Oct 28, 2019, Intro to Robotics
Name: _____________ANSWER KEY_VB____________

PeopleSoft ID: ____

Problem	Score	Possible
1	15	15
2	25	25
3	15	15
4	25	25
5	15	15
6	10	10
Totals	105	105

You may have on your desk:

- Your student ID card
- 1 handwritten 8.5"x11" double-sided crib sheet
- This exam (provided by Professor)

Grading: (problem difficulty) $\times \begin{cases} \text{2 for trying} \\ \text{3 if partially correct} \\ \text{5 if correct} \end{cases}$

As in Appendix A, $\theta = \text{Atan2}(\cos(\theta), \sin(\theta)) = \text{Atan2}(x, y)$

!

Concepts: Covers chapters 1—4, 11.1—11.2

Rotations & transformations

- Composition of rotations about world or current frame
- Construct a homogenous transform

Kinematics

- Assign DH parameters
- Given DH parameters, construct A matrix
- Given two A matrices, construct T matrix

Inverse Kinematics

- Two-argument arc tangent function
- Solve inverse position kinematics for a 3-link arm

Jacobian

- Construct Jacobian given sketch and T matrices

Computer Vision

- Move from camera frame to world frame, reason from image coordinates

Problem 1: _____/15

I. **(5 pt)** Write the matrix product that will give the resulting rotation matrix (*DO NOT perform the matrix multiplications, DO simplify*):

a. Rotate by α about the current z-axis
b. Rotate by β about the world y-axis
c. Rotate by γ about the world z-axis
d. Rotate by Φ about the current z-axis
e. Rotate by θ about the current x-axis
f. Rotate by ψ about the world y-axis

$$R = R_{y,\psi}R_{z,\gamma}R_{y,\beta}R_{z,\alpha}R_{z,\phi}R_{x,\theta} = R_{y,\psi}R_{z,\gamma}R_{y,\beta}R_{z,\alpha+\phi}R_{x,\theta}$$

Page 52: Premultiply if world frame, post-multiply if current

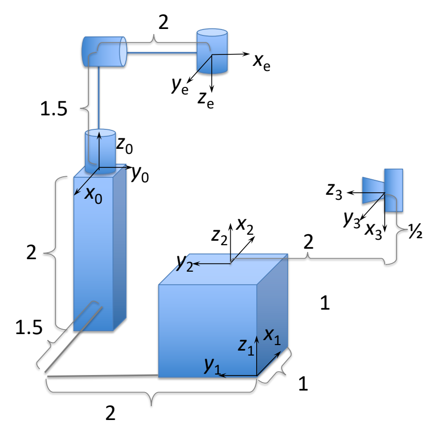
5 pts if correct, 4 pts if z angles are not combined

II. **(5 pt)** Suppose the three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$, and $o_3x_3y_3z_3$ are given, and suppose

$$R_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, R_1^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$ Find the matrix R_3^2

$$R_3^2 = R_1^2 R_3^1 = R_1^2 (R_1^3)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

III. **(5 pt)** Consider diagram at right. The cube front right bottom corner is (1.5,2,-2) meters from the robot base. The cube has 1-meter sides. A frame $o_1x_1y_1z_1$ is fixed to the side of the cube as shown. A second coordinate frame $o_2x_2y_2z_2$ is centered on the top of the cube as shown. A camera is situated 2 meters right and ½ meter above frame 2 with frame $o_3x_3y_3z_3$ attached as shown. Find the **homogenous transform** relating the camera frame to the frame $o_ex_ey_ez_e$, that is, H_3^e:



$$H_3^e = \begin{bmatrix} R_3^e & d_3^e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 1.5 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R_3^e = [x_3^e \quad y_3^e \quad z_3^e]$$

