Week #4 : Integrals - Foundations

Goals:

- Use the definite integral to model and find a solution to a posed area- or accumulation-related problem.
- Scale and add definite integrals, describe the meaning of integral bounds and how to apply them.
- Recognize an anti-derivative of a function.
- Apply the theory of the Fundamental Theorem of Calculus to evaluate simple integrals.
- Distinguish between definite and indefinite integrals and their meaning.

Integration

$\frac{d}{dt} t^2 = 2t$

If we had to summarize the first three weeks of the course, we would say that the focus was on **differentiation**.

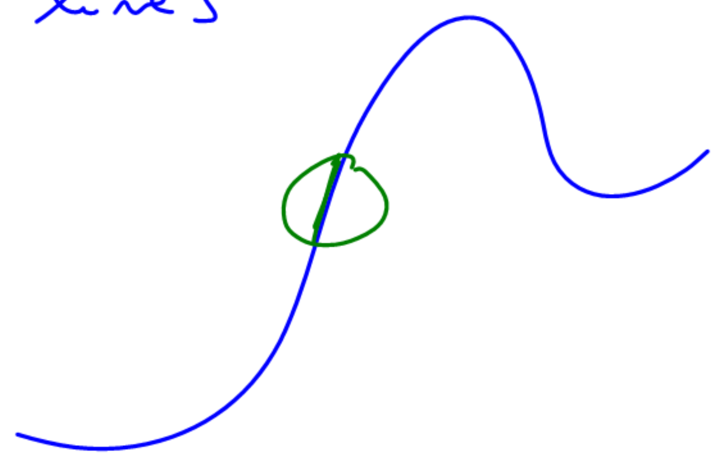
$\frac{d}{dx} x^2 = 2x$

All differentiation problems ask the same basic question: **At what rate does a process change**, and how does that rate of change relate to other characteristics of the process?

The key observation was that **at small scales, rates of change look linear**.

linearization

tangent lines



In the next three weeks of the course we will study **integration**. Again, the analysis will be made possible by the observation that on a very small scale all processes look linear. This time, though, we will use this fact to see how regarding a process as an **accumulation** of infinitely many small linear steps allows us to calculate the accumulated total, even when the rate of accumulation is far from linear. **Integration is always in some way about finding the total at the end of a process of accumulation.**

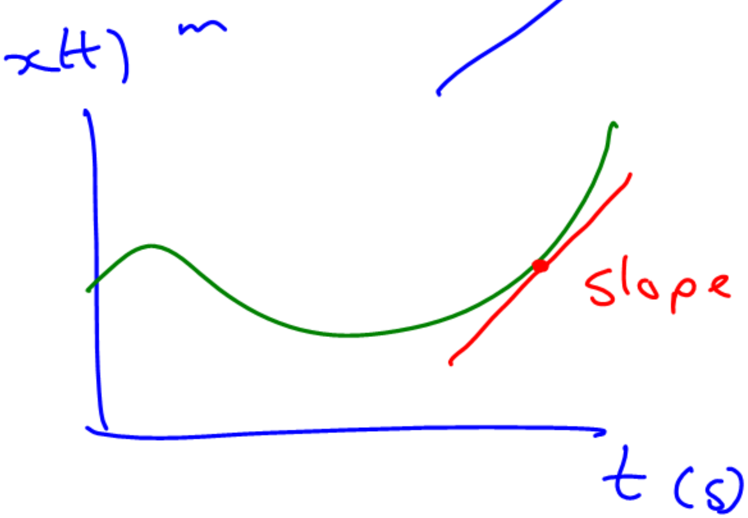
Distance and Velocity

position

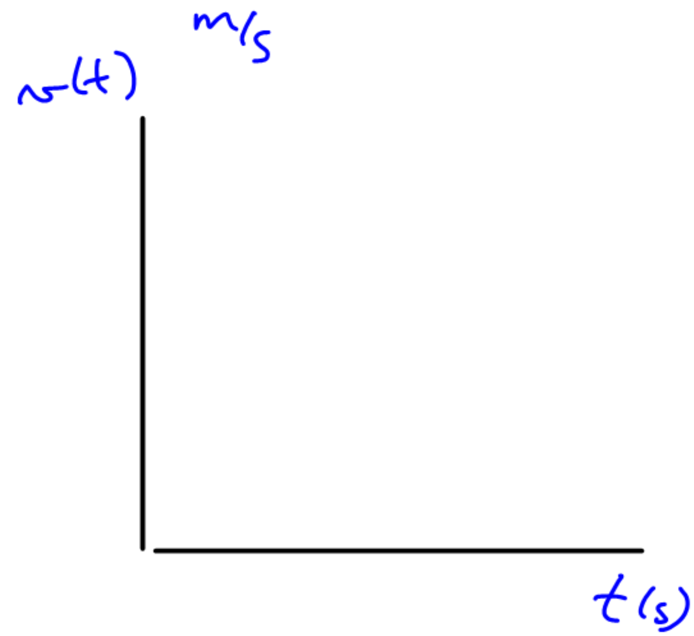
Recall that if we measure distance x as a function of time t, the velocity is determined by differentiating $x(t)$:

Velocity is the **slope** on the **position graph**.

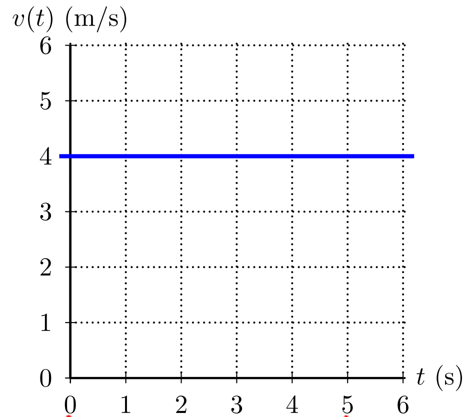
But now suppose we begin with a **graph of the velocity** with respect to time. How can we determine what **distance** will be traveled? Does distance also "appear" in the velocity graph somehow?



slope $\left(\frac{m}{s}\right)$ = velocity



Problem. Consider the graph for the velocity of a particle shown below.



$4 \ m/s$

How far did the particle travel between $t = 0$ and $t = 5$ seconds?

(a) 5 m

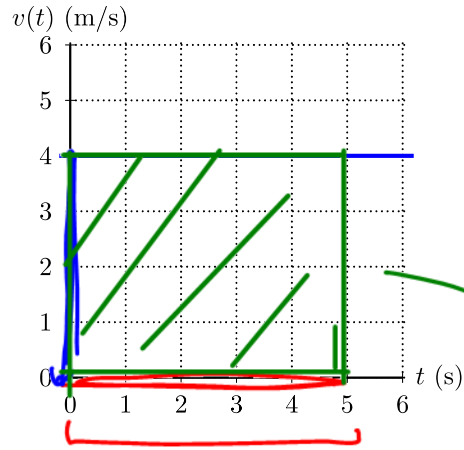
$\text{dist} = \left(4 \ \frac{m}{\cancel{s}}\right)\left(5 \ \cancel{s}\right) = 20 m$

(b) 10 m

(c) 15 m

(d) 20 m

Problem.



dist = 20 m

= (4 m/s)(5 s)

dist = "area" under v(t)

Where does the distance traveled between $t = 0$ and $t = 5$ "appear" on this velocity graph?

over t=0..5.

