1.1.

1. $y' = 2x(1+y^2)$

$f(x) = 2x$

$g(y) = 1+y^2$

$y' = \frac{dy}{dx}$

$\frac{dy}{1+y^2} = 2x \cdot dx \,/\int$

$\int \frac{1}{1+y^2} dy = 2\int x\, dx$

$\operatorname{arctg} y = x^2 + C \Rightarrow y = tg(x^2+C)$

2. $(x^2-1)y' + 2xy^2 = 0$

$y' = \frac{-2xy^2}{x^2-1} \Leftrightarrow \frac{y'}{y^2} = -\frac{2x}{x^2-1}$ $\quad y \neq 0$ $\quad$ sol. singulară $y=0$

$y' = \frac{dy}{dx} \Rightarrow \frac{dy}{y^2} = -\frac{2x}{x^2-1} dx \,/\int$

$\int \frac{1}{y^2} dy = -\int \frac{2x}{x^2-1} dx$

$\frac{y^{-1}}{-1} = -\ln|x^2-1| + c \,/\cdot(-1)$

$y^{-1} = \ln|x^2-1| + c \Rightarrow y = \frac{1}{\ln|x^2-1|+c}$

3. $xy' = y^3 + y$

$\left.\begin{array}{l} y' = \frac{1}{x}(y^3+y) \\ y = \frac{dy}{dx} \end{array}\right\} \Rightarrow \frac{dy}{y^3+y} = \frac{dx}{x} \,/\int \Rightarrow \int \frac{1}{y(y^2+1)} dy = \int \frac{1}{x} dx$

$$\int \frac{1}{y(y^2+1)}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{a}{y} + \frac{by+c}{y^2+1} = \frac{ay^2+a+byy+cy}{y(y^2+1)} = \frac{y^2(a+b)+y\cdot c+a}{y(y^2+1)} \Rightarrow \begin{matrix} a=1 \\ b=-1 \\ c=0 \end{matrix}$$

$$\Rightarrow \int \frac{1}{y}\,dy + \int \frac{-y}{y^2+1} = \ln|y| - \frac{1}{2}\ln|y^2+1|$$

$$\ln|y| - \frac{1}{2}\ln|y^2+1| = \ln|x| + c \qquad c = \ln c_1$$

$$\ln\frac{|y|}{|y^2+1|^{\frac{1}{2}}} = \ln|x|\cdot c \Rightarrow \frac{|y|}{\sqrt{y^2+1}} = |x|\cdot c \quad /()^2$$

$$\frac{y^2}{y^2+1} = x^2\cdot c^2 \Leftrightarrow y^2 = x^2c^2(y^2+1)$$

$$y^2 = y^2x^2c^2 + x^2c^2$$

$$y^2(1-x^2c^2) = x^2c^2$$

$$y^2 = \frac{x^2c^2}{1-x^2c^2} \quad /\sqrt{\ }$$

$$y = \pm\frac{|xc|}{\sqrt{1-x^2c^2}}$$

4. $xy + (2x-1)y' = 0$

$$\left.\begin{matrix} y' = \dfrac{-xy}{2x-1} \\ y' = \dfrac{dy}{dx} \end{matrix}\right\} \Rightarrow \frac{dy}{y} = -\frac{x}{2x-1}\cdot dx \quad /\int \qquad y=0 \text{ sol. singulară}$$

$$\int \frac{1}{y}\,dy = -\int \frac{x}{2x-1}\,dx$$

$$\ln|y| = -\frac{1}{2}\int \frac{2x-1+1}{2x-1}$$

$$\ln|y| = -\frac{1}{2}\left(x + \frac{1}{2}\int \frac{2}{2x-1}\,dx\right)$$

$$\ln|y| = -\frac{1}{2}x - \frac{1}{4}\cdot\ln|2x-1| + c = \ln c_1,\ c_1>0$$

$$e^{\ln|y|} = e^{-\frac{1}{2}x - \frac{1}{4}\ln|2x-1| + c}$$

$$|y| = e^{-\frac{1}{2}x - \frac{1}{4}\ln|2x-1| + c} \Rightarrow |y| = e^{-\frac{1}{2}x + \ln|2x-1|^{-\frac{1}{4}}}\cdot c_1$$

