SUBIECT 4

3. Det. sol. generale pt. ecuațiile:

a) $x \ln x \cdot y' - 2y = \ln x$

I rez. ec. omogenă

$x \ln x \, y' - 2y = 0$

$\frac{1}{y} dy = 2 \frac{1}{x \ln x} dx \ /\int$

$\ln|y| = 2 \ln(\ln x) + c \Rightarrow y_0 = c \ln^2 x$

II det o sol. particulară

$y_p = c(x) \cdot \ln^2 x$

$y_p' = c'(x) \cdot \ln^2 x + c(x) \cdot 2 \ln x \cdot \frac{1}{x}$

înlocuim $\Rightarrow x \ln x \, (c'(x) \ln^2 x + c(x) 2 \ln x \frac{1}{x}) - 2 c(x) \ln^2 x = \ln x$

$x \ln^3 x \, c'(x) = \ln x$

$x \ln^2 x \, c'(x) = 1 \Rightarrow c'(x) = \frac{1}{x \ln^2 x} \ /\int$

$c(x) = \int \frac{1}{x} \cdot \frac{1}{\ln^2 x} dx = \int \frac{1}{t^2} dt = \frac{-1}{t} = -\frac{1}{\ln x}$

not $\ln x = t$

$\Rightarrow dt = \frac{1}{x} dx$

$\Rightarrow y_p = -\ln x$

$\Rightarrow y = -\ln x + c \ln^2 x$

b) $y'' - 2y' + 5y = (8x+4)e^x$

I rez. ec. omogenă: $r^2 - 2r + 5 = 0$

$\Delta = -16 \Rightarrow \sqrt{\Delta} = 4i \Rightarrow r_1 = 1+2i$, $r_2 = 1-2i \Rightarrow \alpha = 1$, $\beta = 2$

$\Rightarrow y_{01} = e^x \cos 2x$

$y_{02} = e^x \sin 2x$

$\Rightarrow y_1 = c_1 e^x \cos 2x + c_2 e^x \sin 2x, \quad c_1, c_2 \in \mathbb{R}$

II det o sol particulară

$(8x+4)e^x \Rightarrow$ avem $e^x(ax+b) = y_p$

$\Rightarrow y_p' = e^x(ax+b) + e^x(a) = e^x(ax+a+b)$

$y_p'' = e^x(ax+a+b) + e^x(a) = e^x(ax+2a+b)$

Înlocuim: $e^x(ax+2a+b) - 2e^x(ax+a+b) + 5e^x(ax+b) = (8x+4)e^x$

$\Rightarrow 4ax + 4b = 8x+4 \Rightarrow \begin{cases} a=2 \\ b=1 \end{cases} \Rightarrow y_p = e^x(2x+1)$

III scriem sol. finală: $y = y_o + y_p = c_1 e^x \cos 2x + c_2 e^x \sin 2x + e^x(2x+1),\ c_1, c_2 \in \mathbb{R}$

4. Det. sol. problemei Cauchy:

$$\begin{cases} (x^2+3)y'' - 2xy' = -\dfrac{(x^2+3)^2}{x^2} \\ y(1) = 0 \\ y'(1) = 4 \end{cases}$$

I rez. ec. omogenă, not $y' = z \Rightarrow \left.\begin{matrix} (x^2+3)z' - 2xz = 0 \\ z' = \dfrac{dz}{dx} \end{matrix}\right\} \Rightarrow \dfrac{dz}{z} = \dfrac{2x}{x^2+3}dx \,\Big/\!\!\int$

$\Rightarrow \ln|z| = \ln|x^2+3| + c \Rightarrow z_o = c(x^2+3),\ c \in \mathbb{R}$

II det. o sol. particulară

$z_p = c(x)(x^2+3)$

$z_p' = c'(x)(x^2+3) + c(x)(2x)$

înlocuim $\Rightarrow (x^2+3)\left(c'(x)(x^2+3) + c(x)\cdot 2x\right) - 2x\,c(x)(x^2+3) = -\dfrac{(x^2+3)^2}{x^2}$

