1 Tutorial Week 9-10: Ito formula and Black-Scholes Formula

1. Let W_t denote a Brownian motion. Ise Ito's formula to compute the differentials of the following stochastic processes:

a) $X_t = e^{\frac{1}{2}t}\cos(W_t)$

b) $X_t = e^{\frac{1}{2}t}\sin(W_t)$

c) $X_t = (W_t + t)\, e^{-W_t - \frac{1}{2}t}$

2. a) Show that the price of the option with payoff

$$h(S_T) = S_T \log\left(\frac{S_T}{K}\right)$$

with $K > 0$ at time t given $S_t = x$ is given by

$$V(t,x) = x\left(\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right)$$

where r is the interest rate and σ the volatility.

b) Compute Greeks of this option, i.e. compute the derivatives $\frac{\partial}{\partial x}V(t,x)$, $\frac{\partial^2}{\partial x^2}V(t,x)$ and $\frac{\partial}{\partial \sigma}V(t,x)$

3. Consider the stock price process S_t under the Black and Scoles assumption, that is, $S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$, where W_t is the Wiener process under the martingale measure $\widetilde{\mathbb{P}}$. Compute the expectation of the Asian underlying, i.e.

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{1}{T}\int_0^T S_t dt\right).$$

Hint: Use that the expectation operator $\mathbb{E}$ and the integral $\int_0^T$ can be interchanged.

2 Solution to Tutorial Week 9-10

1.

a) Recall formulation of Ito formula from lectures:

For the process

$$dx(t) = \phi(t, x(t))\, dt + \sigma(t, x(t))\, dW(t)$$

take a composite function

$$g(t) = G(t, x(t))$$

then its process is

$$\begin{aligned} dg(t) &= \left(G_t(t,x(t)) + G_x(t,x(t))\,\phi(t,x(t)) + \frac{1}{2}G_{xx}(t,x(t))\,\sigma^2(t,x(t))\right)dt \\ &\quad + G_x(t,x(t))\,\sigma(t,x(t))\,dW(t) \end{aligned}$$

so it is a general formula how to differentiate composite functions.

Now, we need to understand "what is what" in $X_t = \exp\left(\frac{1}{2}t\right)\cos\left(W_t\right)$

An obvious way:

$$\begin{aligned}
x(t) &= W\left(t\right)\\
\sigma\left(t, x\left(t\right)\right) &= 1\\
\phi\left(t, x\left(t\right)\right) &= 0\\
g\left(t\right) &= G\left(t, x\left(t\right)\right) = \exp\left(\frac{1}{2}t\right)\cos\left(x\left(t\right)\right)
\end{aligned}$$

Then only three terms remain in Ito formula once we substitute $\sigma = 1$ and $\phi = 0$:

$$dg\left(t\right) = \left(G_t\left(t, x\left(t\right)\right) + \frac{1}{2}G_{xx}\left(t, x\left(t\right)\right)\right)dt + G_x\left(t, x\left(t\right)\right)dW\left(t\right)$$

And we need to find these partial derivatives. The approach is the same for all three cases.

a) $X_t = e^{\frac{1}{2}t}\cos\left(W_t\right)$

$$X_t = G\left(t, W_t\right) = e^{\frac{1}{2}t}\cos\left(W_t\right)$$

$$\begin{aligned}
\frac{\partial G\left(t, W_t\right)}{\partial t} &= \frac{1}{2}e^{\frac{1}{2}t}\cos\left(W_t\right)\\
\frac{\partial G\left(t, W_t\right)}{\partial W} &= -e^{\frac{1}{2}t}\sin\left(W_t\right)\\
\frac{\partial^2 G\left(t, W_t\right)}{\partial W^2} &= -e^{\frac{1}{2}t}\cos\left(W_t\right)
\end{aligned}$$

$$\begin{aligned}
dX_t &= \frac{1}{2}e^{\frac{1}{2}t}\cos\left(W_t\right)dt - e^{\frac{1}{2}t}\sin\left(W_t\right)dW_t - \frac{1}{2}e^{\frac{1}{2}t}\cos\left(W_t\right)dt\\
&= -e^{\frac{1}{2}t}\sin\left(W_t\right)dW_t
\end{aligned}$$

b) $X_t = e^{\frac{1}{2}t}\sin\left(W_t\right)$

$$G\left(t, W_t\right) = e^{\frac{1}{2}t}\sin\left(W_t\right)$$

$$\begin{aligned}
\frac{\partial G\left(t, W_t\right)}{\partial t} &= \frac{1}{2}e^{\frac{1}{2}t}\sin\left(W_t\right)\\
\frac{\partial G\left(t, W_t\right)}{\partial W} &= e^{\frac{1}{2}t}\cos\left(W_t\right)\\
\frac{\partial^2 G\left(t, W_t\right)}{\partial W^2} &= -e^{\frac{1}{2}t}\sin\left(W_t\right)
\end{aligned}$$

$$\begin{aligned}
dX_t &= \frac{1}{2}e^{\frac{1}{2}t}\sin\left(W_t\right)dt + e^{\frac{1}{2}t}\cos\left(W_t\right)dW_t - \frac{1}{2}e^{\frac{1}{2}t}\sin\left(W_t\right)dt\\
&= e^{\frac{1}{2}t}\cos\left(W_t\right)dW_t
\end{aligned}$$

