Mathematical Finance

Binomial Asset Pricing Model

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Semester 1

Outline

We will examine the following issues:

1. The Cox-Ross-Rubinstein Market Model
2. The CRR Call Option Pricing Formula
3. Call and Put Options of American Style
4. Dynamic Programming Approach to American Claims
5. Examples: American Call and Put Options

Introduction

- The **Cox-Ross-Rubinstein market model** (CRR model) is an example of a multi-period market model of the stock price.
- At each point in time, the stock price is assumed to either go 'up' by a fixed factor u or go 'down' by a fixed factor d.
- Only three parameters are needed to specify the binomial asset pricing model: $u > d > 0$ and $r > -1$.
- Note that we do not postulate that $d < 1 < u$.
- The real-world probability of an 'up' movement is assumed to be the same $0 < p < 1$ for each period and is assumed to be independent of all previous stock price movements.

Bernoulli Processes

Definition (Bernoulli Process)

A process $X = (X_t)_{1 \leq t \leq T}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **Bernoulli process** with parameter $0 < p < 1$ if the random variables $X_1, X_2, \dots, X_T$ are independent and have the following common probability distribution

$$\mathbb{P}(X_t = 1) = 1 - \mathbb{P}(X_t = 0) = p.$$

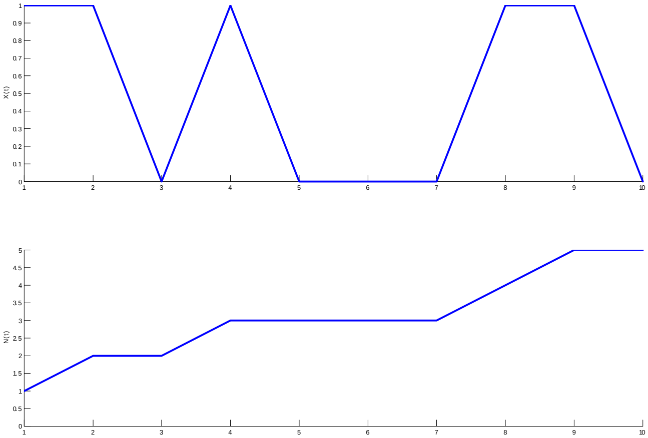
Definition (Bernoulli Counting Process)

The **Bernoulli counting process** $N = (N_t)_{0 \leq t \leq T}$ is defined by setting $N_0 = 0$ and, for every $t = 1, \dots, T$ and $\omega \in \Omega$,

$$N_t(\omega) := X_1(\omega) + X_2(\omega) + \dots + X_t(\omega).$$

The process N is a special case of an **additive random walk**.

Bernoulli Processes



Stock Price

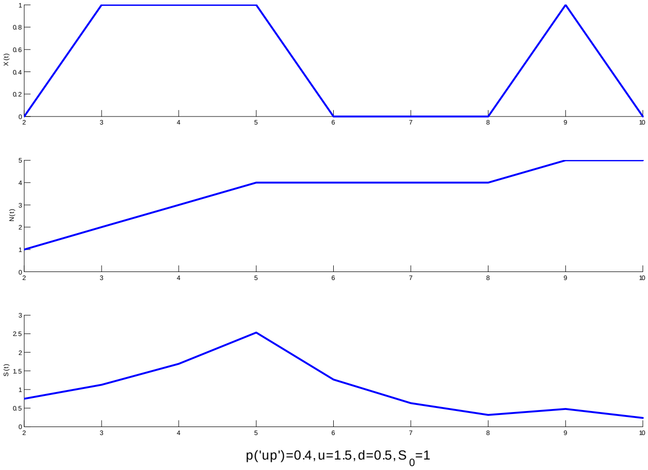
Definition (Stock Price)

The stock price process in the CRR model is defined via an initial value $S_0 > 0$ and, for $1 \leq t \leq T$ and all $\omega \in \Omega$,

$$S_t(\omega) := S_0 u^{N_t(\omega)} d^{\,t-N_t(\omega)}.$$

- The underlying Bernoulli process X governs the 'up' and 'down' movements of the stock. The stock price moves up at time t if $X_t(\omega) = 1$ and moves down if $X_t(\omega) = 0$.
- The dynamics of the stock price can be seen as an example of a **multiplicative random walk**.
- The Bernoulli counting process N counts the up movements. Before and including time t, the stock price moves up N_t times and down $t - N_t$ times.

Stock Price



Stock Price

Let's look closer

$$\begin{aligned} S_t(\omega) &= S_0 u^{N_t(\omega)} d^{\,t-N_t(\omega)} \\ S_{t+1}(\omega) &= S_0 u^{N_{t+1}(\omega)} d^{\,t+1-N_{t+1}(\omega)} \end{aligned}$$

Recall

$$\begin{aligned} N_t(\omega) &= X_1(\omega) + X_2(\omega) + \cdots + X_t(\omega) \\ N_{t+1}(\omega) &= X_1(\omega) + X_2(\omega) + \cdots + X_{t+1}(\omega) = N_t(\omega) + X_{t+1}(\omega) \end{aligned}$$

So that

$$\begin{aligned} S_{t+1} &= S_0 u^{N_{t+1}} d^{\,t+1-N_{t+1}} = S_0 u^{N_t + X_{t+1}} d^{\,t-N_t+1-X_{t+1}} \\ &= S_0 u^{N_t} d^{\,t-N_t} u^{X_{t+1}} d^{\,1-X_{t+1}} = S_t(\omega) u^{X_{t+1}} d^{\,1-X_{t+1}} \end{aligned}$$

If $X_{t+1}(\omega) = 1$ then $S_{t+1}(\omega) = S_t(\omega) u^1 d^{\,0} = S_t(\omega) u$
If $X_{t+1}(\omega) = 0$ then $S_{t+1}(\omega) = S_t(\omega) u^0 d^{\,1} = S_t(\omega) d$

