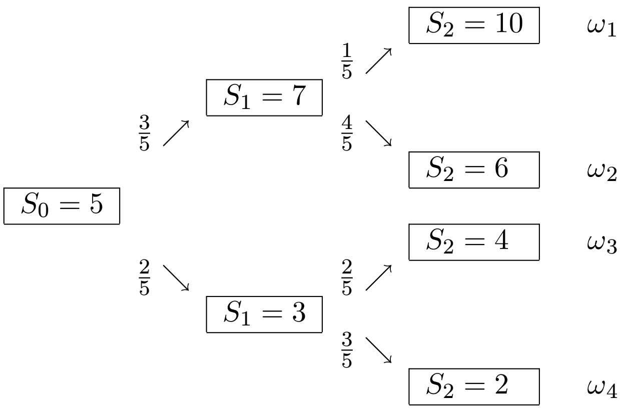
1 Tutorial Week 6: Multi-Period Market Models

1. Consider the following two period market model, with money market account B evolving according to $B_0 = 1$, $B_1 = 1 + r$, $B_2 = (1 + r)^2$ with $r = 0.25$. as well as one stock S evolving according to the diagram:



   (a) Compute the risk neutral probability measure $\mathbb{Q}$ for the model.

   (b) Use the risk neutral measure to compute an arbitrage free price for the Asian option with the payoff at maturity $T = 2$ given by the following formula

$$X = \left(\frac{1}{3}(S_0 + S_1 + S_2) - 4 \right)^+$$

2. Using the same model as in exercise 1

   (a) Compute a replicating strategy for the European Call option

$$X = (S_2 - 7)^+$$

   (b) Compute the price of the Europan call option by using risk neutral probabilities.

2 Tutorial Week 6: Solutions

1. (a) We first focus on the conditional risk neutral probabilities. To compute them, we consider the three embedded single-period two-state models (elementary market models). In each of these models we may use the single-period formula

$$\tilde{p} = \frac{1 + r - d}{u - d}$$

since $r = \frac{1}{4}$ we obtain

$$
\begin{aligned}
\tilde{p}_1 &= \frac{1+\frac{1}{4}-\frac{3}{5}}{\frac{7}{5}-\frac{3}{5}} = \frac{13}{16}, \\
\tilde{p}_2 &= \frac{1+\frac{1}{4}-\frac{6}{7}}{\frac{10}{7}-\frac{6}{7}} = \frac{11}{16} \\
\tilde{p}_3 &= \frac{1+\frac{1}{4}-\frac{2}{3}}{\frac{4}{3}-\frac{2}{3}} = \frac{7}{8}
\end{aligned}
$$

$$
\begin{array}{ccccccc}
 & & & & & \boxed{S_2 = 10} & \mathbb{Q}(\omega_1) = \frac{13}{16}\cdot\frac{11}{16} = \frac{143}{256} \\
 & & & \tilde{p}_2 = \frac{11}{16} & \nearrow & & \\
 & & \boxed{S_1 = 7} & & & & \\
 & \nearrow & & 1-\tilde{p}_2 = \frac{5}{16} & \searrow & & \\
 & \tilde{p}_1 = \frac{13}{16} & & & & \boxed{S_2 = 6} & \mathbb{Q}(\omega_2) = \frac{13}{16}\cdot\frac{5}{16} = \frac{65}{256} \\
\boxed{S_0 = 5} & & & & & & \\
 & & & & & \boxed{S_2 = 4} & \mathbb{Q}(\omega_3) = \frac{3}{16}\cdot\frac{7}{8} = \frac{21}{128} \\
 & 1-\tilde{p}_1 = \frac{3}{16} & \searrow & \tilde{p}_3 = \frac{7}{8} & \nearrow & & \\
 & & \boxed{S_1 = 3} & & & & \\
 & & & 1-\tilde{p}_3 = \frac{1}{8} & \searrow & & \\
 & & & & & \boxed{S_2 = 2} & \mathbb{Q}(\omega_4) = \frac{3}{16}\cdot\frac{1}{8} = \frac{3}{128}
\end{array}
$$

