SISTEME DINAMICE

SEMINAR 4

Probleme atașate ec. dif.

Problema Cauchy

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y^0 \end{cases} \qquad y^0 = \begin{pmatrix} y_1^0 \\ y_2^0 \\ \vdots \\ y_m^0 \end{pmatrix} \in \mathbb{R}^n \text{ vector de valori}, \ x_0 \in \mathbb{R}$$

Exerciții:

① $\begin{cases} (1+e^x) \cdot y \cdot y' - e^x = 0 \\ y(0) = 1 \end{cases}$ → ec. diferențială EVS

$(1+e^x) \cdot y \cdot y' = e^x \Rightarrow y' = \dfrac{e^x}{y(1+e^x)}$; $f(x) = \dfrac{e^x}{1+e^x}$

$g(y) = \dfrac{1}{y}$

$y' = \dfrac{dy}{dx}$

$\Rightarrow y\,dy = \dfrac{e^x}{1+e^x} dx \,/\int \Rightarrow \dfrac{y^2}{2} = \int \dfrac{e^x}{1+e^x} dx \Rightarrow \dfrac{y^2}{2} = \int \dfrac{1}{t} dt = \ln|t|$

/ not $1+e^x = t \to dt = e^x dx$

$\dfrac{y^2}{2} = \ln(1+e^x) + c \Rightarrow y = \pm\sqrt{2\ln(1+e^x) + c}$ → sol. generală

- $y(0) = 1 \Rightarrow \pm\sqrt{2\ln 2 + c} = 1 \Rightarrow 1 = \sqrt{2\ln 2 + c} \ /()^2 \Rightarrow 1 = 2\ln 2 + c \Rightarrow c = 1 - \ln 4$

$\Rightarrow$ Sol. problemei Cauchy: $y = \sqrt{2\ln(1+e^x) + \underbrace{1 - \ln 4}_{c}}$

② $\begin{cases} y_1' = y_1 + y_2 \\ y_2' = -2y_1 + 4y_2 \\ y_1(0) = 0 \\ y_2(0) = 1 \end{cases}$ Sist. de 2 ec. lin. omogene => alegem una și o derivăm

- $y_1'' = y_1' + y_2' = y_1 + y_2 - 2y_1 + 4y_2 = -y_1 + 5y_2$

$\begin{cases} y_1'' = -y_1 + 5y_2 \\ y_1' = y_1 + y_2 \end{cases} \Big| + \Rightarrow y_1'' - 5y_1' + 6y_2 = 0$ (ec. lin. omog. de ord. 2 cu coef. const)

- Ec. caract. $\underline{r^2 - 5r + 6 = 0}$; $r_1 = 3, r_2 = 2$

$\varphi_1 = e^{2x}, \varphi_2 = e^{3x} \Rightarrow \underline{y_1 = c_1 e^{2x} + c_2 e^{3x}}$

$y_2 = y_1' - y_1 = 2c_1 e^{2x} + 3c_2 e^{3x} - c_1 e^{2x} - c_2 e^{3x} = c_1 e^{2x} + 2c_2 e^{3x}$

$\left.\begin{array}{l} y_1(0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \\ y_2(0) = -1 \Rightarrow c_1 + 2c_2 = -1 \Rightarrow c_1 = -1 - 2c_2 \end{array}\right\} \Rightarrow \begin{array}{l} -c_2 = -1 - 2c_2 \\ c_2 = -1, \ c_1 = 1. \end{array}$

Sol. problemei Cauchy este: $\begin{cases} y_1 = e^{2x} - e^{3x} \\ y_2 = e^{2x} - 2e^{3x} \end{cases}$

③ $\begin{cases} \underline{y'' - 5y' + 4y = 0} \\ y(0) = 5 \\ y'(0) = 8 \end{cases}$ → Ec. omogenă liniară cu coef. constanți

Ec caract.: $r^2 - 5r + 4 = 0$; $r_{1/2} \begin{cases} 1 \\ 4 \end{cases}$

$y_1(x) = e^x$

$y_2(x) = e^{4x}$

Sol. generală: $y(x) = c_1 e^x + c_2 e^{4x} \,/'$

$y'(x) = c_1 e^x + 4c_2 e^{4x}$

$y(0)=5 \Rightarrow C_1+C_2=5 \Rightarrow C_1=5-C_2$

$(5-C_2)+4C_2=8 \Rightarrow 5+3C_2=8 \Rightarrow C_2=1$

$C_1=4$

Sol. problemei Cauchy: $y(x)=4e^{x}+e^{4x}$

④ $\begin{cases} \underline{y''+4y=4x} \rightarrow \text{Ec. neomogenă} \\ y(\pi)=0 \\ y'(\pi)=1 \end{cases}$

• Sol. ec. omogene: $y_1''+4y=0$

• Ec. caract.: $r^2+4=0 \Rightarrow r_{1/2}=\pm 2i \qquad r=\alpha+i\beta$

$y_1=e^{0\cdot x}\cdot\cos 2x=\cos 2x$

$y_2=e^{0\cdot x}\sin 2x=\sin 2x$

Sol. ec. omog.: $y_0=C_1y_1+C_2y_2=C_1\cos 2x+C_2\sin 2x$

Sol. particulară:

$f(x)=4x=P_1(x)$ $\boxed{\text{caz I b}}$

$\Rightarrow y_p=xQ_1(x)=x(ax+b)=ax^2+bx$

$y_p''=2a$

$2a+4ax^2+4bx=4x \Rightarrow \begin{cases} b=1 \\ a=0 \end{cases} \Rightarrow y_p=x$

$y=y_0+y_p=x+C_1\cdot\cos 2x+C_2\sin 2x$

$y(\pi)=0 \Rightarrow \pi+C_1=0 \Rightarrow C_1=-\pi$

$y'=1-2C_1\cdot\sin 2x+2C_2\cos 2x$

$y'(\pi)=1+2C_2=1 \Rightarrow C_2=0$

Sol. probl. Cauchy: $y(x)=x-\pi\cos 2x$

Problema bilocală (mai multe sol.)

⑤ $\begin{cases} y'' + \pi^2 y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$ → Ec. lin. omogenă

• Ec. caract.: $r^2 + \pi^2 = 0$

$r_{1/2} = 0 \pm i \cdot \pi$

• Sol. fundamentale: $y_1(x) = e^{0x} \cdot \cos(\pi x)$

$y_2(x) = e^{0x} \cdot \sin(\pi x)$

• Sol. generală: $y = c_1 \cos(\pi x) + c_2 \sin(\pi x)$

$y(0) = c_1 \Rightarrow c_1 = 0$

$y(1) = -c_1 \Rightarrow c_1 = 0$

Sol. problemei bilocale: $y = c_2 \sin(\pi x)$

⑥ $\begin{cases} y'' + y = x \\ y(0) = 1 \\ y(\frac{\pi}{2}) = \frac{\pi}{2} \end{cases}$ → ec. lin. neomog

I • Ec. lin. omog: $y'' + y = 0$

• Ec. caract: $r^2 + 1 = 0 \Rightarrow r_{1/2} = \overset{\alpha \pm i\beta}{0 \pm i} \Rightarrow y_1 = e^{0x} \cdot \cos x$

$y_2 = e^{0x} \cdot \sin x$

Sol. omogenă: $y_0 = c_1 \cos x + c_2 \sin x$

II Det. o sol. particulară a ec. neomog.:

$y'' + y = x$

$f(x) = x$ $\boxed{\text{caz I b}}$

$y_p = x \cdot Q_1(x) = x(ax + b) = ax^2 + bx$

$y'(x) = 2ax + b \Rightarrow y''(x) = 2a$

Înlocuim $\underbrace{2a}_{y''} + \underbrace{ax^2 + bx}_{y} = x \Rightarrow \begin{cases} a = 0 \\ b = 1 \\ 2a = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 1 \end{cases} \Rightarrow \boxed{y_p = x}$

III Sol. gen.: $y = y_0 + y_p = c_1 \cos x + c_2 \sin x + x$

$y(0) = 1 \Rightarrow c_1 = 1$

$y(\frac{\pi}{2}) = \frac{\pi}{2} \Rightarrow c_2 + \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow c_2 = 0$ ⇒ Sol. probl. bilocale: $y = \cos x + x$

Alte tipuri de probleme

⑦ $\begin{cases} x^2 y' \cdot \cos\frac{1}{x} - y \sin\frac{1}{x} = -1 \\ \lim\limits_{x\to\infty} y(x) = 0 \end{cases}$ → Ec. lin. neomog. de ord I $\ /:(x^2 \cdot \cos\frac{1}{x})$

$\Rightarrow y' - y \cdot \frac{tg\frac{1}{x}}{x^2} = -\frac{1}{x^2 \cos\frac{1}{x}}$

Ec. omog: $y' - y \cdot \frac{tg\frac{1}{x}}{x^2} = 0$

! Sol singulară $y = 0$

$y' = \frac{1}{x^2} \cdot tg\frac{1}{x}\, y$

pp. că $y \neq 0 \Rightarrow \left.\begin{array}{l} \frac{y'}{y} = \frac{1}{x^2} \cdot tg\frac{1}{x} \\ y' = \frac{dy}{dx} \end{array}\right\} \frac{1}{y} dy = \frac{1}{x^2} tg\frac{1}{x} dx \ /\int \Rightarrow \ln|y| = \int \frac{1}{x^2} tg\frac{1}{x} dx$

$\ln|y| = -\int -\frac{1}{x^2} tg\frac{1}{x} dx$

not $u = \frac{1}{x} \to du = -\frac{1}{x^2} dx$

$\Rightarrow \ln|y| = -\int tg\, u\, du \Leftrightarrow \ln|y| = -(-\ln|\cos\frac{1}{x}| + c)$

$\Rightarrow \ln|y| = \ln|\cos\frac{1}{x}| + c = \ln|c_1 \cdot \cos\frac{1}{x}|$

$y_o = c\cos\frac{1}{x}$