$|y| = e^{-\frac{1}{2}x} \cdot |2x-1|^{-\frac{1}{3}} \cdot C_1 \Rightarrow y = e^{-\frac{1}{2}x} \cdot |2x-1|^{-\frac{1}{3}} \cdot C_2$

considerăm o const $C_2 \in \mathbb{R}$

5. $y' = k \cdot \frac{y}{x}$, $k \in \mathbb{R}^*$

$\frac{y'}{y} = \frac{k}{x}$ $\quad y=0$ sol. singulară

$y' = \frac{dy}{dx} \Rightarrow \frac{dy}{y} = \frac{k}{x} dx \; / \int \Rightarrow \int \frac{1}{y} dy = k \int \frac{1}{x} dx$

$\ln|y| = k \ln|x| + c$

$\ln|y| = \ln(|x|^k \cdot c_1)$, $c_1 > 0$

$|y| = |x|^k \cdot c_1$

6. $y - xy' = a(1 + x^2 y')$, $a \in \mathbb{R}^*$

$y - xy' = a + ax^2 y'$

$+xy' + ax^2 y' = y - a$

$y'(x + ax^2) = y - a \Rightarrow y' = \frac{y-a}{x(ax+1)}$

$y' = \frac{dy}{dx} \Rightarrow \frac{dy}{y-a} = \frac{dx}{x(ax+1)} \; / \int \Rightarrow \int \frac{1}{y-a} dy = \int \frac{1}{x(ax+1)} dx$

$y = a$ sol. singulară

$\Leftrightarrow \ln|y-a| = \int \left(\frac{1}{x} + \frac{-a}{ax+1}\right) dx \Leftrightarrow \ln|y-a| = \ln|x| - \ln|ax+1| + c$

$\frac{b}{x} + \frac{c}{ax+1} = \frac{bax + b + cx}{x(ax+1)} = \frac{x(ba+c)+b}{x(ax+1)} \Rightarrow \begin{matrix} b=1 \\ c=-a \end{matrix}$

$\Rightarrow |y-a| = \frac{|x|}{|ax+1|} \cdot c_1$, $c_1 > 0$

$y - a = \frac{|x|}{|ax+1|} \cdot c_2 \Rightarrow y = \frac{|x|}{|ax+1|} c_2 + a$

$c_2 \in \mathbb{R}$

(1.2)

1. $2x^2y' = x^2 + y^2$

$y' = \frac{x^2+y^2}{2x^2} = \frac{1}{2} + \frac{1}{2}\left(\frac{y}{x}\right)^2$

not. $z = \frac{y}{x} \;/()' \Rightarrow z'x + zx' = y' \Leftrightarrow z + z'x = y'$

$\Rightarrow \frac{1}{2} + \frac{1}{2}z^2 = z + z'x \Leftrightarrow z' = \frac{1}{x}\left(\frac{1}{2}z^2 - z + \frac{1}{2}\right)$ EVS

$z = \frac{dz}{dx} \Rightarrow \frac{dz}{\frac{1}{2}z^2 - z + \frac{1}{2}} = \frac{dx}{x} \;/\int$

$z=1$ sol. singulară $\Rightarrow y = x$ sol. sing.

