A simple algorithm for r-local and k-sparse unlabeled sensing (DRAFT)

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***Abstract*—The unlabeled sensing problem is to recover an unknown signal from permuted linear measurements. An alternating minimization algorithm is proposed for the widely considered r-local and k-sparse permutation models. We study structured initializations for both models and show that the initialization error can be suitably upper bounded under randomness on the measurement matrix or signal. The proposed algorithm is efficient and experiments on real and synthetic datasets show superior performance compared to baselines.**

***Index Terms*—unlabeled sensing, linear regression without correspondence, r-local, k-sparse**

I. Introduction

THE standard least square problem is $\min_{\mathbf{X}} \quad ||\mathbf{Y}-\mathbf{B}\mathbf{X}||^2$, where $\mathbf{Y} \in \mathbb{R}^{n\times m}$ denotes the matrix of linear measurements of the underlying signal matrix $\mathbf{X}^* \in \mathbb{R}^{d\times m}$. In the unlabeled sensing problem, the "unlabeled" measurements are scrambled by an unknown permutation $\mathbf{P}^*$. Given scrambled measurements $\mathbf{Y}$, the unlabeled sensing model is

$$\mathbf{Y} = \mathbf{P}^*\mathbf{B}\mathbf{X}^* + \mathbf{W}, \tag{1}$$

where $\mathbf{P}^* \in \mathbb{R}^{n\times n}$ is an unknown permutation matrix and $\mathbf{W} \in \mathbb{R}^{n\times m}$ is additive Gaussian noise with $\mathbf{W}_{ij} = \mathcal{N}(0,\sigma^2)$. The main problem is then to estimate the unknown signal $\mathbf{X}^*$ given as input the scrambled measurements $\mathbf{Y}$ and the sensing matrix $\mathbf{B}$. The problem is referred to as single view (multi-view) unlabeled sensing when $m = 1 (m > 1)$.

The work in [1] was the first to consider the unlabeled sensing problem and establish that $n = 2d$ measurements are necessary and sufficient for recovery of $\mathbf{x}^*$. Subsequent works have generalized this result and developed information theoretic inachievability analysis [2], [3]. The challenge in the unlabeled sensing problem is estimating the permutation $\mathbf{P}^*$, which is a combinatorial optimization problem. This has necessitated imposing structure on the underlying permutation where the structure is motivated by practical applications. Some examples of these structures are partially shuffled or k-sparse permutations [4], [5], [6], [7], [8], [9] and r-local or block diagonal permutations [10], [11]. A permutation matrix $\mathbf{P}_k^*$ is k-sparse if it has k off-diagonal elements, i.e., $\langle \mathbf{I}, \mathbf{P}_k^* \rangle = n-k$. A permutation matrix $\mathbf{P}_r^*$ is r-local if it has block-diagonal structure $\mathbf{P}_r^* = (P_1, \cdots, P_s)$. Fig. 1 illustrates the difference two models. In a recent work [11], the two models are also compared by information-theoretic inachiveability.

For single-view unlabeled sensing, algorithms based on branch and bound and expectation maximization are proposed in [12], [13], [14] which are applicable to small problem sizes. A stochastic alternating minimization approach is proposed in [15]. For multi-view unlabeled sensing, the Levsort subspace matching algorithm was proposed in [16]. An approach based on subspace clustering was proposed in [7]. The works [4], [5] propose methods based on bi-convex optimization and robust regression, respectively. A spectral initialization based method was proposed recently in [17].

A. Contributions

We extend our alternating minimization based algorithm for the r-local unlabeled sensing model [10], [11] to the widely considered k-sparse model, Section II. We propose and analyze the initialization to the algorithm, Section III. Specifically, assuming random $\mathbf{B}$ or $\mathbf{X}^*$, we derive upper bounds on the initialization error for both r-local and k-sparse models.

The main part of the analysis is relating the error of a linear least-squares system to a linear system obtained by summing a subset of the rows. This is a problem studied under the topic of sketching linear systems. In contrast to the standard setting for sketching, our original linear system contains an unknown permutation $\mathbf{P}^*$ and our sampling is deterministic resulting from the permutation model. Our analysis uses tools from high-dimensional probability and recent results for random quadratic forms of a positive semi-definite matrix. We remark that the initialization analysis is insightful because the formulated optimization problem is non-convex. We think that the presented analysis is general and might be applicable to other class of under-determined linear inverse problems. Finally, we also compare our algorithm to several existing benchmark methods, Section IV.

B. Applications of unlabeled sensing

The linked linear regression problem [18], [19], [20], or linear regression under blocking, is to fit a linear model by minimizing over $\mathbf{x}$ and $\mathbf{P}$ the objective $\|\mathbf{y} - \mathbf{P}\mathbf{B}\mathbf{x}\|_2^2$, where $\mathbf{P}$ is an unknown r-local permutation. For example, consider a simple statistical model where the weight of an individual depends linearly on age and height. The vector $\mathbf{y} \in \mathbb{R}^n$ contains the weight of $n = 10$ individuals, the matrix $\mathbf{B} \in \mathbb{R}^{n\times d}$, $d = 2$ contains the values of the age, height of the individuals. The 10 individuals comprise 5 males, 5 females, and each record (weight, age, height) is known to belong to a male or a female individual. Letting the first (second) block of 5 rows of $\mathbf{y}, \mathbf{B}$ contain the records of male (female) individuals, the unknown r-local permutation, $r = 5$, assigns each weight value (row of $\mathbf{y}$) to its corresponding age, height (row of $\mathbf{B}$).

For the pose and correspondence problem [21], the rows of $\mathbf{B} \in \mathbb{R}^{n\times d}$ contain the d-dimensional coordinates of

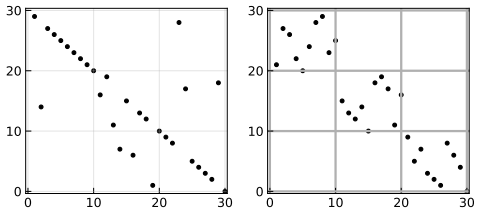


Fig. 1: Left. Sparse (or partially shuffled) permutation considered in [4], [5], [6], [7], with number of shuffles $k = 10$. Right. The r-local permutation structure considered in [10], [11], with block size $r = 10$. In this paper, we propose and analyze a general algorithm that is applicable to both permutation models.

the landmark points of an object. The object is transformed by an unknown linear transformation $\mathbf{X}^*$ and the rows of $\mathbf{Y} = \mathbf{P}^*\mathbf{B}\mathbf{X}^*$ contain the coordinates of the points of the transformed object. The problem is to assign a point in $\mathbf{B}$ to the corresponding point in $\mathbf{Y}$ and recover the unknown transformation $\mathbf{X}^*$. Another application of unlabeled sensing is signal amplitude estimation from binary quantized unlabeled samples [22]. The unknown quantity $x \in \mathbb{R}$ is the signal amplitude and the sensing matrix $\mathbf{B}$ is replaced by a threshold function that models bit quantization. The image correspondence problem [23] and the 1-d unassigned distance geometry problem (uDGP) were recently formulated as unlabeled sensing problems in [11].

