Appendices for The Reachable Set of a Drone: Exploring the Position Isochrones for a Quadcopter

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***Abstract*— These appendices provide a comparison to the reachable set for a Dubin's car and provide the full set of equations for the reachable set of a quadcopter. These give the position isochrones for the quadcopter.**

I. Appendix: Dubins car reachable set

The reader will note similarities between the reachable set of a drone 1 with the reachable set for a Dubins car model 2. For completeness' sake we list here the equations for both sets to aid in comparing the two. The Dubins car is simpler, having no initial velocity, no gravity, only one control input and only single integration.

The Dubins car is a simplified mathematical model of a car that moves on the x, y plane [1]. The car's location is specified at the (x, y) center of the car's rear axle and the orientation θ of the car. The car cannot move sideways because the rear wheels would have to slide rather than roll. The Dubins car model stipulates that the car be moving forward at a constant speed and have a maximum steering angle that translates into a minimum turning radius r.

In 1957, Lester Eli Dubins proved that the shortest path between two (x, y, θ) coordinates for a forward-moving vehicle with a minimum turning radius r is composed entirely of straight lines or no more than three circular arcs of radius r [1]. This section provides equations for the reachable set of (x, y) locations from a starting $(x, y, \theta) = (0, 0, 0)$ coordinate. The boundary of this set is reachable by a circular arc of radius r followed by either a straight path or a circular arc of radius r in the opposite direction. Label a turn to the right at the maximum rate by the letter R, left as L and straight as S; then the optimal paths to the boundary are RS, LS, RL, LR.

If the car has forward velocity of 1 unit per second, the system equations are

$$\frac{dx}{dt} = \cos\theta, \frac{dy}{dt} = \sin\theta, \frac{du}{dt} = u, \tag{1}$$

where u is chosen from the interval $[-1/r_{\text{min}}, 1/r_{\text{min}}]$. For a car starting at $(0, 0, 0)$, define the switching time as τ and arc lengths traveled by the car as $\alpha = \tau/r$ and $\beta = (t-\tau)/r$. Then the car position at time t for each candidate optimal path are:

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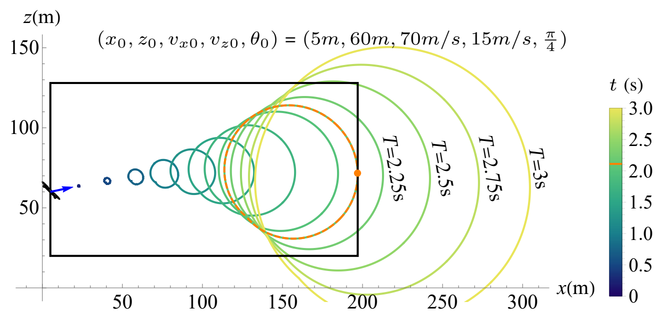

Fig. 1. Contours for a 2D quadcopter model. Here, a fast-moving drone entering the camera frame from the left can first escape at 2.11 s by moving at maximum velocity to the right.

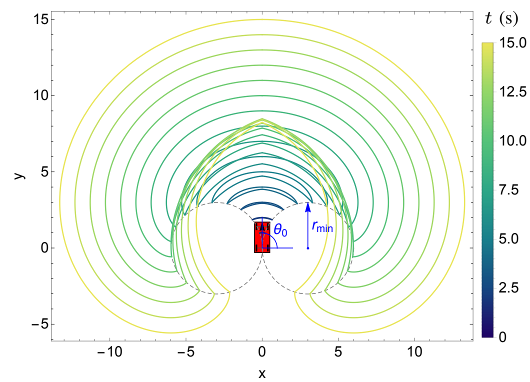

Fig. 2. Contours of a Dubins-car reachable set with $r = 3$, $v = 1$ and $(x_0, y_0, \theta_0) = (0, 0, \pi/2)$, which has two parameters, the minimum turning radius r and the car velocity, and one control variable. The contours are evenly spaced because the model has only one integration.

$$\begin{aligned}
&\text{RS}: r\{+1-\cos\alpha, \sin\alpha\} - (t-\tau)\{\sin\alpha, +\cos\alpha\},\\
&\text{LS}: r\{-1+\cos\alpha, \sin\alpha\} + (t-\tau)\{\sin\alpha, -\cos\alpha\},\\
&\text{RL}: r\{+1-2\cos\alpha+\cos(\alpha-\beta), 2\sin\alpha-\sin(\alpha-\beta)\},\\
&\text{LR}: r\{-1+2\cos\alpha-\cos(\alpha-\beta), 2\sin\alpha-\sin(\alpha-\beta)\}.
\end{aligned}$$

The reachable sets are plotted in Fig. 2. See [2] for additional details, and our interactive demonstration at [3].