(a) The distance traveled is the **slope** between $t = 0$ and $t = 5$

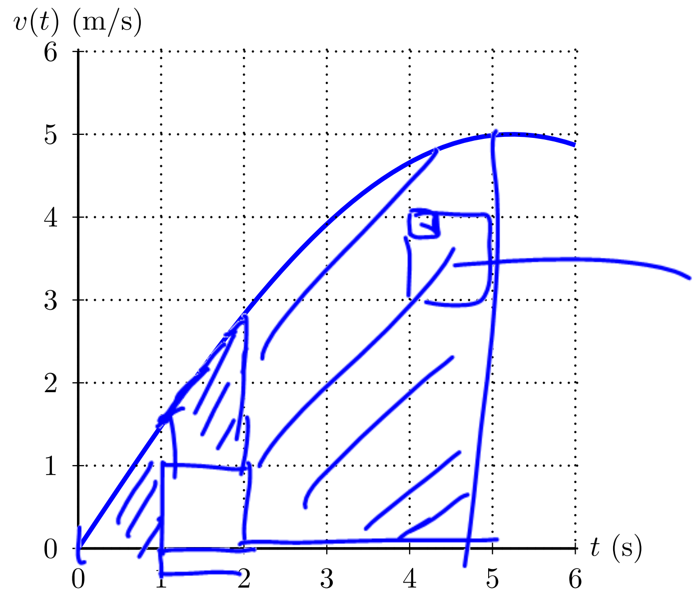
(b) The distance traveled is the **average height** of the graph between $t = 0$ and $t = 5$

(c) The distance traveled is the **area** under the graph between $t = 0$ and $t = 5$.

When the velocity is **constant**, we have the equivalency:

$$\text{dist} = \text{vel} \times \text{time} \qquad \Longleftrightarrow \text{dist} = \text{area under the velocity graph}$$

Problem. What about when the velocity is **not** constant though?



$\text{dist} = \left(\frac{m}{s}\right)(s)$

area = dist travelled.

Do the units of the "area" under this graph still make sense as a distance value?

Yes

Estimating Areas

It appears that

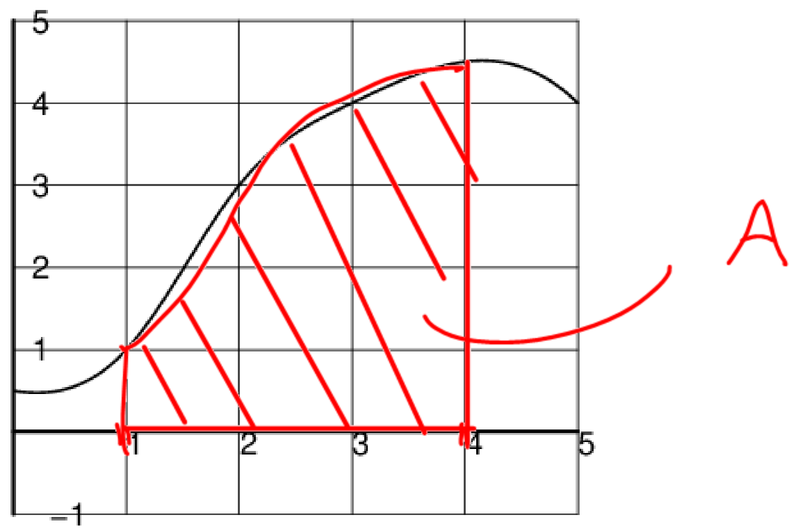
distance traveled = area under the graph of velocity

even when the velocity is changing. This means that we've found two equivalent problems, and can work with whichever version benefits us the most at a particular moment.

Unfortunately, many or most arbitrary areas are essentially impossible to find when the shape isn't a composition of simple shapes (e.g. triangles, rectangles, or circles).

In these cases, we must compute the area using less direct methods. We will start by making an *estimate* of the area under the graph using shapes whose area is easier to calculate.

Problem. Suppose we are trying to find the area underneath the graph of the function $f(x)$ given below between $x = 1$ and $x = 4$. Shade in that region, and call that area A.

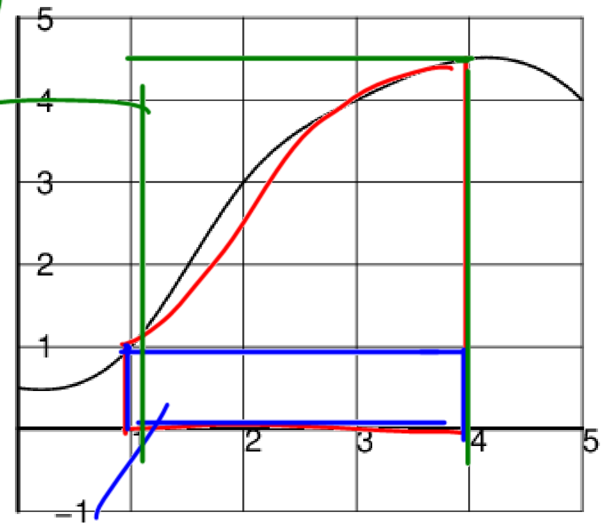


We can make a rough estimate of the area by drawing a rectangle that completely contains the area, or a rectangle that is completely contained by the area.

Problem. Calculate this overestimate and underestimate for the area A.

rect area
$= (3)(4.5)$
$= 13.5$

$3 < A < 13.5$



rect area $= (3)(1) = 3$

The next step is to use smaller rectangles to improve our estimate the area. We can divide the interval from $x = 1$ to $x = 4$ into 3 intervals of width 1, and use different size rectangles on each interval.

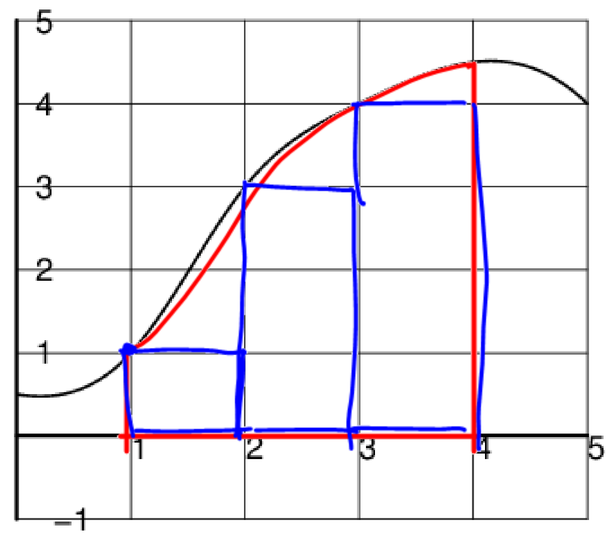
Problem. Estimate the area A by using 3 rectangles of width 1. Use the function value at the *left* edge of the interval as the height of each rectangle.

