Week #2 - Derivatives - Linearization and Applications

Linear Approximations and Tangent Lines

1. Find the equation of the tangent line to the graph of f at (1,1), where f is given by $f(x) = 2x^3 - 2x^2 + 1$.

$f(x) = 2x^3 - 2x^2 + 1$.
In general, the slopes of the function are given by $f'(x) = 6x^2 - 4x$
At the point $(1, 1)$ (which you should check is actually on the graph of $f(x)$!), the slope is
$f'(1) = 6 - 4 = 2$
Using the point/slope formula for a line (or the tangent line formula), a line tangent to the graph of $f(x)$ at the point $(1, 1)$ is

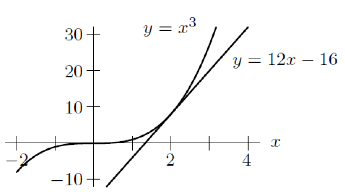
$$\begin{aligned} y &= f'(1)(x-1) + f(1) \\ &= 2(x-1) + 1 \\ \text{or } y &= 2x - 1 \end{aligned}$$

2. (a) Find the equation of the tangent line to $f(x) = x^3$ at $x = 2$.
   (b) Sketch the curve and the tangent line on the same axes, and decide whether using the tangent line to approximate $f(x) = x^3$ would produce *over*- or *under*-estimates of $f(x)$ near $x = 2$.

(a) $f(x) = x^3$, so $f'(x) = 3x^2$.
At $x = 2$, $f(2) = 8$ and $f'(2) = 12$, so the tangent line to $f(x)$ at $x = 2$ is

$$y = 12(x - 2) + 8$$

(b)



From the graph of $y = x^3$, it is clear that the tangent line at $x = 2$ will lie *below* the actual curve. This means that using the tangent line to estimate $f(x)$ values will produce *underestimates* of $f(x)$.

3. Find the equation of the line tangent to the graph of f at $(3, 57)$, where f is given by $f(x) = 4x^3 - 7x^2 + 12$.

Differentiating gives $f'(x) = 12x^2 - 14x$, so $f'(3) = 66$. Thus the equation of the tangent line is $y - 57 = 66(x - 3)$, or $y = 57 + 66(x - 3)$.

4. Given a power function of the form $f(x) = ax^n$, with $f'(3) = 16$ and $f'(6) = 128$, find n and a.

Since $f(x) = ax^n$, $f'(x) = anx^{n-1}$. We know that $f'(3) = (an)3^{n-1} = 16$, and $f'(6) = (an)6^{n-1} = 128$. Therefore,

$$\frac{f'(6)}{f'(3)} = \frac{128}{16} = 8.$$

But

$$\frac{f'(6)}{f'(3)} = \frac{(an)6^{n-1}}{(an)3^{n-1}} = 2^{n-1},$$

so $2^{n-1} = 8$, and so $n = 4$.

Substituting $n = 4$ into the expression for $f'(3)$, we get $4a3^3 = 16$, so $a = \frac{4}{27}$.

5. Find the equation of the line tangent to the graph of f at $(2, 1)$, where f is given by $f(x) = 2x^3 - 5x^2 + 5$.

Differentiating gives $f'(x) = 6x^2 - 10x$, so $f'(2) = 4$. Thus the equation of the tangent line is $y - 1 = 4(x - 2)$, or $y = 1 + 4(x - 2)$.

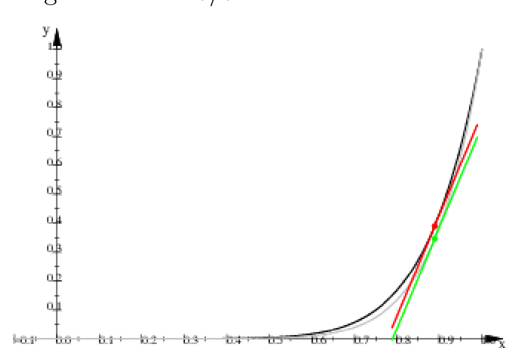
6. Find all values of x where the tangent lines to $y = x^8$ and $y = x^9$ are parallel.

