GRUPA 217

1. Arătați că numărul α este irațional

a) $\alpha = \dfrac{{}^{(\sqrt{3}-\sqrt{2})}2}{\sqrt{3}+\sqrt{2}} = \dfrac{2(\sqrt{3}-\sqrt{2})}{3-2} = 2(\sqrt{3}-\sqrt{2})$

$\alpha = 2(\sqrt{3}-\sqrt{2}) \;/()^2$

$\alpha^2 = 4(3-2\sqrt{6}+2) \Leftrightarrow \alpha^2 = 20-8\sqrt{6} \Leftrightarrow \alpha^2 - 20 = -8\sqrt{6} \;/()^2$

$\Leftrightarrow \alpha^4 - 40\alpha^2 + 400 = 384 \Leftrightarrow \alpha^4 - 40\alpha^2 + 16 = 0$

α - rădăcină a ecuației $\alpha^4 - 40\alpha^2 + 16 = 0$

PP. $\alpha \in \mathbb{Q} \Rightarrow \alpha = \dfrac{p}{q}$, unde $p/16$ și $q/1$

$p \in \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16\}$
$q \in \{\pm 1\}$ $\Rightarrow \dfrac{p}{q} \in \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16\}$ ABSURD!

$\Rightarrow \alpha \notin \mathbb{Q} \Rightarrow \alpha$ - irațional

b) $\alpha = \dfrac{{}^{(\sqrt{3}+\sqrt{2})}3}{\sqrt{3}-\sqrt{2}} = \dfrac{3(\sqrt{3}+\sqrt{2})}{3-2} = 3(\sqrt{3}+\sqrt{2})$

$\alpha = 3(\sqrt{3}+\sqrt{2}) \;/()^2$

$\alpha^2 = 9(3+2\sqrt{6}+2) \Leftrightarrow \alpha^2 = 45+18\sqrt{6} \Leftrightarrow \alpha^2 - 45 = 18\sqrt{6} \;/()^2$

$\alpha^4 - 90\alpha^2 + 2025 = 1944 \Leftrightarrow \alpha^4 - 90\alpha^2 + 81 = 0$

PP. $\alpha \in \mathbb{Q} \Rightarrow \alpha = \dfrac{p}{q}$, unde $p/81$ și $q/1$

$\Rightarrow \dfrac{p}{q} \in \{\pm 1, \pm 3, \pm 9, \pm 27\}$ ABSURD!

$\Rightarrow \alpha \notin \mathbb{Q} \Rightarrow \alpha$ - irațional

2. Determinați mulțimea punctelor limită ale șirului:

a) $x_n = \left(3 \cdot \cos\frac{n\pi}{4}\right)^n$

dăm valori: $n=1 \Rightarrow \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$n=2 \Rightarrow \cos\frac{\pi}{2} = 0$

$n=3 \Rightarrow \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} \to n$ impar

$n=4 \Rightarrow \cos\pi = -1 \to n$ par

$n=5 \Rightarrow \cos\frac{5\pi}{4} = \frac{\sqrt{2}}{2}$

$n=6 \Rightarrow \cos\frac{3\pi}{2} = 0$

$n=7 \Rightarrow \cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$

$n=8 \Rightarrow \cos 2\pi = 1$

după se repetă

$\cos\frac{3\pi}{4} = \cos\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = 0 - \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$

$\cos\frac{5\pi}{4} = \cos\left(\pi+\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\cos\frac{7\pi}{4} = \cos\left(\frac{3\pi}{2}+\frac{\pi}{4}\right) = +\frac{\sqrt{2}}{2}$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$\cancel{\Rightarrow \lim_{n\to\infty} x_n = \begin{cases} \left(3\frac{\sqrt{2}}{2}\right)^n = \left(\frac{\sqrt{18}}{2}\right)^n = \infty \\ (3\cdot 0)^n = 0 \end{cases}}$$

$\cancel{\Rightarrow LIM(x_n) = \{-\infty, +\infty, 0\}}$

$$\Rightarrow \lim_{n\to\infty} x_n = \begin{cases} \left(3\frac{\sqrt{2}}{2}\right)^n = \left(\overset{>1}{\frac{\sqrt{18}}{2}}\right)^n = \infty \\ (3\cdot 0)^n = 0 \\ \left[3\cdot\left(-\frac{\sqrt{2}}{2}\right)\right]^{n\text{-impar}} = \left(\overset{<-1}{-\frac{\sqrt{18}}{2}}\right)^{n\text{- impar}} = -\infty \\ [3\cdot(-1)]^{n\text{-par}} = \infty \\ (3\cdot 1)^n = \infty \end{cases}$$

