1 Tutorial Week 4: Information and Martingales

1. We consider the conditional expectation $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ where σ-field $\mathcal{G}$ is generated by a finite partition $(A_i)_{i\in I}$ of the sample space $\Omega = \{\omega_1, ..., \omega_k\}$.

   Show that the conditional expectation $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ satisfies

$$\sum_{\omega\in G} X(\omega)\mathbb{P}(\omega) = \sum_{\omega\in G} \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})(\omega)\mathbb{P}(\omega), \quad \forall\, G \in \mathcal{G}.$$

2. We consider a discrete-time stochastic process $X = (X_t, t = 0, 1, ...)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. It is assumed thata process X is adapted to filtration $\mathbb{F}$.

   Assume that X has independent increments with respect to $\mathbb{F}$, meaning that for any $t = 0, 1, ..$ the increment $X_{t+1} - X_t$ is independent of the σ-field $\mathcal{F}_t$. Show that the process Y given by the formula

$$Y_t = X_t - \mathbb{E}_{\mathbb{P}}\left(X_t\right),\ t = 0, 1, ..$$

   is a martingale with respect to the filtration $\mathbb{F}$.

3. Consider stochastic process $Z = (Z_t, t = 0, 1, .., T)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with filtration $\mathbb{F} = (\mathcal{F}_t)_{,0\leq t\leq T}$. Let Z be an $\mathcal{F}_T$-measurable. Show that

$$Z_t := \mathbb{E}_{\mathbb{P}}\left(Z|\mathcal{F}_t\right)$$

   is a martingale for $t = 0, 1, .., T$.

2 Tutorial Week 4: Solutions

1. Consider an arbitrary partition $\{A_1, ..., A_m\}$ of the finite space $\Omega = \{\omega_1, ..., \omega_k\}$.

   Let G be an arbitrary event from the σ-field $\mathcal{G}$ generated by partition $\{A_1, ..., A_m\}$. We can count these subsets A_l which constitute G. Namely, then there exists a set $L \subset \{1, 2, ..., m\}$ such that

$$G = \cup_{l\in L} A_l \tag{1}$$

   Moreover, we know Bayes Law: for every $l \in L$ in the event A_l we have that

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}) = \frac{1}{\mathbb{P}(A_l)} \sum_{\omega\in A_l} X(\omega)\mathbb{P}(\omega) \tag{2}$$

   Consequently,

$$\sum_{\omega\in G} X(\omega)\mathbb{P}(\omega) \overset{(1)}{=} \sum_{l\in L}\sum_{\omega\in A_l} X(\omega)\mathbb{P}(\omega) \overset{(2)}{=} \sum_{l\in L} \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})\mathbb{P}(A_l)$$
$$\overset{(3)}{=} \sum_{\omega\in G} \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})(\omega)\mathbb{P}(\omega)$$

where equality (1) holds because of (1) – instead of taking sum over G we take sum over its subsets, equality (2) holds because of the Bayes Law (2) and equality (3) holds since the conditional expectation $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ is the same for all $\omega \in A_l$, namely instead of summing over L subsets which constitute G, we sum over all ω which constitute G.

2. To remind you: An $\mathbb{F}$-adapted process $X = (X_t)_{0\le t\le T}$ on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **martingale** whenever for all $s < t$

$$\mathbb{E}_{\mathbb{P}}(X_t \,|\, \mathcal{F}_s) = X_s.$$

Namely expectation of future value conditional on current information is the current value.

Consider the dates t and $t+1$ for an arbitrary $t = 0, 1, ..$ and let's check that

$$\mathbb{E}_{\mathbb{P}}\left(Y_{t+1}|\mathcal{F}_t\right) = Y_t$$

or, equivalently,

$$\mathbb{E}_{\mathbb{P}}\left(Y_{t+1} - Y_t|\mathcal{F}_t\right) = 0$$

Since increments $X_{t+1} - X_t$ are independent of filtration, then increments plus constant $X_{t+1} - X_t + c$ are also **independent of** $\mathcal{F}_t$, for any $c \in \mathbb{R}$. Therefore, for any $t = 0, 1, ..$

