Week #5 : Integrals - Techniques

Goals:

- Recognize the family of functions that can be solved with the technique of integration by substitution.
- Solve integration problems using the technique of substitution.
- Recognize the family of functions that can be solved with the technique of integration by parts.
- Solve integration problems using the technique of integration by parts.

We now return to the challenge of finding a *formula* for an anti-derivative function. We saw simple cases last week, and now we will extend our methods to handle more complex integrals.

Anti-differentiation by Inspection: The Guess-and-Check Method

$$\int x\,dx = \frac{x^2}{2} + C$$

$\frac{d}{dx}$

antideriv

Often, even if we do not see an anti-derivative immediately, we can make an educated guess and eventually arrive at the correct answer.

Problem. Based on your knowledge of derivatives, what should the anti-derivative of $\cos(3x)$, $\displaystyle\int \cos(3x)\,dx$, look like?

$$\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + C$$

$$\cos(3x) = \frac{1}{3}\cos(3x)\cdot 3 \Leftarrow \frac{d}{dx}\frac{1}{3}\sin(3x)$$

Problem. Find $\int e^{3x-2}\,dx$.

$$\int e^{3x-2}\,dx = \frac{1}{3}e^{3x-2} + C$$

Check: $\frac{1}{3}e^{3x-2}(\underline{3}) = \frac{d}{dx}\frac{1}{3}e^{3x-2}$

$\leftarrow$

$=$

$\underline{\underline{e^{3x-2}}}$

Both of our previous examples had *linear* 'inside' functions. Here is an integral with a *quadratic* 'inside' function:

$\cos(3x)$ $\quad e^{3x-2}$

$$\int x e^{-x^2}\, dx$$

Problem. Evaluate the integral.

$$\int x e^{-x^2}\, dx = \frac{-1}{2} e^{-x^2} + C$$

Check: $x e^{-x^2} = \frac{-1}{2} e^{-x^2} \cdot (-2x) = \frac{-1}{2} \frac{d}{dx} e^{-x^2}$

Why was it important that there be a factor x in front of e^{-x^2} in this integral?

chain rule on exponent $(-x^2)$

forced presence of $x e^{-x^2}$ form

Integration by Substitution

We can formalize the guess-and-check method by defining an *intermediate variable* the represents the "inside" function.

Problem. Show that $\int x^3\sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2}+C.$

integrate

diff te

equal

Check:

$$(4x^3)\frac{1}{6}\left(\frac{3}{2}\right)(x^4+5)^{1/2} = \frac{d}{dx}\left(\frac{1}{6}(x^4+5)^{3/2}+C\right)$$

$$= x^3\sqrt{x^4+5}$$

$$\int x^3\sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2}+C$$

Problem. Relate this result to the **chain rule**.

comes from $\frac{d}{dx}$ (inside) $= \frac{d}{dx}(x^4+5)$

Problem. Now use the **method of substitution** to evaluate $\int x^3 \sqrt{x^4+5}\, dx$

let $w = x^4 + 5$ fixed

diff

so $\dfrac{dw}{\boxed{dx}} = \boxed{4x^3}$

$\dfrac{1}{4x^3}\, dw = dx$ in integral

Rewrite integral using w's, dw's:

$$= \int \cancel{x^3} \sqrt{w} \left(\frac{1}{4\cancel{x^3}} dw \right)$$

$$= \frac{1}{4} \int w^{1/2}\, dw$$ ← much simpler integral

antideriv by inspection

$$= \frac{1}{4} \left(\frac{w^{3/2}}{3/2} + C \right)$$

$$= \frac{2}{3} \left(\frac{1}{4} \right) \left(w^{3/2} \right) + C/4$$

back x's

$$= \frac{1}{6} \left(x^4 + 5 \right)^{3/2} + C/4$$

Steps in the Method Of Substitution

1. Select a simple function $w(x)$ that appears in the integral.
   - Typically, you will also see w' as a **factor** or **multiplier** in the integrand as well.

     deriv of w

   Solve for dx

2. Find $\dfrac{dw}{dx}$ by differentiating. Re-write it in the form $\ldots dw = \ldots dx$

3. Rewrite the integral using only w and dw (no x nor dx). $\int$ dx
   - If you can now evaluate the integral, the substitution was effective.
   - If you cannot remove all the x's, or the integral became harder instead of easier, then either try a different substitution, or a different integration method.