C. Notation and definitions

$\mathbf{P}_r^*$ denotes r-local permutations with block size r. $\mathbf{P}_k^*$ denotes k-sparse permutations with k off-diagonal elements. $\|X\|_{\varphi_1}$ denotes the sub-exponential norm of the sub-exponential random variable X. $\|X\|_{\varphi_2}$ denotes the sub-Gaussian norm of the sub-Gaussian random variable X. $c' \in \mathbb{R}$ denotes an absolute constant. Π_n denotes the set of $n \times n$ permutations. The Hamming distortion d_H between two permutation matrices is the number of mismatches $d_H = \Sigma_i \mathbb{1}(\widehat{\mathbf{P}}(i) \neq \mathbf{P}(i))$.

II. Algorithm

We propose to estimate the signal $\mathbf{X}^*$ and the permutation $\mathbf{P}^*$ by minimizing the forward error in the model (1)

$$\underset{\mathbf{X},\mathbf{P}\in\Pi_n}{\operatorname{argmin}} \; F(\mathbf{X},\mathbf{P}) = \|\mathbf{Y} - \mathbf{P}\mathbf{B}\mathbf{X}\|_F^2. \tag{2}$$

The choice of the objective function is motivated by a result in [3] which upper bounds the Hamming distortion of the estimate $\widehat{\mathbf{P}}$ from the ground truth $\mathbf{P}^*$ in terms of the objective value $F(\mathbf{X},\mathbf{P})$. The result is restated in (63).

We propose alternating minimization (Alt-min) to minimize the objective in (2). The Alt-min updates for $\mathbf{P}, \mathbf{X}$ are

$$\mathbf{P}^{(t)} = \underset{\mathbf{P}\in\Pi_n}{\operatorname{argmin}} \; \langle -\mathbf{Y}\mathbf{X}^{(t)\intercal}\mathbf{B}^\intercal, \mathbf{P}\rangle, \tag{3}$$

$$\mathbf{X}^{(t+1)} = \underset{\mathbf{X}}{\operatorname{argmin}} \; F(\mathbf{X},\mathbf{P}^{(t)}) = (\mathbf{P}^{(t)}\mathbf{B})^\dagger\mathbf{Y}. \tag{4}$$

When the unknown permutation $\mathbf{P}^*$ is r-local, the $\mathbf{P}$ update in (3) decouples along the blocks of $\mathbf{P}^*$ and reduces to the simpler updates in line 9 of Algorithm 1.

The optimization problem in (2) is non-convex as the set of permutation is a discrete (non-convex) set. Therefore, the initialization to the updates (3), (4) has to be chosen carefully. We propose and analyze initializations for both r-local and k-sparse models in the following sections. A key contribution of the proposed algorithm is that the model assumption is incorporated into the initialization.

A. Initialization for the r-local model

For r-local permutation $\mathbf{P}_r^*$, we propose to initialize $\widehat{\mathbf{X}}$ by the *collapsed* initialization (7). Let P_i denote each $r \times r$ block of $\mathbf{P}_r^* = \operatorname{diag}(P_1, \cdots, P_s)$. Let $B_i \in \mathbb{R}^{r\times d}$ denote blocks of the sensing matrix $\mathbf{B} = [B_1; \cdots; B_s]$. The r permuted measurements in each block $[Y_i : P_iB_i]$ can be summed to extract one labelled measurement on $\mathbf{x}^* \in \mathbb{R}^d$

$$\sum_{j=1}^{j=r}[P_iB_i(j)]^\intercal\mathbf{X}^* = \sum_{j=1}^{j=r} Y_i(j) \;\; \forall i \in [s]. \tag{5}$$

Assuming all blocks have the same size r, the number of blocks is $s = n/r$. The s labelled measurements in (5) are represented in the $s \times d$ *collapsed* linear system of equations

$$\widetilde{\mathbf{Y}} = \widetilde{\mathbf{B}}\mathbf{X}^*, \tag{6}$$

where each row $\tilde{\mathbf{b}}_i = \sum_{j=1}^{j=r} B_i(j) \; \forall i \in [s]$ is the sum of the r rows in block B_i of the collapsed matrix $\widetilde{\mathbf{B}} \in \mathbb{R}^{s\times d}$. The proposed initialization $\widehat{\mathbf{X}}$ is the minimum-norm solution to (6)

$$\widehat{\mathbf{X}} = \widetilde{\mathbf{B}}^\dagger\widetilde{\mathbf{Y}}, \tag{7}$$

Substituting the SVD form of $\widetilde{\mathbf{B}}^\dagger = \widetilde{\mathbf{V}}\mathbf{S}^{-1}\widetilde{\mathbf{U}}^\intercal$, where $\widetilde{\mathbf{B}} = \widetilde{\mathbf{U}}\mathbf{S}\widetilde{\mathbf{V}}^\intercal$, shows that $\widehat{\mathbf{X}}$ is the projection of $\mathbf{X}^*$ onto the the row space of $\widetilde{\mathbf{B}}$. Formally,

$$\|\mathbf{X}^* - \widehat{\mathbf{X}}\|_2 = \|(\mathbf{I} - \widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\intercal)\mathbf{X}^*\|_2. \tag{8}$$

For $s \geq d$, $\widehat{\mathbf{X}} = \mathbf{X}^*$. In section III-A, we upper bound the error in initialization $\widehat{\mathbf{X}}$ for the under-determined $s < d$ case.

B. Initialization for the k-sparse model

For the case of k-sparse permutation $\mathbf{P}_k^*$, we propose to initialize $\widehat{\mathbf{P}}^{(0)} = \mathbf{I}$, which sets $\widehat{\mathbf{Y}}^{(0)} = \mathbf{Y} = \mathbf{P}_k^*\mathbf{B}\mathbf{X}^*$. Since the permutation matrix $\mathbf{P}_k^*$ has k off-diagonal elements, the initialization $\widehat{\mathbf{P}}^{(0)} = \mathbf{I}$ is close to the true solution $\mathbf{P}_k^*$ as $d_H(\widehat{\mathbf{P}}^{(0)}, \mathbf{P}_k^*) = k$. The initialization error is upper bounded in section III-B.

III. Initialization analysis

A. r-local model

Assuming random models on either $\mathbf{B}$ or $\mathbf{X}^*$, the results in this section upper bound the error in the proposed initialization (7). Without any assumptions, the worst case error

$$\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 \leq \frac{\|\mathbf{y}\|_2^2 - \sigma_{\min}^2(\mathbf{B})\|\hat{\mathbf{x}}\|_2^2}{\sigma_{\min}^2(\mathbf{B})} \tag{9}$$

