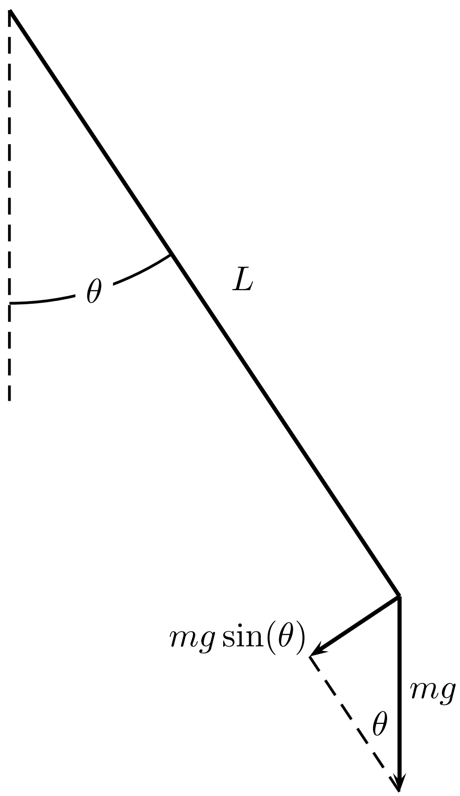
Week #9 : Differential Equations and Engineering

Goals:

- Take problems that can be modeled by differential equations, both first and second order, and give solutions both by hand and MATLAB
- Examine case studies of differential equations applied to engineering problems and reproduce those solutions

Application - Pendulum



Newton's Second Law:

$$\begin{aligned} mL^2\theta'' &= T_g + T_f \\ &= -mLg\sin(\theta) - (\mu L^2 m)\theta' \end{aligned}$$

Solving for θ'': $\theta'' = -\dfrac{g}{L}\sin(\theta) - \mu\theta'$

Problem. Turn this single second-order DE into a pair of first-order DEs.

Problem. Create a new MATLAB function file called `pendulumDE.m`. Start with the first line

```
function dw_dt = pendulumDE(t, w, g, L, mu)
```

In the body of the function, implement the system of differential equations we constructed on the previous page.

$$\frac{dw_1}{dt} = w_2$$
$$\frac{dw_2}{dt} = -\frac{g}{L}\sin(w_1) - \mu w_2$$

Problem. Write a MATLAB that simulates the motion of the pendulum using
$g = 9.8$ m/s^2, $L = 2$ m, $\mu = 0.1$, and
initial amplitude of 0.05 radians (≈ 2.9 degrees).

Application - Period of Pendulum Swings

Galileo famously noticed the consistent period of pendulum swings, even if the amplitude of the swings was changed (so the actual distance travelled was different).

Problem. Compare the periods of the pendulum swings, using a range of initial angles from $\theta_0 = 0.05$ radians up to $\theta_0 = 0.25$ radians (≈ 14 degrees).

However, it turns out that pendulums are **not** perfectly consistent in their period, due to the non-linear term $-\dfrac{g}{L}\sin(\theta)$ in one of the forces: as the amplitudes get bigger, there is a graudal lengthening of the period.

Problem. Compare the periods of the pendulum swings, using a range of initial angles from $\theta_0 = 0.25$ radians up to $\theta_0 = \frac{\pi}{2}$ radians $(= 90$ degrees$)$.

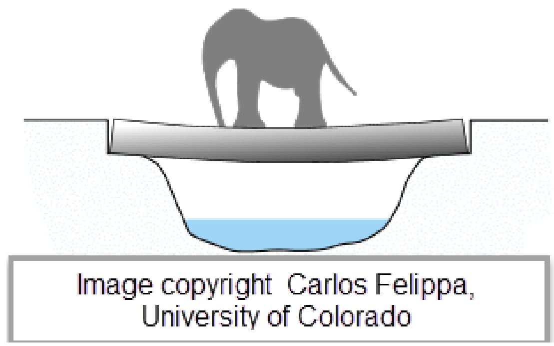
Pendulum - Including an Initial Velocity

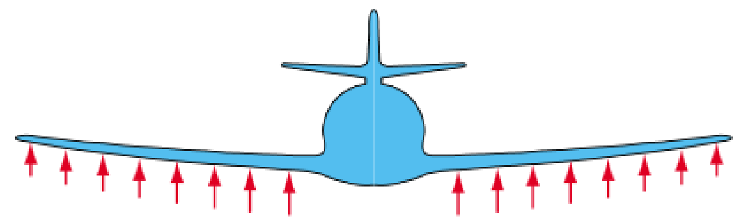
Problem. Write a new simulation script that starts the pendulum swinging from $\theta_0 = -\frac{\pi}{2}$, with no initial velocity. Simulate the motion for this scenario.