Problem. Find $\int \tan(x)\, dx$.

$$= \int \frac{\sin(x)}{\cos(x)}\, dx \qquad = ?$$

move to w's

let $w = \cos(x)$

so $\frac{dw}{\boxed{dx}} = \boxed{-\sin(x)}$

$$\frac{1}{-\sin(x)}\, dw = dx$$

$$= \int \frac{\cancel{\sin(x)}}{w} \left(\frac{1}{-\cancel{\sin(x)}}\, dw \right)$$

$$= \int \frac{-1}{w}\, dw$$

no x's, simpler

antideriv

$$= -\ln|w| + C$$

back to x's

$$= -\ln|\cos(x)| + C$$

Though it is not required unless specifically requested, it can be re-assuring to check the answer.

Problem. Verify that the anti-derivative you found is correct.

$$\frac{d}{dx}\left(-\ln|\cos(x)|+c\right)$$

$$= -\frac{1}{\cos(x)} \cdot (-\sin(x))$$

$$= \frac{\sin(x)}{\cos(x)} = \tan(x) \quad \text{original integrand}$$

Problem. Find $\int x^3 e^{x^4-3} dx$.

let $w = x^4 - 3$

so $\frac{dw}{dx} = 4x^3$

$\Rightarrow \frac{1}{4x^3} dw = dx$

more w's

$= \int \cancel{x^3} e^w \left(\frac{1}{4\cancel{x^3}} dw \right)$

$= \int \frac{1}{4} e^w dw$ simpler, no x's

$= \frac{1}{4} e^w + C$

& back to x's

$= \frac{1}{4} e^{x^4-3} + C$

Check: $\frac{d}{dx}\left(\frac{1}{4} e^{x^4-3} + C \right) = \frac{1}{\cancel{4}} e^{x^4-3} \cdot \cancel{4} x^3$

$= x^3 e^{x^4-3}$ ✓

Problem. For the integral,

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2} dx$$

$\frac{dw}{dx}$ as multiplier

w

both $w = e^x - e^{-x}$ and $w = e^x + e^{-x}$ are seemingly reasonable substitutions.

Question: Which substitution will change the integral into the simpler form?

1. $w = e^x - e^{-x}$ → $\frac{dw}{dx} = e^x + e^{-x}$

"inside" as w /

2. $w = e^x + e^{-x}$ ✓ → $\frac{dw}{dx} = e^x - e^{-x}$

see $\frac{dw}{dx}$ as a multiplier

Problem. Compare both substitutions in practice.

$$I = \int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\, dx$$

with $w = e^x - e^{-x}$

so $\frac{dw}{dx} = e^x + e^{-x}$

or $\left(\frac{1}{e^x + e^{-x}}\right) dw = dx$

$$I = \int \frac{w}{(e^x + e^{-x})^2} \left(\frac{1}{(e^x + e^{-x})} dw\right)$$

$$= \int \frac{w}{(e^x + e^{-x})^3} dw$$

still have x's

cannot be removed w/w's

need to try something

with $w = e^x + e^{-x}$

so $\frac{dw}{dx} = e^x - e^{-x}$

or $\frac{1}{e^x - e^{-x}} dw = dx$

$$I = \int \frac{\cancel{e^x - e^{-x}}}{w^2} \left(\frac{1}{\cancel{e^x - e^{-x}}} dw\right)$$

$$= \int \frac{1}{w^2} dw$$

$\frac{1}{w^2} = w^{-2}$

no x's, simpler

$$= \frac{w^{-1}}{-1} + C$$

$$= -\frac{1}{e^x + e^{-x}} + C$$

Problem. Find $I = \displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}\,dx.$

try: $w = 1 + \cos^2(x)$
↳ complicated $\frac{dw}{dx}$

or

$w = \cos(x)$
↳ simple deriv

let $w = \cos(x)$

so $\dfrac{dw}{dx} = -\sin(x)$

or $\dfrac{-1}{\sin(x)}\,dw = dx$

so

$$I = \int \frac{\cancel{\sin(x)}}{1+w^2}\left(\frac{-1}{\cancel{\sin(x)}}\,dw\right)$$

back to x's

$$= \int \frac{-1}{1+w^2}\,dw = -\arctan(w) + C = -\arctan(\cos(x)) + C$$

Recall $\dfrac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2}$

Using the Method of Substitution for Definite Integrals

If we are asked to evaluate a **definite** integral such as

"Area" under graph

$$\int_{x=0}^{x=\pi/2} \frac{\cos x}{1+\sin x}\, dx \quad , \qquad \Delta x \qquad \to \quad du$$

where a substitution will ease the integration, we have two methods for handling the limits of integration ($x = 0$ and $x = \pi/2$).

a) When we make our substitution, convert both the *variables* x and the *limits* (in x) to the new variable; or

b) do the integration while keeping the limits explicitly in terms of x, writing the final integral back in terms of the original x variable as well, and *then* evaluating.

