MNTC P01 - Week #8 - Second Order Differential Equations

1. Use `ode45` to generate a graph of the solution to the following DEs, over the specified interval, given the initial condition.

   (a) $\frac{dy}{dt} = t^2 + y^2$, $y(0) = 0$, and $0 \leq t \leq 1$.

   (b) $\frac{dy}{dt} = \sin(t) + \cos(y)$, $y(0) = 0$, and $0 \leq t \leq 10$.

   (c) $\frac{dy}{dt} = (1 - y^2) + 0.2\sin(t)$, $y(0) = 0$, and $0 \leq t \leq 20$.

   Solutions are given in q_ode45_examples.m.

2. Consider the single spring/mass system shown below:

where is an external applied force.

Newton's second law gives us the relationship:

$$ma = \sum F = +$$
$$m\ddot{x} = -kx+$$

where k is the spring constant.

(a) By hand, write this second order DE as a system of 1st order DEs, using the new variables $w_1 = x$ and $w_2 = \dot{x}$

(b) Set $m = 0.5$ kg, $k = 10$ N/m, and to zero (no external force). Define the differential equations in MATLAB in `springDE.m`. Use `ode45` to simulate the motion of the spring, given an initial displacement of $x(0) = 0.2$ m, and initial velocity of zero: $\dot{x}(0) = 0$. Generate a plot with

- position against time (do *not* show the velocity), and
- either choosing the time interval used for the `ode45` simulation, or setting the graph's display limits on the graph with `xlim`, to show the first 3 to 4 cycles only.

(c) With the same initial conditions and constants as in (b), simulate the motion of the spring if we now apply an external force of $= \sin(t)$. To do this, you will need to have to add both t and as arguments to the DE. e.g.

```
function dw_dt = springDE2(t, w, m, k, F_ext)
```

Generate a simulation over the time span $t = [0, 40]$ seconds, and plot the position against time.

Explain why the motion looks so disorganized.

(d) Repeat part (c), but with an external force of $= \sin(4t)$. Explain why the motion has cyclic waves in its amplitude.

(a) The first-order system would be:

$$\frac{d}{dt}w_1 = \dot{x} = w_2$$
$$\frac{d}{dt}w_2 = \ddot{x} = \frac{1}{m}(-kx+) = \frac{1}{m}(-kw_1+)$$

(b) The files "springDE1.m" and "q_springSimulation1.m" generate this simulation. In the plot, we see very nice example of harmonic motion.

(c) The files "springDE2.m" and "q_springSimulation2.m" generate this simulation. In the plot, we see that the natural frequency and the regular stimulation by the outside force are quite out of step with each other, resulting in very disorganized motion.

(d) The file "q_springSimulation3.m" generates this simulation. In the plot, we see that the natural frequency and the regular stimulation by the outside force are close to each other (natural frequency is $\omega = \sqrt{\frac{10}{0.5}} \approx 4.5$, rad/s, and the stimulating frequency is at $\omega = 4$ rad/s. This close match of the frequencies leads to the phenonenon called *beats*.

3. In class, we saw the differential equation for the angular motion of a pendulum. Here it is again, with no friction.

$$\begin{aligned}\text{Newton's Second Law: } mL^2\theta'' &= T_g \\ &= -mLg\sin(\theta) \\ \text{Solving for } \theta''\text{: } \theta'' &= -\frac{g}{L}\sin(\theta)\end{aligned}$$

Without simulating the actual motion of the pendulum, we can compute the period, T, using the formula below:

$$T = 4\sqrt{L/g}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity, 9.8 m/s.

For each set of values for L and θ_0,

(a) Use `quad` to find the period of the pendulum oscillations, and

(b) confirm the period by using `ode45` to simulate the motion pendulum for exactly that length of time, and plot a graph of the angular **velocity** against time. The velocity should just reach zero at the end of one cycle.

(i) $L = 2$ m, $\theta_0 = 40^o$,

(ii) $L = 2.5$ m, $\theta_0 = 20^o$.

(iii) $L = 5.0$ m, $\theta_0 = 90^o$.

All of the problems are shown solved in `q_pendulum.m`.

(i) $L = 2$ m, $\theta_0 = 40^o$: $\mathbf{T = 2.9274}$

(ii) $L = 2.5$ m, $\theta_0 = 20^o$: $\mathbf{T = 3.1978}$

(iii) $L = 5.0$ m, $\theta_0 = 90^o$: $\mathbf{T = 5.2974}$

Plots:

4. Newton's law of heating and cooling states that an object with temperature T in an environment at temperature T_{ext} will heat up or cool down according to the differential equation

$$\frac{dT}{dt} = -k(T - T_{ext})$$

Consider a garage used as a workshop. Its insulation and surface area give k a value of 0.1, if time t is measured in hours and the temperatures, T and T_{ext}, are in degrees Celsius.