Algorithm 1 Alt-min

Input: mode $\in \{r\text{-local}, k\text{-sparse}\}$, convergence threshold ϵ

if mode is r-local **then**
  $\widehat{\mathbf{X}} \leftarrow$ collapsed initialization in (7)
  $\widehat{\mathbf{Y}} \leftarrow \mathbf{B}\widehat{\mathbf{X}}$
else
  $\widehat{\mathbf{Y}} \leftarrow \mathbf{Y}$
while relative change $> \epsilon$ **do**
  if mode is r-local **then**
    for $i \in 1 \cdots s$ **do** //s is the number of blocks
      $\widehat{\mathbf{P}}_i \leftarrow \operatorname{argmin}_{\mathbf{P}_i \in \Pi_r} -\langle \mathbf{Y}_i \widehat{\mathbf{Y}}_i^\intercal, \mathbf{P}_i \rangle$
    $\widehat{\mathbf{P}} \leftarrow \operatorname{diag}(\widehat{\mathbf{P}}_1, \cdots, \widehat{\mathbf{P}}_R)$
  else
    $\widehat{\mathbf{P}} = \operatorname{argmin}_{\mathbf{P} \in \Pi_n} -\langle \mathbf{Y}\widehat{\mathbf{Y}}^\intercal, \mathbf{P} \rangle$
  $\widehat{\mathbf{X}} \leftarrow \mathbf{B}^\dagger \widehat{\mathbf{P}}^\intercal \mathbf{Y}$
  $\widehat{\mathbf{Y}} \leftarrow \mathbf{B}\widehat{\mathbf{X}}$
Return $\widehat{\mathbf{P}}, \widehat{\mathbf{X}}$

is achieved for $\mathbf{x}^*$ aligned with the direction of the singular vector corresponding to the smallest singular value $\sigma_{\min}(\mathbf{B})$. Assuming entries of $\mathbf{x}^*$ are zero-mean, independent, sub-Gaussian random variables, Theorem 2.1 in [24] shows that the error $\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2$ is sub-Gaussian. In contrast, Theorem 1 in this work shows a sub-Gaussian error distribution for fixed $\mathbf{x}^*$ and Gaussian $\mathbf{B}$. Theorem 2 shows that the error is sub-Gaussian, assuming fixed $\mathbf{B}$ and sub-Gaussian $\mathbf{x}^*$. Particularly, our result in Theorem 2 does not assume that the entries of $\mathbf{x}^*$ are independently drawn.

Lemma 1. *Let $\hat{\mathbf{x}}$ be as defined in* (7)*. Let $\mathbf{x}^* \in \mathbb{R}^d$ be the fixed unknown vector, and let $\mathbf{B}$ be a Gaussian sensing matrix. For $t > 0$,*

$$\Pr_{\mathbf{B}}\left[\frac{\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2}{\|\mathbf{x}^*\|_2} \geq (1+t)\sqrt{\frac{d-s}{d}}\right] \leq 2\exp(-c't^2(d-s)). \tag{10}$$

The result in (10) confirms that the relative error decreases with increasing number of measurements s in the under-determined system. For example, let $s = 3d/4$, the probability that the relative error exceeds $1/2$ decays with a sub-Gaussian tail. The bound in (10) also sharp because it is the upper tail of the following two-sided inequality

$$\Pr_{\mathbf{B}}\left[\frac{|\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2|}{\|\mathbf{x}^*\|_2} \geq (1+t)\sqrt{\frac{d-s}{d}}\right] \leq 2\exp(-c't^2(d-s)).$$

Proof of Lemma 1. Since $\mathbf{B} \in \mathbb{R}^{n \times d}$ is assumed Gaussian, the scaled collapsed matrix $\widetilde{\mathbf{B}}/r$ (6) is also a Gaussian matrix. The singular vectors of a rectangular Gaussian matrix $\mathbf{M} \in \mathbb{R}^{p \times q}$, $p > q$, span a q-dimensional uniformly random subspace of $\mathbb{R}^p$, see Section 5.2.6 in [25]. Therefore, $\hat{\mathbf{x}} = (\mathbf{I} - \widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\intercal)\mathbf{x}^*$ is the projection of $\mathbf{x}^* \in \mathbb{R}^d$ onto a uniformly random $(d-s)$-dimensional subspace. The result in eq. (10) follows from invoking the Johonson-Lindenstrauss (JL) Lemma (66) with $p = d$ and $q = d - s$.

Theorem 1. *Let $\widehat{\mathbf{X}}$ be as defined in* (7)*. Let $\mathbf{X}^* \in \mathbb{R}^{d \times m}$ be the fixed unknown matrix, and let $\mathbf{B}$ be a Gaussian sensing matrix. For $\log m \leq c't^2(d-s)/2$,*

$$\Pr_{\mathbf{B}}\left[\frac{\|\mathbf{X}^* - \widehat{\mathbf{X}}\|_F}{\|\mathbf{X}^*\|_F} \geq (1+t)\sqrt{\frac{d-s}{d}}\right] \leq 2\exp(-c't^2(d-s)/2). \tag{11}$$

Proof of Theorem 1. Let event $\mathcal{E}$ be defined as the union of events $\mathcal{E}_i$

$$\mathcal{E} = \bigcup_{i=1}^{i=m} \mathcal{E}_i, \text{where}$$

$$\mathcal{E}_i = \{\mathbf{x}^* \mid \|\mathbf{x}_i^* - \hat{\mathbf{x}}_i\|_2^2 \geq \frac{(1+t)^2(d-s)}{d}\|\mathbf{x}_i^*\|_2^2\}. \tag{12}$$

Then

$$= \Pr[\|\mathbf{X}^* - \widehat{\mathbf{X}}\|_F^2 \leq \frac{(1+t)^2(d-s)}{d}\|\mathbf{X}^*\|_F^2] \tag{13}$$

$$= \Pr[\sum_{i=1}^{i=m}\|\mathbf{x}_i^* - \hat{\mathbf{x}}_i\|_2^2 \leq \frac{(1+t)^2(d-s)}{d}\sum_{i=1}^{i=m}\|\mathbf{x}_i^*\|_2^2] \tag{14}$$

$$\geq 1 - \sum_{i=1}^{i=m}\Pr[\mathcal{E}_i] \tag{15}$$

$$\geq 1 - 2\exp(-c't^2(d-s)/2). \tag{16}$$

(14) substitutes $\|\mathbf{X}^*\|_F^2 = \sum_{i=1}^{i=m}\|\mathbf{x}_i^*\|_2^2$. (15) follows from noting that $\Pr[\mathcal{E}^c] = 1 - \Pr[\mathcal{E}]$, and applying the union bound to $\mathcal{E}$, defined in (12). (16) is by substituting (10) and bounding $m\exp(-c't^2(d-s)/2) \leq 1$ for $\log m \leq c't^2(d-s)/2$.

Lemma 2. *Let $\hat{\mathbf{x}}$ be as defined in* (7)*. Let $\mathbf{B}$ be a fixed sensing matrix, and let $\mathbf{x}^* \sim \mathcal{N}(0, \mathbf{I})$ be a zero-mean isotropic Gaussian random vector. For $t \geq 0$,*

$$\Pr_{\mathbf{x}^*}\left[\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 \geq c + 2t(\sqrt{d-s}+1)\right] \leq \exp(-t), \tag{17}$$

where $c = K^2(d - s + \frac{1}{2}\sqrt{d-s})$, and K is as given in (32)*.*

Proof of Lemma 2. The result in (17) follows from applying the Hanson-Wright inequality (65) to (8).

To derive the result in Theorem 2, it is necessary to show that $\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2$ is a sub-exponential random variable. A random variable X is sub-exponential if for $t \geq 0, K_1 > 0$

$$X \sim \text{subExp} \iff \Pr[|X| \geq t] \leq c'\exp(-t/K_1), \tag{18}$$

where $c' \geq 1$. It cannot be concluded that the shifted random variable $\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c$ is sub-exponential because the lower tail probability $\Pr[\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c \leq -t], t > 0$ is not upper bounded. However, our goal is to upper bound the probability that the error *exceeds* a certain value. We therefore define the non-negative random variable $\tilde{\mathbf{e}} - c$, which upper bounds the lower tail as $\{\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c \leq -t, t > 0\} \leq 0$, (19). The definition of $\tilde{\mathbf{e}} - c$ is given in (19), (20). Fig. 2 is a visual description of the definition. We show that $\tilde{\mathbf{e}} - c$ is a sub-exponential random variable in Lemma 3.

