Final Exam

Review / Pre-Test

Explicit integration

- Compute one explicit step of Heun's method and 4th order Runge-Kutta to solve

$$\frac{dy}{dx} = y$$

for the initial condition $(4.5, 1)$ and $\Delta x = 2$

- Analytical solution

$$\ln y = x + C$$

$$C = -4.5$$

$$y = e^{x-4.5}$$

$$y_1 = e^{6.5-4.5} \approx 7.389$$

Explicit Integration

- Euler's method

$$x_1 = x_0 + \Delta x = 4.5 + 2 = 6.5$$

$$y_1 = y_0 + \Delta x \cdot y_0 = 1 + 2 \cdot 1 = 3$$

$$E_r = 59.4\%$$

Explicit Integration

- Heun's Method

$$x_1 = x_0 + \Delta x = 6.5$$

$$\hat{y}_1 = y_0 + \Delta x \cdot y_0 = 1 + 2 = 3$$

$$y_1 = y_0 + \frac{\Delta x}{2}(y_0 + \hat{y}_1) = 1 + 1 + 3 = 5$$

$$E_r = 32.3\%$$

Runge-Kutta method

- Update step:

$$x_{n+1} = x_n + \Delta x$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where k_1 to k_4 are given by:

$$k_1 = \Delta x \cdot f(x_n, y_n)$$

$$k_2 = \Delta x \cdot f\left(x_n + \tfrac{\Delta x}{2}, y_n + \tfrac{k_1}{2}\right)$$

$$k_3 = \Delta x \cdot f\left(x_n + \tfrac{\Delta x}{2}, y_n + \tfrac{k_2}{2}\right)$$

$$k_4 = \Delta x \cdot f(x_n + \Delta x, y_n + k_3)$$

Explicit Integration

- 4^{th} order Runge-Kutta

$$x_1 = 6.5$$
$$k_1 = \Delta x \cdot y_0 = 2$$
$$k_2 = \Delta x \cdot \left(y_0 + \frac{k_1}{2}\right) = 2 \cdot (1 + 1) = 4$$
$$k_3 = \Delta x \cdot \left(y_0 + \frac{k_2}{2}\right) = 2 \cdot (1 + 2) = 6$$
$$k_4 = \Delta x \cdot (1 + k_3) = 2 \cdot (1 + 6) = 14$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_1 = 1 + \frac{1}{6}(2 + 8 + 12 + 14) = 1 + \frac{1}{6} \cdot 36 = 7$$
$$E_r = 5.26\%$$

Interpolation

- Compute the Lagrange interpolating polynomial for the following set of nodes

x	1/3	1/4	1
f(x)	2	-1	7

$$\ell_0 = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) = \left(\frac{x - \frac{1}{4}}{\frac{1}{3} - \frac{1}{4}}\right)\left(\frac{x - 1}{\frac{1}{3} - 1}\right)$$

$$\ell_1 = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) = \left(\frac{x - \frac{1}{4}}{\frac{1}{3} - \frac{1}{4}}\right)\left(\frac{x - 1}{\frac{1}{3} - 1}\right)$$

$$\ell_2 = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right) = \left(\frac{x - \frac{1}{3}}{1 - \frac{1}{3}}\right)\left(\frac{x - \frac{1}{4}}{1 - \frac{1}{4}}\right)$$

