MĂRIRE SD 2021

1. Det. soluțiile generale pt. ecuațiile:

a) $2y^2+(x^2-2xy)\cdot y'=0$

b) $(1-x^2)\cdot y'+2xy=4x$

c) $y''-2y'+5y=10\sin x$

a) $y'=-\frac{2y^2}{x^2-2xy}=\frac{2y^2}{2xy-x^2}=\frac{2y^2}{x^2(2\frac{y}{x}-1)}=2\left(\frac{x}{y}\right)^2\cdot\frac{1}{2\frac{x}{y}-1}=2z^2\cdot\frac{1}{2z-1}$

$z=\frac{y}{x}\Rightarrow y=zx\Rightarrow y'=z'x+z$

$\Rightarrow\quad z'x+z=\frac{2z^2}{2z-1}\Leftrightarrow z'x=\frac{2z^2}{2z-1}-z=\frac{2z^2-2z^2+z}{2z-1}$

$\left.\begin{array}{l} z'x=\frac{z}{2z-1} \\ z'=\frac{dz}{dx}\end{array}\right\}\Rightarrow\int\frac{2z-1}{z}\,dz=\int\frac{1}{x}\,dx$

$2\int\frac{z}{z}-\int\frac{1}{z}=\ln x$

$2z-\ln z=\ln x+c$

$2\frac{y}{x}-\ln\frac{y}{x}=\ln x+c$

b) $(1-x^2)y'=2x(2-y)$

$\left.\begin{array}{l}\frac{y'}{2-y}=\frac{2x}{1-x^2} \\ y'=\frac{dy}{dx}\end{array}\right\}\Rightarrow\frac{dy}{2-y}=dx\cdot\frac{2x}{1-x^2}\quad /\int$

$-\int\frac{1}{y-2}\,dy=-\int\frac{2x}{x^2-1}\,dx$

$-\ln(y-2)=-\ln(x^2-1)+c$

$\frac{1}{y-2}=\frac{c}{x^2-1}\Rightarrow y=\frac{x^2-1}{c}+2$

$y=c(x^2-1)+2$

4. Det. ecuația orbitelor din portretul fazic, situate în cadranul pozitiv, pt. s.

$$\begin{cases} x'(t) = -xy \\ y'(t) = -y+3xy \end{cases}$$

$$\frac{dx}{dy} = \frac{xy}{y-3xy} = \frac{x\cancel{y}}{\cancel{y}(1-3x)} = \frac{x}{1-3x} \; / \int$$

$$\int \frac{3x-1}{x} = -\int dy$$

$$3\cdot x - \ln x = y + c$$

$$y = \ln cx - 3x \quad \text{ec. orbitelor}$$

??

5. Se consideră sistemul $\begin{cases} x'(t) = xy-1 \\ y'(t) = x^2-y^2 \end{cases}$ a) pct de ech b) stabilitatea lor

a)

$$\begin{cases} xy-1=0 \Rightarrow xy=1 \Rightarrow x=\frac{1}{y} \\ x^2-y^2=0 \end{cases} \Rightarrow \left(\frac{1}{y}\right)^2 - y^2 = 0 \Rightarrow y^4 = 1 \Rightarrow \begin{cases} y=1 & x=1 \\ y=-1 & x=-1 \\ y=i & x=\frac{1}{i}=-i \\ y=-i & x=-\frac{1}{i}=i \end{cases}$$

$\Rightarrow$ punctele $X_1^*(1,1)$, $X_2^*(-1,-1)$ de ech.

(x și y trb. $\in \mathbb{R}$)

b) $$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}$$

I pt (1,1) $A = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ $\Rightarrow \det(\lambda I_2 - A) = \begin{vmatrix} \lambda-1 & -1 \\ -2 & \lambda-2 \end{vmatrix} = 0$

$$\lambda^2 + \lambda - 4 = 0$$

$\Delta = 1+16 = 17 \Rightarrow \lambda_{1,2} = -\frac{1 \pm \sqrt{17}}{2} \Rightarrow$ local as. stabil de tip șa

II pt. $(-1,-1)$ $A = \begin{pmatrix} -1 & -1 \\ -2 & 2 \end{pmatrix} \Rightarrow \begin{vmatrix} \lambda+1 & 1 \\ 2 & \lambda-2 \end{vmatrix} = 0$

$(\lambda+1)(\lambda-2) - 2 = 0$

$\lambda^2 - \lambda - 2 - 2 = 0$

$\lambda^2 - \lambda - 4 = 0$

$\Delta = 1 + 16 = \sqrt{17} \Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{17}}{2}$

instabil tip șa

1. c) $y'' - 2y' + 5y = 10 \sin x$

$r^2 - 2r + 5 = 0$

$\Delta = 4 - 4 \cdot 5 = -16 \Rightarrow y_{1,2} = \frac{2 \pm 4i}{2} \Rightarrow y_1 = 1+2i$, $y_2 = 1-2i$

