1 May Exam 2015

1. Consider the discrete-time binomial tree model with three periods of length 1, i.e. $T = 3$ and $t = 0, 1, 2, 3$. In each period the price can move up or down, S_{t+1} is either uS_t or dS_t. Assume that the factor for moving up is $u = 1.3$, the factor for moving down is $d = 0.9$, and that the interest rate is $r = 0.1$. The initial stock price is $S_0 = 100$.

a. Compute the risk neutral probabilities q and $(1-q)$ for moving up and down, and draw the corresponding tree (including the stock prices at all nodes).

b. Compute the price process (i.e. prices at all times and states) for a European *put option* on the stock with strike price $K = 132$ and maturity $T = 3$. Add these prices to the tree in (a).

c. Compute the price process (i.e. prices at all times and states) for an American *put option* on the stock with strike price $K = 132$ and maturity $T = 3$. Add these prices to the tree in (a). Find the rational exercise time for the holder of the American put option.

Briefly explain each different step of the computations.

2. Consider the stock price under the Black-Scholes assumption, i.e.

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

where r denotes the interest rate. Consider an option with payoff

$$h(S_T) = \left(\log\left(\frac{S_T}{K}\right)\right)^2$$

where T is the time of maturity and K is a constant. Decide whether

$$V(t,x) = \exp(-r(T-t))\left[\left(\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right)^2 + \sigma^2(T-t)\right]$$

is the Black Sholes price of the option at time t assuming that $S_t = x$. Explain your arguments.

2 Solution

1.

a.

$$\begin{aligned} q &= \frac{1+r-d}{u-d} = \frac{1+0.1-0.9}{1.3-0.9} = 0.5 \\ 1-q &= 0.5 \end{aligned}$$

$$
\begin{array}{lllllll}
 & & & & & & S_3 = 169.0 \cdot 1.3 = 219.7 \\
 & & & & & \frac{1}{2} \nearrow & \\
 & & & & S_2 = 130 \cdot 1.3 & & \\
 & & & \frac{1}{2} \nearrow & = 169.0 & \frac{1}{2} \searrow & \\
 & & S_1 = 130 & & & & S_3 = 169 \cdot 0.9 = 152.1 \\
 & \frac{1}{2} \nearrow & & \frac{1}{2} \searrow & & \frac{1}{2} \nearrow & \\
S_0 = 100 & & & & S_2 = 90 \cdot 1.3 & & K = 132 \\
 & \frac{1}{2} \searrow & & \frac{1}{2} \nearrow & = 117.0 & \frac{1}{2} \searrow & \\
 & & S_1 = 90 & & & & S_3 = 117 \cdot 0.9 = 105.3 \\
 & & & \frac{1}{2} \searrow & & \frac{1}{2} \nearrow & \\
 & & & & S_2 = 90 \cdot 0.9 & & \\
 & & & & = 81.0 & \frac{1}{2} \searrow & \\
 & & & & & & S_3 = 72.9
\end{array}
$$

b.-c.

$S_0 = 100$
$\pi_0 = \frac{10}{11} \cdot \begin{pmatrix} \frac{1}{2} \cdot 5.5164 \\ +\frac{1}{2} \cdot 23.244 \end{pmatrix} = 13.073$

$\mu_1 = (132 - 100)^+ = 32$

$\mu_1 = \frac{10}{11} \cdot \begin{pmatrix} \frac{1}{2} \cdot 6.8182 \\ +\frac{1}{2} \cdot 42 \end{pmatrix} = 22.19$

exercise

$\frac{1}{2} \nearrow$ $\qquad$ $\frac{1}{2} \searrow$

$S_1 = 130$
$\pi_1 = \frac{10}{11} \cdot \left(\frac{1}{2} \cdot 12.136\right) = 5.5164$
$(132 - 130)^+ = 2;$
$\mu_1 = \frac{10}{11} \cdot \left(\frac{1}{2} \cdot 15.0\right) = 6.8182$ wait

$S_1 = 90$
$\pi_1 = \frac{10}{11} \cdot \begin{pmatrix} \frac{1}{2} \cdot 12.136 \\ +\frac{1}{2} \cdot 39.0 \end{pmatrix} = 23.244$
$\mu_1 = (132 - 90)^+ = 42$
$\mu_1 = \frac{10}{11} \cdot \begin{pmatrix} \frac{1}{2} \cdot 15.0 \\ +\frac{1}{2} \cdot 51 \end{pmatrix} = 30.0$

exercise

$\frac{1}{2} \nearrow$ $\quad$ $\frac{1}{2} \searrow$ $\quad$ $\frac{1}{2} \nearrow$ $\quad$ $\frac{1}{2} \searrow$