Distribution of the Stock Price

- For each $t = 1, 2, \ldots, T$, the random variable N_t follows a **binomial distribution** with parameters p and t.
- Specifically, for every $t = 1, \ldots, T$ and $k = 0, \ldots, t$ we have that

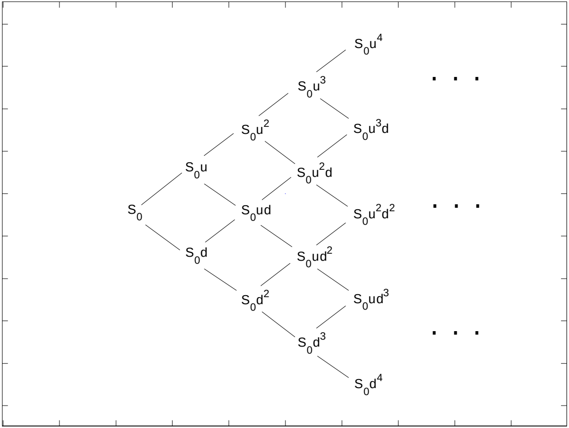
$$\mathbb{P}(N_t = k) = \binom{t}{k} p^k (1-p)^{t-k}.$$

- Hence the probability distribution of the stock price S_t at time t is given by

$$\mathbb{P}(S_t = S_0 u^k d^{t-k}) = \binom{t}{k} p^k (1-p)^{t-k}$$

for $k = 0, 1, \ldots, t$.

Stock Price Lattice



Risk-Neutral Probability Measure

Proposition (7.1)

Assume that $d < 1 + r < u$. Then a probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ is a risk-neutral probability measure for the CRR model $\mathcal{M} = (B, S)$ with parameters p, u, d, r and time horizon T if and only if:

1. $X_1, X_2, X_3, \ldots, X_T$ *are independent under the probability measure* $\tilde{\mathbb{P}}$,
2. $0 < \tilde{p} := \tilde{\mathbb{P}}(X_t = 1) < 1$ *for all* $t = 1, \ldots, T$,
3. $\tilde{p}u + (1 - \tilde{p})d = (1 + r)$,

where X is the Bernoulli process governing the stock price S.

Risk-Neutral Probability Measure

Proposition (7.2)

If $d < 1 + r < u$ then the CRR market model $\mathcal{M} = (B, S)$ is arbitrage-free and complete.

- Since the CRR model is complete, the unique arbitrage free price of any European contingent claim can be computed using the risk-neutral valuation formula

$$\pi_t(X) = B_t \, \mathbb{E}_{\widetilde{\mathbb{P}}} \left(\frac{X}{B_T} \mid \mathcal{F}_t \right)$$

- We will apply this formula to a European call option on a stock.

CRR Call Option Pricing Formula

Proposition (7.3)

The arbitrage free price at time $t = 0$ of the European call option $C_T = (S_T - K)^+$ in the binomial market model $\mathcal{M} = (B, S)$ is given by the CRR call pricing formula

$$C_0 = S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \hat{p}^k (1-\hat{p})^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k}$$

where

$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \hat{p} = \frac{\tilde{p}u}{1+r}$$

and $\hat{k}$ is the smallest integer k such that

$$k \log\left(\frac{u}{d}\right) > \log\left(\frac{K}{S_0 d^T}\right).$$

Proof of Proposition 7.3

Proof.

[Proof of Proposition 7.3]

- The price at time $t = 0$ of the claim $X = C_T = (S_T - K)^+$ can be computed using the risk-neutral valuation under $\tilde{\mathbb{P}}$

$$C_0 = \frac{1}{(1+r)^T} \mathbb{E}_{\tilde{\mathbb{P}}} (C_T) .$$

- Recall that

$$\mathbb{P}(S_T = S_0 u^k d^{T-k}) = \binom{T}{k} p^k (1-p)^{T-k}$$

- $\tilde{p}$ is known, hence

$$C_0 = \frac{1}{(1+r)^T} \sum_{k=0}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k} \max\left(0, S_0 u^k d^{T-k} - K\right) .$$

Proof of Proposition 7.3

Proof.

[Proof of Proposition 7.3]

- Recall we are interested in $\left(S_0 u^k d^{T-k} - K\right)^+$
- Define $\hat{k}$ to be the smallest non-negative integer k such that $S_0 u^k d^{T-k} > K$. If $\hat{k} > T$ then $S_0 u^k d^{T-k} \leq K$ and $C_0 = 0$. Otherwise we have a chance to get some money.
- Note

$$k \log\left(\frac{u}{d}\right) > \log\left(\frac{K}{S_0 d^T}\right) \Leftrightarrow \left(\frac{u}{d}\right)^k > \frac{K}{S_0 d^T} \Leftrightarrow S_0 u^k d^{T-k} - K > 0$$

  so this is just a way to write condition for the first occurence of $S_0 u^k d^{T-k} - K > 0$.
- We define $\hat{k} = \hat{k}(S_0, T)$ as the smallest integer k such that the last inequality is satisfied. If there are less than $\hat{k}$ upward movements the option will expire worthless.

Proof of Proposition 7.3

Proof.

[Proof of Proposition 7.3]

- Therefore, we obtain

$$
\begin{aligned}
C_0 &= \frac{1}{(1+r)^T}\sum_{k=0}^{\hat{k}-1}\binom{T}{k}\tilde{p}^k(1-\tilde{p})^{T-k}(0) \\
&\quad + \frac{1}{(1+r)^T}\sum_{k=\hat{k}}^{T}\binom{T}{k}\tilde{p}^k(1-\tilde{p})^{T-k}\left(S_0u^kd^{T-k}-K\right) \\
&= \frac{S_0}{(1+r)^T}\sum_{k=\hat{k}}^{T}\binom{T}{k}\tilde{p}^k(1-\tilde{p})^{T-k}u^kd^{T-k} \\
&\quad - \frac{K}{(1+r)^T}\sum_{k=\hat{k}}^{T}\binom{T}{k}\tilde{p}^k(1-\tilde{p})^{T-k}
\end{aligned}
$$

□

Proof of Proposition 7.3

Proof.

