SUBIECT 2

3. Det. soluțiile generale pt. ecuațiile:

a) $y' - \frac{1}{\sqrt{1-x^2}} \cdot y = 4x^3 \cdot e^{\arcsin x}$

I det sol. ec. omogene

$$y' - \frac{1}{\sqrt{1-x^2}} y = 0 \iff \left.\begin{array}{l} \frac{y'}{y} = \frac{1}{\sqrt{1-x^2}} \\ y' = \frac{dy}{dx} \end{array}\right\} \Rightarrow \frac{dy}{y} = \frac{dx}{\sqrt{1-x^2}} \Big/ \int \Rightarrow \ln|y| = \arcsin x + c$$

$$y_0 = e^{\arcsin x} \cdot c$$

II det. o sol particulară

$$y_p = c(x) \cdot e^{\arcsin x}$$

$$y_p' = c'(x) e^{\arcsin x} + c(x) \cdot \frac{1}{\sqrt{1-x^2}} \cdot e^{\arcsin x}$$

înlocuim sus $\Rightarrow c'(x) \cancel{e^{\arcsin x}} + \cancel{c(x) \frac{1}{\sqrt{1-x^2}} e^{\arcsin x}} - \cancel{\frac{1}{\sqrt{1-x^2}} c(x) e^{\arcsin x}} = 4x^3 \cancel{e^{\arcsin x}}$

$$c'(x) = 4x^3 \Big/ \int \Rightarrow c(x) = \cancel{4} \frac{x^4}{\cancel{4}} \Rightarrow c(x) = x^4$$

$$\Rightarrow y_p = x^4 \cdot e^{\arcsin x}$$

$$\Rightarrow y = y_0 + y_p = ce^{\arcsin x} + x^4 e^{\arcsin x}$$

b) $y'' + 6y' + 9y = (x+1) e^{-x}$

I rezolvăm ec omogenă $y'' + 6y' + 9y = 0$

$$r^2 + 6r + 9 = 0 \Rightarrow \Delta = 0 \Rightarrow r_{1,2} = -3$$

$$y_1 = e^{-3x} \quad y_2 = x \cdot e^{-3x} \Rightarrow y_0 = c_1 e^{-3x} + c_2 x e^{-3x}, \ c_1, c_2 \in \mathbb{R}$$

II det. o sol particulară avem $e^{-x}(x+1) = e^{-x}(ax+b)$

$$y_p = e^{-x}(ax+b) \Rightarrow y_p' = -e^{-x}(ax+b) + e^{-x} \cdot a$$

$y_p' = +e^{-x}(a-b-ax)$

$y_p'' = -e^{-x}(a-b-ax) + e^{-x}(-a) = e^{-x}(-a+b+ax-a) = e^{-x}(ax-2a+b)$

înlocuim sus:

$$e^{-x}(ax-2a+b) + 6\cdot e^{-x}(a-b-ax) + 9e^{-x}(ax+b) = (x+1)e^{-x}$$

$$e^{-x}(\underline{ax}-2a+b+6a-6b\underline{-6ax}+\underline{9ax}+9b) = e^{-x}(x+1)$$

$$\begin{cases} 4ax = x \Rightarrow 4a = 1 \Rightarrow a = \frac{1}{4} \\ 4a+4b = 1 \Rightarrow 1+4b = 1 \Rightarrow b = 0 \end{cases} \Rightarrow y_p = e^{-x}\cdot\frac{1}{4}x$$

$$\Rightarrow y = y_0 + y_p = C_1 e^{-3x} + C_2\cdot x\cdot e^{-3x} + e^{-x}\cdot\frac{1}{4}x$$

4. Det. sol. problemei Cauchy:

$$\begin{cases} y'' + tg(x)\cdot y' = \cos(x) \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$ not $z = y' \Rightarrow z' + tg x \cdot z = \cos x$

