Week #3 : Derivatives - Applications

Goals:

- Calculate and interpret the first and second derivatives, as well as higher order derivatives.
- Define and calculate Taylor Polynomials.
- Use MATLAB to graph and compare functions with their Taylor polynomial approximations.
- Find and use critical points for global and local optimization problems.
- Use MATLAB optimizers and equation solvers to identify optimal values and critical points.

Second and Higher Derivatives

The information about the graph of a function f provided by the sign of $f'(x)$ and $f''(x)$ on an interval (a, b) is expressed in the following table. (a and b are assumed to be finite.)

$f'(x) > 0$ on (a, b)	f increasing on $[a, b]$
$f'(x) < 0$ on (a, b)	f decreasing on $[a, b]$
$f''(x) > 0$ on (a, b)	f concave up on $[a, b]$
$f''(x) < 0$ on (a, b)	f concave down on $[a, b]$

Problem. Sketch the possible graphs combining different signs of positive and negative first and second derivatives.

Problem. Sketch graphs where the **first** derivative has a zero value.

Problem. Sketch graphs where the **second** derivative has a zero value.

Aside from their graphical interpretation, second derivatives frequently have important physical interpretations in kinematics problems.

Problem. If $x(t) = 4\sin(2t)$ gives the position of a particle at time t, what is particle's **speed** at $t = \dfrac{\pi}{6}$?

For the same particle, what is its **acceleration** at $t = \dfrac{\pi}{6}$?

While their interpretations are not as immediately obvious, it is possible to compute 3rd, 4th, or higher derivatives of function if we want.

Problem. Find the first four derivatives of the function

$$f(x) = 7(2^x) + \ln(x).$$

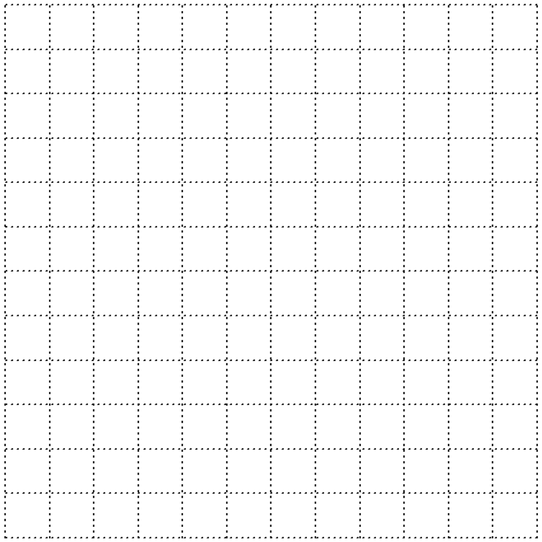
Taylor Polynomials

One application of higher derivative information is to help us build **polynomial approximations** to complicated functions.

Previously we found a formula for linear approximations to functions $f(x)$ around a point $x = a$:

This linear approximation, or tangent line formula, can also be called the **Taylor polynomial of degree 1 approximating** $f(x)$ **near** $x = a$**.**

Problem. Sketch the graph of $\cos(x)$ around $x = 0$, and add its tangent line based at $x = 0$.



The linearization or tangent line is clearly a very limited approximation to this function. What might be a *slightly* more complex form of function that would work better in this case?

Taylor Polynomial of Degree 2

$$f(x) \approx \qquad f(a) \qquad + f'(a)(x-a) \qquad + \frac{f''(a)}{2}(x-a)^2$$

is a *quadratic* approximation to $f(x)$ near $x = a$.

Problem. For values of x close to a do you think this quadratic approximation will be a better or worse approximation than the tangent line? Why?

Problem. Find the quadratic Taylor approximation to $f(x) = \cos(x)$ near $x = 0$.

Problem. Use MATLAB to draw the graph of $\cos(x)$ around $x = 0$, and add both its 1st and 2nd degree Taylor polynomial approximations for x near 0.

There is a very good reason for the particular form of the Taylor polynomial.

Problem. What mathematical features will the original $f(x)$ share with its 2nd degree Taylor approximation at the point $x = a$?