[Proof of Proposition 7.3]

- Therefore, we obtain

$$\begin{aligned} C_0 &= S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \left(\frac{\tilde{p}u}{1+r}\right)^k \left(\frac{(1-\tilde{p})d}{1+r}\right)^{T-k} \\ &\quad - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k} \\ &= S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} (\hat{p})^k (1-\hat{p})^{T-k} \\ &\quad - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k} \end{aligned}$$

where we denote $\hat{p} = \frac{\tilde{p}u}{1+r}$.

□

Put-Call Parity

- Check that $0 < \hat{p} = \frac{\tilde{p}u}{1+r} < 1$ whenever $0 < \widetilde{p} = \frac{1+r-d}{u-d} < 1$.
- Recall that $\mathbb{P}(S_T = S_0 u^k d^{T-k}) = \binom{T}{k} p^k (1-p)^{T-k}$
- For $t = 0$, the price of the call satisfies

$$\begin{aligned} C_0 &= S_0 \sum_{k=\hat{k}}^{T} \binom{T}{k} \hat{p}^k \left(1-\hat{p}\right)^{T-k} \\ &\quad - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k} \\ &= S_0 \, \widehat{\mathbb{P}}(D) - KB(0,T) \, \widetilde{\mathbb{P}}(D) \end{aligned}$$

where $D = \{\omega \in \Omega : S_T(\omega) > K\}$.

Put-Call Parity

- We started with $t = 0$, but the same analysis is true for any t. It is easy to demonstrate that

$$\begin{aligned} C_t \quad &: \quad = B_t \, \mathbb{E}_{\widetilde{\mathbb{P}}} \left(B_T^{-1} (S_T - K)^+ \,|\, \mathcal{F}_t \right) \\ &= \quad S_t \, \widehat{\mathbb{P}}(D \,|\, \mathcal{F}_t) - KB(t, T) \, \widetilde{\mathbb{P}}(D \,|\, \mathcal{F}_t). \end{aligned}$$

- Similarly, for a put option

$$\begin{aligned} P_t \quad &: \quad = B_t \, \mathbb{E}_{\widetilde{\mathbb{P}}} \left(B_T^{-1} (K - S_T)^+ \,|\, \mathcal{F}_t \right) \\ &= \quad -S_t \, \widehat{\mathbb{P}}(\bar{D} \,|\, \mathcal{F}_t) + KB(t, T) \, \widetilde{\mathbb{P}}(\bar{D} \,|\, \mathcal{F}_t) \end{aligned}$$

where $\bar{D} = \{\omega \in \Omega : S_T(\omega) < K\}$.

Put-Call Parity

- Therefore, **put-call parity** holds at any date $t = 0, 1, \dots, T$

$$
\begin{aligned}
C_t - P_t &= S_t\, \widehat{\mathbb{P}}(D \,|\, \mathcal{F}_t) - KB(t,T)\, \widetilde{\mathbb{P}}(D \,|\, \mathcal{F}_t) \\
&\quad + S_t\, \widehat{\mathbb{P}}(\bar{D} \,|\, \mathcal{F}_t) - KB(t,T)\, \widetilde{\mathbb{P}}(\bar{D} \,|\, \mathcal{F}_t) \\
&= S_t \underbrace{\left(\widehat{\mathbb{P}}(D \,|\, \mathcal{F}_t) + \widehat{\mathbb{P}}(\bar{D} \,|\, \mathcal{F}_t)\right)}_{1} \\
&\quad - KB(t,T) \underbrace{\left(\widetilde{\mathbb{P}}(D \,|\, \mathcal{F}_t) + \widetilde{\mathbb{P}}(\bar{D} \,|\, \mathcal{F}_t)\right)}_{1} \\
&= S_t - KB(t,T)
\end{aligned}
$$

where

$$B(t,T) = (1+r)^{-(T-t)}$$

is the price at time t of zero-coupon bond maturing at T.

American Options

- In contrast to a contingent claim of European style, a claim of American style can by exercised by its holder at any date before its expiration date T.

Definition (American Call and Put Options)

An American call (put) option is a contract which gives the holder the right to buy (sell) an asset at any time $t \leq T$ of her/his choice at strike price K.

- In the study of American options, we are concerned with the price process and the 'optimal' exercise policy by its holder.
- If the holder of an American option exercises it at $\tau \in [0, T]$, τ is called the **exercise time**.

American Call Option

Definition

By an **arbitrage free price** of the American call we mean a price process $C_t^a,\ t \le T,$ such that the extended financial market model – that is, a market with trading in riskless bonds, stocks and the American call option – remains arbitrage-free.

By definition, the arbitrage free price of American call option

$$C_t^a = \max_{\tau} \mathbb{E}_{\widetilde{\mathbb{P}}} \left((1+r)^{-(\tau-t)} (S_\tau - K)^+ \,|\, \mathcal{F}_t \right), \quad \forall\, t \le T.$$

and the optimal exercise time

$$\tau_t^* = \min \{ u \ge t \,|\, (S_u - K)^+ \ge C_u^a \}.$$

Proposition (7.4)

The price of an American call option in the CRR arbitrage-free market model with $r \ge 0$ coincides with the arbitrage price of a European call option with the same expiry date and strike price.

Proof of Proposition 7.4

Proof.

[Proof of Proposition 7.4] For simplicity, we assume $r > 0$. Obviously

$$(S_T - K)^+ \geq S_T - K$$

and $C_t^a \geq C_t$ (i.e American call option is worth at least as much as a Eropean option with the same contractual features)

$$\begin{aligned} C_t^a \geq C_t &= B_t \mathbb{E}_{\widetilde{\mathbb{P}}} \left(B_T^{-1} (S_T - K)^+ | \mathcal{F}_t \right) \\ &\geq B_t \mathbb{E}_{\widetilde{\mathbb{P}}} \left(B_T^{-1} (S_T - K) \middle| \mathcal{F}_t \right) \\ &= B_t \mathbb{E}_{\widetilde{\mathbb{P}}} \left(B_T^{-1} S_T \middle| \mathcal{F}_t \right) - B_t B_T^{-1} K \\ &= S_t - \frac{K}{(1+r)^{T-t}}. \end{aligned}$$

□

Proof of Proposition 7.4

Proof.