Lemma 3. *The random variable $\tilde{\mathbf{e}} - c$, defined in* (19)*,* (20)*, is a sub-exponential random variable*

$$\tilde{\mathbf{e}} - c \mid \mathcal{E}_1 \triangleq 0, \tag{19}$$

$$\tilde{\mathbf{e}} - c \mid \mathcal{E}_2 \triangleq \|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c, \tag{20}$$

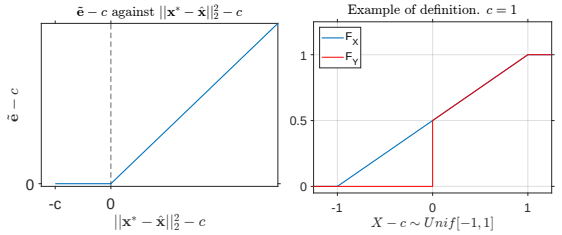


Fig. 2: **Left.** The derived random variable (rv) $\tilde{\mathbf{e}} - c$ is plotted against the error rv $\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c$. Note that i) $\tilde{\mathbf{e}} - c \geq 0$, ii) $\tilde{\mathbf{e}} - c \geq \|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c$, iii) $\tilde{\mathbf{e}} - c \mid \mathcal{E}_1 = 0$, and iv) $\tilde{\mathbf{e}} - c \mid \mathcal{E}_2 = \|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c$. **Right.** Example. The cdfs of $Y(\tilde{\mathbf{e}} - c)$ and $X(\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c) \sim \text{Unif}[-1, 1]$ are plotted.

where events $\mathcal{E}_1, \mathcal{E}_2$ are defined as

$$\mathcal{E}_1 = \{\mathbf{x}^* \mid \|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c \leq 0\}, \tag{21}$$
$$\mathcal{E}_2 = \mathcal{E}_1^c = \{\mathbf{x}^* \mid \|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c > 0\}. \tag{22}$$

Moreover, for $t > 0$

$$\Pr\left[\tilde{\mathbf{e}} - c \geq t\right] = \Pr\left[\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c \geq t\right]. \tag{23}$$

Proof of Lemma 3. We first prove the claim in (23). For $t > 0$,

$$\begin{aligned}
&\Pr[\tilde{\mathbf{e}} - c \geq t] \\
&= \Pr[\tilde{\mathbf{e}} - c \geq t \mid \mathcal{E}_1] \cdot \Pr[\mathcal{E}_1] + \Pr[\tilde{\mathbf{e}} - c \geq t \mid \mathcal{E}_2] \cdot \Pr[\mathcal{E}_2] \\
&= \Pr[\tilde{\mathbf{e}} - c \geq t \mid \mathcal{E}_2] \cdot \Pr[\mathcal{E}_2] \qquad (24) \\
&= \Pr[\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c \geq t \mid \mathcal{E}_2] \cdot \Pr[\mathcal{E}_2] \qquad \text{(By eq. (20))} \\
&= \Pr[\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c \geq t], \qquad (25)
\end{aligned}$$

where (24) follows from $\Pr[\tilde{\mathbf{e}} - c \geq t \mid \mathcal{E}_1] = 0 \, \forall t > 0$. (25) follows from

$$\Pr[\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c \geq t \mid \mathcal{E}_2] \Pr[\mathcal{E}_2] = \Pr[\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c \geq t \cap \mathcal{E}_2],$$

and that the event $\{\mathbf{x}^* \mid \|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c \geq t, t > 0\}$ is a subset of $\mathcal{E}_2$. The claim in (23) is shown in (25). From (17), for $t \geq 0$

$$\Pr\left[\frac{\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c}{2(\sqrt{d-s}+1)} \geq t\right] \leq \exp(-t). \tag{26}$$

From (23), for $t > 0$

$$\Pr\left[\frac{\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c}{2(\sqrt{d-s}+1)} \geq t\right] = \Pr\left[\frac{\tilde{\mathbf{e}} - c}{2(\sqrt{d-s}+1)} \geq t\right], \tag{27}$$

where $\sqrt{d-s} + 1 > 0$ is a positive number since $s < d$, see (8). From (19) and (20), $\tilde{\mathbf{e}} - c$ is non-negative so that $\tilde{\mathbf{e}} - c = |\tilde{\mathbf{e}} - c|$,

$$\Pr\left[\frac{\tilde{\mathbf{e}} - c}{2(\sqrt{d-s}+1)} \geq t\right] = \Pr\left[\frac{|\tilde{\mathbf{e}} - c|}{2(\sqrt{d-s}+1)} \geq t\right]. \tag{28}$$

Combining (27) and (28), for $t > 0$

$$\Pr\left[\frac{|\tilde{\mathbf{e}} - c|}{2(\sqrt{d-s}+1)} \geq t\right] = \Pr\left[\frac{\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2^2 - c}{2(\sqrt{d-s}+1)} \geq t\right]. \tag{29}$$

Substituting (26) in (29), for $t > 0$,

$$\Pr\left[\frac{|\tilde{\mathbf{e}} - c|}{2(\sqrt{d-s}+1)} \geq t\right] \leq \exp(-t) \quad \forall t > 0. \tag{30}$$

Since $\exp(0) = 1$, (30) also holds at $t = 0$

$$\Pr\left[\frac{|\tilde{\mathbf{e}} - c|}{2(\sqrt{d-s}+1)} \geq t\right] \leq \exp(-t) \quad \forall t \geq 0. \tag{31}$$

In (31), we have verified the definition in (18) for $\tilde{\mathbf{e}} - c$ with $K_1 = 2(\sqrt{d-s}+1)$.

Theorem 2. *Let $\mathbf{X}^* \in \mathbb{R}^{d \times m}$ be the unknown matrix such that the columns $\mathbf{x}_i^*$ are identical, independent, zero-mean sub-Gaussian random vectors, i.e., for $\alpha \in \mathbb{R}^d$, $K \geq 0$*

$$\mathbb{E}[\exp(\alpha^\intercal \mathbf{x}_i^*)] \leq \exp(\|\alpha\|_2^2 K^2/2). \tag{32}$$

Let $\widehat{\mathbf{X}}$ be as defined in (7). Let $\mathbf{B}$ be a fixed sensing matrix. For $t \geq 0$,

$$\Pr\Big[\sum_{i=1}^{i=m} \|\mathbf{x}_i^* - \hat{\mathbf{x}}_i\|_2 - mc_2 \geq t\Big] \leq 2\exp\Big(\frac{-c't^2}{m(d-s)}\Big), \tag{33}$$

where $c_2 = (K^2(d-s) + \frac{K^2+1}{2}\sqrt{d-s}+2)^{\frac{1}{2}}$ and c' is an absolute constant.

