3.2.

a) $\begin{cases} y_1' = y_2 \\ y_2' = y_1 \end{cases}$

$y_1' = y_2 \quad |()'$

$y_1'' = y_2' \Rightarrow y_1'' = y_1$

$r^2 - 1 = 0 \Rightarrow r = \pm 1$

$z_1(x) = e^x$

$z_2(x) = e^{-x}$ $\Rightarrow z_0 = c_1 e^x + c_2 e^{-x}$

$y_1'' - y_1 = 0$ (ec. omogenă)

$y_1 = c_1 e^x + c_2 e^{-x} \Rightarrow y_2 = (c_1 e^x + c_2 e^{-x})' = c_1 e^x - c_2 e^{-x}$

b) $\begin{cases} y_1' = y_1 - 5y_2 \\ y_2' = 2y_1 - y_2 \end{cases}$

$y_1'' = y_1' - 5y_2' = y_1 - \cancel{5y_2} - 10y_1 + \cancel{5y_2} = -9y_1$

$y_1'' + 9y_1 = 0$

$r^2 + 9 = 0 \Rightarrow r = \pm 3i \Rightarrow z_0 = c_1 \cos 3x + c_2 \sin(+3x)$

$y_1 = c_1 \cos 3x + c_2 \sin 3x$

$y_2 = \frac{y_1 - y_1'}{5} = \frac{1}{5}(c_1 \cos 3x + c_2 \sin 3x + 3c_1 \sin 3x - 3c_2 \cos 3x)$

$y_1' = -3c_1 \sin 3x + 3c_2 \cos 3x \quad \| = \frac{1}{5}\left((c_1 - 3c_2)\cos 3x + (c_2 + 3c_1)\sin 3x\right)$

c) $\begin{cases} y_1' = y_1 + y_2 \Rightarrow y_2 = y_1' - y_1 \\ y_2' = -2y_1 + 4y_2 \end{cases}$

$$y_1'' = y_1' + y_2' = y_1 + y_2 - 2y_1 + 4y_2 = -y_1 + 5y_2 = -y_1 + 5y_1' - 5y_1 = -6y_1 + 5y_1'$$

$$y_1'' - 5y_1' + 6y_1 = 0$$

$$r^2 - 5r + 6 = 0$$

$$\Delta = 25 - 24 = 1 \Rightarrow r_{1,2} = \frac{5 \pm 1}{2} \Rightarrow \begin{matrix} r_1 = 3 \\ r_2 = 2 \end{matrix} \Rightarrow y_1 = c_1 e^{3x} + c_2 e^{2x}$$

$$y_1' = 3c_1 e^{3x} + 2c_2 e^{2x} \Rightarrow y_2 = 3c_1 e^{3x} + 2c_2 e^{2x} - c_1 e^{3x} - c_2 e^{2x} = e^{3x} 2c_1 + e^{2x} c_2$$

d) $\begin{cases} y_1' = y_1 - y_2 \Rightarrow y_2 = -y_1' + y_1 \\ y_2' = y_1 + y_2 \end{cases}$

$$y_1'' = y_1' - y_2' = y_1 - y_2 - y_2 - y_2 = -2y_2$$

$$y_1'' = 2y_1' - 2y_1 \Rightarrow y_1'' - 2y_1' + 2y_1 = 0$$

$$r^2 - 2r + 2 = 0$$

$$\Delta = 4 - 8 = -4 = (2i)^2 \Rightarrow r_{1,2} = \frac{2 \pm 2i}{2} \Rightarrow \begin{matrix} r_1 = 1 + i \\ r_2 = 1 - i \end{matrix}$$

$$y_1 = c_1 e^x \cos x + c_2 e^x \sin x$$

$$y_1' = c_1(e^x \cos x - e^x \sin x) + c_2(e^x \sin x + e^x \cos x) =$$

$$= (c_2 e^x + c_1 e^x) \cos x + (-c_1 e^x + c_2 e^x) \sin y$$

$$\Rightarrow y_2 = \cos x(-c_2 e^x - \cancel{c_1 e^x} + \cancel{c_1 e^x}) + \sin x(c_1 e^x - \cancel{c_2 e^x} + \cancel{c_2 e^x})$$

$$= -c_2 e^x \cos x + c_1 e^x \sin x$$

$$\begin{cases} y_1' = 7y_1 + y_2 \\ y_2' = 7y_2 \end{cases}$$

$y_2' = 7y_2$

$y_2' = \frac{dy}{dx} \Rightarrow \frac{dy}{7y} = dx \ / \int$

$\frac{1}{7} \ln|y| = x + c$

$y_2^{\frac{1}{7}} = e^{x+c}$

$\sqrt[7]{y_2} = e^x \cdot c \ / ()^7$

$y_2 = c_2 e^{7x}$

$* \ y_1' = 7y_1 + c_2 e^{7x}$

$y_1' - 7y_1 = 0 \Rightarrow y_0 = c_1 e^{7x}$

$y_p = \varphi e^{7x} \ / ()'$

$y_p' = \varphi' e^{7x} + 7\varphi e^{7x}$

înlocuim $\Rightarrow \varphi' e^{7x} + \cancel{7\varphi e^{7x}} - \cancel{7\varphi e^{7x}} = c_2 e^{7x}$