Taylor Polynomials of Higher Degree

Problem. If we wanted a still-better approximation for a function $f(x)$ near a specific point $x = a$, how could we generalize our earlier 1st and 2nd degree Taylor polynomials?

Below is the general formula for the terms in a Taylor polynomial, up to degree n.

$$f(x) \approx \ f(a) \ + f'(a)(x-a) \ + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- $f^{(n)}$ means "the n-th derivative of f".
- $n!$ means "n factorial"

Higher Degree Taylor Polynomials - Example 1

Consider the function $f(x) = \sin(x)$.

Problem. Find the first five derivatives of $f(x)$, and evaluate them at $x = 0$.

Problem. Write out the Taylor poylnomial of degree 5 for $f(x) = \sin(x)$.

Problem. Write out the general form of the Taylor poylnomial of degree n for $f(x) = \sin(x)$.

Problem. Use MATLAB to plot the graph of $f(x) = \sin(x)$ and the Taylor polynomial approximations up to degree 5.

MATLAB Demo of increasing higher degrees.

Higher Degree Taylor Polynomials - Example 2

Consider the function $g(x) = xe^{-x}$.

Problem. Find the first three derivatives of $g(x)$, and evaluate them at $x = 1$.

Problem. Write out the Taylor poylnomial of degree 3 for $g(x) = xe^{-x}$ centered at $x = 1$.

Problem. Use MATLAB to plot the graph of $g(x) = xe^{-x}$ and the Taylor polynomial approximation from degree 1, 2 and 3.

Critical Points

Aside from understanding the shape of functions, derivative information can help us identify and classify interesting points of a function, like the highest and lowest values.

Problem. Sketch graphs which have high and low points.

What do those extreme values have in common?

If $f(x)$ is defined on the interval (a, b), then we call a point c in the interval a **critical point** if:

- $f'(c) = 0$, or
- $f'(c)$ does not exist.

We will also refer to the point $(c, f(c))$ on the graph of $f(x)$ as a critical point. We call the function value $f(c)$ at a critical point c a **critical value**.

Technical Notes:

1. By this definition, $f(c)$ must be **defined** for c to be a critical point.

   Problem. Sketch $f(x) = 1/x$, and decide whether $x = 0$ is a critical point.

