Week #3 : Derivatives - Applications

Goals:

- Calculate and interpret the first and second derivatives, as well as higher order derivatives.
- Define and calculate Taylor polynomials.
- Use MATLAB to graph and compare functions with their Taylor polynomial approximations.
- Find and use critical points for global and local optimization problems.
- Use MATLAB optimizers and equation solvers to identify optimal values and critical points.

Second and Higher Derivatives

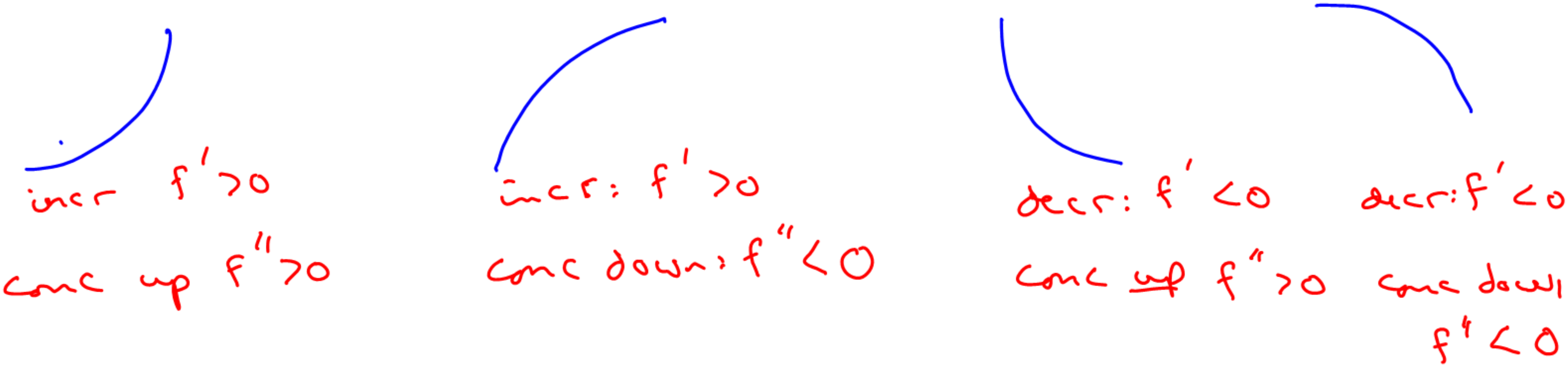
A lot of information about the graph of a function f can be deduced by the sign of $f'(x)$ and $f''(x)$ (the **second derivative** of $f(x)$) on an interval (a, b).

1st derivs

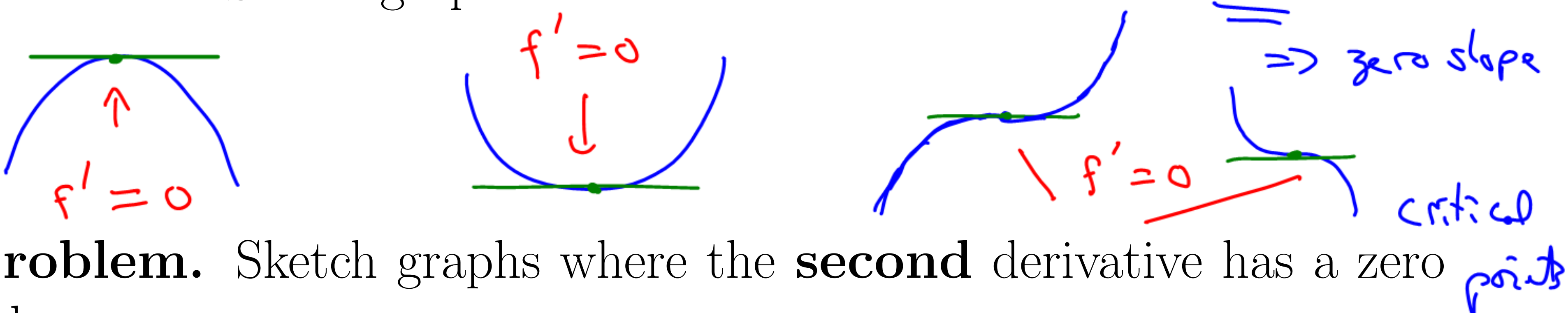
$f'(x) > 0$ on (a, b) pos	f increasing on $[a, b]$	
$f'(x) < 0$ on (a, b) neg	f decreasing on $[a, b]$	slope
$f''(x) > 0$ on (a, b) pos	f concave up on $[a, b]$	
$f''(x) < 0$ on (a, b) neg	f concave down on $[a, b]$	concavity

2nd deriv

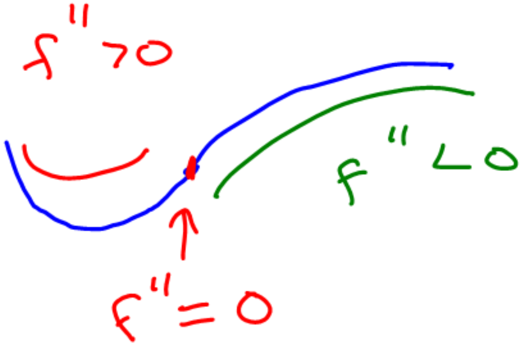
Problem. Sketch the possible graphs combining different signs of positive and negative first and second derivatives.



Problem. Sketch graphs where the **first** derivative has a zero value.



Problem. Sketch graphs where the **second** derivative has a zero value.



point of inflection

Aside from their graphical interpretation, the second derivative frequently has an important physical meaning in kinematics problems.

Problem. If $x(t) = 4\sin(2t)$ gives the position of a particle at time t, what is particle's **speed** at $t = \dfrac{\pi}{6}$?

m

s

$$v = \text{Speed} = \frac{d}{dt}(\text{position}) = \frac{d}{dt}(x) = \frac{dx}{dt} = x' = \dot{x}$$

$$v = \frac{d}{dt}\Big(\underbrace{4\sin(2t)}_{x(t)}\Big) = 4\left(\cos(2t)\cdot 2\right) = 8\cos(2t)$$

At $t = \pi/6 = 8\cos(2(\pi/6)) = 8(1/2) = 4\ \frac{m}{s}$

For the same particle, what is its **acceleration** at $t = \dfrac{\pi}{6}$?

$$a = \frac{d^2}{dt^2}(\text{position}) = \frac{d}{dt}\left(\frac{d}{dt}(\text{pos})\right) = \frac{d}{dt}(v)$$

take 2 deriv's

$$a = \frac{d}{dt}\Big(\underbrace{8\cos(2t)}_{v}\Big) = 8(-\sin(2t)\cdot 2) = -16\sin(2t)\ m/s^2$$

$$a(\pi/6) = -16\sin(2(\pi/6)) = -13.856\ m/s^2$$

While their interpretations are not as immediately obvious, it is also possible to compute 3rd, 4th, or higher derivatives of function if we want.

