Week #6 : Integrals - Modeling

Goals:

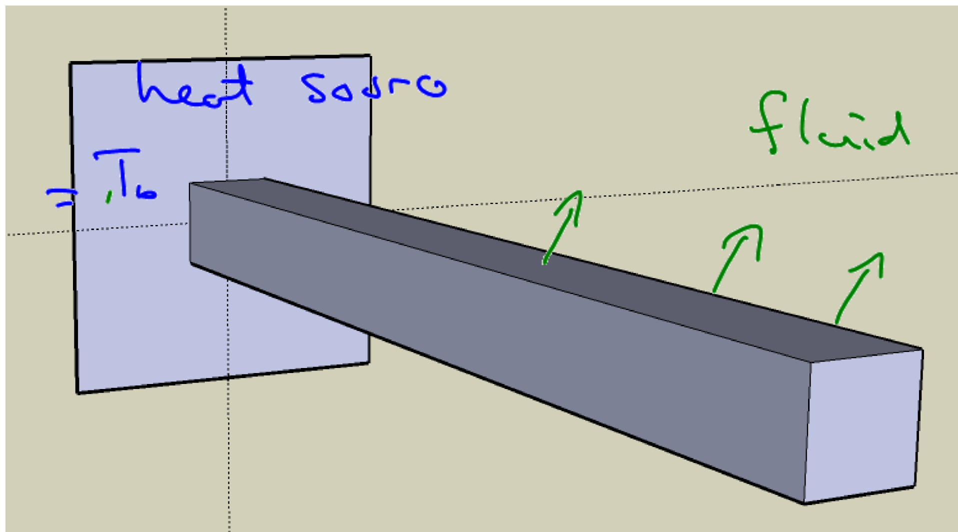
- Use MATLAB to solve a variety of integration problems.
- Use integration to find the average value of a function.
- Use MATLAB to find the average value of a function.
- Applications of Integration.

Heat Transfer and Integrals

A common engineering challenge is to transfer heat generated by a motor or combustion into a nearby fluid. → oil → water air

This transfer is often made more effective by the use of **cooling fins**, which increase the surface area of contact with the fluid.

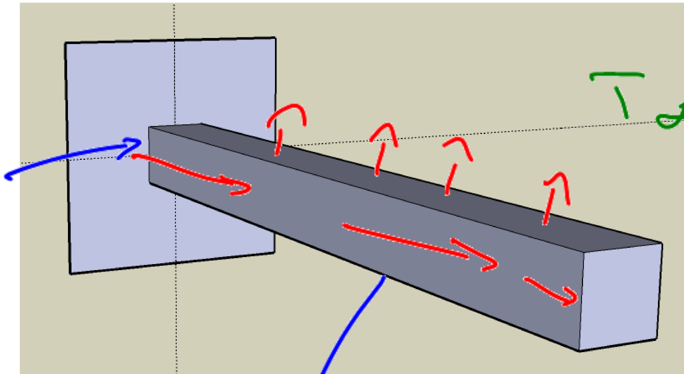
base temp. $= T_0$



If we fix the temperature at the base, we can ask the question **"How quickly is heat radiated out of fin?"**

Problem. What factors affect the rate of heat transfer out of the fin?

Temp diff drive heat transfer



T_b

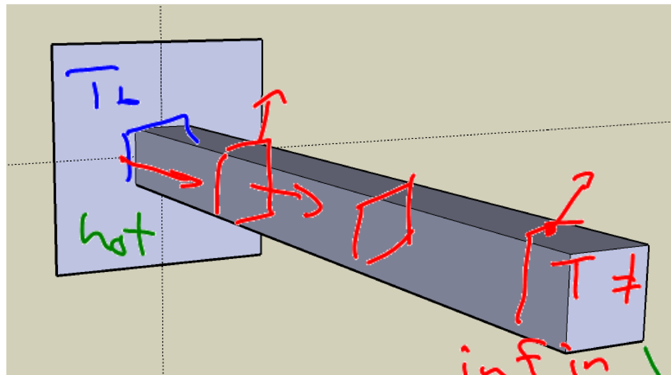
T_∞ environmental temp (fluid temp)

materials
- fin
- fluid

length
cross-section
perimeter / surface

Newton's Law of Heating and Cooling tells us that the rate of heat flow from a metal fin to the environment is proportional to the temperature difference between the fin, T, and the environment, T_∞.

No "the" fin temp b/c it varies along the length of the fin



T_∞

constant

cooler

Problem. What issue is raised when trying to use this rule to compute the rate of heat flow out of the fin we are considering?

Can't easily predict the rate of heat flow based on simple ΔT formulas.

Numerical Integration - Motivation

Problem. Take a small slice of length Δx of the fin. What advantage is there to looking at a small slice, rather than the whole fin at once?

on a small enough slice,
T is $\approx$ constant $\Rightarrow$ can use single formulas for heat transfer

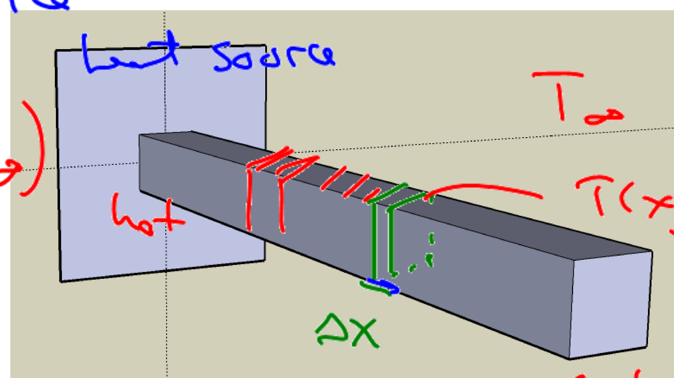
How much heat is lost through that slice?

heat flow $\underbrace{}_{\text{area in slice}}$

$$= h \cdot (\Delta x)\, P\, (T(x) - T_\infty)$$

h: convection coeff

P: perimeter



Give an expression for the total amount of heat lost over the whole fin.