The temperature outside changes during the day, as described by the formula

$$T_{ext} = 10 + 7\cos\left(\frac{\pi}{12}t\right)$$

We now imagine that the power goes out, with the garage at 23^o C at $t = 0$.

(a) Use ode45 and the DE to generate a numerical prediction of the garage's temperature T over time. Graph the solution over a time interval that shows both the initial and long-term behaviour of the temperature.

For the following questions, just use the graph or the numerical prediction of the temperature. You are *not* expected to solve the DE analytically.

(b) How many days does it take for the garage to get into a consistent temperature cycle? (You will need to estimate this by eye.)

(c) How many degrees does the temperature in the building fluctuate by, once the temperature gets into a steady cycle?

(d) Suppose the building were better insulated, so that the rate of heat loss were cut in half. Should k be half as large, or twice as large?

(e) Generate a numerical prediction for the temperature over time in the better-insulated scenario, and produce a graph of the temperature vs time for both scenarios on the same axes.

(f) How large are the temperature fluctuations in the building, now that the extra insulation has been added? Does halving the net heat flow also halve the net temperature fluctuations?

(a) A graph of the temperature over time is shown below:

MATLAB code is available in `q_garageTemp.m`

(b) From the graph, it takes the building roughly 2 days (48 hours) to get into a repeating cycle of temperature variation.

(c) Careful zooming of the graph (or a look at the y values in the ode45 output) give a highest temperature of 12.5 (high) and 7.5 (low), for a net fluctuation of approximately 2.6 degrees per day.

(d) k represents the coefficient of heat flow between the building and the environment. The bigger k is, the *larger* the headflow between the two. Since we're adding insulation, this should *reduce* the heat flow, and so *lower* the value of k.

(e) A graph of the heat change over time, given better insulation, is shown below.

(f) Zooming in on the peaks of the graph, the temperature now fluctuates between approximately 11.3 and 8.8 degrees Celsius, for a range of 2.5 degrees. This *is* roughly half the magnitude of the fluctuations we saw earlier.

5. Show that $y = \frac{2}{3}\mathrm{e}^x + \mathrm{e}^{-2x}$ is a solution of the differential equation $y' + 2y = 2\mathrm{e}^x$.

$$y = \frac{2}{3}\mathrm{e}^x + \mathrm{e}^{-2x} \Rightarrow y' = \frac{2}{3}\mathrm{e}^x - 2\mathrm{e}^{-2x}$$

To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left-hand side of the equation and show that the left-hand side of the equation and show that the left-hand side is equal to the right-hand side.

$$\begin{aligned}\text{LHS} = y' + 2y &= \frac{2}{3}\mathrm{e}^x - 2\mathrm{e}^{-2x} + 2(\frac{2}{3}\mathrm{e}^x + \mathrm{e}^{-2x}) \\ &= \frac{2}{3}\mathrm{e}^x - 2\mathrm{e}^{-2x} + \frac{4}{3}\mathrm{e}^x + 2\mathrm{e}^{-2x} = \frac{6}{3}\mathrm{e}^x = 2\mathrm{e}^x \\ &= \text{RHS}\end{aligned}$$

6. (a) For what values of r does the function $y = \mathrm{e}^{rx}$ satisfy the differential equation $2y'' + y' - y = 0$?
   (b) If r_1 and r_2 are the values of r that you found in part (a), show that every member of the family of functions $y = a\mathrm{e}^{r_1 x} + b\mathrm{e}^{r_2 x}$ is also a solution.

(a)

$$y = \mathrm{e}^{rx} \Rightarrow y' = r\mathrm{e}^{rx} \Rightarrow y'' = r^2\mathrm{e}^{rx}$$

Substituting these expressions into the differential equation $2y'' + y' - y = 0$, we get

$$\begin{aligned} & 2r^2\mathrm{e}^{rx} + r\mathrm{e}^{rx} - \mathrm{e}^{rx} = 0 \\ \Rightarrow\quad & (2r^2 + r - 1)\mathrm{e}^{rx} = 0 \\ \Rightarrow\quad & (2r - 1)(r + 1) = 0 \end{aligned}$$

(since e^{rx} is never zero) $r = \frac{1}{2}$ or -1.