Problem. Find the first four derivatives of the function

$$f(x) = 7e^{-2x} + \ln(x).$$

$$f'(x) = 7e^{-2x}(-2) + \frac{1}{x} \qquad 1^{st} \text{ deriv}$$

slope

$$= -14e^{-2x} + 1/x$$

d/dx

$$\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1}$$

$$= -1x^{-2}$$

$$= -1/x^2$$

$$f''(x) = -14e^{-2x}(-2) + \left(\frac{-1}{x^2}\right)$$

concavity

$$= 28e^{-2x} - 1/x^2 \qquad 2^{nd} \text{ deriv}$$

d/dx

$$\frac{d}{dx} -x^{-2}$$

$$= -(-2)x^{-3}$$

$$f'''(x) = 28e^{-2x}(-2) + \frac{2}{x^3}$$

$$= -56e^{-2x} + 2/x^3$$

$\frac{d}{dx}$

$$f''''(x) = +112e^{-2x} - 6/x^4$$

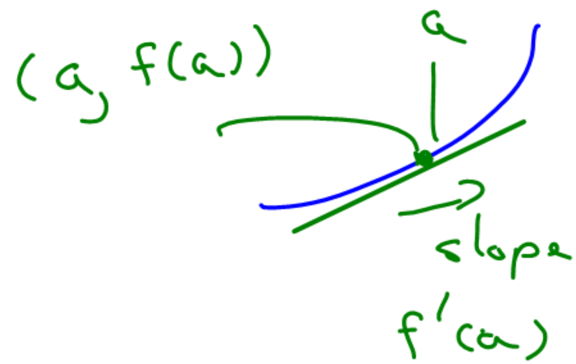
$$= f^{(4)}$$

4 derivs

Taylor Polynomials

One application of higher derivative information is to help us build **polynomial approximations** to other functions.

Previously we found a formula for linear approximations to functions $f(x)$ around a point $x = a$:



$$f(x) \approx y = f'(a)(x-a) + f(a)$$

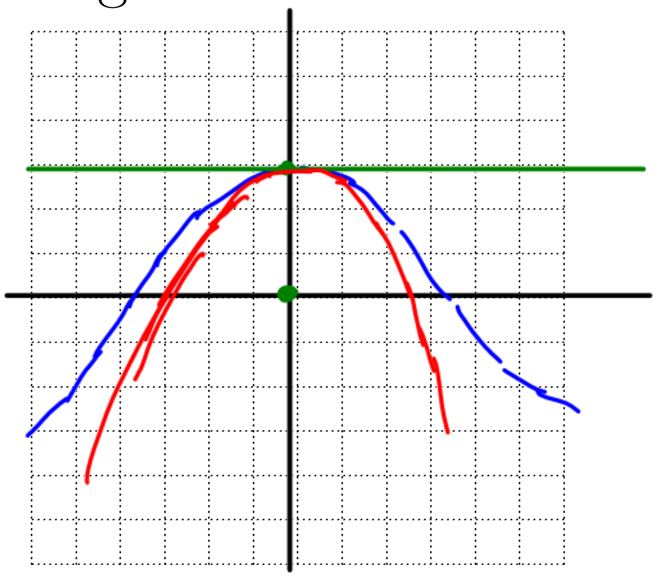
blue curve

formula for tangent line

This linear approximation, or tangent line formula, can also be called the **Taylor polynomial of degree 1 approximating** $f(x)$ **near** $x = a$**.**

linear Taylor polynomial

Problem. Sketch the graph of $\cos(x)$ around $x = 0$, and add its tangent line based at $x = 0$.



tgt line

The linearization or tangent line is clearly a very limited approximation to this function. What might be a *slightly* more complex form of function that would work better in this case?

parabola? same concavity as $\cos(x)$

Taylor Polynomial of Degree 2

$P_2(x)$

$$f(x) \approx \underbrace{f(a) + f'(a)(x-a)}_{\substack{\text{linearization}\\ \text{tgt line}\\ \text{Taylor pol'l degree 1}}} + \overbrace{\frac{f''(a)}{2}(x-a)^2}^{\text{new quadratic term}}$$

is a *quadratic* approximation to $f(x)$ near $x = a$.

Problem. For values of x close to a do you think this quadratic approximation will be a better or worse approximation than the tangent line? Why?

w/ quadratic, we can get a closer fit to shape of the curve we are approximating.

Problem. Find the quadratic Taylor approximation to $f(x) = \cos(x)$ near $x = 0$. at $x = 0$

$f(x) = \cos(x)$ $\big\downarrow \frac{d}{dx}$

$f'(x) = -\sin(x)$ $\big\downarrow \frac{d}{dx}$

$f''(x) = -\cos(x)$

$f(0) = \cos(0) = 1$

$f'(0) = -\sin(0) = 0$

$f''(0) = -\cos(0) = -1$

So $\cos(x) \approx 1 + 0(x-0) + \frac{(-1)}{2}(x-0)^2$

or $P_2(x) \mathbin{/\!/} f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$P_2(x) = 1 - \frac{1}{2}x^2$

Problem. Use MATLAB to draw the graph of $\cos(x)$ around $x = 0$, and add both its 1st and 2nd degree Taylor polynomial approximations for x near 0.

$$f(x) \approx \overbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}^{P_2(x)}$$

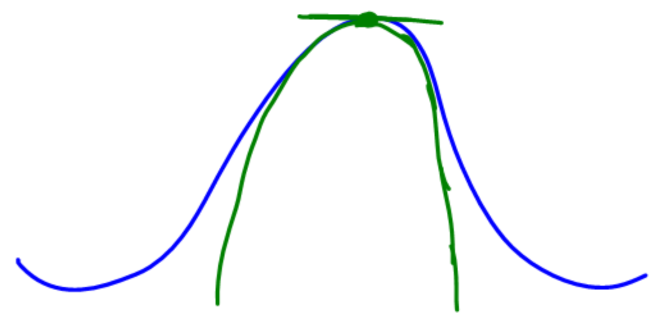
The form of the coefficients in the Taylor polynomial are carefully chosen.

Problem. What mathematical features will the original $f(x)$ share with its 2nd degree Taylor approximation at the point $x = a$?

$f(0) = P_2(0)$ same value

$f'(0) = P_2'(0)$ same slope

$f''(0) = P_2''(0)$ same concavity



at $x=0$

Taylor Polynomials of Higher Degree

Problem. If we wanted a still-better approximation for a function $f(x)$ near a specific point $x = a$, how could we generalize our earlier 1st and 2nd degree Taylor polynomials?

linear → quadratic → cubic → quartic

5th order

degree

Below is the general formula for the terms in a Taylor polynomial, up to degree n.

$$f(x) \approx \overbrace{\overbrace{\underbrace{f(a) + f'(a)(x-a)}_{\substack{\text{linear} \\ \text{tgt line}}} + \frac{f''(a)}{2}(x-a)^2}^{\text{quadratic}} + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n}^{n^{th}\text{ degree}}$$

- $f^{(n)}$ means "the n-th derivative of f".

- $n!$ means "n factorial"

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot \ldots 1 = 120$

$2! = 2 \cdot 1 = 2$

$1! = 1$

Higher Degree Taylor Polynomials - Example 1

Consider the function $f(x) = \sin(x)$.

Problem. Find the first five derivatives of $f(x)$, and evaluate them at $x = 0$.

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = f^{(3)}(x) = -\cos(x)$

$f^{(4)}(x) = -(-\sin(x)) = \sin(x)$

$f^{(5)}(x) = \cos(x)$

$x = 0$

$f(0) = \sin(0) = 0$

$f'(0) = \cos(0) = 1$

$f''(0) = 0$

$f^{(3)}(0) = -1$

$f^{(4)}(0) = 0$

$f^{(5)}(0) = 1$

Problem. Write out the Taylor polynomial of degree 5 for $f(x) = \sin(x)$.

$$\underset{\sin(x)}{f(x)} \approx \overset{f(0)}{\downarrow}\,0 + \overset{f'(0)}{\downarrow}\,1(x-0) + \frac{\overset{f''(0)}{\downarrow}\,0}{2}(x-0)^2 + \frac{\overset{f^{(3)}(0)}{\downarrow}\,(-1)}{3!}(x-0)^3 + \frac{\overset{f^{(4)}(0)}{\downarrow}\,0}{4!}(x-0)^4 + \frac{\overset{f^{(5)}(0)}{\downarrow}\,(1)}{5!}(x-0)^5$$

$$\sin(x) \approx x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \quad \text{near } x=0$$

Problem. Write out the general form of the Taylor polynomial of degree n for $f(x) = \sin(x)$.

$$\sin(x) \cong x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11} + \frac{1}{13!}x^{13} + \cdots$$

Problem. Use MATLAB to plot the graph of $f(x) = \sin(x)$ and the Taylor polynomial approximations up to degree 5.