[Proof of Proposition 7.4] Since $r > 0$ we can then conclude that

$$C_t^a \geq C_t \geq S_t - \frac{K}{(1+r)^{T-t}} > S_t - K$$

for all $t \in 0, 1, ..., T-1$. Hence at all times (except at T) the option value is always strictly greater than its exercise value and exercising prior to maturity is never optimal. At time T these options are the same. □

- The result only holds for non-dividend-paying stock.
- The proof requires $r > 0$. It means that the similar proof for a put option might require $r < 0$, which is not realisic. In fact, we see a number of examples further in the course when it is optimal to exercise put option early.

American Put Option

- Recall that the American put is an option to sell a specified number of shares, which may be exercised at any time before or at the expiry date T.

By definition, the arbitrage free price P_t^a of an American put option equals

$$P_t^a = \max_{\tau} \mathbb{E}_{\widetilde{\mathbb{P}}} \left((1+r)^{-(\tau-t)} (K - S_\tau)^+ \,|\, \mathcal{F}_t \right), \quad \forall\, t \leq T.$$

For any $t \leq T$, the stopping time τ_t^* which realizes the maximum is given by the expression

$$\tau_t^* = \min \{ u \geq t \,|\, (K - S_u)^+ \geq P_u^a \}.$$

Dynamic Programming Recursion

- The stopping time τ_t^* is called the **rational exercise time** of an American put option that is assumed to be still alive at time t.
- By an application of the classic **Bellman principle** (1952), one reduces the optimal stopping problem in Proposition 7.5 to an explicit recursive procedure for the value process.
- We now show how to arrive to the **dynamic programming recursion** for the value of an American put option. The American call option can be treated similarly.
- Note that this is an extension of the backward induction approach to the valuation of European contingent claims.

Dynamic Programming Recursion

Corollary (Bellman Principle)

Let the non-negative adapted process U be defined recursively by setting $U_T = (K - S_T)^+$ and for $t \leq T - 1$

$$U_t = \max \left((K - S_t)^+, \, (1 + r)^{-1} \, \mathbb{E}_{\widetilde{\mathbb{P}}}(U_{t+1} \,|\, \mathcal{F}_t) \right).$$

Then the arbitrage free price P_t^a of the American put option at time t equals U_t and the rational exercise time after time t admits the following representation

$$\tau_t^* = \min \{ u \geq t \,|\, (K - S_u)^+ \geq U_u \}.$$

Therefore, $\tau_T^* = T$ and for every $t = 0, 1, \ldots, T - 1$

$$\tau_t^* = t \mathbb{1}_{\{U_t = (K - S_t)^+\}} + \tau_{t+1}^* \mathbb{1}_{\{U_t > (K - S_t)^+\}}.$$

Dynamic Programming Recursion

- It is also possible to show directly that the price P_t^a satisfies the recursive relationship, for $t \leq T-1$,

$$P_t^a = \max\left\{(K - S_t)^+,\, (1+r)^{-1}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(P_{t+1}^a \,|\, \mathcal{F}_t\right)\right\} \tag{1}$$

  subject to the terminal condition $P_T^a = (K - S_T)^+$.

- In the case of the CRR model, this formula reduces the valuation problem to the simple single-period case.
- To show this we shall argue by contradiction. Assume first that (1) fails to hold for $t = T-1$. If this is the case, one may easily construct at time $T-1$ a portfolio which produces riskless profit at time T. Hence, we conclude that necessarily

$$P_{T-1}^a = \max\left\{(K - S_{T-1})^+,\, (1+r)^{-1}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left((K - S_T)^+ \,|\, \mathcal{F}_T\right)\right\}.$$

- This procedure may be repeated as many times as needed.

American Put Option: Summary

To summarize:

- In the CRR model, the arbitrage pricing of the American put option reduces to the following recursive recipe, for $t \leq T - 1$,

$$P_t^a = \max\left\{(K - S_t)^+,\ (1+r)^{-1}\left(\tilde{p}P_{t+1}^{au} + (1-\tilde{p})P_{t+1}^{ad}\right)\right\}$$

  with the terminal condition

$$P_T^a = (K - S_T)^+.$$

- The quantities P_{t+1}^{au} and P_{t+1}^{ad} represent the values of the American put in the next step corresponding to the upward and downward movements of the stock price starting from a given node on the CRR lattice.

American Call Option: Summary

To summarize:

- In the CRR model, the arbitrage pricing of the American call option reduces to the following recursive recipe, for $t \le T-1$,

$$C_t^a = \max\left\{(S_t - K)^+,\, (1+r)^{-1}\left(\tilde{p}\, C_{t+1}^{au} + (1-\tilde{p})\, C_{t+1}^{ad}\right)\right\}$$

  with the terminal condition

$$C_T^a = (S - K_T)^+.$$

- The quantities C_{t+1}^{au} and C_{t+1}^{ad} represent the values of the American call in the next step corresponding to the upward and downward movements of the stock price starting from a given node on the CRR lattice.

