Week #10: Linear Algebra I

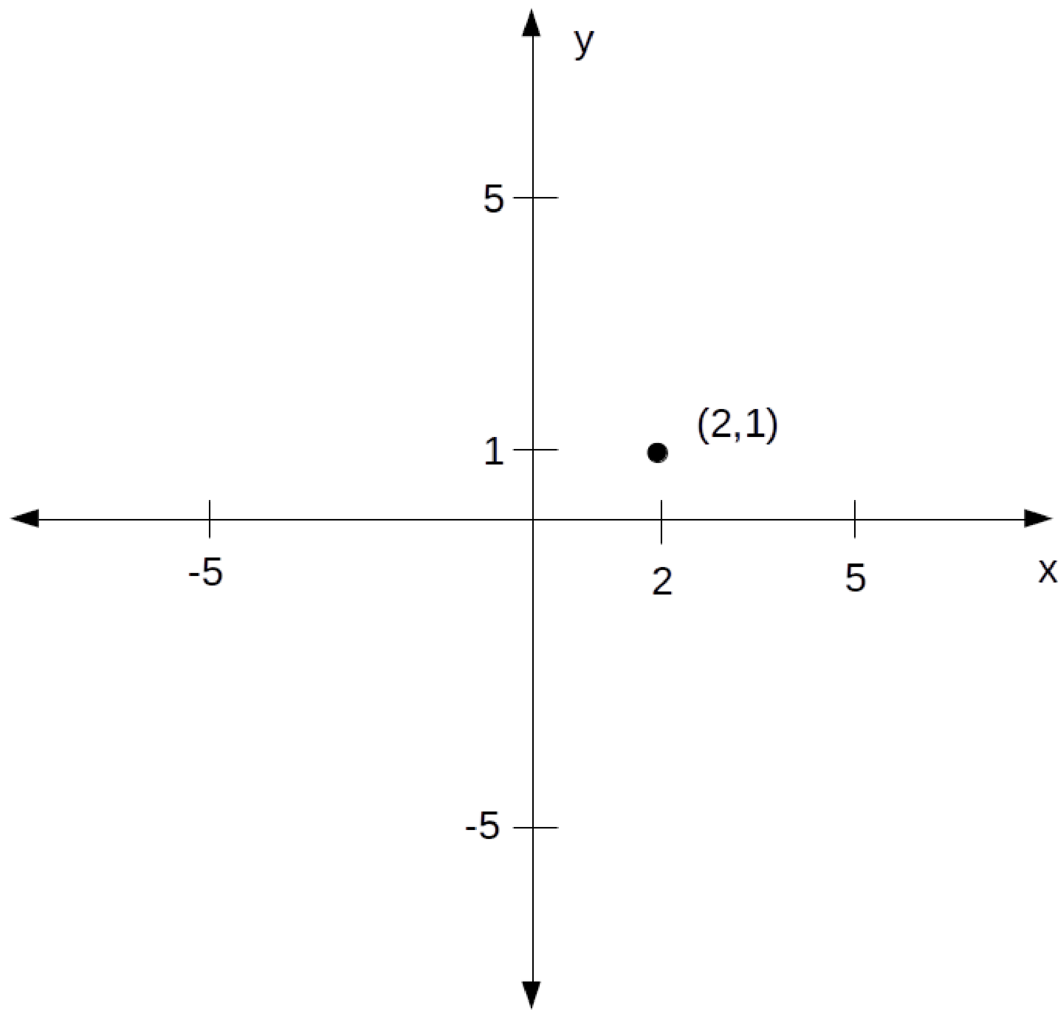
Goals:

- Express vectors, linear combinations, and compute dot products.
- Write information in matrix form in the context of engineering applications.
- Understand the definition of the transpose, and use MATLAB to compute it.
- Use MATLAB to compute the inverse of a matrix.

