1. Determinați inf, sup, min, max pentru mulțimea A

a) $A = \left\{ \frac{p-q}{p+q} \,/\, p, q \in \mathbb{N}^* \right\}$

$\frac{p-q}{p+q} = \frac{p+q-2q}{p+q} = 1 - \frac{2q}{p+q}$

$\frac{2q}{p+q} > 0 \ \forall q, p \in \mathbb{N}^*$

$\Rightarrow 1 - \frac{2q}{p+q} < 1 \ \forall q, p \in \mathbb{N}^*$

$\Rightarrow \sup A = 1$ și $\max A \not\exists$

$q < p+q \Rightarrow \frac{1}{q} > \frac{1}{p+q} \Rightarrow \frac{2q}{p+q} < 2 \Rightarrow -2 < \frac{-2q}{p+q}$

$\downarrow$

$\frac{q}{p+q} < 1 \Rightarrow \frac{2q}{p+q} < 2$

$\Rightarrow -1 < 1 - \frac{2q}{p+q} \Rightarrow -1 = \inf A$ și $\min A \not\exists$

b) $A = \left\{ \frac{p}{p+q} \,/\, p, q \in \mathbb{N}^* \right\}$

$\frac{p}{p+q} = \frac{p+q-q}{p+q} = 1 - \frac{q}{p+q} < 1, \ \forall p, q \in \mathbb{N}^* \Rightarrow \sup A = 1$ și $\max A \not\exists$

$q < p+q, \ \forall p, q \in \mathbb{N}^*$

$\frac{q}{p+q} < 1 \Leftrightarrow -\frac{q}{p+q} > -1 \ /+1 \Leftrightarrow 1 - \frac{q}{p+q} > 0 \Rightarrow \inf A = 0$ și $\min A \not\exists$

2. Determinați mulțimea punctelor limită ale șirului:

a) $x_n = \left(\frac{n+1}{n+3}\right)^{n \cdot \sin\frac{n\pi}{3}} \Rightarrow \lim_{n\to\infty}\left(\frac{n+1}{n+3}\right)^{n\cdot\sin\frac{n\pi}{3}} = (1^\infty) =$

$= \lim\left(1+\frac{-2}{n+3}\right)^{\frac{n+3}{-2}\cdot\frac{-2}{n+3}\cdot n\cdot\sin\frac{n\pi}{3}} = e^{\lim\overbrace{\frac{-2n}{n+3}}^{\to -2}\cdot\sin\frac{n\pi}{3}}$

dăm valori la n pt. $\sin\frac{n\pi}{3}$: $n=1 \Rightarrow \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$n=2 \Rightarrow \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$n=3 \Rightarrow \sin\pi = 0$

$n=4 \Rightarrow \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

$n=5 \Rightarrow \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$

$n=6 \Rightarrow \sin 2\pi = 0$

după se repetă

$\sin\frac{2\pi}{3} = \sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\sin\frac{4\pi}{3} = \sin\left(\pi+\frac{\pi}{3}\right) = 0+\frac{\sqrt{3}}{2}\cdot(-1) = -\frac{\sqrt{3}}{2}$

$\sin\frac{5\pi}{3} = \sin\left(\frac{3\pi}{2}+\frac{\pi}{6}\right) = (-1)\cdot\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$

$\sin(a+b) = \sin a\cos b + \sin b\cos a$

$\Rightarrow \lim_{n\to\infty} x_n = e^{-2\lim\sin\frac{n\pi}{3}}$

$LIM(x_n) = \{e^0, e^{-\sqrt{3}}, e^{\sqrt{3}}\}$

b) $\left(\frac{n+3}{n+1}\right)^{n\cdot\cos\frac{n\pi}{3}} \Rightarrow \lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^{n\cdot\cos\frac{n\pi}{3}} = (1^\infty) =$

$= \lim_{n\to\infty}\left(1+\frac{2}{n+1}\right)^{\frac{n+1}{2}\cdot\frac{2}{n+1}\cdot n\cdot\cos\frac{n\pi}{3}} = e^{\lim_{n\to\infty}\overbrace{\frac{2n}{n+1}}^{\to 2}\cdot\cos\frac{n\pi}{3}}$