References

[1] L. E. Dubins, "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents," *American Journal of mathematics*, vol. 79, no. 3, pp. 497–516, 1957.

[2] E. Cockayne and G. Hall, "Plane motion of a particle subject to curvature constraints," *SIAM Journal on Control*, vol. 13, no. 1, pp. 197–220, 1975.

[3] A. T. Becker and S. Shahrokhi, "Isochrons For A Dubins Car, Wolfram Demonstrations Project," Dec. 2017. [Online]. Available: http://demonstrations.wolfram.com/IsochronsForADubinsCar/

II. Appendix: 2D Drone reachable set equations: $\hat{x}(\hat{t},\hat{t}_R,\hat{t}_T), \hat{z}(\hat{t},\hat{t}_R,\hat{t}_T)$.

For zero initial conditions, curve straight (CS) case when $\hat{t}_R \leq \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t} - \sin(\hat{t})) & \hat{t} \leq \hat{t}_R \\ \hat{t}\underline{u_T} + (\hat{t}_R - \hat{t})\underline{u_T}\cos(\hat{t}_R) + \frac{1}{2}((\hat{t} - \hat{t}_R)^2 - 2)\underline{u_T}\sin(\hat{t}_R) & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \hat{t}\underline{u_T} + (\hat{t}_R - \hat{t})\underline{u_T}\cos(\hat{t}_R) + \frac{1}{2}(\underline{u_T}(\hat{t}_R^2 + 2\hat{t}\hat{t}_T - \hat{t}_T^2 - 2 - 2\hat{t}\hat{t}_R) + \overline{u_T}(\hat{t} - \hat{t}_T)^2)\sin(\hat{t}_R) & \hat{t}_T < \hat{t} \end{cases} \quad (2)$$

$$\hat{z} = \begin{cases} \frac{-\hat{t}^2}{2} + \underline{u_T} - \underline{u_T}\cos(\hat{t}) & \hat{t} \leq \hat{t}_R \\ \frac{-\hat{t}^2}{2} + \underline{u_T} + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\underline{u_T}\cos(\hat{t}_R) + (\hat{t} - \hat{t}_R)\underline{u_T}\sin(\hat{t}_R) & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \frac{-\hat{t}^2}{2} + \underline{u_T} + \frac{1}{2}(\underline{u_T}(-2 - 2\hat{t}\hat{t}_R + \hat{t}_R^2 + 2\hat{t}\hat{t}_T - \hat{t}_T^2) + \overline{u_T}(\hat{t} - \hat{t}_T)^2)\cos(\hat{t}_R) + (\hat{t} - \hat{t}_R)\underline{u_T}\sin(\hat{t}_R) & \hat{t} \geq \hat{t}_T \end{cases} \quad (3)$$

For zero initial conditions, curve straight (CS) case when $\hat{t}_R > \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t} - \sin(\hat{t})) & \hat{t} \leq \hat{t}_T \\ \hat{t}\underline{u_T} - (\hat{t} - \hat{t}_T)(\underline{u_T} - \overline{u_T})\cos(\hat{t}_T) - \overline{u_T}\sin(\hat{t}) + (\overline{u_T} - \underline{u_T})\sin(\hat{t}_T) & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \hat{t}\underline{u_T} + (-\hat{t} + \hat{t}_R)\overline{u_T}\cos(\hat{t}_R) + \frac{1}{2}(\overline{u_T}(-2 + (\hat{t} - \hat{t}_R)^2)\sin(\hat{t}_R) - (\underline{u_T} - \overline{u_T})((\hat{t} - \hat{t}_T)\cos(\hat{t}_T) + \sin(\hat{t}_T)) & \hat{t} > \hat{t}_R \end{cases} \quad (4)$$

$$\hat{z} = \begin{cases} \frac{-\hat{t}^2}{2} + \underline{u_T} - \underline{u_T}\cos(\hat{t}) & \hat{t} \leq \hat{t}_T \\ \frac{-\hat{t}^2}{2} + \underline{u_T} - \overline{u_T}\cos(\hat{t}) - (\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T) + (-\hat{t} + \hat{t}_T)\sin(\hat{t}_T)) & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \frac{-\hat{t}^2}{2} + \underline{u_T} + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\overline{u_T}\cos(\hat{t}_R) + (\hat{t} - \hat{t}_R)\overline{u_T}\sin(\hat{t}_R) - (\underline{u_T} - \overline{u_T})\left(\cos(\hat{t}_T) + (\hat{t}_T - \hat{t})\sin(\hat{t}_T)\right) & \hat{t} \geq \hat{t}_R \end{cases} \quad (5)$$