Problem. Use method (a) (converting both the integral and the limits to the new variable) to evaluate the integral

$$I = \int_{x=0}^{x=\pi/2} \frac{\cos x}{1 + \sin x} dx$$

$w = 1 + \sin(x)$

or

$w = \sin(x)$

let $w = 1 + \sin(x)$

$x = \pi/2 \to w = 1 + \sin(\pi/2)$
$= 2$

so $\frac{dw}{dx} = \cos(x)$

$x = 0 \to w = 1 + \sin(0)$
$w = 1$

or $\frac{1}{\cos(x)} dw = dx$

$$I = \int_{w=1}^{w=2} \frac{\cancel{\cos(x)}}{w} \left(\frac{1}{\cancel{\cos(x)}} dw \right)$$

in terms of w

$$= \int_{w=1}^{w=2} \frac{1}{w} dw = \ln|w| \Big|_1^2 = (\ln|2| + \cancel{c}) - (\ln|1| + \cancel{c})$$

$+c$

simpler, no x's

$= \ln 2 - 0$
$= \ln(2)$

Problem. Use method (b) (converting back to x's to evaluate at the end points) to evaluate

or $w = 1 + \sqrt{x}$

$$\int_{x=9}^{x=64} \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx \ .$$

let $w = \sqrt{x} = x^{1/2}$

so $\dfrac{dw}{dx} = \dfrac{1}{2} x^{-1/2} = \dfrac{1}{2} \dfrac{1}{\sqrt{x}}$

or $2\sqrt{x}\, dw = dx$

so $I = \displaystyle\int_{x=9}^{x=64} \frac{\sqrt{1+w}}{\cancel{\sqrt{x}}} \left(2 \cancel{\sqrt{x}}\, dw\right)$

$= \displaystyle\int_{x=9}^{x=64} 2\sqrt{1+w}\, dw = 2\,(1+w)^{3/2} \Big|_{x=9}^{x=64} = 2\left(1+\sqrt{x}\right)^{3/2} \Big|_{x=9}^{x=64}$

integrate

back to x's

$= 2\left(1+\sqrt{64}\right)^{3/2} - 2\left(1+\sqrt{9}\right)^{3/2}$

$= 2\,(9)^{3/2} - 2\,(4)^{3/2}$

$= 54 - 16 = \boxed{38}$

Integration by Parts

So far in studying integrals we have used

$\int x^2 \, dx = x^3/3 + C$

$\int \sin(x) \, dx$

- direct anti-differentiation, for relatively simple functions, and $= -\cos(x) + C$
- integration by substitution, for some more complex integrals.

$w =$

However, there are many integrals that can't be evaluated with these techniques.

Problem. Try to find $\displaystyle\int xe^{4x}\,dx$.

$* \quad e^{4x}$

Subs!

↓

chain rule

$e^{x^2} \xrightarrow{d/dx} e^{x^2} \cdot 2x$

$w = 4x$

$\frac{dw}{dx} = 4$

→ won't lead to simpler integral

This particular integral can be evaluated with a different integration technique, **integration by parts.** This rule is related to the **product rule** for derivatives.

Problem. Expand

$$\overbrace{\int \overbrace{\frac{d}{dx}}^{\text{deriv}}(uv)\,dx}^{\text{anti deriv}} = \int\left(\frac{du}{dx}\cdot v + u\cdot\frac{dv}{dx}\right)dx$$

Integrate both sides with respect to x and simplify.

back to

$$uv = \int\left(\frac{du}{dx}\,v\right)dx + \int\left(u\,\frac{dv}{dx}\right)dx$$

Express $\int u\frac{dv}{dx}\,dx$ relative to the other terms.

$$\underbrace{\int u\left(\frac{dv}{dx}\right)dx}_{\text{integral we have}} = uv - \int\left(\frac{du}{dx}\,v\right)dx$$

trade for new integral

Integration by Parts
For short, we can remember this formula as

$$\int u\,dv = uv - \int v\,du$$

Integration by parts:

- Choose a part of the integral to be u, and the remaining part to be dv. → includes 'dx'
- **Differentiate** u to get du.
- **Integrate** dv to get v.
- Replace $\displaystyle\int u\ dv$ with $\displaystyle uv - \int v du$.
- Hope/check that the new integral is easier to evaluate.