$S_2 = 130 \cdot 1.3 = 169.0$
$\pi_2 = 0$
$\mu_2 = (132 - 169)^+ = 0$

$S_2 = 90 \cdot 1.3 = 117.0$
$\pi_2 = \frac{10}{11} \cdot \left(\frac{1}{2} \cdot 26.7\right) = 12.136$
$\mu_2 = (132 - 117.0)^+ = 15.0$

exercise

$S_2 = 90 \cdot 0.9 = 81.0$
$\pi_2 = \frac{10}{11} \cdot \begin{pmatrix} \frac{1}{2} \cdot 26.7 \\ +\frac{1}{2} \cdot 59.1 \end{pmatrix} = 39.0$
$\mu_2 = (132 - 81)^+ = 51$

exercise

$\frac{1}{2} \nearrow$ $\quad$ $\frac{1}{2} \searrow$ $\quad$ $\frac{1}{2} \nearrow$ $\quad$ $\frac{1}{2} \searrow$ $\quad$ $\frac{1}{2} \nearrow$ $\quad$ $\frac{1}{2} \searrow$

$S_3 = 219.7$
$\pi_3 = 0$
$\mu_3 = 0$

$S_3 = 152.1$
$\pi_3 = 0$
$\mu_3 = 0$

$K = 132$

$S_3 = 105.3$
$\pi_3 = 132 - 105.3 = 26.7$
$\mu_3 = 26.7$

$S_3 = 72.9$
$\pi_3 = 132 - 72.9 = 59.1$
$\mu_3 = 59.1$

2. Compute

$$V(t,x) = \exp(-r(T-t)) \left[\left(\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t) \right)^2 + \sigma^2 (T-t) \right]$$

$$\begin{aligned}
V_t(t,x) &= r\exp(-r(T-t))\left[\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2+\sigma^2(T-t)\right] \\
&\quad +\exp(-r(T-t))\left[-2\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)\left(r-\frac{1}{2}\sigma^2\right)-\sigma^2\right]
\end{aligned}$$

$$V_x(t,x)=\exp(-r(T-t))\left[2\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)\frac{1}{x}\right]$$

$$V_{xx}(t,x)=\exp(-r(T-t))\left[-2\frac{1}{x^2}\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)+2\frac{1}{x^2}\right]$$

BS equation

$$\begin{aligned}
& rV(t,x)-V_t(t,x)-rxV_x(t,x)-\frac{1}{2}\sigma^2x^2V_{xx}(t,x) \\
={}& r\exp(-r(T-t))\left[\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2+\sigma^2(T-t)\right] \\
& -\left(\begin{array}{c} r\exp(-r(T-t))\left[\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2+\sigma^2(T-t)\right] \\ +\exp(-r(T-t))\left[-2\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)\left(r-\frac{1}{2}\sigma^2\right)-\sigma^2\right]\end{array}\right) \\
& -rx\exp(-r(T-t))\left[2\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)\frac{1}{x}\right] \\
& -\frac{1}{2}\sigma^2x^2\left(\exp(-r(T-t))\left[-2\frac{1}{x^2}\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)+2\frac{1}{x^2}\right]\right)
\end{aligned}$$

$$\begin{aligned}
& rV(t,x)-V_t(t,x)-rxV_x(t,x)-\frac{1}{2}\sigma^2x^2V_{xx}(t,x) \\
={}& \exp(-r(T-t))\left[r\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2+r\sigma^2(T-t)\right] \\
& +\exp(-r(T-t))\left(\begin{array}{c} -r\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2-r\sigma^2(T-t) \\ +2\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)\left(r-\frac{1}{2}\sigma^2\right)+\sigma^2\end{array}\right) \\
& +\exp(-r(T-t))\left[-2r\left(\log\left(\frac{x}{K}\right)+r\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)\right] \\
& +\exp(-r(T-t))\left[+\frac{1}{x^2}\sigma^2x^2\left(\log\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)-\sigma^2\right] \\
={}& 0
\end{aligned}$$

Hence $V(t,x)$ is the price

3 May Exam 2013

1. Consider the discrete-time binomial tree model with three periods of length 1, i.e. $T=3$ and $t=0,1,2,3$. In each period the price can move up or down, S_{t+1} is either uS_t or dS_t.

