1. Justificați cu definiția:

a) $\lim\limits_{n\to\infty} \dfrac{3^n}{2^n+1} = +\infty$

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 : x_n > \varepsilon$

$\dfrac{3^n}{2^n+1} > \varepsilon$ minorez (încerc să-l scot pe n în funcție de ε

NU! adun 2^n și iese

$$\frac{3^n}{2^n+1} > \frac{3^n}{2^n+3^n} = \frac{\cancel{3^n}}{\cancel{3^n}\left[\left(\frac{2}{3}\right)^n+1\right]} = \frac{1}{\left(\frac{2}{3}\right)^n+1}$$

$$\Rightarrow \frac{1}{\left(\frac{2}{3}\right)^n+1} > \varepsilon \;/()^{-1} \iff \left(\frac{2}{3}\right)^n < \varepsilon^{-1} - 1 \;/\ln$$

$$n \ln\frac{2}{3} < \ln(\varepsilon^{-1}-1)$$

$$n < \frac{\ln(\varepsilon^{-1}-1)}{\ln\frac{2}{3}}$$

$$\Rightarrow \text{alegem } n_0 = \left[\frac{\ln(\varepsilon^{-1}-1)}{\ln\frac{2}{3}}\right] - 1$$

b) $\lim\limits_{n\to\infty} \dfrac{2^n}{3^n-1} = 0$

limita FINITĂ de demonstrat

$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 \; |x_n - 0| < \varepsilon$

$\dfrac{2^n}{3^n-1} < \varepsilon$ majorez

$$\frac{2^n}{3^n-1} < \frac{2^n}{3^n-2^n} = \frac{\cancel{2^n}}{\cancel{2^n}\left[\left(\frac{3}{2}\right)^n-1\right]} = \frac{1}{\left(\frac{3}{2}\right)^n-1} < \varepsilon \;/()^{-1}$$

$$\left(\frac{3}{2}\right)^n \; \varepsilon^{-1}+1 \;/\ln \iff n\ln\frac{3}{2} \; \ln(\varepsilon^{-1}+1) \Rightarrow n \; \frac{\ln(\varepsilon^{-1}+1)}{\ln\frac{3}{2}}$$

$$\Rightarrow \text{alegem } n_0 = \left[\frac{\ln(\varepsilon^{-1}+1)}{\ln\frac{3}{2}}\right] + 1$$

2. Calculați limita:

a) $\lim\limits_{n\to\infty} \frac{1\cdot 1! + 2\cdot 2! + \dots + n\cdot n!}{(n+1)!\ln n}$

$(n+1)!\ln n$ - crescător și divergent

$\overset{S-C}{=} \lim\limits_{n\to\infty} \frac{(n+1)(n+1)!}{(n+2)!\ln(n+1) - (n+1)!\ln n} = \lim\limits_{n\to\infty} \frac{n+1}{(n+2)\ln(n+1) - \ln n} \overset{\left(\frac{\infty}{\infty-\infty}\right)}{=}$

$= \lim\limits_{n\to\infty} \frac{\cancel{n}(1+\overset{\to 0}{\frac{1}{n}})}{\cancel{n}\left(\underset{\downarrow 1}{\frac{n+2}{n}}\ln(n+1) - \underset{\downarrow 0}{\frac{\ln n}{n}}\right)} = \frac{1}{\infty} = 0$

=

b) $\lim\limits_{n\to\infty} \frac{\ln\frac{1}{2} + \ln\frac{3}{4} + \dots + \ln\frac{2n-1}{2n}}{\ln n}$

$\ln n$ - crescător și divergent

$\overset{S-C}{=} \lim\limits_{n\to\infty} \frac{\ln\frac{2n+1}{2n+2}}{\ln(n+1) - \ln n} = \lim\limits_{n\to\infty} \frac{\ln\frac{2n+1}{2n+2}}{\ln\frac{n+1}{n}} = \left(\frac{\infty}{\infty}\right) = \lim\limits_{n\to\infty} \frac{1}{\cancel{(n+1)}(2n+1)} \cdot \frac{n\cancel{(n+1)}}{1} = \lim\limits_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$

$\left(\ln\frac{2n+1}{2n+2}\right)' = \frac{\cancel{2n+2}}{2n+1} \cdot \frac{2(2n+2) - 2(2n+1)}{(2n+2)^{\cancel{2}}} = \frac{1}{(n+1)(2n+1)}$

$\left(\ln\frac{n+1}{n}\right)' = \frac{\cancel{n}}{n+1} \cdot \frac{n-(n+1)}{n^{\cancel{2}}} = \frac{1}{(n+1)n}$