For non-zero initial conditions, curve straight (CS) case when $\hat{t}_R \leq \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t}\cos(\theta_0) + \sin(\theta_0) - \sin(\hat{t} + \theta_0)) + v_{\hat{x}_0}\hat{t} & \hat{t} \leq t_R \\ \underline{u_T}(\hat{t}\cos(\theta_0) + (-\hat{t} + \hat{t}_R)\cos(\hat{t}_R + \theta_0) + \sin(\theta_0)) + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\underline{u_T}\sin(\hat{t}_R + \theta_0)) + v_{x_0}\hat{t} & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \underline{u_T}(\hat{t}\cos(\theta_0) + (-\hat{t} + \hat{t}_R)\cos(\hat{t}_R + \theta_0) + \sin(\theta_0)) + \frac{1}{2}(\underline{u_T}(-2 - 2\hat{t}\hat{t}_R + \hat{t}_R^2 + 2\hat{t}\hat{t}_T - \hat{t}_T^2) + & \\ \overline{u_T}(\hat{t} - \hat{t}_T)^2)\sin(\hat{t}_R + \theta_0) + v_{\hat{x}_0}\hat{t} & \hat{t} > \hat{t}_T \end{cases} \quad (6)$$

$$\hat{z} = \begin{cases} \underline{u_T}\cos(\theta_0) - \frac{\hat{t}}{2}(\hat{t} + 2\underline{u_T}\sin(\theta_0)) - \underline{u_T}\cos(\hat{t} + \theta_0) + v_{\hat{z}_0}\hat{t} & \hat{t} \leq \hat{t}_R \\ \underline{u_T}\cos(\theta_0) - \frac{\hat{t}}{2}(\hat{t} + 2\underline{u_T}\sin(\theta_0)) + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\underline{u_T}\cos(\theta_0 + \hat{t}_R) + (\hat{t} - \hat{t}_R)\underline{u_T}\sin(\theta_0 + \hat{t}_R) + v_{\hat{z}_0}\hat{t} & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \underline{u_T}\cos(\theta_0) + \frac{1}{2}(\underline{u_T}(-2 - 2\hat{t}\hat{t}_R + \hat{t}_R^2 + 2\hat{t}\hat{t}_T - \hat{t}_T^2) + \overline{u_T}(\hat{t} - \hat{t}_T)^2)\cos(\theta_0 + \hat{t}_R) + & \\ (\hat{t} - \hat{t}_R)\underline{u_T}\sin(\theta_0 + \hat{t}_R) - \frac{\hat{t}}{2}(\hat{t} + 2(\underline{u_T}\sin(\theta_0 + \hat{t}_R)) + v_{\hat{z}_0}\hat{t} & \hat{t} \geq \hat{t}_T \end{cases} \quad (7)$$

For non-zero initial conditions, curve straight (CS) case when $\hat{t}_R > \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t}\cos(\theta_0) + \sin(\theta_0) - \sin(\hat{t} + \theta_0)) + v_{\hat{x}_0}\hat{t} & \hat{t} \leq \hat{t}_T \\ \underline{u_T}\hat{t}\cos(\theta_0) - (\underline{u_T} - \overline{u_T})\left((\hat{t} - \hat{t}_T)\cos(\theta_0 + \hat{t}_T) + \sin(\hat{t}_T + \theta_0)\right) + \underline{u_T}\sin(\theta_0) - \overline{u_T}\sin(\hat{t} + \theta_0) + v_{\hat{x}_0}\hat{t} & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \underline{u_T}\hat{t}\cos(\theta_0) + \underline{u_T}\sin(\theta_0) + (-\hat{t} + \hat{t}_R)\overline{u_T}\cos(\hat{t}_R + \theta_0) + \frac{1}{2}(\overline{u_T}(-2 + (\hat{t} - \hat{t}_R)^2)\sin(\hat{t}_R + \theta_0) & \\ -(\underline{u_T} - \overline{u_T})((\hat{t} - \hat{t}_T)\cos(\hat{t}_T + \theta_0) + \sin(\hat{t}_T + \theta_0)) + v_{\hat{x}_0}\hat{t} & \hat{t} > \hat{t}_R \end{cases} \quad (8)$$

$$\hat{z} = \begin{cases} \underline{u_T}\cos(\theta_0) - \frac{t}{2}(\hat{t} + 2\underline{u_T}\sin(\theta_0)) - \underline{u_T}\cos(\hat{t} + \theta_0) + v_{\hat{z}_0}\hat{t} & \hat{t} \leq \hat{t}_T \\ \underline{u_T}\cos(\theta_0) - \frac{\hat{t}^2}{2} - \overline{u_T}\cos(\hat{t} + \theta_0) - (\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T + \theta_0) + (\hat{t} - \hat{t}_T)\sin(\hat{t}_T + \theta_0)) & \\ -\hat{t}\underline{u_T}\sin(\theta_0) + v_{\hat{z}_0}\hat{t} & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \underline{u_T}\cos(\theta_0) - \frac{\hat{t}^2}{2} + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\overline{u_T}\cos(\hat{t}_R + \theta_0) + (\hat{t} - \hat{t}_R)\overline{u_T}sin(\hat{t}_R + \theta_0) & \\ -(\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T + \theta_0) + (\hat{t} - \hat{t}_T)\sin(\hat{t}_T + \theta_0)) - \hat{t}\underline{u_T}\sin(\theta_0) + v_{\hat{z}_0}\hat{t} & \hat{t} \geq \hat{t}_R \end{cases} \quad (9)$$