Assume that the factor for moving up is $u = 4/3$, the factor for moving down is $d = 3/4$, and that the interest rate is $r = 0.0$. The initial stock price is $S_0 = 1$.

a. Compute the risk neutral probabilities q and $(1-q)$ for moving up and down, and draw the corresponding tree (including the stock prices at all nodes).

b. Compute the price process (i.e. prices at all times and states) for a European *Call option* on the stock with strike price $K = 1$ and maturity $T = 3$. Add these prices to the tree in (a).

c. Compute the price at time $t = 0$ of the (so called) Australian option $\left(\frac{\frac{1}{T+1}\sum_{t=0}^{T} S_t}{S_T} - K\right)^+$ with $K = 1$. Note: As this option is path dependent, you will not be able to use the recursive method, nor will you be able to use the CRR formula.

3.1 Solution of 1.c

$$
\begin{aligned}
q &= \frac{1+r-d}{u-d} = \frac{1+0-\frac{3}{4}}{\frac{4}{3}-\frac{3}{4}} = \frac{3}{7} \\
1-q &= \frac{4}{7}
\end{aligned}
$$

$$
\begin{array}{ccccccc}
 & & & & & & S_3 = \frac{4}{3}\frac{4}{3}\frac{4}{3} \\
 & & & & & \frac{3}{7}\nearrow & = \frac{64}{27} \\
 & & & & S_2 = \frac{4}{3}\frac{4}{3} = \frac{16}{9} & & \\
 & & & \frac{3}{7}\nearrow & & \frac{4}{7}\searrow & \\
 & & S_1 = \frac{4}{3} & & & & S_3 = \frac{4}{3}\frac{4}{3}\frac{3}{4} = \frac{4}{3} \\
 & \frac{3}{7}\nearrow & & \frac{4}{7}\searrow & & \frac{3}{7}\nearrow & \\
S_0 = 1 & & & & S_2 = \frac{4}{3}\frac{3}{4} = 1 & & K = 1 \\
 & \frac{4}{7}\searrow & & \frac{3}{7}\nearrow & & \frac{4}{7}\searrow & \\
 & & S_1 = \frac{3}{4} & & & & S_3 = \frac{4}{3}\frac{3}{4}\frac{3}{4} = \frac{3}{4} \\
 & & & \frac{4}{7}\searrow & & \frac{3}{7}\nearrow & \\
 & & & & S_2 = \frac{3}{4}\frac{3}{4} = \frac{9}{16} & & \\
 & & & & & \frac{4}{7}\searrow & \\
 & & & & & & S_3 = \frac{3}{4}\frac{3}{4}\frac{3}{4} = \frac{27}{64}
\end{array}
$$

The question asks to price it at time 0 only! The easiest way is to compute it directly: expectation of future payoff, but realise that payoff is path-dependent. So, you cannot use the recoursive formula given in lectures. But you are still using general Risk Neutral Valuation formula!

	$\sum_{t=0}^{3} S_t$	$\frac{1}{(T+1)S_T} \sum_{t=0}^{3} S_t$	$\left(\frac{1}{(T+1)S_T} \sum_{t=0}^{3} S_t - 1\right)^+$	RN prob
uuu	$1+\frac{4}{3}+\frac{16}{9}+\frac{64}{27}=\frac{175}{27}$	$\frac{175}{27}\frac{1}{4}\frac{27}{64}=\frac{175}{256}$	0	$\left(\frac{3}{7}\right)^3$
uud	$1+\frac{4}{3}+\frac{16}{9}+\frac{4}{3}=\frac{49}{9}$	$\frac{49}{9}\frac{1}{4}\frac{3}{4}=\frac{49}{48}$	$\frac{49}{48}-1=\frac{1}{48}$	$\left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)$
udu	$1+\frac{4}{3}+1+\frac{4}{3}=\frac{14}{3}$	$\frac{14}{3}\frac{1}{4}\frac{3}{4}=\frac{7}{8}$	0	$\left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)$
udd	$1+\frac{4}{3}+1+\frac{3}{4}=\frac{49}{12}$	$\frac{49}{12}\frac{1}{4}\frac{4}{3}=\frac{49}{36}$	$\frac{49}{36}-1=\frac{13}{36}$	$\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)$
duu	$1+\frac{3}{4}+1+\frac{4}{3}=\frac{49}{12}$	$\frac{49}{12}\frac{1}{4}\frac{3}{4}=\frac{49}{64}$	0	$\left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)$
dud	$1+\frac{3}{4}+1+\frac{3}{4}=\frac{7}{2}$	$\frac{7}{2}\frac{1}{4}\frac{4}{3}=\frac{7}{6}$	$\frac{7}{6}-1=\frac{1}{6}$	$\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)$
ddu	$1+\frac{3}{4}+\frac{9}{16}+\frac{3}{4}=\frac{49}{16}$	$\frac{49}{16}\frac{1}{4}\frac{4}{3}=\frac{49}{48}$	$\frac{49}{48}-1=\frac{1}{48}$	$\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)$
ddd	$1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}=\frac{175}{64}$	$\frac{175}{64}\frac{1}{4}\frac{64}{27}=\frac{175}{108}$	$\frac{175}{108}-1=\frac{67}{108}$	$\left(\frac{4}{7}\right)^3$