1 error, 4 pts, 2 errors, 3 pts

Problem 2: _____/25 Rotation matrices

Let $R_{YZX}=\begin{bmatrix} c_\alpha c_\theta & -c_\beta s_\alpha + c_\alpha s_\beta s_\theta & s_\alpha s_\beta + c_\alpha c_\beta s_\theta \\ s_\alpha c_\theta & c_\alpha c_\beta + s_\alpha s_\beta s_\theta & -c_\alpha s_\beta + c_\beta s_\alpha s_\theta \\ -s_\theta & s_\beta c_\theta & c_\beta c_\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

If r_{11} and r_{21} are not both zero, then $r_{31} \neq \pm 1$, $-s_\theta = r_{31}$, and $c_\theta = \pm\sqrt{1-r_{31}^2}$.
If c_θ>0 (c_θ positive), then

a. **(5 pt)** $\alpha = \text{Atan2}(r_{11}, r_{21})$ (remember $\theta = \text{Atan2}(\cos(\theta), \sin(\theta))$)

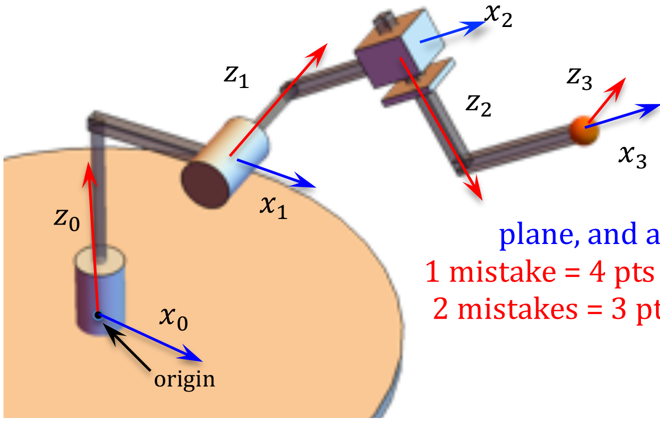
b. **(5pt)** $\beta = \text{Atan2}(r_{33}, r_{32})$

c. **(15 pts)** Matrix Identification, state **Yes** or **No**. +1 for each correctly listed **(B version swaps row 3 and 1)**

	valid so(k)	valid SE(n)	valid SO(n)
$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	Yes	No	Yes
$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	No	Yes	Yes
$\begin{bmatrix} 0 & -1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix}$	No	No	Yes
$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	No	Yes	Yes
$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	No	No	Yes

Problem 3: _____/15, Forward Kinematics

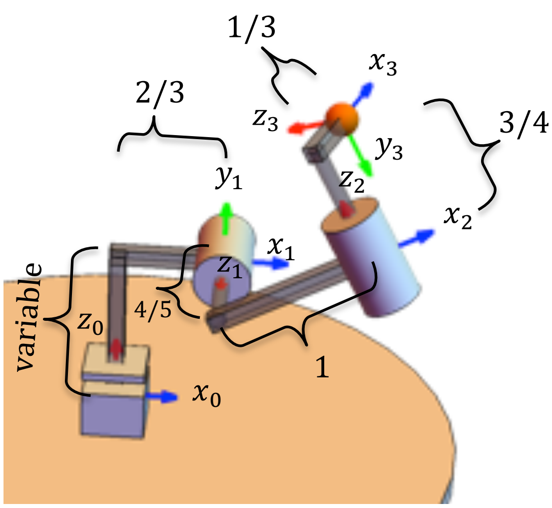
a.) **(5 pt)** For the 3-link robot below, **draw** and **label** the *z* and *x*-axes according to the DH convention



This is one possible answer. x_0 can be at any orientation in the plane, and all z axes can be inverted.

1 mistake = 4 pts
2 mistakes = 3 pts

b.) **(5pt)** Give the DH parameters for this PRR robot.