MATLAB Demo of even higher degree Taylor approximations to $\sin(x)$.

Higher Degree Taylor Polynomials - Example 2

Consider the function $g(x) = xe^{-x}$.

Problem. Find the first three derivatives of $g(x)$, and evaluate them at $x = 1$.

$$g(x) = x \cdot e^{-x}$$

$$g'(x) = 1e^{-x} + x(-e^{-x})$$
$$= (1-x)e^{-x}$$

$$g''(x) = (-1)e^{-x} + (1-x)(-e^{-x})$$
$$= (-x)e^{-x}$$

$$g'''(x) = (-1)e^{-x} + (-x)(-e^{-x})$$
$$= (-1+x)e^{-x}$$
$$= (x-1)e^{-x}$$

at $x=1$

$$g(1) = e^{-1}$$

$$g'(1) = 0$$

$$g''(1) = -1e^{-1}$$

$$g'''(1) = 0$$

Problem. Write out the Taylor polynomial of degree 3 for $g(x) = xe^{-x}$ centered at $x = 1$.

$$\underset{xe^{-x}}{g(x)} \approx \underset{g(1)}{e^{-1}} \cdot 1 \quad \frac{\underset{g'(1)}{0}}{1}(x-1) + \frac{\underset{g''(1)}{(-e^{-1})}}{2}(x-1)^2 + \frac{\underset{g'''(1)}{0}}{3!}(x-1)^3$$

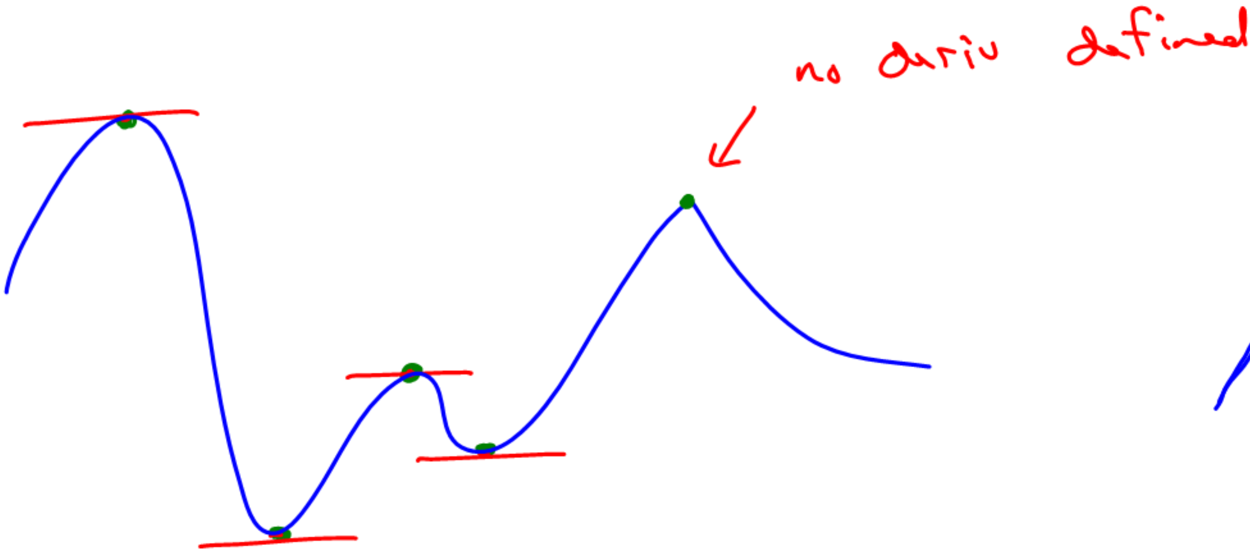
$$= e^{-1} + \left(\frac{-e^{-1}}{2}\right)(x-1)^2$$

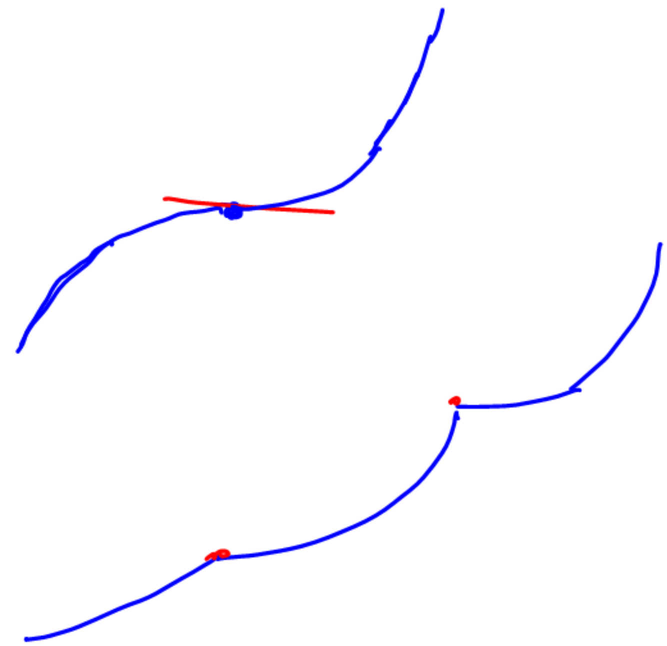
Problem. Use MATLAB to plot the graph of $g(x) = xe^{-x}$ and the Taylor polynomial approximation from degree 1, 2 and 3.

Critical Points

Aside from understanding the shape of functions, derivative information can help us identify and classify interesting points of a function, like the highest and lowest values.

Problem. Sketch graphs which have high and low points.





What do those extreme values have in common?

To help identify local extreme values, we introduce the definition of a critical point.

If $f(x)$ is defined on the interval (a, b), then we call a point c in the interval a **critical point** if:

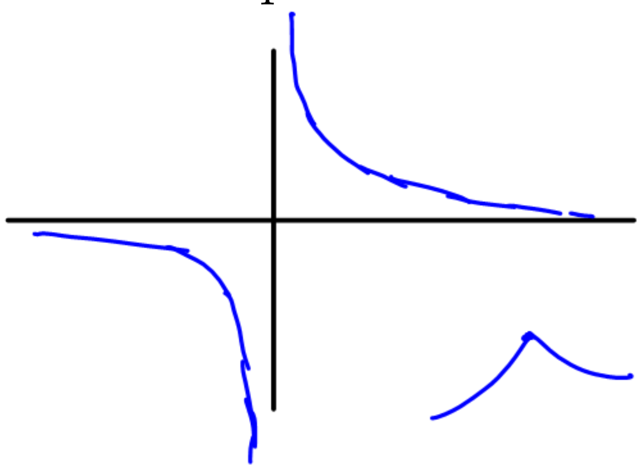
- $f'(c) = 0$, or   slope = 0
- $f'(c)$ does not exist.   cusp

We will also refer to the point $(c, f(c))$ on the graph of $f(x)$ as a critical point. We call the function value $f(c)$ at a critical point c a **critical value**.

Technical Notes:

1. By this definition, $f(c)$ must be **defined** for c to be a critical point.

   Problem. Sketch $f(x) = 1/x$, and decide whether $x = 0$ is a critical point.