Use the parameters $g = 9.8$ m/s^2, $L = 2$ m, and $\mu = 0.1$.

Problem. If we keep the initial angle at $-\frac{\pi}{2}$ (pendulum out horizontally), experiment and find the initial velocity that will push the pendulum "over the top".

Deformation of a Loaded Beam





The shape of a beam under load is defined by the differential equation

$$EIy^{(4)} = p(x)$$

where

- $y(x)$ is the deflection (distance away from a straight line),
- $p(x)$ is the loading in N/m at point x along the beam,
- E is the modulus of elasticity of the beam (depends on material), and
- I is the moment of inertia of the beam (depends on beam shape and size)

Also relevant are the properties

$$V_0 = \text{shear}(0) = \int_0^L p(x)\, dx \text{ and}$$

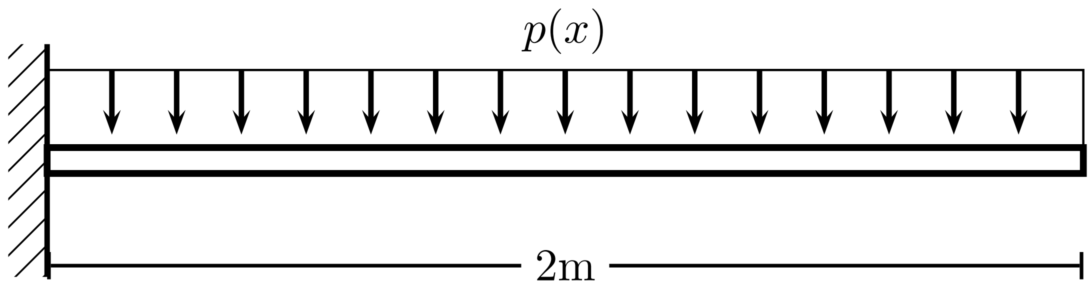
$$M_0 = \text{torque}(0) = \int_0^L x\, p(x)\, dx$$

Cantilevered Beam Under Uniform Load

Under a uniform loading (constant force per unit length), a *cantilevered beam* which is $L = 2$ m long, made out of a pine "2 by 4" satisfies

$p(x) = 100$ N/m, (or roughly 10 kg applied to each meter)

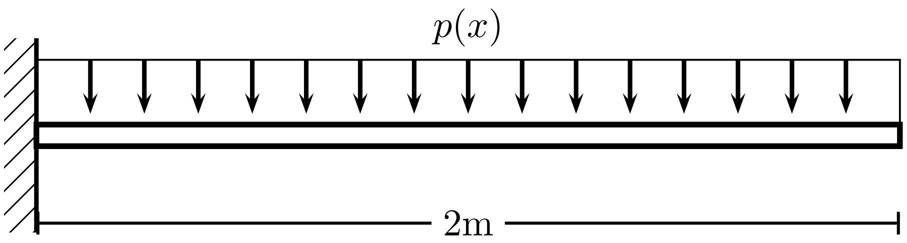
$I = 2.23 \times 10^{-6}\ \mathrm{m}^4, \qquad E = 9.1 \times 10^{9}\ \mathrm{N/m}^2,$



and the initial conditions

$y(0) = 0$, $y'(0) = 0$, $y''(0) = V_0$, and $y'''(0) = M_0$.

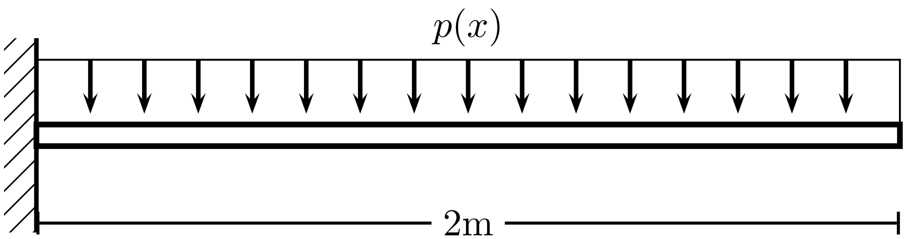
Problem. Find the amount of deflection of the beam at the tip under this load, using **multiple integrals**.