Let $f(x) = x^8$ and let $g(x) = x^9$. The two graphs have parallel tangent lines at all x where $f'(x) = g'(x)$.

$$\begin{aligned} f'(x) &= g'(x) \\ 8x^7 &= 9x^8 \\ 8x^7 - 9x^8 &= 0 \\ x^7(8 - 9x) &= 0 \end{aligned}$$

hence, $x = 0$ or $x = \frac{8}{9}$.

The point at $x = 0$ is easy to visualize (both graphs are flat there). Here is a graph showing the parallel tangents at $x = 8/9$.



7. Consider the function $f(x) = 9 - e^x$.

   (a) Find the slope of the graph of $f(x)$ at the point where the graph crosses the x-axis.

   (b) Find the equation of the tangent line to the curve at this point.

   (c) Find the equation of the line perpendicular to the tangent line at this point. (This is the *normal* line.)

(a) $f(x) = 9 - e^x$ crosses the x-axis where $0 = 9 - e^x$, which happens when $e^x = 9$, so $x = \ln 9$. Since $f'(x) = -e^x$, $f'(\ln 9) = -9$.

(b) $y = -9(x - \ln(9))$.

(c) The slope of the normal line is the negative reciprocal of the slope of the tangent, so $y = \frac{1}{9}(x - \ln(9))$.

8. Consider the function $y = 2^x$.

   (a) Find the tangent line based at $x = 1$, and find where the tangent line will intersect the x axis.

   (b) Find the point on the graph $x = a$ where the tangent line will pass through the origin.

(a) We find the linearization using $f(x) = 2^x$, so $f'(x) = 2^x \ln(2)$ (non-e exponential derivative rule).

At the point $x = 1$, $f(1) = 2^1 = 2$ and $f'(1) = 2^1 \ln(2)$, so the linear approximation to $f(x)$ is

$L(x) = 2 + (2\ln(2))(x - 1)$.

Solving for where this line intersects the x axis (or the $y = 0$ line), we find the x intercept is approximately -0.4427.

(b) This question is more general. Instead of asking for a linearization at a specific point, it is asking "at what point would the linearization pass through the origin?" Let us give the point a name: $x = a$ (as opposed to $x = 1$ used in part (a)).

From the function and the derivatives, the linearization at the point $x = a$ is given by:

$$L_a(x) = \underbrace{2^a}_{f(a)} + \underbrace{2^a \ln(2)}_{f'(a)}(x - a)$$

That is true in general, but we want the point $x = a$ where the linearization will go through $(0, 0)$, i.e. for which $L_a(0) = 0$:

$$\begin{aligned} 0 &= 2^a + 2^a \ln(2)(0 - a) \\ \text{Solving for } a, \quad 0 &= 2^a(1 - a\ln(2)) \\ 0 &= 1 - a\ln(2) \\ a\ln(2) &= 1 \\ a &= \frac{1}{\ln(2)} \approx 1.442 \end{aligned}$$

At that x point, the graph of $y = 2^x$'s tangent line will pass exactly through the origin.

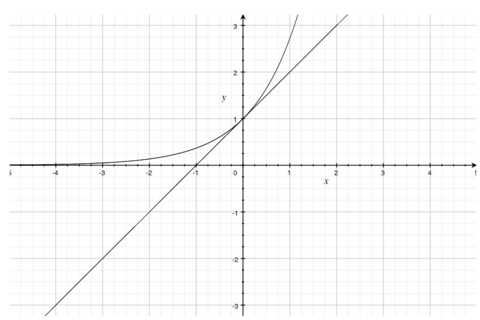
9. (a) Find the tangent line approximation to $f(x) = e^x$ at $x = 0$.

   (b) Use a sketch of $f(x)$ and the tangent line to determine whether the tangent line produces over- or under-estimates of $f(x)$.

   (c) Use your answer from part (b) to decide whether the statement $e^x \geq 1 + x$ is always true or not.