Total heat flow (n slices)

$$= \sum_{i=1}^{n} h P (T(x_i) - T_\infty)\, \Delta x \quad + \text{end area.}$$

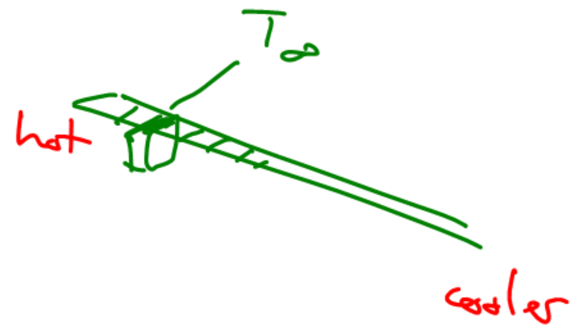
Integration

As soon as you see any sum of the form $\sum \ldots \Delta x$, you should be thinking "integral"!

Problem. For our fin example, net heat flow to environment is given by:

L = length of fin

$$Q = \int_0^L h P (T(x) - T_\infty)\, dx$$



If, for simplicity, we assume $T_\infty = 0$ °C, our target integral becomes:

$$Q = \int_0^L h P \, T(x)\, dx$$

Finding the Temperature Distribution

To evaluate the integral, we first need to find the temperature distribution along the fin. Without getting into all the gory details, tables or other methods will lead to the following formula for the temperature along the fin:

length L

convection coeff

perimetr

$$T(x) = \frac{T_b\left(\cosh(m(L-x)) + \frac{h}{mk}\right)(\sinh(m(L-x)))}{\cosh(mL) + \frac{h}{mk}\sinh(mL)}, m = \sqrt{\frac{hP}{kA_c}}$$

conduction coeff

cross sectional area

Problem. What are these new $\cosh(x)$ and $\sinh(x)$ functions?

"cosh", "shin" → hyperbolic trig functions

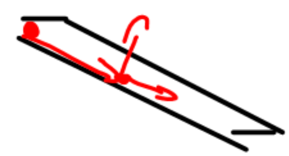
$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

What does the m parameter capture in the physics of the scenario?

