Seminar 9

2.1.> Să se găsească un exemplu de 2 subgrupuri ale unui grup a căror reuniune NU este un subgrup.

$2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\} \leq (\mathbb{Z}, +)$

$3\mathbb{Z} = \{3k \mid k \in \mathbb{Z}\} \leq (\mathbb{Z}, +)$

De ce $2\mathbb{Z} \cup 3\mathbb{Z} \leq (\mathbb{Z}, +)$?

$$\underbrace{4}_{\substack{\in 2\mathbb{Z} \\ \in 2\mathbb{Z}\cup 3\mathbb{Z}}} + \underbrace{3}_{\substack{\in 3\mathbb{Z} \\ \in 2\mathbb{Z}\cup 3\mathbb{Z}}} = 7 \to \notin 2\mathbb{Z} \cup 3\mathbb{Z}$$

2.1.59 Fie $(G,+)$ grup abelian $H, K \leq G$. S.ș.a.c. $\langle H \cup K \rangle = H + K$, unde

$H + K = \{x + y \mid x \in H, y \in K\}$.

$$\langle H \cup K \rangle = H + K \Leftrightarrow \begin{cases} 1)\ H + K \leq G \\ 2)\ H \cup K \leq H + K \\ 3)\ \text{dacă } L \leq G \end{cases}$$

a.î. $H \cup K \leq L$ atunci $H + K \leq L$ a

1) $H + K \leq G$.

I. $H, K \leq G \Rightarrow 0 \in H, K$

$$\left.\begin{array}{l} 0 = \underset{\in H}{0} + \underset{\in K}{0} \end{array}\right| \Rightarrow 0 \in H + K$$

II $\forall x, y \in H + K$. Vrem $x + y \in H + K$

$\Downarrow$

$\exists a \in H, b \in K$ a.î. $x = a + b$.

$\exists c \in H, d \in K$ a.î. $y = c + d$

$x+y=(a+b)+(c+d) \quad \underline{\text{+ com, asoc}}$

$=\underbrace{(a+c)}_{\in H\leq G}+\underbrace{(b+d)}_{\in K\leq G} \in H+K$

$\underline{\text{iii}}$. $\forall\, x\in H+K$ vrem $-x\in H+K$

$\Downarrow$

$\exists\, a\in H,\ b\in K$ aî $x=a+b$.

$-x=(-a)+(-b)$

$$\left.\begin{array}{l} a\in H \xRightarrow{H\leq G} -a\in H \\ b\in K \xRightarrow{K\leq G} -b\in K \end{array}\right\} \Rightarrow -x\in H+K$$

2) $H\cup K\subseteq H+K$

Fie $x\in H\cup K$. Vrem $x\in H+K$

$\Downarrow$

$x\in H$ sau $x\in K$

$\underline{\text{Caz I}}$ $x\in H$ Vrem $x\in H+K$

$x=\underset{\in H}{x}+\underset{\in K\leq G}{0}$

$\underline{\text{Caz II}}$ $x\in K$

$x=\underset{\in H}{0}+\underset{\in K}{x} \Rightarrow x\in H+K$

II. Dacă $L\leq G$ aî $H\cup K\subseteq L$

Vrem $\underbrace{H+K}_{\leq G}\subseteq\underbrace{L}_{\leq G} \Leftrightarrow H+K\leq L$

Fie $x\in H+K$. Vrem $x\in L$

$\Downarrow$

$\exists\, h\in H$ și $k\in K$ aî $x=h+k$.

$$\left.\begin{array}{l} h\in H\subseteq H\cup K\subseteq L \\ k\in K\subseteq H\cup K\subseteq L \end{array}\right\} \Rightarrow h+k\in L \text{ pt că } L \text{ p.s. } (L\leq G) \Rightarrow x\in L$$

2.1.61. Fie $n, m \in \mathbb{Z}$. S.a.c. (a) $n\mathbb{Z} \subseteq m\mathbb{Z} \Leftrightarrow m | n$.

(b) $n\mathbb{Z} \cap m\mathbb{Z} = K\mathbb{Z}$ unde $K = \underset{\Downarrow}{\text{lcm}}(n,m)$
cmmmmc.

(c) $n\mathbb{Z} + m\mathbb{Z} = d\mathbb{Z}$, unde $d = \underset{\Downarrow}{\text{gcd}}(n,m)$
cmmmmd.

(a) "$\Rightarrow$" Știm $n\mathbb{Z} \subseteq m\mathbb{Z}$. Vrem $m|n$

$n = n \cdot 1 \in n\mathbb{Z} \xRightarrow{n\mathbb{Z} \subseteq m\mathbb{Z}} n \in m\mathbb{Z} \Rightarrow \exists K \in \mathbb{Z}$ a.î. $n = mK$
$\Downarrow$
$m | n$

"$\Leftarrow$" Știm $m|n$. Vrem $n\mathbb{Z} \subseteq m\mathbb{Z}$

Fie $x \in n\mathbb{Z}$. Vrem $x \in m\mathbb{Z}$
$\Downarrow$
$\exists K \in \mathbb{Z}$ a.î. $x = n \cdot K$.
$m|n \Rightarrow \exists t \in \mathbb{Z}$ a.î. $n = m \cdot t$

$\Big\} \Rightarrow x = m\underbrace{(\underset{\in\mathbb{Z}}{t} \cdot \underset{\in\mathbb{Z}}{K})}_{\in\mathbb{Z}}$ $\exists$

$\Rightarrow x \in m\mathbb{Z}$

(b) Vrem $n\mathbb{Z} \cap m\mathbb{Z} = K \cdot \mathbb{Z}$, $K = \text{lcm}(n,m)$
$\uparrow$
dem prin dublă incluziune

Dem că $n \cdot \mathbb{Z} \cap m \cdot \mathbb{Z} \subseteq K \cdot \mathbb{Z}$

Fie $x \in n \cdot \mathbb{Z} \cap m \cdot \mathbb{Z}$. Vrem $x \in K \cdot \mathbb{Z}$
$\Downarrow$
$x \in n\mathbb{Z}$ și $x \in m\mathbb{Z}$
$\Downarrow$ $\Downarrow$
$\underbrace{n | x \qquad m | x}$
$\Rightarrow x$ este multiplu comun $\cancel{a}$ n și m.

Fie $m, n \in \mathbb{N}$, $d = \gcd(m, n)$

c) $\Rightarrow m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z}$

$d \in d\mathbb{Z} \Rightarrow d \in m\mathbb{Z} + n\mathbb{Z}$
$\Downarrow$
$d \cdot 1$

$\exists\, s, t \in \mathbb{Z}$ a.î. $d = m \cdot s + n \cdot t$

În part. ~~regula~~, $\gcd(m, n) = 1 \Rightarrow \exists\, s, t \in \mathbb{Z}$ a.î. $1 = m \cdot s + n \cdot t$

"$\Leftarrow$" Știm că $\exists\, s, t \in \mathbb{Z}$ a.î. $1 = m \cdot s + n \cdot t$

Vrem că $\gcd(m, n) = 1$.

$\gcd(m, n) = d \Rightarrow d \mid m$ și $d \mid n$.

$$\begin{array}{cc} \Downarrow & \Downarrow \\ d \mid m \cdot s & d \mid n \cdot t \\ \hline \multicolumn{2}{c}{\Downarrow} \\ \multicolumn{2}{c}{d \mid m \cdot s + n \cdot t} \\ \multicolumn{2}{c}{\Downarrow} \\ \multicolumn{2}{c}{d \mid 1 \Rightarrow d = 1} \end{array}$$

$\Rightarrow \gcd(m, n) = 1$.