$\varphi' e^{7x} = c_2 e^{7x} \ / : e^{7x}$

$\varphi' = c_2 \ / \int \Rightarrow \varphi = c_2 x \Rightarrow y_p = c_2 x e^{7x}$

$\Rightarrow y_1 = c_1 e^{7x} + c_2 x e^{7x}$

f) $\begin{cases} y_1' = 2y_1 - y_2 \Rightarrow y_2 = 2y_1 - y_1' \\ y_2' = 3y_1 - 2y_2 \end{cases}$ $\Rightarrow$

$y_1'' = 2y_1' - y_2' = 2(2y_1 - y_2) - (3y_1 - 2y_2) = 4y_1 - \cancel{2y_2} - 3y_1 + \cancel{2y_2} = y_1$

$y_1'' - y_1 = 0$

$r^2 - 1 = 0 \Rightarrow r = \pm 1 \Rightarrow y_1 = c_1 e^x + c_2 e^{-x}$

$y_1' = c_1 e^x - c_2 e^{-x}$

$\Rightarrow y_2 = 2c_1 e^x + 2c_2 e^{-x} - c_1 e^x + c_2 e^{-x} =$

$= c_1 e^x + 3c_2 e^{-x}$

3.3

a) $\begin{cases} y_1' = y_1 + y_2 \\ y_2' = -2y_1 + 4y_2 \end{cases}$

$y_1 = c_1 \cdot e^{\lambda x}$

$y_2 = c_2 e^{\lambda x}$

$Y(x) = \begin{pmatrix} c_1 e^{\lambda x} \\ c_2 e^{\lambda x} \end{pmatrix}$ $\quad Y'(x) = \lambda \begin{pmatrix} c_1 e^{\lambda x} \\ c_2 e^{\lambda x} \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$

$Y' - AY = 0 \Leftrightarrow \lambda \begin{pmatrix} c_1 e^{\lambda x} \\ c_2 e^{\lambda x} \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda x} \\ c_2 e^{\lambda x} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\left(\lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right) e^{\lambda x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \left(\lambda I_2 - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \right) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad / : e^{\lambda x}$

$$\Rightarrow \left[\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\right] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = 0 \Rightarrow (\lambda - 1)(\lambda - 4) + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\Delta = 25 - 24 = 0 \Rightarrow \lambda_{1,2} = \frac{5 \pm 1}{2} \Rightarrow \lambda_1 = 3, \quad \lambda_2 = 2$$

$$\lambda_1 : \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - c_2 \\ 2c_1 - 2c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = c_2$$

alegem $c_1 = c_2 = 1$

$$\Rightarrow Y_1 = e^{2x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2x} \\ e^{2x} \end{pmatrix}$$

$$\lambda_2 : \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2c_1 - c_2 \\ 2c_1 - c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2c_1 = c_2$$

alegem $c_1 = 1 \Rightarrow c_2 = 2$

$$\Rightarrow Y_2 = e^{3x} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^{3x} \\ 2e^{3x} \end{pmatrix}$$

$$\Rightarrow \underset{\substack{\downarrow \\ \text{mat. sol.}}}{U} = (Y_1 \quad Y_2) = \begin{pmatrix} e^{2x} & e^{3x} \\ e^{2x} & 2e^{3x} \end{pmatrix} \Rightarrow \begin{cases} y_1 = c_1 e^{2x} + c_2 e^{3x} \\ y_2 = c_1 e^{2x} + 2 c_2 e^{3x} \end{cases}$$

y_1

y_2

b) $\begin{cases} y_1' = y_1 - 5y_2 \\ y_2' = 2y_1 - y_2 \end{cases}$ $\quad \det|\lambda I_2 - A| = 0$ $\quad y_1 = c_1 e^{\lambda x}$, $y_2 = c_2 e^{\lambda x}$

$$\det\left[\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix}\right] = 0 \Leftrightarrow \begin{vmatrix} \lambda - 1 & +5 \\ -2 & \lambda + 1 \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda+1) + 10 = 0$$

$$\lambda^2 - 1 + 10 = 0$$

$$\lambda^2 = -9 \Rightarrow \lambda_1 = 3i, \quad \lambda_2 = -3i$$

! $e^{(\alpha + i\beta)x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$

$$\lambda_1 : \begin{pmatrix} 3i-1 & +5 \\ -2 & -3i+1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(c_1, c_2) - vect proprii