$$\sqrt{\frac{hP}{kA_c}}$$

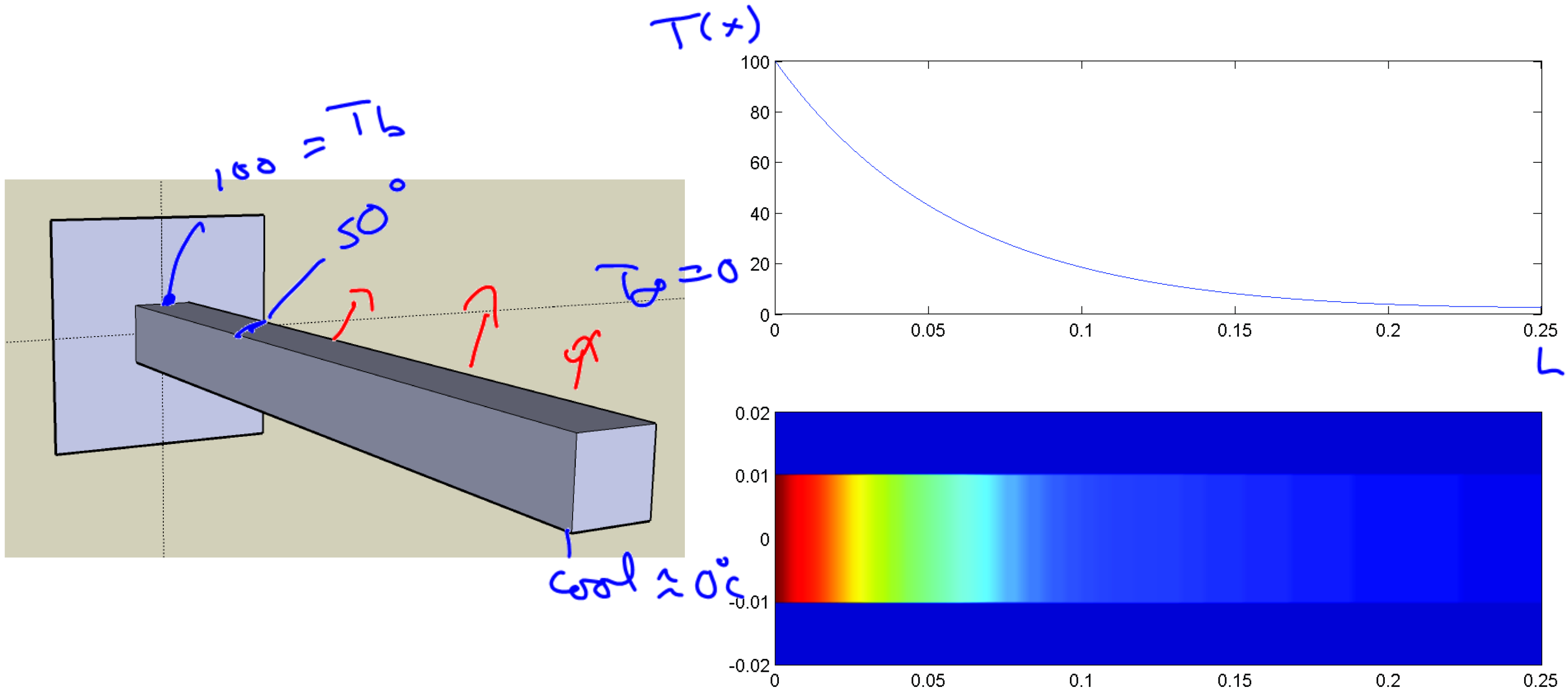
hP ← heat flow to environ

kA_c ← heat flow within the fin

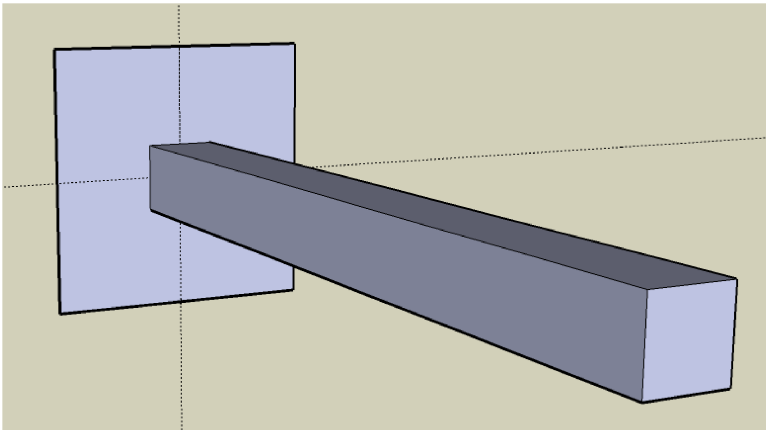


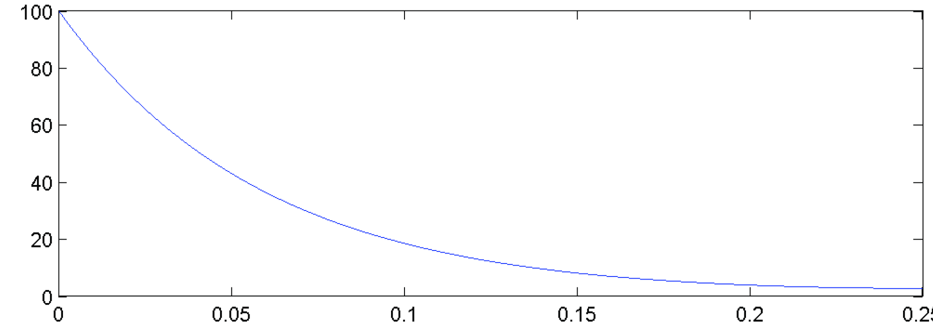
Graphically

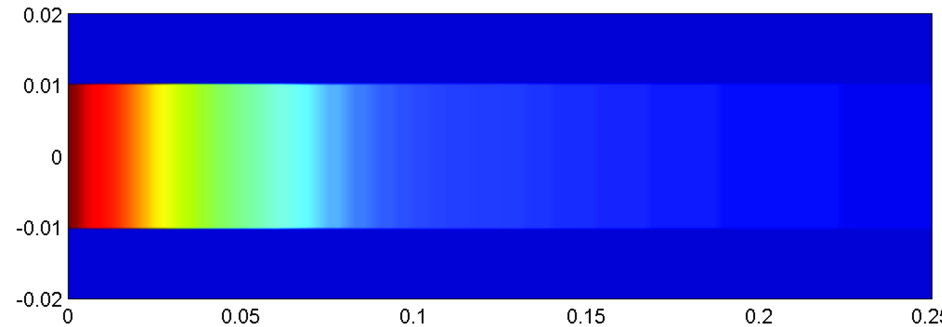
Once we have the temperature distribution, we can graph it along the length of the fin to see if it makes sense.



$T(x) = \ldots$ (messy)







Our next step is to see if we can use this temperature distribution, $T(x)$, to compute the rate of heat transfer by integration:

$$Q = \int_0^L \underbrace{h \; P}_{\text{const.}} \; \underbrace{T(x)}_{\text{complicated messy}} \; dx \qquad (T_\infty \text{ assumed} = 0)$$

Computing Total Heat Loss

$T_\infty = 0$

Now, we have addressed one challenge in our problem: we know the steady-state temperature along the fin. Next, we want to compute the net rate of heat flow out, or the cooling ability of the fin.

The heat flow out of the fin is given by

$$Q = \int_0^L \overbrace{h\ P}^{\text{consts}}\ \underbrace{T(x)}_{\text{complicated}}\ dx \qquad (T_\infty \text{ assumed} = 0)$$

Our first approach, if it is possible, should be direct anti-differentiation (think $\int x^2\ dx = \frac{1}{3}x^3$).

$$Q = \int_0^L h\ P\ T(x)\ dx \qquad (T_\infty \text{ assumed} = 0)$$

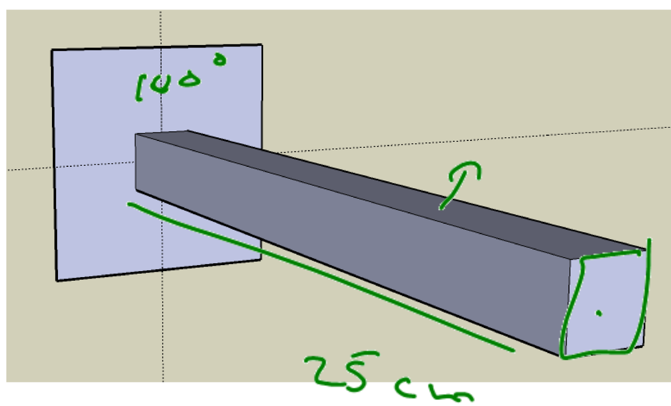
For this problem, given the earlier temperature we found, $T(x)$, we **can** evaluate the integral exactly:

$$m = \sqrt{\frac{hP}{k A_c}}$$

$$Q = \frac{1}{2} \frac{T_b\, P\, h\, (m\, k\, exp(2\, m\, L) - m\, k + h\, exp(2\, m\, L) + h - 2\, h\, exp(m\, L)}{m(\cosh(m\, L)\, m\, k + h\, \sinh(m\, L))\, exp(-m\, L)}$$

Evaluated with appropriate constants for the material, base temp, etc. we would obtain the final value

$$Q = 2.363 \text{ J/s}$$



Comments on Anti-Derivatives

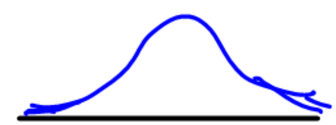
Through this last step, we reached what would be the important engineering goal: obtaining the **numerical value** for the integral. When we compute integrals analytically, by using anti-derivatives, we are doing the best possible thing.

- Integrals give exact values.
- Integrals can be re-used immediately with different constants.

Unfortunately, actually computing the numerical value of an integral using antiderivatives isn't always an option:

bell curve

- Some functions don't have antiderivatives. $\int e^{-x^2}\,dx$



- Sometimes we don't have a function, but only data.

formula

- Sometimes we forget how to find the anti-derivative!

can't find

Numerical Quadrature /Integration

πr^2

The word *quadrature* comes from the Greek challenge of trying to *square the circle*, or finding the **area** (in square units) of the round circle.

When you hear **quadrature** think **numerical integration**.