total rects areas

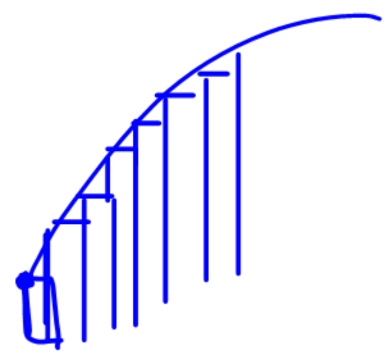
$= (1)(1) + (1)(3) + (1)(4)$

$= 8 \ u^2$

$8 < A$



We can repeat this process for any number of rectangles, and we expect that our estimation of the area will get better the more rectangles we use. The method we used above, choosing for the height of the rectangles the function at the left edge, is called the **left hand sum**, and is denoted LEFT(n) if we use n rectangles.



as # intervals / # rectangles incr

$\Rightarrow$ better estimate

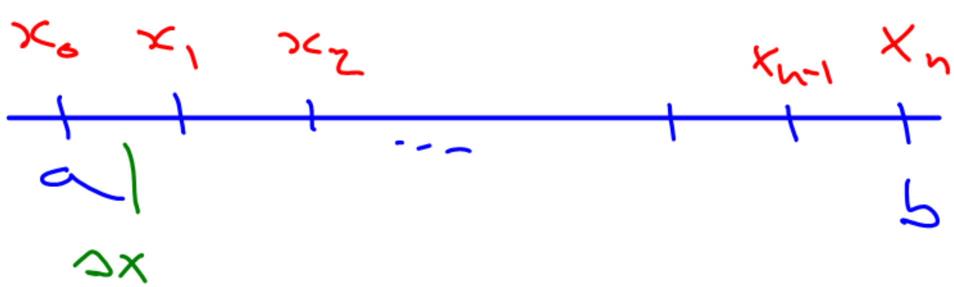
Last page: LEFT(3) = 8 u^2

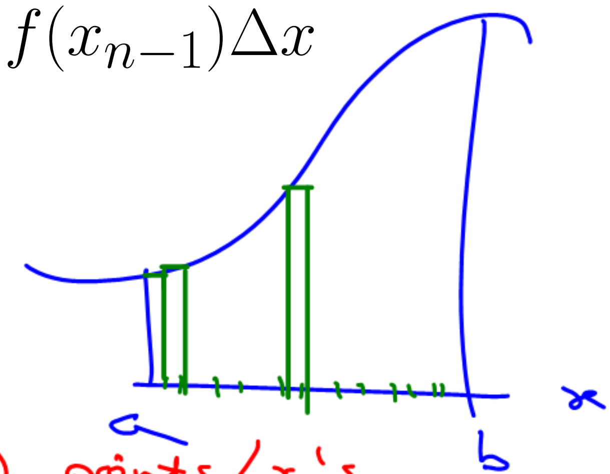
Generalizing the Area Calculation

Suppose we are trying to estimate the area under the function $f(x)$ from $x = a$ to $x = b$ via the left hand sum with n rectangles.

- the **width** of each rectangle will be $\Delta x = \dfrac{b-a}{n}$.
- If we label the endpoints of the intervals to be $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, then the **heights** of the rectangles will be $f(x_i)$'s, and
- the formula for the left hand sum/area will be

$$\begin{aligned} \text{LEFT}(n) &= \overbrace{f(x_0)}^{\text{height}}\overbrace{\Delta x}^{\text{width}} + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x \\ &= \sum_{i=1}^{n} f(x_{i-1})\Delta x. \end{aligned}$$





(n+1) points/x's for n intervals

Aside: Summation Notation

The capital Greek letter sigma, $\sum$, is used as a short form notation for long sums. E.g.

$$\sum_{i=1}^{n} \boxed{x_i}$$

Sequence $\quad i=1 \quad 2 \quad 3 \quad \ldots \quad i=n$

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n$$

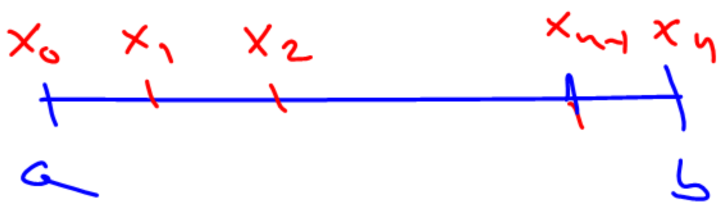
Problem. Translate the following into more traditional sums:

$i=1 \quad 2 \quad 3 \quad 4 \quad \dots \quad 99 \quad 100$

$$\sum_{i=1}^{100} i = 1 + 2 + 3 + 4 + \dots + 99 + 100$$

$n=1 \quad 2 \quad 3 \quad 4 \quad \dots \quad 9 \quad 10$

$$\sum_{n=1}^{10} (n^2 + n) = (1^2+1) + (2^2+2) + (3^2+3) \dots + (9^2+9) + (10^2+10)$$



$i=1 \quad i=2 \quad \dots \quad i=n$

$$\sum_{i=1}^{n} \overbrace{f(x_{i-1})}^{h}\, \overbrace{\Delta x}^{w} = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x$$

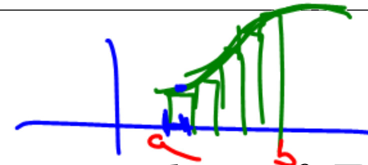
area approximation

Summation notation lends itself very nicely to translations between hand-written sums and MATLAB computations. A `for` loop can be mapped easily on to a sum.

Problem. Implement each of the sums below as a MATLAB loop and compute the total.

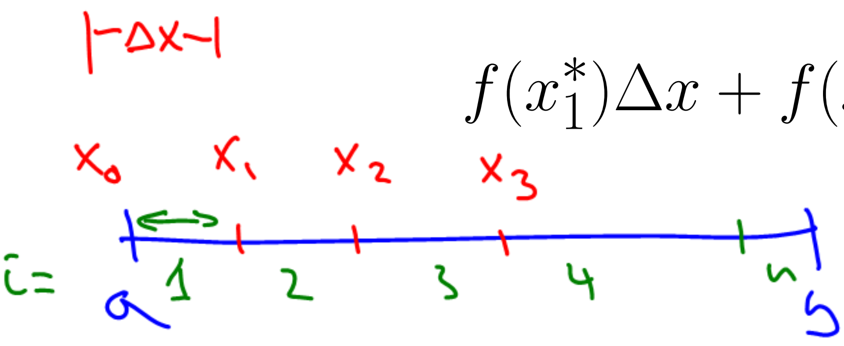
$$\sum_{i=1}^{100} i = \quad 5050$$

$$\sum_{n=1}^{10} n^2 + n = \quad 440$$



Riemann Sums

Area estimations like LEFT(n) are examples of **Riemann sums**, after the mathematician Bernhard Riemann (1826-1866) who formalized many of the techniques of calculus. The general form for a Riemann Sum is



$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

$$= \sum_{i=1}^{n} \overbrace{\underbrace{f(x_i^*)}_{h}\underbrace{\Delta x}_{w}}$$

area of one rectangle

where each x_i^* is some point in the interval $[x_{i-1}, x_i]$. For LEFT(n), we choose the left hand endpoint of the interval, so $x_i^* = \underline{x_{i-1}}$.

