example (5.4 Continued)

- The same price process as above
- At time t=1 a marketing survey is conducted which will be either favorable $\{\omega_1, \omega_2\}$ or unfavorable $\{\omega_3, \omega_4\}$
- Moreover, in either case the risky security will possibly take one or two distict values. Hence the relevant partition at time $t = 1$ is

  $$\mathcal{P}_1 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$$

  and $\mathcal{F}_1 = 2^\Omega$.
- Investors can distinguish ω_1 from ω_2 as well as from ω_3 and ω_4 at time 1.
- They can look into the future, as they know what prices will be at time $t = 2$.
- Information flow is different in this case, although the price processes are the same.

σ-Field

- The model of information structure can be described using sequence of partitons, where each partition becomes **finer**. Or it can be described using **trees**.
- The concept of a σ-field can be used to describe the amount of information available at a given moment.
- Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all natural numbers.

Definition (σ-Field)

A collection $\mathcal{F}$ of subsets of Ω is called a σ-**field** (or a σ-**algebra**) whenever

1. Subsets cover the whole space, $\Omega \in \mathcal{F}$.
2. Complement set exists and belongs to the same collection of subsets, if $A \in \mathcal{F}$ then $A^c := \Omega \setminus A \in \mathcal{F}$.
3. Operation defined for every two members: If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ then $A \bigcup B \in \mathcal{F}$.

σ-Field

- It follows that the empty set belongs to $\mathcal{F}$: $\emptyset = \Omega^c \in \mathcal{F}$
- So, we have defined algebra: zero element is there, the inverse element is there, and an operation is defined.
- Moreover, it is a field as two operations (sum=union and multiplication=intersection) are defined, existence of inverse operation. Most of the literature uses 'σ-algebra' rather than 'σ-field'.

Interpretation of a σ-Field

- Many different σ-fields can be defined on one set Ω.
- The set of information has to contain all possible states, so that we postulate that Ω belongs to each σ-field.
- Any set $A \in \mathcal{F}$ is interpreted as an observed **event**.
- If an event $A \in \mathcal{F}$ is given, that is, some collection of states is given, then the remaining states can also be identified and thus the complement A^c is also an event.
- The idea of a σ-field is to model a certain level of information.
- We will later introduce a concept of an increasing flow of information, formally represented by an ordered family of σ-fields.

Probability Measure

Definition (Probability Measure)

A map $\mathbb{P} : \mathcal{F} \to [0,1]$ is called a **probability measure** if

1. $\mathbb{P}(\Omega) = 1$.
2. For any sequence $A_i, i \in \mathbb{N}$ of pairwise disjoint events we have

$$\mathbb{P}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

- Note that $\mathbb{P}(\emptyset) = 0$ due to the definition of probability measure.
- By convention, the probability of all possibilities is 1.
- Probability should satisfy σ-additivity.

Example: σ-Fields

Example (5.1)

We take $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and we define the σ-fields:

$$\begin{aligned}
\mathcal{F}_1 &= \{\emptyset, \Omega\} \\
\mathcal{F}_2 &= (\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \\
&\quad \{\omega_1, \omega_2, \omega_4\}, \{\omega_3, \omega_4\}) \\
\mathcal{F}_3 &= 2^{\Omega} \quad \text{(the class of all subsets of } \Omega\text{)}.
\end{aligned}$$

Note that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$, that is, the information increases:

- $\mathcal{F}_1$: no information, except for the set Ω.
- $\mathcal{F}_2$: partial information, since we cannot distinguish between the occurrence of either ω_1 or ω_2.
- $\mathcal{F}_3$: full information, since events $\{\omega_1\}$, $\{\omega_2\}$, $\{\omega_3\}$ and $\{\omega_4\}$ can be observed.