(b) Let $r_1 = \frac{1}{2}$ and $r_2 = -1$, so we need to show that every member of the family of functions $y = a\mathrm{e}^{x/2} + b\mathrm{e}^{-x}$ is a solution of the differential equation $2y'' + y' - y = 0$.

$$\begin{aligned} & y = a\mathrm{e}^{x/2} + b\mathrm{e}^{-x} \\ \Rightarrow\quad & y' = \frac{1}{2}a\mathrm{e}^{x/2} - b\mathrm{e}^{-x} \\ \Rightarrow\quad & y'' = \frac{1}{4}a\mathrm{e}^{x/2} + b\mathrm{e}^{-x} \end{aligned}$$

$$\begin{aligned}\text{LHS} &= 2y'' + y' - y \\ &= 2(\frac{1}{4}a\mathrm{e}^{x/2} + b\mathrm{e}^{-x}) + (\frac{1}{2}a\mathrm{e}^{x/2} - b\mathrm{e}^{-x}) \\ &\quad - (a\mathrm{e}^{x/2} + b\mathrm{e}^{-x}) \\ &= \frac{1}{2}a\mathrm{e}^{x/2} + 2b\mathrm{e}^{-x} + \frac{1}{2}a\mathrm{e}^{x/2} - b\mathrm{e}^{-x} \\ &\quad - a\mathrm{e}^{x/2} - b\mathrm{e}^{-x} \\ &= (\frac{1}{2}a + \frac{1}{2}a - a)\mathrm{e}^{x/2} + (2b - b - b)\mathrm{e}^{-x} \\ &= 0 \\ &= \text{RHS}\end{aligned}$$

7. (a) For what values of k does the function $y = \cos(kt)$ satisfy the differential equation $4y'' = -25y$?
   (b) For those values of k, verify that every member of the vamily of functions $y = A \sin kt + B \cos kt$ is also a solution.

(a)

$$y = \cos kt \;\Rightarrow\; y' = -k \sin kt \;\Rightarrow\; y'' = -k^2 \cos kt$$

Substituting expressions into the differential equation $4y'' = -25y$, we get

$$\begin{aligned} & 4(-k^2 \cos kt) = -25(\cos kt) \\ \Rightarrow\; & (25 - 4k^2) \cos kt = 0 (\text{for all } t) \\ \Rightarrow\; & 25 - 4k^2 = 0 \\ \Rightarrow\; & k^2 = \frac{25}{4} \;\Rightarrow\; k = \pm\frac{5}{2} \end{aligned}$$

(b)

$$\begin{aligned} & y = A \sin kt + B \cos kt \\ \Rightarrow\; & y' = Ak \cos kt - Bk \sin kt \\ \Rightarrow\; & y'' = -Ak^2 \sin kt - Bk^2 \cos kt \end{aligned}$$

The given differential equation $4y'' = -25y$ is equivalent to $4y'' + 25y = 0$. Thus,

$$\begin{aligned} \text{LHS} &= 4y'' + 25y \\ &= 4(-Ak^2 \sin kt - Bk^2 \cos kt) \\ &\quad + 25(A \sin kt + B \cos kt) \\ &= -4Ak^2 \sin kt - 4Bk^2 \cos kt \\ &\quad + 25A \sin kt + 25B \cos kt \\ &= (25 - 4k^2)A \sin kt + (25 - 4k^2)B \cos kt \\ &= 0 \quad \text{since } k^2 = \frac{25}{4} \end{aligned}$$

8. (a) What can you say about a solution of the equation $y' = -y^2$ just by looking at the differential equation?
   (b) Verify that all members of the family $y = 1/(x + C)$ are solutions of the equation in part (a).
   (c) Can you think of a (very simple) solution of the differential equation $y' = -y^2$ that is not a member of the family in part (b)?
   (d) Find the solution to the initial-value problem

$$y' = -y^2 \qquad y(0) = 0.5$$

(a) Since the derivative of $y' = -y^2$ is always negative (or 0 if $y = 0$), the function y must be decreasing (or equal to 0) on any interval on which it is defined.

(b) $y = \frac{1}{x+C} \Rightarrow y' = -\frac{1}{(x+C)^2}$.

LHS $= y' = -\frac{1}{(x+C)^2} = -\left(\frac{1}{x+C}\right)^2 = -y^2 =$ RHS

(c) $y = 0$ is a solution of $y' = -y^2$ that is not a member of the family in part (b).

(d) If $y(x) = \frac{1}{x+C}$, then $y(0) = \frac{1}{0+C} = \frac{1}{C}$. Since $y(0) = 0.5$, $\frac{1}{C}\frac{1}{2} \Rightarrow C = 2$, so $y = \frac{1}{x+2}$

Numerical ODE Solutions With MATLAB

9. Create a plot for the solution to the differential equation $y' - \frac{y^2}{x^3} = 0$ if y(2) = 1. Include a large enough `xspan` to see the long-term behaviour.