(a) $f(x) = e^x$, so $f'(x) = e^x$ as well.
To build the tangent line at $x = 0$, we use $a = 0$ as our reference point: $f(0) = e^0 = 1$, and $f'(0) = e^0 = 1$. The tangent line is therefore
$l(x) = 1(x - 0) + 1 = x + 1$

(b)



Since the exponential graph is concave up, it curves upwards away from the graph. This means that the linear approximation will always be an underestimate of the original function.

(c) Since the linear function will always underestimate the value of e^x, we can conclude that

$$1 + x \leq e^x$$

, and they will be equal only at the tangent point, $x = 0$.

10. The speed of sound in dry air is

$$f(T) = 331.3\sqrt{1 + \frac{T}{273.15}}\,\text{m/s}$$

where T is the temperature in degrees Celsius. Find a linear function that approximates the speed of sound for temperatures near 0^o C.

$$f(T) = 331.3\sqrt{1 + \frac{T}{273.15}}$$

$$f'(T) = 331.3\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{1+\frac{T}{273.15}}}\right)\frac{1}{273.15}$$

so at $T = 0^o$ C, $\quad f(0) = 331.3 \quad f'(0) = \dfrac{331.3}{(2)(273.15)} \approx 0.606$

Thus the speed of sound for air temperatures around 0^o C is

$f(T) \approx 0.606(T-0)+331.3$, or $f(t) \approx 0.606T+331.3$ m/s

11. Find the equations of the tangent lines to the graph of $y = \sin(x)$ at $x = 0$, and at $x = \pi/3$.
    (a) Use each tangent line to approximate $\sin(\pi/6)$.
    (b) Would you expect these results to be equally accurate, given that they are taken at equal distances on either side of $\pi/6$? If there is a difference in accuracy, can you explain it?

(a) At $x = 0$, the tangent line is defined by $f(0) = 0$ and $f'(0) = 1$, so

$$y = 1(x-0)+0 = x$$

is the tangent line to $f(x)$ at $x = \dfrac{\pi}{3}$.

At $x = \dfrac{\pi}{3}$, the tangent line is defined by $(\pi/3) = \frac{\sqrt{3}}{2}$ and $f'(\pi/3) = \frac{1}{2}$, so

$$y = \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}$$

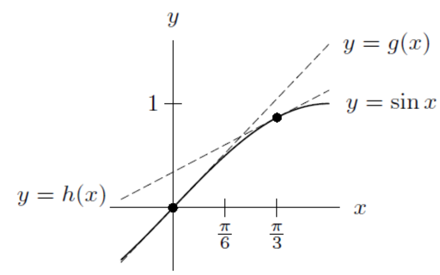
is the tangent line to $f(x)$ at $x = 0$.

The estimates of each tangent line at the point $x = \pi/6$ would be

- Based on $x = 0$ tangent line, $f(x) \approx x$, so $f(\pi/6) \approx \pi/6 \approx 0.5236$.
- Based on $x = \pi/3$ tangent line, $f(x) \approx \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}$, so $f(\pi/6) \approx \frac{1}{2}\left(\pi/6 - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2} \approx 0.6042$
- The **actual** value of $f(x) = \sin(x)$ at $x = \pi/6$ is $\sin(\pi/6) = 0.5$.

(b) From these calculations, the estimate obtained by using the tangent line based at $x = 0$ gives the more accurate prediction for $f(x)$ at $x = \pi/6$

A sketch might help explain these results.



In the interval $x \in [0, \pi/6]$, the function stays very close to linear (i.e. does not curve much), which means that the tangent line stays a good approximation for a relatively long time.

The function is most curved/least linear around its peak, so the linear approximation around $x = \pi/3$ is less accurate even over the same Δx.