The two common scenarios where we need *numerical* integration will be:

- **Formula** for $f(x)$ known, want $\displaystyle\int_a^b f(x)dx$
- **Data** for $f(x)$ collected at $f(x_i)$, want $\displaystyle\int_a^b f(x)dx$

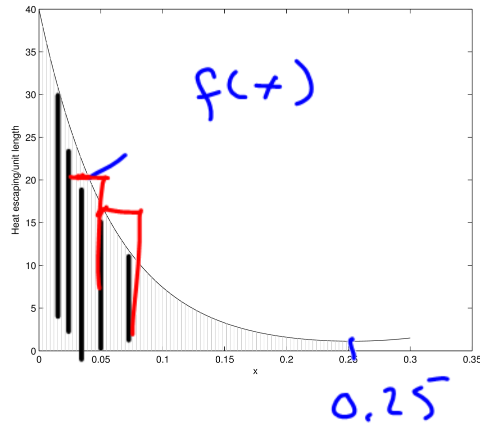
We will study the formula case initially, because it is easier to experiment with. We will continue to use our cooling fin example, where

$$Q = \int_0^L h\ P\ T(x)\ dx = 2.36269950112023\ \mathrm{J/s}$$

const variable

Graphically, $Q = \int_0^L \underbrace{h\;P\;T(x)}_{f(x)}\;dx$ is the area shown below:

$$= \int_0^L f(x)\,dx$$



Numerical integration is performed by:

- separating the desired interval into *panels*, and then
- on each *panel*, evaluating the integrand, $f(x)$ one or more times and those values are combined in some way to *estimate* the area of the panel.

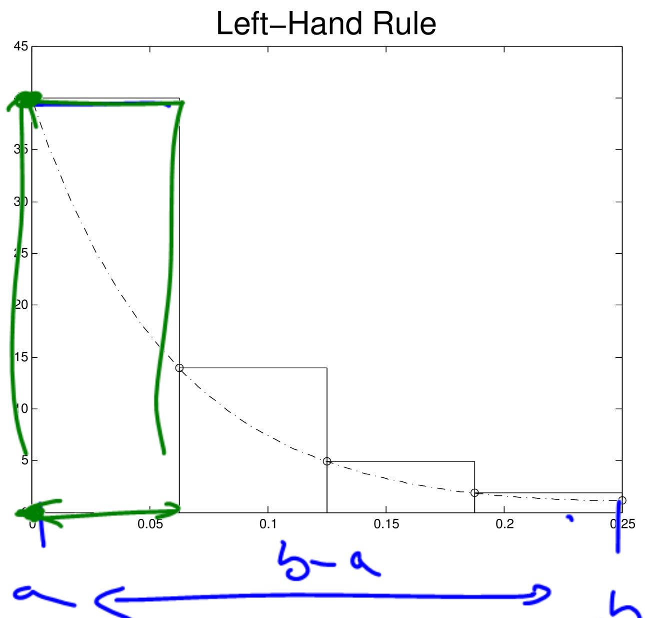
Left-Hand Sum

The simplest quadrature rule is one we have already seen: the LEFT(n) sum.

intervals panels

- Divide the interval into n panels, width $\Delta x = (b-a)/n$
- Evaluate the function at the left end point, $f(x_{i-1})$, on each panel.
- Compute the area of rectangles, $\displaystyle\sum_{i=1}^{N} f(x_{i-1}) \cdot \Delta x$ or

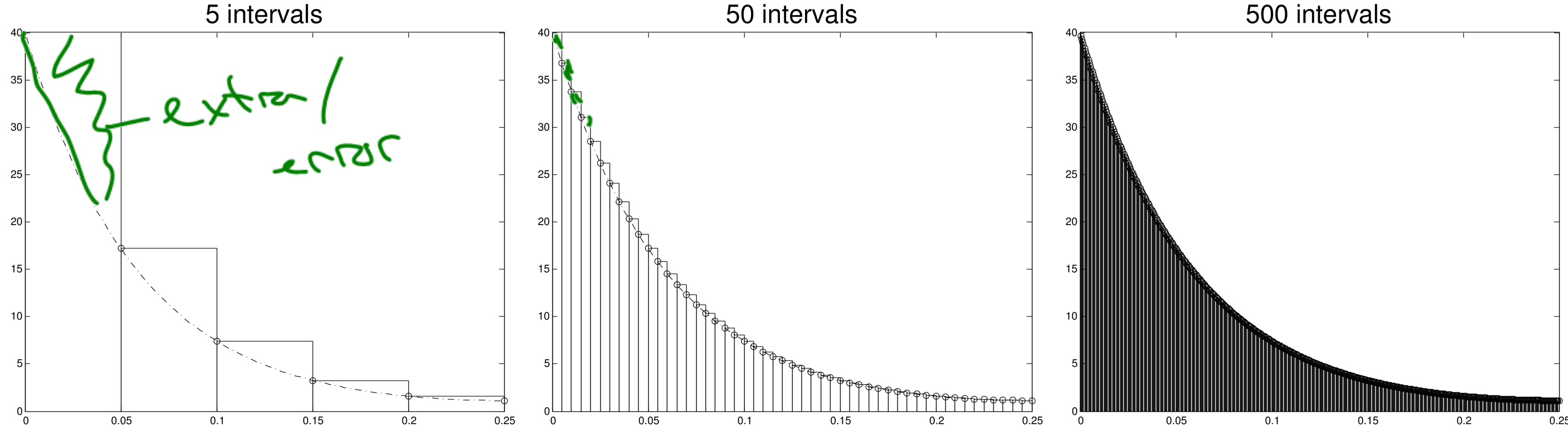
$\int_a^b f(x) \cdot dx \approx$ height · width



n = 4 example

Quadrature Principles

We are approximating a complex shape with simpler shapes for which we can compute the area. The **more panels** we use, the **more accurate** the area estimate will be:

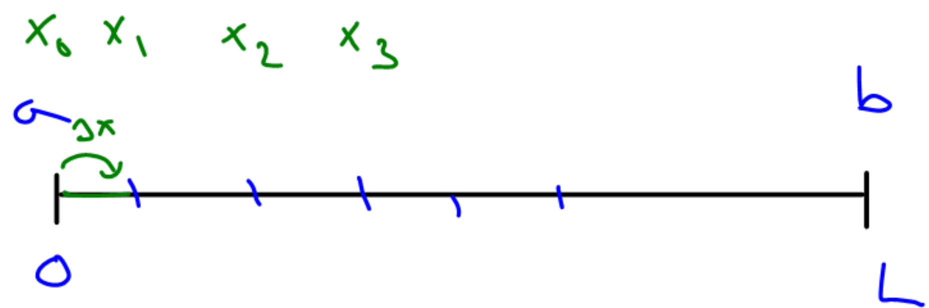


By using enough panels, we can reduce the error to any level we like, but with the trade-off that it takes longer to compute.

Problem. Download the file week06CoolingFin.m, and extend it so it plots the graph of the temperature along the fin.

