Mathematical Finance

Introduction

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Semester 1

Practicalities

- Lectures on Monday
- *Preliminary* lecture slides are already uploaded to Moodle and available to download
  - I revise slides before each lecture, most likely on Sundays, so a new update might appear just before the lecture
  - such changes are likely to be minor so it does not matted which set of slides you will bring to the lecture. If anything major happens I will send an email to all participants.
  - the split by weeks is very preliminary, we may go slower or faster
- Most of Tutorials are at the end of the week, Thursday and Friday
  - Brief solutions to tutorial questions will be posted on Moodle after tutorials, most likely during the following weekend.
- Contact
  - *Office Hours* Tuesday 3-5
  - By appointment: send me an email and we will agree convenient time

Recommended readings:

- Pliska, S. R. (1997). *Introduction to Mathematical Finance: Discrete Time Models.* Blackwell Publishing.
- Van der Hoek, J. and Elliott, R. E. (2006). *Binomial Models in Finance.* Springer.
- Hull, J. C. (2002). *Options, Futures, and Other Derivatives.* 5th Edition. Prentice Hall.
- Shreve, S. E. (2004). *Stochastic Calculus for Finance. Volume 1: The Binomial Asset Pricing Model.* Springer.
- Seydel, R. (2009). *Tools for Computational Finance.* 4th Edition. Springer.
- Musiela, M. and Rutkowski, M. (2005). *Martingale Methods in Financial Modelling.* 2nd Edition. Springer.

Goals of the Course

- Our main aim is to compute the *fair price* of a derivative security
- Of course, we will need to *define a concept* of fair price
- We will need to learn how to model various features, for example how to model the randomness inherent in securities prices
- We will introduce and use a lot of new concepts, for example
  - arbitrage
  - trading strategies and hedging strategies
  - risk-neutral probability measure
  - attainable claims
  - complete markets
- We first start working with the simplest possible model, and introduce all notions, which the model can explain.
- We then enrich the model with some more realistic features and go again over the key results, to see whether they change.
- Gradually, we start working with 'proper' discrete-time models and then with continuous time models.

We Will Cover

- Discrete-Time Models
- Continuous-Time Models

Discrete and Continuous Time Finance

- In the paper *The Pricing of Options and Corporate Liabilities* by Fischer Black and Myron Scholes published in 1973, the authors developed ideas, which are nowadays widely used by practitioners to price derivative securities.

- The full theory of security markets requires some knowledge of stochastic calculus and numerical methods for partial differential equations (PDEs).

- We therefore *start* with discrete-time financial models and get a good grasp of fundamental ideas of arbitrage pricing *before* doing continuous-time models.

- In practice, discrete time models *are* used to price derivatives. Continuous-time models *have to be solved numerically*, and it is always *discretisation*, i.e. approximation by discrete-time models.

We Will Cover

- European Type Options
- American Type Options

The structure of the course

- Elementary Market Model (one period, two assets, two states of nature)
- Single-period market model (one period, many assets, many states of nature)
- Modelling uncertainty (filtration and conditioning)
- Generalisation of all results in Multi-period market model
- Binomial model as a special case of Multi-period market model
- Applications: European and American type of options in binomial models
- Continuous time models: Black-Scholes formula
- Extensions

Long and Short Positions

- A financial security (or asset) is a contract which records some entitlement or obligation and which can be traded at financial markets.
- Some examples of financial securities: stocks, bonds, options, swaps, and futures.
- An agent takes a so-called long (short) position in the asset if the agent owns (owes) it.
- At modern financial markets, it may not be necessary to own an asset to sell. The strategy of selling a borrowed asset is called short-selling.
  - One can borrow a number of stocks and sells them to undertake other investments. The agent buys them back and returns to the original owner at a predetermined time.

Short Selling Explained

- *Short Selling is NOT allowed*

Investor has money M at time 0 and can invest into shares and bonds at time 1.

time 0: Money $M \rightarrow\rightarrow\rightarrow$ buy $\rightarrow\rightarrow\rightarrow$ time 1:
$\begin{cases} \text{share 1} \\ \text{share 2} \\ ... \\ \text{share } N \\ \text{bond (bank account)} \end{cases}$

$M\alpha_1 \rightarrow$ share 1

$M\alpha_2 \rightarrow$ share 2

...

