MNTC P01 - Week #5 - Integrals - Techniques

Substitution Integrals

To practice computing integrals using substitutions, do as many of the problems from this section as you feel you need. The problems trend from simple to the more complex.

Note: In the solutions to these problems, we always show the substitution used. On a test, if you can compute the antiderivative in your head, you do *not* need to go through all the steps shown here. They are included in these solutions as learning & comprehension aid.

1. $\displaystyle\int te^{t^2}\,dt$
2. $\displaystyle\int e^{3x}\,dx$

   NOTE: we will not show the differentiation check for any later questions, as the process is always the same, and you should be comfortable enough with the derivative rules to do the checking independently. If you are uncertain about any problem, contact your instructor.
3. $\displaystyle\int e^{-x}\,dx$
4. $\displaystyle\int 25e^{-0.2t}\,dt$
5. $\displaystyle\int t\cos(t^2)\,dt$
6. $\displaystyle\int \sin(2x)\,dx$
7. $\displaystyle\int \sin(3-t)\,dt$
8. $\displaystyle\int xe^{-x^2}\,dx$
9. $\displaystyle\int (r+1)^3\,dr$
10. $\displaystyle\int y(y^2+5)^8\,dy$
11. $\displaystyle\int t^2(t^3-3)^{10}\,dt$
12. $\displaystyle\int x^2(1+2x^3)^2\,dx$
13. $\displaystyle\int x(x^2+3)^2\,dx$
14. $\displaystyle\int x(x^2-4)^{7/2}\,dx$
15. $\displaystyle\int y^2(1+y)^2\,dy$
16. $\displaystyle\int (2t-7)^{73}\,dt$
17. $\displaystyle\int \frac{1}{y+5}\,dy$
18. $\displaystyle\int \frac{1}{\sqrt{4-x}}\,dx$
19. $\displaystyle\int (x^2+3)^2\,dx$
20. $\displaystyle\int x^2e^{x^3+1}\,dx$
21. $\displaystyle\int \sin\theta(\cos\theta+5)^7\,d\theta$
22. $\displaystyle\int \sqrt{\cos(3t)}\sin(3t)\,dt$
23. $\displaystyle\int \sin^6\theta\cos\theta\,d\theta$
24. $\displaystyle\int \sin^3\alpha\cos\alpha d\alpha$
25. $\displaystyle\int \sin^6(5\theta)\cos(5\theta)\,d\theta$
26. $\displaystyle\int \tan(2x)\,dx$
27. $\displaystyle\int \frac{(\ln z)^2}{z}\,dz$
28. $\displaystyle\int \frac{e^t+1}{e^t+t}\,dt$
29. $\displaystyle\int \frac{y}{y^2+4}\,dy$
30. $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$
31. $\displaystyle\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy$
32. $\displaystyle\int \frac{1+e^x}{\sqrt{x+e^x}}\,dx$
33. $\displaystyle\int \frac{e^x}{2+e^x}\,dx$
34. $\displaystyle\int \frac{x+1}{x^2+2x+19}\,dx$
35. $\displaystyle\int \frac{t}{1+3t^2}\,dt$

36. $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\ dx$

37. $\displaystyle\int \frac{(t+1)^2}{t^2}\ dt$

38. $\displaystyle\int \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}\ dx$

39. $\displaystyle\int_0^{\pi} \cos(x+\pi)\ dx$

40. $\displaystyle\int_0^{1/2} \cos(\pi x)\ dx$

$$\begin{aligned}\int_0^{1/2} \cos(\pi x)\ dx &= \frac{1}{\pi}\sin(\pi x)\Big|_0^{1/2} \\ &= \frac{1}{\pi}\left[\sin(\pi/2) - \sin(0)\right] = \frac{1}{\pi}[1-0] = \frac{1}{\pi}\end{aligned}$$

41. $\displaystyle\int_0^{\pi/2} e^{-\cos(\theta)}\sin(\theta)\ d\theta$

42. $\displaystyle\int_1^2 2xe^{x^2}\ dx$

43. $\displaystyle\int_1^8 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\ dx$

44. $\displaystyle\int_{-1}^{e-2} \frac{1}{t+2}\ dt$

45. $\displaystyle\int_1^4 \frac{\cos\sqrt{x}}{\sqrt{x}}\ dx$

46. $\displaystyle\int_0^2 \frac{x}{(1+x^2)^2}\ dx$

47. If appropriate, evaluate the following integrals by substitution. If substitution is not appropriate, say so, and do not evaluate.