Example: Probability Measure

Example (5.1 Continued)

- We define the probability measure $\mathbb{P}$ on the σ-field $\mathcal{F}_2$

$$\mathbb{P}\left(\{\omega_1, \omega_2\}\right) = \frac{2}{3}, \quad \mathbb{P}\left(\{\omega_3\}\right) = \frac{1}{6}, \quad \mathbb{P}\left(\{\omega_4\}\right) = \frac{1}{6}.$$

- The σ-additivity of $\mathbb{P}$ leads to

$$\mathbb{P}\left(\{\omega_1, \omega_2\} \cup \{\omega_3\} \cup \{\omega_4\}\right) = 1 = \mathbb{P}\left(\Omega\right).$$

- Note that $\mathbb{P}$ is not yet defined on the σ-field $\mathcal{F}_3 = 2^\Omega$ and in fact the extension of $\mathbb{P}$ from $\mathcal{F}_2$ to $\mathcal{F}_3$ is not unique.
- For any $\alpha \in [0, 2/3]$ we may set

$$\mathbb{P}_\alpha\left(\{\omega_1\}\right) = \alpha = \frac{2}{3} - \mathbb{P}_\alpha\left(\{\omega_2\}\right).$$

Partitions

Definition

Let I be some index set. Assume that we are given a collection $(B_i)_{i\in I}$ of subsets of Ω. Then the smallest σ-field containing this collection is denoted by $\sigma\left((B_i)_{i\in I}\right)$ and is called the σ-field **generated** by the collection $(B_i)_{i\in I}$.

Definition (Partition)

By a **partition** of Ω, we mean any collection $\mathcal{P} = (A_i)_{i\in I}$ of non-empty subsets of Ω such that the sets A_i are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$ whenever $i \neq j$ and $\bigcup_{i\in I} A_i = \Omega$.

Lemma

A partition $\mathcal{P} = (A_i)_{i\in I}$ generates a σ-field $\mathcal{F}$ if every set $A \in \mathcal{F}$ can be written as a union of some of the A_is, that is, $A = \bigcup_{j\in J} A_j$ for some subset $J \subset I$.

Partition Associated with a σ-Field

Definition (Partition Associated with $\mathcal{F}$)

A **partition of Ω associated with** a σ-field $\mathcal{F}$ is a collection of non-empty sets $A_i \in \mathcal{F}$ for some $i \in I$ such that

1. $\Omega = \bigcup_{i \in I} A_i$.
2. The sets A_i are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$.
3. For each $A \in \mathcal{F}$ there exists $J \subseteq I$ such that $A = \bigcup_{i \in J} A_i$.

Lemma

For any σ-field $\mathcal{F}$ of subsets of a finite state space Ω, a partition associated with this σ-field always exists and is unique.

Partition Associated with a σ-Field

Further properties of partitions:

- If Ω is **countable** then for any σ-field $\mathcal{F}$ there exists a unique partition $\mathcal{P}$ of Ω associated with $\mathcal{F}$. It is also clear that this partition generates $\mathcal{F}$, so that $\mathcal{F} = \sigma(\mathcal{P})$
- The sets A_i in a partition must be smallest, specifically, if $\mathcal{F} = \sigma(\mathcal{P})$ and $A \in \mathcal{F}$ is such that $A \subseteq A_i$ then $A = A_i$.
- The probability of any $A \in \mathcal{F}$ equals the sum of probabilities of A_is in the partition generating $\mathcal{F}$, specifically,

$$A = \bigcup_{i \in J} A_i \quad \Rightarrow \quad \mathbb{P}(A) = \sum_{i \in J} \mathbb{P}(A_i).$$

Example: Partition Associated with a σ-Field

Example (5.2)

- Consider the σ-field $\mathcal{F}_2$ introduced in Example 5.1.
- The unique partition associated with $\mathcal{F}_2$ is given by

$$\mathcal{P}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}.$$

- Define the probabilities

$$\mathbb{P}(\{\omega_1, \omega_2\}) = \frac{2}{3}, \quad \mathbb{P}(\{\omega_3\}) = \frac{1}{6}, \quad \mathbb{P}(\{\omega_4\}) = \frac{1}{6}.$$

- Then for each event $A \in \mathcal{F}_2$ the probability of A can be easily evaluated, for instance

$$\mathbb{P}(\{\omega_1, \omega_2, \omega_4\}) = \mathbb{P}(\{\omega_1, \omega_2\}) + \mathbb{P}(\{\omega_4\}) = \frac{5}{6}.$$