$c'(x) = -\dfrac{1}{x^2} \Big/\!\!\int \Rightarrow c(x) = \dfrac{1}{x} \Rightarrow z_p = \dfrac{1}{x}\cdot(x^2+3)$

$z = z_o + z_p = c(x^2+3) + \dfrac{1}{x}(x^2+3)$

$y' = z \Rightarrow y' = c_1(x^2+3) + \dfrac{1}{x}(x^2+3) \Big/\!\!\int$

$y = c_1\dfrac{x^3}{3} + 3c_1 x + \dfrac{x^2}{2} + 3\ln x + c_2$

$$\begin{cases} y(1)=0 \\ y'(1)=4 \end{cases} \Rightarrow \begin{cases} \frac{1}{3}c_1 + 3c_1 + \frac{1}{2} + c_2 = 0 \\ c_1 \cdot 4 + 1 \cdot 4 = 4 \Rightarrow c_1 = 0 \end{cases} \Rightarrow c_2 = -\frac{1}{2}$$

$$\Rightarrow y = \frac{x^2}{2} + 3\ln x - \frac{1}{2}$$

5. Se consideră pb. Cauchy $\begin{cases} y' = 3x^2 + y^2 \\ y(0) = 1 \end{cases}$.

a) ec. integrală Volterra

b) formula șirului aprox. succesive și pt. funcția de start $y_0(x) \equiv 1$ calculează primele 2 aproximații succesive

ecuația Volterra: $y = y_0 + \int_{x_0}^{x} f(s, y(s))\,ds$

șirul aproximațiilor succesive: $y_{m+1} = y_0 + \int_{x_0}^{x} f(s, y_m(s))\,ds$

a) $y_0(x) \equiv 1$, $x_0 = 1$

$f(x,y) = 3x^2 + y^2$

$y = 1 + \int_0^x (3s^2 + y(s)^2)\,ds$

b) $y_{m+1} = y_0 + \int_{x_0}^{x} f(s, y_m(s))$

$y_1(x) = 1 + \int_{x_0}^{x} f(s,1)\,ds = 1 + \int_0^x (3s^2 + 1)\,ds = 1 + \left(\cancel{3}\frac{s^3}{\cancel{3}} + s\right)\Big|_0^x = 1 + x^3 + x$

$y_2(x) = 1 + \int_0^x f(s, 1+x+x^3)\,ds$

$y_2(x) = 1 + \int_0^x 3s^2 + (1+s+s^3)^2 = 1 + \int_0^x (3s^2 + 1 + s^2 + s^6 + 2s + 2s^3 + 2s^4) =$

$= 1 + \left(s + \cancel{2}\frac{s^2}{\cancel{2}} + 4\frac{s^3}{3} + \cancel{2}\frac{s^4}{\cancel{4}2} + 2\frac{s^5}{5} + \frac{s^7}{7}\right)\Big|_0^x =$

$= 1 + x + x^2 + \frac{4}{3}x^3 + \frac{1}{2}x^4 + \frac{2}{5}x^5 + \frac{1}{7}x^7$

6. $\begin{cases} x'(t) = -y(t) \quad /()' \\ y'(t) = 4x(t) \end{cases} \Rightarrow x'' = -y' \Rightarrow x'' = -4x \Leftrightarrow x'' + 4x = 0$

$r^2 + 4 = 0 \Rightarrow r_{1,2} = \pm 2$

$\alpha = 0 \quad \beta = 2$

$x = C_1 \cos 2x + C_2 \sin 2x$

a) fluxul

b) portretul fazic. Stabilește tipul punctului de ech. (0,0)

a) $x(0) = \eta_1$

$y(0) = \eta_2$

$x' = -2C_1 \sin 2x + 2C_2 \cos 2x$

$x' = -y \Rightarrow y = -x' = 2C_1 \sin 2x - 2C_2 \cos 2x$

$x(0) = \eta_1 \Rightarrow C_1 = \eta_1$

$y(0) = \eta_2 \Rightarrow -2C_2 = \eta_2 \Rightarrow C_2 = -\dfrac{\eta_2}{2}$

$\varphi(t, \eta_1, \eta_2) = \left(\eta_1 \cos 2x - \dfrac{\eta_2}{2} \sin 2x,\ 2\eta_1 \sin 2x + \eta_2 \cos 2x\right)$