Proof of Theorem 2. Since $\tilde{\mathbf{e}} - c$ is a sub-exponential random variable (Lemma 3), by (67), $\sqrt{\tilde{\mathbf{e}} - c}$ is a sub-Gaussian random variable. To apply Hoeffding's inequality to the sum $\sum_i \sqrt{\tilde{\mathbf{e}}_i}$, we first upper bound the expectation $\mathbb{E}[\sqrt{\tilde{\mathbf{e}}}]$ and the sub-Gaussian norm $\|\sqrt{\tilde{\mathbf{e}}}\|_{\varphi_2}$. For non-negative random variable X, $\mathbb{E}[X] = \int_0^\infty \Pr[X > t]dt$. Applied to $\tilde{e} - c/2(\sqrt{d-s}+1)$,

$$\mathbb{E}\left[\frac{\tilde{\mathbf{e}} - c}{2(\sqrt{d-s}+1)}\right] \leq \int_0^\infty \Pr\left[\frac{\tilde{\mathbf{e}} - c}{2(\sqrt{d-s}+1)} \geq t\right] dt \tag{34}$$
$$\leq \int_0^\infty \exp(-t)dt = 1. \tag{35}$$

In (35), we have substituted the tail probability upper bounds from (31). By (35), and substituting c from (17), the expectation is upper bounded as

$$\mathbb{E}[\tilde{\mathbf{e}}] \leq 2(\sqrt{d-s}+1) + c = K^2(d-s) + \frac{K^2+1}{2}\sqrt{d-s} + 2. \tag{36}$$

By Jensen's inequality $\mathbb{E}[\sqrt{\tilde{\mathbf{e}}}] \leq \sqrt{\mathbb{E}[\tilde{\mathbf{e}}]}$, and therefore

$$\mathbb{E}[\sqrt{\tilde{\mathbf{e}}}] \leq (K^2(d-s) + \frac{K^2+1}{2}\sqrt{d-s}+2)^{1/2}. \tag{37}$$

For sub-Gaussian X, $\|X\|_{\varphi_2} \leq c'\mathbb{E}[X]$ (Proposition 2.7.1 (ii), (iv) [25]). By (67), the sub-Gaussian norm squared is upper bounded as

$$\begin{aligned}
\|\tilde{\mathbf{e}}\|_{\varphi_2}^2 = \|\tilde{\mathbf{e}}\|_{\varphi_1} &\leq c'(K^2(d-s) + \frac{K^2+1}{2}\sqrt{d-s}+2)^{1/2} \\
&\leq c'K\sqrt{d-s}. \qquad (38)
\end{aligned}$$

Given the upper bound on the sub-Gaussian norm in (38), we apply Hoeffding's inequality (68) to the sum $\sum_i \sqrt{\tilde{\mathbf{e}}_i}$

$$\Pr[\sum_{i=1}^{i=m}(\sqrt{\tilde{\mathbf{e}}_i} - \mathbb{E}[\sqrt{\tilde{\mathbf{e}}_i}]) \geq t] \leq 2\exp(\frac{-c't^2}{mK\sqrt{d-s}}), \tag{39}$$

where c' is an absolute constant. Let c_2 denote

$$c_2 = (K^2(d-s) + \frac{K^2+1}{2}\sqrt{d-s}+2)^{1/2}. \tag{40}$$

Substituting(37) into (39)

$$\Pr\big[\sum_{i=1}^{i=m}\sqrt{\tilde{\mathbf{e}}_i} - mc_2 \geq t\big] \leq 2\exp\big(\frac{-c't^2}{mK\sqrt{d-s}}\big). \quad (41)$$

By definition in (19), (20) (also see Fig. 2), $\|\mathbf{x}_i^* - \hat{\mathbf{x}}_i\|_2 \leq \sqrt{\tilde{\mathbf{e}}_i}\ \forall\ i \in [m]$. Therefore,

$$\Pr[\sum_{i=1}^{i=m}\|\mathbf{x}_i^* - \hat{\mathbf{x}}_i\|_2 - mc_2 \geq t] \leq \Pr[\sum_{i=1}^{i=m}\sqrt{\tilde{\mathbf{e}}_i} - mc_2 \geq t] \leq 2\exp(\frac{-c't^2}{mK\sqrt{d-s}}). \quad (42)$$

In (42), we have substituted the bound in (41). We conclude with the final result

$$\Pr[\sum_{i=1}^{i=m}\|\mathbf{x}_i^* - \hat{\mathbf{x}}_i\|_2 - mc_2 \geq t] \leq 2\exp(\frac{-c't^2}{mK\sqrt{d-s}}). \quad (43)$$

Remark. Assuming Gaussian $\mathbf{B}$, Theorem 1 shows the probability that the relative error in (7) exceeds $\sqrt{(d-s)/d}$ decays at sub-Gaussian rate. Assuming sub-Gaussian $\mathbf{x}^*$ instead, Theorem 2 shows the probability that the error exceeds $(K^2(d-s)+\frac{K^2+1}{2}\sqrt{d-s}+2)^{\frac{1}{2}}$ also decays at a sub-Gaussian rate. Both results agree that the estimate (7) improves as the number of measurements s in the under-determined system (6) increases. The quantity K in (33) depends linearly on the sub-Gaussian norm of $\mathbf{x}^*$ and can be estimated in practice. For $\mathbf{x}^* \sim \mathcal{N}(0, \mathbf{I})$, $K = 1$. For $\mathbf{x}^*$ uniformly distributed on the Euclidean ball, $K = c'$, where c' is an absolute constant.

B. k-sparse model

For the standard (un-permuted) linear inverse problem $\mathbf{y} = \mathbf{B}\mathbf{x}^* + \mathbf{w}$, the error in the solution $\|\mathbf{x}^* - \hat{\mathbf{x}}\|_2$ is upper bounded in terms of the noise $\|\mathbf{w}\|_2$ and $\sigma_{\min}(\mathbf{B})$, the smallest singular value of $\mathbf{B}$. In this section, we bound the error in the proposed initialization for the k-sparse permutation model in terms of the value of the objective function F and $\sigma_{\min}(\mathbf{B})$.

Lemma 4. *Let $\mathbf{P}_k^* \in \mathbb{R}^{n\times n}$ be the fixed unknown permutation matrix. Let $\mathbf{x}^*$ be the fixed unknown vector. Assuming Gaussian $\mathbf{B}$, let $\mathbf{y}^* = \mathbf{B}\mathbf{x}^*$, $\hat{\mathbf{y}}^{(0)} = \mathbf{P}_k^*\mathbf{B}\mathbf{x}^*$. For $k = 1, \cdots, n-1$, and $t \geq 0$,*

$$\Pr\big[\|\mathbf{y}^* - \hat{\mathbf{y}}^{(0)}\|_2^2 \geq 2\|\mathbf{y}^*\|_2^2 - 2\|\mathbf{x}^*\|_2^2\big(n - k - c_1\sqrt{t} + 6t\big)\big] \leq 7\exp(-t), \quad (44)$$

where $c_1 = 2\sqrt{n-k} + 4\sqrt{3k}$.