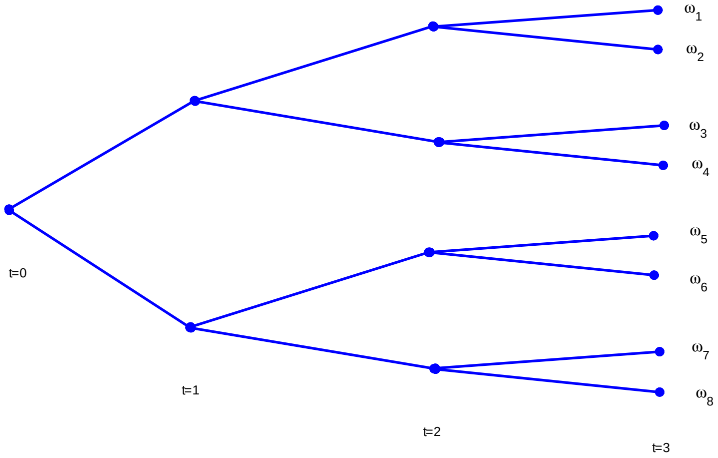
Random Variable and Measurability

Let $\mathcal{F}$ be an arbitrary σ-field of subsets of Ω. In the next definition, we do not assume that the sample space is discrete.

Definition ($\mathcal{F}$-Measurability)

A map $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$**-measurable** (measurable with respect to algebra $\mathcal{F}$) if the function $\omega \to X(\omega)$ is constant on any subset in the partition corresponding to $\mathcal{F}$. Equivalently, for every real number x the subset $\{\omega \in \Omega : X(\omega) = x\}$ is an element of algebra $\mathcal{F}$.
If X is $\mathcal{F}$-measurable then X is called a **random variable** on $(\Omega, \mathcal{F})$.

Example



Example

Example

We take $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}$ and we define the σ-fields:

$$\begin{aligned} \mathcal{F}_0 &= \{\varnothing, \Omega\} \\ \mathcal{F}_1 &= (\varnothing, \Omega, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}) \end{aligned}$$

suppose

$$X(\omega) = \begin{cases} 6, & \omega = \omega_1, \omega_2, \omega_3 \text{ or } \omega_4 \\ 8, & \omega = \omega_5, \omega_6, \omega_7 \text{ or } \omega_8 \end{cases}$$

and

$$Y(\omega) = \begin{cases} 1, & \omega = \omega_1, \omega_3, \omega_5 \text{ or } \omega_7 \\ 0, & \omega = \omega_2, \omega_4, \omega_6 \text{ or } \omega_8 \end{cases}$$

Then X is measurable with respect to $\mathcal{F}_1$ (as $\omega \to X(\omega)$ is constant on any subset in the partition) but Y is not (as $\omega \to X(\omega)$ is not constant on any subset in the partition).

Information Flow: Filtration

- In a typical application, the information about random events increases over time.
- To model the information flow, we introduce the concept of a **filtration**.
- The following definition covers the cases of the discrete and continuous time.

Definition (Filtration)

A family $(\mathcal{F}_t)_{0 \leq t \leq T}$ of σ-fields on Ω is called a **filtration** if $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$. For brevity, we denote $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$.

- We interpret the σ-field $\mathcal{F}_t$ as the information available to an agent at time t. In particular, $\mathcal{F}_0$ represents the information available at time 0, that is, the initial information.
- We assume that the information accumulated over time can only grow, so that we never forget anything!

Stochastic Process

Definition (Stochastic Process)

A **stochastic process** is a real-valued function $S_n(t, \omega) : \{0, 1, ..., T\} \times \Omega \to \mathbb{R}$. For each fixed $\omega \in \Omega$ the function $t \to S_n(t, \omega)$ is called **sample path**. For each fixed t the function $\omega \to S_n(t, \omega)$ is a random variable. A stochastic process S_n is said to be $\mathbb{F}$-**adapted** if for every $t = 0, 1, \ldots, T$ the random variable X_t is $\mathcal{F}_t$-measurable.

