SUBIECTUL B

1. Justificați cu definiția valoarea limitei:

a) $\lim\limits_{n\to\infty} \frac{\sqrt{n}-1}{n+1} = 0$

limita FINITĂ de demonstrat

$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 \ |x_n - 0| < \varepsilon$

$\left|\frac{\sqrt{n}-1}{n+1}\right| < \varepsilon$ → încerc să-l scot pe n în funcție de ε

$n > 0 \Rightarrow \frac{\sqrt{n}-1}{n+1} < \varepsilon$

~~minorez~~ majorez

$\frac{\sqrt{n}-1}{n+1} < \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} = n^{-\frac{1}{2}}$

$n^{-\frac{1}{2}} < \varepsilon \ /()^{-2}$

$n > \varepsilon^{-2} \Rightarrow$ alegem $n_0 = [\varepsilon^{-2}] + 1 \in \mathbb{N}$

b) $\lim\limits_{n\to\infty} \frac{n+1}{\sqrt{n}+1} = +\infty$

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 : x_n > \varepsilon$ (respectiv $x_n < -\varepsilon$)

$\Rightarrow \frac{n+1}{\sqrt{n}+1} > \varepsilon$ → încerc să-l scot pe n în funcție de ε

minorez

$\frac{n+1}{\sqrt{n}+1} > \frac{n}{\sqrt{n}+1} = \frac{n(\sqrt{n}-1)}{n-1} > \frac{(n-1)(\sqrt{n}-1)}{n-1} = \sqrt{n}-1$

$\sqrt{n} - 1 > \varepsilon$

$\sqrt{n} > \varepsilon + 1 \ /()^2$

$n > (\varepsilon+1)^2 \Rightarrow$ alegem $n_0 = [(\varepsilon+1)^2] + 1$

2. Calculați limita

a) $$\lim_{x\to\infty} \frac{\ln\left(\frac{\pi}{2}-\operatorname{arctg} x\right)}{\ln x} = \left(\frac{-\infty}{\infty}\right) \overset{l'H}{=} \lim_{x\to\infty} \frac{\frac{1}{\frac{\pi}{2}-\operatorname{arctg} x}\cdot\frac{-1}{1+x^2}}{\frac{1}{x}} =$$

$$= -\lim \frac{x}{(1+x^2)\left(\frac{\pi}{2}-\operatorname{arctg} x\right)} = -\lim \frac{x\cdot\frac{1}{1+x^2}}{\frac{\pi}{2}-\operatorname{arctg} x} = -\lim \frac{\frac{x}{x\left(\frac{1}{x}+x\right)}}{\frac{\pi}{2}-\operatorname{arctg} x} =$$

$$= -\lim \frac{\frac{1}{x+\frac{1}{x}}}{\frac{\pi}{2}-\operatorname{arctg} x} = \left(\frac{0}{0}\right) \overset{l'H}{=} -\lim \frac{+\left(x+\frac{1}{x}\right)^{-2}\cdot\left(1-\frac{1}{x^2}\right)}{+\frac{1}{1+x^2}} =$$

$$= -\lim \frac{1+x^2}{\left(\frac{x^2+1}{x}\right)^2} = -\lim \cancel{(1+x^2)}\cdot\frac{x^2}{(x^2+1)^{\cancel{2}}} = -\lim \frac{x^2}{x^2+1} = -1$$

b) $$\lim_{x\searrow 0} \frac{\ln\left(\frac{\pi}{2}-\arccos x\right)}{\ln x} = \left(\frac{-\infty}{-\infty}\right) \overset{l'H}{=} \lim_{x\searrow 0} \frac{\frac{1}{\frac{\pi}{2}-\arccos x}\cdot\frac{-1}{\sqrt{1-x^2}}}{\frac{1}{x}} =$$

$$= -\lim_{x\searrow 0} \frac{x}{\left(\frac{\pi}{2}-\arccos x\right)\cdot\sqrt{1-x^2}} = -\lim_{x\searrow 0} \frac{\frac{x}{\sqrt{1-x^2}}}{\frac{\pi}{2}-\arccos x} = -\lim_{x\searrow 0} \frac{x\cdot(1-x^2)^{-\frac{1}{2}}}{\frac{\pi}{2}-\arccos x} = \left(\frac{0}{0}\right) =$$

↳ il izolăm

$$\overset{l'H}{=} -\lim \frac{(1-x^2)^{-\frac{1}{2}} + x\cdot\left(-\frac{1}{2}\right)\cdot(1-x^2)^{-\frac{3}{2}}\cdot(-2x)}{-\frac{1}{\sqrt{1-x^2}}} = -\frac{1}{-1} = 1$$

3. Determinați $x \in \mathbb{R}$ astfel încât:

a) $\sum_{n=1}^{\infty} \frac{1}{(1+x)^{2n}} = \frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{1}{(1+x)^{2n}} = -1 + \sum_{n=0}^{\infty} \frac{1}{(1+x)^{2n}} = -1 + \sum_{n=0}^{\infty} \left[\frac{1}{(1+x)^2}\right]^n$$

$\sum_{n=0}^{\infty} a^n$ - conv dacă $a \in (-1,1)$ și $\Rightarrow \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

$$a = \frac{1}{(1+x)^2} \Rightarrow -1 + \frac{1}{1 - \frac{1}{(1+x)^2}} = \frac{1}{2}$$

$$-1 + \frac{1}{\frac{x^2+2x}{(1+x)^2}} = \frac{1}{2} \Leftrightarrow -1 + \frac{(1+x)^2}{x^2+2x} = \frac{1}{2}$$

$$\Leftrightarrow \frac{\cancel{-x^2} - \cancel{2x} + \cancel{x^2} + \cancel{2x} + 1}{x^2+2x} = \frac{1}{2}$$

$$\Leftrightarrow x^2 + 2x = 2 \Leftrightarrow x^2 + 2x - 2 = 0$$

$$\Delta = 4 + 8 = 12$$

$$x_{1,2} = \frac{-2 \pm 2\sqrt{3}}{2} \Rightarrow \boxed{x_1 = -1 + \sqrt{3}} \quad \boxed{x_2 = -1 - \sqrt{3}}$$

b) $\sum_{n=2}^{\infty} \frac{1}{(x-1)^n} = 2$

ideea de sus $\Rightarrow -1 - \frac{1}{x-1} + \sum_{n=0}^{\infty} \left(\frac{1}{x-1}\right)^n = 2$

$$-1 - \frac{1}{x-1} + \frac{1}{1 - \frac{1}{x-1}} = 2 \Leftrightarrow \frac{-x+1-1}{x-1} + \frac{x-1}{x-2} = 2$$

$$\Leftrightarrow \frac{-x}{x-1} + \frac{x-1}{x-2} = 2 \Leftrightarrow \frac{\cancel{-x^2} + \cancel{2x} + \cancel{x^2} - \cancel{2x} + 1}{x^2 - 2x - x + 2} = 2$$

$$\Leftrightarrow 2x^2 - 6x + 4 = 1 \Leftrightarrow 2x^2 - 6x + 3 = 0$$

$$\Delta = 36 - 24 = 12 \Rightarrow x_{1,2} = \frac{6 \pm 2\sqrt{3}}{4} \Rightarrow \boxed{x_1 = \frac{3+\sqrt{3}}{2}} \quad \boxed{x_2 = \frac{3-\sqrt{3}}{2}}$$