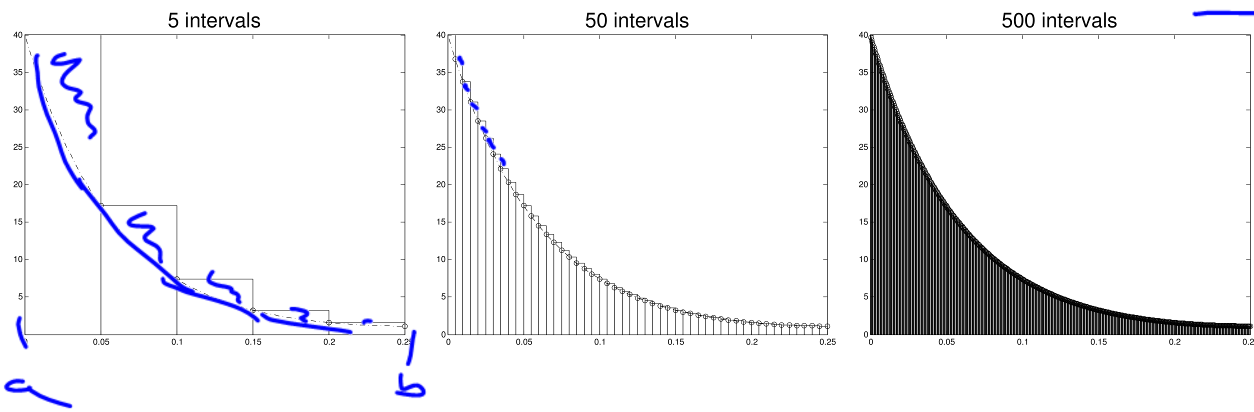
The common property of all these approximations is that they involve

- a sum of rectangular areas, with
- widths (Δx), and
- heights $(f(x_i^*))$

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

There are other Riemann Sums that give slightly better estimates of the area underneath a graph, but they often require extra computation.

We observed that as we increase the number of rectangles used to approximate the area under a curve, our estimate of the area under the graph becomes more accurate. We can see that based on an example like the graph below:



→ # intervals
→ ∞

This implies that to obtain the **exact area**, we should use a **limit** on our Riemann sums.

exact

The area underneath the graph of $f(x)$ between $x = a$ and $x = b$ is

$$\text{equal to } \lim_{n\to\infty} \text{LEFT}(n) = \lim_{n\to\infty} \left(\underbrace{\sum_{i=1}^{n} f(x_{i-1})\Delta x}_{\text{Riemann Sum}}, \right) \quad \text{where}$$

more and more intervals

$$\Delta x = \frac{b-a}{n}.$$

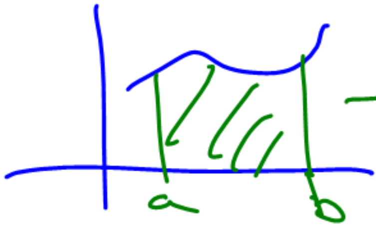
→ smaller

This limit, which gives the **exact value of the accumulation of** $f(x)$ is called the **definite integral** of $f(x)$ from a to b, and is equal to the area under curve* whenever $f(x)$ is a non-negative continuous function.

$$\int_a^b f(x)\, dx$$

The Definite Integral

The definite integral of $f(x)$ between $x = a$ and $x = b$ is denoted by the symbol



$h \cdot w$

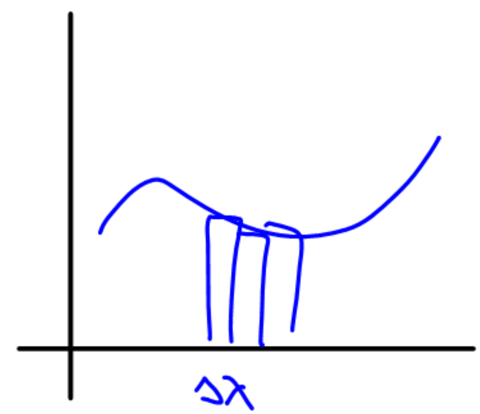
$$\int_a^b f(x)\,dx \qquad = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

We call a and b the **limits of integration** and $f(x)$ the **integrand**. Note that this notation shares the same common structure with Riemann sums:

- a sum ($\int$ sign)
- widths (dx), and
- heights ($f(x)$)

Problem. Write the definite integral representing the area underneath the graph of $\underbrace{f(x) = x + \cos x}_{f(x)}$ between $\underbrace{x = 2}_{a}$ and $\underbrace{x = 4}_{b}$.

$$\text{area} = \int_{x=2}^{x=4} \left(x + \cos(x)\right) \, dx$$



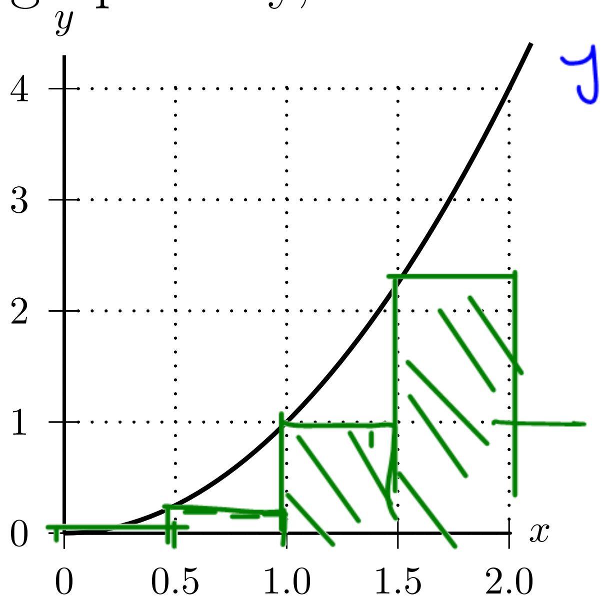
Problem. Write an integral that represents the area under the graph of $f(x) = x^2$ on the interval $x \in [0, 2]$.

a b

$$\text{area} = \int_0^2 x^2 \, dx$$

intervals

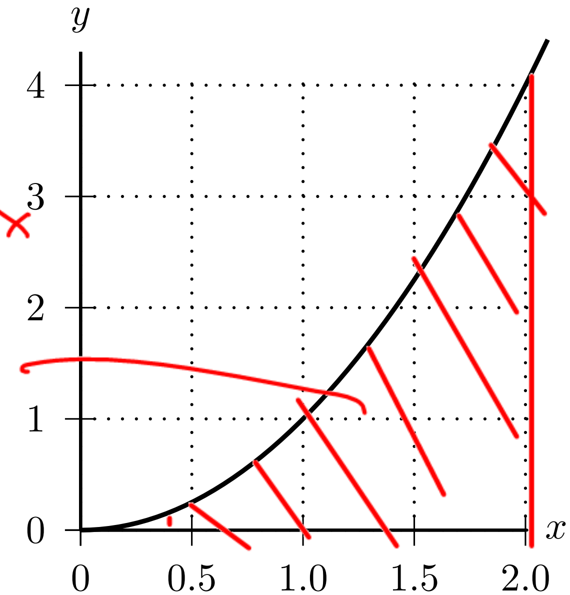
Problem. Sketch how the LEFT(4) estimate would be represented graphically, and how it differs from the integral value.