Example (5.3)

Consider once again the elementary market model. The stochastic process of the stock price is S_0 and S_1 on $\Omega = \{\omega_1, \omega_2\}$ and the filtration is $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and

$$\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1\}, \{\omega_2\}\} = 2^{\Omega}.$$

Note that $\mathcal{F}_0$ is the initial information, which means that the investor knows only all possible states.

Example 5.4: Stock Price Model

$$\begin{tikzcd}
 & & \boxed{S_2 = 1.8} & \omega_1 & \mathbb{P}\left(\omega_1\right) = 3/16 \\
 & \boxed{S_1 = 1.3} \arrow[ur] \arrow[dr] & & & \\
 & & \boxed{S_2 = 1.2} & \omega_2 & \mathbb{P}\left(\omega_2\right) = 5/16 \\
\boxed{S_0 = 1} \arrow[uur] \arrow[ddr] & & & & \\
 & & \boxed{S_2 = 1.1} & \omega_3 & \mathbb{P}\left(\omega_3\right) = 4/16 \\
 & \boxed{S_1 = 0.7} \arrow[ur] \arrow[dr] & & & \\
 & & \boxed{S_2 = 0.5} & \omega_4 & \mathbb{P}\left(\omega_4\right) = 4/16
\end{tikzcd}$$

Example: Stock Price Model

Example (5.4 Continued)

- At time zero the investor observes $S_0 = 1$, and has no clue about the true state, so $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- At $t = 2$ the investor observes S_2 and knows the true state ω. $\mathcal{F}_2 = 2^{\Omega}$ since the partition generating $\mathcal{F}_2$ is

$$\mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}.$$

- At $t = 1$, investor observes $S_1 = 1.3$ or $S_1 = 0.7$ so it is either $\{\omega_1, \omega_2\}$ or $\{\omega_3, \omega_4\}$. The relevant partition is $\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$
- Note that $\emptyset, \Omega$ must always be there as Ω is a union of all possible sets and $\emptyset$ is 'zero' and complement to Ω. In other words, $\mathcal{F}$ would not be an algebra/field.
- Note that stochastic process S_n is adapted to the filtration.

Filtration Generated by a Stochastic Process

- A filtration construced in previous example is said to be **generated** by the stochastic process.
- It is the coarsest one possible, i.e. the various algebras have the fewest possible subsets such that the stochastic process under discussion is adapted.
- Notation: Let $X = (X_t)_{0 \leq t \leq T}$ be a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \leq t \leq T}$ is the filtration **generated** by the process X.
- But there is another way to construct the securities market model, see the next slide.

Example: Stock Price Model

Example (5.4 Continued)

- The same price process as above
- At time t=1 a marketing survey is conducted which will be either favorable $\{\omega_1, \omega_2\}$ or unfavorable $\{\omega_3, \omega_4\}$
- Moreover, in either case the risky security will possibly take one or two distict values. Hence the relevant partition at time $t = 1$ is

$$\mathcal{P}_1 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$$

  and $\mathcal{F}_1 = 2^\Omega$.
- Investors can distinguish ω_1 from ω_2 as well as from ω_3 and ω_4 at time 1.
- They can look into the future, as they know what prices will be at time $t = 2$.
- The filtration in this example is different from the one in the previous example, although the price processes are the same.

Conditional Expectation in Elementary Probability Theory

Sample space Ω is finite.

Definition

The **conditional expectation** of discrete random variable X given the event A is denoted $\mathbb{E}(X|\,A)$ and defined in terms of conditional probability distribution $\mathbb{P}(X = x\,|A)$ by

$$\mathbb{E}(X|\,A) = \sum_x x\,\mathbb{P}(X = x|A)$$

Bayes Law

$$\mathbb{P}(X = x|A) = \frac{\mathbb{P}\left(\{X = x\} \cap A\right)}{\mathbb{P}(A)}$$