$EIy^{(4)} = 100$
$y(0) = 0$, $y'(0) = 0$,
$y''(0) = V_0$, and $y'''(0) = M_0$.

Cantilevered Beam Under Uniform Load - Differential Equation

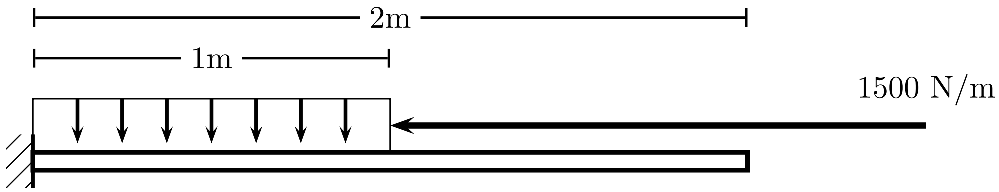
Problem. Find the amount of deflection of the beam at the tip under this load, using a **differential equation solver**.



$EIy^{(4)} = 100$
$y(0) = 0$, $y'(0) = 0$,
$y''(0) = V_0$, and $y'''(0) = M_0$.

Problem. If the maximum allowable deflection in such a beam is only 0.2 cm (say in a building code), what would the maximum uniform load be?

Cantilevered Beam - Non-Uniform Load



$$EIy^{(4)} = 100$$
$$y(0) = 0,\ y'(0) = 0,$$
$$y''(0) = V_0, \text{ and } y'''(0) = M_0.$$

Problem. Use a differential equation solver to generate a plot of the deflection of the beam shown above, for a 2×10 wood beam: $I = 4.1 \times 10^{-5}$ m^4, and $E = 9 \times 10^9$ N/m.

Application - Simple Tailings Pond

Problem. Consider a tailings pond that initially contains 10 million litres of fresh water. Water containing an undesirable chemical flows into the pond at the rate of 5 million litres per year; the mixture in the pond flows out at the same rate. The concentration $\gamma(t)$ of chemical in the incoming water varies periodically with time according to the expression $\gamma(t) = 2 + \sin(2t)\ \mathrm{g} \cdot \mathrm{L}^{-1}$.

1. Construct a mathematical model of this flow process.

2. Determine the amount of chemical in the pond at any time.

3. Describe the effect of the variation in the incoming concentration.

Application - Tailings Pond With Sediment

Consider another tailings pond, where the the inflow contains sediments that will settle out of the water.

In this pond, the volume is 40,000 cubic meters.

- Water is flowing in and out of the pond at a rate of 2,000 cubic meters per day.
- The water flowing into the pond contains 2 g of toxic chemical per cubic meter.
- The inflow water also contains

Problem. Sketch a diagram of this scenario.

Write a differential equation that describes

Application- Interconnected Tanks

Consider the tanks shown below, which shows water flowing between the tanks, and the concentration of a salt solution coming in. Within each tank, the water/salt solution is kept well mixed.

Problem. If both tanks start with no salt, sketch what you expect will happen to the concentration within each tank over time.

Problem. Create a system of differential equations that dictate how the two tank concentrations will evolve over time.