$\Rightarrow y_o = c_1 e^x \cos 2x + c_2 e^x \sin 2x$

aflăm o sol particulară:

avem $10 \sin x \Rightarrow y_p = a \cos x + b \sin x$

$y_p' = -a \sin x + b \cos x$

$y_p'' = -a \cos x - b \sin x$

înlocuim $\Rightarrow -a\cos x - b \sin x - 2(-a \sin x + b \cos x) + 5(a \cos x + b \sin x) = 10 \sin x$

$\cos x(-a - 2b + 5a) + \sin x(-b + 2a + 5b) = 10 \sin x$

$\begin{cases} 4a - 2b = 0 \\ 4b + 2a = 10 \end{cases} \underset{:2}{} \Leftrightarrow$ $2a - b = 0 \Rightarrow b = 2a$

$4(2a) + 2a = 10$

$10a = 10 \Rightarrow a = 1$

$b = 2$

$\Rightarrow y_p = \cos x + 2 \sin x$

$\Rightarrow y = \cos x + 2 \sin x + c_1 e^x \cos 2x + c_2 e^x \sin 2x$

2. Det. sol. problemei bilocale

$$\begin{cases} \sin x\, y'' - \cos x\, y' = \sin^2 x \\ y(0) = 1 \\ y(\frac{\pi}{2}) = 2 \end{cases}$$

not $y' = z \Rightarrow \sin x\, z' - \cos x\, z = \sin^2 x \quad /:\sin$

$$z' - ctg\, x \cdot z = \sin x$$

I rez. ec. omogenă

$$\left.\begin{array}{l} z' = z \cdot ctg\, x \\ z' = \frac{dz}{dx} \end{array}\right\} \Rightarrow \frac{dz}{z} = dx\, ctg\, x \ /\int$$

$$\ln|z| = \ln \sin x + c$$

$$z = c \sin x$$

II aflăm o sol particulară

$$z_p = c(x) \cdot \sin x$$

$$z_p' = c'(x) \cdot \sin x + c(x) \cos x$$

înlocuim $\Rightarrow \cancel{\sin x} \cdot c'(x) + \cancel{c(x) \cdot \sin x \cos x} - \cancel{c x \sin x \cos x} = \sin^{\cancel{2}} x$

$$c'(x) = 1 \ /\int \Rightarrow c(x) = x$$

$$z_p = x \sin x$$

$$\Rightarrow z = c \sin x + x \sin x$$

$y' = z \Rightarrow \quad y = \int c \sin x + \underbrace{\int x \sin x}_{} = -c \cdot \cos x + \sin x - x \cos x$

$$I_1 = \int x \sin x = \int \overset{f}{x} \cdot \overset{g'}{(-\cos x)'} = x(-\cos x) + \int \cos x = -x \cos x + \sin x$$

Fie $x(t)>0$ mărimea unei populații ce se dezvoltă conform

$$x' = x(1-x) - a\cdot x$$ unde $a>0$ parametru real. Precizați evoluția.

6. $\begin{cases} y' = -4x^3 + 3xy^2 \\ y(0) = 1 \end{cases}$

a) ec. Volterra, form. şirului aprox succesive

b) f. de start $y_0(x) \equiv 1$ primele 2 aprox

a) $y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\, ds$ ec. Volterra

$y_{n+1} = y_0 + \int_{x_0}^{x} f(s, y_n(s))\, ds$ şirul aprox succesive

$x_0 = 0 \quad y_0 = 1$

$\Rightarrow y(x) = 1 + \int_0^x -4s^3 + 3sy(s)^2\, ds = 1 - \cancel{4}\frac{x^4}{\cancel{4}} + \int_0^x 3sy(s)^2$

b) $y_{n+1}(x) = 1 - x^4 + \int_0^x 3sy(s)^2\, ds$ form. şirului aprox. succesive

$y_1 = 1 - x^4 + \int_0^x 3s\, ds = 1 - x^4 + 3\frac{x^2}{2}$

$y_2 = 1 - x^4 + \int_0^x 3s\left(1 + \frac{3}{2}s^2 - s^4\right)^2 ds =$

$= 1 - x^4 + \int_0^x 3s\left(1 + \frac{9}{4}s^4 - s^{16} + 3s^2 - 3s^6 - 2s^4\right) ds$

$= 1 - x^4 + \frac{3x^2}{2} + \frac{3 \cdot 9}{4}\frac{x^6}{6} - 3\frac{s^{18}}{18} + 9\frac{s^4}{4} - 9\frac{s^8}{8} - 6\frac{s^6}{6}$