We conclude that the unique risk-neutral probability measure $\mathbb{Q}$ for the model satisfies $\mathbb{Q} = (q_1, q_2, q_3, q_4) = \left(\frac{143}{256}, \frac{65}{256}, \frac{21}{128}, \frac{3}{128}\right)$. Of course, $\frac{143}{256} + \frac{65}{256} + \frac{21}{128} + \frac{3}{128} = 1$.

(b) We search for the arbitrage price process of an Asian option with the following payoff at time 2

$$
Y = \left(\frac{1}{3}(S_0 + S_1 + S_2) - 4\right)^+ = (Y(\omega_1), Y(\omega_2), Y(\omega_3), Y(\omega_4)) = \left(\frac{10}{3}, 2, 0, 0\right)
$$

These are prices $\pi_2(Y)$.

It is obvious that the model is complete and so the option price can be replicated. This means that the unique arbitrage price can be computed by a direct application of the risk-neutral probability measure found in part (a). We compute conditional expectation

$$
\pi_1(Y) = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left(Y \mid \mathcal{F}_1^S\right) = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left(Y \mid S_1\right)
$$

and the expected value

$$
\pi_0(Y) = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left(\pi_1(Y)\right)
$$

We obtain

$$
\begin{array}{ccccc}
 & & & & \boxed{\pi_2 = \frac{10}{3}} \\
 & & & \frac{11}{16} \nearrow & \\
 & \frac{13}{16} \nearrow & \boxed{\pi_1 = \frac{4}{5}\left(\frac{10}{3}\cdot\frac{11}{16} + 2\cdot\frac{5}{16}\right) = \frac{7}{3}} & & \\
 & & & \frac{5}{16} \searrow & \\
 & & & & \boxed{\pi_2 = 2} \\
\boxed{\pi_0 = \frac{4}{5}\left(\frac{7}{3}\cdot\frac{13}{16} + 0\cdot\frac{3}{16}\right) = \frac{91}{60}} & & & & \\
 & & & & \boxed{\pi_2 = 0} \\
 & & & \frac{7}{8} \nearrow & \\
 & \frac{3}{16} \searrow & \boxed{\pi_1 = \frac{4}{5}\left(0\cdot\frac{7}{8} + 0\cdot\frac{1}{8}\right) = 0} & & \\
 & & & \frac{1}{8} \searrow & \\
 & & & & \boxed{\pi_2 = 0}
\end{array}
$$

Alternatively,

$$\pi_0 = \frac{1}{(1+r)^2}\mathbb{E}_{\mathbb{Q}}(\pi_2) = \left(\frac{4}{5}\right)^2\left(\frac{143}{256}\cdot\frac{10}{3} + \frac{65}{256}\cdot 2 + \frac{21}{128}\cdot 0 + \frac{3}{128}\cdot 0\right) = \frac{91}{60}$$

2. Risk neutral probabilities are computed in the previous exercise. They only depend on the price process and the interest rate. To remind

$$
\begin{array}{cccccc}
 & & & & \boxed{S_2 = 10} & \mathbb{Q}(\omega_1) = \frac{13}{16}\cdot\frac{11}{16} = \frac{143}{256} \\
 & & & \tilde{p}_2 = \frac{11}{16} \quad \nearrow & & \\
 & & \nearrow \quad \boxed{S_1 = 7} & & & \\
 & & & 1 - \tilde{p}_2 = \frac{5}{16} \quad \searrow & & \\
 & \tilde{p}_1 = \frac{13}{16} & & & \boxed{S_2 = 6} & \mathbb{Q}(\omega_2) = \frac{13}{16}\cdot\frac{5}{16} = \frac{65}{256} \\
\boxed{S_0 = 5} & & & & & \\
 & & & & \boxed{S_2 = 4} & \mathbb{Q}(\omega_3) = \frac{3}{16}\cdot\frac{7}{8} = \frac{21}{128} \\
 & 1 - \tilde{p}_1 = \frac{3}{16} & \searrow & \tilde{p}_3 = \frac{7}{8} \quad \nearrow & & \\
 & & \boxed{S_1 = 3} & & & \\
 & & & 1 - \tilde{p}_3 = \frac{1}{8} \quad \searrow & & \\
 & & & & \boxed{S_2 = 2} & \mathbb{Q}(\omega_4) = \frac{3}{16}\cdot\frac{1}{8} = \frac{3}{128}
\end{array}
$$