12. Find the **quadratic** polynomial $g(x) = ax^2 + bx + c$ which best fits the function $f(x) = e^x$ at $x = 0$, in the sense that

$$g(0) = f(0), g'(0) = f'(0), \text{ and } g''(0) = f''(0)$$

$$\begin{aligned} f(x) &= e^x \\ \text{so } f'(x) &= e^x \\ \text{and } f''(x) &= e^x \\ \text{Evaluating at } x &= 0, \\ f(0) = f'(0) &= f''(0) = 1 \end{aligned}$$

Comparing with the derivatives of the quadratic,

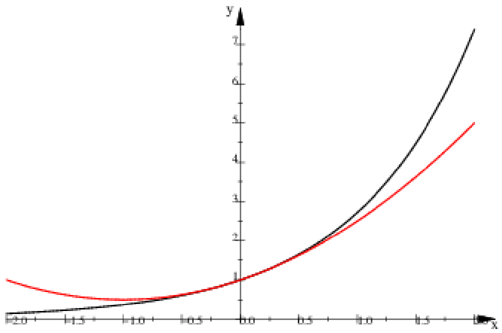
$$\begin{aligned} g(x) &= ax^2 + bx + c \\ \text{so } g'(x) &= 2ax + b \\ \text{and } g''(x) &= 2a \\ \text{Evaluating at } x &= 0, \\ g(0) = c, g'(0) &= b, \text{ and } g''(0) = 2a \end{aligned}$$

For $g(x)$ to fit the shape of $f(x)$ near $x = 0$, we would then pick $c = 1$, $b = 1$ and $2a = 1$ or $a = 0.5$, so

$$g(x) = 0.5x^2 + x + 1$$

would be the best fit quadratic to $f(x) = e^x$ near $x = 0$.

Here is a graph of the two functions, $f(x) = e^x$ in black, $g(x) = 0.5x^2 + x + 1$ in red. Notice how similar they look near their intersection.



13. Consider the graphs of $y = \sin(x)$ (regular sine graph), and $y = ke^{-x}$ (exponential decay, but scaled vertically by k).

    If $k \geq 1$, the two graphs will intersect. What is the smallest value of k for which two graphs will be *tangent* at that intersection point?

Let $f(x) = \sin(x)$ and $g(x) = ke^{-x}$. They intersect when $f(x) = g(x)$, and they are tangent at that intersection if $f'(x) = g'(x)$ as well. Thus we must have

$$\sin(x) = ke^{-x} \qquad \text{and} \ \cos(x) = -ke^{-x}$$

We can't solve either equation on its own, but we can divide one by the other:

$$\begin{aligned} \frac{\sin(x)}{\cos(x)} &= \frac{ke^{-x}}{-ke^{-x}} \\ \tan(x) &= -1 \\ x &= \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots \end{aligned}$$

Since we only need one value of k, we try the first value, $x = 3\pi/4$.

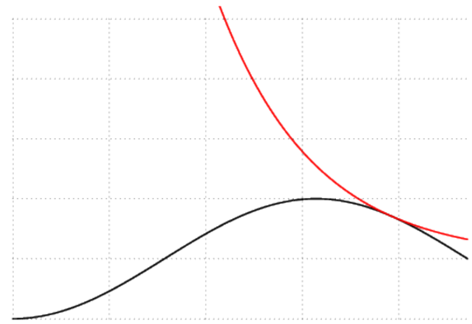
$$\begin{aligned} \sin(3\pi/4) &= ke^{-3\pi/4} \\ \frac{1}{\sqrt{2}}e^{3\pi/4} &= k \\ k &\approx 7.46 \end{aligned}$$

We confirm our answer by verifying both the values and derivatives are equal at $x = 3\pi/4$,

$$\begin{aligned} \sin(3\pi/4) &= 7.46e^{-3\pi/4} \approx 0.7071 \text{ (same } y\text{: intersection)} \\ \text{and } \cos(3\pi/4) &= -7.46e^{-3\pi/4} \approx -0.7071 \text{ (same derivative)} \end{aligned}$$

The actual point of tangency is at $(x, y) = \left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

A sketch is shown below.



14. (a) Show that $1 + kx$ is the local linearization of $(1 + x)^k$ near $x = 0$.

    (b) Someone claims that the square root of 1.1 is about 1.05. Without using a calculator, is this estimate about right, and how can you decide using part (a)?