Using MATLAB, estimate the integral $Q = \int_0^L hP\ T(x)\ dx$ with the left-hand rule,

$$Q \approx \sum_{i=1}^{n} \overbrace{hP\ T(x_{i-1})}^{f(x_{i-1})}\ \underline{\underline{\Delta x}}$$



$$\Delta x = \frac{L-0}{n}$$

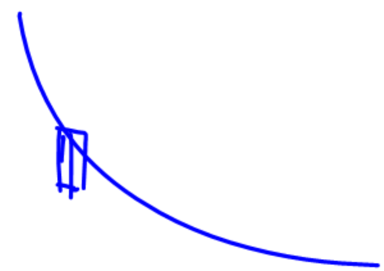
$$x_{i-1} = \text{xim1} = a + (i-1)\Delta x$$

$$i=1 \rightarrow \quad x_0 = \underset{0}{a} + (1-1)\Delta x$$

Problem. Add a statement that displays the number of intervals used, n, and the resulting error in the LEFT(n) integral estimate. Experiment by doubling the number of intervals and seeing the resultant reduction in error.

When you double the number of intervals, what happens to the error?

2x intervals $\rightarrow$ $\frac{1}{2}$ error.



Note: If you are using computer software for modeling in your career, it would be a good idea to get familiar with *numerical methods* and *numerical analysis* concepts.

Built-In Integration in MATLAB

Previously, we have used the LEFT(n) approximation to estimate the value of an integral, both by hand and using MATLAB to speed up the computation.

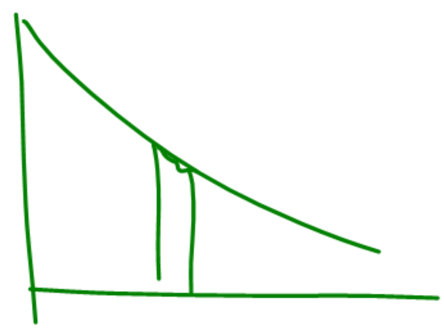
As you might expect, numerical integration is something we can do using built-in MATLAB functions, instead of writing our own LEFT(n) rule.

Problem. Look up the functions `quad` and `integral` in MATLAB help.

Problem. Starting with week06CoolingFin.m, use **`quad`** or **`integral`** to evaluate the rate of heat transfer from the cooling fin. Compare it with the exact integral value,

$$Q = \int_0^L \overbrace{h\ P\ T(x)}^{f(x)}\, dx = 2.36269950112023 \text{ J/s}$$

Experiment with additional options that could be used to increase the accuracy of the numerical integral estimate.



Numerical Integration in MATLAB - Examples

We now have two techniques we can try when evaluating definite integrals like $\int_a^b f(x)\,dx$ where the formula for $f(x)$ is given:

- The Fundamental Theorem of Calculus: finding an anti-derivative $F(x)$ then evaluating $F(b) - F(a)$; or

- Numerical integration tools.

LEFT(n)

"integral" in MATLAB

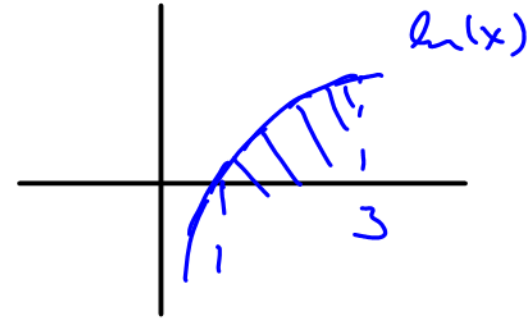
Problem. Use the Fundamental Theorem of Calculus to evaluate the integral

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

subs? ×

by parts

$$I = \int_1^3 \ln(x)\, dx.$$



$$u = \ln(x) \qquad \int dv = \int dx$$

$$\Big\downarrow \tfrac{d}{dx}$$

$$du = \frac{1}{x}\,dx \longleftarrow v = x$$

$$I = \ln(x)\cdot x \Big|_1^3 - \int_1^3 dx = \left(\ln(x)\cdot x - x\right)\Big|_1^3$$

$$= \left(3\ln(3) - 3\right) - \left(1\,\cancel{\ln(1)}^{\,0} - 1\right)$$

$$= 3\ln(3) - 2$$

Use the built-in numerical integration tools in MATLAB to estimate the value of

$$\int_1^3 \ln(x)\, dx.$$

Problem. Use the Fundamental Theorem of Calculus to evaluate the integral

subs ✓

$$\int_0^\pi x^2 \cos(x^3)\, dx.$$

let $w = x^3$

so $\frac{dw}{dx} = 3x^2$

or $\frac{1}{3x^2}\, dw = dx$

$$= \int_{x=0}^{x=\pi} \cancel{x^2} \cos(w) \left(\frac{1}{3\cancel{x^2}}\, dw\right)$$

$$= \frac{1}{3} \int_{x=0}^{x=\pi} \cos(w)\, dw$$

simple!

$$= \frac{1}{3} \sin(w) \Big|_{x=0}^{x=\pi} = \frac{1}{3} \sin(x^3) \Big|_0^\pi$$

$$= 1/3 \left(\sin(\pi^3) - \underbrace{\cancel{\sin(0)}}_{0}\right)$$

$$= 1/3 \sin(\pi^3)$$

Use the built-in numerical integration tools in MATLAB to estimate the value of

$$\int_0^\pi x^2 \cos(x^3)\, dx.$$

Problem. Try to use the Fundamental Theorem of Calculus to evaluate the integral

$$\int_{-2}^{2} e^{-x^2}\, dx.$$

subs

by parts

. . . .

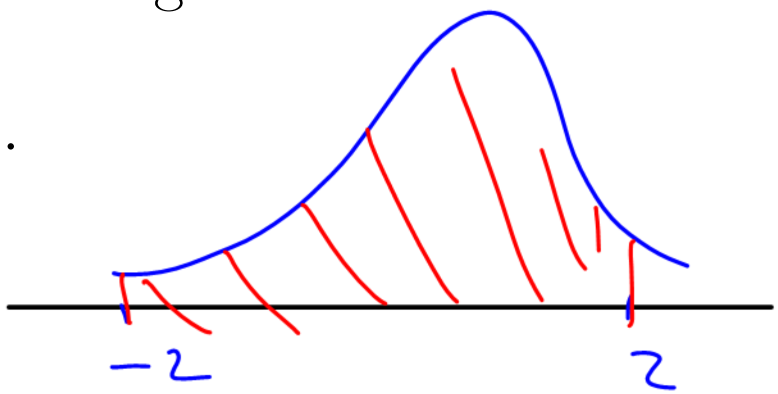
no simple antiderivative of e^{-x^2}

Problem. Use MATLAB to plot the graph of $f(x) = e^{-x^2}$ on the interval $x \in [-2, 2]$.