$\Updownarrow$

$(a_1 + ib_1, a_2 + ib_2)$

$\Downarrow$

$a_1 = 1 \quad a_2 = \frac{1}{5}$

$b_1 = 0 \quad b_2 = -\frac{3}{5}$

$\alpha = 0$

$\beta = 3$

$$\begin{pmatrix} (3i-1)c_1 + 5c_2 \\ -2c_1 + (-3i+1)c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3ic_1 - c_1 + 5c_2 = 0 \\ -2c_1 - 3ic_2 + c_2 = 0 \end{cases}$$ (avem o inf. sol)

alegem $\underline{\underline{c_1 = 1}} \Rightarrow 3i - 1 + 5c_2 = 0$

$5c_2 = 1 - 3i \Rightarrow \underline{\underline{c_2 = \frac{1-3i}{5}}}$

$$\Rightarrow z = e^{3ix}\begin{pmatrix} 1 \\ \frac{1-3i}{5} \end{pmatrix} = \begin{pmatrix} e^{3ix} \\ \frac{1-3i}{5}e^{3ix} \end{pmatrix} = \begin{pmatrix} \cos 3x + i\sin 3x \\ \frac{1-3i}{5}(\cos 3x + i\sin 3x) \end{pmatrix} =$$

$\downarrow$

de aici z_1 și z_2

$$= \begin{pmatrix} \cos 3x + i\sin 3x \\ \frac{1-3i}{5}\cos 3x + \frac{1-3i}{5} i\sin 3x \end{pmatrix} = \begin{pmatrix} \cos 3x + i\sin 3x \\ \frac{\cos 3x}{5} - \frac{3i\cos 3x}{5} + \frac{i\sin 3x}{5} + 3\sin 3x \end{pmatrix}$$

$$\begin{pmatrix} \cos 3x \\ \frac{1}{5}(\cos 3x) + \frac{3}{5}\sin 3x \end{pmatrix} + i \begin{pmatrix} \sin 3x \\ -\frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \end{pmatrix}$$

$$\downarrow \qquad\qquad \downarrow$$

$$z_1 \qquad\qquad z_2$$

$$U = (z_1 \quad z_2) = \begin{pmatrix} \cos 3x & \sin 3x \\ \frac{1}{5}(\cos 3x) + \frac{3}{5}\sin 3x & -\frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \end{pmatrix}$$

$\to y_1$

$\to y_2$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = U \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos 3x & \sin 3x \\ \frac{1}{5}(\cos 3x) + \frac{3}{5}\sin 3x & -\frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow y_1 = c_1 \cos 3x + c_2 \sin 3x$$

$$y_2 = \frac{c_1}{5}\cos 3x + \frac{c_1}{5}\sin 3x - \frac{3c_2}{5}\cos 3x + \frac{c_2}{5}\sin 3x$$

c) $\begin{cases} y_1' = y_1 + y_2 \\ y_2' = -2y_1 + 4y_2 \end{cases}$ $\det|\lambda I_2 - A| = 0$ e același cu a

d) $\begin{cases} y_1' = y_1 - y_2 \\ y_2' = y_1 + y_2 \end{cases}$ $\det|\lambda I_2 - A| = 0$

$\Rightarrow \det\left(\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right) = 0$

$\begin{vmatrix} \lambda-1 & 1 \\ -1 & \lambda-1 \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 1 + 1 = 0$

$\Delta = 4 - 8 = -4 = (2i)^2$

$\lambda_{1,2} = \frac{2 \pm 2i}{2} \Rightarrow \lambda_1 = 1+i$

$\lambda_2 = 1-i$

$\lambda_1: \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$

$\begin{pmatrix} \alpha_1 i + \alpha_2 \\ -\alpha_1 - i\alpha_2 \end{pmatrix} = 0 \Rightarrow \alpha_1 i + \alpha_2 = 0 \Rightarrow \alpha_2 = -\alpha_1 i$

alegem $\alpha_1 = 1 \Rightarrow \alpha_2 = -i$

$\Rightarrow z = z_1 + i z_2$

$z = e^{(1+i)x}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} e^{(1+i)x} \\ -ie^{(1+i)x} \end{pmatrix} = e^x\begin{pmatrix} \cos x + i\sin x \\ -i\cos x + \sin x \end{pmatrix} = \overbrace{\underbrace{e^x\begin{pmatrix} \cos x \\ \sin x \end{pmatrix}}_{z_1} + e^x i \underbrace{\begin{pmatrix} \sin x \\ -\cos x \end{pmatrix}}_{z_2}}^{U}$ y_1