(a)

$$\begin{aligned} & f(x) = (1 + x)^k \quad && f'(x) = k(1 + x)^{k-1} \\ \text{so at } x = 0, \quad & f(0) = 1^k = 1 && f'(0) = k(1^{k-1}) = k \end{aligned}$$

so the tangent line at $x = 0$ will be

$$y = k(x - 0) + 1 \text{ or } y = 1 + kx$$

(b) As an estimate for the square root of 1.1, we could note that $\sqrt{1.1} = (1 + 0.1)^{1/2}$. This matches exactly the form of $f(0.1)$ if we choose $k = \frac{1}{2}$. From our linearization above,

$$f(0.1) \approx 1 + \frac{1}{2}(0.1) = 1.05$$

so yes, a good approximation for $\sqrt{1.1}$ is 1.05. (Calculator gives the value of ≈ 1.0488.

15. (a) Find the local linearlization of e^x near $x = 0$.

    (b) Square your answer to part (a) to find an approximation to e^{2x}.

    (c) Compare your answer in part (b) to the actual linearization to e^{2x} near $x = 0$, and discuss which is more accurate.

(a) e^x has a tangent line/local linearization near $x = 0$ of $y = x + 1$ (slope 1, point (0, 1)).

(b) Multiplying this approximation by itself, we get $(e^x)(e^x)$ or $e^{2x} \approx (x + 1)(x + 1) = x^2 + 2x + 1$

(c) To compare with the actual linearization of $g(x) = e^{2x}$, we find its derivative and value at $x = 0$,

$$\begin{aligned} g(x) &= e^{2x} \qquad & g(0) &= 1 \\ g'(x) &= 2e^{2x} & g'(0) &= 2 \end{aligned}$$

so a linearization of $g(x) = e^{2x}$ near $x = 0$ is $y = 2(x - 0) + 1$ or

$$y = 2x + 1$$

Note that his is the same as the approximation we obtained before, except that our product version had an additional term, x^2.

These approximations give the same straight-line estimate of the function, but I would expect the first (multiplication) version to be more accurate because it contains more information (the squared term that the pure linear approximation was missing).

We will see more of this idea in Taylor polynomials and Taylor series.

16. (a) Show that $1 - x$ is the local linearization of $\dfrac{1}{1+x}$ near $x = 0$.

    (b) From your answer to part (a), show that near $x = 0$,

$$\frac{1}{1+x^2} \approx 1 - x^2.$$

    (c) Without differentiating, what do you think the derivative of $\dfrac{1}{1+x^2}$ is at $x = 0$?

(a) Let $f(x) = 1/(1 + x)$. Then $f'(x) = \frac{-1}{(1+x)^2}$.
At $x = 0$, $f(0) = 1$ and $f'(0) = -1$.
So near $x = 0$, $f(x) \approx -1x + 1 = -x + 1$

(b) For small x values (i.e. x near zero), we can approximate $1/(1+x)$ with $1-x$. Replace the variable x with y (because the name doesn't matter),

$$1/(1 + y) \approx 1 - y$$

If we choose y small but equal to x^2, then

$$\frac{1}{1+x^2} \approx 1 - x^2$$

(c) The linearization of $1/(1 + x^2)$ is the linear part of $1 - x^2$, or just 1. Since the derivative at $x = 0$ is the coefficient for x in the linear part, this means $\dfrac{d}{dx}\dfrac{1}{1+x^2}$ at $x = 0$ must equal zero.

17. (a) Find the local linearization of

$$f(x) = \frac{1}{1+2x}$$

    near $x = 0$.

    (b) Using your answer to (a), what quadratic function would you expect to approximate $g(x) = \dfrac{1}{1+2x^2}$?

    (c) Using your answer to (b), what would you expect the derivative of $\frac{1}{1+2x^2}$ to be even without doing any differentiation?

(a) We know $f(0) = 1$ and $f'(0) = -\frac{2}{(1+2(0))^2} = -2$, so the local linearization is $f(x) \approx 1 - 2x$.

(b) Next, $g(x) = f(x^2)$, so we expect that $g(x) \approx 1 - 2x^2$.

(c) Noting that the approximation we found in (b) is downward opening parabola with vertex on the y-axis, we expect that the derivative of $g(x)$ at $x = 0$ will be zero.