* indicates variable

Link	r_i	α_i	d_i	θ_i
1	2/3	$\pi/2$	d_1^*	0
2	1	$-\pi/2$	4/5	θ_2^*
3	1/3	$-\pi/2$	3/4	θ_3^*

1 error, 4 pts, 2 or 3 errors, 3 pts

c.) **(5pt)** Compute the transformation matrix A_2 using the DH parameters:

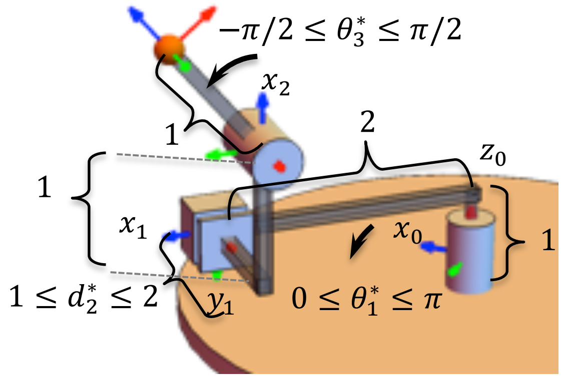
* indicates variable

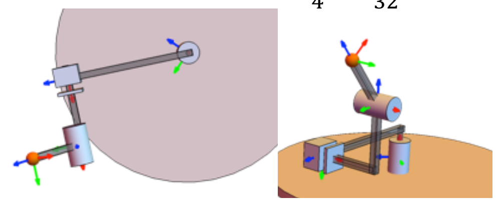
Link	r_i	α_i	d_i	θ_i
1	3	45°	5	θ_1^*
2	5	90°	2	θ_2^*
3	7	90°	4	θ_3^*
4	9	−90°	d_4^*	−90°

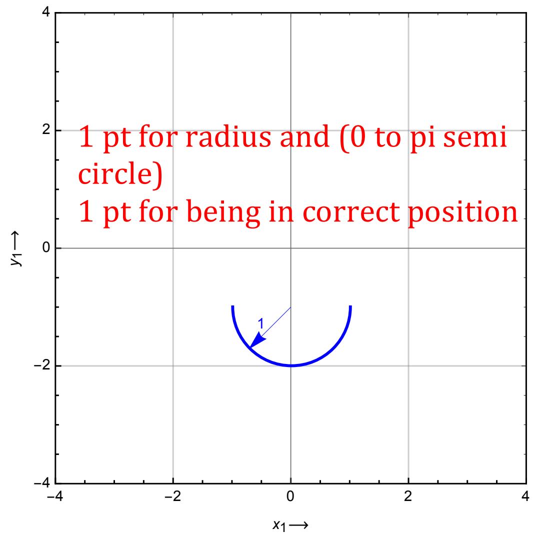
$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 5c_2 \\ s_2 & 0 & -c_2 & 5s_2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

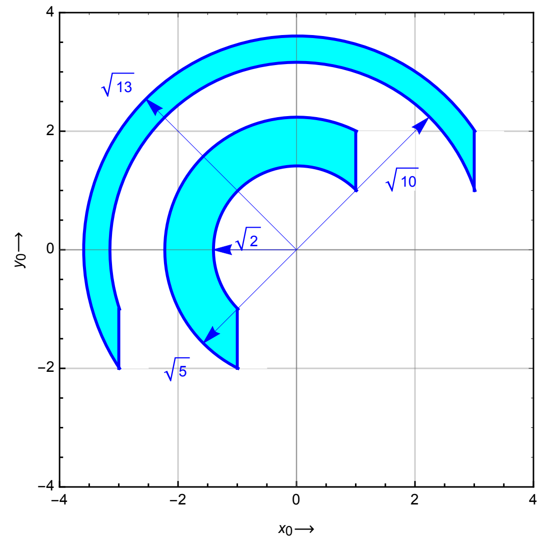
1 error, 4 pts, 2 or 3 errors, 3 pts

Problem 4: _____/25
RPR robot for inverse kinematics.
Shown: $(\theta_1, d_2, \theta_3) = (\frac{\pi}{4}, 1, \frac{7\pi}{32})$









1 pt for each circle (4) of correct radius
1 point shading outer
1 point shading inner
1 point for starting angle
1 point ending angle

a.) **(2pt)** draw & shade x_1, y_1 cross-section of manipulator's *workspace* at $z_1 = 1$. Label all radii.

b.) **(8pt)** draw & shade x_0, y_0 cross-section of manipulator's *workspace* at $z_0 = 2$. Label all radii.

