1 Tutorial Week 8: American-type Contingent Claims in CRR Model

Consider the CRR binomial model with the initial stock price $S_0 = 9$, interest rate $r = 0.01$ and the volatility $\sigma = 0.1$.

1. Use the CRR-parametrization for u and d, that is

$$u = e^{\sigma\sqrt{\Delta t}}, \ d = \frac{1}{u}$$

with time increment $\Delta t = 1$.

a) Compute the price of European Put option with strike price $K = 10$, maturity time $T = 2$.

b) Compute an American put option with strike price $K = 10$, maturity time $T = 2$.Will the option be exercised before maturity?

c) For the issuer of the American put option, find the replicating strategy up to the rational exercise time.

2. Use the JR-parametrization for u and d, that is

$$\begin{aligned} u &= e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} \\ d &= e^{\left(r-\frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}} \end{aligned}$$

with time increment $\Delta t = 1$.

a) Compute the price of European Put option with strike price $K = 10$, maturity time $T = 2$.

b) Compute an American put option with strike price $K = 10$, maturity time $T = 2$.Will the option be exercised before maturity?

c) For the issuer of the American put option, find the replicating strategy up to the rational exercise time.

2 Solutions to Tutorial week 8

1. The CRR parameterisation for u and d with $\Delta t = 1$ gives

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} = e^{\sigma} = 1.105171 \\ d &= \frac{1}{u} = 0.904837 \end{aligned}$$

Consequently,

$$\tilde{p} = \frac{1 + r - d}{u - d} = 0.524938$$

Denote μ price of European Put Option, π price of American Put Option.

Denote π^w price of waiting for one more period, π^e is price of exercising

$S_2 = u^2 S_0 = 9e^{2\cdot 0.1} = 10.993$
$\pi_2 = (10 - 10.993)^+ = 0$
$\mu_2 = (10 - 10.993)^+ = 0$

$S_1 = uS_0 = 9e^{0.1} = 9.946\,5$
$\pi_1^w = \frac{1}{1+0.01}(0.524938 \cdot 0$
$+ (1 - 0.524938) \cdot 1)$
$= 0.470\,36$
$\pi_1^e = (10 - 9.946\,5)^+ = 0.053\,5$
$\pi_1 = \max(0.470\,36, 0.053\,5)$
$= 0.470\,36$
wait
$\mu_1 = \frac{1}{1+0.01}(0.524938 \cdot 0$
$+ (1 - 0.524938) \cdot 1)$
$= 0.470\,36$

$S_0 = 9$
$\pi_0^w = \frac{1}{1+0.01}(0.524938 \cdot 0.470\,36$
$+ (1 - 0.524938) \cdot 1.856\,5)$
$= 1.117\,7$
$\pi_0^e = (10 - 9)^+ = 1$
$\pi_0 = \max(1.117\,7, 1) = 1.117\,7$
wait
$\mu_0 = \frac{1}{1+0.01}(0.524938 \cdot 0.470\,36$
$+ (1 - 0.524938) \cdot 1.757\,4)$
$= 1.071\,1$

$S_2 = udS_0 = 9e^{0.1}e^{-0.1} = 9.0$
$\pi_2 = (10 - 9)^+ = 1$
$\mu_2 = (10 - 9)^+ = 1$

$S_1 = dS_0 = 9e^{-0.1} = 8.143\,5$
$\pi_1^w = \frac{1}{1+0.01}(0.524938 \cdot 1$
$+ (1 - 0.524938) \cdot 2.631\,4)$
$= 1.757\,4$
$\pi_1^e = (10 - 8.143\,5)^+ = 1.856\,5$
$\pi_1 = \max(1.757\,4, 1.856\,5)$
$= 1.856\,5$
exercise
$\mu_1 = \frac{1}{1+0.01}(0.524938 \cdot 1$
$+ (1 - 0.524938) \cdot 2.631\,4)$
$= 1.757\,4$

$S_2 = d^2 S_0 = 9e^{-2\cdot 0.1} = 7.368\,6$
$\pi_2 = (10 - 7.368\,6)^+ = 2.631\,4$
$\mu_2 = (10 - 7.368\,6)^+ = 2.631\,4$