Sketch the graphical interpretation of the integral

$$\boxed{\text{area}} = \int_{-2}^{2} e^{-x^2}\, dx.$$

very well defined.



Use the built-in numerical integration tools in MATLAB to estimate the value of

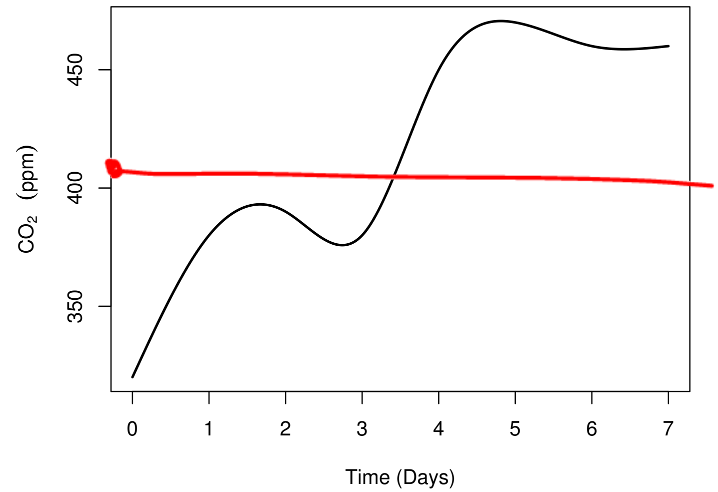
$$\int_{-2}^{2} e^{-x^2}\, dx.$$

$\boxed{1.764162781524843}$

Applications of Integrals - Average Value

Problem. If the following graph describes the level of CO_2 in the air in a mine over a week, estimate the *average level* of CO_2 over that period.

of a function

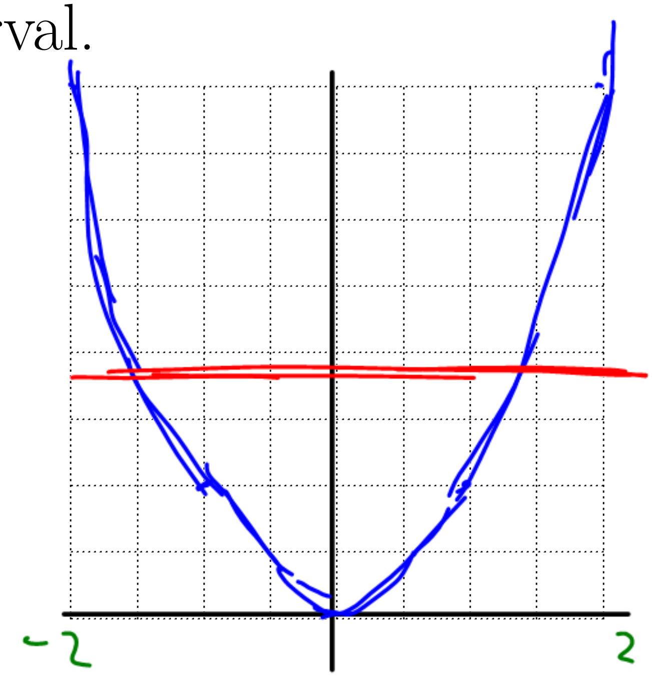


Problem. Give the units of the average CO_2 level.

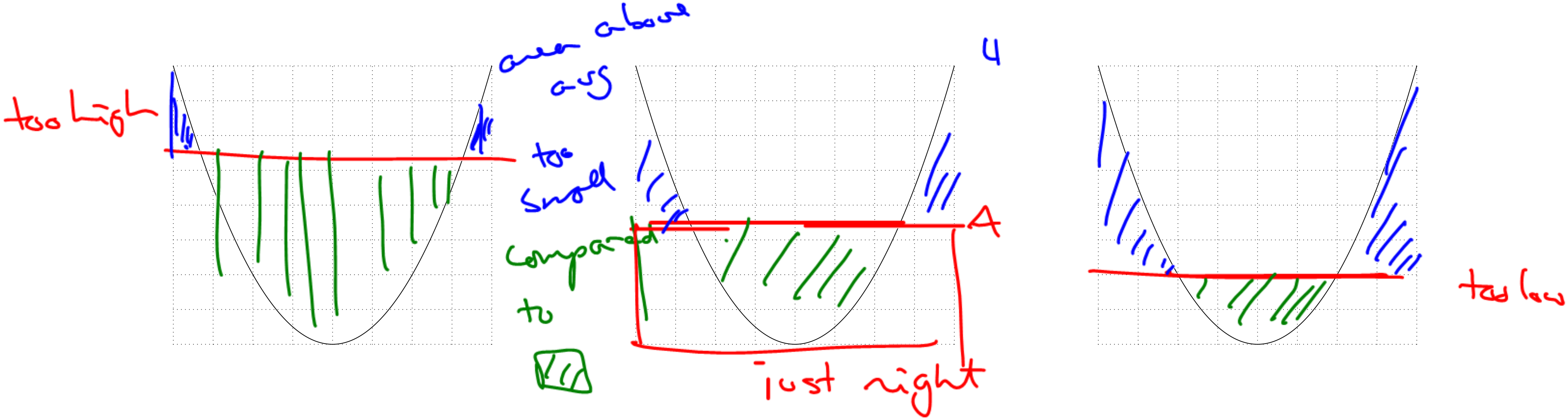
↳ ppm.

Problem. Sketch the graph of $f(x) = x^2$ from $x = -2$ to 2, and estimate the average value of f on that interval.

avg of f(x)
will be a single y value



Problem. What makes the single "average" f value different or distinct from other possible f values?