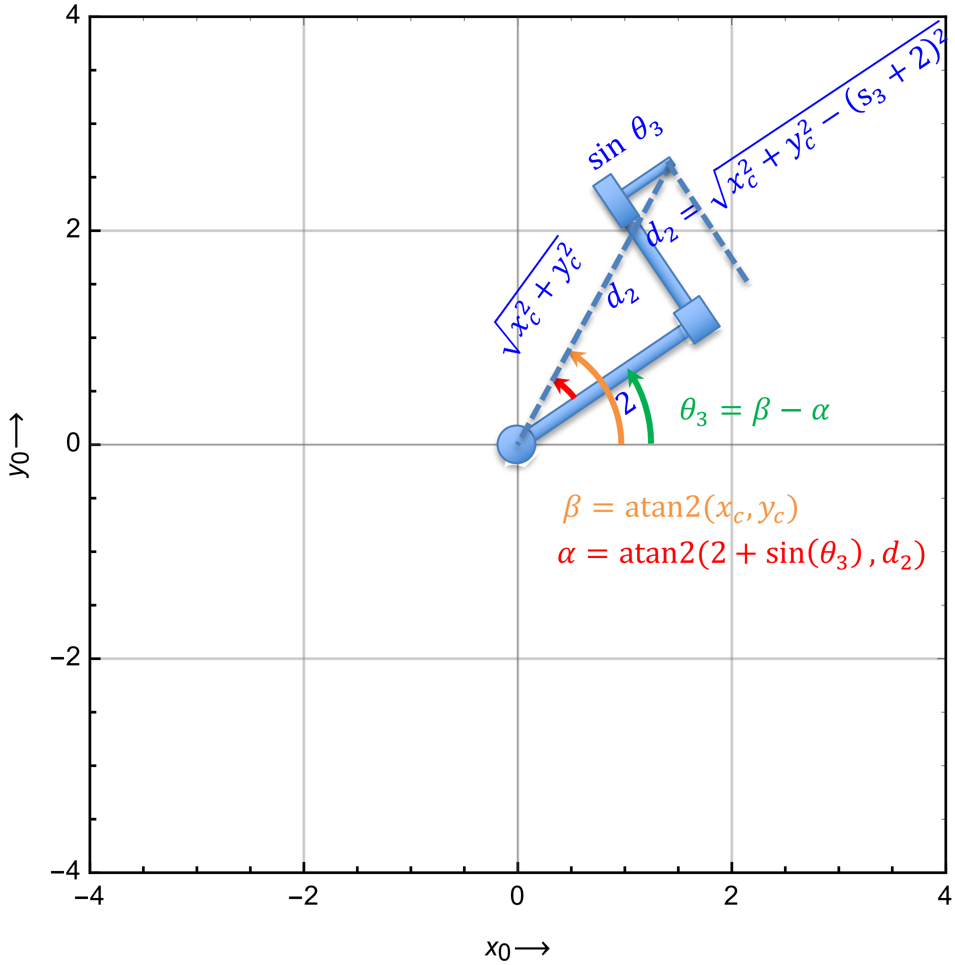
What joint variables place the end-effector at the point $[x_c, y_c, z_c]$ specified in the frame $o_0 x_0 y_0 z_0$? Assume the point is reachable and that $0 \le \theta_3^* \le \pi/2$.

c.) **(5pt)** $\theta_3^* = \mathrm{acos}(z_c - 2)$ or $\mathrm{atan}(z_c - 2, \sqrt{1 - (z_c - 2)^2})$ since $0 \le \theta_3^* \le \pi/2$
Note that this only depends on z_c