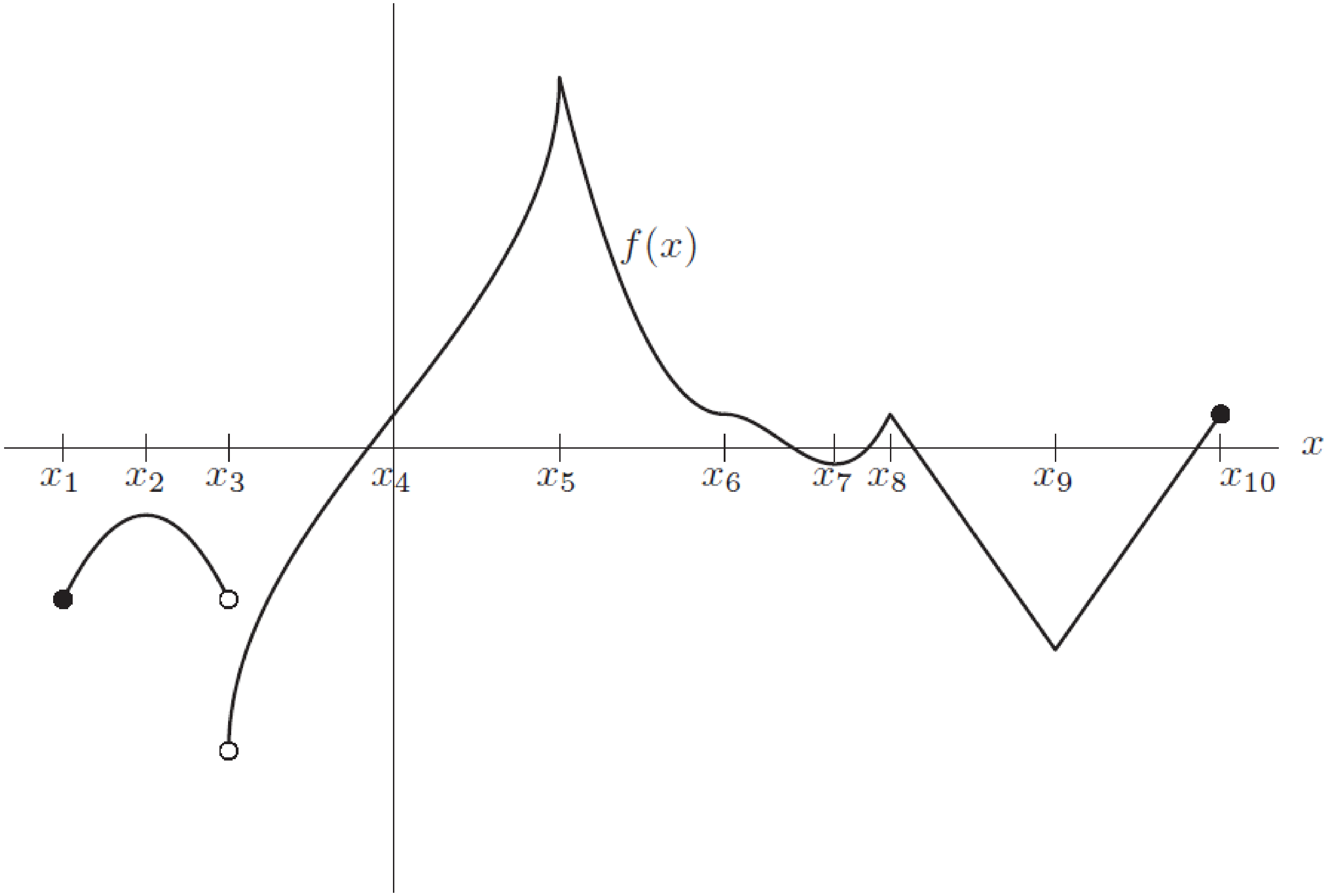
   Sketch $g(x) = |x|$, and decide whether $x = 0$ is a critical point.

2. By the definition, if a function is defined on a closed interval, the endpoints of interval **cannot** be critical points.

   Problem. Sketch the graph of $f(x) = \sqrt{x}$ and decide whether $x = 0$ is a critical point.

Problem. Sketch the graph of $g(x) = \sqrt[3]{x}$ and decide whether $x = 0$ is a critical point.

Problem. Identify all the critical points on the graph below.



Classifying Critical Points

We will now formalize two ways to determine if a critical point is a local min, max, or neither. This avoids the need for a sketch of the graph.

First Derivative Test

One way to decide whether at a critical point there is a local maximum or minimum is to examine the sign of the derivative on opposite sides of the critical point. This method is called the **first derivative test**. *Complete this table:*

	f' sign left of c	f' sign right of c	Sketch
local minimum at c			
local maximum at c			
neither local max nor min			

Example: *Find the critical points of the function* $f(x) = 2x^3 - 9x^2 + 12x + 3$. *Use the first derivative test to show whether each critical point is a local maximum or a local minimum.*

Using your answer to the preceding question, determine the number of real solutions of the equation $2x^3 - 9x^2 + 12x + 3 = 0$.

Second Derivative Test

You may also use the Second Derivative Test to determine if a critical point is a local minimum or maximum.

- The first derivative test uses the **first** derivative **around** the critical point.
- The second derivative test uses the **second** derivative **at** the critical point.
- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

Example: *A function f has derivative $f'(x) = \cos(x^2) + 2x - 1$. Does it have a local maximum, a local minimum, or neither at its critical point $x = 0$?*

Global vs. Local Optimization

Reading: Section 4.2

The first and second derivative tests only give us *local* information in most cases. However, if there are multiple local maxima or minima, we usually want the **global** max or min. The ease of determining when we have found the global max or min of a function depends strongly on the properties of the question.