Problem. Use this property to find a general expression for the **average value of** $f(x)$ **on the interval** $x \in [a, b]$.

avg y satisfies $\int_a^b \left[A - f(x)\right] dx = 0$

rewrite $\int_a^b \overset{\text{const}}{A}\, dx = \int_a^b f(x)\, dx$

$\underset{\text{width}}{(b-a)}\ \underset{\text{height}}{A} = \underbrace{\int_a^b f(x)\, dx}_{\text{area}}$ under curved $f(x)$

$$A = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Problem. Find the exact average of $f(x) = x^2$ on the interval $x \in [-2, 2]$ using this formula.

a b

$$A = \frac{1}{2-(-2)} \int_{-2}^{2} x^2 \, dx$$

$$= \frac{1}{4} \left(x^3/3 \right) \Big|_{-2}^{2}$$

length of interval $\longrightarrow$ 4

$$= \frac{1}{12} \left(2^3 - (-2)^3 \right)$$

$$= \frac{1}{12} (8 + 8) = \frac{16}{12} = \frac{4}{3}$$

Problem. Use MATLAB to generate a graph of $f(x) = x^2$, compute the average value using the built-in integration tools, and then add a line at the average value. Use the same $x \in [-2, 2]$ interval.

Average Value of a Function on $[a, b]$

The average value of a function $f(x)$ on the interval $[a, b]$ is given by

$$\text{average height} \leftarrow A = \underbrace{\frac{1}{b-a}}_{1/\text{interval length}} \underbrace{\int_a^b f(x)\, dx}_{\text{area under the graph}}$$

$$(\text{width})(\text{avg height}) = \int_a^b f(x)\, dx$$

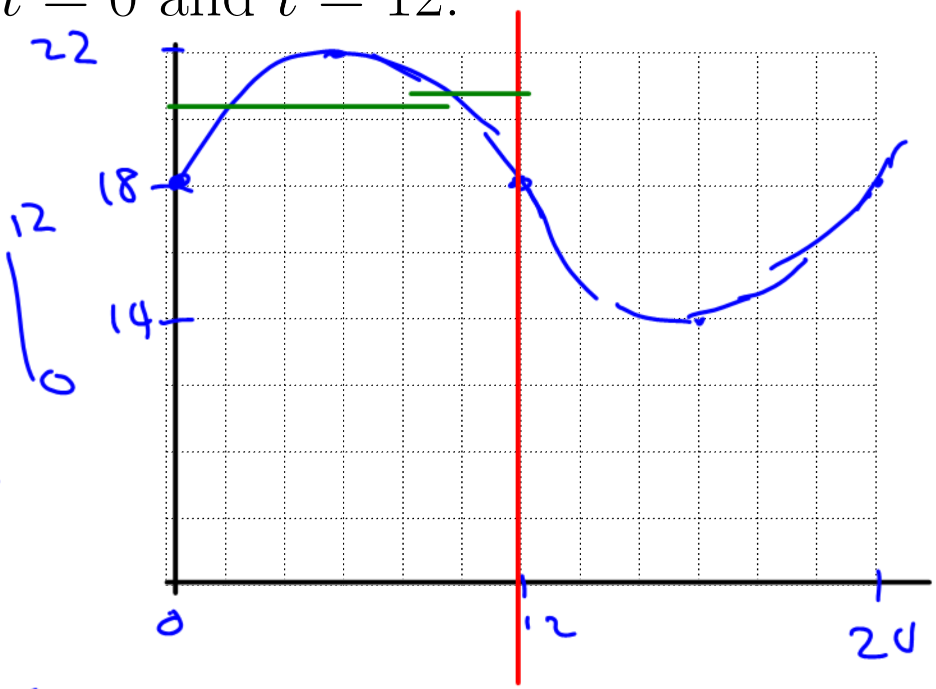
The temperature in a house is given by

$$H(t) = \underbrace{18}_{\text{vert'l shift}} + \underbrace{4}_{\text{amplitude}}\sin(\pi t/12),$$

$$\text{period} = \frac{2\pi}{\pi/12} = (2)(12) = 24 \text{ hours}$$

where t is in hours and H is degrees C.

Problem. Sketch the graph of $H(t)$ from $t = 0$ to $t = 12$, then find the average temperature between $t = 0$ and $t = 12$.



$$A = \frac{1}{12-0}\int_0^{12} 18 + 4\sin\left(\frac{\pi t}{12}\right)dt$$

$$\xrightarrow{d/dt}$$

$$= \frac{1}{12}\left(18t + 4\left(-\cos\left(\frac{\pi t}{12}\right)\right)\frac{12}{\pi}\right)\Bigg|_0^{12}$$

$$= \frac{3}{2}t - \frac{4}{\pi}\left(\cos\left(\frac{\pi t}{12}\right)\right)\Bigg|_0^{12}$$

$$= \left(\frac{3}{2}(12) - \frac{4}{\pi}\underbrace{\cos\left(\frac{12\pi}{12}\right)}_{=\cos(\pi)\,=-1}\right) - \left(0 - \frac{4}{\pi}\underbrace{\cos(0)}_{=1}\right)$$

$$= 18 + \frac{4}{\pi} + \frac{4}{\pi}$$

$$\approx 20.546\ ^\circ C$$

Problem. Repeat your analysis for the temperature function

$$H(t) = 18 + 4\sin(\pi t/12)$$

in MATLAB.

Include a graph of $H(t)$ over $t = 0 \ldots 12$ and a line showing the average value.

Average Value from Data

One of the major consumers of electricity in a mining operation is running the crushers. Being able to track the efficiency and power demand of these machines is therefore an important component of managing mine operations.

Sensors can record the power draw (in MW) over time, and record them to a spreadsheet.

Problem. Download the file week06CrusherData.csv and either open it in Excel, or double-click on it in the MATLAB File Viewer.

The data in this file contains data for one 24 hour period, in 2 columns:

- Time (in hours after midnight), and
- Power (in MW) for the crusher.

MATLAB can import Comma Separated Variable (CSV) files easily, though it does require some new-to-us tools for manipulating matrices.

Problem. Explain what each line in this MATLAB script does, and what each variable will contain.

```
M = csvread('week06CrusherData.csv')
```

← assign

t P

M =

```
t = M(:, 1);
```

all rows col

t =

```
P = M(:, 2);
```

col

all rows

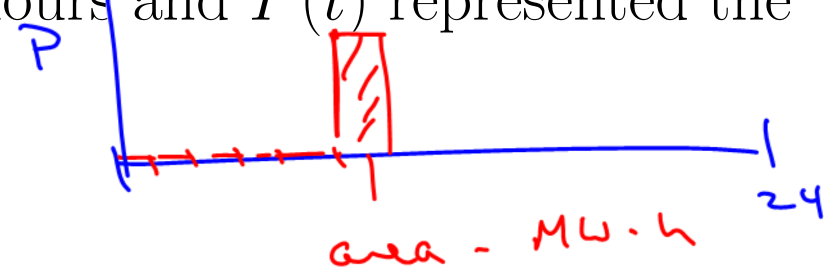
P =

```
plot(t, P);
```

x , y

energy → MW·h

Problem. How would the total ~~power~~ consumption over the day be represented mathematically, if t is in hours and $P(t)$ represented the power being used at hour t?



$$\sum P(t) \cdot \Delta t$$

MW · h

$$\hookrightarrow \int_0^{24} P(t)\,dt = \text{Total energy consumed over 24 hours (in MW·h)}$$

Can we compute this value using the Fundamental Theorem of Calculus?