I rez. ec. omogenă

$$\left.\begin{array}{l} z' + z\, tg x = 0 \\ z' = \frac{dz}{dx} \end{array}\right\} \Rightarrow \frac{dz}{z} = -tg x \cdot dx \,/\int$$

$$\ln z = \ln(\cos x) + C \Rightarrow z_0 = c\cos x$$

II aflăm o sol. particulară

$$\left.\begin{array}{l} z_p = c(x)\cdot\cos x \\ z_p' = c'(x)\cdot\cos x - c(x)\cdot\sin x \end{array}\right\} \Rightarrow tg x\,(\cos x\cdot c(x)) + c'(x)\cos x - \cancel{c(x)\sin x} = \cos x$$

$$c'(x) = 1 \,/\int \Rightarrow c(x) = x$$

înlocuim $\Rightarrow z_p = x\cos x \Rightarrow z = x\cos x + c\cos x$

înlocuim înapoi $y \Rightarrow y' = \overbrace{x\cos x}^{I_1} + c\cos x \,/\int \Rightarrow y = c\sin x + x\sin x + \cos x + c_1$

$$I_1 = \int \overset{f}{x}\,\overset{g'}{\cos x} = x\cdot\sin x - \int \sin x = x\sin x + \cos x$$

$y(0)=1 \Rightarrow C_1+1=1 \Rightarrow C_1=0$

$y'(0)=0 \Rightarrow C\cdot 1=0 \Rightarrow C=0$ $\Rightarrow y = x\sin x+\cos x$

5. Se consideră problema Cauchy $\begin{cases} y'=x^2-y^2 \\ y(0)=1 \end{cases}$. Scrieți ecuația integrală Volterra echivalentă cu problema Cauchy, formula șirului aproximațiilor succesive și pt. funcția de start $y_0(x)\equiv 1$ calculați primele 2 aproximații succesive.

$y=1+\int_0^x (s^2-y(s)^2)\,ds$ ec. Volterra

$x_0=0 \quad y_0=1$

$f(x,y)=x^2-y^2$

$y_{n+1}=y_0+\int_{x_0}^x f(s,y_n(s))\,ds$

$y_1=y_0+\int_0^x f(s,y_0)=1+\int_0^x (s^2-1)=1+\left(\frac{s^3}{3}-s\right)\Big|_0^x=1+\frac{x^3}{3}-x$

$y_2=y_0+\int_0^x f(s,y_1)=1+\int_0^x \left[s^2-s\left(1-s+\frac{s^3}{3}\right)^2\right]ds=$

$=1+\int_0^x s^2-\left(1+s^2+\frac{s^6}{9}-2s-2\frac{s^4}{3}+2\frac{s^3}{3}\right)=$

$=1+\int_0^x \left(\cancel{s^2}-1-\cancel{s^2}-\frac{s^6}{9}+2s+\frac{2}{3}s^4-\frac{2}{3}s^3\right)=$

$=1+\int\left(1-\frac{s^6}{9}+2s+\frac{2}{3}s^4-\frac{2}{3}s^3\right)=1+x-\frac{9}{7}x^7+x^2+\frac{10}{3}x^5-\frac{8}{3}x^5$

6. Se consideră sistemul $\begin{cases} x'(t) = x(t) \\ y'(t) = 3y(t) \end{cases}$. Se cere:

a) Să se det. fluxul

b) Să se det. portretul fazic și să se precizeze stabilitatea și tipul punctului de ech. $(0,0)$.

a) $\begin{cases} x' = x \\ y' = 3y \end{cases}$

$x(0) = \eta_1$

$y(0) = \eta_2$

$\left.\begin{matrix} x' = x \\ x' = \frac{dx}{dt} \end{matrix}\right\} \Rightarrow \frac{dx}{x} = dt \ /\int$

$\ln|x| = t + c \Rightarrow x = c_1 e^{t}$

$\left.\begin{matrix} y' = 3y \\ y' = \frac{dy}{dt} \end{matrix}\right\} \Rightarrow \frac{dy}{3y} = dt \ /\int \Rightarrow \ln|y| = 3t + c \Rightarrow y = c_2 \cdot e^{3t}$

$\begin{cases} x = c_1 e^{t} \\ y = c_2 e^{3t} \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \Rightarrow \begin{cases} x = \eta_1 e^{t} \\ y = \eta_2 e^{3t} \end{cases}$