Example: American Call Option

Example (7.1)

- We consider here the CRR binomial model with the horizon date $T = 2$ and the risk-free rate $r = 0.2$.
- The stock price S for $t = 0$ and $t = 1$ equals

$$S_0 = 10, \quad S_1^u = 13.2, \quad S_1^d = 10.8.$$

- Let X^a be the American call option with maturity date $T = 2$ and the following payoff process

$$g(S_t, t) = (S_t - K_t)^+.$$

- The strike K_t is **variable** and satisfies

$$K_0 = 9, \quad K_1 = 9.9, \quad K_2 = 12.$$

Example: American Call Option

Example (7.1 Continued)

- We will first compute the arbitrage price $\pi_t(X^a)$ of this option at times $t = 0, 1, 2$ and the rational exercise time τ_0^*.
- Subsequently, we will compute the replicating strategy for X^a up to the rational exercise time τ_0^*.
- We start by noting that the unique risk-neutral probability measure $\widetilde{\mathbb{P}}$ satisfies

$$\widetilde{p} = \frac{1 + r - d}{u - d} = \frac{(1 + r)S_0 - S_1^d}{S_1^u - S_1^d} = \frac{12 - 10.8}{13.2 - 10.8} = 0.5$$

- The second exhibit represents the price of the call option.

$$\begin{tikzcd}
 & & \boxed{S_2 = 13.2 \cdot \frac{13.2}{10} = 17.424} \\
 & \boxed{S_1 = 13.2} \arrow[ur, "0.5"] \arrow[dr, "0.5"] & \\
\boxed{S_0 = 10} \arrow[ur, "0.5"] \arrow[dr, "0.5"] & & \boxed{S_2 = 13.2 \cdot \frac{10.8}{10} = 14.256} \\
 & \boxed{S_1 = 10.8} \arrow[ur, "0.5"] \arrow[dr, "0.5"] & \\
 & & \boxed{S_2 = 10.8 \cdot \frac{10.8}{10} = 11.664}
\end{tikzcd}$$

$S_0 = 10$

$\pi_0^w = \frac{1}{1.2}\left(\frac{3.3}{2} + \frac{0.94}{2}\right) = 1.7$

$\pi_0^e = 10 - 9 = 1$

$\pi_0 = \max(1.7, 1) = 1.7667$

$\Longrightarrow$ wait

$S_1 = 13.2$

$\pi_1^w = \frac{1}{1.2}\left(\frac{5.4}{2} + \frac{2.2}{2}\right) = 3.2$

$\pi_1^e = 13.2 - 9.9 = 3.3$

$\pi_1 = \max(3.2, 3.3) = 3.3$

$\Longrightarrow$ exercise

$S_1 = 10.8$

$\pi_1^w = \frac{1}{1.2}\left(\frac{2.2}{2} + \frac{0}{2}\right) = 0.94$

$\pi_1^e = 10.8 - 9.9 = 0.9$

$\pi_1 = \max(0.94, 0.9) = 0.94$

$\Longrightarrow$ wait

$S_2 = 17.424$

$\pi_2 = (17.4 - 12)^+ = 5.424$

$S_2 = 14.256$

$\pi_2 = (14.2 - 12)^+ = 2.256$

$S_2 = 11.664$

$\pi_2 = (11.6 - 12)^+ = 0$

Example: Replicating Strategy

Example (7.1 Continued)

- **Holder.** The rational holder should exercise the American option at time $t = 1$ if the stock price rises during the first period. Otherwise, the option should be held till time 2. Hence $\tau_0^* : \Omega \to \{0, 1, 2\}$ equals

$$\tau_0^*(\omega) = 1 \quad \text{for} \quad \omega \in \{\omega_1, \omega_2\}$$
$$\tau_0^*(\omega) = 2 \quad \text{for} \quad \omega \in \{\omega_3, \omega_4\}$$

- **Issuer.** We now take the position of the issuer of the option. At $t = 0$, we need to solve

$$1.2\,\phi_0^0 + 13.2\,\phi_0^1 = 3.3$$
$$1.2\,\phi_0^0 + 10.8\,\phi_0^1 = 0.94$$

Hence $(\phi_0^0, \phi_0^1) = (-8.067, 0.983)$ for all ω.

Example: Replicating Strategy

Example (7.1 Continued)

- If the stock price rises during the first period, the option is exercised and thus we do not need to compute the strategy at time 1 for $\omega \in \{\omega_1, \omega_2\}$.
- If the stock price falls during the first period, we solve

$$1.2\,\widetilde{\phi}_1^0 + 14.256\,\phi_1^1 = 2.256$$
$$1.2\,\widetilde{\phi}_1^0 + 11.664\,\phi_1^1 = 0$$

- Hence $(\widetilde{\phi}_1^0, \phi_1^1) = (-8.46, 0.8704)$ for $\omega \in \{\omega_3, \omega_4\}$.
- Note that $\widetilde{\phi}_1^0 = -8.46$ is the amount of cash borrowed at time 1, rather than the number of units of the savings account B.
- The replicating strategy $\phi = (\phi^0, \phi^1)$ is defined at time 0 for all ω and it is defined at time 1 on the event $\{\omega_3, \omega_4\}$ only.

$\boxed{-}$

$\boxed{V_1^u = 3.3}$

$\boxed{-}$

$\boxed{(\phi_0^0, \phi_0^1) = (-8.067, 0.983)}$

$\boxed{V_2^{du} = 2.256}$

$\boxed{(\widetilde{\phi}_1^0, \phi_1^1) = (-8.46, 0.8704)}$

$\boxed{V_2^{dd} = 0}$

Example: American Call Option vs. European Call Option

Example (7.1)

- Consider the same setup but $K_t = 12$ and does not depend on time. The same strike price is for the European Call option.
- We start by noting that the unique risk-neutral probability measure $\widetilde{\mathbb{P}}$ satisfies $\widetilde{p} = \frac{1+r-d}{u-d} = \frac{(1+r)S_0 - S_1^d}{S_1^u - S_1^d} = \frac{12-10.8}{13.2-10.8} = 0.5$

$$
\begin{tikzcd}
 & & \boxed{S_2 = 13.2 \cdot \frac{13.2}{10} = 17.424} \\
 & \boxed{S_1 = 13.2} \arrow[ur, "0.5"] \arrow[dr, "0.5"] & \\
\boxed{S_0 = 10} \arrow[ur, "0.5"] \arrow[dr, "0.5"] & & \boxed{S_2 = 13.2 \cdot \frac{10.8}{10} = 14.256} \\
 & \boxed{S_1 = 10.8} \arrow[ur, "0.5"] \arrow[dr, "0.5"] & \\
 & & \boxed{S_2 = 10.8 \cdot \frac{10.8}{10} = 11.664}
\end{tikzcd}
$$