For this first example of use MATLAB to build a numerical solution to a DE, we will show the full listing of a script that generates a solution to the given differential equation. In later solutions, we will only include the key lines for the MATLAB script.

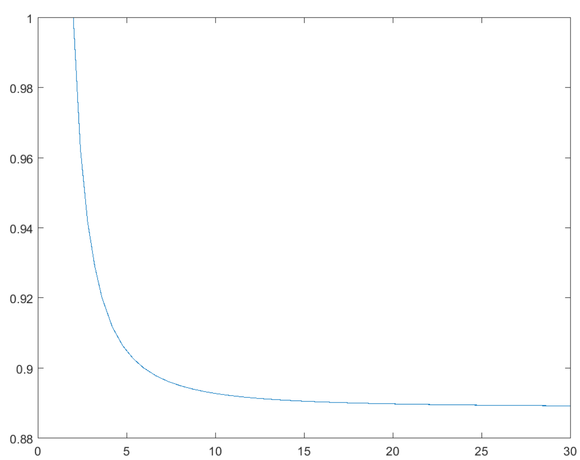
Notes:

- We set `xspan` to start at 2 in the line `xspan = [2, 30]`. This is used because the solution MATLAB is generating will start at the coordinates x_0 = first element of `xspan`, and y_0 = `y0` in the code, and our initial condition is $x = 2$, $y = 1$.
- We find the second value in the time span with some trial and error. Any value larger than 15 or 20 would be sufficient to show the long-term trend in the solution.

```
% ode45 solution to y' = - y^2/x^3, y(1) = 1
close all;
xspan = [2, 30];  % must start at x=2, from y(2) = 1
y0 = 1; % = y value at the start of xspan; y(2) = 1
[x, y] = ode45( @(x, y) -y.^2./x.^3, xspan, y0);  % have MATLAB solve the DE
plot(x, y);
```

Link to the MATLAB code:
W07DE01.m

Here is the graph of the solution.



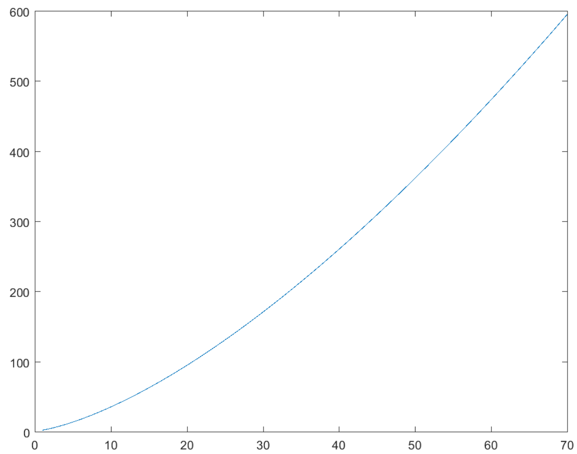
10. Create a plot for the solution to the differential equation $(2y - 4)y' - 3x^2 = 4x - 4$, if y(1) = 3.

To generate a first-order DE solution in MATLAB, the differential equation must be written first in the form $y' = \ldots$.

$$
\begin{aligned}
(2y - 4)y' - 3x^2 &= 4x - 4 \\
(2y - 4)y' &= 3x^2 + 4x - 4 \\
y' &= \frac{(3x^2 + 4x - 4)}{(2y - 4)}
\end{aligned}
$$

Link to the MATLAB code:
W07DE02.m

Here is the graph of the solution.

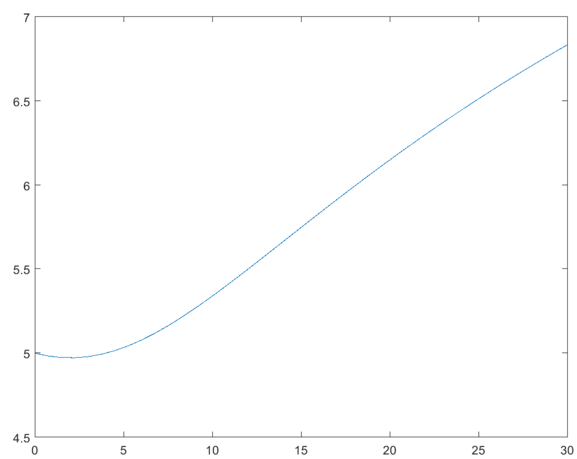


11. Create a plot for the solution to the differential equation $y' = e^{-y}(2t - 4)$ if $y(0) = 5$

This DE is already in the form $y' = \ldots$, so we can input it into MATLAB as-is. Note that the independent variable in this example is t, so we will use that in MATLAB instead of the variable x.