$M\alpha_N \rightarrow$ share N

$M\alpha_{N+1} \rightarrow$ bonds
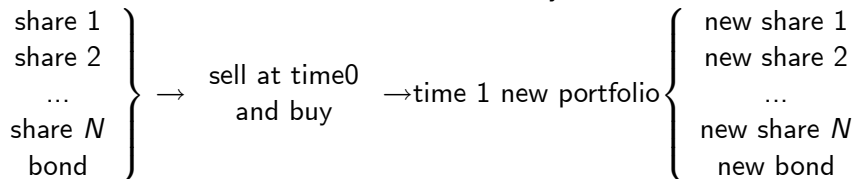
$$0 \leq \alpha_1, \alpha_2, ..., \alpha_N, \alpha_{N+1} \leq 1$$

$$\alpha_1 + \alpha_2 + ... + \alpha_N + \alpha_{N+1} = 1$$

Short Selling Explained

- *Short Selling is allowed*

Investor has shares and bonds and can sell any of them at time 0.

$$
\left. \begin{array}{c} \text{share 1} \\ \text{share 2} \\ \text{...} \\ \text{share } N \\ \text{bond} \end{array} \right\} \rightarrow \begin{array}{c} \text{sell at time0} \\ \text{and buy} \end{array} \rightarrow \text{time 1 new portfolio} \left\{ \begin{array}{c} \text{new share 1} \\ \text{new share 2} \\ \text{...} \\ \text{new share } N \\ \text{new bond} \end{array} \right.
$$

Short Selling Explained

- *Short Selling is allowed*

Investor has shares and bonds and can sell any of them at time 0.

share 1
bond $\left.\right\}$ → sell bond at time 0 and buy more shares 1 at time 1

share 1
bond $\left.\right\}$ → sell shares at time 0 and buy more bonds at time 1

As we are selling, coefficients α_i which describe proportion of money invested into assets will be negative (compare with the case when no short selling is allowed).

Exchanges and OTC Markets

- If a large number of traders short sell a certain stock, it is very likely that the stock price will fall. This has drawn some criticism in the last couple of years and restrictions on short-selling were introduced in some countries.

- Two types of financial markets:
  - Exchanges (stock exchange, options exchange, futures exchange, etc.)
  - Over-the-counter (OTC) markets.

- A securities exchange provides an organised forum for trading securities whose contracts are standardized. Buyers and sellers do not directly contact each other and all trades are done by the market makers. Prices of securities traded on exchanges are widely disseminated.

Exchanges

- Some examples of exchanges and securities traded on them:
  - ▸ Equity markets: stocks and shares.
  - ▸ Bond market: zero-coupon bonds (ZCBs), government bonds and corporate bonds.
  - ▸ Futures market: futures contracts.
  - ▸ Foreign exchange (FX) market: foreign currencies, options on foreign currencies.
  - ▸ Options market: equity options and interest rate options.
  - ▸ Commodity market: oil and metals futures.

OTC Markets

- OTC markets are less organised and two institutions such as a bank and an investment company may be simply involved.
- Prices of OTC traded securities are generally hard to obtain.
- Some examples of OTC markets and most typical securities traded on them:
  - Interest rate market (mostly OTC): caps, floors, swaps and swaptions.
  - Exotic options market: barrier options, lookback options, compound options.
  - Credit market (mostly OTC): credit derivatives such as CDSs and CDOs.

Reasons of Trading in Securities

- Profit
  - A trader believing that the price of a security will rise may follow the idea of buying low and selling high. If he believes that the price of a security will decline then short-selling is the right strategy.

- Protection
  - For example, a company dependent on imports may prefer to fix the exchange rate in advance and apply it to a trade later. To this end, one can purchase a futures contract on the exchange rate. It is also possible to enter an OTC forward contract on the exchange rate.

- Hedging
  - To reduce the risk that a trader's portfolio is exposed to, by holding a certain position in related securities.

- Diversification
  - Risks affecting specific securities behave in different manner and thus they may cancel out.

Bear and Bull Markets

- Market prices of traded securities are driven by the forces of **supply** and **demand** (that is, the willingness to sell and buy, respectively).
- In over-supply or under-demand, security prices will generally fall. This is referred to as the **bear market**.
- In under-supply or over-demand, security prices will generally rise. This is called the **bull market**.
- Some factors, such as: market information, rumour mill and human psychology, cause **random fluctuations** in supply and demand and hence the market price as well.
- Prices that satisfy supply and demand are called to be in **equilibrium** (at least, in the academic literature).