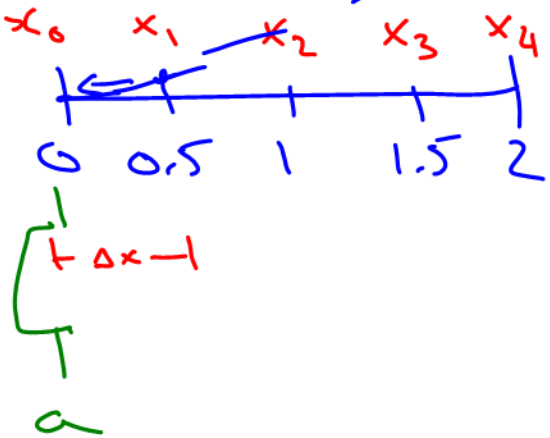
$y = x^2$

LEFT(4)

$$\int_0^2 x^2 dx$$



Problem. Use a LEFT sum with $n = 4$ to estimate $\int_0^2 x^2\,dx$.



$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$$

x	$f(x) = x^2$
$x_0 = 0$	0
$x_1 = 0.5$	0.25
$x_2 = 1$	1
$x_3 = 1.5$	2.25

$$\underbrace{\int_0^2 x^2\,dx}_{\text{exact}} \cong f(0)\cdot\Delta x + f(0.5)\,\Delta x + f(1)\,\Delta x + f(1.5)\,\Delta x$$

$$= 0.5\,(3.5)$$

$$= \underline{1.75}$$

Repeat your LEFT estimate, but using MATLAB.

1.75 $\qquad \sum_{i=1}^{n} f(x_{i-1})\,\Delta x \qquad (x_{i-1})^2$

$x_i = a + i\cdot\Delta x \qquad x_{i-1} = a + (i-1)\Delta x$

Repeat the calculation again in MATLAB, but now with 100 intervals.

2.2668 $\qquad$ [exact $= 2.6\dot{6}$]

Students often ask why instructors discuss Riemann sums, if they just lead to definite integrals.

The Role of Riemann sums

1. They are needed to say what we mean by an integral.

   definition

2. They enable us to decide which integral is appropriate in a word problem.

   word problem $\longrightarrow \sum (f-) \cdot (\Delta x) \longrightarrow \int f(t)\,dt$

   $\Delta t \longleftarrow$

3. They can also be used, with a finite number of intervals, to give an approximate value of the integral.

$$\int_0^2 x^2\,dx$$

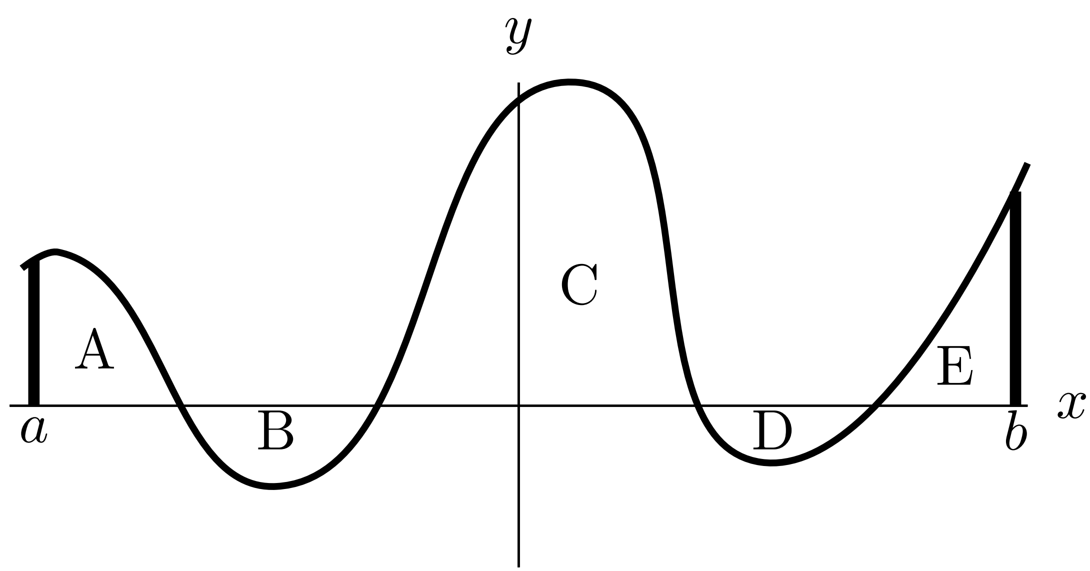
$$\int_a^b f(x)\,dx = \text{area under } f(x)$$

if f positive

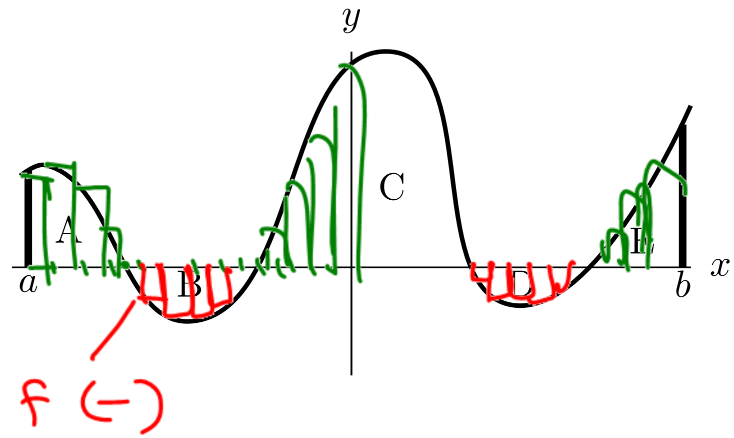
Negative Integral Values

So far we have only dealt with the areas under/integrals of **positive** functions. Will the definite integral still be equal to the area underneath the graph if $f(x)$ is always negative? What happens if $f(x)$ crosses the x-axis several times?

Problem. Suppose that $f(x)$ has the graph shown below, and that A, B, C, D, and E are the areas of the regions shown.



$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\,\Delta x$$



If we were to partition $[a, b]$ into small subintervals and construct a corresponding Riemann sum, then the first few terms in the Riemann sum would correspond to the region with area A, the next few to B, etc.

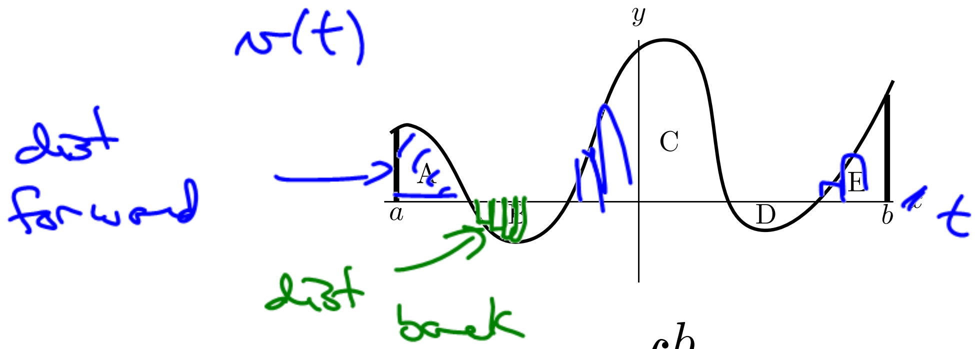
Problem. Which of these sets of terms have positive values?