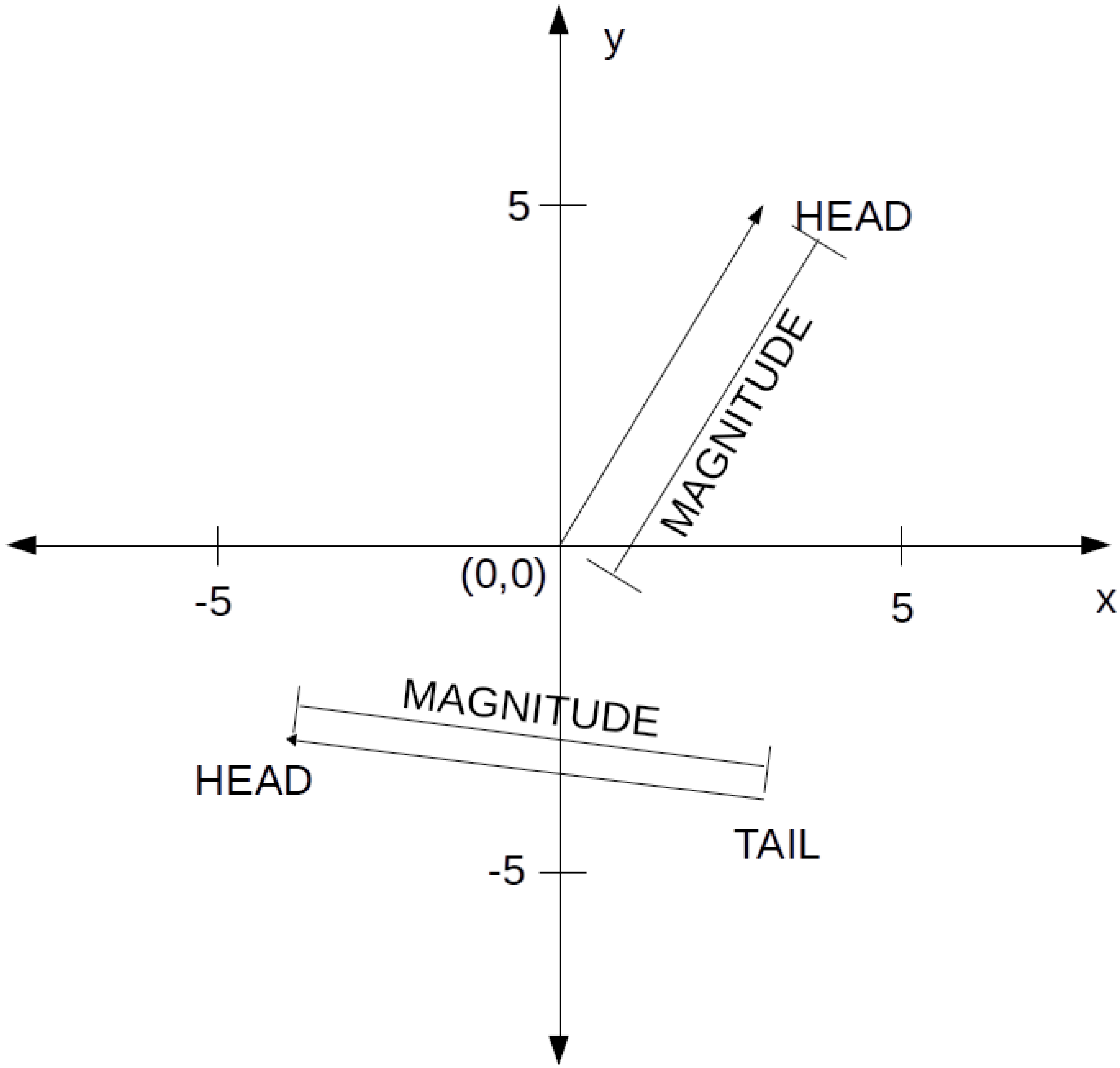
Introduction to Vectors

In a physical context we often need to describe measurable quantities such as pressure, mass, and speed, and these objects can be completely described by a single number known as the **magnitude**. However there are other quantities which help us describe the world around us such as force, velocity and acceleration that require more than magnitude to describe them. They also need a **direction**. A **vector** is a magnitude (a number describing how much, how fast, etc) in combination with a direction. Vectors are denoted by letters with arrows above them, such as $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{u}$, $\overrightarrow{v}$.

When discussing direction, we need to decide on a frame of reference. We will be using the Cartesian coordinate system. A *point* in the two-dimensional Cartesian plane, denoted as $\mathbb{R}^2$ is denoted by an ordered pair (x,y) of real numbers, which we call *coordinates*. So for example, (2,1) can be represented as a point on a grid like so:



Consider the point (x,y) in $\mathbb{R}^2$ (meaning any point on the plane that we can draw). If we draw a directed line segment from the **origin** which is the point (0,0), to (x,y), we get the following picture:
Notice there is an arrow pointing at (x,y). We call this the **head** of the vector and the **tail** is at the origin, indicating a direction. A vector's tail does not need to start at the origin, as it can be seen in the next picture. The **magnitude** of the vector is its length from (0,0) to (x,y).


y
5
HEAD
MAGNITUDE
(0,0)
-5
5
x
MAGNITUDE
HEAD
TAIL
-5

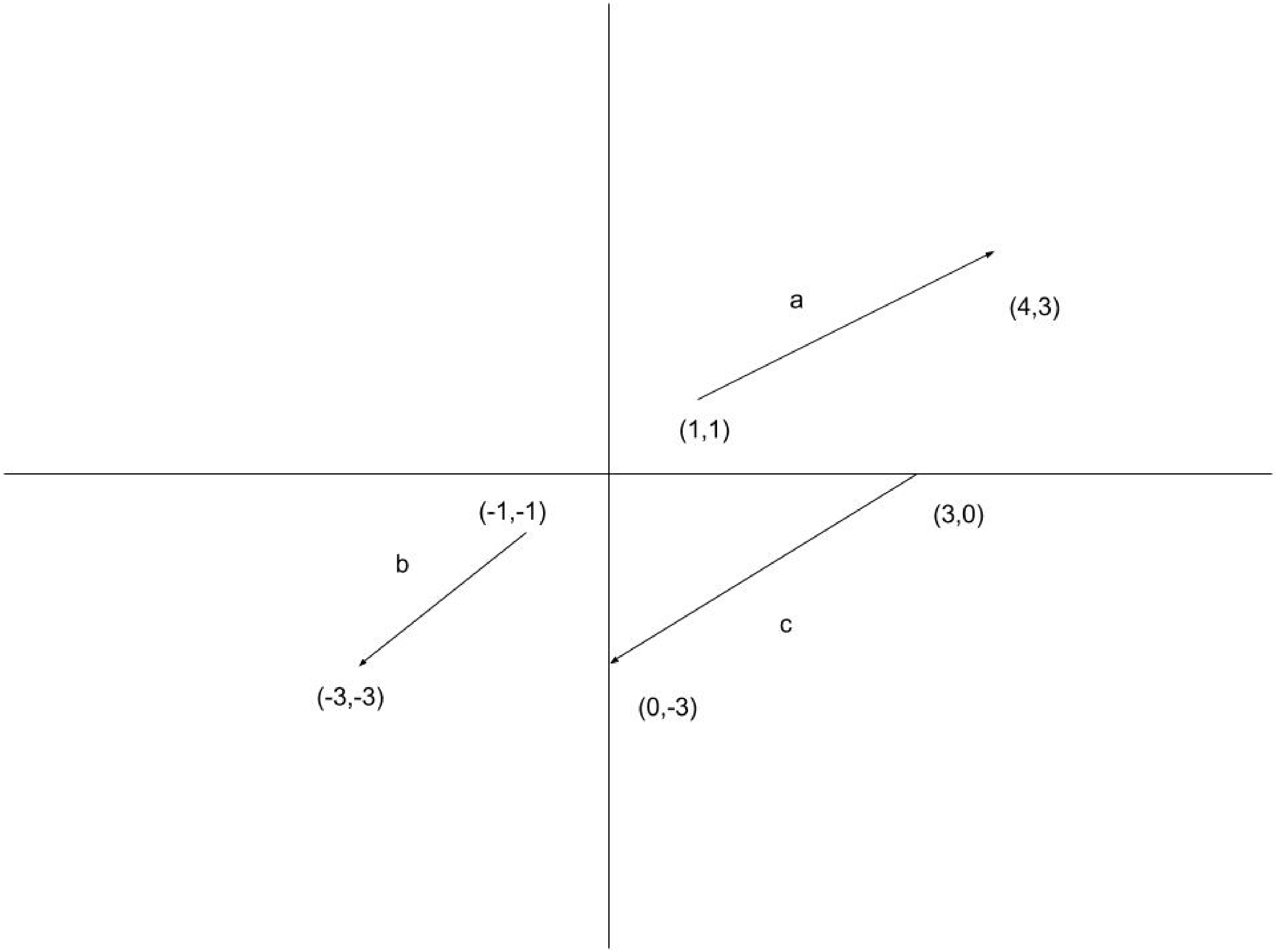
To find the magnitude of a vector, we first find its **components**. If the head of the vector is at the point (x', y') and the tail of the vector is at the point (x,y), then the components of the vector are $x' - x$ and $y' - y$, and we represent them in the following way:

$$\begin{bmatrix} x' - x \\ y' - y \end{bmatrix}$$

Once you have found the components, the formula for finding the magnitude of the vector is:

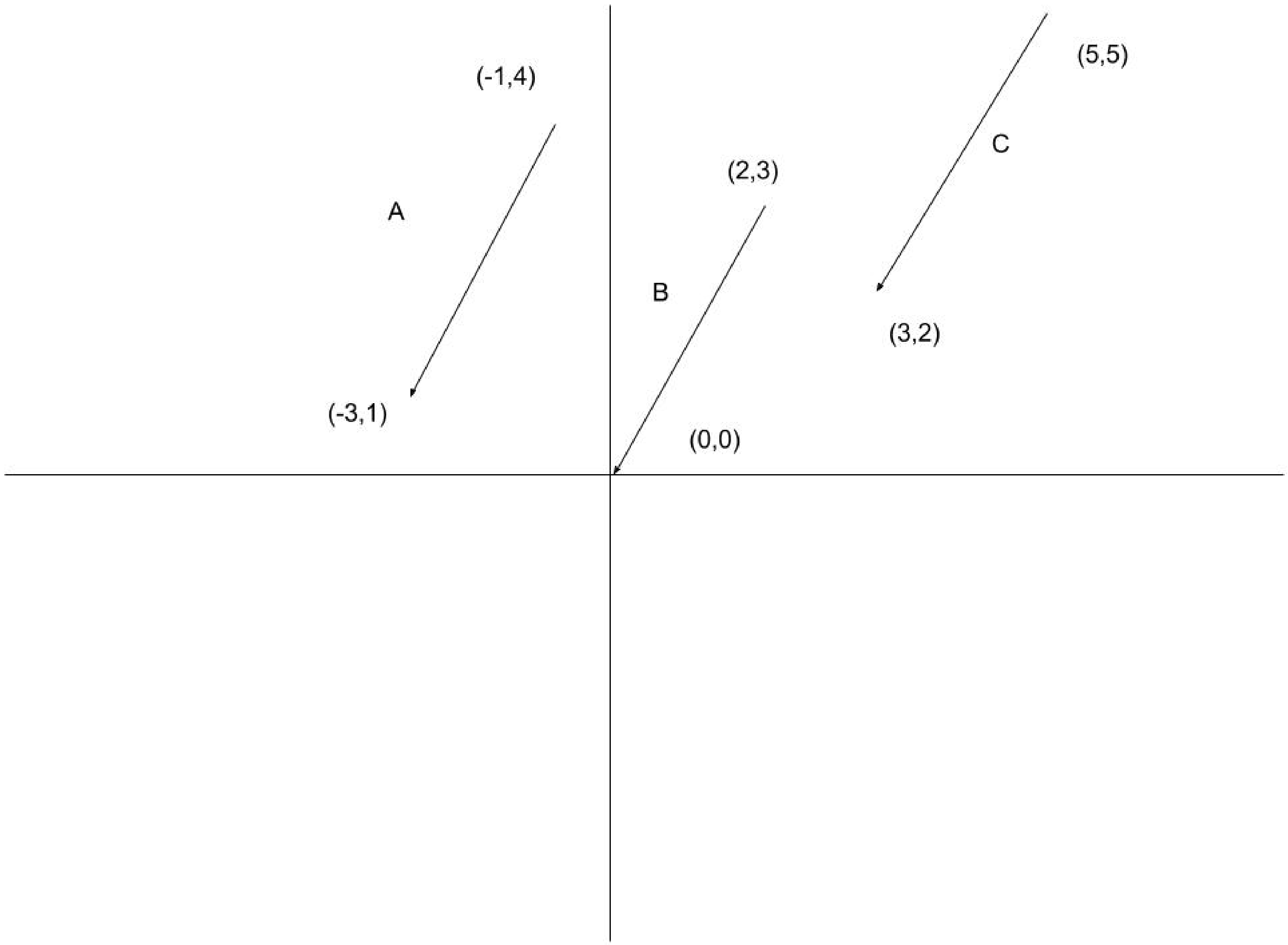
$$\text{magnitude} = \sqrt{(x' - x)^2 + (y' - y)^2}$$

Draw and find the magnitude of the following vectors: $\overrightarrow{a}$ has the tail at (1,1) and the head at (4,3), $\overrightarrow{b}$ has the tail at (-1,-1) and the head at (-3,-3), and $\overrightarrow{c}$ has the tail at (3,0) and the head at (0,-3). We use the notation $||\overrightarrow{a}||$ to represent the magnitude of $\overrightarrow{a}$.



$$||a|| = \sqrt{13},\ ||b|| = \sqrt{8},\ ||c|| = \sqrt{18}$$

An interesting fact to note is that two vectors are **equal** if their components are equal. Consider the vector $\overrightarrow{A}$ going from (-1,4) to (-3,1) (tail to head), $\overrightarrow{B}$ going from (2,3) to (0,0) and $\overrightarrow{C}$ going from (5,5) to (3,2). All of these vectors occupy the different regions of $\mathbb{R}^2$, yet we can define them as equal since they have the same components. A **position vector** is a vector whose tail starts at the origin, and if a vector is given with just its components, you may assume it is a position vector. Try drawing the vectors and see if they have the same magnitude and direction.



The magnitudes are all $\sqrt{13}$.

Vector Addition and Scalar Multiplication

If we have two vectors in $\mathbb{R}^2$ with components

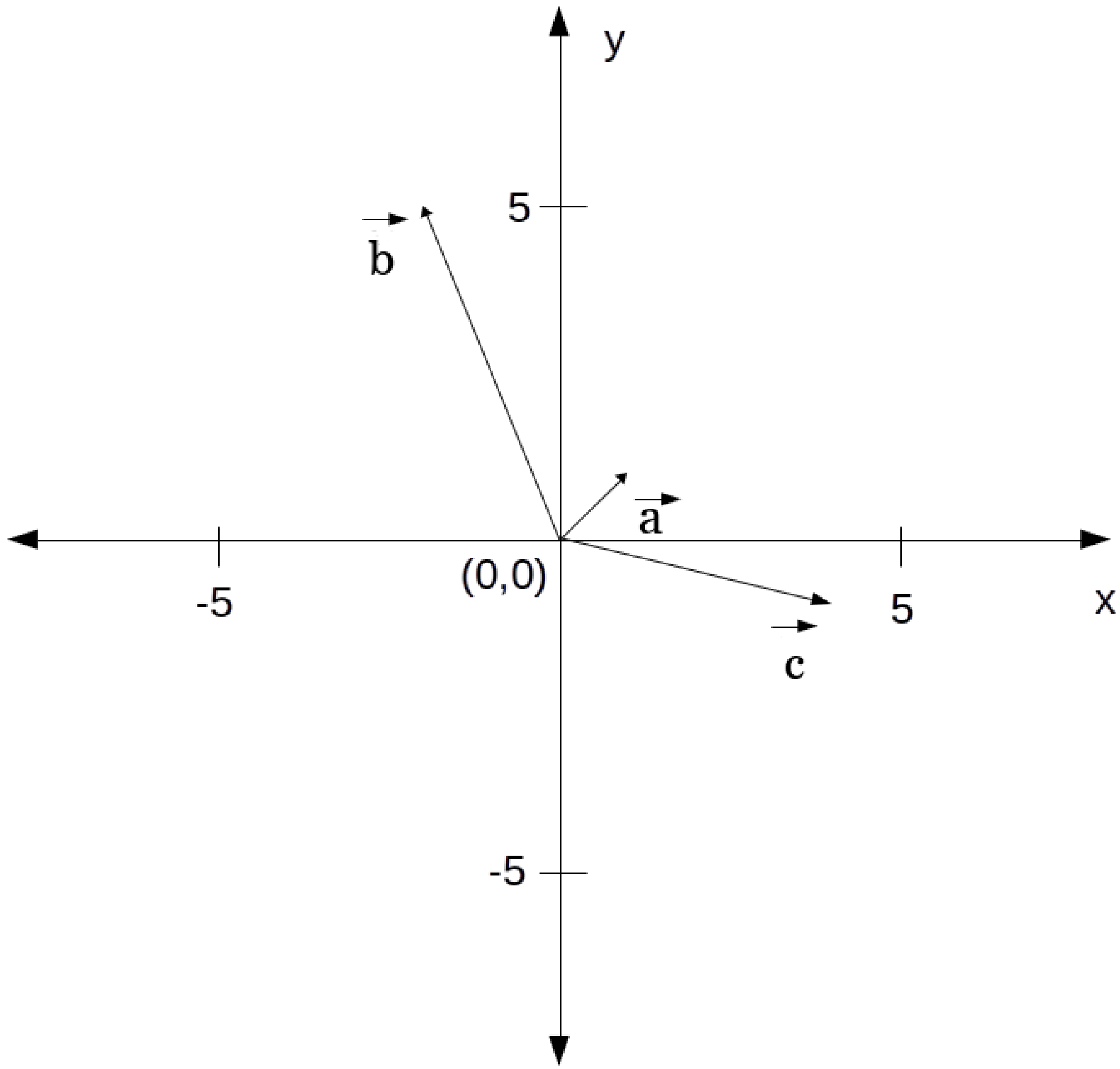
$$\overrightarrow{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ and } \overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