Link to the MATLAB code:
W07DE03.m

Here is the graph of the solution.



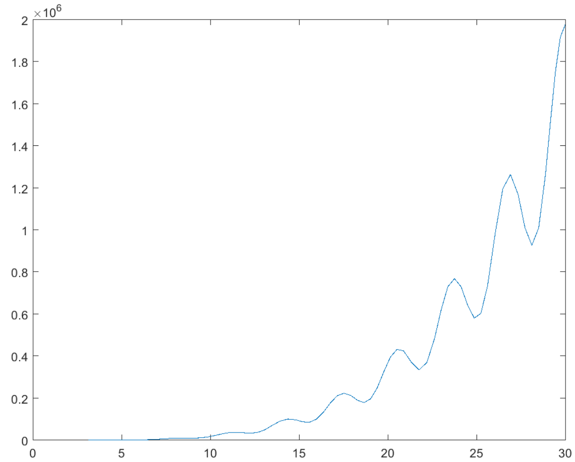
12. Create a plot for the solution to the differential equation $ty' - 2y = t^5 \sin(2t) - t^3 + 4t^4$, if $y(\pi) = \frac{3}{2}\pi^4$

To generate a first-order DE solution in MATLAB, the differential equation must be written first in the form $y' = \ldots$.

$$\begin{aligned} ty' - 2y &= t^5 \sin(2t) - t^3 + 4t^4 \\ ty' &= 2y + t^5 \sin(2t) - t^3 + 4t^4 \\ y' &= \frac{1}{t}(2y + t^5 \sin(2t) - t^3 + 4t^4) \end{aligned}$$

Link to the MATLAB code:
W07DE04.m

Here is the graph of the solution.



Note that in this example, because of the $\sin(2t)$ introducing an oscillation in the system, the solution won't look at simple as some of the other examples.

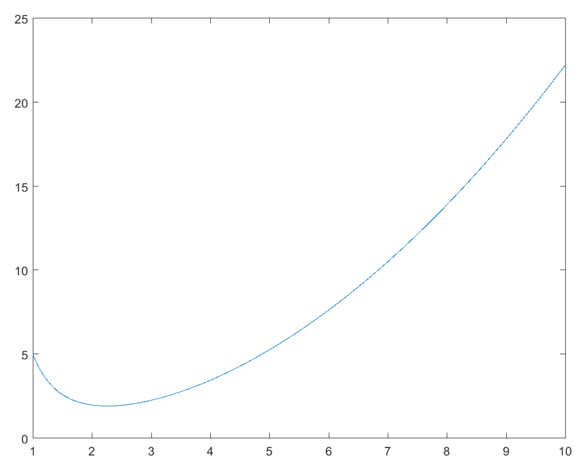
13. Create a plot for the solution to the differential equation $ty' + 2y = t^2 - t + 1$, if $y(1) = 0.5$.

To generate a first-order DE solution in MATLAB, the differential equation must be written first in the form $y' = \ldots$.

$$\begin{aligned} ty' + 2y &= t^2 - t + 1 \\ ty' &= -2y + t^2 - t + 1 \\ y' &= \frac{1}{t}(-2y + t^2 - t + 1) \end{aligned}$$

Link to the MATLAB code:
W07DE05.m

Here is the graph of the solution.



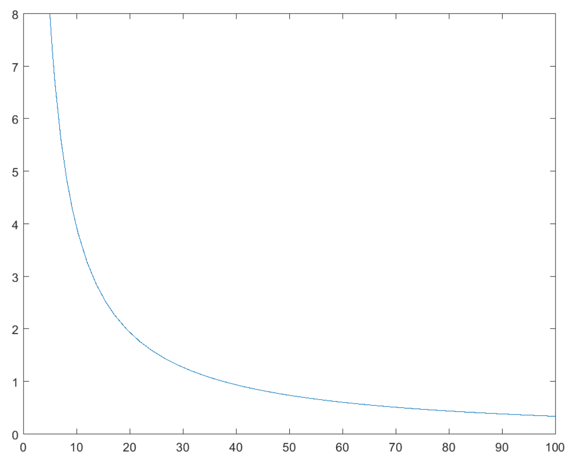
14. Create a plot for the solution to the differential equation $2xy^2 + 4 = 2(3 - x^2y)y'$ if $y(5) = 8$.

