2022

2. Det. soluțiile generale ale ecuațiilor

a) $x^2y' = xy + y^2$

$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2$

not $\frac{y}{x} = z \Rightarrow y = xz \Rightarrow y' = z + xz'$

$\Rightarrow \cancel{z} + xz' = \cancel{z} + z^2 \Rightarrow \frac{z'}{z^2} = \frac{1}{x}$

$z' = \frac{dz}{dx}$

$\Rightarrow \frac{dz}{z^2} = \frac{dx}{x} \,/\!/\int$

$-\frac{1}{z} = \ln x + c$

$z = -\frac{1}{\ln x c}$

$\frac{y}{x} = -\frac{1}{\ln x c} \Rightarrow y = -\frac{x}{\ln x c}, c \in \mathbb{R}$

b) $y'' - 4y' + 8y = (25x - 5)e^x$

I rez. ec. omogenă

$r^2 - 4r + 8 = 0 \Rightarrow \Delta = 16 - 32 = -16 \Rightarrow \sqrt{\Delta} = 4i \Rightarrow r_{1,2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$

$\alpha = 2$

$\beta = 2$

$y_1 = e^{2x} \cos 2x$

$y_2 = e^{2x} \sin 2x$

$y_1 = c_1 e^{2x} \cos 2x + c_2 e^{2x} \sin 2x$

II aflăm o sol. particulară: avem $e^x(25x - 5) \Rightarrow e^x(ax + b)$

$y_p = e^x(ax + b)$

$y_p' = e^x(a) + e^x(ax + b) = e^x(ax + a + b)$

$y_p'' = e^x(ax + a + b) + e^x(a) = e^x(ax + 2a + b)$

înlocuim $\Rightarrow e^x(ax + 2a + b) - 4e^x(ax + a + b) + 8e^x(ax + b) = e^x(25x - 5)$

$\Rightarrow 5ax - 2a + 5b = 25x - 5 \Rightarrow a = 5$

$-10 + 5b = -5 \Rightarrow 5b = 5 \Rightarrow b = 1$

$\Rightarrow y = y_o + y_p$

$\Rightarrow y_p = e^x(5x + 1)$

3. Det. soluția problemei Cauchy

$$\begin{cases} xy'' + y' = 4x \\ y(1) = 1 \\ y'(1) = 4 \end{cases}$$

not $y' = z \Rightarrow xz' + z = 4x$

I rez. ec. omogenă $xz' + z = 0$

$$\left.\begin{aligned} \frac{z'}{z} &= -\frac{1}{x} \\ z' &= \frac{dz}{dx} \end{aligned}\right\} \Rightarrow \frac{dz}{z} = -\frac{dx}{x} \Big/ \int \Rightarrow \ln|z| = -\ln|x| + c$$

$$z_0 = \frac{1}{x} c$$

II căutăm o sol particulară $z_p = \frac{1}{x} \cdot c(x)$

$$z_p' = \frac{c'(x) \cdot x - c(x) \cdot 1}{x^2}$$

înlocuim sus $\frac{c'(x) \cdot x - c(x)}{x} + \frac{c(x)}{x} = 4x \;/ \cdot x$

$$x c'(x) = 4x^2$$

$$c'(x) = 4x \;/\int \Rightarrow c(x) = 4\frac{x^2}{2} \Rightarrow c(x) = 2x^2$$

$$\Rightarrow z_p = 2x$$

$$\Rightarrow z = z_0 + z_p = \frac{c}{x} + 2x$$

$$y' = \frac{c}{x} + 2x \;/\int \Rightarrow y = c_1 \cdot \ln x + x^2 + c_2$$

$$y(1) = 1 \Rightarrow 1 + c_2 = 1 \Rightarrow c_2 = 0$$

$$y'(1) = 4 \Rightarrow c_1 + 2 = 4 \Rightarrow c_1 = 2$$

$$\Rightarrow y = 2\ln x + x^2$$

Se consideră sistemul. Să se studieze stabilitatea lui $x^*(0,0)$ utilizând funcția de tip Lyapunov $V(x,y)=x^2+y^2$.

$$\begin{cases} x'(t) = -x+e^x y \\ y'(t) = -e^x x - y \end{cases}$$

$f_1 = -x+e^x y$

$f_2 = -e^x x - y$

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -1+e^x \cdot y & e^x \\ -e^x(x+1) & -1 \end{pmatrix}$$

$J_f(0,0) = 1-1 = 0$ nu se aplică T. stab. în prima aprox.

$V(x,y)=x^2+y^2$, $D=\mathbb{R}^2$

$V(0,0)=0$, $V(x,y)>0 \ \forall\, x,y \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\dot V(x,y) = \frac{\partial V}{\partial x}\cdot f_1 + \frac{\partial V}{\partial y}\cdot f_2 = 2x\cdot(-x+e^x y) + 2y(-e^x x - y) = -2x^2-2y^2 < 0 \ \forall\, x,y\in\mathbb{R}^2$$

$\Rightarrow x^*(0,0)$ este local as. stabil

5. Să se determine și să se studieze stabilitatea punctelor de ech. pt. ec.

$x' = ax^2-x^3-a+x$ unde a parametru real.

