MNTC P01 - Week #6 - Integrals - Modeling

Numerical Integration

As part of this assignment, you should be able to reproduce the LHR rule calculations in MATLAB using a loop. You should know how to adapt it to handle either data from a file, or a function defined by a formula, as the case requires.

1. **Theory** Consider the problem of estimating the general form of the integral

$$\int_a^b f(x)\ dx$$

   (a) Assume $f(x)$ is a smooth and continuous function. For our the Left-Hand sum, LEFT(n), by what factor do we reduce the error if we use 10 times the number of intervals?

   (b) Evaluate the integral $\int_0^6 \cos(x)\ dx$ exactly, using anti-derivatives and the Fundamental Theorem of Calculus.

   (c) Confirm your answers to part (a) by finding the change in the error for LEFT(n) for the same integral, $\int_0^6 \cos(x)\ dx$ using $n = 20$, $n = 200$, and $n = 2000$. Find the error with each n value, and then compute the ratio of the errors.

(a) If $f(x)$ is a smooth function, if we use 10 times the intervals, the error for LHR's estimate will drop by a factor of $\frac{1}{10}$

(b) We can evaluate the integral based on our earlier study of derivatives. The only thing to be careful of is that the input values will be in radians (not degrees).

$$\int_0^6 \cos(x)\ dx = \sin(x)\Big|_0^6 = \sin(6) - \sin(0) = -0.279415498$$

(c) Here are the errors and ratios.

(d)

$\int_0^6 \cos(x)\ dx$	$n = 20$ error	$N = 200$ error	Ratio
LHR	0.0080732234	0.00061840218	$0.0766 \approx \frac{1}{10}$

2. Write code that will find the LEFT(n)integral estimate of the following integrals:

   (a) $\int_0^5 x^3 - 5\ dx$ (exact value: $\frac{5^4}{4} - 25$)

   (b) $\int_1^{10} 4\log_{10}(x)\ dx$ (exact value: $\frac{(-36 + 40\ln(2) + 40\ln(5))}{\ln(2) + \ln(5)}$

   (c) $\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\ dx$ (exact value not known; approximately 0.95450)

   For all integral estimates, use $n = 100$ intervals.

3. Use the `integral` function to estimate the following integrals, and print the estimates with 8 digits after the decimal. Use the default accuracy for the `quad` function.

   (a) $\int_0^5 x^3 - 5\ dx$ (exact value: $\frac{5^4}{4} - 25$)

   (b) $\int_1^{10} 4\log_{10}(x)\ dx$ (exact value: $\frac{(-36 + 40\ln(2) + 40\ln(5))}{\ln(2) + \ln(5)}$

   (c) $\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\ dx$ (exact value not known; approximately 0.95450)

Code that does this would be e.g.

```
f = @(x) x.^3 - 5; % note: our function has to work with _vectors_ of x values
% so we need to use '.^' instead of just '^' for the exponent/power
I = integral(f, 0, 5);
format long;   % have MATLAB print out more digits
I
```

The results are

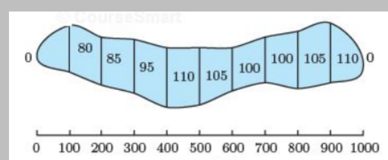
(a) function: `f = @(x) x.^3 - 5;`
estimated integral: 131.25000000

(b) function: `f = @(x) 4*log10(x);`
estimated integral: 24.36539860

(c) function: `f = @(x) (1/sqrt(2*pi)) * exp(-x.^2/2);`
estimate integral: 0.95449979

Note that all these values are within 10^{-7} of the exact values (except the last, which we only know to 10^{-5}).

4. The width (in feet) of the golf course fairway was measured at 100-foot intervals as indicated on the figure. Estimate the square footage of of the fairway.



NOTE: In this problem, because we are only given data and not a function, the only technique we have that would work is LEFT(n). We **can't** use the Fundamental Theorem or the `integral` function because both of those require a formula for the function that we don't have.

5. When a pendulum oscillates, with maximum deviation angle θ_0, the period of the pendulum is given by

$$T = 4\sqrt{L/g}\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity, 9.8 m/s.

Compute and compare the period of a pendulum with

- $L = 2$, $\theta_0 = 40^o$,
- $L = 2$, $\theta_0 = 20^o$.
- $L = 2.5$, $\theta_0 = 40^o$,
- $L = 2.5$, $\theta_0 = 20^o$.

Describe how significant the effect of maximum swing angle θ_0 is on the period of a pendulum, compared to the effect of the pendulum length.

Full solution is available in `q_pendulumPeriod.m`.

The periods computed are show below. "quad" was used for the integration.

L = 2.0 m, theta0 = 40.0 deg, period = 2.9274 s.
L = 2.0 m, theta0 = 20.0 deg, period = 2.8602 s.
L = 2.5 m, theta0 = 40.0 deg, period = 3.2729 s.
L = 2.5 m, theta0 = 20.0 deg, period = 3.1978 s.

From these results, it is clear that the angle has a minimal effect on the period of the oscillations, compared with the effect of the length. This insensitivity of the period to the oscillation angle explains why pendulum clocks and metronomes do not need a specific swing angle to be fairly accurate, but *do* need a specific length.

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