SUBIECTUL D

1. Determinați $\inf A$, $\sup A$, $\min A$ și $\max A$ pentru mulțimea:

a) $A = \left\{ \frac{2^n \cdot n^n}{(2n-3)!!} \middle| n \in \mathbb{N}, n \geq 2 \right\}$

vedem cum arată șirul la limită, $n \geq 2$

$$\lim_{n\to\infty} \frac{\overbrace{2^n \cdot n^n}^{n \text{ termeni}}}{\underbrace{(2n-3)!!}_{n-1 \text{ termeni}}} = \lim_{n\to\infty} \frac{2n}{1} \cdot \frac{2n}{3} \cdot \frac{2n}{5} \cdot \ldots \cdot \frac{2n}{2n-3} \cdot 2n = \infty$$

$\Rightarrow \sup A = \infty$ și $\max A \nexists$

monotonie

$$\frac{x_{n+1}}{x_n} = \frac{2^{\cancel{n+1}} \cdot (n+1)^{n+1}}{\underbrace{\cancel{(2n-1)!!}}_{2n-1}} \cdot \frac{\cancel{(2n-3)!!}}{\cancel{2^n} \cdot n^n} = \underbrace{\left(\frac{n+1}{n}\right)^n}_{>1} \cdot \underbrace{\frac{2(n+1)}{2n-1}}_{>1} > 1 \Rightarrow x_{n+1} > x_n$$

$\Rightarrow x_n$ - crescător

$\Rightarrow x_0$ - margine inferioară

$n=2 \Rightarrow x_0 = \frac{4 \cdot 4}{1} = 16 \Rightarrow \inf A = \min A = 16$

b) $A = \left\{ \frac{(2n-2)!!}{3^n \cdot n^n} \middle| n \in \mathbb{N}, n \geq 2 \right\}$

$$\lim_{n\to\infty} \frac{\overbrace{(2n-2)!!}^{n-1 \text{ termeni}}}{\underbrace{(3 \cdot n)^n}_{n \text{ termeni}}} = \lim_{n\to\infty} \frac{2}{3n} \cdot \frac{4}{3n} \cdot \ldots \cdot \frac{2n-2}{3n} \cdot \frac{1}{3n} = 0$$

monotonie

$$\frac{x_{n+1}}{x_n} = \frac{\overset{2n}{\cancel{(2n)!!}}}{3^{\cancel{n+1}} (n+1)^{n+1}} \cdot \frac{\cancel{3^n} \cdot n^n}{\cancel{(2n-2)!!}} = \underbrace{\left(\frac{n}{n+1}\right)^n}_{<1,\ >0} \cdot \underbrace{\frac{2n}{3(n+1)}}_{<1} < 1 \Rightarrow x_n \text{- descrescător}$$

x_0 - marg. $\Rightarrow n=2$: $\frac{\cancel{2}}{9 \cdot \cancel{4}2} = \frac{1}{18} \Rightarrow \inf A = 0$, $\min A \nexists$

$\sup A = \frac{1}{18} = \max A$

2. Calculați limita:

a) $$\lim_{n\to\infty} \frac{\sqrt{1+2^2}+\sqrt{1+3^2}+\dots+\sqrt{1+n^2}}{1+n^2} \overset{SC}{=} \lim \frac{\sqrt{n^2+2n+1+1}}{\cancel{n^2}+2n+1+\cancel{1}-\cancel{n^2}-\cancel{1}} =$$

$1+n^2$ - cresc. și divergent

$$= \lim_{n\to\infty} \frac{\cancel{n}\sqrt{1+\overset{\to 0}{\frac{2}{n}}+\overset{\to 0}{\frac{2}{n^2}}}}{\cancel{n}(2+\underset{\to 0}{\frac{1}{n}})} = \frac{1}{2}$$

b) $$\lim_{n\to\infty} \frac{1+\sqrt{2}+\sqrt{3}+\dots+\sqrt{n}}{n\sqrt{n}} \overset{S.C.}{=} \lim \frac{\sqrt{n+1}}{(n+1)\sqrt{n+1}-n\sqrt{n}}$$

$n\sqrt{n}$ - cresc. și divergent

facem fact. comun $(+\sqrt{n}-\sqrt{n})$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}}{(n+1)\sqrt{n+1}-(n+1)\sqrt{n}+\sqrt{n}} \overset{\sqrt{n+1}+\sqrt{n}}{=} \lim_{n\to\infty} \frac{\sqrt{n+1}}{(n+1)(\sqrt{n+1}-\sqrt{n})+\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})}{(n+1)(n+1-n)+\sqrt{n}(\sqrt{n+1}+\sqrt{n})} =$$

$$= \lim_{n\to\infty} \frac{n+1+\sqrt{n}\cdot\sqrt{n+1}}{n+1+n+\sqrt{n}\cdot\sqrt{n+1}} = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\overset{\to 0}{\frac{1}{n}}+\overset{\to 1}{\sqrt{\frac{n^2+n}{n^2}}}\right)}{\cancel{n}\left(2+\underset{\to 0}{\frac{1}{n}}+\underset{\to 1}{\sqrt{\frac{n^2+n}{n^2}}}\right)} = \frac{2}{3}$$

3. Studiați convergența seriei cu termeni pozitivi:

studiați convergența seriei cu term. pozitivi :

a) $\sum\limits_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$

$D = \lim\limits_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim \frac{e^{\frac{1}{n}}}{n^2} \cdot \frac{(n+1)^2}{e^{\frac{1}{n+1}}} = \lim e^{\frac{1}{n} - \frac{1}{n+1}} \cdot \left(\frac{n+1}{n}\right)^2 = \lim\limits_{n\to\infty} e^{\frac{1}{n^2+n}} \cdot \left(\frac{n+1}{n}\right)^2 =$

$R = \lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim \cancel{n} \cdot e^{\frac{1}{n^2+n}} \cdot \frac{(n+1)^2}{n^{\cancel{2}}} - n =$

$= \lim\limits_{n\to\infty} \frac{n^{\cancel{2}}\left[e^{\frac{1}{n^2+n}}\left(1 + \underbrace{\frac{2}{n}}_{\to 0} + \underbrace{\frac{1}{n^2}}_{\to 0}\right) - 1\right]}{\cancel{n}} = (\infty \cdot 0)$

compar cu $\sum\limits_{n=1}^{\infty} \frac{e}{n^2} = e\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ - convergentă

$e^{\frac{1}{n}} \leq e \Rightarrow \sum\limits_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$ - convergentă

b) $\sum\limits_{n=1}^{\infty} \frac{e^{-\frac{1}{n}}}{n^{\cancel{4}}} = \sum\limits_{n=1}^{\infty} \frac{1}{e^{\frac{1}{n}} \cdot n^{\cancel{4}}}$

compar cu $\frac{1}{e} \cdot \sum \frac{1}{n}$ - divergentă

$e^{\frac{1}{n}} \leq e$

$\frac{1}{e^{\frac{1}{n}}} \geq \frac{1}{e}$ $\Rightarrow \sum\limits_{n=1}^{\infty} \frac{e^{-\frac{1}{n}}}{n}$ - divergentă