For zero initial conditions, curve curve (CC) case when $\hat{t}_R \leq \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t} - \sin(\hat{t})) & \hat{t} \leq \hat{t}_R \\ \underline{u_T}(\hat{t} + 2(-\hat{t} + \hat{t}_R)\cos(\hat{t}_R) - \sin(\hat{t}) + \sin(\hat{t} - 2\hat{t}_R) + \sin(\hat{t}_R)) & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \underline{u_T}\ \hat{t} + 2(-\hat{t} + \hat{t}_R)\underline{u_T}\cos(\hat{t}_R) + (\hat{t} - \hat{t}_T)(\underline{u_T} - \overline{u_T})\cos(2\hat{t}_R - \hat{t}_T) - \underline{u_T}\sin(\hat{t}) & \\ +\overline{u_T}\sin(\hat{t} - 2\hat{t}_R) + \underline{u_T}\sin(\hat{t}_R) - \underline{u_T}\sin(2\hat{t}_R - \hat{t}_T) + \overline{u_T}\sin(2\hat{t}_R - \hat{t}_T) & \hat{t} > \hat{t}_T \end{cases} \quad (10)$$

$$\hat{z} = \begin{cases} \frac{-\hat{t}^2}{2} + \underline{u_T}\left(1 - \cos(\hat{t})\right) & \hat{t} \leq \hat{t}_R \\ \frac{-\hat{t}^2}{2} + \underline{u_T}\left(1 - \cos(\hat{t} - 2\hat{t}_R) + 2(\hat{t} - \hat{t}_R)\sin(\hat{t}_R)\right) & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \frac{-\hat{t}^2}{2} + \underline{u_T}\left(1 - \frac{\overline{u_T}}{\underline{u_T}}\cos(\hat{t} - 2\hat{t}_R) + 2(\hat{t} - \hat{t}_R)\sin(\hat{t}_R)\right) - (\underline{u_T} - \overline{u_T})\left(\cos(2\hat{t}_R - \hat{t}_T)\right. & \\ \left.+(\hat{t} - \hat{t}_T)\sin(2\hat{t}_R - \hat{t}_t)\right) & \hat{t} \geq \hat{t}_T \end{cases} \quad (11)$$

For zero initial conditions, curve curve (CC) case when $\hat{t}_R > \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t} - \sin(\hat{t})) & \hat{t} \leq \hat{t}_T \\ \underline{u_T}\hat{t} - (\hat{t} - \hat{t}_T)(\underline{u_T} - \overline{u_T})\cos(\hat{t}_T) - (\underline{u_T} + \overline{u_T})\sin(\hat{t}) + \overline{u_T}\sin(\hat{t}_T) & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \underline{u_T}\hat{t} - (\hat{t} - \hat{t}_T)(\underline{u_T} - \overline{u_T})\cos(\hat{t}_T) + \overline{u_T}(\sin(\hat{t} - 2\hat{t}_R) + \sin(\hat{t}_T)) + 2(-\hat{t} + \hat{t}_R)\overline{u_T}\cos(\hat{t}_R) - \underline{u_T}\sin(\hat{t}) & \hat{t} > \hat{t}_R \end{cases} \quad (12)$$

$$\hat{z} = \begin{cases} \frac{-\hat{t}^2}{2} + \underline{u_T} - \underline{u_T}\cos(\hat{t}) & \hat{t} \leq \hat{t}_T \\ \frac{-\hat{t}^2}{2} + \underline{u_T} - \overline{u_T}\cos(\hat{t}) \qquad - (\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T) + (\hat{t}_T - \hat{t})\sin(\hat{t}_T)) & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \frac{-\hat{t}^2}{2} + \underline{u_T} - \overline{u_T}\cos(\hat{t} - 2\hat{t}_R) - (\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T) + (\hat{t}_T - \hat{t})\sin(\hat{t}_T)) + 2(\hat{t} - \hat{t}_R)\overline{u_T}\sin(\hat{t}_R) & \hat{t} \geq \hat{t}_R \end{cases} \quad (13)$$