Therefore

$$\mathbb{E}(X|\,A) = \sum_x x\,\frac{\mathbb{P}\left(\{X\left(\omega\right) = x\} \cap A\right)}{\mathbb{P}(A)} = \sum_{\omega \in A} \frac{X\left(\omega\right)\mathbb{P}(\omega)}{\mathbb{P}(A)}\,.$$

Conditional Expectation in Elementary Probability Theory

$$
\begin{tikzcd}
 & & \boxed{S_2 = 1.8} & \omega_1 & \mathbb{P}\left(\omega_1\right) = 1/4 \\
 & \boxed{S_1 = 1.3} \arrow[ur] \arrow[dr] & & & \\
 & & \boxed{S_2 = 1.2} & \omega_2 & \mathbb{P}\left(\omega_2\right) = 1/4 \\
\boxed{S_0 = 1} \arrow[uur] \arrow[ddr] & & & & \\
 & & \boxed{S_2 = 1.1} & \omega_3 & \mathbb{P}\left(\omega_3\right) = 1/4 \\
 & \boxed{S_1 = 0.7} \arrow[ur] \arrow[dr] & & & \\
 & & \boxed{S_2 = 0.5} & \omega_4 & \mathbb{P}\left(\omega_4\right) = 1/4
\end{tikzcd}
$$

Suppose $\mathbb{P}\left(\omega_i\right) = \frac{1}{4}$. Then $\mathbb{P}(S_2 = 1.8 | \, S_1 = 1.3) = \frac{1/4}{1/4+1/4} = 1/2$.
Therefore $\mathbb{E}(S_2 | \, S_1 = 1.3) = 1.8 \cdot 1/2 + 1.2 \cdot 1/2 = 1.5$.

Conditional Expectations

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite (or countable) probability space.
- Let X be an arbitrary $\mathcal{F}$-measurable random variable.
- Assume that $\mathcal{G}$ is a σ-field which is contained in $\mathcal{F}$.
- Let $(A_i)_{i \in I}$ be the unique partition associated with $\mathcal{G}$.
- Our next goal is to define the **conditional expectation** $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$, that is, the conditional expectation of a random variable X with respect to a σ-field $\mathcal{G}$.

Conditional Expectation

We work with stochastic processes defined on a filtered probability space need to define $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$. It is convenient to use $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ as a sum of all conditional expectations of the form $\mathbb{E}_{\mathbb{P}}(X|\,A)$ as the event A runs through the algebra $\mathcal{G}$.
We can see that

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})\mathbb{1}_A = \mathbb{E}_{\mathbb{P}}(X|\,A) \text{ for all } A \in \mathcal{P}$$

where $\mathcal{P}$ is partition of Ω that corresponds to $\mathcal{G}$.

Definition (Conditional Expectation)

The **conditional expectation** $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ of X with respect to $\mathcal{G}$ is defined as the random variable which satisfies, for every $\omega \in A_i$,

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})(\omega) = \sum_{i \in I} \frac{1}{\mathbb{P}(A_i)}\,\mathbb{E}_{\mathbb{P}}(X\mathbb{1}_{A_i})\mathbb{1}_{A_i}$$

This generalises the definition of conditional expectation to probability spaces when Ω is not finite.

Properties of Conditional Expectation

- $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ is $\mathcal{G}$-measurable random variable
- $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ is the best estimate of X given the information represented by the σ-field $\mathcal{G}$.
- The following identity uniquely characterises the conditional expectation (in addition to $\mathcal{G}$-measurability):

$$\sum_{\omega \in G} X(\omega)\mathbb{P}(\omega) = \sum_{\omega \in G} \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})(\omega)\mathbb{P}(\omega), \quad \forall\, G \in \mathcal{G}.$$

- One can represent this equality using (discrete) integrals: for every $G \in \mathcal{G}$,

$$\int_G X \, d\mathbb{P} = \int_G \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}) \, d\mathbb{P}, \quad \forall\, G \in \mathcal{G}.$$

Properties of Conditional Expectation

Proposition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be endowed with sub-*σ*-fields* $\mathcal{G}$ *and* $\mathcal{G}_1 \subset \mathcal{G}_2$ *of* $\mathcal{F}$. *Then*

