# MNTC P01 - Week #3 - Derivatives - Applications

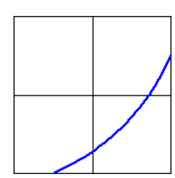
## Taylor Polynomials

For reference, the general formula for the Taylor polynomial centered at x = a is:

$$P(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^n(a)}{n!}(x-a)^n$$

where $n!$ means "n factorial", or $n \times (n-1) \times (n-2) \times ...2 \times 1$. E.g. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

1. Suppose $g$ is a function which has continuous derivatives, and that $g(5) = -3, g'(5) = -2,\ g''(5) = 1,\ g'''(5) = -5$.
   (a) What is the Taylor polynomial of degree 2 for $g$ near 5?
   (b) What is the Taylor polynomial of degree 3 for $g$ near 5?
   (c) Use the two polynomials that you found in parts (a) and (b) to approximate $g(5.1)$.
2. Find the Taylor polynomial of degree $n = 4$ for $x$ near the point $a = \frac{\pi}{4}$ for the function $\cos(4x)$.
   Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around $x = \pi/4$.
3. Find the fifth order Taylor polynomial for $\sin(x)$ near $x = \pi/4$.
   Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around $x = \pi/4$.
4. Find the Taylor polynomial of degree 3 around the point $x = -4$ of $f(x) = \sqrt{5+x}$.
   Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around the reference point.
5. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = 1$ for $f(x) = \ln(x+1)$.
   Use MATLAB to graph the function and both the Taylor polynomials on a reasonable interval around the reference point.
6. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = \frac{\pi}{6}$ for $f(x) = \sin(x)$.
   Use MATLAB to graph the function and both the Taylor polynomials on a reasonable interval around the reference point.
7. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = 7$ for $f(x) = \frac{1}{1+x}$.
   Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around the reference point.
8. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = 2$ for $f(x) = e^{-x} + e^{-2x}$.
   Use MATLAB to graph the function and both the Taylor polynomials on a reasonable interval around the reference point.
9. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = \frac{\pi}{4}$ for $f(x) = \tan(x)$.
   Use MATLAB to graph the function and both the Taylor polynomials on a reasonable interval around the reference point.
10. Find the second-degree Taylor polynomial $P_2(x)$ for the function $f(x) = \sqrt{15+x^2}$ at the number $x = 1$.
    Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around $x = \pi/4$.
11. Find the second-degree Taylor polynomial for $f(x) = 2x^2 - 3x + 8$ about $x = 0$. What do you notice about your polynomial?
12. Suppose that $P_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for the function $f$ about $x = 0$. What can you say about the signs of $a$, $b$, $c$ if $f$ has the graph given below?
    Note that the central lines are the $x$ and $y$ axes.



13. The function $f(x)$ is approximated near $x = 0$ by the second degree Taylor polynomial $P_2(x) = 3x - 3 + 8x^2$.
    Give values the values of $f(0)$, $f'(0)$, and $f''(0)$.

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## Optimization Introduction

14. Let $f(x) = x^2 - 10x + 13$, and consider the interval $[0, 10]$.

    (a) Find the critical point $c$ of $f(x)$ and compute $f(c)$.

    (b) Compute the value of $f(x)$ at the endpoints of the interval $[0, 10]$.

    (c) Determine the global min and max of $f(x)$ on $[0, 10]$.

    (d) Find the global min and max of $f(x)$ on $[0, 1]$. (Note: not the same interval as before)

15. Find the maximum and minimum values of the function $f(x) = \dfrac{\ln(x)}{x}$ on the interval [1,3].

16. Find the minimum and maximum values of $y = \sqrt{10}\theta - \sqrt{5}\sec\theta$ on the interval $[0, \frac{\pi}{3}]$.