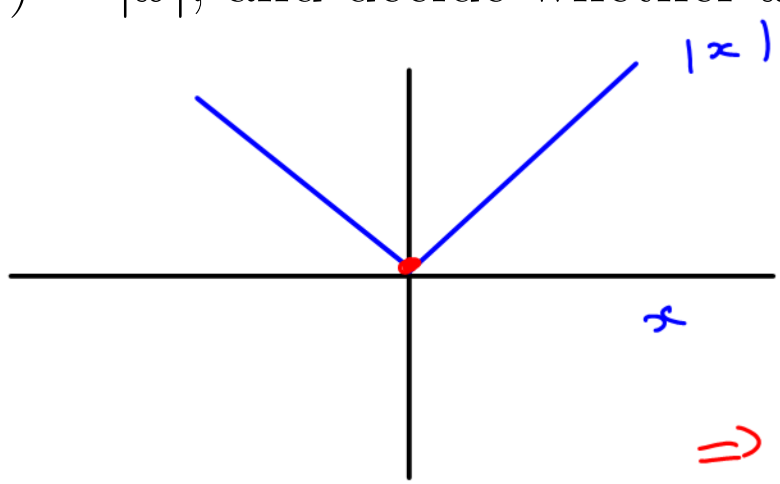


$f'(0)$ is not defined

Is $x=0$ a critical point

b/c $f(0) = \frac{1}{0}$ is also not defined

   Sketch $g(x) = |x|$, and decide whether $x = 0$ is a critical point.



$|0| = 0$ is defined

$g'(0)$ is not defined

$\Rightarrow x = 0$ is a critical point for $g(x)$

2. By the definition, if a function is defined on a closed interval, the endpoints of interval **cannot** be critical points. ↳ include end points

Problem. Sketch the graph of $f(x) = \sqrt{x}$ and decide whether $x = 0$ is a critical point.

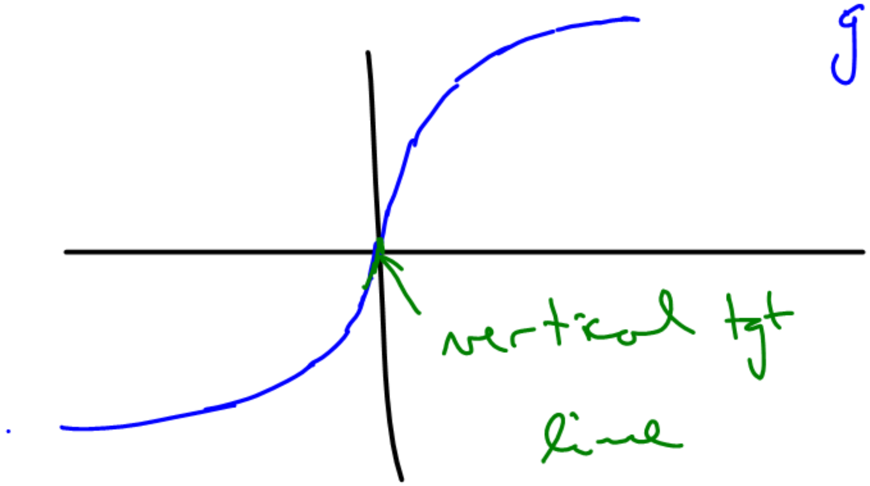
only defined for $x \geq 0$

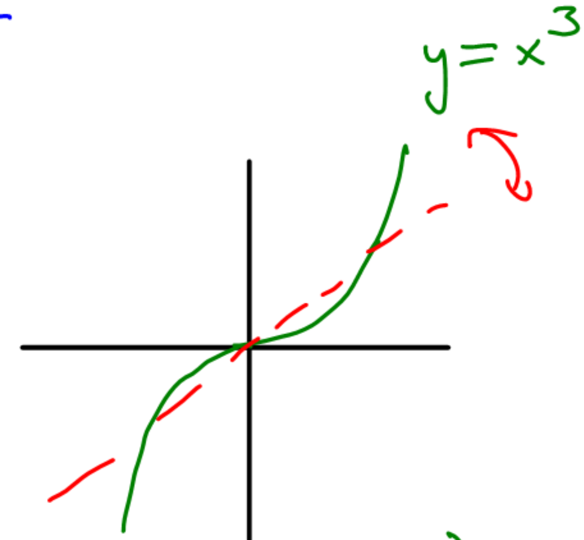
not a critical point

b/c no tgt line based on both sides of $x=0$

Problem. Sketch the graph of $g(x) = \sqrt[3]{x}$ and decide whether $x = 0$ is a critical point.

$g(x) = \sqrt[3]{x}$





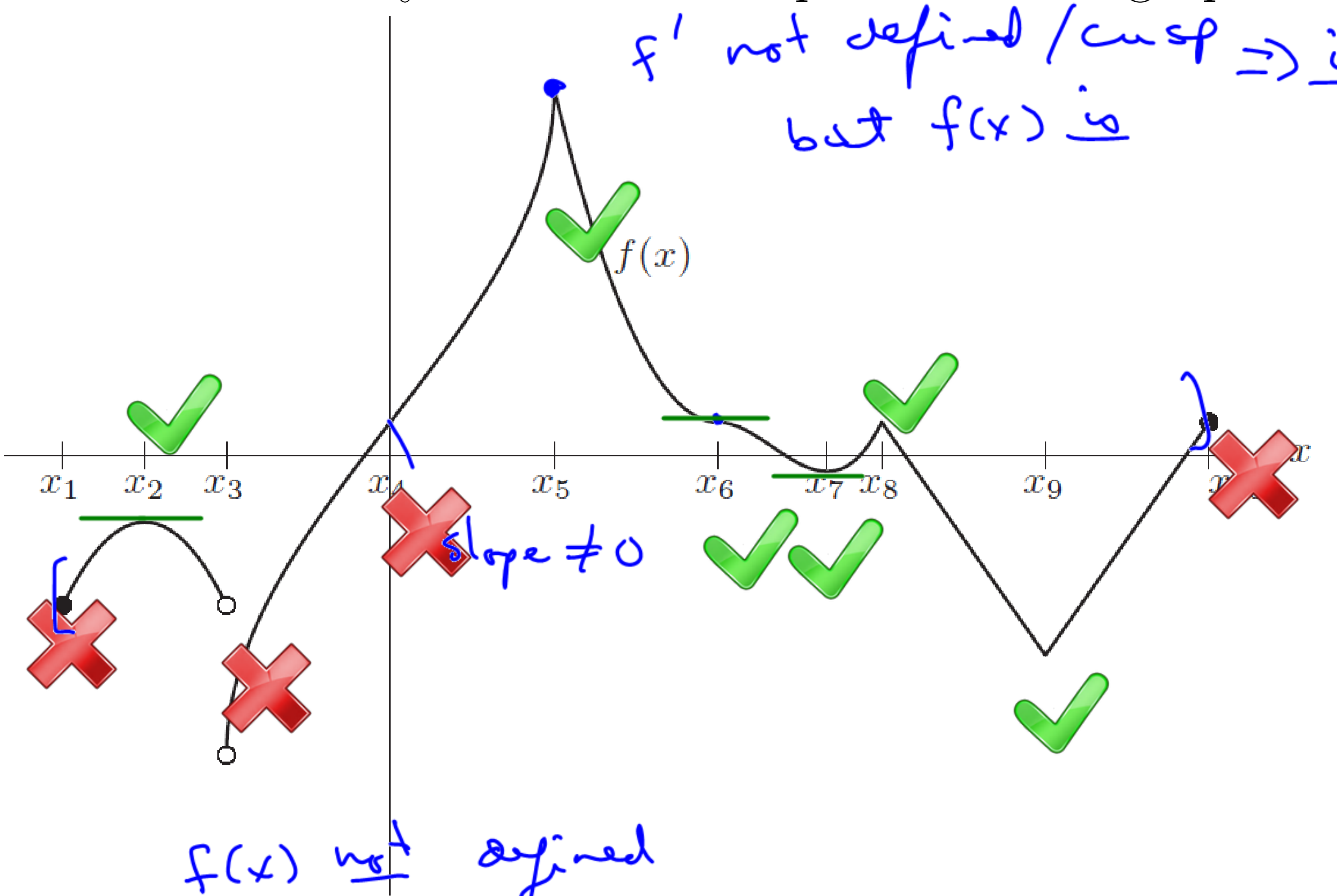
$\Rightarrow$ slope is undefined $\left(\sqrt[3]{0} = 0\right)$

+ function <u>is</u> defined

$\Rightarrow$ $x = 0$ <u>is</u> a critical point

Problem. Identify all the critical points on the graph below.