Local vs Global Extrema

A **local max** occurs at $x = c$ when $f(c) > f(x)$ for x values near c.

A **global max** occurs at $x = c$ if $f(c) \geq f(x)$ for **all** values of x in the domain. It is possible to have several global maxima if the function reaches its peak value at more than one point.

Corresponding definitions apply for local and global minima.

Example: *Give an example of a simple function with multiple global maxima.*

Example: *Give an example of a simple function with a single global maximum, but no global minimum.*

Example: *Give an example of a simple function with neither a global maximum nor a global minimum.*

Example: *Earlier we worked with the function* $f(x) = 2x^3 - 9x^2 + 12x + 3$. *If we limit the function to the interval* $x \in [0, 2.5]$, *what are the* **global max** *and* **global minimum** *values on that interval?*

Global Extrema on Closed Intervals

A continuous function on a closed interval will **always** have a global max and a global min value. These values will occur at either

- a critical point *or*
- an end point of the interval.

To find which value is the global extrema, you can compute the original function's values at all the critical points and end points, and select the point with the highest/lowest value of the function.

Global Extrema on Open Intervals

A function defined on an open interval may or may not have global maxima or minima.

If you are trying to demonstrate that a point is a global max or min, and you are working with an open interval, including the possible interval $(-\infty, \infty)$, proving that a particular point is a global max or min requires a careful argument. A recommendation is to look at either:

- values of f when x approaches the endpoints of the interval, or $\pm\infty$, as appropriate; or
- if there is only one critical point, look at the sign of f' on either side of the critical point.

With that information, you can often construct an argument about a particular point being a global max or min.

Example: *Determine whether the function* $f(x) = 2x^3 - 9x^2 +$

$12x + 3$ *has a global max and/or min.*

Example: *Determine whether the function* $f(x) = (x-2)^4$ *has a global max/and or min.*

Optimization

Reading: Section 4.4

An optimization problem is one in which we have to find the maximum or minimum value of some quantity. In principle, we already know how to find the maximum and minimum values of a function if we are given a formula for the function and the interval on which the maximum or minimum is sought. Usually the hard part in an optimization problem is interpreting the word problem in order to find the formula of the function to be optimized.

Example: *A farmer wants to build a rectangular enclosure to contain livestock. The farmer has 120 meters of wire fencing with which to build a fence, and one side of the enclosure will be part of the side an already existing building (so there is no need to put up fence on that side). What should the dimensions of each side be to maximize the area of the enclosure?*

What is the quantity to be maximized in this example?

What are the variables in this question, and how are they related? You may want to draw a picture.

Express the quantity to be optimized in terms of the variables. Try to eliminate all but one of the variables.

What is the domain on which the one remaining variable makes sense?

Use the techniques learned earlier in the course to maximized the function on this domain. Give reasons explaining why the answer you found is the **global** *maximum.*

(continued)

Example: *(Storage Container)*
A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is to be twice its width. Material for the base costs \$10.00 per m^2, and material for the sides costs \$6.00 per m^2. Determine the cost of the material for the cheapest such container.

Example (Taken from 2004 Dec Exam)

A fisher is in a boat at point A, which is 2 km from the nearest point on the shoreline. He is to go to a lighthouse at point B, which is 3 km down the coast (see figure below).

If the fisher can row at 4 km per hour, and walk at 5 km per hour, find an expression for $T(x)$, the travel time if the fisher lands the boat x km down the shore from the nearest approach.

If the fisher can row at 4 km per hour, and walk at 5 km per hour, how far from the point B should he land the boat to minimize the time it takes to get to the lighthouse? Make sure to indicate how you know your answer is the global minimum.

Often, the numerical values in an optimization problem are somewhat arbitrary, or estimated using best guesses. It is often more important to discover the response in the solution to a *range* of possible problem values. In that vein, we now suppose the fisher has a motor that will drive his boat at a speed of v km per hour.

If the fisher's walking speed is still 5 km per hour, for what values of v will it be fastest to simply drive the boat directly to the lighthouse (i.e. do no walking)?