17. Find the maximum and minimum values of the function $f(x) = x - \dfrac{125x}{x+5}$ on the interval [0,21].

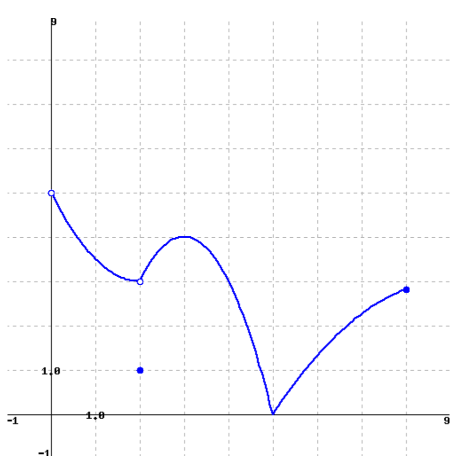
18. The function $f(x) = -2x^3 + 21x^2 - 36x + 10$ has one local minimum and one local maximum. Find their $(x, y)$ locations.

19. A University of Rochester student decided to depart from Earth after his graduation to find work on Mars. Before building a shuttle, he conducted careful calculations. A model for the velocity of the shuttle, from liftoff at t = 0 s until the solid rocket boosters were jettisoned at t = 80 s, is given by

$$v(t) = 0.001094333t^3 - 0.08215t^2 + 28.6t - 4.3$$

(in feet per second). Using this model, estimate the global maximum value and global minimum value of the **acceleration** of the shuttle between liftoff and the jettisoning of the boosters.

20. Use the given graph of the function on the interval $(0, 8]$ to answer the following questions.




    (a) Where does the function $f$ have a local maximum?

    (b) Where does the function $f$ have a local minimum?

    (c) What is the global maximum of $f$?

    (d) What is the global minimum of $f$?

21. Find the global maximum and minimum values of the following function on the given interval.
    $f(t) = 4t\sqrt{4 - t^2}$, $[-1, 2]$

22. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle $\theta$ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}$$

where $\mu$ is a positive constant called the coefficient of friction and where $0 \leq \theta \leq \pi/2$. Find the value for $\tan\theta$ which minimizes the force. Your answer may depend on W and $\mu$.

23. Find the exact global maximum and minimum values of the function $f(t) = \frac{3t}{8+t^2}$ if its domain is all real numbers.

24. A ball is thrown up on the surface of a moon. Its height above the lunar surface (in feet) after $t$ seconds is given by the formula

$$h = 217t - \frac{7}{4}t^2.$$

    (a) Find the time that the ball reaches its maximum height.

    (b) Find the maximal height attained by the ball.

25. In a certain chemical reaction, substance $A$ combines with substance $B$ to form substance $Y$. At the start of the reaction, the quantity of $A$ present is $a$ grams, and the quantity of $B$ present is $b$ grams. Assume $a < b$ and $y \leq a$. At time $t$ seconds after the start of the reaction, the quantity of $Y$ present is $y$ grams. For certain types of reactions, the rate of the reaction, in grams/sec, is given by

$$\text{Rate} = k(a - y)(b - y),$$

where $k$ is a positive constant.

    (a) Sketch a graph of the rate against $y$.

    (b) For what values of $y$ is the rate non-negative?

    (c) Use your graph to find the value of $y$ at which the rate of the reaction is fastest.

26. At what value(s) of $x$ on the curve $y = 1 + 250x^3 - 3x^5$ does the tangent line have the largest slope?

## Optimization With MATLAB

For Questions 27-32, use MATLAB to:

- generate a graph of the given function on the domain shown, and
- use the `fminbnd` function to find the global maximum and global minimum of the function on that interval.

27. $f(x) = 7e^{7x^3-7x}$, on $-1 \le x \le 0$

28. $f(x) = 7x - 21\ln(x)$, on $[1, 4]$

29. $f(x) = 4e^{-x} - 4e^{-2x}$, on $[0, 1]$

30. $f(x) = 7x - 14\cos(x)$, on $[-\pi, \pi]$

31. $f(x) = (3\cos x)/(20 + 10\sin x)$, $0 \le x \le 2\pi$

32. $f(t) = \frac{10}{t} + 4, 0 < t \le 1$
Note the open end due to the $0 < t$ instead of $0 \le t$.

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## Optimization Word Problems

33. Some airlines have restrictions on the size of items of luggage that passengers are allowed to take with them. Suppose that one has a rule that the sum of the length, width and height of any piece of luggage must be less than or equal to 192 cm. A passenger wants to take a box of the maximum allowable volume.