A, C, E → pos f(x) → pos contributions to integral

Which of these sets have negative values?

B, D → neg f(x)

→ neg contributions to integral



Problem. Express the integral $\int_a^b f(x)dx$ in terms of the (positive) areas A, B, C, D, and E.

areas → positive
integrals → positive
or
negative

pos

$$\int_a^b f(x)\,dx = A - B + C - D + E$$

If f were to represent velocity over time, what would the "negative areas" represent?

pos $v(t)$ → moving forwards

neg $v(t)$ → " backwards

The Fundamental Theorem of Calculus

We started our discussion of integration by stressing the fact that an integral problem is always at heart a problem in which something **accumulates**.

We then converted that accumulation problem into the problem of computing the **area under a graph**, and representing both using the notation

h · w

$\sum f(x_i) \cdot \Delta x$

$$\int_a^b f(x)\, dx.$$

So far, we don't know how to evaluate that exactly: we can only use our approximation based on a Riemann Sum,

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(x_i*)\, dx$$

finite

If we want to evaluate this integral exactly, we need a new tool.

The new idea is one of the most important discoveries in calculus: the connection between integrals and the **inverse of differentiation**.

Fundamental Theorem of Calculus

If f is continuous and F is an **anti-derivative** of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

little f

capital F

Problem. Explain in words how we would use this to evaluate an integral over the interval $x = a \ldots b$.

- start w/ given f
- find anti-derivative F (satisfying $F' = f$)
- plug x=b, x=a into F, subtract

Using the Fundamental Theorem of Calculus

To use the Fundamental Theorem of Calculus, we need to know formulas for the anti-derivatives of functions. We already know quite a few.

Problem. Complete the following table of basic anti-derivatives by asking yourself the question, "$f(x)$ is the derivative of what function, $F(x)$?".

diff'te

Function $f(x)$		Anti-derivative $F(x)$
x^n $(n \neq -1)$	$\leftarrow \frac{d}{dx}$	$\frac{1}{n+1}x^{n+1}$
$\frac{1}{x}$	$\leftarrow \frac{d}{dx}$	$\ln\|x\|$
e^x	$\leftarrow \frac{d}{dx}$	e^x
$\cos x$	$\leftarrow \frac{d}{dx}$	$\sin(x)$
$\sin x$	$\leftarrow \frac{d}{dx}$	$-\cos(x)$
$\sec^2 x$	$\leftarrow \frac{d}{dx}$	$\tan(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\leftarrow \frac{d}{dx}$	$\arcsin(x)$
$\frac{1}{1+x^2}$	$\leftarrow \frac{d}{dx}$	$\arctan(x)$

add one to power, divide by new power

include absolute values

$\int x^3\,dx$

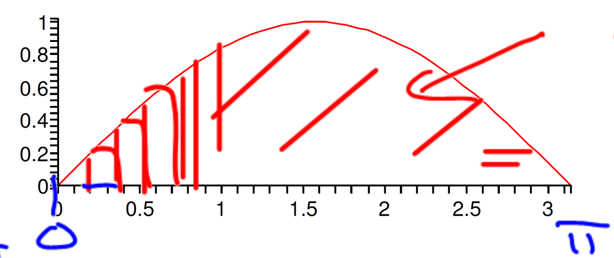
$\int_a^b \frac{1}{x}\,dx$

$\int_a^b \cos(x)\,dx \rightarrow \sin(b) - \sin(a)$

$\int \frac{1}{\sqrt{1-x^2}}\,dx$

Fundamental Theorem - Simple Examples

Problem. Use the Fundamental Theorem to calculate the area under one section of the graph of $\sin x$, the part from 0 to π.



area exactly

$2\ u^2$

exact area $= \int_0^{\pi} \underbrace{\sin(x)}_{f(x)}\, dx$

$\frac{d}{dx}\left(-\cos(x)\right) = \sin(x)$ $= -(-\sin(x))$

$F(x) \leftarrow f$

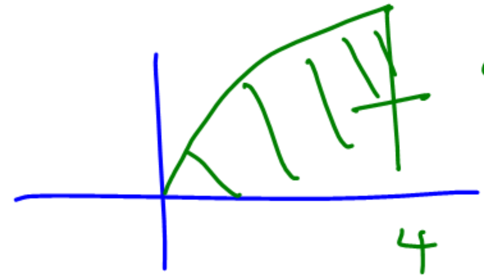
antideriv

$= \underbrace{(-\cos(\pi))}_{F(b)} - \underbrace{(-\cos(0))}_{F(a)}$

$= -(-1) - (-1)$

$= 1 + 1 = 2$

Problem. Calculate $\displaystyle\int_0^4 \sqrt{x}\,dx$.



area $= \frac{16}{3}\,u^2$

$$= \int_{\underbrace{0}_{a}}^{\overbrace{4}^{b}} \underbrace{x^{1/2}}_{f}\,dx$$

$$= \underbrace{\frac{4^{3/2}}{3/2}}_{F(b)} - \underbrace{\cancel{\frac{0^{3/2}}{3/2}}}_{F(a)}$$

$$= \frac{2}{3}\,8 = \frac{16}{3}$$

$$\frac{d}{dx}\underbrace{\frac{x^{3/2}}{3/2}}_{F} = \underbrace{x^{1/2}}_{f}$$

add one to power,
divide by
new power

$$\frac{1}{\cancel{3/2}}\left(\cancel{3/2}\,x^{1/2}\right)$$

$$= x^{1/2} = \sqrt{x}$$

Notation

The expression $F(b) - F(a)$ comes up so often that there is a special notation for it. It is written as

$$F(x)\Big|_a^b \quad \text{or} \quad [F(x)]_a^b$$

$$\int_0^4 x^{1/2}\,dx = \frac{x^{3/2}}{3/2}\Big|_0^4 = \frac{4^{3/2}}{3/2} - \frac{0^{3/2}}{3/2}$$

Problem. Calculate $\displaystyle\int_{\pi/4}^{\pi/3} \underbrace{\sec^2(\theta)}_{f}\, d\theta$.

anti differentiation

$$= \tan(\theta)\Big|_{\pi/4}^{\pi/3}$$

sub in b, a

$$= \underbrace{\tan(\pi/3)}_{F(b)} - \underbrace{\tan(\pi/4)}_{F(a)}$$

$$= \sqrt{3} - 1$$

$$\frac{d}{d\theta}\underbrace{\tan(\theta)}_{F} = \underbrace{\sec^2\theta}_{f}$$

Properties of Integrals

Before we go on to refine our skill at calculating integrals, we should first reflect on some basic properties of integrals that derive from their origins as limits of more and more accurate Riemann sums.

$f(x_i)\,\Delta x$

1. If $a > b$ then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

$\Delta x(+)$

2. If $a = b$ then $\int_a^b f(x)\,dx = 0$.

zero area

ab zero width

$\Delta x\ (-)$

3. $\int_a^b c\,dx = c\,(b-a)$.

rect area $= c\,(b-a)$

4. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

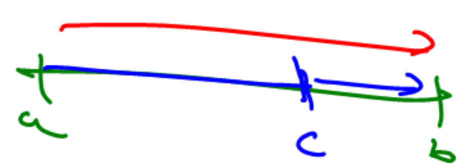
mult out front

linearity of integrals

5. $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$.

split

6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.