$S_0 = 10$

$\pi_0^w = \frac{1}{1.2}\left(\frac{3.2}{2} + \frac{0.94}{2}\right) = 1.6$

$\pi_0^e = (10 - 12)^+ = 0$

$\pi_0 = \max(1.6, 0) = 1.60$

$\Longrightarrow$ wait

$S_1 = 13.2$

$\pi_1^w = \frac{1}{1.2}\left(\frac{5.4}{2} + \frac{2.2}{2}\right) = 3.2 = \pi_1^{Eur.call}$

$\pi_1^e = 13.2 - 12 = 1.2$

$\pi_1 = \max(3.2, 1.2) = 3.2$

$\Longrightarrow$ wait

$S_1 = 10.8$

$\pi_1^w = \frac{1}{1.2}\left(\frac{2.2}{2} + \frac{0}{2}\right) = 0.94 = \pi_1^{Eur.call}$

$\pi_1^e = (10.8 - 12)^+ = 0$

$\pi_1 = \max(0.94, 0) = 0.94$

$\Longrightarrow$ wait

$S_2 = 17.424$

$\pi_2 = (17.4 - 12)^+ = 5.424$

$S_2 = 14.256$

$\pi_2 = (14.2 - 12)^+ = 2.256$

$S_2 = 11.664$

$\pi_2 = (11.6 - 12)^+ = 0$

Derivation of u and d from r and σ

- the parameters r and S_0 can be observed in real financial markets, but the parameters u and d are only a model idealization and can not be directly determined by observation of real world data
- people trading on stock markets study a different parameter, which they call **volatility** and which reflects a property of the corresponding continuous time model, also known as the Black-Scholes model
- From the market data for stock prices, one can estimate the stock price volatility σ per one time unit (typically, one year).
- Note that until now we assumed that $t = 0, 1, 2, \ldots, T$, which means that $\Delta t = 1$. In general, the length of each period can be any positive number smaller than 1. We set $n = T/\Delta t$.

Derivation of u and d from r and σ

Two widely used conventions for obtaining u and d from σ and r are:

- **The Cox-Ross-Rubinstein (CRR) parametrisation**:

$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = \frac{1}{u}.$$

- **The Jarrow-Rudd (JR) parameterisation**:

$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}.$$

The CRR parameterisation

Proposition (7.5)

Assume that $B_{k\Delta t} = (1 + r\Delta t)^k$ *for every* $k = 0, 1, \ldots, n$ *and* $u = \frac{1}{d} = e^{\sigma\sqrt{\Delta t}}$ *in the CRR model. Then the risk-neutral probability measure* $\widetilde{\mathbb{P}}$ *satisfies*

$$\widetilde{\mathbb{P}}\left(S_{t+\Delta t} = S_t u \,|\, S_t\right) = \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t} + o(\sqrt{\Delta t})$$

provided that Δt *is sufficiently small.*

Proof.

[Proof of Proposition 7.5] The risk-neutral probability measure for the CRR model is given by

$$\widetilde{p} = \widetilde{\mathbb{P}}\left(S_{t+\Delta t} = S_t u \,|\, S_t\right) = \frac{1 + r\Delta t - d}{u - d}$$

$\square$

The CRR parameterisation

Proof.

[Proof of Proposition 7.5] Under the CRR parametrisation, we obtain

$$\widetilde{p} = \frac{1 + r\Delta t - d}{u - d} = \frac{1 + r\Delta t - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

The Taylor expansions up to the second order term are

$$\begin{aligned} e^{\sigma\sqrt{\Delta t}} &= 1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t + o(\Delta t) \\ e^{-\sigma\sqrt{\Delta t}} &= 1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t + o(\Delta t) \end{aligned}$$

□

The CRR parameterisation

Proof.

[Proof of Proposition 7.5] By substituting the Taylor expansions into the risk-neutral probability measure, we obtain

$$
\begin{aligned}
\widetilde{p} &= \frac{1 + r\Delta t - \left(1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t\right) + o(\Delta t)}{\left(1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t\right) - \left(1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2}{2}\Delta t\right) + o(\Delta t)} \\
&= \frac{\sigma\sqrt{\Delta t} + \left(r - \frac{\sigma^2}{2}\right)\Delta t + o(\Delta t)}{2\sigma\sqrt{\Delta t} + o(\Delta t)} \\
&= \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t} + o(\sqrt{\Delta t})
\end{aligned}
$$

as was required to show. $\square$

The CRR parameterisation

- To summarise, for Δt sufficiently small, we get

$$\widetilde{p} = \frac{1 + r\Delta t - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \approx \frac{1}{2} + \frac{r - \frac{\sigma^2}{2}}{2\sigma}\sqrt{\Delta t}.$$

- Note that $1 + r\Delta t \approx e^{r\Delta t}$ when Δt is sufficiently small.
- Hence the risk-neutral probability measure can also be represented as follows

$$\widetilde{p} \approx \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

- More formally, we can define $\hat{r}$ such that $(1 + \hat{r})^n = e^{rT}$ for a fixed T and $n = T/\Delta t$ then $\hat{r} \approx r\Delta t$ since $\ln(1 + \hat{r}) = r\Delta t$ and $\ln(1 + \hat{r}) \approx \hat{r}$ when $\hat{r}$ is close to zero.

The Binomial Asset Pricing Model

- in applications one generally chooses a small time unit $\Delta t = t_i - t_{i-1}$ and uses $B_t = (1 + r\Delta t)^{\left(\frac{t}{\Delta t}\right)} \approx e^{rt}$
- though simple by its construction, the binomial market model is a very powerful tool in practical applications, if one chooses Δt very small, in fact one then obtains a reasonably good approximation of trading in continuous time
- the CRR options pricing formula then approximates the Black-Scholes options pricing formula (see Black and Scholes (1973) and later in this module). In fact, the Black-Scholes options pricing formula can be derived by letting the time increments in the CRR options pricing formula go to zero.