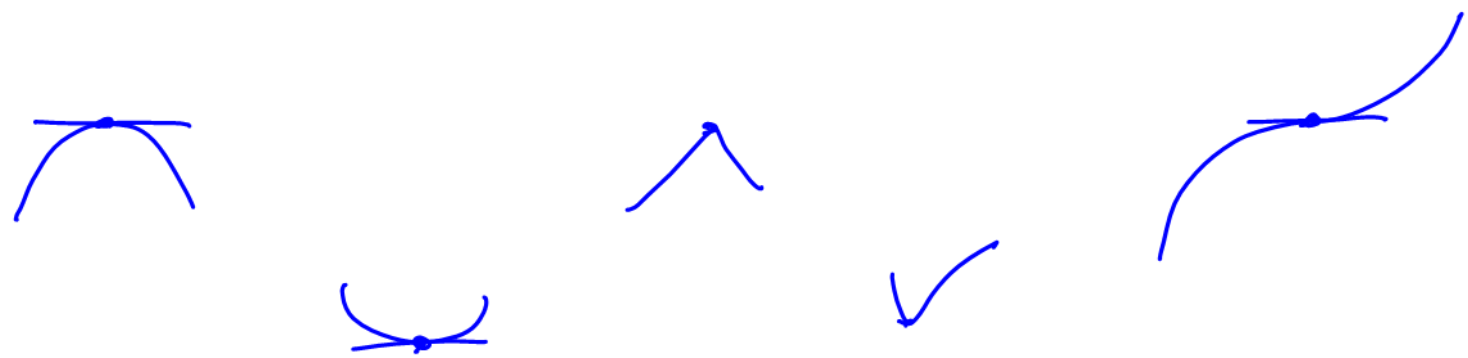
f' not defined / cusp ⇒ is a critical point
but f(x) is



f(x) not defined

Classifying Critical Points

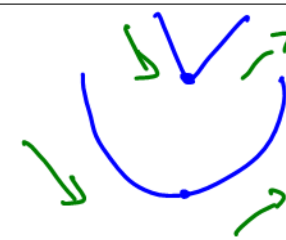
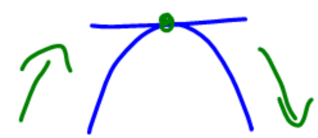
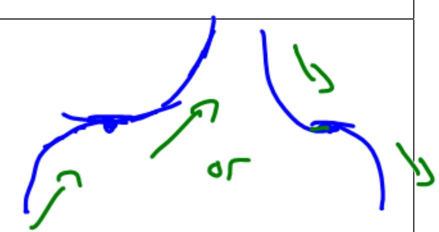
We will now formalize two ways to determine if a critical point is a local min, max, or neither. This avoids the need for a sketch of the graph.



First Derivative Test

One indicator of the type of critical points is the sign of the derivative on each side of the critical point. This classification method is called the **first derivative test**.

Problem. Complete this table:

	f' sign left of c	f' sign right of c	Sketch of $f(x)$
local minimum at c	$(-)$	$(+)$	
local maximum at c	$(+)$	$(-)$	
neither local max nor min	$(-)$ or $(+)$	$(-)$ $(+)$	

Problem. Find the critical points of the function

$$f(x) = 2x^3 - 9x^2 + 12x + 3.$$

Use the first derivative test to show whether each critical point is a local maximum or a local minimum.