Problem. Use integration by parts to evaluate $\int \underbrace{x}_{u} \underbrace{e^{4x}\,dx}_{dv}$.

let $u = x$ $\quad \downarrow d/dx$

$\frac{du}{\cancel{dx}} = 1\,dx$

$\int 1\,dv = \int e^{4x}\,dx$ $\quad \downarrow$ integrate

$v = \frac{e^{4x}}{4}$

(no +C req'd: would end up cancelling later)

product: complicated

new integral

$$\underbrace{\int x \cdot e^{4x}\,dx}_{\text{orig'l integral}} = (x)\left(\frac{e^{4x}}{4}\right) - \overbrace{\int \left(\frac{e^{4x}}{4}\right) dx}^{\text{new integral}}$$

$$u \cdot v - \int v \cdot du$$

simpler

tidy

$$= \frac{1}{4} x e^{4x} - \frac{1}{4} \int e^{4x}\,dx$$

evaluate new integral

$$= \frac{1}{4} x e^{4x} - \frac{1}{4}\left(\frac{e^{4x}}{4}\right) + C$$

Problem. Verify that your anti-derivative is correct.

$$\frac{d}{dx}\left(\frac{1}{4}xe^{4x} - \frac{1}{4}\left(\frac{e^{4x}}{4}\right) + C\right)$$

$$= \frac{1}{4}\left[1\cdot e^{4x} + x\cdot\left(4e^{4x}\right)\right] - \frac{1}{16}\left(4e^{4x}\right)$$

$$= \cancel{\frac{1}{4}e^{4x}} + xe^{4x} - \cancel{\frac{1}{4}e^{4x}}$$

$$= xe^{4x}$$

Confirms $\displaystyle\int xe^{4x}\,dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C$

Integration By Parts - Examples

Guidelines for selecting u and dv

- Ensure you can actually integrate the dv part by itself, then
- Try to select u and dv so that either
  - u' is simpler than u or
  - $\int dv$ is simpler than dv

or not more complicated

$\int dv =$

$$\int e^{2x}\,dx \to \frac{1}{2}e^{2x}$$

$$\frac{d}{dx}x = 1 \quad \text{simpler}$$

Problem. Find $\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv}$.

let $u = x$ $\qquad \int dv = \int \cos(x)\, dx$

$\downarrow d/dx \qquad\qquad \downarrow$ integrate

$du = 1\, dx \qquad v = \sin(x)$

simpler!

so $\int \underbrace{x}_{u} \underbrace{\cos(x)\, dx}_{dv} = \underbrace{x}_{u} \cdot \underbrace{\sin(x)}_{v} - \int \underbrace{\sin(x)}_{v} \cdot \underbrace{dx}_{du}$ integrate

$= x\sin(x) - (-\cos(x)) + C$

$= x\sin(x) + \cos(x) + C$

Problem. Now evaluate the slightly more challenging integral

$$\int x^2 \cos x \; dx$$

let $u = x^2$ $\quad\quad \int dv = \int \cos(x)\, dx$

$\downarrow d/dx$ $\quad\quad$ integrate

$du = 2x\, dx$ $\quad\quad v = \sin(x)$

so $\displaystyle\int x^2 \cos(x)\, dx = x^2 \sin(x) - \int \sin(x)(2x)\, dx$

$\downarrow$ tidy

$\displaystyle = x^2 \sin(x) - 2 \int \underbrace{x \cdot \sin(x)\, dx}$

new integral, simpler but not simple enough to do w/out technique.

$$\int x^2 \cos x \, dx$$

$$I_2 = \int x \sin(x) dx$$

by parts

$u = x \quad \int dv = \int \sin(x)\, dx$

$\downarrow d/dx \quad \downarrow$ integrate

$du = 1 \cdot dx \longleftarrow v = -\cos(x)$

so $\displaystyle\int x \sin(x) . dx = \underbrace{x}_{u} \cdot \underbrace{(-\cos(x))}_{v} \underbrace{- \int - \cos(x)\, dx}_{-\int v \, du}$ tidy

$$= -x \cos(x) + \int \cos(x)\, dx$$

integrate

$$= -x \cos(x) + \sin(x)$$

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) - 2 \int x \sin(x) dx$$

$$= x^2 \sin(x) - 2\left(-x \cos(x) + \sin(x)\right) + C$$

Integration By Parts - Definite Integrals

When using integration by parts to evaluate *definite* integrals, you need to apply the limits of integration to the **entire** anti-derivative that you find.

Problem. Evaluate $\displaystyle\int_0^{\pi} x \sin 4x \; dx$

Don't forget that dv does not require any other factors besides dx. That can help when there is only a single factor in the integrand.

Problem. Find the area under the graph of $\ln x$ between $x = 1$ and $x = 2$.

General integration advice:

- Look for a substitution in your integral first - they are the simplest method to use, and usually the most obvious.
- Only try integration by parts if substitution fails.
- With all methods, you may need to **experiment** with your choice of u, dv, or your substitution.