↳ antiderive → sub in limits of integration.

No formula for $P(t)$.

All we have is measured $P(t)$.

Problem. Write out the Riemann sum approximation to the integral.

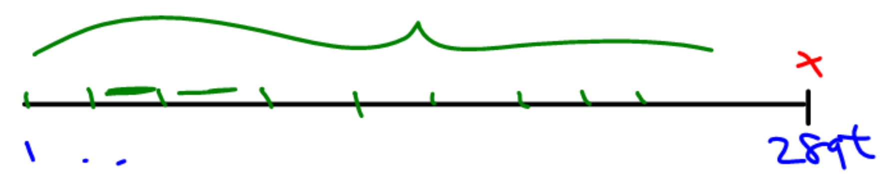
n - find # intervals

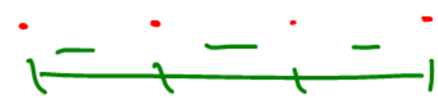
$$\text{Total Energy} \approx \sum_{i=1}^{n} P(t_{i-1})\,\Delta t$$

got

find

288 intervals





4 points,
3 intervals

Use MATLAB to estimate the total power consumed during the 24 recorded hours.

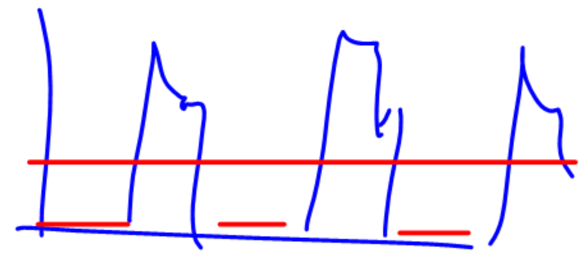
61,001 MW·h

Problem. Use our earlier formula for average values to estimate the **average** power consumption rate over 24 hours.

↳ MW

$$A = \underbrace{\frac{1}{24}}_{\text{div by \# hours}} \underbrace{\int_0^{24} P(t)\,dt}_{\text{total energy}} = \frac{1}{24}(61.001)$$

$$\cong 2.54 \text{ MW}$$



Given the data format in MATLAB, find the average power consumption rate in a different (more traditional) way.

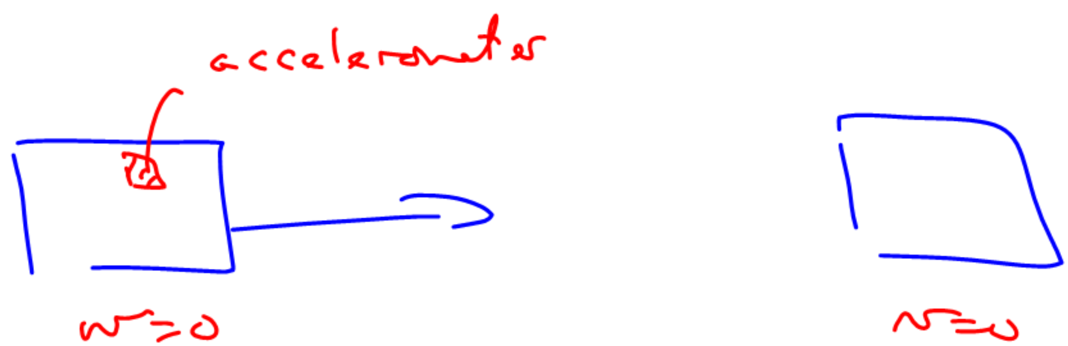
Plot both the power consumption over time, and the average power usage graphically.

Integration and Averages from Data - Acceleration

Accelerometers are now relatively inexpensive sensors that can be incorporated in machinery and can be used to infer changes in velocity and changes in position.

In testing a new collection accelerometer sensors, you attach them to a cart and record the acceleration over a 6 minute test interval, running the cart at several different motor powers as the cart moves from a minehead to a dropoff area. The cart starts at rest, and is stopped (zero velocity) before the end of the recording.

The acceleration data is recorded in the file week06CartAcceleration.csv



Problem. In MATLAB, load the data from week06CartAcceleration.csv and plot the acceleration against time graph.
Time is recorded in seconds, and acceleration in m/s^2.
Add a black line at $a = 0$ so you can see forwards vs backwards accelerations more easily.

Describe in words what the motion of the cart would have looked like, based just on the acceleration records.

accel once quickly / short time
coasted . . .
two weaker accel, then one very sharp accel.
long coast
braking

Problem. Write an integral that represents the total change in velocity for the cart over the six minutes.

Net Change Thm

accel

$$\Delta v = \int_0^{360} \underbrace{\underset{(m/s)/s}{v'(t)}\ \underset{\cdot\, s}{dt}}_{m/s} = \int_0^{360} a(t)\cdot dt$$

Can we compute this value using the Fundamental Theorem of Calculus?

We only have measured values for 'a', but we would need a formula for F.T.C.

Problem. Write out the Riemann sum approximation to the integral that computes the change in velocity.

$$\Delta v = \int_0^{360} a(t)\,dt \approx \sum a(t_i)\,\Delta t$$

Use MATLAB to estimate the total change in velocity during the recording hours.

Δv from $\sum = -0.005$ m/s
over 6 minutes

Comment on the accuracy of the sensors.

Know $\Delta v = 0$ based on
the experiment.

Problem. Use our earlier formula for average values to estimate the **average** acceleration over the recording time.

$a \cong -1.4 \times 10^{-5}$

Based on the scenario that was given, should this value be zero as like the net change in velocity? Why or why not?

expect 0 = average accel $= \dfrac{1}{\text{time span}} \Delta v$

$= \dfrac{1}{\text{time span}} \underbrace{\int_0^{360} a(t)\,dt}_{\Delta v \text{ expect} = 0}$