$f(x) = 2x^3 - 9x^2 + 12x + 3$

$f'(x) = 6x^2 - 18x + 12$

Crit points $\rightarrow$ f' undefined $\times$

$\rightarrow$ $f' = 0$ Set $f'(x) = 0$, solve for x

Set $0 = 6x^2 - 18x + 12$

$\frac{0}{6} = \frac{\cancel{6}}{\cancel{6}}(x^2 - 3x + 2)$

2 critical points

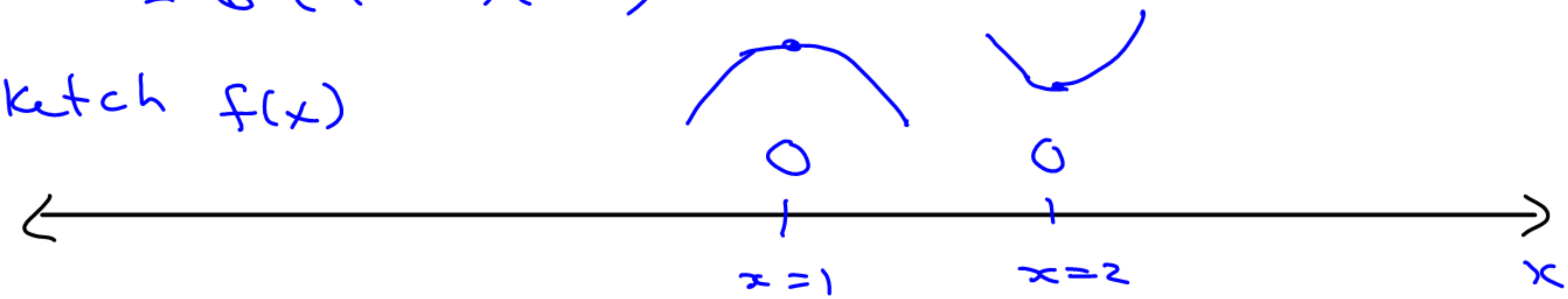
$0 = x^2 - 3x + 2$

factor $0 = (x-2)(x-1)$

$x = 2$ or $x = 1$

$$f'(x) = 6x - 18x + 12 = 6(x^2 - 3x + 2)$$
$$= 6(x-2)(x-1)$$

Sketch $f(x)$

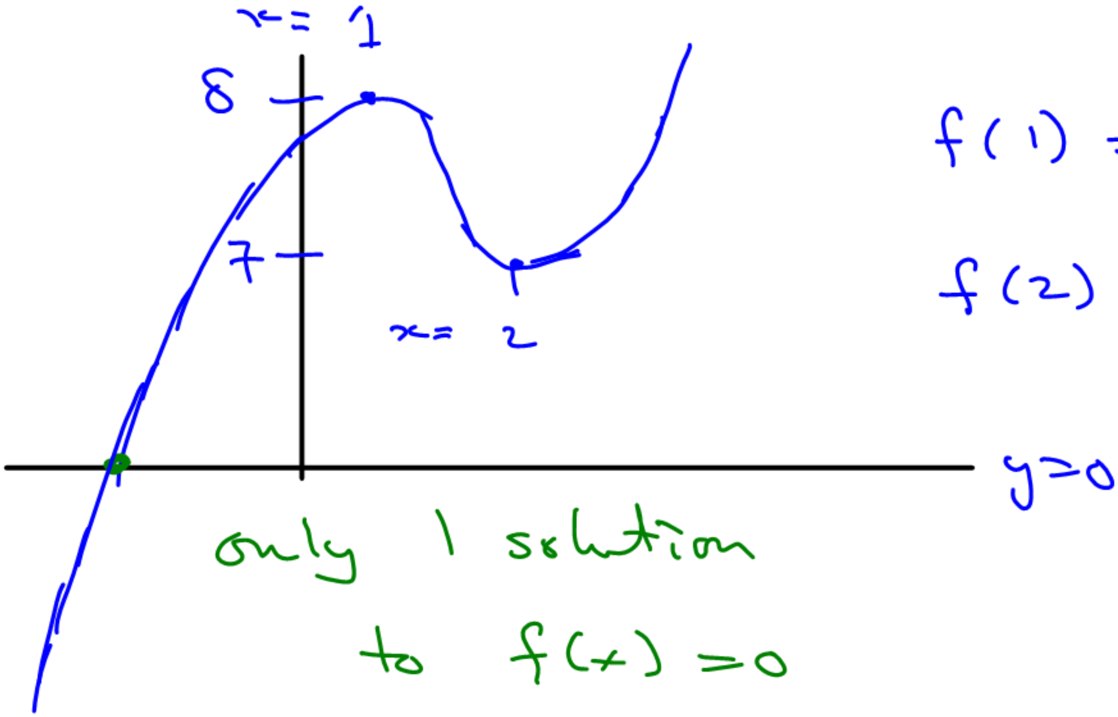


sign of	$(x-2)$	$(-)$	$(-)$	$(+)$
	$(x-1)$	$(-)$	$(+)$	$(+)$
$f' =$	$6(x-2)(x-1)$	$(+)$	$(-)$	$(+)$

$x=1$ is a local <u>max</u>

$x=2$ is a local <u>min</u>

Problem. Using your answer to the preceding question, determine the number of real solutions of the equation $2x^3 - 9x^2 + 12x + 3 = 0$.



$f(1) = 2 - 9 + 12 + 3 = 8$

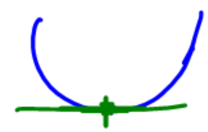
$f(2) = \dots = 7$

Second Derivative Test

You may also use the Second Derivative Test to determine if a critical point is a local minimum or maximum.

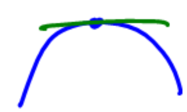
- The first derivative test uses the **first** derivative **around** the critical point.
- The second derivative test uses the **second** derivative **at** the critical point.
- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.

concave up



- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

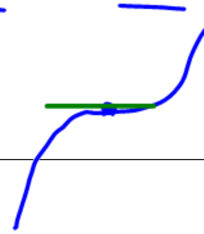
concave down



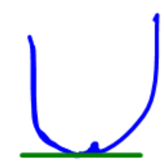
- If $f'(c) = 0$ and $f''(c) = 0$ then the test is inconclusive.

f'' is undefined

x^5



x^4



Problem. A function f has derivative $f'(x) = \cos(x^2) + 2x - 1$. Does it have a local maximum, a local minimum, or neither at its critical point $x = 0$?

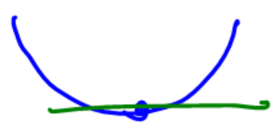
at $x = 0$, $f'(0) = \cos(0) + 0 - 1$

$= 1 - 1 = 0$ $\quad$ $x=0$ is a critical point

$f''(x) = -\sin(x^2)\cdot 2x + 2$

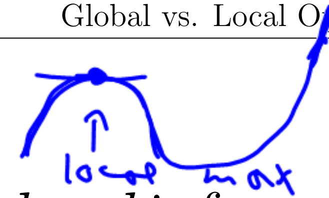
At $x = 0$, $f''(0) = -\sin(0)\cdot 0 + 2$

$= 2$ $\quad$ f concave up at $x=0$



$x=0$ is a local min

Global vs. Local Optimization



The first and second derivative tests only give us *local* information in most cases. However, if there are multiple local maxima or minima, we usually want the **global** max or min. The ease of determining when we have found the global max or min of a function depends strongly on the properties of the question.

Local vs Global Extrema

A **local max** occurs at $x = c$ when $f(c) > f(x)$ for x values near c.

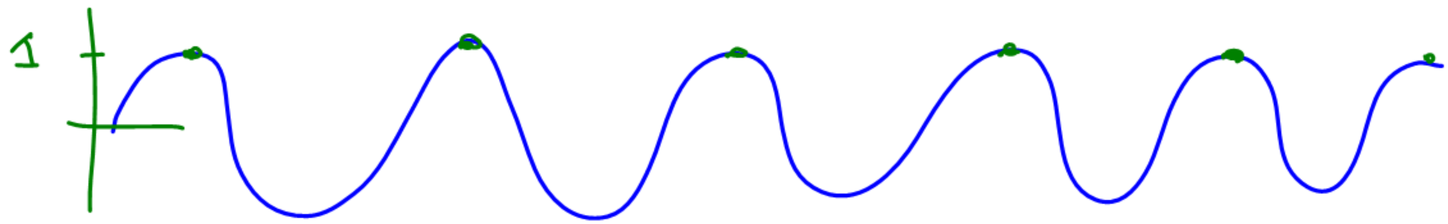
A **global max** occurs at $x = c$ if $f(c) \geq f(x)$ for **all** values of x in the domain.

It is possible to have several global maxima if the function reaches its peak value at more than one point.

Corresponding definitions apply for local and global minima.

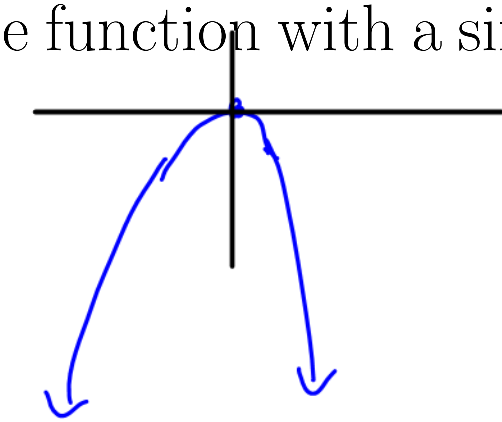
Problem. Give an example of a simple function with multiple global maxima.

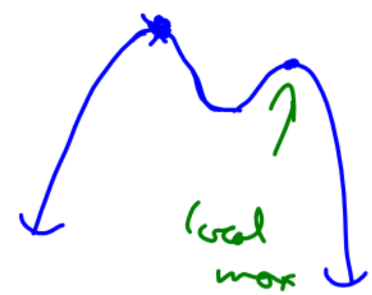
sin(x)



Problem. Give an example of a simple function with a single global maximum, but no global minimum.

$y = -x^2$





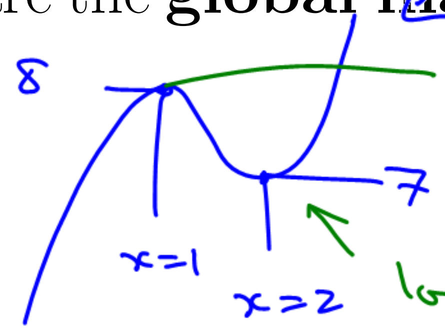
Problem. Give an example of a simple function with neither a global maximum nor a global minimum.

$y = x^3$

Earlier we worked with the function

$$f(x) = 2x^3 - 9x^2 + 12x + 3.$$

Problem. If we limit the function to the interval $x \in [0, 2.5]$, what are the **global max** and **global minimum** values on that interval?