$Y(x) = U \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^x\cos x & e^x\sin x \\ e^x\sin x & -e^x\cos x \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{cases} y_1 = e^x\cos x\, c_1 + e^x\sin x\, c_2 \\ y_2 = e^x\sin x\, c_1 - e^x\cos x\, c_2 \end{cases}$

$$\begin{cases} y_1' = 7y_1 + y_2 \\ y_2' = 7y_2 \end{cases}$$

$$\det(\lambda I_2 - A) = 0 \Leftrightarrow \det\left(\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}7 & 1\\ 0 & 7\end{pmatrix}\right) = 0$$

$$\begin{vmatrix}\lambda-7 & -1\\ 0 & \lambda-7\end{vmatrix} = 0 \Leftrightarrow (\lambda-7)^2 = 0$$

$$\Rightarrow \underline{\underline{\lambda = 7}}$$

ord. mult. 2

$$\Rightarrow \begin{cases} (A - \lambda I_2)\,u_1 = 0 \\ (A - \lambda I_2)\,u_2 = u_1 \end{cases}$$

$$\begin{cases} \left[\begin{pmatrix}7 & 1\\ 0 & 7\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right] u_1 = 0 \\ \left[\begin{pmatrix}7 & 1\\ 0 & 7\end{pmatrix} - \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right] u_2 = u_1 \end{cases} \Leftrightarrow \begin{cases} \begin{pmatrix}7-\lambda & 1\\ 0 & 7-\lambda\end{pmatrix} u_1 = 0 \\ \begin{pmatrix}7-\lambda & 1\\ 0 & 7-\lambda\end{pmatrix} u_2 = u_1 \end{cases} \Leftrightarrow \begin{cases} \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} u_1 = 0 \\ \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} u_2 = u_1 \end{cases}$$

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha_{11}\\ \alpha_{12}\end{pmatrix} = 0 \Rightarrow \alpha_{12} = 0 \text{ și } \alpha_{11} \in \mathbb{R} \Rightarrow u_1 = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha_{21}\\ \alpha_{22}\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \Rightarrow \alpha_{22} = 1 \text{ și } \alpha_{21} \in \mathbb{R} \Rightarrow u_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\Rightarrow Y_1 = e^{\lambda x} u_1 = e^{7x}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}e^{7x}\\ 0\end{pmatrix}$$

$$Y_2 = e^{\lambda x}(x u_1 + u_2) = e^{7x}\left(x\begin{pmatrix}1\\ 0\end{pmatrix} + \begin{pmatrix}1\\ 1\end{pmatrix}\right) = e^{7x}\begin{pmatrix}x+1\\ 1\end{pmatrix} = \begin{pmatrix}e^{7x}(x+1)\\ e^{7x}\end{pmatrix}$$

$$Y = \begin{pmatrix}e^{7x} & e^{7x}(x+1)\\ 0 & e^{7x}\end{pmatrix} \Rightarrow \begin{aligned} y_1 &= c_1 e^{7x} + c_2 e^{7x}(x+1) \\ y_2 &= c_2 e^{7x} \end{aligned}$$

f) $\begin{cases} y_1' = 2y_1 - y_2 \\ y_2' = 3y_1 - 2y_2 \end{cases}$

$\det(\lambda I_2 - A) = 0$

$\det\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\right] = 0$

$\begin{vmatrix} \lambda-2 & 1 \\ -3 & \lambda+2 \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4 + 3 = 0 \Leftrightarrow \lambda^2 = +1 \Rightarrow \lambda = \pm 1$

λ_1: $\begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2$

alegem $\alpha_1 = 1 \Rightarrow \alpha_2 = 1$

$\Rightarrow Y_1 = e^{\overset{\lambda}{1}x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^x \\ e^x \end{pmatrix}$

$\lambda_2 = \begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_2 = 3\alpha_1$

alegem $\alpha_1 = 1 \Rightarrow \alpha_2 = 3$

$\Rightarrow Y_2 = e^{-x}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} e^{-x} \\ 3e^{-x} \end{pmatrix}$

$U = (Y_1 \ Y_2) = \begin{pmatrix} e^x & e^{-x} \\ e^x & 3e^{-x} \end{pmatrix} \Rightarrow \begin{cases} y_1 = c_1 e^x + c_2 e^{-x} \\ y_2 = c_1 e^x + 3c_2 e^{-x} \end{cases}$