then the **sum** of the vectors $\overrightarrow{a}$ and $\overrightarrow{y}$ is

$$\overrightarrow{a} + \overrightarrow{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

So if we have vectors

$$\overrightarrow{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \overrightarrow{b} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \text{ and } \overrightarrow{c} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$



Then

$$\overrightarrow{a} + \overrightarrow{b} = \begin{bmatrix} 1 + (-2) \\ 1 + 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

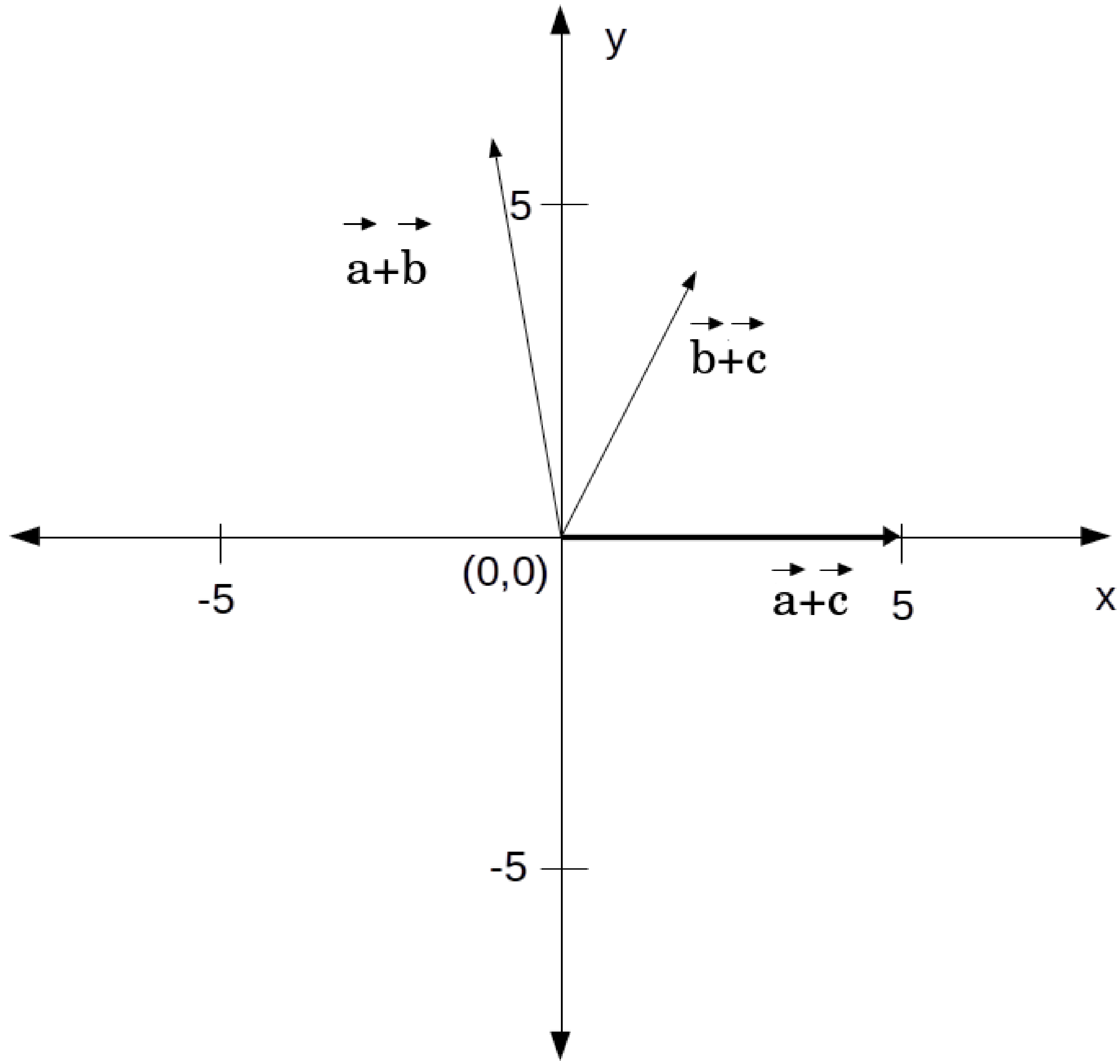
Or we could have

$$\overrightarrow{b} + \overrightarrow{c} = \begin{bmatrix} -2 + 4 \\ 5 + (-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