$\Rightarrow \int \frac{1}{\frac{1}{2}z^2 - z + \frac{1}{2}}\,dz = \int \frac{1}{x}\,dx \Leftrightarrow \int \frac{1}{\frac{1}{2}z^2 - z + \frac{1}{2}}\,dz = \ln|x| + c$

$2\int \frac{1}{z^2 - 2z + 1}\,dz = \ln|x| + c$

$2\int \frac{1}{(z-1)^2}\,dz = \ln|x| + c$

$2\,\frac{(z-1)^{-1}}{-1} = \ln|x| + c$

$\frac{1}{z-1} = \frac{1}{-2}(\ln|x| + c) \Rightarrow z - 1 = \frac{-2}{\ln|x| + c}$

$z = \frac{-2}{\ln|x| + c} + 1$

$z = \frac{y}{x} \Rightarrow \frac{-2}{\ln|x|+c} + 1 = \frac{y}{x} \Rightarrow y = x\left(\frac{-2}{\ln|x|+c} + 1\right)$

$y' = -\frac{x+y}{y} = -\left(\frac{x}{y}+1\right) \Rightarrow$ not $z = \frac{y}{x} \Rightarrow z + z'x = y'$

$y' = -\left(\frac{1}{z}+1\right)$

$\frac{-1-z^2-z}{1/z}$

$\Rightarrow z + z'x = -\frac{1}{z} - 1 \Leftrightarrow z'x = -\frac{1}{z} - z - 1 \Leftrightarrow z' = \frac{1}{x}\left(-\frac{1}{z} - z - 1\right)$

$z' = \frac{dz}{dx} \Rightarrow \frac{dz}{-\frac{(z^2+z+1)}{z}} = \frac{dx}{x} \Big/ \int \Rightarrow -\int \frac{z}{z^2+z+1}\, dz = \ln|x| + c$

$-\frac{1}{2}\int \frac{2z+1-1}{z^2+z+1}\, dz = \ln|x|+c$

$\frac{z^2+z+1}{z} = 0$ sol singulară

nu are sol $\in \mathbb{R}$

$-\frac{1}{2}\left(\int\frac{2z+1}{z^2+z+1}\, dz - \int\frac{1}{z^2+z+1}\right) = \ln|x|+c$

$-\frac{1}{2} \ln|z^2+z+1| + \frac{1}{2}\int \frac{1}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}} = \ln|x|+c$

$-\frac{1}{2}\ln|z^2+z+1| + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}\cdot \operatorname{arctg} 2\frac{z+\frac{1}{2}}{\sqrt{3}} = \ln|x|+c$

$-\frac{1}{2}\ln|z^2+z+1| + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2z+1}{\sqrt{3}} = \ln|x|+c$

apoi înlocuim cu $\frac{y}{x}$

3. $y' = e^{\frac{y}{x}} + \frac{y}{x} = e^{z} + z$

not $z' = \frac{y}{x} \Rightarrow z + z'x = y'$

$\left.\begin{array}{l} e^{z} + z = z + z'x \Rightarrow z' = \frac{1}{x}\cdot e^{z} \\ z = \frac{dz}{dx} \end{array}\right\} \Rightarrow \frac{dz}{e^{z}} = \frac{dx}{x} \Big/ \int \Rightarrow \int \frac{1}{e^{z}}\, dz = \int \frac{1}{x}\, dx$

$-e^{-z} = \ln|x|+c$

$e^{-z} = -\ln|x|-c$

$\ln e^{-z} = \ln(-\ln|x|-c)$

$-z = \ln(-\ln|x|-c)$

$z = -\ln(-\ln|x|-c)$

$z = \frac{y}{x} \Rightarrow y = -x\ln(-\ln|x|-c)$

4. $xy' = \sqrt{x^2-y^2}+y$

$$y' = \sqrt{\frac{x^2-y^2}{x^2}} + \frac{y}{x} = \sqrt{1-\left(\frac{y}{x}\right)^2} + \frac{y}{x}$$

not $z = \frac{y}{x} \Rightarrow z'x + z = y'$

$$\sqrt{1-z^2} + \cancel{z} = z'x + \cancel{z} \Leftrightarrow \sqrt{1-z^2}\,\frac{1}{x} = z'$$

$$z' = \frac{dz}{dx} \Rightarrow \frac{dz}{\sqrt{1-z^2}} = \frac{1}{x}dx \ /\int$$