(a) We start by noting that the European call option has the following payoff at time 2:

$$Y = (S_2 - 7)^+ = (Y(\omega_1), Y(\omega_2), Y(\omega_3), Y(\omega_4)) = (3, 0, 0, 0)$$

To compute the replicationg strategy for the claim Y we proceed by backward induction We first consider the replicating portfolio for Y at time 1 on the event $A_1 = \{\omega_1, \omega_2\}$. Let φ^0 stands for the amount of cash in the savings account and let φ^1 be the number of shares held. Then we need to solve

$$
\begin{aligned}
\left(1 + \frac{1}{4}\right)^2 \varphi^0 + 10\varphi^1 &= 3 \\
\left(1 + \frac{1}{4}\right)^2 \varphi^0 + 6\varphi^1 &= 0
\end{aligned}
$$

where we assumed that we buy those bonds whic price at time zero was equal to one (*This is what we did in lectures!*). We find $\left(\varphi^0,\varphi^1\right)=\left(-\frac{72}{25},\frac{3}{4}\right)$. The value of this portfolio on event A_1 (time 1, event $\{\omega_1,\omega_2\}$) *after portfolio rebalancing* equals

$$V_1(\varphi)=-\frac{72}{25}\left(1+\frac{1}{4}\right)+\frac{3}{4}\cdot 7=\frac{33}{20}$$

We next consider the replicating portfolio for Y at time 1 on the event $A_2=\{\omega_3,\omega_4\}$. Let φ^0 stands for the amount of cash in the savings account and let φ^1 be the number of shares held. Then we need to solve

$$\begin{aligned}\left(1+\frac{1}{4}\right)^2\varphi^0+4\varphi^1 &= 0\\ \left(1+\frac{1}{4}\right)^2\varphi^0+2\varphi^1 &= 0\end{aligned}$$

we find $\left(\varphi^0,\varphi^1\right)=(0,0)$. The value of this portfolio on event A_2 (time 1, event $\{\omega_3,\omega_4\}$) equals

$$V_1(\varphi)=0+0\cdot 3=0$$

Finally, we consider the replicating portfolio for $V_0(\varphi)$ at time 0 on the event $A_0=\{\omega_1,\omega_2,\omega_3,\omega_4\}$. Let φ^0 stands for the amount of cash in the savings account and let φ^1 be the number of shares held. Then we need to solve

$$\begin{aligned}\left(1+\frac{1}{4}\right)\varphi^0+7\varphi^1 &= \frac{33}{20}\\ \left(1+\frac{1}{4}\right)\varphi^0+3\varphi^1 &= 0\end{aligned}$$

we find $\left(\varphi^0,\varphi^1\right)=\left(-\frac{99}{100},\frac{33}{80}\right)$. The value of this portfolio on event A_0 (time 0, event $\{\omega_1,\omega_2,\omega_3,\omega_4\}$) equals

$$V_1(\varphi)=-\frac{99}{100}+\frac{33}{80}\cdot 5=\frac{429}{400}$$

Important Note:

In what is above (and in lectures we computed everything assuming we invest into multi-period bonds which price at time 0 was eqial to 1. What we usually do is to assume that the bond issuer only issues 1-period bonds, and so every period they buy bonds that were issued a period ago, and issue new bonds. So the price of bonds in *previous* period is always 1. This, perhaps, is eqsier way of solving things. We need to modify the replicating portfolio for Y at time 1 on the event $A_1=\{\omega_1,\omega_2\}$.