Proof of Lemma 4. Under the k-sparse assumption on $\mathbf{P}^*$, the known vector $\mathbf{y}$ has k shuffled entries. For $\widehat{\mathbf{P}}^{(0)} = \mathbf{I}$, *i.e.*, $\hat{\mathbf{y}}^{(0)} = \mathbf{y}$, the forward error is

$$\|\mathbf{y}^* - \hat{\mathbf{y}}^{(0)}\|_2^2 = 2\|\mathbf{y}^*\|_2^2 - 2\langle\mathbf{y}^*, \hat{\mathbf{y}}^{(0)}\rangle, \quad (45)$$

where $\mathbf{y}^* = \mathbf{B}\mathbf{x}^*$ is the unknown vector of unshuffled measurements. Since $\|\mathbf{y}^*\|_2 = \|\hat{\mathbf{y}}^{(0)}\|_2$, and initialization $\hat{\mathbf{y}}^{(0)} = \mathbf{P}_k^*\mathbf{B}\mathbf{x}^*$ is known, the only unknown term in (45) is the inner product $\langle\mathbf{y}^*, \hat{\mathbf{y}}^{(0)}\rangle$. The scaled inner product $\langle\mathbf{y}^*, \hat{\mathbf{y}}^{(0)}\rangle/\|\mathbf{x}^*\|_2^2$ is expanded as

$$\frac{\langle\mathbf{y}^*, \hat{\mathbf{y}}^{(0)}\rangle}{\|\mathbf{x}^*\|_2^2} = \frac{1}{\|\mathbf{x}^*\|_2^2}\sum_{i=1}^{i=n}\mathbf{b}_i^\intercal\mathbf{x}^*\mathbf{b}_{\mathbf{P}^*(i)}^\intercal\mathbf{x}^* \quad (46)$$
$$= \underbrace{\frac{1}{\|\mathbf{x}^*\|_2^2}\sum_{i=1}^{i=n-k}(\mathbf{b}_i^\intercal\mathbf{x}^*)^2}_{\triangleq T_1} + \underbrace{\frac{1}{\|\mathbf{x}^*\|_2^2}\sum_{i=n-k+1}^{n}\mathbf{b}_i^\intercal\mathbf{x}^*\mathbf{b}_{\mathbf{P}^*(i)}^\intercal\mathbf{x}^*}_{\triangleq T_2}.$$

We have assumed in (46), *wlog*, that the last k rows of $\mathbf{P}_k^*$ are shuffled. Assuming Gaussian $\mathbf{B}$, $(\mathbf{b}_i^\intercal\mathbf{x}^*/\|\mathbf{x}^*\|_2)^2 \sim \chi^2$ is Chi-squared distributed with 1 degree of freedom. T_1, defined in (46), is the sum of $n-k$ Chi-square random variables, and is bounded, using (71), as

$$\Pr\big[T_1 \leq n - k - 2\sqrt{(n-k)t}\big] \leq \exp(-t). \quad (47)$$

The product random variables in T_2 are distributed as the difference of two independent χ^2 random variables Z_i^1, Z_i^2

$$\frac{\mathbf{b}_i^\intercal\mathbf{x}^*\mathbf{b}_{\mathbf{P}(i)}^\intercal\mathbf{x}^*}{\|\mathbf{x}^*\|_2^2} \sim \frac{1}{2}Z_i^1 - \frac{1}{2}Z_i^2\ \ \forall i \in k+1, \cdots, n. \quad (48)$$

The random variables (rv) in (48) are not mutually independent, but each rv depends on, at most, two rvs. To see this, let permutation $\mathbf{P}$ such that $\mathbf{P}(i) \mapsto j$, then $\mathbf{b}_i^\intercal\mathbf{x}^*\mathbf{b}_j^\intercal\mathbf{x}^*$ is not independent of

$$\mathbf{b}_j^\intercal\mathbf{x}^*\mathbf{b}_{\mathbf{P}(j)}^\intercal\mathbf{x}^*,\ \ \mathbf{b}_{\mathbf{P}^\intercal(i)}^\intercal\mathbf{x}^*\mathbf{b}_i^\intercal\mathbf{x}^*. \quad (49)$$

The k rvs in (48) can therefore be partitioned into three sets P, Q, R such that the rvs within each set are independent. Let k_1 be the number of rvs in set P. The sum T_P, defined as

$$T_P \triangleq \sum_{i\in P}\mathbf{b}_i^\intercal\mathbf{x}^*\mathbf{b}_{\mathbf{P}(i)}^\intercal\mathbf{x}^* = \sum_{i\in P}Z_i^1 - \sum_{i\in P}Z_i^2, \quad (50)$$

is upper bounded in probability as

$$\Pr[T_P \geq 4\sqrt{k_1 t} + 2t] \leq 2\exp(-t). \quad (51)$$

(51) follows from applying the union bound to probabilities p_1, p_2.

$$p_1 = \Pr\big[\sum_{i\in P}Z_i^1 \leq k_1 - 2\sqrt{k_1 t}\big] \leq \exp(-t), \quad (52)$$

$$p_2 = \Pr\big[\sum_{i\in P}Z_i^2 \geq k_1 + 2\sqrt{k_1 t} + 2t\big] \leq \exp(-t), \quad (53)$$

and bounding p_1, p_2 using tail inequalities (71), (70), respectively. Defining T_Q, T_R similarly to (50), and applying the union bound as in (52), (53) gives

$$\Pr\big[T_2 \geq 4(\sqrt{k_1 t} + \sqrt{k_2 t} + \sqrt{k_3 t}) + 6t\big] \leq 6\exp(-t), \quad (54)$$

where $T_2 = T_P + T_Q + T_R$. Since $k_1 + k_2 + k_3 = k$, and $\sqrt{k_1} + \sqrt{k_2} + \sqrt{k_3} \leq \sqrt{3k}$,

$$\Pr\big[T_2 \geq 4\sqrt{3kt} + 6t\big] \leq 6\exp(-t). \quad (55)$$

Applying the union bound to (47), (55) gives the result in (44).

Lemma 5. *For the same assumptions as in Lemma 4, $k = 1, \cdots, n-1$, and $t \geq 0$,*

$$\Pr\big[\sigma^2_{\min}(\mathbf{B})\|\mathbf{x}^* - \hat{\mathbf{x}}^{(1)}\|_2^2 \geq 2\|\mathbf{y}^*\|_2^2 - 2\|\mathbf{x}^*\|_2^2\big(n-k-c_1\sqrt{t}-6t\big) - F^{(1)}\big] \leq 7e^{-t}, \tag{56}$$

where $\hat{\mathbf{x}}^{(1)} = \mathbf{B}^\dagger\hat{\mathbf{y}}^{(0)}$, $F^{(1)} = \|\mathbf{y} - \mathbf{B}\hat{\mathbf{x}}^{(1)}\|_2^2$, and $c_1 = 2\sqrt{n-k} + 4\sqrt{3k}$.

Proof of Lemma 5. Let $\mathbf{e}$ denote the error term such that

$$\hat{\mathbf{y}}^{(0)} = \mathbf{B}\hat{\mathbf{x}}^{(1)} + \mathbf{e}, \tag{57}$$

where $\hat{\mathbf{x}}^{(1)} = \mathbf{B}^\dagger\hat{\mathbf{y}}^{(0)}$, $\hat{\mathbf{y}}^{(0)} = \mathbf{P}_k^*\mathbf{B}\mathbf{x}^*$, and $\mathbf{e}$ is orthogonal to the range space of $\mathbf{B}$, i.e., $\mathbf{e} \perp \mathcal{R}(\mathbf{B})$. Substituting (57) and using Pythogras' theorem below

$$\begin{aligned}&\|\mathbf{y}^* - \hat{\mathbf{y}}^{(0)}\|_2^2 \triangleq \|\mathbf{B}\mathbf{x}^* - \mathbf{P}_k^*\mathbf{B}\mathbf{x}^*\|_2^2\\ =&\|\mathbf{B}(\mathbf{x}^* - \hat{\mathbf{x}}^{(1)}) - \mathbf{e}\|_2^2 = \|\mathbf{B}(\mathbf{x}^* - \hat{\mathbf{x}}^{(1)})\|_2^2 + \|\mathbf{e}\|_2^2.\end{aligned} \tag{58}$$

Substituting the lower bound $\sigma^2_{\min}(\mathbf{B})\|\mathbf{x}^* - \hat{\mathbf{x}}^{(1)}\|_2^2 \leq \|\mathbf{B}(\mathbf{x}^* - \hat{\mathbf{x}}^{(1)})\|_2^2$ in (58),

$$\sigma^2_{\min}(\mathbf{B})\|\mathbf{x}^* - \hat{\mathbf{x}}^{(1)}\|_2^2 \leq \|\mathbf{y}^* - \hat{\mathbf{y}}^{(0)}\|_2^2 - \|\mathbf{e}\|_2^2. \tag{59}$$

(60) follows by substituting the probability upper bound for $\|\mathbf{y}^* - \hat{\mathbf{y}}^{(0)}\|_2^2$ from (44) into (59), and noting that $\|\mathbf{e}\|_2^2 = F^{(1)} = F(\widehat{\mathbf{P}}^{(0)}, \hat{\mathbf{x}}^{(1)})$.