$\Rightarrow LIM(x_n) = \{0, +\infty, -\infty\}$

Calculați derivata de ordin n a funcției:

a) $f(x) = \frac{x^2}{e^{2x+1}} = x^2 \cdot e^{-(2x+1)}$

folosim formula lui Leibnitz

$f(x) = x^2$ - funcții indefinit derivabile pe $\mathbb{R}$

$g(x) = e^{-(2x+1)}$

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k \cdot f^{(k)} \cdot g^{(n-k)}$$

$$\Rightarrow (f\cdot g)^n = C_n^0 \cdot x^2 \cdot \left[e^{-(2x+1)}\right]^{(n)} + C_n^1 \cdot 2x \cdot \left[e^{-(2x+1)}\right]^{(n-1)} + C_n^2 \cdot 2 \cdot \left[e^{-(2x+1)}\right]^{(n-2)} + 0$$

$$= x^2 \cdot \left[e^{-(2x+1)}\right]^{(n)} + n \cdot 2x \cdot \left[e^{-(2x+1)}\right]^{n-1} + \frac{n(n-1)}{\cancel{2}} \cdot \cancel{2} \cdot \left[e^{-(2x+1)}\right]^{(n-2)}$$

calculăm derivata de ordin n pentru $g(x) = e^{-(2x+1)}$

$g'(x) = -2 \cdot e^{-(2x+1)} = (-1)^1 \cdot 2^1 \cdot e^{-(2x+1)}$

$g''(x) = -2 \cdot (-2) \cdot e^{-(2x+1)} = (-1)^2 \cdot 2^2 \cdot e^{-(2x+1)}$

$\vdots$

$p(n): g^{(n)}(x) = (-1)^n \cdot 2^n \cdot e^{-(2x+1)}$

inducție: I verificare : $p(1) = (-1)^1 \cdot 2^1 \cdot e^{-(2x+1)}$ „A"

II demonstrație

$p(n) \rightarrow p(n+1)$

$p(n+1): g^{(n+1)} = (-1)^{n+1} \cdot 2^{n+1} \cdot e^{-(2x+1)}$

$\left[(-1)^n \cdot 2^n \cdot e^{-(2x+1)}\right]' = (-1)^{n+1} \cdot 2^{n+1} \cdot e^{-(2x+1)}$

$(-2) \cdot (-1)^n \cdot 2^n \cdot e^{-(2x+1)} = (-1)^{n+1} \cdot 2^{n+1} \cdot e^{-(2x+1)}$

$(-1)^{n+1} \cdot 2^{n+1} \cdot e^{-(2x+1)} = (-1)^{n+1} \cdot 2^{n+1} \cdot e^{-(2x+1)}$ „A"

$\Rightarrow p(n)$: adevărat

$\Downarrow$

$$(f\cdot g)^{(n)} = e^{-(2x+1)}\left[x^2 \cdot (-1)^n \cdot 2^n + 2x \cdot (-1)^{n-1} \cdot 2^{n-1} + n(n-1) \cdot (-1)^{n-2} \cdot 2^{n-2}\right]$$

b) $f(x) = \frac{(2x-1)^2}{e^x} = (2x-1)^2 \cdot e^{-x} = (4x^2-4x+1)\cdot e^{-x}$

formula lui Leibnitz

$\Rightarrow C_n^0 (4x^2-4x+1)\cdot(e^{-x})^{(n)} + C_n^1 (8x-4)\cdot(e^{-x})^{(n-1)} + C_n^2 \cdot 8 \cdot (e^{-x})^{(n-2)} =$

$= (4x^2-4x+1)(e^{-x})^{(n)} + n\cdot(8x-4)\cdot(e^{-x})^{(n-1)} + \frac{n(n-1)}{\cancel{2}}\cdot \cancel{8}4 \cdot (e^{-x})^{(n-2)}$

calculăm derivata de ordin n a lui $e^{-x} = g(x)$

$g' = -e^{-x} = (-1)^1 \cdot e^{-x}$

$g'' = e^{-x} = (-1)^2 \cdot e^{-x}$

$\Rightarrow g^{(n)} = (-1)^{(n)} \cdot e^{-x}$ + inducție ca la a

$\Rightarrow (f\cdot g)^{(n)} = e^{-x}\left[(4x^2-4x+1)(-1)^n + n(8x-4)\cdot(-1)^{n-1} + 4n(n-1)(-1)^{n-2}\right]$