MNTC P01 - Week #6 - Integrals - Modeling

Numerical Integration

As part of this assignment, you should be able to reproduce the LHR rule calculations in MATLAB using a loop. You should know how to adapt it to handle either data from a file, or a function defined by a formula, as the case requires.

1. **Theory** Consider the problem of estimating the general form of the integral

$$\int_a^b f(x)\ dx$$

   (a) Assume $f(x)$ is a smooth and continuous function. For our the Left-Hand sum, LEFT(n), by what factor do we reduce the error if we use 10 times the number of intervals?

   (b) Evaluate the integral $\displaystyle\int_0^6 \cos(x)\ dx$ exactly, using anti-derivatives and the Fundamental Theorem of Calculus.

   (c) Confirm your answers to part (a) by finding the change in the error for LEFT(n) for the same integral, $\displaystyle\int_0^6 \cos(x)\ dx$ using $n = 20$, $n = 200$, and $n = 2000$. Find the error with each n value, and then compute the ratio of the errors each time you use $10\times$ as many intervals.

2. For each of the following integrals,

   - Evaluate the integral exactly,
   - use MATLAB to compute the LEFT(1000) integral estimate, and
   - comment on the agreement between the results.

   (a) $\displaystyle\int_0^5 x^3 - 5\ dx$

   (b) $\displaystyle\int_1^{10} \log_{10}(x)\ dx$

   (c) $\displaystyle\int_{-1}^{1} x^2 e^{x^3}\ dx$

3. Use the `integral` function to estimate the following integrals, and print the estimates with at least 8 digits after the decimal. Use the default accuracy for the `integral` function. Note that the exact values for these integrals were computed in the previous question, so we can compare the `integral` estimate to the exact values.

   (a) $\displaystyle\int_0^5 x^3 - 5\ dx$ (exact value: $\dfrac{5^4}{4} - 25$)

   (b) $\displaystyle\int_1^{10} 4\log_{10}(x)\ dx$ (exact value: $\dfrac{(-36 + 40\ln(2) + 40\ln(5))}{\ln(2) + \ln(5)}$

   (c) $\displaystyle\int_{-1}^{1} x^2 e^{x^3}\ dx$

4. When a pendulum oscillates, with maximum deviation angle θ_0, the period of the pendulum is given by

$$T = 4\sqrt{L/g}\int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

   where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity, 9.8 m/s.

   Compute and compare the period of a pendulum with

   - $L = 2$, $\theta_0 = 40^o$,
   - $L = 2$, $\theta_0 = 20^o$.

- $L = 2.5$, $\theta_0 = 40^o$,
- $L = 2.5$, $\theta_0 = 20^o$.

Describe how significant the effect of maximum swing angle θ_0 is on the period of a pendulum, compared to the effect of the pendulum length.

5. When underground mining operations are in progress, one concern the monitoring or predicting the subsidence of the rock between the mine and the surface, as “the differential settlement and horizontal strain developed during subsidence tend to be critical in terms of structural damage”[1]

   If the rock is considered isomorphic in character, then complete subsidence s at a point on the surface is given by an integral of the form

$$s = \int_0^R p(r)\ dr$$

   where R is the effective radius of influence, and $p(r)$ is an empirically derived “influence function”.

   Consider the influence function $p(r) = \dfrac{1}{R^2} e^{-\pi r^2/R^2}$ and use an appropriate technique to evaluate the integral

$$s = \int_0^R p(r)\ dr$$

   with $R = 100$.

6. The x coordinate of the center of mass of a object can be computed using the formula

$$\bar{x} = \frac{\int_a^b x \cdot f(x)\ dx}{\int_a^b f(x)\ dx}$$

   where $f(x)$ is the height (or mass) at the point x.