Theorem 3. *Let $\mathbf{P}_k^* \in \mathbb{R}^{n\times n}$ be the fixed unknown permutation matrix. Assuming Gaussian $\mathbf{B}$, let $\mathbf{X}^*$ be the fixed unknown matrix. Let $\mathbf{Y}^* = \mathbf{B}\mathbf{X}^*$, and $\widehat{\mathbf{Y}}^{(0)} = \mathbf{P}_k^*\mathbf{B}\mathbf{X}^*$. For $k = 1, \cdots, n-1$, and $t \geq \log m^2$,*

$$\Pr\big[\sigma^2_{\min}(\mathbf{B})\|\mathbf{X}^* - \widehat{\mathbf{X}}^{(1)}\|_F^2 \geq 2\|\mathbf{Y}^*\|_F^2 - 2\|\mathbf{X}^*\|_F^2\big(n-k-c_1\sqrt{t}-6t\big) - F^{(1)}\big] \leq 7e^{-t}, \tag{60}$$

where $\widehat{\mathbf{X}}^{(1)} = \mathbf{B}^\dagger\widehat{\mathbf{Y}}^{(0)}$, $F^{(1)} = \|\mathbf{Y} - \mathbf{B}\widehat{\mathbf{X}}^{(1)}\|_F^2$, and $c_1 = 2\sqrt{n-k} + 4\sqrt{3k}$.

An immediate upper bound that does not depend on the number of diagonal entries of $\mathbf{P}_k^*$ is

$$\sigma^2_{\min}(\mathbf{B})\|\mathbf{X}^* - \widehat{\mathbf{X}}^{(1)}\|_F^2 \leq 4\|\mathbf{Y}^*\|_F^2 - F^{(1)}. \tag{61}$$

Instead, (60) shows that the probability that the error exceeds $\|\mathbf{Y}^*\|_F^2 - c$, $c = \Omega(n-k)$, and $n-k$ is the number of diagonal entries in $\mathbf{P}_k^*$, decays exponentially. Furthermore, the term $\|\mathbf{Y}^*\|_F^2$ is known, and $\|\mathbf{X}^*\|_F$ can be lower bounded as $\|\mathbf{X}^*\|_F^2 \leq \|\mathbf{Y}^*\|_F^2/\sigma^2_{\max}(\mathbf{B})$.

Proof of Theorem 3. The proof follows by applying the union bound to (56).

IV. Results

Benchmarks. We compare against six benchmark methods. We refer to these methods as 'ℓ_2-regularized' [5],'ds+' [5], 'Spectral' [17], 'Biconvex' [4], 'RLUS' [10], and 'Stochastic Alt-min' [15]. The 'ℓ_2-regularized' method considers the k-sparse permutation model. The method incorporates the model assumption by imposing a row-wise group sparse penalty on $\|\widehat{\mathbf{Y}}_{i,:}\|_2$, where $\widehat{\mathbf{Y}} = \mathbf{B}\widehat{\mathbf{X}}$. The other benchmark methods are discussed in the following paragraphs.

Figs. 3a, 3b. We compare the proposed algorithm for the r-local and k-sparse permutation models to benchmarks. To adapt the 'Spectral', 'ℓ_2-regularized', and 'Biconvex' methods to the r-local model, we add a constraint enforcing the permutation estimates to be r-local. The results show that the proposed algorithm recovers $\mathbf{P}^*$ with decreasing block size r and number of shuffles k. This observation confirms the conclusions of Theorems 1,2,3 as the initialization to the algorithm improves with lower values of r and k. The proposed algorithm is also the only algorithm applicable to both models.

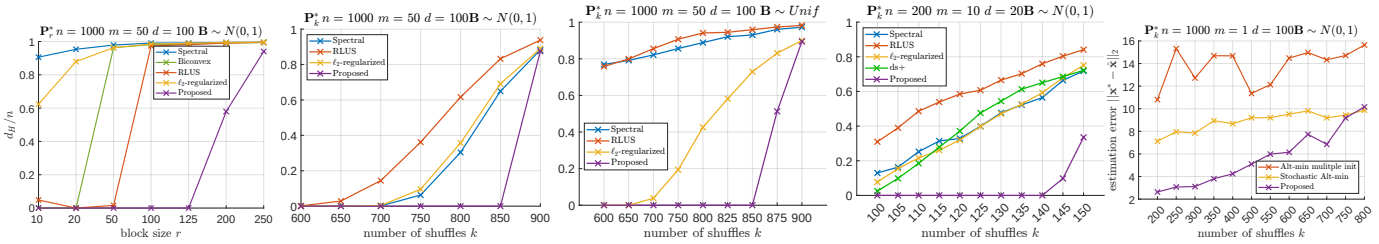
Fig. 3c. The entries of the design matrix $\mathbf{B}$ are sampled i.i.d. from the uniform $[0, 1]$ distribution. Compared to the case for Gaussian $\mathbf{B}$ (Fig. 3b), the performance of the 'Spectral' and 'RLUS' methods deteriorates significantly. This is because both algorithms consider quadratic measurements $\mathbf{Y}\mathbf{Y}^\intercal$. Specifically, the 'Spectral' method is based on the spectral initialization technique [26] which assumes that the design matrix is Gaussian. In contrast, the performance of the proposed algorithm and the 'ℓ_2-regularized' method does not deteriorate.

Fig. 3d. The 'ds+' algorithm [5] considers the convex relaxation of (2) by minimizing the objective over the set of doubly stochastic matrices. Assuming an upper bound on the number of shuffles k is known, 'ds+' also constrains $\langle \mathbf{I}, \mathbf{P}\rangle \geq n-k$. Each iteration of 'ds+' minimizes a linear program, which greatly increases the run-time of the algorithm. The results show the proposed algorithm outperforms 'ds+', possibly because the proposed algorithm optimizes the objective directly over the set of permutations.

Fig. 3e. We compare to the method in [15] which considers the $m = 1$ single-view setup and proposes stochastic alternating minimization (S.alt-min) to optimize (2). The runtime for S.alt-min is 50 times the run-time of the proposed algorithm because S.alt-min updates $\mathbf{P}$ multiple times in each iteration and retains the best update. [15] also proposes alt-min with multiple initializations for $\widehat{\mathbf{P}}^{(0)}$. The results in Fig. 3e show that the proposed algorithm (alt-min with $\widehat{\mathbf{P}}^{(0)} = \mathbf{I}$ initialization), outperforms both S.alt-min and alt-min with multiple initializations.