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}}\left(Y_{t+1} - Y_t|\mathcal{F}_t\right) \overset{\text{substitute Y}}{=} \mathbb{E}_{\mathbb{P}}\left(X_{t+1} - X_t - \mathbb{E}_{\mathbb{P}}\left(X_{t+1}\right) + \mathbb{E}_{\mathbb{P}}\left(X_t\right)|\mathcal{F}_t\right) \\ & \overset{\text{remove conditioning}}{=} \mathbb{E}_{\mathbb{P}}\left(X_{t+1} - X_t - \mathbb{E}_{\mathbb{P}}\left(X_{t+1}\right) + \mathbb{E}_{\mathbb{P}}\left(X_t\right)\right) \\ = \; & \mathbb{E}_{\mathbb{P}}X_{t+1} - \mathbb{E}_{\mathbb{P}}X_t - \mathbb{E}_{\mathbb{P}}\left(X_{t+1}\right) + \mathbb{E}_{\mathbb{P}}\left(X_t\right) = 0 \end{aligned}$$

In the last line expectations of expectation is the original expectation.

Now, note that $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ as $\mathcal{F}_{t+1}$ contains more information

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\left(Y_{t+2}|\mathcal{F}_t\right) &= \mathbb{E}_{\mathbb{P}}\left(Y_{t+2} - Y_{t+1} + Y_{t+1}|\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(Y_{t+2} - Y_{t+1}|\mathcal{F}_t\right) + \mathbb{E}_{\mathbb{P}}\left(Y_{t+1}|\mathcal{F}_t\right) \\ &\overset{\text{tower property}}{=} \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}\left(Y_{t+2} - Y_{t+1}|\mathcal{F}_{t+1}\right)|\mathcal{F}_t\right) + \mathbb{E}_{\mathbb{P}}\left(Y_{t+1}|\mathcal{F}_t\right) \\ &= 0 + Y_t \end{aligned}$$

So we can continue and show that for any time $t+k$

$$\mathbb{E}_{\mathbb{P}}\left(Y_{t+k}|\mathcal{F}_t\right) = Y_t$$

This proves that the process Y is a martingale.

3. We have

$$Z_t := \mathbb{E}_{\mathbb{P}}\left(Z|\mathcal{F}_t\right)$$

$$Z_{t+1} := \mathbb{E}_{\mathbb{P}}\left(Z|\mathcal{F}_{t+1}\right)$$

As before, we need to show

$$\mathbb{E}_{\mathbb{P}}\left(Z_{t+1}|\mathcal{F}_t\right) = Z_t$$

Indeed, note that $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ so we can use the tower property and trivial conditioning ($\mathcal{F}_{t+1}$ is information at the next period of time, nothing today depends on it):

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left(Z_{t+1}|\mathcal{F}_t\right) \overset{\text{substitute } Z_{t+1}}{=} \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}\left(Z|\mathcal{F}_{t+1}\right)|\mathcal{F}_t\right) \\
&\overset{\text{tower property}}{=} \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}\left(Z|\mathcal{F}_t\right)|\mathcal{F}_{t+1}\right) \\
&\overset{\text{trivial}}{=} \mathbb{E}_{\mathbb{P}}\left(Z|\mathcal{F}_t\right) \overset{\text{definition of Z}}{=} Z_t
\end{aligned}$$

The proof is identical for any $t+k$

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left(Z_{t+k}|\mathcal{F}_t\right) \overset{\text{substitute } Z_{t+1}}{=} \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}\left(Z|\mathcal{F}_{t+k}\right)|\mathcal{F}_t\right) \\
&\overset{\text{tower property}}{=} \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}\left(Z|\mathcal{F}_t\right)|\mathcal{F}_{t+k}\right) \\
&\overset{\text{trivial}}{=} \mathbb{E}_{\mathbb{P}}\left(Z|\mathcal{F}_t\right) \overset{\text{definition of Z}}{=} Z_t
\end{aligned}$$

Properties of Conditional Expectation

Proposition 1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be endowed with sub-σ-fields $\mathcal{G}$ and $\mathcal{G}_1 \subset \mathcal{G}_2$ of $\mathcal{F}$. Then*

1. ***Tower property:*** *If $X : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable r.v. then*

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}_1) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}_2)\,|\,\mathcal{G}_1\right) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}_1)\,|\,\mathcal{G}_2\right).$$

2. ***Taking out what is known:*** *If $X : \Omega \to \mathbb{R}$ is a $\mathcal{G}$-measurable r.v. and $Y : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable r.v. then*
$$\mathbb{E}_{\mathbb{P}}(XY|\,\mathcal{G}) = X\,\mathbb{E}_{\mathbb{P}}(Y|\,\mathcal{G}).$$

3. ***Trivial conditioning:*** *If $X : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable r.v. independent of $\mathcal{G}$ then*

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}) = \mathbb{E}_{\mathbb{P}}(X).$$