Problem. Find the value of the integral $\int_{-2}^{2} (4x^2 - 3e^x)\, dx$

$$= \left(4\int_{-2}^{2} x^2\, dx\right) - 3\left(\int_{-2}^{2} e^x\, dx\right)$$

limits

anti-deriv

$$= \left[4\,\frac{x^3}{3} - 3e^x\right]\Bigg|_{-2}^{2}$$

sub in

$$= \left(\frac{4}{3}(2^3) - 3e^2\right) - \left(\frac{4}{3}(-2)^3 - 3e^{-2}\right)$$

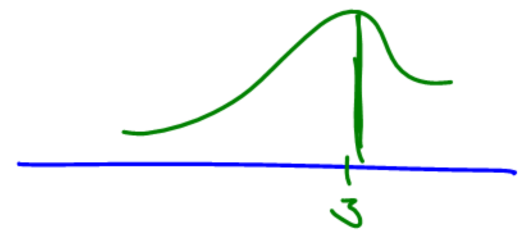
$$F(2) \quad - \quad F(-2)$$

$$= \frac{32}{3} - 3e^2 - \left(-\frac{32}{3}\right) + 3e^{-2}$$

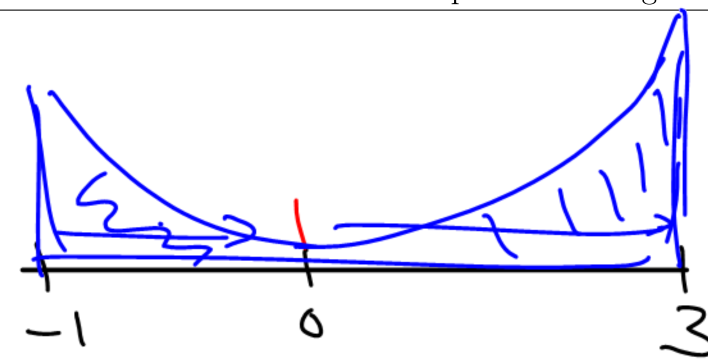
$$= \frac{64}{3} - 3e^2 + 3e^{-2}$$

Problem. Evaluate $\displaystyle\int_3^3 \sin(x^{10})\,dx$

$= 0$ zero area



zero width



Problem. Compute both

$$\int_{-1}^{3} x^2\,dx, \text{ and}$$

anti deriv

$$= \left.\frac{x^3}{3}\right|_{-1}^{3}$$

sub in

$$= \left(\frac{3^3}{3}\right) - \left(\frac{(-1)^3}{3}\right) = 9 + 1/3$$

equal

$$\int_{-1}^{\boxed{0}} x^2\,dx + \int_{\boxed{0}}^{3} x^2\,dx.$$

$$= \left.\frac{x^3}{3}\right|_{-1}^{0} + \left.\frac{x^3}{3}\right|_{0}^{3}$$

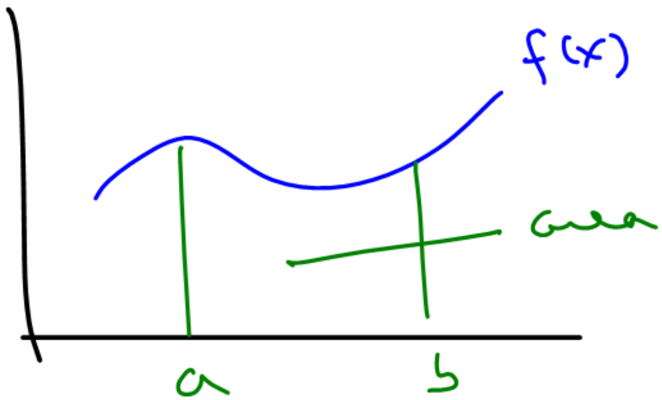
$$= \left(\cancel{0} - \left(\frac{(-1)^3}{3}\right)\right) + \left(\frac{3^3}{3} - \cancel{\frac{0^3}{3}}\right) = 9 + 1/3$$

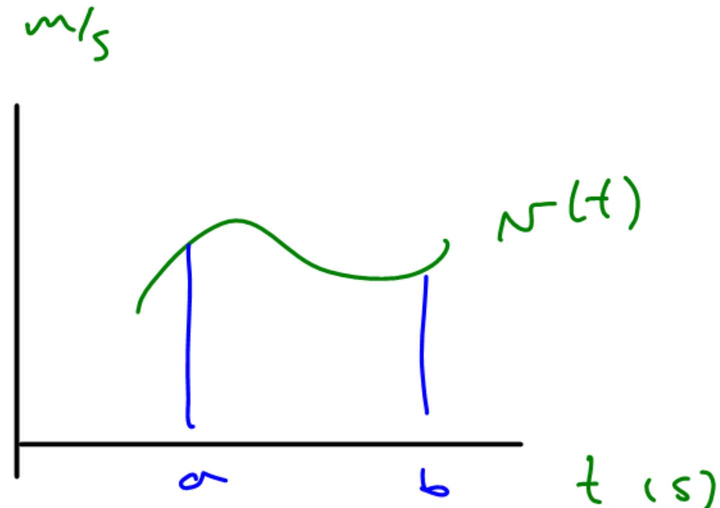
Explain the relationship between the two answers with a sketch.

Net Change Theorem

F want, but we have F'

Note that we create an anti-derivative $F(x)$, we are building it such that $f(x) = F'(x)$. This means that f gives the rate of change of F. Notice that this observation was made much earlier, when we started our discussion of integration: when an integral is associated with a process of **accumulation** then the **rate of accumulation** is always precisely the integrand.





$$\int_a^b \underset{h}{f(x)}\,\underset{w}{dx}$$

$$\int_a^b \underset{\text{rate (m/s)}}{v(t)}\;\underset{\text{time (s)}}{dt} = \text{total/net change in position}$$

no formula — know

Consider $F(x)$ as the quantity we are tracking, so F' is its rate of change. Another statement of the Fundamental Theorem of Calculus Part 2 would then be

$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a). = \Delta F}$$

net change in F.

"The integral of a rate of change is the total change". Textbooks refer to this the **Net Change Theorem**.

