Week #2 : Derivatives - Linearization

Goals:

- Describe the meaning and value of linearization
- Apply the technique of linearization to solve a variety of nonlinear equations
- Use MATLAB to graph and compare functions with their linearizations
- Use MATLAB to implement Newton's method
- Calculate and interpret the first and second derivatives, as well as higher order derivatives

Linear Approximations

You should now feel comfortable in finding the derivative of a wide variety of functions with formulas.

In this section, we will explore how the derivatives you can compute can be tied back to understanding the behaviour of the original function.

We will start by returning to the definition of the derivative, based on the $\dfrac{\text{rise}}{\text{run}}$ formula for slopes:

$$f'(x) = \frac{d}{dx}f = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For example, if $y = f(x)$, then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

Problem. What is the relationship between f' and Δy, Δx for merely *small* delta values?

Now sketch a graph, and label the two points in the Δx difference 'x' and 'a'. What expression do we obtain for $f(x)$?

We want to have a name for the RHS of this approximation:

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

Problem. What are some names for this linear function?

Problem. Consider the tangent line approximation to the graph of $f(x) = e^x$ at $(0, 1)$, where we know that $f'(0) = 1$.
A good approximation for $e^{0.5}$ is therefore:

A. $e^{0.5} \approx 0.5$

B. $e^{0.5} \approx 1 + e^{.5}$

C. $e^{0.5} \approx 1 + .5$

Problem. A rock formation with high density ores was identified using gravity measurements; the formation is roughly cubic in shape. The edge each side of the cube was found to be 370 m, with a possible error in measurement of 10 m. Use the ideas of linear approximations to estimate the maximum possible error (positive or negative) in computing the **volume** of the formation.

Problem. What are the trade-offs of using the linear approximation to obtain the above error estimate, compared to a direct calculation of the possible volumes with the error measurements?

Problem. Sketch the function $f(x) = \dfrac{1}{x}$.

Problem. If we drew a tangent line to $f(x)$ at $x = 4$, what range we would expect for the slope there?

A. Slope above 1.

B. Slope between 0 and 1.

C. Slope between 0 and -1.

D. Slope below -1.

Problem. Find the linearization of $f(x) = \dfrac{1}{x}$ at $a = 4$.

Find the equation of the tangent line to the hyperbola $y = \dfrac{1}{x}$ at $x = 4$.

Problem. Sketch the graph of $y = \dfrac{1}{x}$, and its tangent line at $x = 4$.

The $\sin(x)$ Approximation

One of the most commonly-used approximation in physics is the relationship

$$\sin(x) \approx x$$

Problem. Derive this relationship using linearization.

What is the fine-print that should **always** be associated with this approximation?

Sketch the graphs of $y = \sin(x)$ and $y = x$. Focus on the domain $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$.

Below are more detailed calculations relating $\sin(x)$ and x.
By filling in some or all of the missing values, determine the range of angles which the *relative error* in the approximation $\sin(x) \approx x$ is **less than 1%**. State your answer in degrees and radians.

x (degrees)								
x (rad)	0.0500	0.1000	0.1500	0.2000	0.2500	0.3000	0.3500	0.4000
$\sin(x)$	0.0500	0.0998	0.1494	0.1987	0.2474	0.2955	0.3429	0.3894
abs err	0.0000	0.0002	0.0006	0.0013	0.0026	0.0045	0.0071	0.0106
rel err								

Just for fun, put your calculator into degree mode, and see whether $\sin(x) \approx x$ still holds for small x.