ăm valori la n pt $\cos\frac{n\pi}{3}$: $n=1 \Rightarrow \cos\frac{\pi}{3}=\frac{1}{2}$

$n=2 \Rightarrow \cos\frac{2\pi}{3}=-\frac{1}{2}$

$n=3 \Rightarrow \cos\pi=-1$

$n=4 \Rightarrow \cos\frac{4\pi}{3}=-\frac{1}{2}$

$n=5 \Rightarrow \cos\frac{5\pi}{3}=\frac{\sqrt{3}}{2}$

$n=6 \Rightarrow \cos 2\pi=1$

după se repetă

$$\cos\frac{2\pi}{3}=\cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=0-1\cdot\frac{1}{2}=-\frac{1}{2}$$

$$\cos\frac{4\pi}{3}=\cos\left(\pi+\frac{\pi}{3}\right)=-1\cdot\frac{1}{2}+0=-\frac{1}{2}$$

$$\cos\frac{5\pi}{3}=\cos\left(\frac{3\pi}{2}+\frac{\pi}{6}\right)=0-(-1)\cdot\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}$$

$\cos(a+b)=\cos a\cdot\cos b-\sin a\sin b$ $\qquad \lim x_n=e^{2\lim\limits_{n\to\infty}\cos\frac{n\pi}{3}}$

$$\Rightarrow LIM(x_n)=\{e, e^{-1}, e^{-2}, e^{\sqrt{3}}\}$$

3. Calculați limita

a) $$\lim_{x\to 0}\frac{x^3-\sin^3 x}{x^5}=\lim_{x\to 0}\frac{(x-\sin x)(x^2+x\sin x+\sin^2 x)}{x^5}=$$

$$=\lim_{x\to 0}\frac{x-\sin x}{x^3}\cdot\lim_{x\to 0}\frac{\cancel{x^2}\left[1+\underbrace{\frac{\sin x}{x}}_{\to 1}+\underbrace{\left(\frac{\sin x}{x}\right)^2}_{\to 1}\right]}{\cancel{x^2}}=$$

$$=3\cdot\lim_{x\to 0}\frac{x-\sin x}{x^3}\left(\frac{0}{0}\right)\overset{l'H}{=}3\cdot\lim_{n\to 0}\frac{1-\cos x}{3x^2}\left(\frac{0}{0}\right)\overset{l'H}{=}3\lim_{n\to 0}\frac{\sin x}{6x}\left(\frac{0}{0}\right)\overset{l'H}{=}3\cdot\lim\frac{-\cos x}{6}=3\cdot\frac{-1}{6}=-\frac{1}{2}$$

b) $\lim\limits_{x\to 0} \dfrac{x^3 - tg^3 x}{x^5} = \lim\limits_{x\to 0} \dfrac{(x - tg x)}{x^3} \cdot \lim\limits_{x\to 0} \dfrac{(x^2 + x tg x + tg^2 x)}{x^2} =$

$= \lim\limits_{x\to 0} \dfrac{x - tg x}{x^3} \cdot \lim \left[1 + \overset{\nearrow 1}{\dfrac{tg x}{x}} + \overset{\nearrow 1}{\left(\dfrac{tg x}{x}\right)^2}\right] = 3 \cdot \lim\limits_{x\to 0} \dfrac{x - tg x}{x^3} \left(\dfrac{0}{0}\right) \underset{l'H}{=}$

$= 3 \cdot \lim\limits_{x\to 0} \dfrac{1 - \dfrac{1}{\cos^2 x}}{3x^2} \left(\dfrac{0}{0}\right) \underset{l'H}{=} \dfrac{-(-2)\cdot(\cos x)^{-3}\cdot(-\sin x)}{6x} = \lim\limits_{x\to 0} \dfrac{-2 \sin x \cdot \cos^3 x}{6x} \ \dfrac{0}{0} \underset{l'H}{=}$

$= \lim\limits_{x\to 0} \dfrac{-\cancel{2}(\overset{\nearrow 1}{\cos^4 x} - 3\overset{\nearrow 0}{\sin^2 x} \cdot \cos^2 x)}{\cancel{6}\ 3} = -\dfrac{1}{3}$