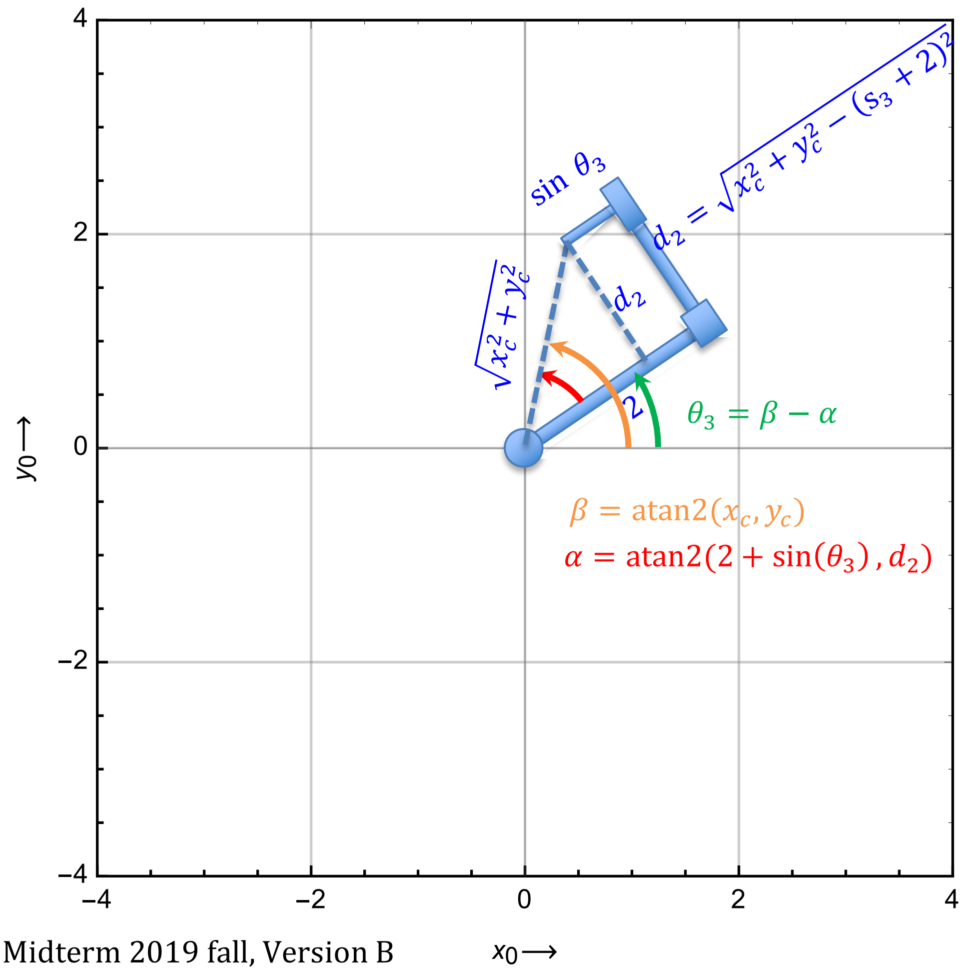
d.) **(5pt)** $d_2^* = \sqrt{x_c^2 + y_c^2 - (\sin(\theta_3) + 2)^2}$

See diagrams on next two pages

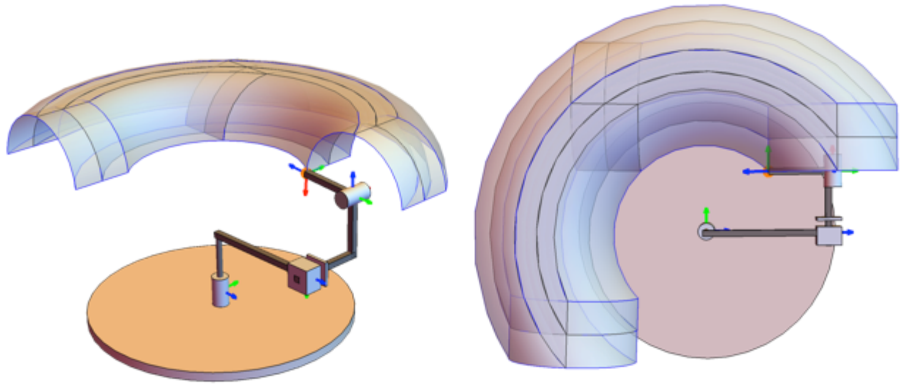
e.) **(5pt)** $\theta_1^* = \mathrm{atan}(x_c, y_c) - \mathrm{atan}(2 + \sin(\theta_3), d_2)$
At least 3 points if answer includes $\mathrm{atan}(x_c, y_c)$