$$\begin{cases} 1-z^2=0 \\ z=\pm 1 \text{ sol singulară} \\ y=\pm x \end{cases}$$

$$\int \frac{1}{\sqrt{1-z^2}}\,dz = \ln|x|+c$$

$$\arcsin z = \ln|x|+c \Rightarrow z = \sin(\ln|x|+c)$$

$$\Rightarrow y = x\sin(\ln|x|+c)$$

5. $y' = \frac{y}{x} + tg\,\frac{y}{x}$

not $z = \frac{y}{x} \Rightarrow z'x+z=y'$ $\Rightarrow \cancel{z} + tg\,z = z'x + \cancel{z}$

$tg\,z=0$ sol singulară

$$\left.\begin{array}{l} z' = \frac{1}{x}(tg\,z) \\ z' = \frac{dz}{dx} \end{array}\right\} \Rightarrow \frac{dz}{tg\,z} = \frac{dx}{x} \ /\int$$

$$\int \frac{1}{tg\,z}\,dz = \ln|x|+c$$

$$\int \frac{\overset{=\sin'}{\cos z}}{\sin z}\,dz = \ln|x|+c$$

$$\ln|\sin z| = \ln|x|\cdot c_1, \quad c_1>0$$

$$|\sin z| = |x|\cdot c_1$$

$$z = \arcsin(|x|\cdot c_1)$$

$$y = x\arcsin(|x|\cdot c_1)$$

$$x - y\cdot\cos\left(\frac{y}{x}\right) + x\cos\left(\frac{y}{x}\right) y' = 0$$

$$y' = \left(y\cdot\cos\frac{y}{x} - x\right)\cdot\frac{1}{x\cos\left(\frac{y}{x}\right)}$$

$$y' = \frac{y}{x} - \frac{1}{\cos\frac{y}{x}}$$

$$\left.\begin{array}{l} y' = \frac{y}{x} - \frac{1}{\cos\frac{y}{x}} \\ \text{not } z = \frac{y}{x} \Rightarrow z'x + z = y' \end{array}\right\} \Rightarrow \cancel{z} - \frac{1}{\cos z} = z'x + \cancel{z}$$

$$\left.\begin{array}{l} z' = -\frac{1}{x\cos z} \\ z' = \frac{dz}{dx} \end{array}\right\} \Rightarrow dz\cdot\cos z = -\frac{1}{x}dx \,/\int$$

$$+\sin z = -\ln|x| + c$$

$$z = +\arcsin(-\ln|x| + c)$$

$$y = x\arcsin(-\ln|x| + c)$$

1.3

1. $y' + y\cdot tg(x) = \frac{1}{\cos(x)}$

$$y' + y\cdot tg(x) = 0$$

$$\left.\begin{array}{l} y' = -y\,tg(x) \\ y' = \frac{dy}{dx} \end{array}\right\} \Rightarrow \frac{dy}{-y} = tgx\cdot dx \,/\int$$

$$-\int\frac{1}{y}dy = \int tg\,x\,dx$$

$$-\ln|y| = -\int\frac{\overset{=-\cos'}{\sin x}}{\cos x}dx$$

$$\ln|y| = \ln|\cos x| + c$$

$$|y| = |\cos x|\cdot c_1$$

$$y_0 = |\cos x|\,c_2$$

$$y_p = |\cos x|\cdot\varphi \,/()'$$

$$y_p' = \varphi'\cos x - \varphi\sin x$$

$$\varphi'\cos x - \varphi\sin x + \cos x\varphi\cdot tg x = \frac{1}{\cos x}$$

$$\varphi'\cos x - \cancel{\varphi\sin x} + \cancel{\varphi\sin x} = \frac{1}{\cos x}$$

$$\varphi' = \frac{1}{\cos^2 x} \,/\int$$

$$\varphi = \int\frac{1}{\cos^2 x} \Rightarrow \varphi = tgx$$

$$y_p = \cos x\cdot tgx = \sin x$$

$$\Rightarrow y = \sin x + \cos x\cdot c_2$$

2. $y' + \frac{2}{x} \cdot y = x^3$

I $y' + \frac{2}{x} y = 0$

$y' = -2 \frac{y}{x}$

not $z = \frac{y}{x} \Rightarrow z'x + z = y'$ $\Bigg\}$ $\Rightarrow z'x + z = -2z$