To generate a first-order DE solution in MATLAB, the differential equation must be written first in the form $y' = \ldots$. We start by switching both sides of the equation to put y' on the left.

$$\begin{aligned} 2(3 - x^2y)y' &= 2xy^2 + 4 \\ y' &= \frac{2xy^2 + 4}{2(3 - x^2y)} \end{aligned}$$

Link to the MATLAB code:
W07DE06.m

Here is the graph of the solution.



15. Find the particular solution to the IVP $y'' + 4y = 2x$, $y(0) = 1$, $y'(0) = 2$.

$y_c = c_1 \cos(2x) + c_2 \sin(2x)$

$y_p = A + Bx$
$y_p' = B$
$y_p'' = 0$

Substituting into the DE gives $A = \frac{1}{2}$ and $B = 0$, so the general solution is

$$y = y_c + y_p = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x}{2}$$

To find c_1 and c_2, use the initial conditions:

$$\begin{aligned} y(0) = 1: &\qquad 1 = c_1 + 0 + 0 \\ y'(0) = 2: &\qquad 2 = 0 + 2c_2 + \frac{1}{2} \end{aligned}$$

Solving gives

$$y = 1 \cdot \cos(2x) + \frac{3}{4} \sin(2x) + \frac{x}{2}$$

16. Find the particular solution to the IVP $y'' + 9y = \sin(2x)$, $y(0) = 1$, $y'(0) = 0$.

$y_c = c_1 \cos(3x) c_2 \sin(3x)$

$y_p = A\cos(2x) + B\sin(2x)$. Solving for A and B gives $A = 0$ and $B = \frac{1}{5}$. The general solution is

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{5} \sin(2x)$$

Using the initial conditions, we can solve for $c_1 = 1$ and $c_2 = -\frac{2}{15}$

$$y = \cos(3x) - \frac{2}{15} \sin(3x) + \frac{1}{5} \sin(2x)$$

Applications

17. For a spring/mass system with $m = 1$ kg, $c = 6$ N/(m/s), and $k = 45$, approximately what frequency of external forcing would produce the largest amplitude steady-state vibration?

The natural frequency is found using the auxiliary equation $r^2+6r+45 = 0$, giving $r = -3\pm 6i$, so $x_c = c_1e^{-3t}\cos(6t)+c_2e^{-3t}\sin(6t)$

Since this system's natural or intrinsic oscillations are at 6 rad/s, the spring/mass will show the largest amplitude response to an outside force if that outside force also has a frequency close to 6 rad/s.

18. For a spring/mass system with $m = 1$ kg, $c = 10$ N/(m/s), and $k = 650$, approximately what frequency of external forcing would produce the largest amplitude steady-state vibration?

$m = 1, c = 10, k = 650, F_0 = 100$, so

$$x'' + 10x' + 650x = \cos(\omega t)$$

The natural frequency is found using the auxiliary equation $r^2 + 10r + 650 = 0$, giving $r = -5 \pm 25i$, so $x_c = c_1e^{-5t}\cos(25t) + c_2e^{-5t}\sin(25t)$

It will show its largest amplitude response to stimuli with frequencies close to 25 rad/s.

19. Consider the undamped spring model $y'' + \omega_0^2 y = \cos(\omega t)$.
    (a) Solve $y'' + \omega_0^2 y = \cos(\omega t)$ where $\omega^2 \neq \omega_0^2$.
    (b) What does this predict will happen to the amplitude of the oscillations as the stimulus frequence ω is brought closer and closer to the natural frequency ω_0?

(a) The characteristic equation is $r^2 + \omega_0^2 = \left(r - \omega_0\sqrt{-1}\right)\left(r + \omega_0\sqrt{-1}\right)$, so the general solution to the corresponding homogeneous equation is $C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$. Using the method of undetermined coefficients, we consider a solution of the form $y_g(t) := A_1\cos(\omega t) + A_2\sin(\omega t)$ for some constants A_1 and A_2. It follows that

$$\begin{aligned}\cos(\omega t) &= y_g'' + \omega_0^2 y_g = \left(-A_1\omega^2\cos(\omega t) - A_2\omega^2\sin(\omega t)\right) + \omega_0^2\left(A_1\cos(\omega t) + A_2\sin(\omega t)\right)\\ &= (\omega_0^2 - \omega^2)\left(A_1\cos(\omega t) + A_2\sin(\omega t)\right)\end{aligned}$$

Since $A_1 = \frac{1}{\omega_0^2-\omega^2}$ and $A_2 = 0$, the general solution is

$$C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{1}{(\omega_0^2 - \omega^2)}\cos(\omega t)\,.$$

(b) As the stimulus frequency ω approaches the natural frequency ω_0, the amplitude given by $\frac{1}{(\omega_0^2-\omega^2)} \to \infty$. This corresponds to the result of resonance, when $\omega = \omega_0$, and the solutions change to $y_c = At\cos(\omega t)$, which has an linearly growing (unbounded) amplitude.