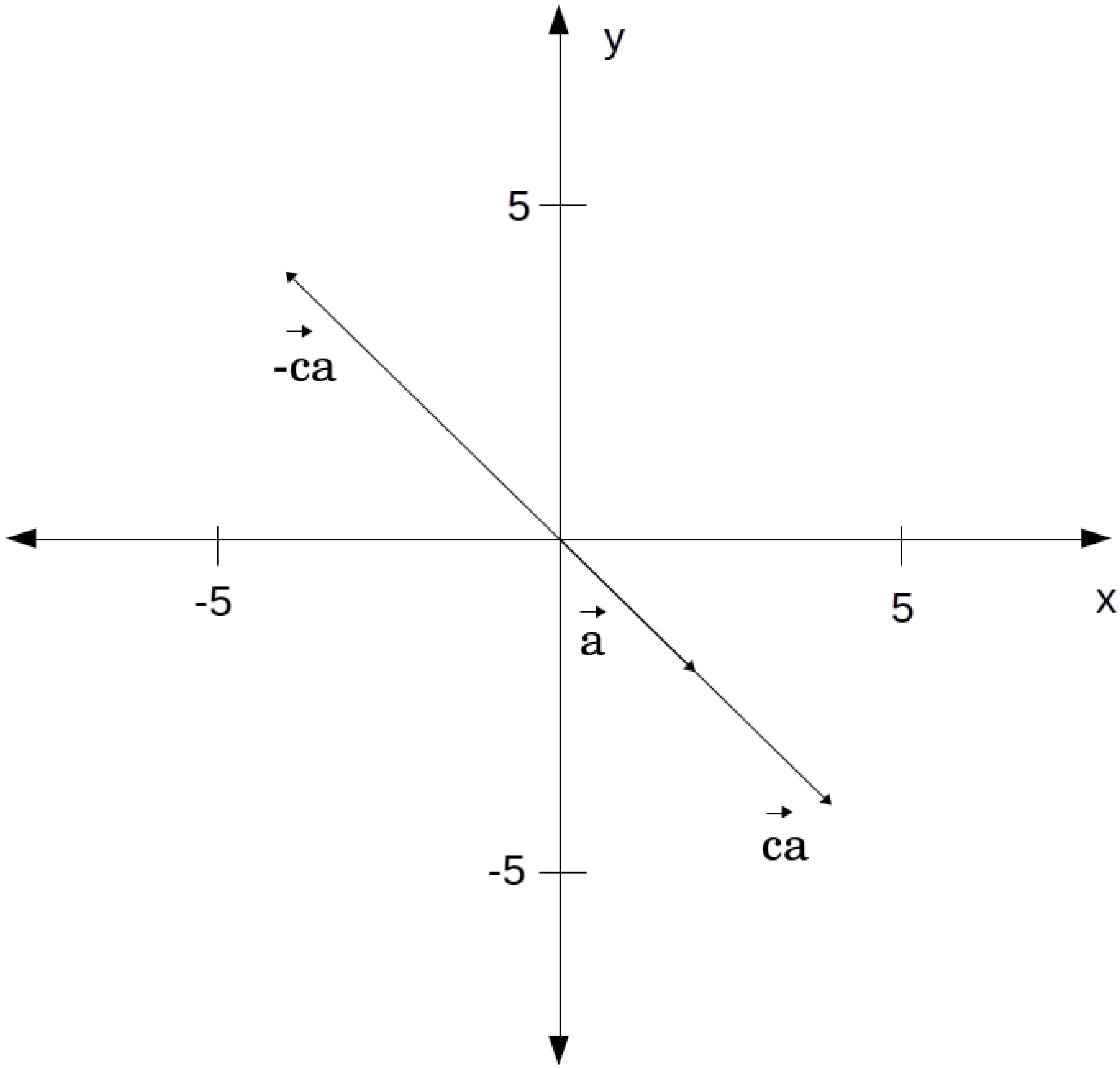
And

$$\overrightarrow{a} + \overrightarrow{c} = \begin{bmatrix} 1 + 4 \\ 1 + (-1) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

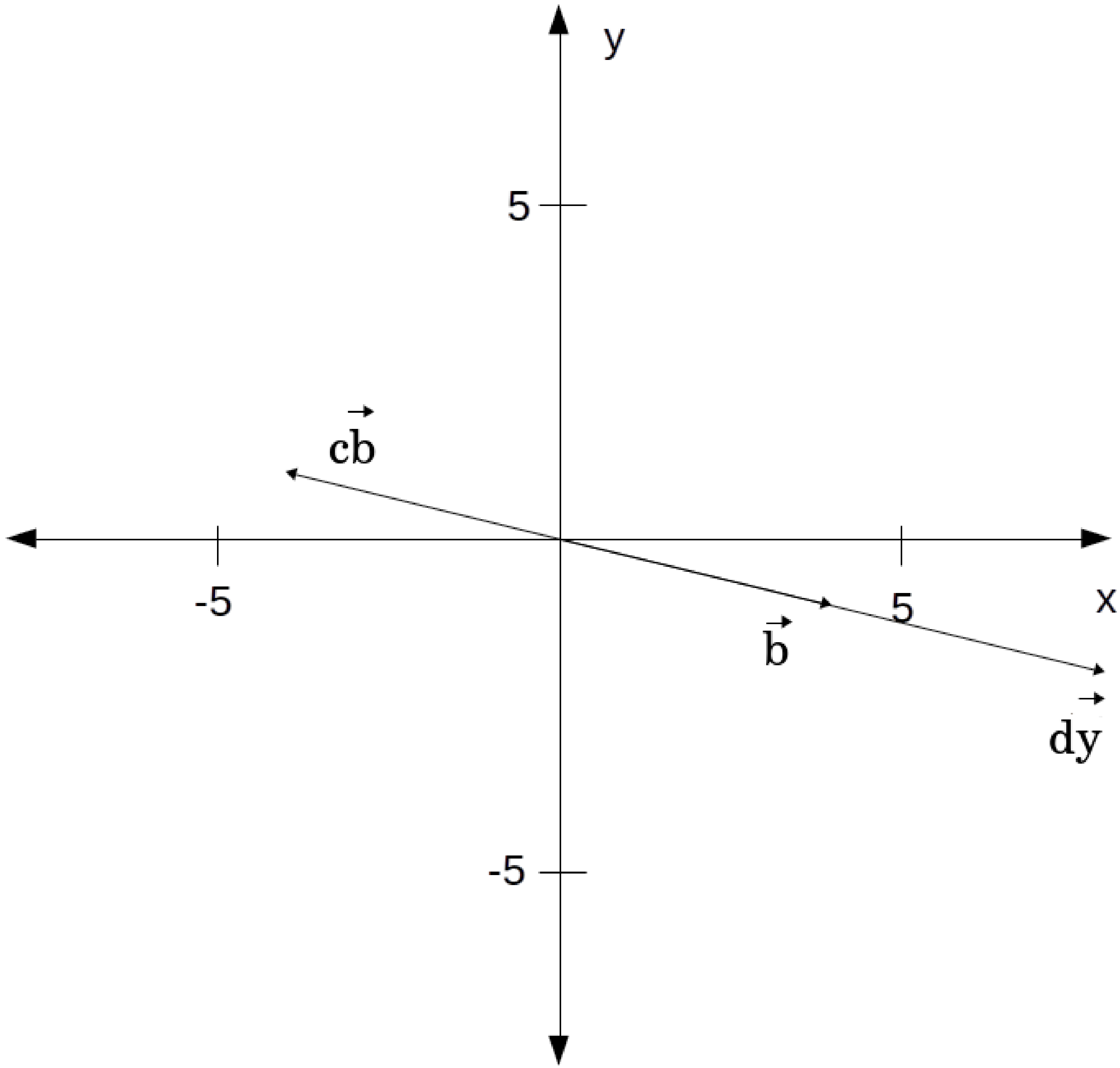
We can interpret vector addition geometrically.