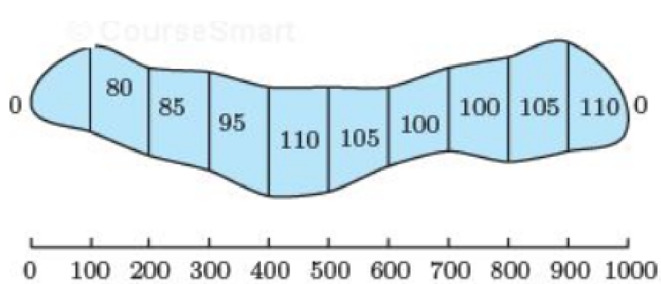
   For the following functions, use MATLAB, with the `integral` function and other tools, to

   - plot the graph of $f(x)$,
   - compute the x center of mass of the object, and
   - draw the center of mass as a vertical line on the same graph.

   (a) $f(x) = x^3$ on $x = [1, 4]$.

   (b) $f(x) = xe^{-x}$ on $x = [0, 10]$.

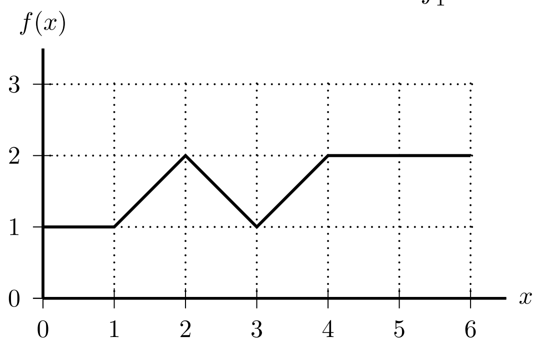
7. The width (in feet) of the golf course fairway was measured at 100-foot intervals as indicated on the figure. Estimate the square footage of of the fairway, using any appropriate means taught in the course.



Average Value

[1]SME Mining Engineering Handbook, 3rd edition, page 632.

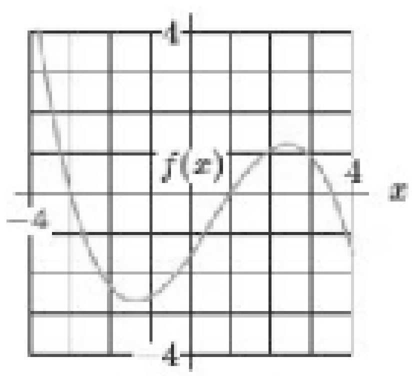
8. (a) Using the graph shown below, find $\int_1^6 f(x)\,dx$.



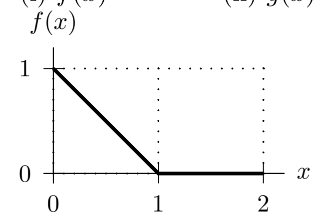
(b) What is the average value of f on [1, 6]?

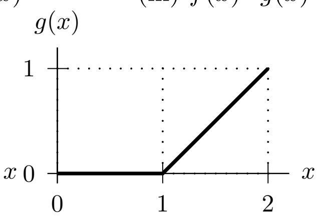
9. (a) Using the graph below, estimate $\int_{-3}^{3} f(x)\,dx$.

(b) Which of the following average values of $f(x)$ is larger?
(i) Between $x = -3$ and $x = 3$, or
(ii) Between $x = 0$ and $x = 3$?



10. (a) Using the graphs of $f(x)$ and $g(x)$ shown below, find the average value on $0 \le x \le 2$ of
(i) $f(x)$ (ii) $g(x)$ (iii) $f(x) \cdot g(x)$





(b) Is the statement that

"Average(f) $\cdot$ Average(g) = Average($f \cdot g$)"

true or not? Explain your answer.

11. (a) Without computing any integrals, explain why the average value of $f(x) = \sin x$ on $[0, \pi]$ must be between 0.5 and 1.

(b) Compute the exact average of $\sin x$ on $[0, \pi]$.

(c) Use MATLAB to plot the graph of $\sin(x)$ on $[0, \pi]$, and draw the average value on the graph as well.

12. (a) What is the average value of $f(x) = \sqrt{1 - x^2}$ over the interval $0 \le x \le 1$? (Hint: we don't have the integration tools to evaluate the integral of $f(x)$ from first principles, but can use MATLAB to evaluate it.)

(b) Use MATLAB to plot the graph of $f(x)$ on $[0, 1]$, and draw the average value on the graph as well.

(c) How can you tell whether this average value is more or less than 0.5 without doing any calculations?