Newton's Method

18. Consider the equation $e^x + x = 2$. This equation has a solution near $x = 0$. By replacing the left side of the quation by its linearization near $x = 0$, find an approximate value for the solution.

    (In other words, perform one step of Newton's method, starting at $x = 0$.)

Our equation is

$$\underbrace{e^x + x}_{f(x)} = 2$$

To find the linearization of $f(x)$ near $x = 0$, we need f and its derivative f', both evaluated at $x = 0$.

$$\begin{aligned} f(x) &= e^x + x & \text{so } f(0) &= e^0 + 0 = 1 \\ f'(x) &= e^x + 1 & \text{so } f'(0) &= e^0 + 1 = 2 \end{aligned}$$

The linearization is then

$$e^x + x \approx \underbrace{(2)}_{f'(0)}(x-0) + \underbrace{1}_{f(0)} = 2x + 1$$

We now replace the original (unsolvable) equation

$$\begin{aligned} & & e^x + x &= 2 \\ &\text{with the simpler approximation:} & 2x + 1 &= 2 \end{aligned}$$

Solving this second version is straightforward, yielding $x = 0.5$. This is actually a fair approximation to the solution, since $e^{0.5}+0.52.149$ which is close to what the equation RHS is, 2.

(If we continued our linearizations and their approximations, we would get values even closer to the real solution, which is (to 4 decimal place) 0.4429.)

19. Use Newton's Method with the equation $x^2 = 2$ and initial value $x_0 = 3$ to calculate x_1, x_2, x_3 (the next three solution estimates generated by Newton's method).

Moving everything to the left side, we get

$$\underbrace{x^2 - 2}_{f(x)} = 0$$

Differentiating, we have $f'(x) = 2x$. Therefore:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 2}{2x_0} = 3 - \frac{3^2 - 2}{2 * 3} = 1.83333$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.83333 - \frac{1.83333^2 - 2}{2 \times 1.83333} \approx 1.46212$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.46212 - \frac{1.46212^2 - 2}{2 \times 1.46212} \approx 1.415$$

This sequence provides successive approximations to the exactly solution,which would equal

$\sqrt{2} \approx 1.4142$

20. Use Newton's Method with the function $x^3 = 5$ and initial value $x_0 = 1.5$ to calculate x_1, x_2, x_3 (the next three solution estimates generated by Newton's method).

Moving everything to the left side, we get

$$\underbrace{x^3 - 5}_{f(x)} = 0$$

Differentiating, we have $f'(x) = 3x^2$. Therefore:

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = x_0 - \dfrac{x_0^3 - 5}{3x_0^2} = 1.5 - \dfrac{1.5^3 - 5}{3 * 1.5^2} =$ 1.74074

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.74074 - \frac{1.74074^3 - 5}{3 \times 1.74074^2} \approx 1.71052$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.71052 - \frac{1.71052^3 - 5}{3 \times 1.71052^2} \approx 1.70998$$

This sequence provides successive approximations to the third root of 5, $\sqrt[3]{5} \approx 1.70997$

21. Use Newton's Method to approximate $4^{\frac{1}{3}}$ and compare with the value obtained from a calculator.

    (Hint: write out a simple equation that $4^{\frac{1}{3}}$ would satisfy, and use Newton's method to solve that.)

We need to find an approximation to $4^{\frac{1}{3}}$ using Newton's Method.
We let $f(x) = x^3 - 4$, thus $f'(x) = 3x^2$.
A good initial guess could be $x_0 = 8^{1/3} = 2$. Thus:

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2^3 - 4}{3 \times 2^2} = 1.66667$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.66667 - \frac{1.66667^3 - 4}{3 \times 1.66667^2} \approx 1.59111$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.59111 - \frac{1.59111^3 - 4}{3 \times 1.59111^2} \approx 1.58741$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.58741 - \frac{1.58741^3 - 4}{3 \times 1.58741^2} \approx 1.5874$

Using a calculator to find $4^{\frac{1}{3}}$, we get: $4^{\frac{1}{3}} \approx 1.58740$.