If $\overrightarrow{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is a vector and c is a scalar (meaning a real number), then the **scalar** multiple $c\overrightarrow{a}$, meaning every component of $\overrightarrow{a}$ is multiplied by c, is $c\overrightarrow{a} = \begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix}$. If $c > 0$, then $c\overrightarrow{a}$ is in the same direction as $\overrightarrow{a}$. If $c < 0$, then $c\overrightarrow{a}$ is in the opposite direction as $\overrightarrow{a}$.

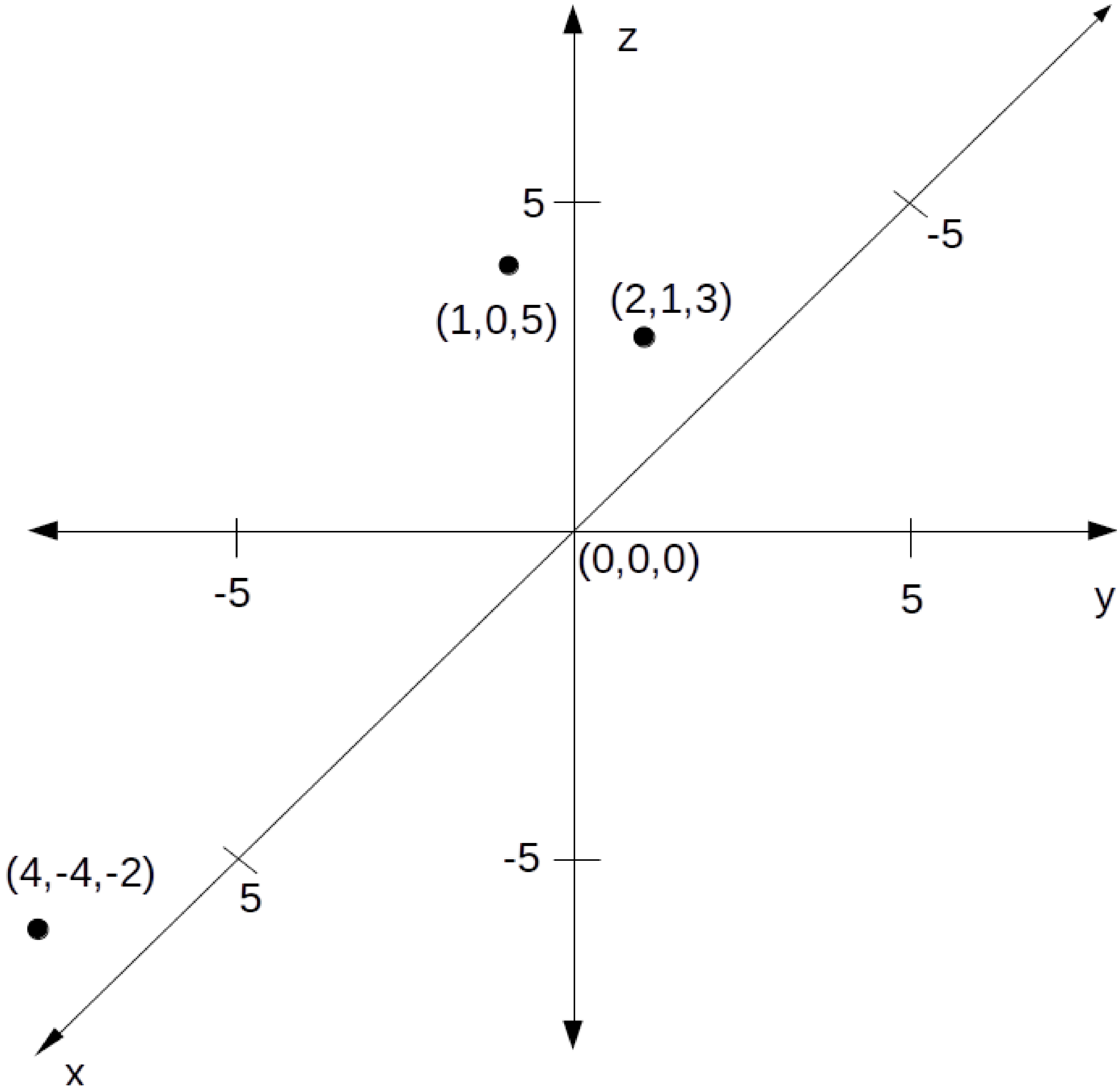

y
5
-5
-5
5
x
$\vec{a}$
$-c\vec{a}$
$c\vec{a}$

For example, if $c = -1$, $d = 2$ and $\overrightarrow{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (Recall that $\overrightarrow{b}$ is a position vector, whose tail starts at the origin), then $c\,\overrightarrow{b} = \begin{bmatrix} (-1)\cdot(4) \\ (-1)\cdot(-1) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $d\,\overrightarrow{b} = \begin{bmatrix} (2)\cdot(4) \\ (2)\cdot(-1) \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$ We get the following picture in the next slide.


y
5
-5
5
x
-5
$\vec{cb}$
$\vec{b}$
$\vec{dy}$

3-Dimensional Vectors

3-dimensional (also known as 3-space) vectors exist in $\mathbb{R}^3$, meaning that we are now dealing another axis, the z-axis. Just like in $\mathbb{R}^2$, there is an origin where all of the axes meet, (0,0,0). Points in $\mathbb{R}^3$ are represented by an ordered triplet (x, y, z). The points (2,1,3), (4, -4, -2), and (1,0,5) would be drawn like so:


z
5
-5
(1,0,5)
(2,1,3)
(0,0,0)
-5
5
y
(4,-4,-2)
5
-5
x

Components for 3-dimensional vectors are defined in the same way as 2-dimensional vectors, except that now there are three of them. So for a vector $\overrightarrow{u}$ whose tail starts at the point (x,y,z) and has its head at the point (x',y',z'), the components of $\overrightarrow{u}$ are:

$$\begin{bmatrix} x' - x \\ y' - y \\ z' - z \end{bmatrix}$$

A 3-dimensional vector whose components are given without information about the head or the tail is a position vector, and you can assume its tail starts at the origin.

All of the rules for vector addition and scalar multiplication we presented for 2-dimensional vectors are the same for the 3-dimensional versions. If we have position vector $\overrightarrow{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ then the magnitude of $\overrightarrow{a}$ is

$$||\overrightarrow{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Keep in mind that we get this formula from the fact that the tail of $\overrightarrow{a}$ is at the origin (0,0,0).

Find the magnitude of the following vectors:

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and

$$\vec{v} = \begin{bmatrix} -4 \\ -2 \\ -1 \end{bmatrix}$$

and

$$\vec{w} = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}$$

$$||u|| = \sqrt{14},\ ||v|| = \sqrt{21},\ ||w|| = \sqrt{50}$$

Unit Vectors

A vector is called a **unit vector** if it has a length of 1. Often, we want to find the unit vector of some given vector $\overrightarrow{a}$. You can find the unit vector using the following formula.

$$\overrightarrow{u} = \frac{1}{||\overrightarrow{a}||}\overrightarrow{a}$$

First, find the length of $\overrightarrow{a}$, then divide every component of $\overrightarrow{a}$ by that length. This is called **normalizing** a vector.