$z' = -3z \cdot \frac{1}{x}$

$z' = \frac{dz}{dx}$ $\Bigg\}$ $\Rightarrow \frac{dz}{-3z} = \frac{1}{x} dx \,/\!\int$

$z=0$ sol sing $\Rightarrow y = 0$

$-\frac{1}{3} \int \frac{1}{z} dz = \int \frac{1}{x} dx$

$-\frac{1}{3} \ln|z| = \ln|x| + c$

$\ln|z|^{-\frac{1}{3}} = \ln|x| \cdot c_1, \; c_1 > 0$

$|z|^{-\frac{1}{3}} = |x| \cdot c_1$

$\frac{1}{\sqrt[3]{z}} = x \cdot c_1 \;/()^{-1}$

$\sqrt[3]{z} = \frac{1}{x c_1} \;/()^3$

$z = \frac{1}{x^3 c_1^3} \Rightarrow y_0 = \frac{1}{x^2 c_1^3} = \frac{1}{x^2 C}$

$y_p = \frac{1}{x^2 \varphi} \;/()' \Rightarrow y_p' = \frac{-(2x\varphi + x^2\varphi')}{x^4 \varphi^2} =$

$= \frac{-2\varphi + x\varphi'}{x^3 \varphi^2}$

$\frac{-2\varphi + x\varphi'}{x^3 \varphi^2} + \frac{2}{x} \cdot \frac{1}{x^2 \varphi} = x^3$

$\cancel{-2\varphi} + x\varphi' + \cancel{2\varphi} = x^6 \varphi^2$

$\varphi' = x^5 \varphi^2 \rightarrow$ am greșit

$\varphi' = x^5 \,/\!\int$

$\varphi = \frac{x^6}{6}$

$\Rightarrow y_p = \frac{1}{x^2 \cdot \frac{x^6}{6}} = \frac{6}{x^8} \Rightarrow y = \frac{6}{x^8} + \frac{1}{x^2 C}$

3. $y' + 2xy = 2xe^{-x^2}$

I $y' + 2xy = 0$

$$\left.\begin{array}{l} y' = -2xy \\ y' = \frac{dy}{dx} \end{array}\right\} \Rightarrow \frac{dy}{y} = -2x\,dx \;/\int$$

$$\ln|y| = -\cancel{2}\frac{x^2}{\cancel{2}} + c$$

$$\ln|y| = -x^2 + c \Rightarrow |y| = e^{-x^2+c} \Rightarrow y = \pm\left(e^{-x^2} \cdot e^{c}\right) = c \cdot e^{-x^2}$$

II $y_p = \varphi \cdot e^{-x^2} \;/()'$

$$y_p' = -e^{-x^2} \cdot 2x \cdot \varphi + \varphi' e^{-x^2}$$

$$\cancel{-2x\varphi e^{-x^2}} + \varphi' e^{-x^2} + \cancel{2x \cdot \varphi \cdot e^{-x^2}} = 2xe^{-x^2}$$

$$\varphi' = \frac{1}{\cancel{e^{-x^2}}} \cdot 2x\cancel{e^{-x^2}} \Rightarrow \varphi = \cancel{2}\frac{x^2}{\cancel{2}} \Rightarrow \varphi = x^2$$

