Exam 2

Review / Pre-Test

Random number generators

- What modulus would be used to optimize the period for a 5-bit multiplicative linear congruential generator?

  You can use the Mersenne prime $2^5 - 1 = 31$, which will allow you to select any 5-bit positive seed.

- What are the first 5 values with a seed of $x_0 = 5$ if the multiplier $a = 17$?

  $5 \times 17 = 85\ mod\ 31 = 23$
  $23 \times 17 = 391\ mod\ 31 = 19$
  $19 \times 17 = 323\ mod\ 31 = 13$
  $13 \times 17 = 221\ mod\ 31 = 4$
  $4 \times 17 = 68\ mod\ 31 = 6$

- How many planes would these numbers occupy in a 3D space?
  - $3 = (k - 1)$, where $k = 4$
  - Therefore, the MLCG will create $(31)^{1/4} \approx 2 - 3$ planes

Finite differences

- Calculate the derivative of $\frac{1}{\sin x}$ at $x = 1.77$ using the first forward and central difference operators and a step size of $h = 0.1$.
- Provide the relative error
- Use 3 digits of precision

$$\frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\frac{\cos x}{\sin^2 x}$$

analytical solution = .206
forward difference = .3 $(E_r = 45.6\%)$
central difference = .2 $(E_r = 2.91\%)$

- What happens when you half the step size? How can you fix this problem?

forward difference = .2 $(E_r = 2.91\%)$
central difference = .2 $(E_r = 2.91\%)$

Scaled Partial Pivoting

- Solve the following matrix using scaled partial pivoting. Make sure to keep track of a scale vector and index vector. Represent the matrix as it would be represented using the algorithm.

$$A = \begin{bmatrix} -1 & -5 & -7 \\ 4 & -5 & 4 \\ 1 & 5 & -1 \end{bmatrix} \qquad s = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix} \quad l = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$l = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & \frac{-25}{4} & -6 \\ 4 & -5 & 4 \\ 0 & \frac{25}{4} & -2 \end{bmatrix}$$

$$l = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & 0 & -8 \\ 4 & -5 & 4 \\ 0 & \frac{25}{4} & -2 \end{bmatrix}$$

- What is the determinant?

$$(-8)\left(\frac{25}{4}\right)(4) = -200$$

Matrix conditioning

- Approximately how many digits of precision are lost calculating:

$$x + \frac{1}{2}y = 1$$

$$\frac{1}{2}x + \frac{1}{3}y = 3$$

- Compute the LU Decomposition of $\mathrm{A} = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/12 \end{bmatrix}$$

- Compute the inverse of A

$$A^{-1} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix}$$

- Compute the condition number

$$\kappa(A) = \|A\| \times \left\|A^{-1}\right\| = \|A\|_e \times \left\|A^{-1}\right\|_e = \frac{\sqrt{58}}{6} \times 2\sqrt{58} = \frac{58}{3} \approx 19.3$$

- The number of digits lost is approximately k when $\kappa(A) = 10^k$

  so, approximately 1-2 digits

Horner’s Algorithm

- Calculate $\mathrm{f(x)} = \sin(x)\cos(x) + e^x$ for $x = 0.1$ using Horner’s algorithm to an error of $O(x^4)$ and 3 digits of precision.

  - Since $x < 1$, calculate the Maclaurin expansion:

$$f(x) = \sin(x)\cos(x) + e^x$$
$$f'(x) = \cos^2(x) - \sin^2 x + e^x$$
$$f''(x) = -4\cos(x)\sin(x) + e^x$$
$$f^{(3)}(x) = 4\sin^2(x) - 4\cos^2(x) + e^x$$

$$f(x) \approx 1 + 2x + \frac{x^2}{2} - \frac{x^3}{2}$$

  - Calculate using synthetic division:

0.1 |	-0.5	0.5	2	1
		-0.05	0.045	0.205
	-0.5	0.45	2.05	1.21

Reformulation

- Calculate $\cos(\pi + 0.05) + 1$ to maximum precision using 3 significant digits. What is the relative error?

  - Reformulate (step 1)

    $$\cos(\pi)\cos(0.05) - \sin(\pi)\sin(0.05) + 1$$
    $$-\cos(0.05) + 1$$

  - Reformulate (step 2)

    $$[1 - \cos(0.05)] \times \frac{1+\cos(0.05)}{1+\cos(0.05)}$$

    $$\frac{1-\cos^2(0.05)}{1+\cos(0.05)} = \frac{\sin^2(0.05)}{1+\cos(0.05)}$$

  - Calculate
    - exact = .00117
    - computed = .001 $(E_r = 14.5\%)$
    - reformulated = .00125 $(E_r = 6.84\%)$