Problem. State what this means if $F(t)$ represents the position of an object at time t.

position

$$\int_a^b \underbrace{F'(t)}_{\text{vel}}\,dt = \text{net change in position over time } t = a \ldots b.$$

Problem. Suppose water is flowing into/out of a tank at a rate given by $r(t) = 200 - 10t$ L/min, where *positive* rates indicate flow *in*. By how much does the water level in the tank change during the first 45 minutes after $t = 0$?

net change in V

have $F' = \frac{dV}{dt}$

Net Change Theorem

$r(t) = 200 - 10t$

$$\Delta V = V(b) - V(a) = \int_{t=0}^{t=45} \left(\frac{dV}{dt}\right) dt$$

$$= \int_0^{45} (\underbrace{200 - 10t}_{L/min}) \underbrace{dt}_{min}$$

$$= 200t - \frac{10t^2}{2} \Big|_0^{45} = \left((200)(45) - \frac{10(45)^2}{2}\right) - (0) = -1125$$

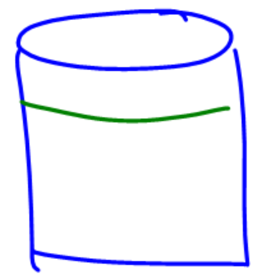
Volume will drop by 1125 L in 45 mins.

$$\frac{d}{dt}\left(200t - \frac{10t^2}{2}\right)$$

$$= 200 - \frac{10(\cancel{2}t)}{\cancel{2}}$$

$$= 200 - 10t$$

Problem. What is an assumption you would have to make about the initial amount of water in the tank for this to make sense?



$V(0)$ larger than 1125 L

over 45 min,

V drops by 1125 L,

-1125 is the <u>change</u> in V

To get amount at end, $t=45$

amount at $t=45$

$$\underbrace{\boxed{V(45)} - V(0)}_{\substack{\text{net change} \\ = -1125}} = \underbrace{\int_a^b r(t)\,dt}$$

Final amount = net change + initial amount

Problem. If the velocity of a particle is given by $v(t) = \cancel{t^3}\ t^3$, find $\int_{-1}^{2} v(t)\, dt$.

net change in pos $= \int_{-1}^{2} t^3\, dt = \frac{t^4}{4}\Big|_{-1}^{2} = \left(\left(\frac{2^4}{4}\right) - \frac{(-1)^4}{4}\right)$

$\uparrow$ rate of change of pos

$= \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$

How should this value be interpreted, based on the Net Change Theorem?

over time $t = -1 \ldots 2$,

travel $15/4$ m away from initial position

The Indefinite Integral

The second part of the Fundamental Theorem of Calculus,

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F' = f(x)$

or F was an anti deriv of $f(x)$

focuses the problem of evaluating an integral to the step of finding an anti-derivative F.

To isolate that anti-derivative step, we define a related notation called the **indefinite integral**, and written as

$$\int f(x)\,dx \left(= F(x)\right).$$

no limits of integration

- A **definite integral** has the form $\int_a^b f(x)\,dx$.
  - It represents area under f(x) b/w x=a..b
    net change in F b/w x=a..b
  - Its value is a number (no x's)

- An **indefinite integral** has the form $\int f(x)\,dx$.
  - It represents an anti-derivative of f(x)
  - Its value is a function x's in it.

Problem. Find $\displaystyle\int \left(\sqrt{t} - \frac{3}{\sqrt{t}}\right)\, dt$.

indefinite

$$= \int \left(t^{1/2} - 3t^{-1/2}\right) dt$$

$$= \frac{t^{3/2}}{3/2} - 3\,\frac{t^{1/2}}{1/2} + C$$

check: $\displaystyle\frac{d}{dt}\left(\frac{t^{3/2}}{3/2} - 3\,\frac{t^{1/2}}{1/2} + C\right) = \frac{\cancel{3/2}\, t^{1/2}}{\cancel{3/2}} - 3\,\frac{\cancel{1/2}\, t^{-1/2}}{\cancel{1/2}} + 0$

$$= \sqrt{t} - 3/\sqrt{t}$$

Note: When you are asked to find an indefinite integral, as in the previous question, it is important to add the "$+C$" to indicate that there is more than one anti-derivative, and that all of them differ from each other by a constant.

Finding Anti-Derivatives - Guess and Check

Finding antiderivatives is surprisingly difficult. You would think that if we know how to find derivatives then we should know how to find antiderivatives. In fact, however, the latter is much more difficult, as illustrated next.

$$\int t^{1/2}\,dt$$

$$= \frac{t^{3/2}}{3/2} + C$$

Problem. Calculate $\int \frac{1}{x^3+1}\,dx$ $\overset{\text{possible}}{=}$ $\ln|x^3+1|+C$ ✗

Check: $\frac{d}{dx}\left(\ln|x^3+1|\right) = \frac{1}{x^3+1}\cdot 3x^2$ not equal to our integrand, $\frac{1}{x^3+1}$

no (simple, closed form) integral for $\int \frac{1}{x^3+1}\,dx$

Guess and Check

Next week we will study the "Substitution Rule", or "Method of Substitution", the first of a list of techniques for finding anti-derivatives. To prepare us for the Method of Substitution, let's explain an informal method often called "guess and check".

Problem. Find $\int \cos(5x)\,dx$.

Possible

$$= -\frac{1}{5}\sin(5x) + C$$

Check: $\frac{d}{dx}\left(-\frac{1}{5}\sin(5x) + C\right) = -\frac{1}{\cancel{5}}\left(-\cos(5x)\cdot\cancel{5}\right) + 0$

$$= \cos(5x)$$

integrate

$$\cos(5x) \qquad -\frac{1}{5}\sin(5x) + C$$

diff'te

Problem. Find $\int \cos(x^2)\,dx$.

possible

$$= \frac{-1}{2x}\sin(x^2) + C$$

quotient rule

Check: $$\frac{d}{dx}\left(\frac{-1}{2x}\sin(x^2) + C\right) = -\left(-\cos(x^2)\cdot 2x\right)$$

$$= \underline{\underline{2x}}\,\cos(x^2)$$

not const

$$\int \cos(x^2)\,dx$$ has no simple closed form solution.

Problem. Now what if we were given the problem $\int x\cos(x^2)\,dx$?

possible

$$\int x\cos(x^2)\,dx = -\frac{1}{2}\sin(x^2)+C$$

Check

$$\frac{d}{dx}\left(-\frac{1}{2}\sin(x^2)+C\right) = -\frac{\cancel{1}}{\cancel{2}}\left(-\cos(x^2)(\cancel{2}x)\right)$$

$$= \cancel{2}x\cos(x^2)$$

const

So why does "guess and check" work for $\int x\cos(x^2)\,dx$ but not for $\int \cos(x^2)\,dx$?

no x mult

not solvable

mult

deriv of inside.

non linear → chain rule

Problem. Which of the following anti-differentiations would you predict can be evaluated by the guess-and-check method?

1. $\int x^2 e^{x^3}\,dx$ 2. $\int x^2 e^{x^2}\,dx$ 3. $\int x e^{x^2}\,dx$

$\frac{d}{dx}\left(e^{x^3}\right) = 3\left(x^2 e^{x^3}\right)$

$\frac{d}{dx} e^{x^2} = 2\left(x e^{x^2}\right)$

A. 1

B. 2

check $\frac{d}{dx}\left(\_\_ \right) =$

C. 3

D. 1 and 3

$\int x \cos(x^2)\,dx$

d/dx

E. 1 and 2

Problem. Calculate $\int \cos(x)\, e^{\sin(x)}\, dx.$ $\overset{\text{possible}}{=} e^{\sin(x)} + C$

d/dx

$$\frac{d}{dx}\left(e^{\sin(x)} + C\right) = e^{\sin(x)} \cdot \cos(x)$$
$$= \cos(x)\, e^{\sin(x)}$$
$$= \text{original integrand.}$$

Next week we will turn these ideas into a formal procedure called "Substitution".