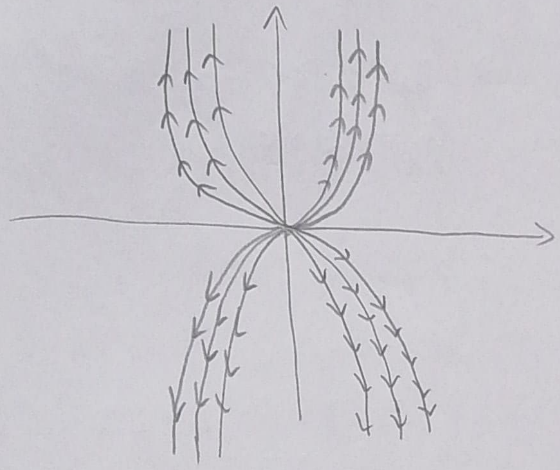
$\varphi(t, \eta_1, \eta_2) = (\eta_1 e^{t}, \eta_2 e^{3t})$

$\left.\begin{matrix} \varphi : I_{max} \cdot \mathbb{R}^2 \to \mathbb{R}^2 \\ I_{max} = \mathbb{R} \end{matrix}\right\} \Rightarrow \varphi : \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{cases} x = \eta_1 e^{t} \quad /()^3 \\ y = \eta_2 e^{3t} \end{cases} \Rightarrow \begin{cases} x^3 = \eta_1^3 e^{3t} \\ y = \eta_2 e^{3t} \end{cases} \Big/ (:) \to \frac{x^3}{y} = \frac{\eta_1^3}{\eta_2} \Rightarrow y = \frac{x^3 \eta_2}{\eta_1^3}$$

vrem să scăpăm de t

$y = x^3 \cdot c,\ c \in \mathbb{R}$



analiză pe cadrane:

$\underline{\text{I}}$ $x(t) > 0$, $y(t) > 0$ $\Rightarrow$ $x' = x > 0$, $y' = 3y > 0$ $\Rightarrow$

$\underline{\text{III}}$ $x(t) < 0$, $y(t) < 0$ $\Rightarrow$ $x' = x < 0$, $y' = 3y < 0$ $\Rightarrow$

la fel pt. $\underline{\text{II}}$ și $\underline{\text{IV}}$

vedem cum e punctul $(0,0)$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \qquad \det(\lambda I_2 - A) = 0 \Rightarrow \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 3 \end{vmatrix} = 0$$

$(\lambda - 1)(\lambda - 3) = 0$

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases} \Rightarrow x^*(0,0)$ pct. de ech. instabil de tip nod

7. Să se determine punctele de ech. și să se studieze stabilitatea acestora:

$$\begin{cases} x'(t) = y^3 + 1 \\ y'(t) = x^2 + y \end{cases}$$

$$\begin{cases} y^3 = -1 \Rightarrow y = -1 \\ x^2 + y = 0 \end{cases} \Rightarrow x = \pm 1 \Rightarrow x_1^*(1,-1),\ x_2^*(-1,-1) \text{ pct. de echilibru}$$

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 3y^2 \\ 2x & 1 \end{pmatrix}$$

I $(1,-1) = J_f(1,-1) = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$

$\det(\lambda I_2 - A) = 0$

$\begin{vmatrix} \lambda & -3 \\ -2 & \lambda-1 \end{vmatrix} = 0 \Rightarrow \lambda(\lambda-1) - 6 = 0$

$\lambda^2 - \lambda - 6 = 0 \Rightarrow \Delta = 25$

$\Rightarrow \lambda_1 = 3 > 0$ pct instabil, tip şa

$\lambda_2 = -2 < 0$

II $(-1,-1)$

$J_f(-1,-1) = \begin{pmatrix} 0 & 3 \\ -2 & 1 \end{pmatrix}$

$\det(\lambda I_2 - A) = 0$

$\begin{vmatrix} \lambda & -3 \\ 2 & \lambda-1 \end{vmatrix} = 0 \Rightarrow \lambda(\lambda-1) + 6 = 0$

$\lambda^2 - \lambda + 6 = 0 \quad \Delta = 23$

$\lambda_1 = \frac{1 + i\sqrt{23}}{2}$ instabil, tip focus

$\lambda_2 = \frac{1 - i\sqrt{23}}{2}$