20. Consider the equation for the spring/mass system with $m = 1$ kg, $c = 4$ N/(m/s) and $k = 4$ N/m, and which is being forced by an external periodic force of $10\cos(3t)$ N:

$$x'' + 4x' + 4x = 10\cos(3t)$$

Find the formula for the steady-state oscillations, and find their amplitude.

The system is damped, so the x_c solution will have negative exponentials in it. i.e. that part of the solution will be **transient** because its contribution $\to 0$ over time. That is why x_c is referred to as $x_{\text{transient}}$ or x_{tr} throughout the problems in this section.

Since the RHS is pure cos/sine, its differential family will not overlap with the exp sin/cos in x_c. This means x_p will be a simple linear combination of $\cos(3t)$ and $\sin(3t)$.

Let $x_p = A\cos(3t) + B\sin(3t)$.
$x_p' = -3A\sin(3t) + 3B\cos(3t)$

$x_p'' = -9A\cos(3t) - 9B\sin(3t)$

Substituting into the DE,

$$(-9A\cos(3t) - 9B\sin(3t)) + 4(-3A\sin(3t) + 3B\cos(3t)) + 4(A\cos(3t) + B\sin(3t)) = 10\cos(3t)$$

$$\begin{array}{rlr} \text{sine coeffs:} & -9B - 12A + 4B = 0 & -12A - 5B = 0 \\ \text{cos coeffs:} & -9A + 12B + 4A = 10 & -5A + 12B = 10 \end{array}$$

$$\begin{aligned} -60A - 25B &= 0 \\ -60A + 144B &= 120 \end{aligned}$$

$$-169B = -120$$

Solving gives $A = \dfrac{-50}{169}$, $B = \dfrac{120}{169}$, so

$$x_p = x_{sp} = \frac{-50}{169}\cos(3t) + \frac{120}{169}\sin(3t)$$

The amplitude of the sum $A\cos(bt) + B\sin(bt)$ is given by $C = \sqrt{A^2 + B^2}$, so the amplitude here is $\sqrt{\left(\dfrac{-50}{169}\right)^2 + \left(\dfrac{120}{169}\right)^2} \approx$ 0.77 meters or 77 cm.

21. Consider the equation for the spring/mass system

$$x'' + 3x' + 5x = 4\cos(5t)$$

Find the formula for the steady-state oscillations, and find their amplitude.

Again, because there is damping, x_c will have negative exponentials, and will fade away with time (transient part of the solution). Also, there is no overlap between the functions in the transient x_c and steady-state x_p, so we can set the form of x_p as a linear combination of $\cos(5t)$ and $\sin(5t)$.

Let $x_p = A\cos(5t) + B\sin(5t)$ so
$x_p' = -5A\sin(5t) + 5B\cos(5t)$
$x_p'' = -25A\cos(5t) - 25B\sin(5t)$

Substituting into the DE,

$$(-25A\cos(5t) - 25B\sin(5t)) + 3(-5A\sin(5t) + 5B\cos(5t)) + 5(A\cos(5t) + B\sin(5t)) = 4\cos(5t)$$

$$\begin{array}{rlr} \text{sine coeffs:} & -25B - 15A + 5B = 0 & -15A - 20B = 0 \\ \text{cos coeffs:} & -25A + 15B + 5A = 4 & -20A + 15B = 4 \end{array}$$

$$\begin{aligned} -60A - 80B &= 0 \\ -60A + 45B &= 12 \end{aligned}$$

$$-125B = 12$$

Solving gives $A = \dfrac{16}{125}$, $B = \dfrac{-12}{125}$ so

$$x_p = x_{sp} = \frac{16}{125}\cos(5t) - \frac{12}{125}\sin(5t)$$

The amplitude of the sum $A\cos(bt)+B\sin(bt)$ is given by $C = \sqrt{A^2+B^2}$, so the amplitude here is $\sqrt{\left(\frac{16}{125}\right)^2 + \left(\frac{-12}{125}\right)^2} =$ 0.16 m, or 16 cm.

22. The charge at a point in an electrical circuit is called $Q(t)$. For a particular series circuit with a resistor, a conductor, and a capacitor, $Q(t)$ satistifies the differential equation

$$2Q'' + 60Q' + 1/0.0025Q = 100e^{-t}$$

(a) Given the initial conditions that $Q(0) = 0$, $Q'(0) = 0$, find the particular solution for $Q(t)$.