The rational holder should exercise the American put option at time t=1 whenever the stock price falls during the first period. Otherwise, he should not exercise the option till time 2.

$$\begin{aligned}\tau_0^*(\omega) &= 1 \quad \text{for} \quad \omega \in \{\omega_3, \omega_4\}\\ \tau_0^*(\omega) &= 2 \quad \text{for} \quad \omega \in \{\omega_1, \omega_2\}\end{aligned}$$

At $t = 0$, we need to solve

$$\begin{aligned}1.01\,\phi_0^0 + 9.946\,5\,\phi_0^1 &= 0.470\,36\\ 1.01\,\phi_0^0 + 8.143\,5\,\phi_0^1 &= 1.856\,5\end{aligned}$$

$$\begin{aligned}\phi_0^1 = \frac{0.470\,36 - 1.856\,5}{(9.946\,5\ - 8.143\,5)} &= -0.768\,80\\ \phi_0^0 \quad = \quad \frac{1.856\,5}{1.01} - \frac{8.143\,5}{1.01}\frac{(0.470\,36 - 1.856\,5)}{(9.946\,5\ - 8.143\,5)} &= 8.036\,8\end{aligned}$$

Hence $(\phi_0^0, \phi_0^1) = (-0.768\,80, 8.036\,8)$ for all ω.

If the stock price falls during the first period, the option is exercised and thus we do not need to compute the strategy at time 1 for $\omega \in \{\omega_3, \omega_4\}$

If the stock price rises during the first period, we solve

$$\begin{aligned}1.01\widetilde{\phi}_1^0 + 10.993\,\phi_1^1 &= 0\\ 1.01\,\widetilde{\phi}_1^0 + 9.0\,\phi_1^1 &= 1\end{aligned}$$

$$\begin{aligned}\phi_1^1 &= -\frac{1}{(10.993\ - 9.0)} = -0.501\,76\\ \widetilde{\phi}_1^0 &= \frac{1}{1.01} + \frac{9.0}{1.01}\frac{1}{(10.993\ - 9.0)} = 5.461\,2\end{aligned}$$

Hence $(\widetilde{\phi}_1^0, \phi_1^1) = (5.461\,2, -0.501\,76)$ for $\omega \in \{\omega_1, \omega_2\}$.

2. The JR parameterisation for u and d with $\Delta t = 1$ gives

$$\begin{aligned}u &= e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} = e^{\left(r - \frac{\sigma^2}{2}\right) + \sigma} = e^{\left(0.01 - \frac{0.1^2}{2}\right) + 0.1} = 1.110\,7\\ d &= e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}} = e^{\left(0.01 - \frac{0.1^2}{2}\right) - 0.1} = 0.909\,37\end{aligned}$$

Consequently,

$$\tilde{p} = \frac{1 + r - d}{u - d} = \frac{1 + 0.01 - 0.909\,37}{1.110\,7 - 0.909\,37} = 0.499\,83$$

Denote μ price of European Put Option, π price of American Put Option.

Denote π^w price of waiting for one more period, π^e is price of exercising

$S_2 = u^2 S_0 = 9e^{2\left(\left(0.01-\frac{0.1^2}{2}\right)+0.1\right)}$
$= 11.103$
$\pi_2 = (10 - 11.103)^+ = 0$
$\mu_2 = (10 - 11.103)^+ = 0$

$S_1 = uS_0 = 9e^{\left(0.01-\frac{0.1^2}{2}\right)+0.1}$
$= 9.9964$
$\pi_1^w = \frac{1}{1+0.01}(0.49983 \cdot 0$
$+ (1 - 0.49983) \cdot 0.9095)$
$= 0.4504$
$\pi_1^e = (10 - 9.9964)^+ = 0.0036$
$\pi_1 = \max(0.4504, 0.0036)$
$= 0.4504$
wait
$\mu_1 = \frac{1}{1+0.01}(0.49983 \cdot 0$
$+ (1 - 0.49983) \cdot 0.9095)$
$= 0.4504$