Perfect Markets

- Assumption 1. Markets are frictionless, i.e., no transaction costs, no taxes or other costs and there are no penalties for short selling.
- Assumption 2. Infinitely divisible security prices. It is possible to buy or sell any (also non-integer) quantity of any security.
- Assumption 3. All traders have access to the same information. The Efficient Market Hypothesis (EMH) holds:
  - Weak-form: Current security prices reflect all past market prices and data. Technical analysis is of no benefit.
  - Semi-strong form: Current security prices reflect all publicly available information. Fundamental analysis is of no benefit.
  - Strong form: Current security prices reflect all available information. Inside information is of no benefit.

Arbitrage-free Markets

- Assumption 4. Markets are assumed to be arbitrage-free:

  - **Arbitrage** is the guarantee of certain future profit without any current investment. Much of the modern financial theory holds under the assumption that efficient financial markets should forbid such an arbitrage opportunity.

  - It follows from the **no-arbitrage principle** that if two financial securities have the same pattern of future cash-flows, they should have the same price today. This property is commonly known as the **law of one price**.

- Standing assumptions 1. to 4. provide a framework in which we can develop mathematical models for **primary securities** and propose pricing methods for **derivative securities**.

- Assumptions 1. to 3. will be implicitly taken into account when describing **trading**. We will thus mainly focus on the no-arbitrage principle when dealing with **valuation**.

European Call Option

Definition. A **European call option** is a financial security, which gives its buyer the right (but not the obligation) to buy an asset at a future time $T > 0$ for a price $K > 0$, which is known as the strike price.

- The underlying asset, the **expiration** (or maturity) date T and the **strike** (or exercise) price K are specified in the contract.
- We assume that the underlying asset is one share of the stock with price S_t at time t.

At expiry T, a rational holder of a European call option should proceed as follows:

- If the stock price $S_T > K$, he should buy the stock at time T for the price K from the seller of the option (an option is **exercised**) and immediately sell it on the market for the market price S_T, leading to a positive payoff of $S_T - K$.

Call Option Payoff

- If $S_T \leq K$, then it does not make sense to buy the stock for the price K from the seller, since it can be bought for a lower price on the market. In that case, the holder should waive his right to buy (an option is **abandoned**) and this leads to a payoff of 0.

- These arguments show that a European call option is formally equivalent to the following **random payoff** C_T at time T

$$C_T := \max(S_T - K, 0) = (S_T - K)^+$$

where we denote $x^+ = \max(x, 0)$ for any real number x. Note that the payoff C_T is **path-independent**, meaning that it only depends on the stock price at expiry date T, and not on the whole sample path S_t, $t \in [0, T]$ of the stock price.

European Put Option

Definition. A **European put option** is a financial security, which gives its buyer the right (but not the obligation) to sell an asset at a future time $T > 0$ for a price , which is known as the strike price.

One can show that a European put option on the stock S is formally equivalent to the following random payoff P_T at time T:

$$P_T := \max(K - S_T, 0) = (K - S_T)^+$$

It is also easy to see that $C_T - P_T = S_T - K$

- European call and put options are actively traded on organised exchanges.
- A pertinent theoretical question thus arises:
  - What should be the 'fair' prices C_t and P_t of European call and put options prior to the expiry date T?

Interest Rates and Zero-Coupon Bonds

- All participants of financial markets have access to **riskless cash** through borrowing and lending from the money market.
- An investor who borrows cash must pay the loan back to the lender with **interest** at some time in the future. An investor who lends cash will receive from the borrower the nominal value and the interest on the loan at some time in the future.
- We assume throughout that the **borrowing rate** is equal to the **lending rate**.

Definition (Zero-Coupon Bond) The unit **zero-coupon bond** maturing at T is a financial security returning to its holder one unit of cash at time T. We denote by $B(t, T)$ the bond price at time $t \in [0, T]$. Hence $B(T, T) = 1$.

Discretely Compounded Interest

- The set of dates is $\{0, 1, 2, ...\}$. Let a real number $r > -1$ represent the simple interest rate over each period $[t, t+1]$ for $t = 0, 1, 2, ...$ One unit of cash invested at time 0 in the money market account yields the following amount at time $t = 0, 1, 2, ...$

$$B_t = (1 + r)^t.$$

- From the law of one price, it follows that

$$B(t, T) = \frac{B_t}{B_T} = (1 + r)^{-(T-t)}.$$

- In general, if $r(t) > -1$ is the deterministic simple interest rate over the time period $[t, t+1]$ then we obtain, for every $t = 0, 1, 2, ...,$

$$B_t = \prod_{u=0}^{t-1} (1 + r(u)), \quad B(t, T) = \prod_{u=t}^{T-1} (1 + r(u))^{-1}$$