    (a) If the length and width are to be equal, what should the dimensions be?

    (b) In this case, what is the volume?

    (c) If the length is be twice the width, what should the dimensions be?

    (d) In this case, what is the volume?

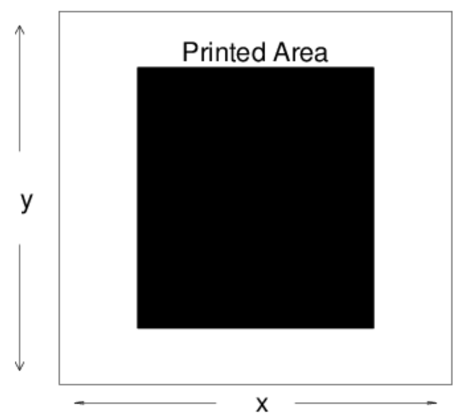
    Include units in all your answers.

34. A wire 3 meters long is cut into two pieces. One piece is bent into a square for a frame for a stained glass ornament, while the other piece is bent into a circle for a TV antenna.

    (a) To reduce storage space, where should the wire be cut to **minimize** the total area of both figures?

    (b) Where should the wire be cut to **maximize** the total area?

35. A printed poster is to have a total area of 799 square inches with top and bottom margins of 6 inches and side margins of 4 inches. What should be the dimensions of the poster so that the printed area be as large as possible? Let $x$ denote the width of the poster and let $y$ denote the length.




    (a) Write the function of $x$ and $y$ that you need to maximize.

    (b) Express that function in terms of $x$ alone.

    (c) Find the critical points of the function.

    (d) Use the second derivative test to verify that $f(x)$ has a maximum at this critical point

    (e) Find the optimal dimensions of the poster, and the resulting area. Include units.

36. A box with an open top has vertical sides, a square bottom, and a volume of 32 cubic meters. If the box has the least possible surface area, find its dimensions.

37. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle with sides of length 2 if one side of the rectangle lies on the base of the triangle.

38. Find the minimum distance from the parabola

$$x - y^2 = 0$$

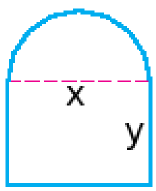
to the point (0,3).

39. I have enough pure silver to coat 2 square meters of surface area. I plan to coat a sphere and a cube.

(a) Allowing for the possibility of all the silver going onto one of the solids, what dimensions should they be if the total volume of the silvered solids is to be a maximum?

(b) Now allowing for the possibility of all the silver going onto one of the solids, what dimensions should they be if the total volume of the silvered solids is to be a minimum?

40. Suppose that 241 ft of fencing are used to enclose a corral in the shape of a rectangle with a semicircle whose diameter is a side of the rectangle as the following figure:




Find the dimensions of the corral with maximum area.

41. A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$4 per square foot and the metal for the sides costs \$6 per square foot. Find the dimensions that minimize cost if the box has a volume of 35 cubic feet.

42. Centerville is the headquarters of Greedy Cablevision Inc. The cable company is about to expand service to two nearby towns, Springfield and Shelbyville. There needs to be cable connecting Centerville to both towns. The idea is to save on the cost of cable by arranging the cable in a Y-shaped configuration. Centerville is located at $(8, 0)$ in the $xy$-plane, Springfield is at $(0, 5)$, and Shelbyville is at $(0, -5)$. The cable runs from Centerville to some point $(x, 0)$ on the $x$-axis where it splits into two branches going to Springfield and Shelbyville. Find the location $(x, 0)$ that will minimize the amount of cable between the 3 towns and compute the amount of cable needed. Justify your answer.

(a) What function of $x$ needs to be minimized to solve this problem?

(b) Find the critical points of $f(x)$.

(c) Use the second derivative test to verify that $f(x)$ has a minimum at this critical point.

(d) Compute the minimum amount of wire needed.

43. A cylinder is inscribed in a right circular cone of height 4 m and radius (at the base) equal to 3.5 m. What are the dimensions of such a cylinder which has maximum volume?