$S_0 = 9$
$\pi_0^w = \frac{1}{1+0.01}(0.49983 \cdot 0.4504$
$+ (1 - 0.49983) \cdot 1.8156)$
$= 1.122$
$\pi_0^e = (10 - 9)^+ = 1$
$\pi_0 = \max(1.122, 1) = 1.122$
wait
$\mu_0 = \frac{1}{1+0.01}(0.49983 \cdot 0.4504$
$+ (1 - 0.49983) \cdot 1.7166)$
$= 1.0730$

$S_2 = udS_0$
$= 9e^{\left(0.01-\frac{0.1^2}{2}\right)+0.1} e^{\left(0.01-\frac{0.1^2}{2}\right)-0.1}$
$= 9e^{2\left(0.01-\frac{0.1^2}{2}\right)} = 9.0905$
$\pi_2 = (10 - 9.0905)^+ = 0.9095$
$\mu_2 = (10 - 9.0905)^+ = 0.9095$

$S_1 = dS_0 = 9e^{\left(0.01-\frac{0.1^2}{2}\right)-0.1}$
$= 8.1844$
$\pi_1^w = \frac{1}{1+0.01}(0.49983 \cdot 0.9095$
$+ (1 - 0.49983) \cdot 2.5574)$
$= 1.7166$
$\pi_1^e = (10 - 8.1844)^+ = 1.8156$
$\pi_1 = \max(1.7166, 1.8156)$

$= 1.8156$
exercise
$\mu_1 = \frac{1}{1+0.01}(0.49983 \cdot 0.9095$
$+ (1 - 0.49983) \cdot 2.5574)$
$= 1.7166$

$S_2 = d^2 S_0 = 9e^{2\left(\left(0.01-\frac{0.1^2}{2}\right)-0.1\right)}$
$= 7.4426$
$\pi_2 = (10 - 7.4426)^+ = 2.5574$
$\mu_2 = (10 - 7.4426)^+ = 2.5574$

The rational holder should exercise the American put option at time t=1 whenever the stock price falls during the first period. Otherwise, he should not exercise the option till time 2.

$$\begin{aligned}\tau_0^*(\omega) &= 1 \quad \text{for} \quad \omega \in \{\omega_3, \omega_4\} \\ \tau_0^*(\omega) &= 2 \quad \text{for} \quad \omega \in \{\omega_1, \omega_2\}\end{aligned}$$

At $t = 0$, we need to solve

$$\begin{aligned}1.01\,\phi_0^0 + 9.996\,4\,\phi_0^1 &= 0.450\,4 \\ 1.01\,\phi_0^0 + 8.184\,4\phi_0^1 &= 1.815\,6\end{aligned}$$

$$\phi_0^1 = \frac{0.450\,4 - 1.815\,6}{(9.996\,4\ - 8.184\,4)} = -0.753\,42$$

$$\phi_0^0 \quad = \quad \frac{1.815\,6}{1.01} - \frac{8.184\,4}{1.01}\,\frac{(0.450\,4 - 1.815\,6)}{(9.996\,4\ - 8.184\,4)} = 7.902\,9$$

Hence $(\phi_0^0, \phi_0^1) = (-0.753\,42, 7.902\,9)$ for all ω.

If the stock price falls during the first period, the option is exercised and thus we do not need to compute the strategy at time 1 for $\omega \in \{\omega_3, \omega_4\}$

If the stock price rises during the first period, we solve

$$\begin{aligned}1.01\widetilde{\phi}_1^0 + 11.103\,\phi_1^1 &= 0 \\ 1.01\,\widetilde{\phi}_1^0 + 9.090\,5\,\phi_1^1 &= 0.909\,5\end{aligned}$$

$$\begin{aligned}\phi_1^1 &= -\frac{0.909\,5}{(11.103\ - 9.090\,5)} = -0.451\,93 \\ \widetilde{\phi}_1^0 &= \frac{0.909\,5}{1.01} + \frac{9.090\,5}{1.01}\,\frac{0.909\,5}{(11.103 - 9.090\,5)} = 4.968\end{aligned}$$

Hence $(\widetilde{\phi}_1^0, \phi_1^1) = (4.968, -0.451\,93)$ for $\omega \in \{\omega_1, \omega_2\}$.