Appendices for The Reachable Set of a Drone: Exploring the Position Isochrones for a Quadcopter

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***Abstract*— These appendices provide a comparison to the reachable set for a Dubin's car and provide the full set of equations for the reachable set of a quadcopter. These give the position isochrones for the quadcopter.**

I. Appendix: Dubins car reachable set

The reader will note similarities to the reachable set for a Dubins car model. For completeness' sake we list that set here to aid in comparing to the reachable set for a drone. The Dubins car is simpler, having no initial velocity, no gravity, only one control input and only single integration.

The Dubins car is a simplified mathematical model of a car that moves on the x, y plane [**?**]. The car's location is specified at the (x, y) center of the car's rear axle and the orientation θ of the car. The car cannot move sideways because the rear wheels would have to slide rather than roll. The Dubins car model stipulates that the car be moving forward at a constant speed and have a maximum steering angle that translates into a minimum turning radius r.

In 1957, Lester Eli Dubins proved that the shortest path between two (x, y, θ) coordinates for a forward-moving vehicle with a minimum turning radius r is composed entirely of straight lines or no more than three circular arcs of radius r [**?**]. This section provides equations for the reachable set of (x, y) locations from a starting $(x, y, \theta) = (0, 0, 0)$ coordinate. The boundary of this set is reachable by a circular arc of radius r followed by either a straight path or a circular arc of radius r in the opposite direction. Label a turn to the right at the maximum rate by the letter R, left as L and straight as S; then the optimal paths to the boundary are RS, LS, RL, LR.

If the car has forward velocity of 1 unit per second, the system equations are

$$\frac{dx}{dt} = \cos\theta, \frac{dy}{dt} = \sin\theta, \frac{du}{dt} = u, \tag{1}$$

where u is chosen from the interval $[-1/r_{\min}, 1/r_{\min}]$. For a car starting at $(0, 0, 0)$, define the switching time as τ and arc lengths traveled by the car as $\alpha = \tau/r$ and $\beta = (t - \tau)/r$. Then the car position at time t for each candidate optimal path are:

RS: $r\{+1 - \cos\alpha, \sin\alpha\} - (t - \tau)\{\sin\alpha, + \cos\alpha\}$,
LS: $r\{-1 + \cos\alpha, \sin\alpha\} + (t - \tau)\{\sin\alpha, - \cos\alpha\}$,
RL: $r\{+1 - 2\cos\alpha + \cos(\alpha - \beta), 2\sin\alpha - \sin(\alpha - \beta)\}$,
LR: $r\{-1 + 2\cos\alpha - \cos(\alpha - \beta), 2\sin\alpha - \sin(\alpha - \beta)\}$.

The reachable sets are plotted in Fig. 1. See [**?**] for additional details, and our interactive demonstration at [**?**].

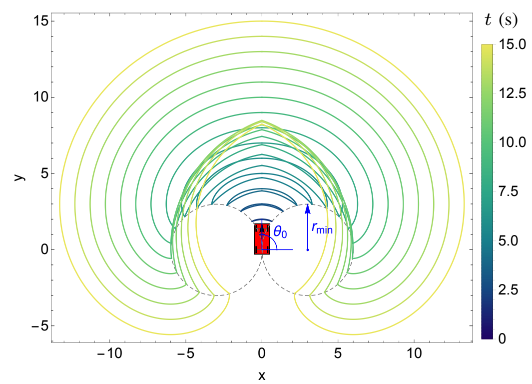

Fig. 1. Contours of a Dubins-car reachable set with $r = 3$, $v = 1$ and $(x_0, y_0, \theta_0) = (0, 0, \pi/2)$, which has two parameters, the minimum turning radius r and the car velocity, and one control variable. The contours are evenly spaced because the model has only one integration.

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II. APPENDIX: 2D DRONE REACHABLE SET EQUATIONS: $\hat{x}(\hat{t}, \hat{t}_R, \hat{t}_T), \hat{z}(\hat{t}, \hat{t}_R, \hat{t}_T)$.