For example, if we want to normalize the vector $\overrightarrow{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, first we find the length of $\overrightarrow{b}$.

$$||\overrightarrow{b}|| = \sqrt{4^2 + 3^2} = 5$$

So the unit vector $\overrightarrow{u}$ of $\overrightarrow{b}$ is

$$\overrightarrow{u} = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

Find the unit vectors of the following vectors

$$\overrightarrow{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and

$$\overrightarrow{v} = \begin{bmatrix} -4 \\ -2 \\ -1 \end{bmatrix}$$

and

$$\overrightarrow{w} = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}$$

$$\overrightarrow{u}_u = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$$

and

$$\overrightarrow{v}_u = \begin{bmatrix} \frac{-4}{\sqrt{21}} \\ -\frac{2}{\sqrt{21}} \\ \frac{-1}{\sqrt{21}} \end{bmatrix}$$

and

$$\overrightarrow{w}_u = \begin{bmatrix} \frac{3}{\sqrt{50}} \\ \frac{5}{\sqrt{50}} \\ \frac{-4}{\sqrt{50}} \end{bmatrix}$$

The Standard Basis

The standard basis is a set of unit vectors for a given space. For 2-dimensional space, the standard basis is made up of the vectors

$$\overrightarrow{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \overrightarrow{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

in 3-dimensional space the standard basis is given as

$$\overrightarrow{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \overrightarrow{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Any 2 or 3 dimensional vector can be represented as a combination of the standard basis. For example

$$\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = 2\overrightarrow{i} + 3\overrightarrow{j} - 4\overrightarrow{k}$$

Represent the following vectors using a combination of the standard basis.

$$\overrightarrow{a} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \overrightarrow{b} = \begin{bmatrix} 4 \\ -2 \end{bmatrix} \quad \overrightarrow{c} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{a} = 2\overrightarrow{i} + \overrightarrow{k}$$

$$\overrightarrow{b} = 4\overrightarrow{i} - 2\overrightarrow{j}$$

$$\overrightarrow{a} = -1\overrightarrow{i} - 1\overrightarrow{j}$$

The Dot Product

Consider two vectors

$$\overrightarrow{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ and } \overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

in $\mathbb{R}^2$. The **dot product** is defined as

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_1 b_1 + a_2 b_2$$

and if $\overrightarrow{u}$ and $\overrightarrow{v}$ are 3-space vectors so that

$$\overrightarrow{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } \overrightarrow{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

then the dot product is

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

The dot product is important because it allows us to find the angle between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, using the following formula

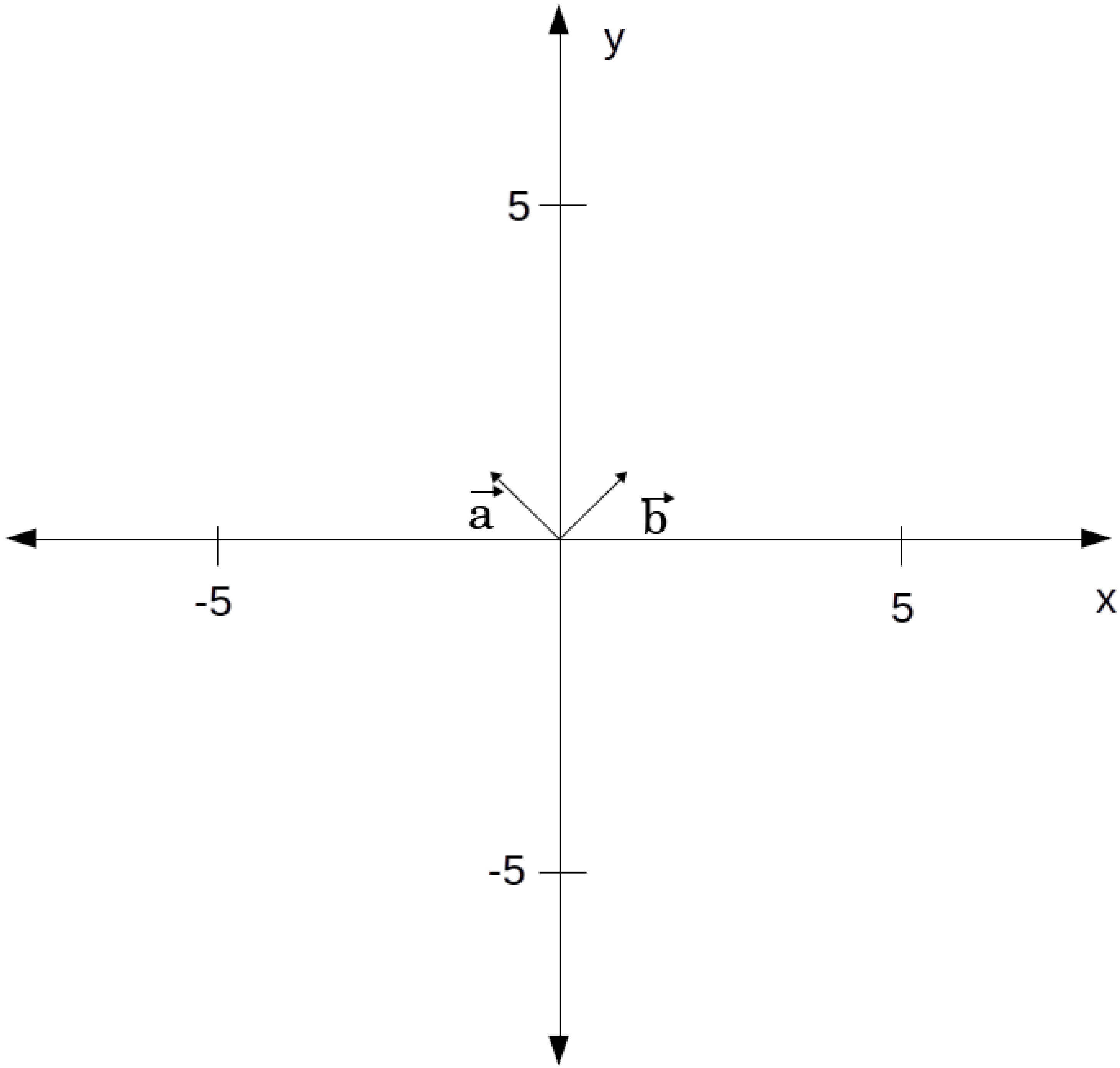
$$cos(\theta) = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{||\overrightarrow{a}||||\overrightarrow{b}||}$$

One of the most useful properties of the dot product comes from the following fact. Recall that the $\cos(90)^\circ = 0$. So if the dot product of two vectors equals 0, that means they must be at a right angle to each other! Two vectors that are at a right angle in three dimensional space are called **orthogonal**.