Let φ^0 stands for the amount of cash in the savings account and let φ^1 be the number of shares held. Then we need to solve

$$\begin{aligned}\left(1+\frac{1}{4}\right)\varphi^0+10\varphi^1 &= 3\\ \left(1+\frac{1}{4}\right)\varphi^0+6\varphi^1 &= 0\end{aligned}$$

We find $\left(\varphi^0, \varphi^1\right) = \left(-\frac{18}{5}, \frac{3}{4}\right)$ so you see it is only the weight on bonds is affected. The value of this portfolio on event A_1 (time 1, event $\{\omega_1, \omega_2\}$) *after portfolio rebalancing* equals

$$V_1(\varphi) = -\frac{18}{5} + \frac{3}{4} \cdot 7 = \frac{33}{20}$$

so you see that the value is the same as above.
For A_2 (time 1, event $\{\omega_3, \omega_4\}$) we will have

$$\begin{aligned}\left(1 + \frac{1}{4}\right)\varphi^0 + 4\varphi^1 &= 0 \\ \left(1 + \frac{1}{4}\right)\varphi^0 + 2\varphi^1 &= 0\end{aligned}$$

with $\left(\varphi^0, \varphi^1\right) = (0, 0)$.
A_0 is unaffected.
It is easier to work with truly one-period bonds.

(b) Risk neutral probabilities as in the previous example. We search for the arbitrage price process of the European call option with the following payoff at time 2

$$Y = (S_2 - 7)^+ = (Y(\omega_1), Y(\omega_2), Y(\omega_3), Y(\omega_4)) = (3, 0, 0, 0)$$

These are prices $\pi_2(Y)$.
It is obvious that the model is complete and so the option price can be replicated. This means that the unique arbitrage price can be computed by a direct application of the risk-neutral probability measure found in part (a). We compute conditional expectation

$$\pi_1(Y) = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left(Y \mid \mathcal{F}_1^S\right) = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}(Y \mid S_1)$$

and the expected value

$$\pi_0(Y) = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}(\pi_1(Y))$$

We obtain

$$\begin{tikzcd}
 & & \boxed{\pi_2 = 3} \\
 & \boxed{\pi_1 = \frac{4}{5}\left(3 \cdot \frac{11}{16} + 0 \cdot \frac{5}{16}\right) = \frac{33}{20}} \arrow[ur, "\frac{11}{16}"] \arrow[dr, "\frac{5}{16}"'] & \\
 & & \boxed{\pi_2 = 0} \\
\boxed{\pi_0 = \frac{4}{5}\left(\frac{33}{20} \cdot \frac{13}{16} + 0 \cdot \frac{3}{16}\right) = \frac{429}{400}} \arrow[uur, "\frac{13}{16}"] \arrow[ddr, "\frac{3}{16}"'] & & \\
 & & \boxed{\pi_2 = 0} \\
 & \boxed{\pi_1 = \frac{4}{5}\left(0 \cdot \frac{7}{8} + 0 \cdot \frac{1}{8}\right) = 0} \arrow[ur, "\frac{7}{8}"] \arrow[dr, "\frac{1}{8}"'] & \\
 & & \boxed{\pi_2 = 0}
\end{tikzcd}$$

Alternatively,

$$\pi_0 = \frac{1}{(1+r)^2}\mathbb{E}_{\mathbb{Q}}\left(\pi_2\right) = \left(\frac{4}{5}\right)^2\left(\frac{143}{256}\cdot 3 + \frac{65}{256}\cdot 0 + \frac{21}{128}\cdot 0 + \frac{3}{128}\cdot 0\right) = \frac{429}{400}$$