For zero initial conditions, curve straight (CS) case when $\hat{t}_R \leq \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t} - \sin(\hat{t})) & \hat{t} \leq \hat{t}_R \\ \hat{t}\underline{u_T} + (\hat{t}_R - \hat{t})\underline{u_T}\cos(\hat{t}_R) + \frac{1}{2}((\hat{t} - \hat{t}_R)^2 - 2)\underline{u_T}\sin(\hat{t}_R) & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \hat{t}\underline{u_T} + (\hat{t}_R - \hat{t})\underline{u_T}\cos(\hat{t}_R) + \frac{1}{2}(\underline{u_T}(\hat{t}_R^2 + 2\hat{t}\hat{t}_T - \hat{t}_T^2 - 2 - 2\hat{t}\hat{t}_R) + \overline{u_T}(\hat{t} - \hat{t}_T)^2)\sin(\hat{t}_R) & \hat{t}_T < \hat{t} \end{cases} \tag{2}$$

$$\hat{z} = \begin{cases} \dfrac{-\hat{t}^2}{2} + \underline{u_T} - \underline{u_T}\cos(\hat{t}) & \hat{t} \leq \hat{t}_R \\ \frac{-\hat{t}^2}{2} + \underline{u_T} + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\underline{u_T}\cos(\hat{t}_R) + (\hat{t} - \hat{t}_R)\underline{u_T}\sin(\hat{t}_R) & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \frac{-\hat{t}^2}{2} + \underline{u_T} + \frac{1}{2}(\underline{u_T}(-2 - 2\hat{t}\hat{t}_R + \hat{t}_R^2 + 2\hat{t}\hat{t}_T - \hat{t}_T^2) + \overline{u_T}(\hat{t} - \hat{t}_T)^2)\cos(\hat{t}_R) + (\hat{t} - \hat{t}_R)\underline{u_T}\sin(\hat{t}_R) & \hat{t} \geq \hat{t}_T \end{cases} \tag{3}$$

For zero initial conditions, curve straight (CS) case when $\hat{t}_R > \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t} - \sin(\hat{t})) & \hat{t} \leq \hat{t}_T \\ \hat{t}\underline{u_T} - (\hat{t} - \hat{t}_T)(\underline{u_T} - \overline{u_T})\cos(\hat{t}_T) - \overline{u_T}\sin(\hat{t}) + (\overline{u_T} - \underline{u_T})\sin(\hat{t}_T) & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \hat{t}\underline{u_T} + (-\hat{t} + \hat{t}_R)\overline{u_T}\cos(\hat{t}_R) + \frac{1}{2}(\overline{u_T}(-2 + (\hat{t} - \hat{t}_R)^2)\sin(\hat{t}_R) - (\underline{u_T} - \overline{u_T})((\hat{t} - \hat{t}_T)\cos(\hat{t}_T) + \sin(\hat{t}_T)) & \hat{t} > \hat{t}_R \end{cases} \tag{4}$$

$$\hat{z} = \begin{cases} \dfrac{-\hat{t}^2}{2} + \underline{u_T} - \underline{u_T}\cos(\hat{t}) & \hat{t} \leq \hat{t}_T \\ \frac{-\hat{t}^2}{2} + \underline{u_T} - \overline{u_T}\cos(\hat{t}) - (\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T) + (-\hat{t} + \hat{t}_T)\sin(\hat{t}_T)) & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \frac{-\hat{t}^2}{2} + \underline{u_T} + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\overline{u_T}\cos(\hat{t}_R) + (\hat{t} - \hat{t}_R)\overline{u_T}\sin(\hat{t}_R) - (\underline{u_T} - \overline{u_T})\left(\cos(\hat{t}_T) + (\hat{t}_T - \hat{t})\sin(\hat{t}_T)\right) & \hat{t} \geq \hat{t}_R \end{cases} \tag{5}$$

For non-zero initial conditions, curve straight (CS) case when $\hat{t}_R \leq \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t}\cos(\theta_0) + \sin(\theta_0) - \sin(\hat{t} + \theta_0)) + v_{\hat{x}_0}\hat{t} & \hat{t} \leq t_R \\ \underline{u_T}(\hat{t}\cos(\theta_0) + (-\hat{t} + \hat{t}_R)\cos(\hat{t}_R + \theta_0) + \sin(\theta_0)) + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\underline{u_T}\sin(\hat{t}_R + \theta_0)) + v_{x_0}\hat{t} & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \underline{u_T}(\hat{t}\cos(\theta_0) + (-\hat{t} + \hat{t}_R)\cos(\hat{t}_R + \theta_0) + \sin(\theta_0)) + \frac{1}{2}(\underline{u_T}(-2 - 2\hat{t}\hat{t}_R + \hat{t}_R^2 + 2\hat{t}\hat{t}_T - \hat{t}_T^2) + & \\ \overline{u_T}(\hat{t} - \hat{t}_T)^2)\sin(\hat{t}_R + \theta_0) + v_{\hat{x}_0}\hat{t} & \hat{t} > \hat{t}_T \end{cases} \tag{6}$$