Interpolation

$$\ell_0 = -\frac{9}{2}(x-1)(4x-1)$$

$$\ell_1 = \frac{16}{3}(x-1)(3x-1)$$

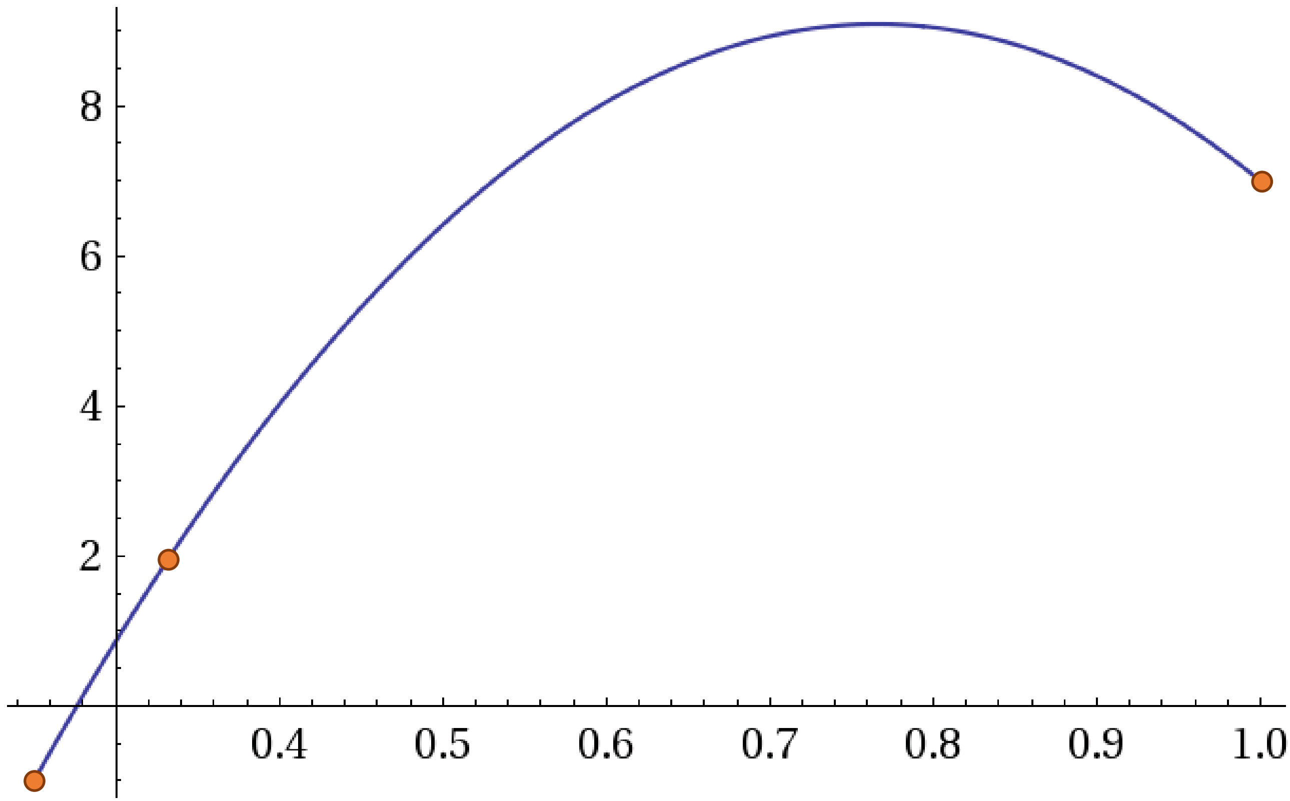
$$\ell_2 = \frac{1}{6}(3x-1)(4x-1)$$

$$p(x) = \ell_0 y_0 + \ell_1 y_1 + \ell_2 y_2$$

$$p(x) = -38x^2 + \frac{349}{6}x - \frac{79}{6}$$

Interpolation

$$p(x) = -38x^2 + \frac{349}{6}x - \frac{79}{6}$$



Numerical integration

- Use Simpson's Rule to calculate

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

with partition points at $x = 0, 0.5, 1$

- Analytical

The antiderivative of $\frac{1}{1+x^2}$ is $\tan^{-1} x$ so

$$\int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

Numerical integration

- Simpson's Rule states:

$$\int_a^b f(x)dx \approx \frac{1}{6}\Delta x\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

and we have values at $x = 0, 0.5, 1$, so

$$\int_0^1 \frac{1}{1+x^2}dx \approx \frac{1}{6}[f(0) + 4f(0.5) + f(1)]$$
$$\approx \frac{1}{6}\left[\frac{1}{1+0^2} + \frac{4}{1+(0.5)^2} + \frac{1}{1+1^2}\right]$$

$$\approx \frac{1}{6}\left[1 + \frac{16}{5} + \frac{1}{2}\right] = \frac{47}{60} \approx 0.783$$

$$E_r = 0.25\%$$

Numerical integration

- How does this compare to the trapezoid rule?

$$\int_a^b f(x)dx \approx \frac{1}{2}\Delta x \sum_{i=1}^{n} [f(x_{i-1}) + f(x_i)]$$

and we have values at $x = 0, 0.5, 1$, so

$$\int_0^1 \frac{1}{1+x^2}dx \approx \frac{1}{2}\cdot\frac{1}{2}[f(0) + f(0.5)] + \frac{1}{2}\cdot\frac{1}{2}[f(0.5) + f(1)]$$

$$\approx \frac{1}{4}\left[\frac{1}{1+0^2} + \frac{1}{1+(0.5)^2}\right] + \frac{1}{4}\left[\frac{1}{1+(0.5)^2} + \frac{1}{1+1^2}\right]$$

$$\approx \frac{1}{4}\left[1 + \frac{8}{5} + \frac{1}{2}\right] = \frac{1}{4}\cdot\frac{31}{10} = \frac{31}{20} \approx 0.775$$

$$E_r = 1.3\%$$

Convert to base-10

$$1\ 1\ 1\ 1\ .\ 1\ 0\ 1\ 1$$

$$1\ 0\ 0\ 1\ .\ 0\ 0\ 0\ 1$$

Floating point precision

- You are writing a controller for a thermostat that uses an 8-bit floating point unit. The format uses a 4-bit mantissa with a 3-bit exponent and 1 sign bit. There is an implied 1 in the mantissa. The thermostat will generally be operating between $32°$C and $63°$C. How much precision can you expect in your calculations?

- format: $(-1)^s\, 0\,.\,1\, m_1\, m_2\, m_3\, m_4 \times 2^{e_1\, e_2\, e_3}$
- since $32_{10} = 100000_2$

  the exponent must be 6_{10}
- So, the smallest increment we can represent using this number format is

$$0.10001_2 \times 2^6 - 0.10000_2 \times 2^6 = 34 - 32 = 2$$

  so our precision will be $\pm 1°C$

Catastrophic cancellation

- Where is the catastrophic cancellation?

$$\frac{\sin x}{x - \sqrt{x^2 - 1}}$$

- Reformulate

$$\frac{\sin x}{x - \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}$$

$$\frac{\sin x \left(x + \sqrt{x^2 - 1}\right)}{x^2 - x^2 + 1}$$

$$\sin x \left(x + \sqrt{x^2 - 1}\right)$$

Catastrophic cancellation

- Where is the catastrophic cancellation?

$$\frac{1}{102} - \frac{1}{101}$$

- Reformulate

Using the MVT: $f(b) - f(a) = (b - a)f'(\theta)$

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$a = 101,\ b = 102, \theta = 101.5$$

$$f(b) - f(a) \approx (1) \cdot f'(101.5) = -\frac{1}{101.5^2}$$

- What is the relative error of your evaluation (using 3 significant figures)

$$-\frac{1}{101.5^2} \approx -\frac{1}{101^2} \approx \frac{1}{10400} \approx -.98 \times 10^{-4}$$

the exact value is $-.961 \times 10^{-4}$, so $E_r = .978\%$