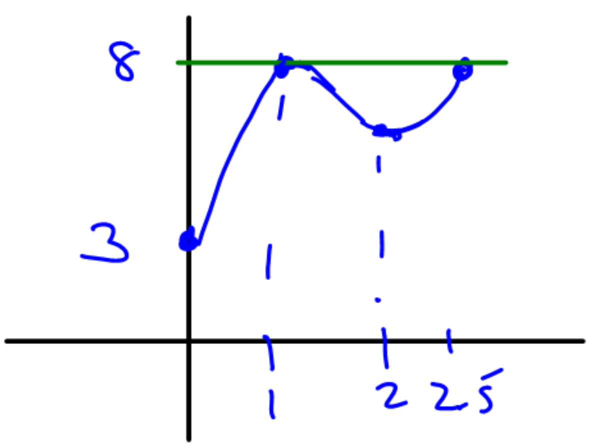


local max

local min

no global min
no global max

if we use all x's.



$f(0) = 3$

$f(1) = 8$

$f(2) = 7$

$f(2.5) = 8$

$x = 0$ is global min
$y = 3$

global max is $y = 8$ at $x = 1, 2.5$

Confirm your answer using MATLAB to plot the graph of $f(x)$.

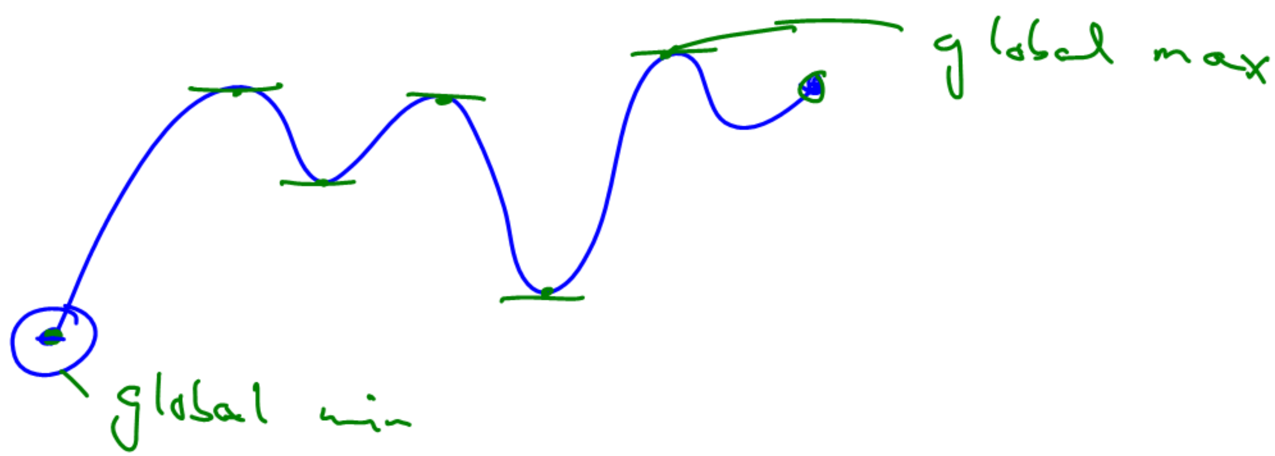
Global Extrema on Closed Intervals

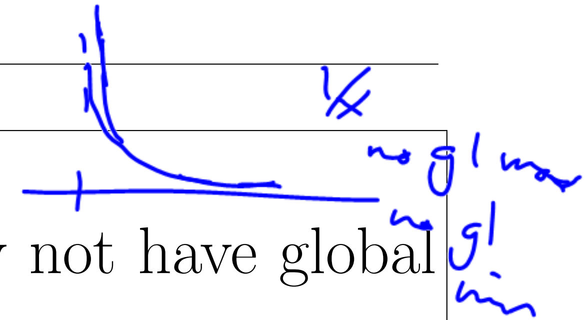
include end points

A continuous function on a closed interval will **always** have a global max and a global min value. These values will occur at either

- a critical point *or*
- an end point of the interval.

To find which value is the global extrema, you can compute the original function's values at all the critical points and end points, and select the point with the highest/lowest value of the function.





Global Extrema on Open Intervals

A function defined on an open interval may or may not have global maxima or minima.

If you are trying to demonstrate that a point is a global max or min, and you are working with an open interval, including the possible interval $(-\infty, \infty)$, proving that a particular point is a global max or min requires a careful argument. A recommendation is to look at either:

- values of f when x approaches the endpoints of the interval, or $\pm\infty$, as appropriate; or
- if there is only one critical point, look at the sign of f' on either side of the critical point.

With that information, you can often construct an argument about a particular point being a global max or min.

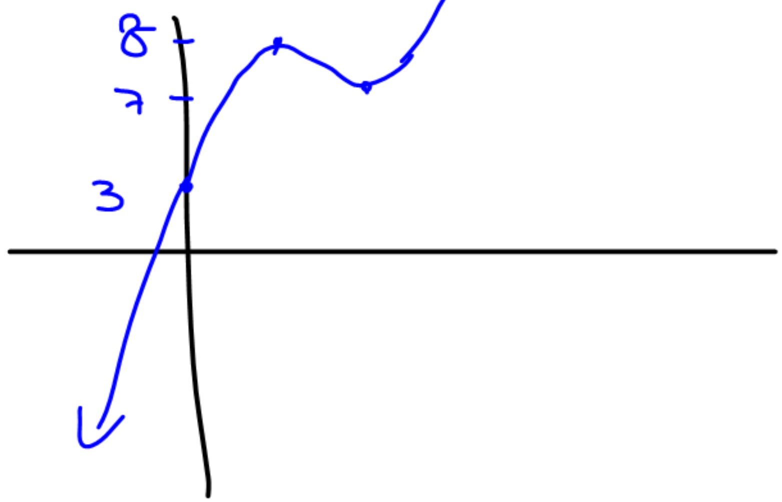
Problem. Determine whether the function

$$f(x) = 2x^3 - 9x^2 + 12x + 3$$

has a global max and/or min.

neither of those w/all x's

(open interval $(-\infty, \infty)$)



no $\quad (x-2)^4 \to \infty$ as $x \to \infty$

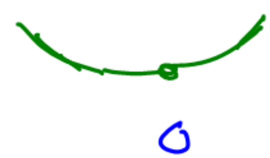
Problem. Determine whether the function $f(x) = (x-2)^4$ has a global max and/or min.

$$f'(x) = 4(x-2)^3(1)$$
$$= 4(x-2)^3$$

all x's (open interval)

Critical points $\quad$ Set $f' = 0$:

$$0 = 4(x-2)^3$$
$$0 = (x-2)^3$$
$$(x-2) = 0$$
$$\boxed{x = 2}$$ is the only critical point



sketch of $f(x)$

local max or min?

sign of $f'(x)$ $\quad (-) \quad (+)$

$\Rightarrow x = 2$ is a local min (First deriv test)

$\Rightarrow$ global min at $x = 2$ b/c slopes are consistently $(-)$ left, $(+)$ right for all x's!

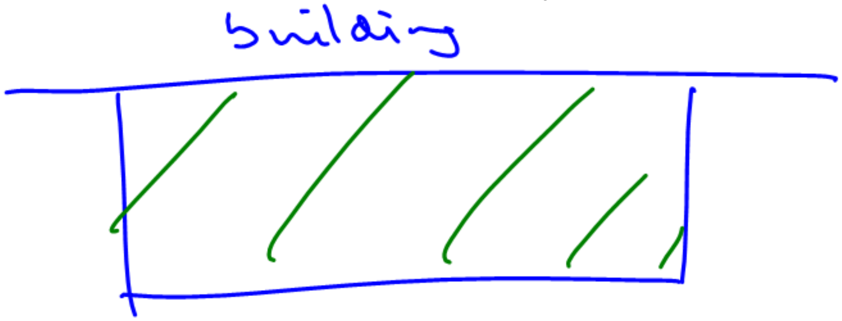
Optimization

critical points

An optimization problem is one in which we have to find the maximum or minimum value of some quantity. In principle, we already know how to find the maximum and minimum values of a function if we are given a formula for the function and the interval on which the maximum or minimum is sought. Usually the hard part in an optimization problem is interpreting the word problem in order to find the formula of the function to be optimized.