(b) Knowing that $I = Q'$ is the current in the circuit, find an expression for $I(t)$.

(a) Solving for Q_c, we solve $2r^2 + 60r + 400 = 0$, for $r = -20, -10$

$$Q_c = c_1e^{-20t} + c_2e^{-10t}$$

Solving for Q_p, we assume $Q_p = Ae^{-t}$. Subbing into the DE gives $2A-60A+400A = 100$ or $A = 100/342 = 50/171$.

$$Q_p = \frac{50}{171}e^{-t}$$

This gives the general solution

$$Q = Q_c + Q_p = c_1e^{-20t} + c_2e^{-10t} + \frac{50}{171}e^{-t}$$

Solving for c_1 and c_2 given $Q(0) = 0$ and $Q'(0) = 0$, gives

$$\begin{aligned} 0 &= c_1 + c_2 + \frac{50}{171} \\ 0 &= -20c_1 - 10c_2 - \frac{50}{171} \end{aligned}$$

Solving gives $c_1 = 45/171 = 5/19$, $c_2 = -95/171 = -5/9$, so

$$Q = \frac{5}{19}e^{-20t} + \frac{-5}{9}e^{-10t} + \frac{50}{171}e^{-t}$$

(b) Differentiating $Q(t)$ to find I we obtain

$$I = \frac{-100}{19}e^{-20t} + \frac{50}{9}e^{-10t} - \frac{50}{171}e^{-t}$$

23. A cantilevered beam which is $L = 2$ m long, made out of a pine "2 by 4" has $I = 2.23 \times 10^{-6}$ m^4, $E = 9.1 \times 10^9$ N/m^2, and its deformation satisfies the differential equation

$$EIy^{(4)} = p(x)$$

where $p(x)$ is the loading on the beam in N/m, and boundary conditions $y(0) = 0$, $y'(0) = 0$, $y''(2) = 0$, and $y'''(2) = 0$.

(a) Find the formula for $y(x)$ if the load applied is *triangular* (increases linearly), given by $p(x) = 20x$ N/m. I.e. as x increases, moving away from the mounting point of the beam, the load increases, until at the tip of the beam ($x = 2$), the load is 40 N/m.

(b) Find the amount of deflection of the beam at the tip under this load.

(a) Since the corresponding homogeneous DE is simply $EIy^{(4)} = 0$, with roots to the characteristic equation of zero (repeated four times), $y_c = c_1 + c_2x + c_3x^2 + c_4x^3$.

Since the RHS $= 20x$, y_p would usually have been $A + Bx$. However, this is not linearly independent with the functions in y_c, so we boost by powers of x until it is, so we choose $y_p = Ax^4 + Bx^5$.

Subbing y_p into the DE and solving for A and B,

$$EI\left(\underbrace{24A + 120Bx}_{y^{(4)}}\right) = 20x$$

$$\text{const coeff's:} \quad A = 0 \text{ and}$$

$$x \text{ coeff's:} \quad 120EIB = 20, \text{ or } B = \frac{1}{6EI}$$

$$\text{so } y_P = \frac{1}{6EI}x^5$$

This means the general solution is

$$y = \underbrace{c_1 + c_2x + c_3x^2 + c_4x^3}_{y_c} + \underbrace{\frac{1}{6EI}x}_{y_p}$$

To solve for $c_1, \ldots, c_4$, we use the boundary conditions.

$$\begin{aligned} y(0) = 0 &\implies 0 = c_1 \\ y'(0) = 0 &\implies 0 = c_2 \\ y''(2) = 0 &\implies 0 = 2c_3 + 6c_4(2) + \frac{1}{6EI}20(2)^3 \\ y'''(2) = 0 &\implies 0 = 6c_4 + \frac{1}{6EI}60(2)^2 \end{aligned}$$

Solving gives $c_4 = \dfrac{-20}{3EI}$ and $c_3 = \dfrac{80}{3EI}$, so the particular solution, or the formula for the beam deflection at x is

$$y = \frac{1}{EI}\left(\frac{80}{3}x^2 - \frac{20}{3}x^3 + \frac{1}{6}x^5\right)$$

(b) The deflection at the tip, $x = 2$, is simply the value of $y(2)$:

$$y(2) = \frac{1}{EI}\left(\frac{80}{3}2^2 - \frac{20}{3}2^3 + \frac{1}{6}2^5\right) \approx 0.00289$$

This means the tip of the beam deflects by 0.00289 m or roughly 3 mm under this load.