$$\hat{z} = \begin{cases} \underline{u_T}\cos(\theta_0) - \dfrac{\hat{t}}{2}(\hat{t} + 2\underline{u_T}\sin(\theta_0)) - \underline{u_T}\cos(\hat{t} + \theta_0) + v_{\hat{z}_0}\hat{t} & \hat{t} \leq \hat{t}_R \\ \underline{u_T}\cos(\theta_0) - \frac{\hat{t}}{2}(\hat{t} + 2\underline{u_T}\sin(\theta_0)) + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\underline{u_T}\cos(\theta_0 + \hat{t}_R) + (\hat{t} - \hat{t}_R)\underline{u_T}\sin(\theta_0 + \hat{t}_R) + v_{\hat{z}_0}\hat{t} & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \underline{u_T}\cos(\theta_0) + \frac{1}{2}(\underline{u_T}(-2 - 2\hat{t}\hat{t}_R + \hat{t}_R^2 + 2\hat{t}\hat{t}_T - \hat{t}_T^2) + \overline{u_T}(\hat{t} - \hat{t}_T)^2)\cos(\theta_0 + \hat{t}_R) + & \\ (\hat{t} - \hat{t}_R)\underline{u_T}\sin(\theta_0 + \hat{t}_R) - \frac{\hat{t}}{2}(\hat{t} + 2(\underline{u_T}\sin(\theta_0 + \hat{t}_R)) + v_{\hat{z}_0}\hat{t} & \hat{t} \geq \hat{t}_T \end{cases} \tag{7}$$

For non-zero initial conditions, curve straight (CS) case when $\hat{t}_R > \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t}\cos(\theta_0) + \sin(\theta_0) - \sin(\hat{t} + \theta_0)) + v_{\hat{x}_0}\hat{t} & \hat{t} \leq \hat{t}_T \\ \underline{u_T}\hat{t}\cos(\theta_0) - (\underline{u_T} - \overline{u_T})\left((\hat{t} - \hat{t}_T)\cos(\theta_0 + \hat{t}_T) + \sin(\hat{t}_T + \theta_0)\right) + \underline{u_T}\sin(\theta_0) - \overline{u_T}\sin(\hat{t} + \theta_0) + v_{\hat{x}_0}\hat{t} & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \underline{u_T}\hat{t}\cos(\theta_0) + \underline{u_T}\sin(\theta_0) + (-\hat{t} + \hat{t}_R)\overline{u_T}\cos(\hat{t}_R + \theta_0) + \frac{1}{2}(\overline{u_T}(-2 + (\hat{t} - \hat{t}_R)^2)\sin(\hat{t}_R + \theta_0) & \\ -(\underline{u_T} - \overline{u_T})((\hat{t} - \hat{t}_T)\cos(\hat{t}_T + \theta_0) + \sin(\hat{t}_T + \theta_0)) + v_{\hat{x}_0}\hat{t} & \hat{t} > \hat{t}_R \end{cases} \tag{8}$$

$$\hat{z} = \begin{cases} \underline{u_T}\cos(\theta_0) - \dfrac{t}{2}(\hat{t} + 2\underline{u_T}\sin(\theta_0)) - \underline{u_T}\cos(\hat{t} + \theta_0) + v_{\hat{z}_0}\hat{t} & \hat{t} \leq \hat{t}_T \\ \underline{u_T}\cos(\theta_0) - \frac{\hat{t}^2}{2} - \overline{u_T}\cos(\hat{t} + \theta_0) - (\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T + \theta_0) + (\hat{t} - \hat{t}_T)\sin(\hat{t}_T + \theta_0)) & \\ -\hat{t}\underline{u_T}\sin(\theta_0) + v_{\hat{z}_0}\hat{t} & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \underline{u_T}\cos(\theta_0) - \frac{\hat{t}^2}{2} + \frac{1}{2}(-2 + (\hat{t} - \hat{t}_R)^2)\overline{u_T}\cos(\hat{t}_R + \theta_0) + (\hat{t} - \hat{t}_R)\overline{u_T}sin(\hat{t}_R + \theta_0) & \\ -(\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T + \theta_0) + (\hat{t} - \hat{t}_T)\sin(\hat{t}_T + \theta_0)) - \hat{t}\underline{u_T}\sin(\theta_0) + v_{\hat{z}_0}\hat{t} & \hat{t} \geq \hat{t}_R \end{cases} \tag{9}$$