   (a) $\displaystyle\int x\sin(x^2)\ dx$

   (b) $\displaystyle\int x^2\sin(x)\ dx$

   (c) $\displaystyle\int \frac{x^2}{1+x^2}\ dx$

   (d) $\displaystyle\int \frac{x}{(1+x^2)^2}\ dx$

   (e) $\displaystyle\int x^3e^{x^2}\ dx$

   (f) $\displaystyle\int \frac{\sin(x)}{2+\cos(x)}\ dx$

48. Find the exact area under the graph of $f(x) = xe^{x^2}$ between $x = 0$ and $x = 2$.

49. Find the exact area under the graph of $f(x) = 1/(x+1)$ between $x = 0$ and $x = 2$.

50. Find $\displaystyle\int 4x(x^2+1)\ dx$ using two methods:

   (a) Do the multiplication first, and then antidifferentiate.

   (b) Use the substitution $w = x^2 + 1$.

   (c) Explain how the expressions from parts (a) and (b) are different. Are they both correct?

51. (a) Find $\displaystyle\int \sin\theta\cos\theta\ d\theta$

   (b) You probably solved part (a) by making the substitution $w = \sin\theta$ or $w = \cos\theta$. (If not, go back and do it that way.) Now find $\displaystyle\int \sin\theta\cos\theta\ d\theta$ by making the other substitution.

   (c) There is yet another way of finding this integral which involves the trigonometric identities:

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta.\end{aligned}$$

   Find $\displaystyle\int \sin\theta\cos\theta d\theta$ using one of these identities and then the substitution $w = 2\theta$.

   (d) You should now have three different expressions for the indefinite integral $\displaystyle\int \sin\theta\cos\theta d\theta$. Are they really different? Are they all correct? Explain.

Substitution Integrals - Applications

52. Let $f(t)$ be the rate of flow, in cubic meters per hour, of a flooding river at time t in hours. Give an integral for the total flow of the river:

   (a) Over the 3-day period, $0 \le t \le 72$ (since t is measured in hours).

   (b) In terms of time T in **days** over the same 3-day period.

53. Oil is leaking out of a ruptured tanker at the rate of $r(t) = 50e^{-0.02t}$ thousand liters per minute.

   (a) At what rate, in liters per minute, is oil leaking out at $t = 0$? At $t = 60$?

(b) How many liters leak out during the first hour?

54. If we assume that wind resistance is proportional to velocity, then the downward velocity, v, of a body of mass m falling vertically is given by

$$v = \frac{mg}{k}(1 - e^{-kt/m})$$

where g is the acceleration due to gravity and k is a constant. Find the height of the body, h, above the surface of the earth as a function of time. Assume the body starts at height h_0.

55. The rate at which water is flowing into a tank is $r(t)$ gallons/minute, with t in minutes.

    (a) Write an expression approximating the amount of water entering the tank during the interval from time t to time $t + \Delta t$, where Δt is small.

    (b) Write a Riemann sum approximating the total amount of water entering the tank between $t = 0$ and $t = 5$. Then write an exact expression for this amount.

    (c) By how much has the amount of water in the tank changed between $t = 0$ and $t = 5$ if $r(t) = 20e^{0.02t}$?

    (d) If $r(t)$ is as in part (c), and if the tank contains 3000 gallons initially, find a formula for $Q(t)$, the amount of water in the tank at time t.

56. After a spill of radioactive iodine, measurements at $t = 0$ showed the ambient radiation levels at the site of the spill to be four times the maximum acceptable limit. The level of radiation from an iodine source decreases according to the formula

$$R(t) = R_0 e^{-0.004t}$$

where R is the radiation level (in millirems/ hour) at time t in hours and R_0 is the initial radiation level (at t =0).

    (a) How long will it take for the site to reach an acceptable level of radiation?

    (b) Engineers look up the safe limit of radiation and find it to be 0.6 millirems/hour. How much total radiation (in millirems) will have been emitted by the time found in part (a)?