1. **Tower property:** *If* $X : \Omega \to \mathbb{R}$ *is an* $\mathcal{F}$*-measurable random variable, then*

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}_1) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}_2)\,|\,\mathcal{G}_1\right) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}_1)\,|\,\mathcal{G}_2\right).$$

2. **Taking out what is known:** *If* $X : \Omega \to \mathbb{R}$ *is a* $\mathcal{G}$*-measurable random variable and* $Y : \Omega \to \mathbb{R}$ *is an* $\mathcal{F}$*-measurable random variable, then*

$$\mathbb{E}_{\mathbb{P}}(XY|\,\mathcal{G}) = X\,\mathbb{E}_{\mathbb{P}}(Y|\,\mathcal{G}).$$

3. **Trivial conditioning:** *If* $X : \Omega \to \mathbb{R}$ *is an* $\mathcal{F}$*-measurable random variable independent of* $\mathcal{G}$, *then*

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}) = \mathbb{E}_{\mathbb{P}}(X).$$

Example 5.5: Conditional Expectation

$$
\begin{tikzcd}
 & & \boxed{S_2 = 9} & \omega_1 & \mathbb{P}(\omega_1) = 6/25 \\
 & \boxed{S_1 = 8} \arrow[ur, "\frac{3}{5}"] \arrow[dr, "\frac{2}{5}"'] & & & \\
 & & \boxed{S_2 = 6} & \omega_2 & \mathbb{P}(\omega_2) = 4/25 \\
\boxed{S_0 = 5} \arrow[uur, "\frac{2}{5}"] \arrow[ddr, "\frac{3}{5}"'] & & & & \\
 & & \boxed{S_2 = 6} & \omega_3 & \mathbb{P}(\omega_3) = 6/25 \\
 & \boxed{S_1 = 4} \arrow[ur, "\frac{2}{5}"] \arrow[dr, "\frac{3}{5}"'] & & & \\
 & & \boxed{S_2 = 3} & \omega_4 & \mathbb{P}(\omega_4) = 9/25
\end{tikzcd}
$$

Example: Conditional Expectation

Example (5.5)

The underlying probability space is given by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

- At time $t = 0$, the stock price is known and unique value is $S_0 = 5$. Hence the σ-field $\mathcal{F}_0^S$ is the trivial σ-field.
- At time $t = 1$, the stock can take two possible values so that $\mathcal{F}_1^S = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$.
- At time $t = 2$, we have $\mathcal{F}_1^S = 2^\Omega$.

Example: Conditional Expectation

Example (5.5 Continued)

$$\mathbb{P}(S_2 = 9 \,|\, A_1) = \frac{\mathbb{P}(\{\omega \in A_1\} \cap \{S_2(\omega) = 9\})}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\omega_1)}{\mathbb{P}(\{\omega_1, \omega_2\})} = \frac{\frac{6}{25}}{\frac{2}{5}} = \frac{3}{5}$$

$$\mathbb{P}(S_2 = 6 \,|\, A_1) = \frac{\mathbb{P}(\{\omega \in A_1\} \cap \{S_2(\omega) = 6\})}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\omega_2)}{\mathbb{P}(\{\omega_1, \omega_2\})} = \frac{\frac{4}{25}}{\frac{2}{5}} = \frac{2}{5}$$

$$\mathbb{P}(S_2 = 3 \,|\, A_1) = \frac{\mathbb{P}(\{\omega \in A_1\} \cap \{S_2(\omega) = 3\})}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_1, \omega_2\})} = 0$$

$$\mathbb{P}(S_2 = 9 \,|\, A_2) = \frac{\mathbb{P}(\{\omega \in A_2\} \cap \{S_2(\omega) = 9\})}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(\{\omega_3, \omega_4\})} = 0$$

$$\mathbb{P}(S_2 = 6 \,|\, A_2) = \frac{\mathbb{P}(\{\omega \in A_2\} \cap \{S_2(\omega) = 6\})}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\omega_3)}{\mathbb{P}(\{\omega_3, \omega_4\})} = \frac{\frac{6}{25}}{\frac{3}{5}} = \frac{2}{5}$$

$$\mathbb{P}(S_2 = 3 \,|\, A_2) = \frac{\mathbb{P}(\{\omega \in A_2\} \cap \{S_2(\omega) = 3\})}{\mathbb{P}(A_2)} = \frac{\mathbb{P}(\omega_4)}{\mathbb{P}(\{\omega_3, \omega_4\})} = \frac{\frac{9}{25}}{\frac{3}{5}} = \frac{3}{5}$$