If we let $\overrightarrow{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, are these two vectors orthogonal? We can take the dot product to find out!

$$\overrightarrow{a} \cdot \overrightarrow{b} = 1 \cdot 1 + 1 \cdot (-1) = 1 - 1 = 0$$

Since the dot product between these two vectors is zero, they are at a right angle to one another. If we draw the picture, we can see the algebra matches what the picture tells us.


y
5
-5
5
-5
x
$\vec{a}$
$\vec{b}$

Try out these pairs of vectors to see if they are orthogonal:

$$\overrightarrow{a} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \text{ and } \overrightarrow{b} = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

$$\overrightarrow{c} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \text{ and } \overrightarrow{d} = \begin{bmatrix} -5 \\ -5 \\ -2 \end{bmatrix}$$

$$\overrightarrow{e} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \text{ and } \overrightarrow{f} = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}$$

$\overrightarrow{a} \cdot \overrightarrow{b}$ not orthogonal
$\overrightarrow{c} \cdot \overrightarrow{d}$ not orthogonal
$\overrightarrow{e} \cdot \overrightarrow{f}$ orthogonal

What is a Matrix?

An $m \times n$ (you say "m by n") **matrix** is a rectangular block of real numbers that has m rows and *columns*. It is standard to give a matrix a name like a variable or a vector, but to indicate it is a matrix we use a capital, bold-faced letter. For example,

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 7 & 2 \\ 1 & -2 \end{pmatrix}$$

The matrix $\mathbf{A}$ has 3 rows and 2 columns, so we would say that $\mathbf{A}$ is a 3×2 (3 by 2) matrix. If we are discussing more than one matrix, we say **matrices**.

Matrices of Interest

A matrix is called **square** if it has the same number of rows as columns, and is called **rectangular** if the number of rows is different than the number of columns.

The **identity matrix of size n**, denoted $\mathbf{I}_n$ is a square matrix that has n rows and n columns with the property that every number in the matrix is 0 except for the **diagonal**, which has all 1's.

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ \mathbf{I}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that the identity matrix is composed of the standard basis for any sized dimensional space.

Writing Information in Matrix Form

A matrix is a useful way of collecting and displaying information. If you have information that can be represented as a few (or many) vectors, combining them all into matrix form allows you to do all sorts of operations on that matrix that can give you quick answers to certain kinds of questions. What sort of questions can be answered using a matrix and linear algebra will be discussed in the next week's notes.

For now, let's focus on turning information into a matrix. Consider the following example: A mining company has two mines. One's day operation at mine #1 produces ore that contains 30 metric tons of copper and 600 kg of silver, while one day's operation at mine #2 produces ore that contains 40 metric tones of copper and 380 kg of silver. Let $v_1 = \begin{pmatrix} 30 \\ 600 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 40 \\ 380 \end{pmatrix}$. Then, v_1 and v_2 represent the output per day at mines #1 and #2, respectively. If we wanted to represent the total amount being produced by both mines per day, we can place the vectors together in a matrix $\mathbf{M}$ like so.

$$\mathbf{M} = \begin{pmatrix} 30 & 40 \\ 600 & 380 \end{pmatrix}$$

Once we have a matrix, there are a couple of operations that will be useful for analyzing them.

The Transpose of a Matrix

Once we've established our matrix, we can find it's **transpose** using MATLAB.
Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

The transpose of $\mathbf{A}$, denoted $\mathbf{A}^T$, is the matrix that has the rows of $\mathbf{A}$ as its columns. Using $\mathbf{A}$ as it is defined above, we can see that

$$\mathbf{A}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Notice that the first row of $\mathbf{A}$ is the first column of $\mathbf{A}^T$, the second row becomes the second column and the third row becomes the third column. Order matters greatly here, so be careful if you are doing this by hand. MATLAB has a built in function to find the transpose of a matrix.

Then once you have entered your matrix into MATLAB (I called mine $\mathbf{A}$), to find the transpose of your matrix simply enter the command:
transpose(A)
Be sure to include the correct name of your matrix if you called it something different.

Use MATLAB to find the transposes of the following matrices:

$$\mathbf{B} = \begin{pmatrix} 2 & -1 & 0 \\ 7 & 2 & 2 \\ 1 & -2 & -3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 4 & -10 & 3 \\ 0 & 0 & 8 \\ 2 & 1 & 11 \end{pmatrix} \quad \mathbf{D} = \begin{pmatrix} -7 & 9 & 16 \\ 5 & 4 & 3 \\ -6 & 3 & 6 \end{pmatrix}$$

$$\mathbf{B}^T = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 2 & -2 \\ 0 & 2 & -3 \end{pmatrix} \ \mathbf{C}^T = \begin{pmatrix} 4 & 0 & 2 \\ -10 & 0 & 1 \\ 3 & 8 & 11 \end{pmatrix} \ \mathbf{D}^T = \begin{pmatrix} -7 & 5 & -6 \\ 9 & 4 & 3 \\ 16 & 3 & 6 \end{pmatrix}$$

The Inverse of a Matrix

The inverse of a matrix **A** is any matrix **A'** that has the property that

$$\mathbf{A'A} = \mathbf{I} = \mathbf{AA'}$$

Where **I** is the identity matrix we learned about earlier. Finding the inverse by hand can be a very time consuming process, especially with a large matrix. Thankfully, we have MATLAB.

To find the inverse of a matrix, enter the matrix into MATLAB, be sure to remember what name you give it. Enter the matrix **A**, or another matrix of your choosing.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

into MATLAB. To find the inverse, enter inv(A) into the command line. If you entered **A** into MATLAB, you should get

$$\mathbf{A'} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Note! Not all matrices have inverses. There are certain conditions that a matrix has to have in order to be **invertible**, meaning it has an inverse. We will cover some of those conditions in the next week, and what an inverse is used for in an engineering context.

Practice entering these matrices into MATLAB and finding both their transpose and inverse.

$$\mathbf{D} = \begin{pmatrix} 2 & -11 & 3 \\ 1 & -2 & 1 \\ 1 & -2 & -5 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} -2 & 7 & 16 \\ 0 & 0 & 8 \\ -4 & 21 & -8 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 3 & 8 & 0 \\ 0 & -1 & 1 \\ -2 & 2 & 2 \end{pmatrix}$$

$$\mathbf{D}^{-1} = \begin{pmatrix} -0.286 & 1.452 & 0.119 \\ -0.143 & 0.310 & -0.024 \\ 0.000 & 0.167 & -0.167 \end{pmatrix} \quad \mathbf{E}^{-1} = \begin{pmatrix} -1.500 & 3.500 & 0.500 \\ -0.286 & 0.714 & 0.143 \\ 0.000 & 0.125 & 0.000 \end{pmatrix}$$

$$\mathbf{F}^{-1} = \begin{pmatrix} 0.143 & 0.571 & -0.286 \\ 0.071 & -0.214 & 0.107 \\ 0.071 & 0.786 & 0.107 \end{pmatrix}$$