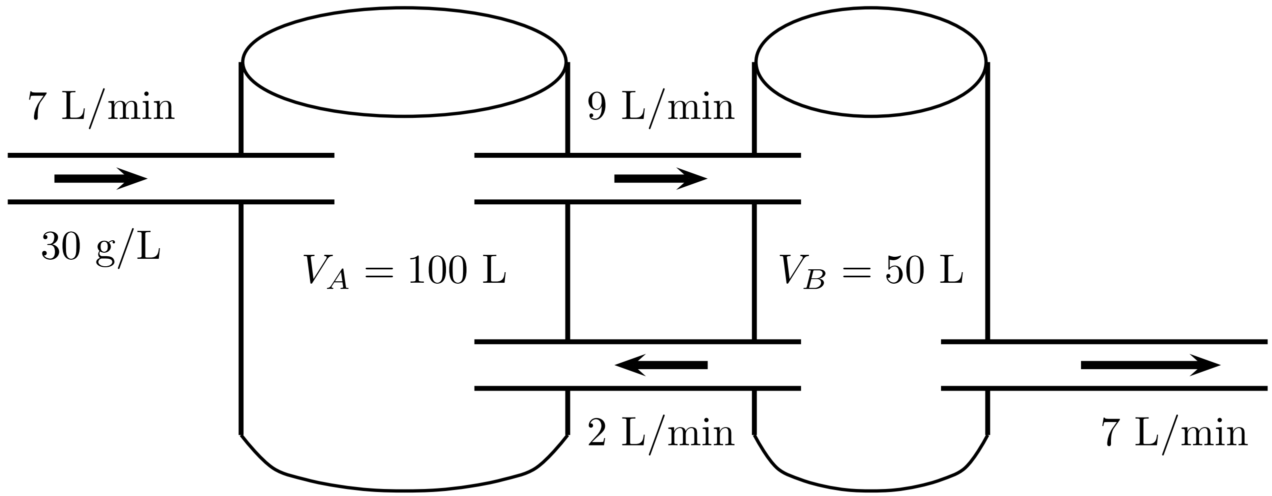
Problem. Predict the exact salt concentrations over time by solving the system of linear differential equations

$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \end{bmatrix} = \begin{bmatrix} \frac{-1}{10} & 0 \\ \frac{1}{20} & \frac{-1}{20} \end{bmatrix} \begin{bmatrix} c_A \\ c_B \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$c_A = -30e^{-t/10} + 30;$$

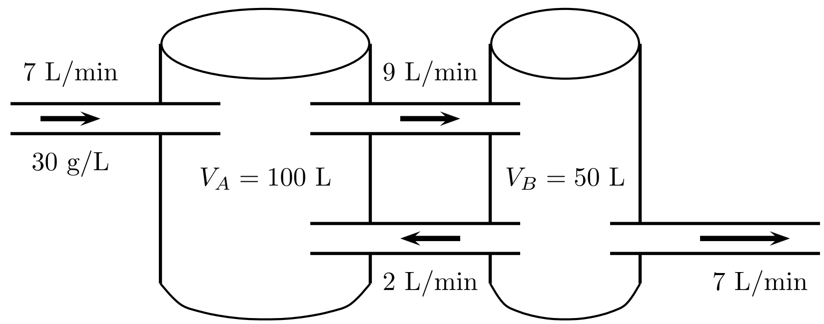
$$c_B = 30e^{-t/10} - 60e^{-t/20} + 30$$

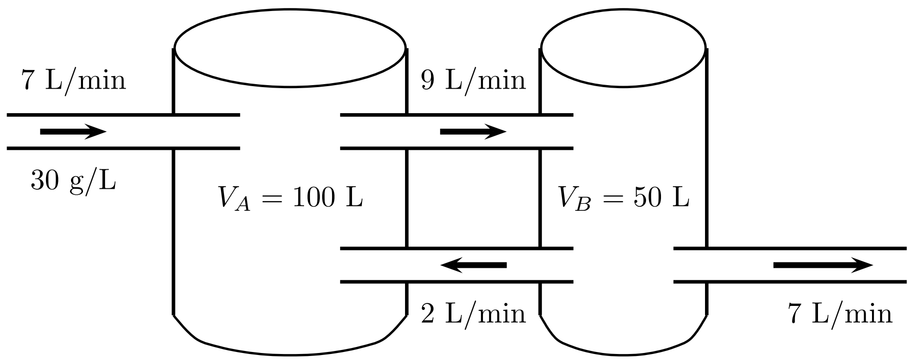
Consider the more complicated tank arrangement shown below.



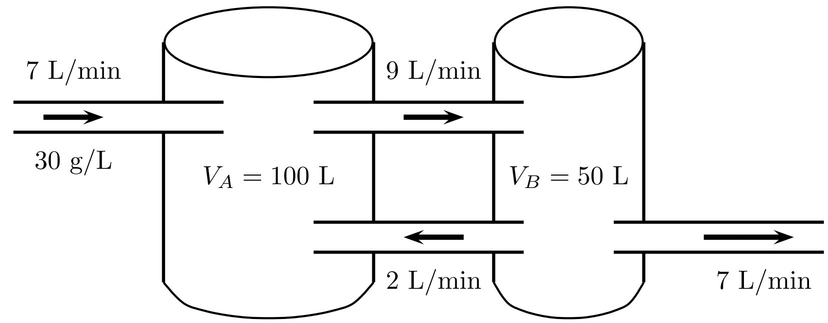
Problem. Given that the initial concentrations are
$c_A(0) = 0$ g/L and $c_B(0) = 90$ g/L,
sketch what you would predict for the concentration in each tank over time.