Example: Conditional Expectation

Example (5.5 Continued)

We have

$$\mathbb{E}_{\mathbb{P}}(S_2 \,|\, \mathcal{F}_1^S)(\omega) = 9 \cdot \frac{3}{5} + 6 \cdot \frac{2}{5} + 3 \cdot 0 = \frac{39}{5} \quad \text{for } \omega \in A_1$$

$$\mathbb{E}_{\mathbb{P}}(S_2 \,|\, \mathcal{F}_1^S)(\omega) = 9 \cdot 0 + 6 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{21}{5} \quad \text{for } \omega \in A_2$$

and thus

$$\mathbb{E}_{\mathbb{P}}(S_2 | \mathcal{F}_1^S) = \begin{cases} 39/5 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 21/5 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases}$$

Note that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(S_2 | \mathcal{F}_1^S)\right) &= \frac{39}{5} \cdot \frac{2}{5} + \frac{21}{5} \cdot \frac{3}{5} = \frac{141}{25} \\ \mathbb{E}_{\mathbb{P}}(S_2) &= 9 \cdot \frac{6}{25} + 6 \cdot \frac{4}{25} + 6 \cdot \frac{6}{25} + 3 \cdot \frac{9}{25} = \frac{141}{25} \end{aligned}$$

Change of a Probability Measure

- Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures on $(\Omega, \mathcal{F})$.
- Let the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ be

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = L(\omega)$$

meaning that L is $\mathcal{F}$-measurable and, for every $A \in \mathcal{F}$,

$$\int_A X \, d\mathbb{Q} = \int_A XL \, d\mathbb{P}.$$

- If Ω is finite then this equality becomes

$$\sum_{\omega \in A} X(\omega)\, \mathbb{Q}(\omega) = \sum_{\omega \in A} X(\omega) L(\omega)\, \mathbb{P}(\omega).$$

- Random variable L is strictly positive and $\mathbb{E}_{\mathbb{P}}(L) = 1$.
- Equality $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(XL)$ holds for any $\mathbb{Q}$-integrable random variable X (it suffices to take $A = \Omega$).

Abstract Bayes formula

Lemma (5.1 Bayes Formula)

Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$ and let X be a $\mathbb{Q}$-integrable random variable. Then the Bayes formula holds

$$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{G}) = \frac{\mathbb{E}_{\mathbb{P}}(XL \mid \mathcal{G})}{\mathbb{E}_{\mathbb{P}}(L \mid \mathcal{G})}.$$

Martingales

Martingales are stochastic processes representing **fair games**.

Definition (Martingale)

An $\mathbb{F}$-adapted process $X = (X_t)_{0 \le t \le T}$ on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **martingale** whenever for all $s < t$

$$\mathbb{E}_{\mathbb{P}}(X_t \,|\, \mathcal{F}_s) = X_s.$$

To establish the equality above, it suffices to check that for every $t = 0, 1, \ldots, T-1$

$$\mathbb{E}_{\mathbb{P}}(X_{t+1} \,|\, \mathcal{F}_t) = X_t.$$

Martingales

Definition

An $\mathbb{F}$-adapted process $X = (X_t)_{0 \le t \le T}$ on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **supermartingale** whenever for all $s < t$

$$\mathbb{E}_{\mathbb{P}}(X_t \,|\, \mathcal{F}_s) \le X_s.$$

Definition

An $\mathbb{F}$-adapted process $X = (X_t)_{0 \le t \le T}$ on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **submartingale** whenever for all $s < t$

$$\mathbb{E}_{\mathbb{P}}(X_t \,|\, \mathcal{F}_s) \ge X_s.$$