$f(x) = ax^2-x^3-a+x = x^2(a-x)-(a-x) = (a-x)(x^2-1)$

$f(x)=0 \Rightarrow (a-x)(x^2-1)=0 \Rightarrow x^*=a$ pct. de ech.

x	$-\infty$		a		∞
$f(x)$	+ + + + +	$\rightarrow$	0	— — — $\leftarrow$	

$\Rightarrow$ pct. local asimptotic stabil

6. Se consideră problema Cauchy. Ec. integrală Volterra, formula șirului aprox. succesive pt. funcția de start $y_0(x) \equiv 1$ primele 2.

$$\begin{cases} y' = 3x^2 + xy^2 \\ y(0) = 1 \end{cases}$$

$x_0 = 0,\ y_0 = 1 \quad f(x,y) = 3x^2 + xy^2$

ec. Volterra: $y = 1 + \int_0^x (3s^2 + s y(s)^2)\, ds$

$$y_{m+1} = y_0 + \int_{x_0}^{x} f(s, y_m(s))\, ds$$

$$y_1 = y_0 + \int_0^x f(s,1) = 1 + \int_0^x (3s^2 + s)\, ds = 1 + \left(s^3 + \frac{s^2}{2}\right)\Big|_0^x =$$

$$= 1 + x^3 + \frac{x^2}{2}$$

$$y_2 = 1 + \int_0^x f\left(s, 1 + s^3 + \frac{s^2}{2}\right) ds =$$

$$= 1 + \int \left(3s^2 + s\left(1 + s^3 + \frac{1}{2}s^2\right)^2\right) =$$

$$= 1 + \int \left[3s^2 + s\left(1 + s^6 + \frac{1}{4}s^4 + 2s^3 + s^2 + s^5\right)\right] =$$

$$= 1 + \int \left(3s^2 + s + s^{10} + \frac{1}{4}s^5 + 2s^4 + s^3 + s^6\right) =$$

$$= 1 + \left(\cancel{3}\frac{s^3}{\cancel{3}} + \frac{s^2}{2} + \frac{s^{11}}{11} + \frac{1}{4}\frac{s^6}{6} + 2\frac{s^5}{5} + \frac{s^4}{4} + \frac{s^7}{7}\right)\Big|_0^x =$$

$$= 1 + x^3 + \frac{x^2}{2} + \frac{x^{11}}{11} + \frac{x^6}{24} + \frac{2}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{7}x^7$$

Se consideră sistemul $\begin{cases} x'(t) = -4y(t) \\ y'(t) = x(t) \end{cases}$

a) fluxul

b) portret fazic și stabilitatea lui $(0,0)$

a) $x(0) = \eta_1$

$y(0) = \eta_2$

$x'(t) = -4y(t) \Rightarrow x'' = -4y' = -4x$

$x'' + 4x = 0$

$r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow x = c_1 \cos 2x + c_2 \sin 2x$

$x' = -2c_1 \sin 2x + 2c_2 \cos 2x$

$x' = -4y \Rightarrow y = \frac{x'}{-4} \Rightarrow y = \frac{1}{2} c_1 \sin 2x - \frac{1}{2} c_2 \cos 2x$

$x(0) = \eta_1 \Rightarrow c_1 = \eta_1$

$y(0) = \eta_2 \Rightarrow -\frac{1}{2} c_2 = \eta_2 \Rightarrow c_2 = -2\eta_2$

$\Rightarrow \varphi(t, \eta_1, \eta_2) = (\eta_1 \cos 2x - 2\eta_2 \sin 2x, \frac{1}{2}\eta_1 \sin 2x + \eta_2 \cos 2x)$ $\frac{1}{4x}$

$\left.\begin{array}{l} \varphi : I_{max} \times \mathbb{R}^2 \to \mathbb{R}^2 \\ I_{max} = \mathbb{R} \end{array}\right\} \Rightarrow \varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

b) $A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$ $\det|\lambda I_2 - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -4 \\ 1 & \lambda \end{vmatrix} = 0$

$\lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4$ local stabil

$\lambda = \pm 2i$ de tip centru

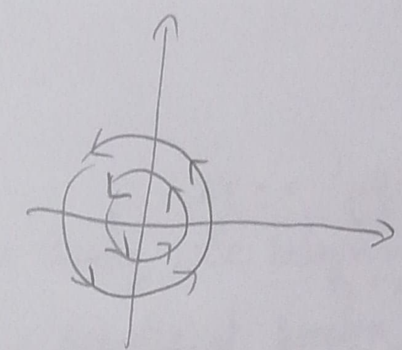
$$\begin{cases} \dfrac{dx}{dt} = -4y \\[2mm] \dfrac{dy}{dt} = x \end{cases}$$

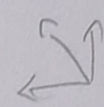
$(:)$

$\Rightarrow \dfrac{dx}{dy} = -\dfrac{4y}{x}$

$x\,dx = -4y\,dy \,/\int$

$\dfrac{x^2}{2} = -4\dfrac{y^2}{2} \Leftrightarrow \dfrac{x^2}{2} = -2y^2 \Rightarrow \dfrac{x^2}{2} + 2y^2 = 0$





1. $\begin{cases} x'(t) = r \cdot x \\ x(0) = x_0 \end{cases}$

$x_0 = 1000$

$x(10) = 50\,000$

$\dfrac{dx}{dt} = rx \Rightarrow \dfrac{dx}{x} = r\,dt \,/\int$

$\ln x = rt + c \Rightarrow x = e^{rt+c} \Rightarrow x = c e^{rt}$

$x_0 = 1000 \Rightarrow x(0) = 1000 \Rightarrow c = 1000$

$x(10) = 50\,000 \Rightarrow 1000 \cdot e^{10r} = 50000$

$\Rightarrow 10r = \ln 50$

$r = \dfrac{\ln 50}{10}$