57. David is learning about catalysts in his Chemistry course. He has read the definition:

> Catalyst: A substance that helps a reaction to go faster without being used up in the reaction.

In today's Chemistry lab exercise, he has to add a catalyst to a chemical mixture that produces carbon dioxide. When there is no catalyst, the carbon dioxide is produced at a rate of 8.37×10^{-9} moles per second. When C moles of the catalyst are present, the carbon dioxide is produced at a rate of $(6.15 \times 10^{-8})C + 8.37 \times 10^{-9}$ moles per second.

The reaction begins at exactly 10:00 a.m. One minute later, at 10:01 sharp, David starts to add the catalyst at a constant rate of 0.5 moles per second.

How much carbon dioxide is produced between 10:00 (sharp) and 10:05?

Integration by Parts

58. For each of the following integrals, indicate whether integration by substitution or integration by parts is more appropriate. Do not evaluate the integrals.

    (a) $\int x \sin(x)\, dx$

    (b) $\int \frac{x^2}{1 + x^3}\, dx$

    (c) $\int xe^{x^2}\, dx$

    (d) $\int x^2 \cos(x^3)\, dx$

    (e) $\int \frac{1}{\sqrt{3x + 1}}\, dx$

    (f) $\int x^2 \sin x\, dx$

    (g) $\int \ln x\, dx$

To practice computing integrals by parts, do as many of the problems from this section as you feel you need. The problems trend from simple to the more complex.

For Questions #59 to #82, evaluate the integral.

59. $\int t \sin t\, dt$

60. $\int te^{5t}\, dt$

61. $\int pe^{-0.1p}\, dp$

62. $\int (z + 1)e^{2z}\, dz$

63. $\int \ln x \, dx$

64. $\int y \ln y \, dy$

65. $\int x^3 \ln x \, dx$

66. $\int q^5 \ln(5q) \, dq$

67. $\int t^2 \sin t \, dt$

68. $\int x^2 \cos(3x) \, dx$

69. $\int (\ln t)^2 \, dt$

70. $\int t^2 e^{5t} \, dt$

71. $\int y\sqrt{y+3} \, dy$

72. $\int (t+2)\sqrt{2+3t} \, dt$

73. $\int (p+1)\sin(p+1) \, dp$

74. $\int \frac{z}{e^z} \, dz$

75. $\int \frac{\ln x}{x^2} \, dx$

76. $\int \frac{y}{\sqrt{5-y}} \, dy$

77. $\int \frac{t+7}{\sqrt{5-t}} \, dt$

78. $\int x(\ln x)^2 \, dx$

79. $\int \arcsin(w) \, dw$

80. $\int \arctan(7x) \, dx$

81. $\int x \arctan(x^2) \, dx$

82. $\int x^3 e^{x^2} \, dx$

83. $\int_1^5 \ln t \, dt$

84. $\int_3^5 x \cos x \, dx$

85. $\int_0^{10} z e^{-z} \, dz$

86. $\int_1^3 t \ln(t) \, dt$

87. $\int_0^1 \arctan(y) \, dy$

88. $\int_0^5 \ln(1+t) \, dt$

89. $\int_0^1 \arcsin z \, dz$

90. $\int_0^1 x \arcsin(x^2) \, dx$

91. Find the area under the curve $y = te^{-t}$ on the interval $0 \leq t \leq 2$.

92. Find the area under the curve $f(z) = \arctan z$ on the interval $[0, 2]$.

93. Use integration by parts twice to find $\int e^x \sin(x) \, dx$.

94. Use integration by parts twice to find $\int e^y \cos(y) \, dy$.

95. Use integration by parts to show that

$$\int x^n \cos(ax) \, dx = \frac{1}{a} x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) \, dx$$

96. The concentration, C, in ng/ml, of a drug in the blood as a function of the time, t, in hours since the drug was administered is given by

$$C = 15te^{-0.2t}.$$

The area under the concentration curve is a measure of the overall effect of the drug on the body, called the *bioavailability*. Find the bioavailability of the drug between $t = 0$ and $t = 3$.

97. During a surge in the demand for electricity, the rate, r, at which energy is used can be approximated by

$$r = te^{-at}$$

where t is the time in hours and a is a positive constant.

(a) Find the total energy, E, used in the first T hours. Give your answer as a function of a.

(b) What happens to E as $T \to \infty$?