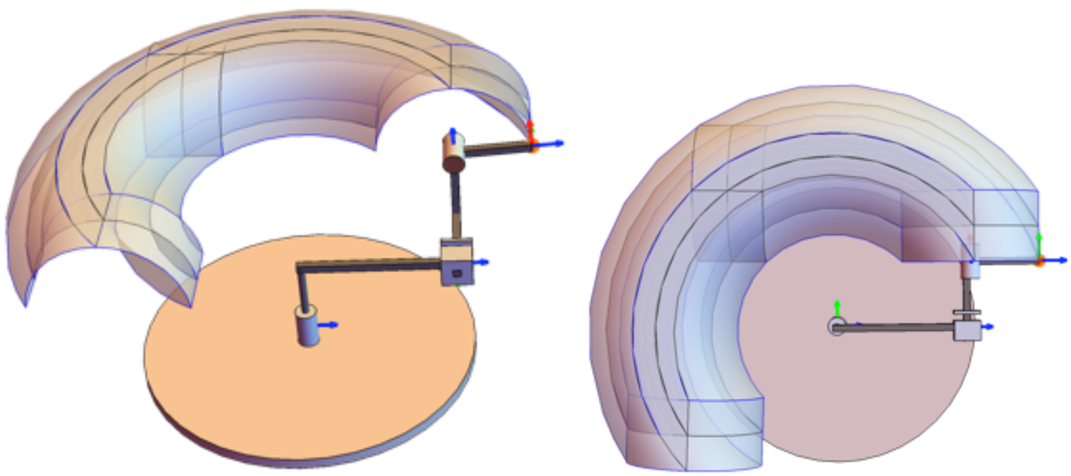
$\sin\theta_3$
$d_2 = \sqrt{x_c^2 + y_c^2 - (s_3 + 2)^2}$
$\sqrt{x_c^2 + y_c^2}$
d_2
2
$\theta_3 = \beta - \alpha$
$\beta = \text{atan2}(x_c, y_c)$
$\alpha = \text{atan2}(2 + \sin(\theta_3), d_2)$
$x_0 \longrightarrow$
$y_0 \longrightarrow$



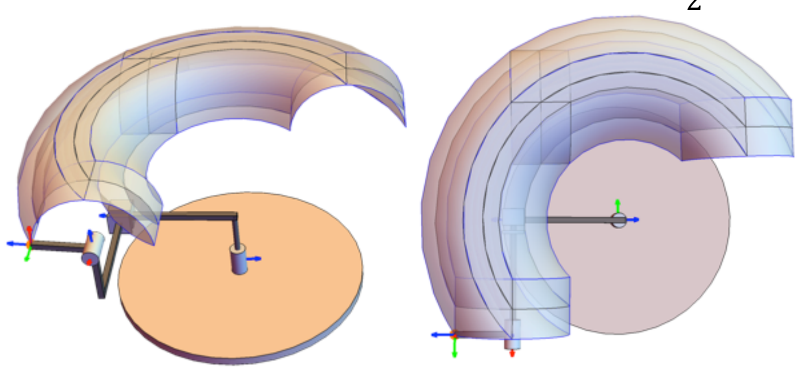
$\sin\theta_3$
$d_2 = \sqrt{x_c^2 + y_c^2 - (s_3 + 2)^2}$
$\sqrt{x_c^2 + y_c^2}$
d_2
2
$\theta_3 = \beta - \alpha$
$\beta = \text{atan2}(x_c, y_c)$
$\alpha = \text{atan2}(2 + \sin(\theta_3), d_2)$
$x_0 \longrightarrow$
$y_0 \longrightarrow$