The Binomial Asset Pricing Model

To implement a numerical algorithm for the evaluation of options and contingent claims we introduce the notation $M = \frac{T}{\Delta_t}$ and

$$S_{ji} := S_0 u^j d^{i-j},\ i = 0, ..., M, j = 0, ...i$$

The numerical values S_{ji} represent the price of the stock at time i, when counting from initial time 0 and the stock has gone up j times.

- let X be a contingent claim. We can compute an arbitrage free price for X at time $i\Delta t$ via the formula

$$V_i = e^{ri\Delta t}\mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{-rT}X \mid \mathcal{F}_i\right)$$

- by Definition of the conditional expectation this random variable is $\mathcal{F}_i$-measurable, meaning that it depends on information gathered up to time $i\Delta t$

The Binomial Asset Pricing Model

- If X is of the type $h(S_M)$, meaning that the payoff only depends on the terminal value of the stock, then one can use the independence of the $\frac{S_i}{S_{i-1}}$'s and show that V_i depends on this information only through the price of the stock S_i at time $i\Delta t$.
- The latter reflects the Markov property of the stock price process in the binomial model. We will later see this property occurring in more complicated models.

We are now able to write the random variable V_i as a function of S_i. We then denote

$$V_{ji} := \text{ price of } X \text{ at time } i \text{ if stock price is } S_{ji}$$

The Binomial Asset Pricing Model

At terminal time T ($i = M$) the contingent claim pays off:

$$V_{jM} = h\left(S_{jM}\right)$$

for all $j = 0, ..., M$. We use this as the initialization of our scheme and work backward in time.

- using the tower property of the conditional expectation, one derives that

  $$V_i = e^{-r\Delta t}\mathbb{E}_{\widetilde{\mathbb{P}}}\left(V_{i+1} \mid \mathcal{F}_i\right)$$

  and therefore one obtains the backward step

  $$V_{ji} = e^{-r\Delta t}\left(\widetilde{p}V_{j+1,i+1} + \left(1 - \widetilde{p}\right)V_{j,i+1}\right)$$

  This is so-called Bellman equation.
- using our initialization we now iterate the backward step in order to compute V_{00}, the price of the contingent claim at present time 0

Binomial Asset Pricing Model: Matlab

```
function [price, lattice] = LatticeEurCall(S0,K,r,T,sigma,N)
deltaT = T/N;
u=exp(sigma * sqrt(deltaT));
d=1/u;
p=(exp(r*deltaT) - d)/(u-d);
lattice = zeros(N+1,N+1);
for i=0:N
lattice(i+1,N+1)=max(0 , S0*(u^i)*(d^(N-i)) - K);
end
for j=N-1:-1:0
for i=0:j
lattice(i+1,j+1) = exp(-r*deltaT) * ...
(p * lattice(i+2,j+2) + (1-p) * lattice(i+1,j+2));
end
end
price = lattice(1,1);
```

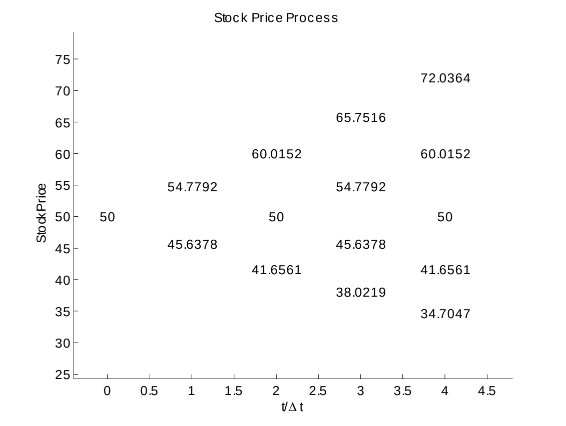
Example: American Put Option

Example (7.2 – CRR Parameterisation)

- Let the annualized variance of logarithmic returns be $\sigma^2 = 0.1$.
- The interest rate is set to $r = 0.1$ per annum.
- Suppose that the current stock price is $S_0 = 50$.
- We examine European and American put options with strike price $K = 53$ and maturity $T = 4$ months (i.e. $T = \frac{1}{3}$).
- The length of each period is $\Delta t = \frac{1}{12}$, that is, one month.
- Hence $n = \frac{T}{\Delta t} = 4$ steps.
- We adopt the CRR parameterisation to derive the stock price.
- Then $u = 1.0956$ and $d = 1/u = 0.9128$.
- We compute $1 + r\Delta t = 1.00833 \approx e^{r\Delta t}$ and $\widetilde{p} = 0.5228$.

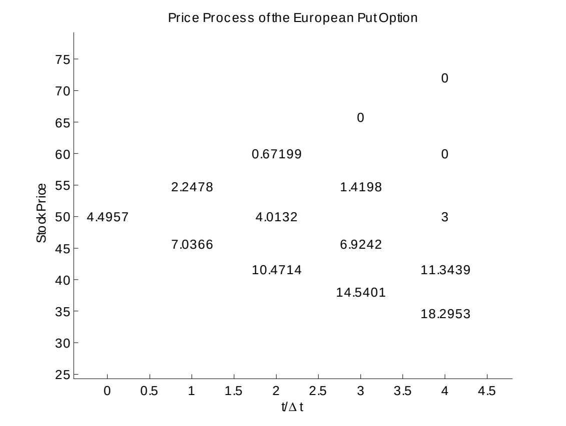
Example: American Put Option

Example (7.2 Continued)



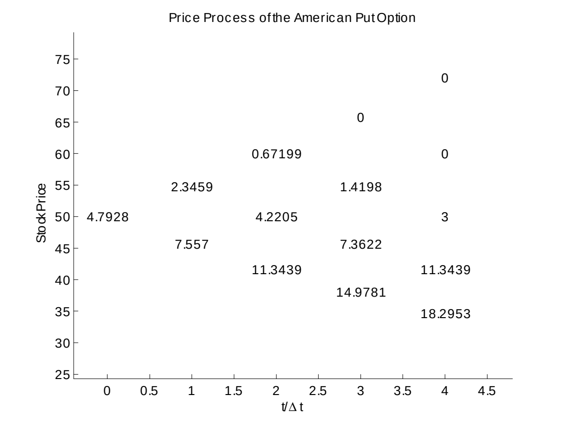
Example: American Put Option

Example (7.2 Continued)



