ST425 - Project

1) $$f(x) = \begin{cases} \dfrac{\alpha \beta^{\alpha}}{(x+\beta)^{\alpha+1}} , & x \geq 0 \\ 0 , & x < 0 \end{cases}$$

a) $$F(x) = \int_0^x f(u)\, du$$

$$= \int_0^x \frac{\alpha \beta^\alpha}{(u+\beta)^{\alpha+1}}\, du = \int_0^x \alpha \beta^\alpha (u+\beta)^{-\alpha-1}\, du$$

$$= \alpha \beta^\alpha \left[\frac{(u+\beta)^{-\alpha}}{-\alpha} \right]_0^x$$

$$= \alpha \beta^\alpha \left[\frac{1}{-\alpha (x+\beta)^\alpha} - \frac{1}{-\alpha \beta^\alpha} \right]$$

$$= 1 - \frac{1}{\cancel{\beta^\alpha}} \left(\frac{\beta}{x+\beta} \right)^{\alpha} , \quad x \geq 0$$

$$E(X) = \int_0^\infty x f(x)\, dx$$

$$= \int_0^\infty x \cdot \alpha \beta^\alpha (x+\beta)^{-\alpha-1}\, dx$$

$$\alpha\beta^{\alpha}\int_{0}^{\infty} x\,(x+\beta)^{-\alpha-1}\,dx$$

Let $y = x+\beta$

$x = y-\beta$

$$\alpha\beta^{\alpha}\int_{\beta}^{\infty} (y-\beta)\,y^{-\alpha-1}\,dy$$

$$\alpha\beta^{\alpha}\int_{\beta}^{\infty} y^{-\alpha} - \beta y^{-\alpha-1}\,dy$$

$$\alpha\beta^{\alpha}\left[\frac{y^{-\alpha+1}}{1-\alpha} - \beta\frac{y^{-\alpha}}{-\alpha}\right]_{\beta}^{\infty}$$

$$\alpha\beta^{\alpha}\left[0-\left(\frac{\beta^{1-\alpha}}{1-\alpha}+\frac{\beta}{\alpha\beta^{\alpha}}\right)\right]$$

$$\alpha\beta^{\alpha}\left[-\frac{\beta^{1-\alpha}}{1-\alpha}-\frac{\beta}{\alpha\beta^{\alpha}}\right]$$

$$-\frac{\alpha\beta}{1-\alpha}-\beta$$

$$\frac{-\alpha\beta-\beta(1-\alpha)}{1-\alpha} = \frac{\beta}{\alpha-1},\ \alpha>1$$

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$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_0^\infty x^2 \frac{\alpha\beta^\alpha}{(x+\beta)^{\alpha+1}} dx$$

$$\alpha\beta^\alpha \int_0^\infty x^2 (x+\beta)^{-\alpha-1} dx$$

Let $y = x+\beta$

$x = y-\beta$

$$\alpha\beta^\alpha \int_\beta^\infty (y-\beta)^2 y^{-\alpha-1} dy$$

$$\alpha\beta^\alpha \int_\beta^\infty (y^2 - 2\beta y + \beta^2) y^{-\alpha-1} dy$$

$$\alpha\beta^\alpha \int_\beta^\infty y^{-\alpha+1} - 2\beta y^{-\alpha} + \beta^2 y^{-\alpha-1} dy$$

$$\alpha\beta^\alpha \left[\frac{y^{-\alpha+2}}{-\alpha+2} - 2\beta \frac{y^{-\alpha+1}}{-\alpha+1} + \beta^2 \frac{y^{-\alpha}}{-\alpha} \right]_\beta^\infty$$

$$\alpha\beta^\alpha \left[0 - \left(\frac{\beta^{-\alpha+2}}{-\alpha+2} - 2\frac{\beta^{-\alpha+2}}{-\alpha+1} + \frac{\beta^{-\alpha+2}}{-\alpha} \right) \right]$$

$$\alpha\beta^2 \left[\frac{1}{\alpha-2} + \frac{2}{1-\alpha} + \frac{1}{\alpha} \right]$$

$$\cancel{\alpha}\beta^2 \left[\frac{\alpha(1-\alpha)+2\alpha(\alpha-2)+(\alpha-2)(1-\alpha)}{\cancel{(\alpha)}(\alpha-2)(1-\alpha)}\right]$$

$$\beta^2\left[\frac{\cancel{\alpha}-\cancel{\alpha^2}+2\cancel{\alpha^2}-4\cancel{\alpha}+\cancel{\alpha}-\cancel{\alpha^2}-2+2\cancel{\alpha}}{(\alpha-2)(1-\alpha)}\right]$$

$$\frac{-2\beta^2}{(\alpha-2)(1-\alpha)}$$

$$V(X) = E(X^2) - E(X)^2$$

$$= \frac{\cancel{-}2\beta^2}{(\alpha-2)(-1+\alpha)} - \frac{\beta^2}{(\alpha-1)^2}$$

$$= \frac{2\beta^2(\alpha-1) - \beta^2(\alpha-2)}{(\alpha-1)^2(\alpha-2)}$$

$$= \frac{2\beta^2\alpha - 2\beta^2 - \beta^2\alpha + 2\beta^2}{(\alpha-1)^2(\alpha-2)}$$

$$= \frac{\beta^2\alpha}{(\alpha-1)^2(\alpha-2)}$$

$\alpha > 2$ $\qquad$ $Var(X) > 0$

and integral to be definite.

Median of $X = a$ when

$$P(X \leq a) = 0.5$$

$$F_X(a) = 0.5$$

$$1 - \left(\frac{\beta}{a+\beta}\right)^{\alpha} = 0.5$$

$$\left(\frac{\beta}{a+\beta}\right)^{\alpha} = 0.5$$

$$\frac{\beta}{a+\beta} = 0.5^{1/\alpha}$$

$$\beta = 0.5^{1/\alpha}(a+\beta)$$

$$\beta = a\,0.5^{1/\alpha} + \beta\,0.5^{1/\alpha}$$

$$\frac{\beta(1-0.5^{1/\alpha})}{0.5^{1/\alpha}} = a$$

$$\therefore \text{Median of } X = \frac{\beta(1-0.5^{1/\alpha})}{0.5^{1/\alpha}}$$

b) Inversion Method.

Let $u = F_X$

$$u = 1 - \left(\frac{\beta}{x+\beta}\right)^{\alpha}$$

$$\left(\frac{\beta}{x+\beta}\right)^{\alpha} = 1 - u$$

$$\frac{\beta}{x+\beta} = (1-u)^{1/\alpha}$$

$$\beta = (1-u)^{1/\alpha}\left[x+\beta\right]$$

$$\beta = x(1-u)^{1/\alpha} + \beta(1-u)^{1/\alpha}$$

$$\frac{\beta\left[1-(1-u)^{1/\alpha}\right]}{(1-u)^{1/\alpha}} = x$$

$$x = \beta\left[\frac{\cancel{\beta}\,1-(1-u)^{1/\alpha}}{(1-u)^{1/\alpha}}\right]$$

$u \sim unif(0,1)$

$1-u \sim unif(0,1)$

$$\therefore x = \beta\left[\frac{1-u^{1/\alpha}}{u^{1/\alpha}}\right]$$

Generate value for x.

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u = runif(10000)

x =  100000*(1 - u^(1/3)) / u^(1/3)

x.
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