$$\pi_0=\frac{1}{(1+r)^3}\mathbb{E}_{\mathbb{Q}}(\pi_3)=1\cdot\left(\begin{array}{c}\left(\frac{3}{7}\right)^2\left(\frac{4}{7}\right)\cdot\frac{1}{48}+\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)\cdot\frac{13}{36}+\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)\cdot\frac{1}{6}\\+\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)\cdot\frac{1}{48}+\left(\frac{4}{7}\right)^3\cdot\frac{67}{108}\end{array}\right)=\frac{7213}{37\,044}$$

Digging deeper we can price in all times, using general Risk Neutral Valuation formula but we must take into account history up to time t. If at time $0<t\leq T$ the option is traded, then its price depends on history at times $0,1,\ldots t-1$

$S_3=\frac{4}{3}\frac{4}{3}\frac{4}{3}=\frac{64}{27}$

$\frac{3}{7}\nearrow$ $\pi_3(u,u,u)=0$

$S_2=\frac{4}{3}\frac{4}{3}=\frac{16}{9}$

$\pi_1=\frac{3}{7}\frac{1}{84}+\frac{4}{7}\frac{13}{63}=\frac{31}{252}$

$\frac{3}{7}\nearrow$ $\pi_2=\frac{3}{7}0+\frac{4}{7}\frac{1}{48}=\frac{1}{84}$ $\quad\frac{4}{7}\searrow$ $\pi_3(u,u,d)=\frac{1}{48}$

$S_1=\frac{4}{3}$ $\qquad S_3=\frac{4}{3}\frac{4}{3}\frac{3}{4}=\frac{4}{3}$

$\frac{3}{7}\nearrow$ $\quad\frac{4}{7}\searrow$ $\pi_2(ud)=\frac{3}{7}0+\frac{4}{7}\frac{13}{36}=\frac{13}{63}$ $\quad\frac{3}{7}\nearrow$ $\pi_3(d,u,u)=0$

$S_0=1$ $\qquad S_2=\frac{4}{3}\frac{3}{4}=1$ $\qquad \pi_3(u,d,u)=0$

$\pi_0=\frac{3}{7}\frac{31}{252}+\frac{4}{7}\frac{1315}{5292}=\frac{7213}{37\,044}$

$\frac{4}{7}\searrow$ $\quad\frac{3}{7}\nearrow$ $\pi_2(du)=\frac{3}{7}0+\frac{4}{7}\frac{1}{6}=\frac{2}{21}$ $\quad\frac{4}{7}\searrow$ $\pi_3(u,d,d)=\frac{13}{36}$

$S_1=\frac{3}{4}$ $\qquad S_3=\frac{4}{3}\frac{3}{4}\frac{3}{4}=\frac{3}{4}$

$\pi_1=\frac{3}{7}\frac{2}{21}+\frac{4}{7}\frac{157}{432}=\frac{1315}{5292}$ $\quad\frac{4}{7}\searrow$ $\quad\frac{3}{7}\nearrow$ $\pi_3(d,u,d)=\frac{1}{6}$

$S_2=\frac{3}{4}\frac{3}{4}=\frac{9}{16}$ $\qquad \pi_3(d,d,u)=\frac{1}{48}$

$\pi_2=\frac{3}{7}\frac{1}{48}+\frac{4}{7}\frac{67}{108}=\frac{157}{432}$ $\quad\frac{4}{7}\searrow$

$S_3=\frac{3}{4}\frac{3}{4}\frac{3}{4}=\frac{27}{64}$

$\pi_3(d,d,d)=\frac{67}{108}$