$(\theta_1, d_2, \theta_3) = (0, 1, -\frac{\pi}{2})$



$(\theta_1, d_2, \theta_3) = (0, 1, \frac{\pi}{2})$



$(\theta_1, d_2, \theta_3) = (\pi, 2, \frac{\pi}{2})$

Problem 5: _____/15

Calculate the manipulator Jacobian of the 2-link RR arm at the position $o_2 = o_c$.

$$T_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 & 4c_1 \\ s_1 & c_1 & 0 & 4s_1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_2^0 = \begin{bmatrix} c_{1+2} & 0 & -s_{1+2} & 4c_1 + 2\,c_{1+2} \\ s_{1+2} & 0 & c_{1+2} & 4s_1 + 2\,s_{1+2} \\ s_2 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a. **(3 pts)** Write out the J matrix in terms of z_i and o_i.

$$J = \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & z_1 \end{bmatrix}$$

-1 per entry

b. **(2 pt)** Write out the z_i and o_i values needed for part a.

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, o_1 = \begin{bmatrix} 4c_1 \\ 4s_1 \\ 4 \end{bmatrix}, o_2 = \begin{bmatrix} 4c_1 + 2\,c_{1+2} \\ 4s_1 + 2\,s_{1+2} \\ 9 \end{bmatrix}$$

-2 if one missing, -4 if 2 missing.

c. **(10 pts)** Write out the J matrix. Calculate the cross products.

$$J = \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & z_1 \end{bmatrix} = \begin{bmatrix} -4\,s_1 - 2\,s_{1+2} & -2\,s_{1+2} \\ 4\,c_1 + 2\,c_{1+2} & 2\,c_{1+2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

-1 for each error

Problem 6: _____/10, Computer Vision

a. **(5 pt)** frames $o_1x_1y_1z_1$ and $o_0x_0y_0z_0$ are related by homogenous transformation H_0^1. A particle has position $[4,2,-6]^T$ relative to frame $o_0x_0y_0z_0$. What is the position of the particle in frame $o_1x_1y_1z_1$?

$$H_0^1 = \begin{bmatrix} 0 & 0 & 1 & -8 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^1 = \begin{bmatrix} R_0^1 & d_0^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -8 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad p^0 = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$$

$$p^1 = R_0^1 p^0 + d_0^1,$$

$$p^1 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix} + \begin{bmatrix} -8 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ -2 \end{bmatrix} + \begin{bmatrix} -8 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -14 \\ -2 \\ 3 \end{bmatrix}$$ -1 per bad entry

b. **(5 pt)** For a camera with focal length $\lambda = 4$, suppose the optical axis is parallel with the world y-axis, the camera x-axis is parallel to the negative world z-axis, (and thus camera y-axis parallel to negative world x-axis) and the center of projection has world coordinates $[3,-9,6]^T$. (if tried, min score 2)

Compute $[18,3,18]^w$ → $(-12,-15,12)^c$ -1 per bad entry

And convert these to image plane coordinates → $(u,v) = (-4,-5)$ -1 per bad entry

$$\begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} -(z^w - 6) \\ -(x^w - 3) \\ y^w + 9 \end{bmatrix}, \quad k \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} u \\ v \\ \lambda \end{bmatrix}, \quad k = \frac{\lambda}{z^c}$$

$(18,3,18)^w$ → $(-12,-15,12)^c$

$$k = \frac{\lambda}{z^c} = \frac{4}{12} = \frac{1}{3},$$

$$(u,v) = \left(\frac{-12}{3}, \frac{-15}{3}\right) = (-4,-5)$$