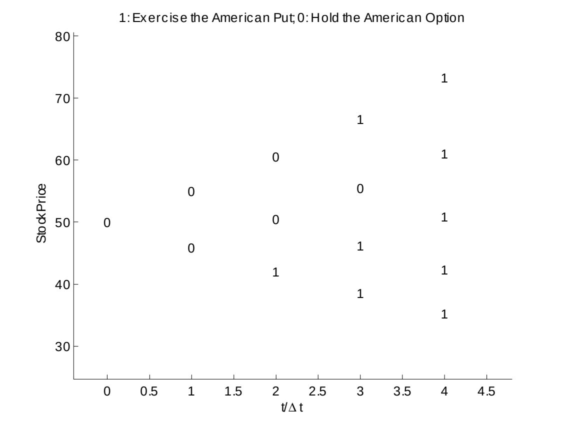
Example: American Put Option

Example (7.2 Continued)



Example: American Put Option

Example (7.2 Continued)



The JR parameterisation

The next result deals with the Jarrow-Rudd parametrisation.

Proposition (7.9)

Let $B_{k\Delta t} = (1 + r\Delta t)^k$ *for* $k = 0, 1, \ldots, n$. *We assume that*

$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}}$$

and

$$d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}.$$

Then the risk-neutral probability measure $\widetilde{\mathbb{P}}$ *satisfies*

$$\widetilde{\mathbb{P}}\left(S_{t+\Delta t} = S_t u \,|\, S_t\right) = \frac{1}{2} + o(\Delta t)$$

provided that Δt *is sufficiently small.*

The JR parameterisation

Proof.

[Proof of Proposition 7.9] Under the JR parametrisation, we have

$$\widetilde{p} = \frac{1 + r\Delta t - d}{u - d} = \frac{1 + r\Delta t - e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}}{e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} - e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}}.$$

The Taylor expansions up to the second order term are

$$\begin{aligned} e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}} &= 1 + r\Delta t + \sigma\sqrt{\Delta t} + o(\Delta t) \\ e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}} &= 1 + r\Delta t - \sigma\sqrt{\Delta t} + o(\Delta t) \end{aligned}$$

and thus

$$\widetilde{p} = \frac{1}{2} + o(\Delta t).$$

□

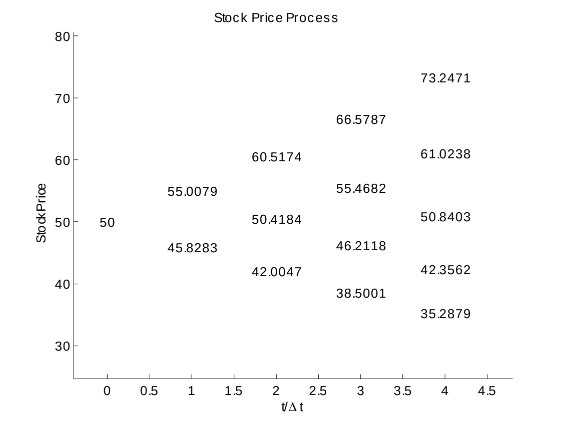
Example: American Put Option

Example (7.3 – JR Parameterisation)

- We consider the same problem as in Example 7.2, but with parameters u and d computed using the JR parameterisation. We obtain $u = 1.1002$ and $d = 0.9166$.
- As before, $1 + r\Delta t = 1.00833 \approx e^{r\Delta t}$, but $\widetilde{p} = 0.5$.
- We compute the price processes for the stock, the European put option, the American put option and we find the rational exercise time.
- When we compare with Example 7.2, we see that the results are slightly different than before, although it appears that the rational exercise policy is the same.
- The CRR and JR parameterisations are both set to approach the Black-Scholes model.
- For Δt sufficiently small, the prices computed under the two parametrisations will be very close to one another.

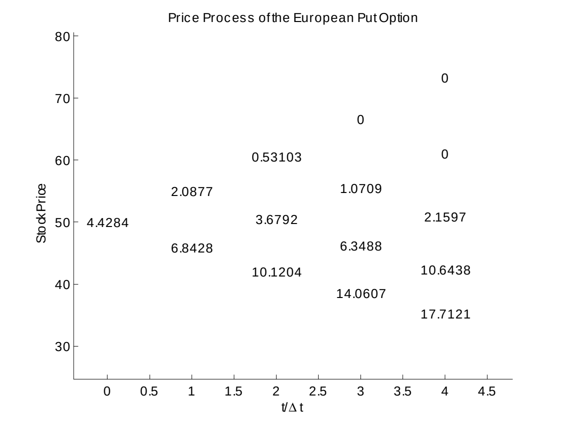
Example: American Put Option

Example (7.3 Continued)



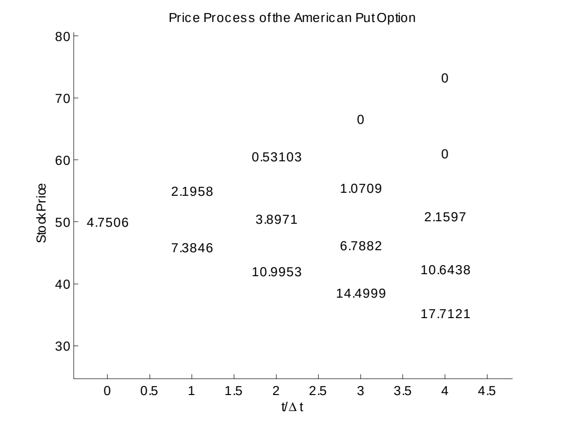
Example: American Put Option

Example (7.3 Continued)



Example: American Put Option

Example (7.3 Continued)



Example: American Put Option

Example (7.3 Continued)