1 Tutorial Week 3: Single-Period Market Models

1. We consider the single-period market model $\mathcal{M} = (B, S^1, S^2)$ and we assume that the state space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ let the interest rate be $r = \frac{1}{9}$. Stock prices at time $t = 0$ are given by $S_0^1 = 5$ and $S_0^2 = 10$. Random stock prices at time $t = 1$ are given by the following table

	ω_1	ω_2	ω_3	ω_4
S_1^1	$\frac{60}{9}$	$\frac{60}{9}$	$\frac{40}{9}$	$\frac{20}{9}$
S_1^2	$\frac{40}{3}$	$\frac{80}{9}$	$\frac{80}{9}$	$\frac{120}{9}$

   (a) Compute explicitly the random variables $V_1(x, \phi)$, $G_1(x, \phi)$, $\widehat{V}_1(x, \phi)$ and $\widehat{G}_1(x, \phi)$.

   (b) Does $G_1(x, \phi)$ (or $\widehat{G}_1(x, \phi)$) depend on the initial endowment x?

2. Consider the market model $\mathcal{M} = (B, S)$ with $k = 3$, $n = 1$, $r = \frac{1}{9}$, $S_0 = 5$ and the random stock price S_1 given by the table

	ω_1	ω_2	ω_3
S_1	$\frac{60}{9}$	$\frac{40}{9}$	$\frac{30}{9}$

   Find the class $\mathbb{M}$ of all risk neutral probability measures for this market model by making use of Definition of Risk Neutral Probability Measure.

2 Tutorial Week 3: Solutions

1. (a) For any trading strategy $(x, \phi) = (x, \phi^1, \phi^2)$ we have

$$V_1(x, \phi) = \left(x - \sum_{j=1}^{2} \phi^j S_0^j \right) (1 + r) + \sum_{j=1}^{2} \phi^j S_1^j$$

or more explicitly

$$V_1(x, \phi)(\omega_i) = \begin{cases} \frac{10}{9}x + \frac{10}{9}\phi^1 + \frac{20}{9}\phi^2, & i = 1 \\ \frac{10}{9}x + \frac{10}{9}\phi^1 - \frac{20}{9}\phi^2, & i = 2 \\ \frac{10}{9}x - \frac{10}{9}\phi^1 - \frac{20}{9}\phi^2, & i = 3 \\ \frac{10}{9}x - \frac{30}{9}\phi^1 + \frac{20}{9}\phi^2, & i = 4 \end{cases}$$

Since

$$G_1(x, \phi) = V_1(x, \phi) - V_0(x, \phi) = V_1(x, \phi) - x$$

we obtain

$$G_1(x, \phi)(\omega_i) = \begin{cases} \frac{1}{9}x + \frac{10}{9}\phi^1 + \frac{20}{9}\phi^2, & i = 1 \\ \frac{1}{9}x + \frac{10}{9}\phi^1 - \frac{20}{9}\phi^2, & i = 2 \\ \frac{1}{9}x - \frac{10}{9}\phi^1 - \frac{20}{9}\phi^2, & i = 3 \\ \frac{1}{9}x - \frac{30}{9}\phi^1 + \frac{20}{9}\phi^2, & i = 4 \end{cases}$$

Next,

$$\widehat{V}_1(x, \phi) = (1 + r)^{-1} V_1(x, \phi) = \frac{9}{10} V_1(x, \phi)$$

so that

$$\widehat{V}_1(x,\phi)(\omega_i) = \begin{cases} x+\phi^1+2\phi^2, & i=1 \\ x+\phi^1-2\phi^2, & i=2 \\ x-\frac{1}{9}\phi^1-2\phi^2, & i=3 \\ x-3\phi^1+2\phi^2, & i=4 \end{cases}$$

Finally,

$$\widehat{G}_1(x,\phi) = \widehat{V}_1(x,\phi) - \widehat{V}_0(x,\phi) = \widehat{V}_1(x,\phi) - x$$

and thus

$$\widehat{G}_1(x,\phi)(\omega_i) = \begin{cases} \phi^1+2\phi^2, & i=1 \\ \phi^1-2\phi^2, & i=2 \\ -\phi^1-2\phi^2, & i=3 \\ -3\phi^1+2\phi^2, & i=4 \end{cases}$$

(b) It is clear that $G_1(x,\phi)$ depends on the initial wealth x but $\widehat{G}_1(x,\phi)$ does not, so that for any $x,y \in \mathbb{R}$ and arbitrary $\phi \in \mathbb{R}^n$ we have $\widehat{G}_1(x,\phi) = \widehat{G}_1(y,\phi)$.

2. In view of the definition we need to find all probability measures $\mathbb{Q}$ on the space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ so that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ (that is $\mathbb{Q}(\omega_i) > 0$ for $i = 1,2,3$) and

$$\mathbb{E}_{\mathbb{Q}}\left(\Delta\widehat{S}_1\right) = 0$$

where

$$\Delta\widehat{S}_1 = \widehat{S}_1 - \widehat{S}_0 = \frac{9}{10}\left(\frac{60}{9}, \frac{40}{9}, \frac{30}{9}\right) - (5,5,5) = (1,-1,-2)$$

Let us denote $\mathbb{Q} = (q_1, q_2, q_3)$. Then we search for a solution (q_1,q_2,q_3) of the system

$$\begin{cases} 0 < q_i < 1, & i = 1,2,3 \\ q_1 - q_2 - 2q_3 = 0 \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

This yields

$$\begin{cases} q_1 = \frac{1}{2}q_3 + \frac{1}{2} \\ q_2 = -\frac{3}{2}q_3 + \frac{1}{2} \end{cases}$$

The condition $0 < q_i < 1$ is satisfied for $i = 1,2,3$ whenever $0 < q_3 < \frac{1}{3}$

Hence the set of all risk neutral probability measures for $\mathcal{M}$ can be represented as follows

$$\mathbb{M} = \left\{ \mathbb{Q} = \left(\frac{1}{2}, \frac{1}{2}, 0\right) + q_3\left(\frac{1}{2}, -\frac{3}{2}, 1\right), q_3 \in \left(0, \frac{1}{3}\right) \right\}$$