Problem. Construct the differential equation for the salt concentration in each tank, and write it in matrix form.

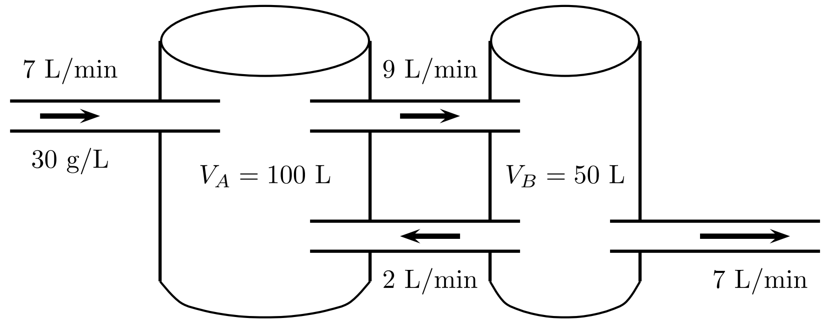



7 L/min
30 g/L
$V_A = 100$ L
9 L/min
2 L/min
$V_B = 50$ L
7 L/min

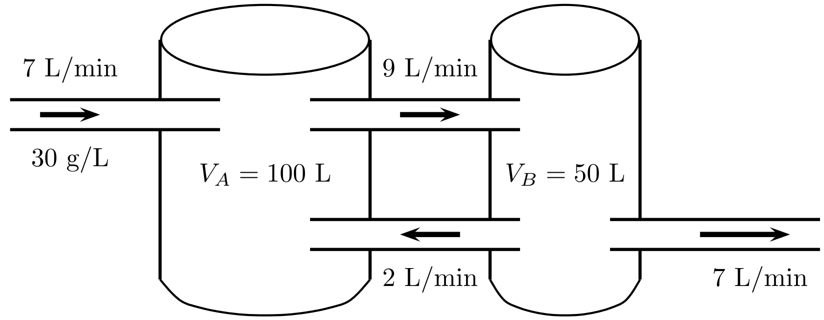
Problem. Predict the exact salt concentrations over time by solving the system of linear differential equations



$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \end{bmatrix} = \begin{bmatrix} -0.09 & 0.02 \\ 0.18 & -0.18 \end{bmatrix} \begin{bmatrix} c_A \\ c_B \end{bmatrix} + \begin{bmatrix} 2.1 \\ 0 \end{bmatrix}$$

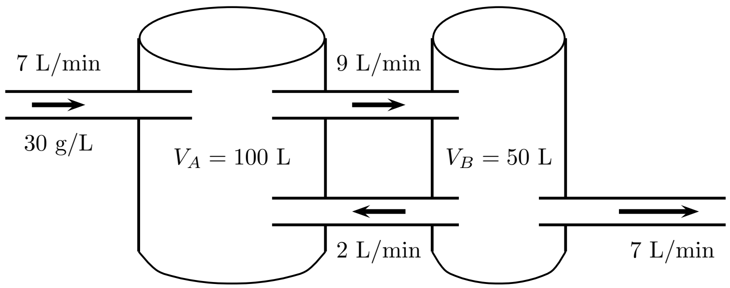


$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \end{bmatrix} = \begin{bmatrix} -0.09 & 0.02 \\ 0.18 & -0.18 \end{bmatrix}\begin{bmatrix} c_A \\ c_B \end{bmatrix} + \begin{bmatrix} 2.1 \\ 0 \end{bmatrix}$$



$$\frac{d}{dt}\begin{bmatrix} c_A \\ c_B \end{bmatrix} = \begin{bmatrix} -0.09 & 0.02 \\ 0.18 & -0.18 \end{bmatrix}\begin{bmatrix} c_A \\ c_B \end{bmatrix} + \begin{bmatrix} 2.1 \\ 0 \end{bmatrix}$$

Solution



$$c_A = -8(2)e^{-0.06t} - 14(1)e^{-0.21t} + 30$$
$$c_B = -8(3)e^{-0.06t} - 14(-6)e^{-0.21t} + 30$$