3. Calculați derivata de ordin n a funcției

a) $f(x) = x\cdot\ln(x-1)$, $x>1$

$u(x) = x$

$v(x) = \ln(x-1)$ - indefinit derivabile pe $(1,\infty)$

formula lui Leibnitz $\sum\limits_{k=0}^{n} C_n^k\, u(x)^{(k)} \cdot v(x)^{(n-k)} \Rightarrow$

$\Rightarrow C_n^0 \cdot x \cdot [\ln(x-1)]^{(n)} + C_n^1 [\ln(x-1)]^{(n-1)} + 0 = x\ln(x-1)^{(n)} + n[\ln(x-1)]^{(n-1)}$

...ivata de ordin n a lui $\ln(x-1) = g(x)$

$$g' = \frac{1}{x-1} = \frac{(-1)^0 \cdot 1}{(x-1)^1}$$

$$g'' = \frac{-1}{(x-1)^2} = \frac{(-1)^1 \cdot 1}{(x-2)^2}$$

$$g''' = \frac{2}{(x-1)^3} = \frac{(-1)^2 \cdot 1 \cdot 2}{(x-1)^3}$$

$$g^{(4)} = \frac{-2 \cdot 3}{(x-1)^4} = \frac{(-1)^3 \cdot 1 \cdot 2 \cdot 3}{(x-1)^4}$$

$$\Rightarrow g^{(n)} = \frac{(-1)^{n-1} \cdot 1 \cdot 2 \cdot \ldots \cdot (n-1)}{(x-1)^n} \quad \forall n \geq 2$$

inducție I verificare : $g'' = \frac{(-1)^1 \cdot 1}{(x-1)^2}$ „A"

II demonstrație: $p(n) \to p(n+1)$

$$p(n+1): \; g^{(n+1)} = \frac{(-1)^n \cdot 2 \cdot 3 \cdot \ldots \cdot n}{(x-1)^{n+1}}$$

$$\left(g^{(n)}\right)' = \left(\frac{(-1)^{n-1} \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}{(x-1)^n}\right)' = \frac{-\left((-1)^{n-1} \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n \cdot x-1)^{n-1}\right)}{(x-1)^{\cancel{2n}\, n+1}}$$

$$= g^{(n+1)}$$

$\Rightarrow p(n)$ - adevărat

$$\Rightarrow (v \cdot u)^{(n)} = x \cdot \frac{(-1)^{n-1} \cdot 1 \cdot 2 \cdot \ldots \cdot (n-1)}{(x-1)^n} + n \cdot \frac{(-1)^{n-2} \cdot 1 \cdot 2 \cdot \ldots \cdot (n-2)}{(x-1)^{n-1}}$$

b) $f(x) = x \cdot \sqrt{x-1}$ (tot cu Leibnitz)

facem inducția pt deriv. de ordin n a lui $\sqrt{x-1} = (x-1)^{\frac{1}{2}} = g(x)$

$$g' = \frac{1}{2}(x-1)^{-\frac{1}{2}}$$

$$g'' = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot (x-1)^{-\frac{3}{2}}$$

$$g''' = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdot (x-1)^{-\frac{5}{2}}$$

$$g^{(4)} = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdot \left(-\frac{5}{2}\right) \cdot (x-1)^{-\frac{7}{2}}$$

$$\Rightarrow g^{(n)} = (-1)^{n-1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n} \cdot (x-1)^{-\frac{2n-1}{2}}$$

$n \geq 2$

inducție:

I verificare $n=2 \Rightarrow g'' = (-1)^1 \cdot \frac{1}{2^2} \cdot (x-1)^{-\frac{3}{2}}$ „A"

II demonstrația:

$p(n) \rightarrow p(n+1)$

$p(n+1): g^{(n+1)} = (-1)^n \cdot \frac{1 \cdot 2 \cdot \ldots \cdot (2n-1)}{2^{n+1}} \cdot (x-1)^{-\frac{2n+1}{2}}$

$(g^{(n)})' = \left[(-1)^{n-1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n} \cdot (x-1)^{-\frac{2n-1}{2}}\right]' =$

$= (-1) \cdot \frac{2n-1}{2} \cdot (-1)^{n-1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots (2n-3)}{2^n} \cdot (x-1)^{-\frac{2n-1}{2}-1} = g^{(n+1)}$

$\Rightarrow p(n)$ - adevărat

$\Rightarrow \sum_{k=0}^{n} C_n^k \, u(x)^{(k)} \cdot v(x)^{(n-k)} = C_n^0 \, x \cdot (v(x))^{(n)} + C_n^1 \cdot (v(x))^{(n-1)} =$

$= x \cdot (-1)^{n-1} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)}{2^n} \cdot (x-1)^{-\frac{2n-1}{2}} + n \cdot (-1)^{n-2} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-5)}{2^{n-1}}$