V. Conclusion

In this paper, we propose and analyze the alternating minimization algorithm with structured initialization as applied to the unlabeled sensing problem. Under randomness on either the measurement matrix or the underlying signal, we upper bound the initialization error for k-sparse and r-local permutation models. Compared to competing baselines, numerical experiments show that the algorithm is scalable and attains superior performance in recovering the underlying permutation and signal. As part of future work, we plan to i) characterize the rate of convergence, and ii) analyze the second-order convergence properties of the proposed algorithm.



(a) (b) (c) (d) (e)

Fig. 3: **Synthetic simulations.** $\mathbf{Y} = \mathbf{P}^*\mathbf{B}_{n\times d}\mathbf{X}^*_{d\times m} + \mathbf{W}$. The entries of $\mathbf{X}^*$ are drawn from the normal distribution. The permutation matrix $\mathbf{P}_r^*(\mathbf{P}_k^*)$ is sampled uniformly from the set of permutations $\Pi_r(\Pi_n)$. The results are averaged over 15 Monte-Carlo runs. (a). Figure plots the fractional Hamming distortion d_H/n against block size r. (b), (c). Figure plots d_H/n against number of shuffles k. (d). Figure plots the estimation error against k.

VI. Appendix

Lemma 5 in [3]. *For $1 < d < n$, any two permutation matrices $\widehat{\mathbf{P}}$,$\mathbf{P}^*$ such that $d_H(\widehat{\mathbf{P}}, \mathbf{P}^*) = h$ and $t \geq hn^{-\frac{2n}{n-d}}$,*

$$\Pr_{\mathbf{B}}[\|\mathbf{y} - \widehat{\mathbf{P}}\mathbf{B}\hat{\mathbf{x}}\|_2^2 \leq t\|\mathbf{x}^*\|_2^2] \leq 2\max\big(\exp(-n\log\frac{n}{2}),$$
$$6\exp\big(\frac{-h}{10}\big[\log\big(\frac{h}{tn^{\frac{2n}{n-d}}}\big) + \frac{tn^{\frac{2n}{n-d}}}{h} - 1\big]\big)\big), \tag{62}$$

where $\hat{\mathbf{x}} = (\widehat{\mathbf{P}}\mathbf{B})^\dagger\mathbf{y}$ and $\mathbf{y} = \mathbf{P}^*\mathbf{B}\mathbf{x}^*$.

The result in (62) upper bounds, in terms of the Hamming distortion $h = d_H(\widehat{\mathbf{P}}, \mathbf{P}^*)$, the probability that the forward error for the least squares solution $\hat{\mathbf{x}} = (\widehat{\mathbf{P}}\mathbf{B})^\dagger\mathbf{y}$ is less than $t\|\mathbf{x}^*\|_2^2$. Let $i^* = \underset{i\in[m]}{\operatorname{argmin}} \|\mathbf{Y}_{:,i} - \widehat{\mathbf{P}}\mathbf{B}\widehat{\mathbf{X}}_{:,i}\|_2^2$ and t^* such that $F(\widehat{\mathbf{X}}_{:,i}, \widehat{\mathbf{P}}) = t^*\|\mathbf{X}^*_{:,i^*}\|_2^2$. Then

$$\Pr_{\mathbf{B}}[d_H(\widehat{\mathbf{P}}, \mathbf{P}^*) \geq h] \leq 2\max\big(\exp(-n\log\frac{n}{2}),$$
$$6\exp\big(\frac{-h}{10}\big[\log\big(\frac{h}{tn^{\frac{2n}{n-d}}}\big) + \frac{t^*n^{\frac{2n}{n-d}}}{h} - 1\big]\big)\big). \tag{63}$$

Hanson Wright Inequality. *From Theorem 2.1 in [27], let $\Sigma = \mathbf{A}^\intercal\mathbf{A}$ be a positive semi-definite matrix. Let $\mathbf{x} = (x_1, \cdots, x_d)$ be a zero-mean sub-Gaussian random vector. i.e., for $\alpha \in \mathbb{R}^d$, $K \geq 0$*

$$\mathbb{E}[\exp(\alpha^\intercal\mathbf{x}^*)] \leq \exp(\|\alpha\|_2^2K^2/2). \tag{64}$$

For $t \geq 0$,

$$\Pr[\|\mathbf{A}\mathbf{x}\|_2^2 \geq K^2(tr(\Sigma) + 2\sqrt{tr(\Sigma^2)t} + 2t\|\Sigma\|)] \leq e^{-t}. \tag{65}$$

To derive the result in (17), set $\mathbf{A} = (\mathbf{I} - \widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^\intercal)$, as in (8). Since $\mathbf{A} \in \mathbb{R}^{d\times d}$ is a $(d-s)$-dimensional projection matrix, the matrix $\Sigma = \mathbf{A}^\intercal\mathbf{A} = \mathbf{A}\mathbf{A} = \mathbf{A}$ is also a projection matrix, $tr(\Sigma) = tr(\Sigma^2) = tr(\mathbf{A}) = d - s$ and $\|\Sigma\| = 1$, and upper bounding $2\sqrt{tr(\Sigma^2)t} \leq 2t\sqrt{tr(\Sigma^2)} + \frac{1}{2}\sqrt{tr(\Sigma^2)}$.

Johnson Lindenstrauss Lemma. *Let P be a projection in $\mathbb{R}^p$ onto a uniformly distributed random q-dimensional subspace. Let $z \in \mathbb{R}^n$ be a fixed point and $t > 0$. Then, with probability at least $1 - 2\exp(-ct^2q)$,*

$$(1-t)\sqrt{\frac{q}{p}}\|z\|_2 \leq \|Pz\|_2 \leq (1+t)\sqrt{\frac{q}{p}}\|z\|_2. \tag{66}$$

Lemma 2.7.6 in [25]. *A random variable X is sub-Gaussian if and only if X^2 is sub-exponential. Moreover, the sub-Gaussian norm $\|X\|_{\varphi_2}$ and the sub-exponential norm $\|X\|_{\varphi_1}$ of X are such that*

$$\|X^2\|_{\varphi_1} = \|X\|_{\varphi_2}^2. \tag{67}$$

Hoeffdings' inequality. Theorem 2.6.2 in [25]. *Let $X_1, \cdots, X_m$ be independent, sub-Gaussian random variables. Then, for every $t \geq 0$,*

$$\Pr[|\sum_{i=1}^{i=n} X_i - \mathbb{E}[X_i]| \geq t] \leq 2\exp\big(-\frac{ct^2}{\sum_{i=1}^{i=m}\|X_i\|_{\varphi_2}^2}\big), \tag{68}$$

where $\|X_i\|_{\varphi_2}$ denotes the sub-Gaussian norm of X_i, and c is an absolute constant.

Sub-exponential norm definition. *The sub-exponential norm a random variable X is defined as*

$$\|X\|_{\varphi_1} \triangleq \inf\{t > 0 \mid \mathbb{E}[\exp(|X|/t)] \leq 2\}. \tag{69}$$

Tail inequality for χ_D^2 distributed random variables [28]. *Let Z_D be a χ^2 statistic with D degrees of freedom. For any positive t,*

$$\Pr[Z_D \geq D + 2\sqrt{Dt} + 2t] \leq \exp(-t), \tag{70}$$
$$\Pr[Z_D \leq D - 2\sqrt{Dt}] \leq \exp(-t). \tag{71}$$

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