c) $X_t = (W_t + t)\, e^{-W_t - \frac{1}{2}t}$

$$G(t, W_t) = (W_t + t)\, e^{-W_t - \frac{1}{2}t}$$

$$\begin{aligned}
\frac{\partial G(t, W_t)}{\partial t} &= e^{-W_t - \frac{1}{2}t} - \frac{1}{2}(W_t + t)\, e^{-W_t - \frac{1}{2}t} \\
\frac{\partial G(t, W_t)}{\partial W} &= e^{-W_t - \frac{1}{2}t} - (W_t + t)\, e^{-W_t - \frac{1}{2}t} \\
\frac{\partial^2 G(t, W_t)}{\partial W^2} &= -e^{-W_t - \frac{1}{2}t} - e^{-W_t - \frac{1}{2}t} + (W_t + t)\, e^{-W_t - \frac{1}{2}t}
\end{aligned}$$

$$\begin{aligned}
dX_t &= \left(e^{-W_t - \frac{1}{2}t} - \frac{1}{2}(W_t + t)\, e^{-W_t - \frac{1}{2}t}\right) dt + \left(e^{-W_t - \frac{1}{2}t} - (W_t + t)\, e^{-W_t - \frac{1}{2}t}\right) dW_t \\
&\quad + \frac{1}{2}\left(-e^{-W_t - \frac{1}{2}t} - e^{-W_t - \frac{1}{2}t} + (W_t + t)\, e^{-W_t - \frac{1}{2}t}\right) dt \\
&= (1 - W_t - t)\, e^{-W_t - \frac{1}{2}t} dW_t
\end{aligned}$$

2. a) First, check that

$$V(T, S_T) = S_T\left(\log\left(\frac{S_T}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - T)\right) = S_T \log\left(\frac{S_T}{K}\right) = h(S_T)$$

so the terminal condition is verified

Second, check that

$$V(t, x) = x\left(\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)\right)$$

satisfies Black-Scholes equation

$$rV(t, x) - \frac{\partial}{\partial t}V(t, x) = \frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2}V(t, x) + rx\frac{\partial}{\partial x}V(t, x),$$

$$\begin{aligned}
\frac{\partial V(t, x)}{\partial t} &= -x\left(r + \frac{1}{2}\sigma^2\right) \\
\frac{\partial V(t, x)}{\partial x} &= \left(\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)\right) + x\frac{1}{x} = \log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t) + 1 \\
\frac{\partial^2 V(t, x)}{\partial x^2} &= \frac{1}{x}
\end{aligned}$$

$$rx\left(\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)\right) + x\left(r + \frac{1}{2}\sigma^2\right) = \frac{\sigma^2 x^2}{2}\frac{1}{x} + rx\left(\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t) + 1\right)$$

$$rx\log\left(\frac{x}{K}\right) + rx\left(r + \frac{1}{2}\sigma^2\right)(T - t) + x\left(r + \frac{1}{2}\sigma^2\right) = \frac{\sigma^2}{2}x + rx\log\left(\frac{x}{K}\right) + rx\left(r + \frac{1}{2}\sigma^2\right)(T - t) + rx$$

which is easily verified.

b) Compute Greeks of this option

$$\begin{aligned}
\Delta &= \frac{\partial V(t,x)}{\partial x} = \log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) + 1 \\
\Theta &= \frac{\partial V(t,x)}{\partial t} = -x\left(r + \frac{1}{2}\sigma^2\right) \\
\Gamma &= \frac{\partial^2 V(t,x)}{\partial x^2} = \frac{1}{x} \\
\rho &= \frac{\partial V(t,x)}{\partial r} = x(T-t) \\
vega &= \frac{\partial V(t,x)}{\partial \sigma} = x\sigma(T-t)
\end{aligned}$$

3.

$$\begin{aligned}
\mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{1}{T}\int_0^T S_t dt\right) &= \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{1}{T}\int_0^T S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) dt\right) \\
&= \frac{S_0}{T}\int_0^T \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\left(r-\frac{1}{2}\sigma^2\right)t} e^{\sigma W_t}\right) dt \\
&= \frac{S_0}{T}\int_0^T e^{\left(r-\frac{1}{2}\sigma^2\right)t}\mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma W_t}\right) dt \\
&= \frac{S_0}{T}\int_0^T e^{\left(r-\frac{1}{2}\sigma^2\right)t}\mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_0)} e^{\sigma(W_0)}\right) dt
\end{aligned}$$

Now, $W_0 = 0$ and $(W_t - W_0)\,\tilde{}\,N\left(0, \sqrt{t}\right)$ so that

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{1}{T}\int_0^T S_t dt\right) = \frac{S_0}{T}\int_0^T e^{\left(r-\frac{1}{2}\sigma^2\right)t}\mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_0)}\right) dt$$

We need to compute $\mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_0)}\right)$
$(W_t - W_0)\,\tilde{}\,N\left(0, \sqrt{t}\right)$, or $\sigma\left(W_t - W_0\right)\,\tilde{}\,N\left(0, \sigma\sqrt{t}\right)$
We have general formula for the mean of log-normal distribution

Recall that

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\mu + \sigma Z}\right) = e^{\mu + \frac{1}{2}\sigma^2}$$

where $Z\,\tilde{}\,N\left(0,1\right)$
We have

$$\sigma\left(W_t - W_0\right) = 0 + \sigma\sqrt{t}Z$$

and

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma\sqrt{t}Z}\right) = e^{\frac{1}{2}\sigma^2 t}$$

so that

$$\begin{aligned}
\mathbb{E}_{\widetilde{\mathbb{P}}}\left(\frac{1}{T}\int_0^T S_t dt\right) &= \frac{S_0}{T}\int_0^T e^{\left(r-\frac{1}{2}\sigma^2\right)t} e^{\frac{1}{2}\sigma^2 t} dt \\
&= \frac{S_0}{T}\int_0^T e^{rt} dt \\
&= \frac{S_0}{Tr}\left(e^{rT}-1\right)
\end{aligned}$$