$$y_p = x^2 \cdot e^{-x^2}$$

$$\Rightarrow y = y_0 + y_p = ce^{-x^2} + x^2 e^{-x^2} = e^{-x^2}(c + x^2)$$

4. $xy' - y + x = 0$

$$xy' - y = -x \;/ : x \neq 0$$

$$y' - \frac{y}{x} = -1$$

$$\left.\begin{array}{l} y' - \frac{y}{x} = 0 \\ y' = \frac{dy}{dx} \end{array}\right\} \Rightarrow \frac{dy}{y} = \frac{1}{x}dx \;/\int$$

$$\ln|y| = \ln|x| + c$$

$$|y| = (|x| \cdot c_1),\; c_1 > 0$$

$$y = |x| \cdot c_2,\; c_2 \in \mathbb{R} \quad y = xc_2$$

$$y_p = x \cdot \varphi \;/()' \Rightarrow y_p' = \varphi + x \cdot \varphi'$$

$$x \cdot (\varphi + x\varphi') - x\varphi + x = 0$$

$$\cancel{x\varphi} + x^2\varphi' - \cancel{x\varphi} + x = 0$$

$$\varphi' = -\frac{x}{x^{\cancel{2}}} = -\frac{1}{x} \;/\int \Rightarrow \varphi = -\ln|x|$$

$$\Rightarrow y_p = -x\ln|x|$$

$$\Rightarrow y = -x\ln|x| + x \cdot c$$

5. $y' - y = \sin x$???

I $y' - y = 0$

$$\left.\begin{matrix} y' = y \\ y' = \frac{dy}{dx} \end{matrix}\right\} \Rightarrow \frac{dy}{y} = dx$$

$$\ln|y| = \cancel{x} + c \Rightarrow |y| = e^{x+c}$$

$$|y| = e^x \cdot e^c$$

$$y = c_1 e^x$$

II $y_p = \varphi e^x \ /()' \Rightarrow y_p' = \varphi' e^x + \varphi e^x$

$$\varphi' e^x + \cancel{\varphi e^x} - \cancel{\varphi e^x} = \sin x$$

$$\varphi' = \frac{\sin x}{e^x} \ /\int \Rightarrow \varphi = \int \frac{\sin x}{e^x}$$ to be continued

6. $y' + \frac{x}{1-x^2} y = x + \arcsin(x) \Rightarrow x \in [-1,1] \Rightarrow 1 - x^2 > 0$

I $y' + \frac{x}{1-x^2} y = 0$

$$\left.\begin{matrix} y' = -\frac{x}{1-x^2} y \\ y' = \frac{dy}{dx} \end{matrix}\right\} \Rightarrow \frac{dy}{y} = -\frac{x}{1-x^2} dx \ /\int \Rightarrow \ln|y| = +\frac{1}{2}\ln|1-x^2| + c$$

$$|y| = |1-x^2|^{\frac{1}{2}} \cdot C_1, \ C_1 > 0$$

$$y = \sqrt{\underbrace{|1-x^2|}_{>0}} \cdot C_2, \ C_2 \in \mathbb{R}$$

II $y_p = \sqrt{1-x^2} \cdot \varphi \ /()' \Rightarrow y_p' = \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} \cdot \varphi + \sqrt{1-x^2} \cdot \varphi'$

$$\cancel{-\frac{x}{\sqrt{1-x^2}} \cdot \varphi} + \sqrt{1-x^2}\,\varphi' + \cancel{\frac{x}{1-x^2} \sqrt{1-x^2} \cdot \varphi} = x + \arcsin(x)$$

$$\varphi' = \frac{1}{\sqrt{1-x^2}}\left(x + \arcsin(x)\right) \Rightarrow \varphi = \int \frac{1}{\sqrt{1-x^2}}\left(x + \arcsin(x)\right) dx =$$

$$= \int \left(\frac{x}{\sqrt{1-x^2}} + \frac{\arcsin(x)}{\sqrt{1-x^2}}\right) = -\sqrt{1-x^2} + \int \frac{\arcsin x}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \int (\arcsin(x))' \cdot \arcsin x \, dx =$$

$$= -\sqrt{1-x^2} + \frac{(\arcsin x)^2}{2} \ldots$$