For zero initial conditions, curve curve (CC) case when $\hat{t}_R \leq \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t} - \sin(\hat{t})) & \hat{t} \leq \hat{t}_R \\ \underline{u_T}(\hat{t} + 2(-\hat{t} + \hat{t}_R)\cos(\hat{t}_R) - \sin(\hat{t}) + \sin(\hat{t} - 2\hat{t}_R) + \sin(\hat{t}_R)) & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \underline{u_T}\ \hat{t} + 2(-\hat{t} + \hat{t}_R)\underline{u_T}\cos(\hat{t}_R) + (\hat{t} - \hat{t}_T)(\underline{u_T} - \overline{u_T})\cos(2\hat{t}_R - \hat{t}_T) - \underline{u_T}\sin(\hat{t}) & \\ +\overline{u_T}\sin(\hat{t} - 2\hat{t}_R) + \underline{u_T}\sin(\hat{t}_R) - \underline{u_T}\sin(2\hat{t}_R - \hat{t}_T) + \overline{u_T}\sin(2\hat{t}_R - \hat{t}_T) & \hat{t} > \hat{t}_T \end{cases} \tag{10}$$

$$\hat{z} = \begin{cases} \dfrac{-\hat{t}^2}{2} + \underline{u_T}\left(1 - \cos(\hat{t})\right) & \hat{t} \leq \hat{t}_R \\ \frac{-\hat{t}^2}{2} + \underline{u_T}\left(1 - \cos(\hat{t} - 2\hat{t}_R) + 2(\hat{t} - \hat{t}_R)\sin(\hat{t}_R)\right) & \hat{t}_R < \hat{t} \leq \hat{t}_T \\ \frac{-\hat{t}^2}{2} + \underline{u_T}\left(1 - \frac{\overline{u_T}}{\underline{u_T}}\cos(\hat{t} - 2\hat{t}_R) + 2(\hat{t} - \hat{t}_R)\sin(\hat{t}_R)\right) - (\underline{u_T} - \overline{u_T})\left(\cos(2\hat{t}_R - \hat{t}_T)\right. & \\ \left.+(\hat{t} - \hat{t}_T)\sin(2\hat{t}_R - \hat{t}_t)\right) & \hat{t} \geq \hat{t}_T \end{cases} \tag{11}$$

For zero initial conditions, curve curve (CC) case when $\hat{t}_R > \hat{t}_T$:

$$\hat{x} = \begin{cases} \underline{u_T}(\hat{t} - \sin(\hat{t})) & \hat{t} \leq \hat{t}_T \\ \underline{u_T}\hat{t} - (\hat{t} - \hat{t}_T)(\underline{u_T} - \overline{u_T})\cos(\hat{t}_T) - (\underline{u_T} + \overline{u_T})\sin(\hat{t}) + \overline{u_T}\sin(\hat{t}_T) & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \underline{u_T}\hat{t} - (\hat{t} - \hat{t}_T)(\underline{u_T} - \overline{u_T})\cos(\hat{t}_T) + \overline{u_T}(\sin(\hat{t} - 2\hat{t}_R) + \sin(\hat{t}_T)) + 2(-\hat{t} + \hat{t}_R)\overline{u_T}\cos(\hat{t}_R) - \underline{u_T}\sin(\hat{t}) & \hat{t} > \hat{t}_R \end{cases} \tag{12}$$

$$\hat{z} = \begin{cases} \dfrac{-\hat{t}^2}{2} + \underline{u_T} - \underline{u_T}\cos(\hat{t}) & \hat{t} \leq \hat{t}_T \\ \frac{-\hat{t}^2}{2} + \underline{u_T} - \overline{u_T}\cos(\hat{t}) \qquad - (\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T) + (\hat{t}_T - \hat{t})\sin(\hat{t}_T)) & \hat{t}_T < \hat{t} \leq \hat{t}_R \\ \frac{-\hat{t}^2}{2} + \underline{u_T} - \overline{u_T}\cos(\hat{t} - 2\hat{t}_R) - (\underline{u_T} - \overline{u_T})(\cos(\hat{t}_T) + (\hat{t}_T - \hat{t})\sin(\hat{t}_T)) + 2(\hat{t} - \hat{t}_R)\overline{u_T}\sin(\hat{t}_R) & \hat{t} \geq \hat{t}_R \end{cases} \tag{13}$$