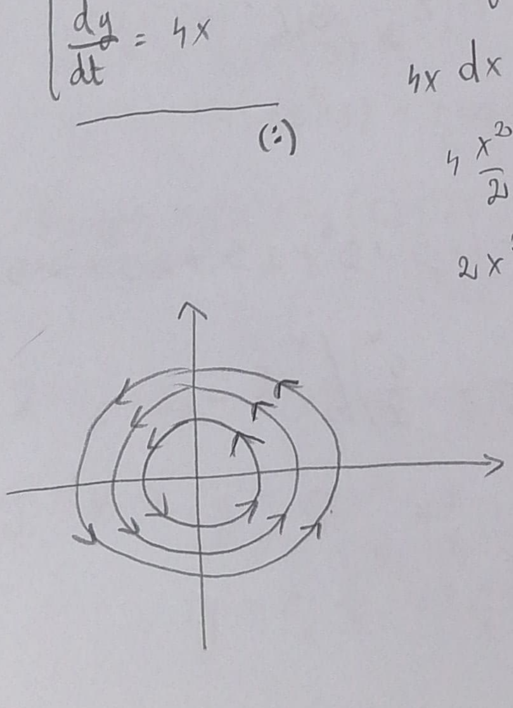
$\left.\begin{array}{l} \varphi : I_{max} \times \mathbb{R}^2 \to \mathbb{R}^2 \\ I_{max} = \mathbb{R} \end{array}\right\} \Rightarrow \varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

b) $\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = 4x \end{cases}$ $(:)$ $\Rightarrow \dfrac{dx}{dy} = -\dfrac{y}{4x}$

$4x\, dx = -y\, dy \quad / \int$

$4\dfrac{x^2}{2} = -\dfrac{y^2}{2} + C$

$2x^2 = -\dfrac{1}{2} y^2 + C$



$\left.\begin{array}{l} x > 0 \\ y > 0 \end{array}\right\} \Rightarrow \begin{array}{l} x' = -y < 0 \\ y' = 4x > 0 \end{array} \Rightarrow$

$$\begin{cases} x'(t) = -x + xy \\ y'(t) = -4y + 8xy \end{cases}$$ punctele de ech + stabilitatea

$$\begin{cases} -x + xy = 0 \Rightarrow x = xy \Rightarrow y = 1 \text{ și } x \neq 0 \\ -4y + 8xy = 0 \Rightarrow 4y = 8xy \Rightarrow x = \frac{1}{2} \text{ și } y \neq 0 \end{cases}$$

dacă $x = 0$ și $y = 0$ se verifică

$\Rightarrow X_1^*(0,0)$, $X_2^*\left(\frac{1}{2}, 1\right)$

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} y-1 & x \\ 8y & 8x-4 \end{pmatrix}$$

pt. punctul $(0,0) \Rightarrow J_f = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} = A$ $\quad \det(\lambda I_2 - A) = \begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda+4 \end{vmatrix}$

$\Rightarrow \lambda_1 = -1$, $\lambda_2 = -4$ $\Rightarrow$ local as. stabil de tip nod

pt. punctul $\left(\frac{1}{2}, 1\right) \Rightarrow J_f = \begin{pmatrix} 0 & \frac{1}{2} \\ 8 & 0 \end{pmatrix} = A$ $\quad \det(\lambda I_2 - A) = \begin{vmatrix} \lambda & -\frac{1}{2} \\ -8 & \lambda \end{vmatrix} =$

$= \lambda^2 - 4 \Rightarrow \lambda = \pm 2 \Rightarrow$ instabil tip șa