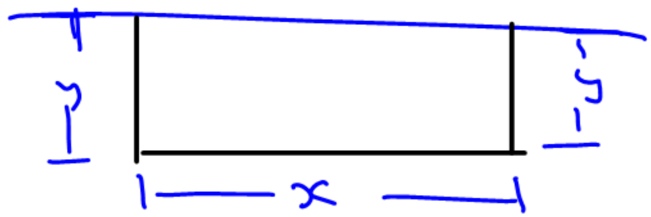
Problem. A site manager wants to build a rectangular security fence to surround a site. The manager has 480 meters of wire fencing with which to build a fence, and one side of the enclosure will be part of the side an already existing building (so there is no need to put up fence on that side). What should the dimensions of each side be to maximize the area enclosed?

What is the quantity to be maximized in this example?



Area

Problem. What are the variables in this question, and how are they related? You may want to draw a picture.



(1) $2y + x = 480$ m of available fencing.

Area

Express the quantity to be optimized in terms of the variables. Try to eliminate all but one of the variables.

(1) Isolate y

$$2y = 480 - x$$

$$y = \frac{(480 - x)}{2}$$

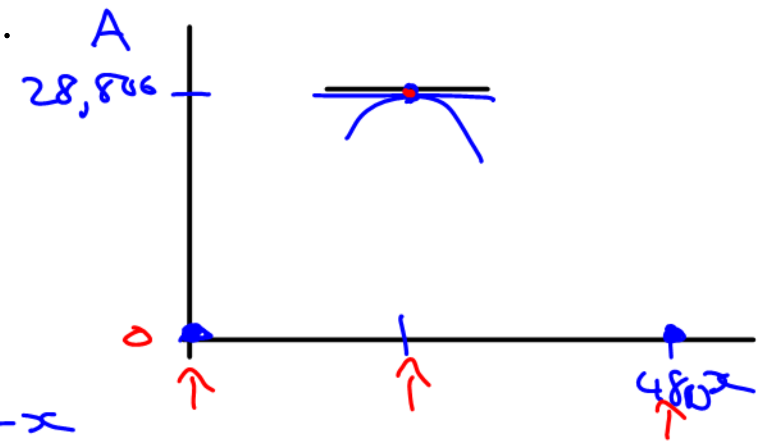
$$A = x \cdot y$$

$$A(x) = x\left(\frac{480 - x}{2}\right)$$

What is the domain on which the one remaining variable makes sense?

acceptable x values $\quad x \geq 0 \quad x \leq 480$

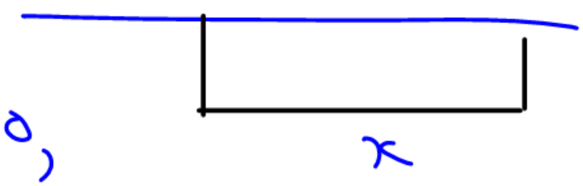
Problem. Use the techniques learned earlier in the course to maximize the enclosed by the fence. Give reasons explaining why the answer you found is the **global** maximum.



$$A = x\left(\frac{480-x}{2}\right) = 240x - \frac{x^2}{2}$$

Look for critical points:

$$\frac{dA}{dx} \text{ or } A' = 240 - \frac{2x}{2} = 240 - x$$



Find critical points by setting $A'(x) = 0$, solving for x

$$0 = 240 - x \qquad \boxed{x = 240} \quad \text{only critical point.}$$

$$A(240) = 240(240) - \frac{(240)^2}{2} = 28,800 \text{ m}^2$$

On a closed interval, global max occurs at endpoint or a critical point. must be a global max.

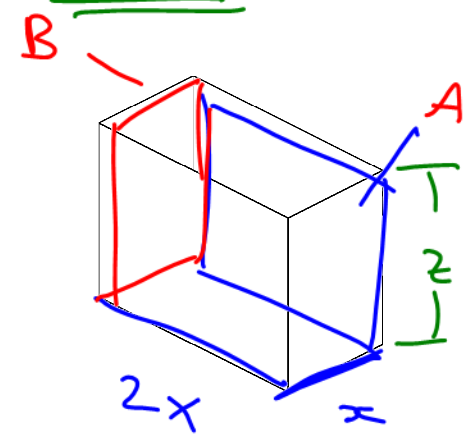
$\Rightarrow x = 240$, $A = 28,800$

(continued)

Problem. Confirm your solution using a MATLAB graph of your function for the enclosed area.

Problem. (Storage Container)
A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is to be twice its width. Material for the base costs \$10.00 per m^2, and material for the sides costs \$6.00 per m^2. Determine the cost of the material for the cheapest such container.

Goal: Minimize cost



Cost = base + cost sides

$= \text{base} + 2(A) + 2(B)$

$= \frac{\$10}{m^2}(x)(2x) + 2(\$6\cdot(2x)(z)) + 2(\$6\cdot x\cdot z)$

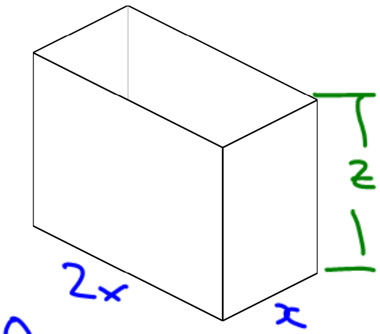
$= 20x^2 + 24xz + 12xz$

$= 20x^2 + 36xz$

$$C(x) = 20x^2 + 36xz$$

$$= 20x^2 + 36x\left(\frac{5}{x^2}\right) \qquad 180(x^{-1})$$

$$= 20x^2 + \frac{180}{x}$$

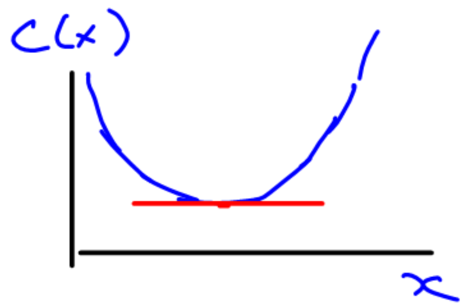


Vol = 10

$$10 = (x)(2x)(z)$$

$$10 = 2x^2 z$$

$$z = \frac{10}{2x^2} = \frac{5}{x^2}$$



Find critical point of $C(x)$

Find $C'(x) = 40x - \dfrac{180}{x^2}$

Set $C'(x) = 0$: $\quad 0 = 40x - \dfrac{180}{x^2}$

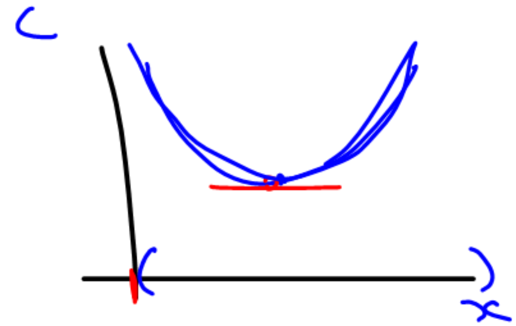
$$\frac{18\cancel{0}}{x^2} = 4\cancel{0}x$$

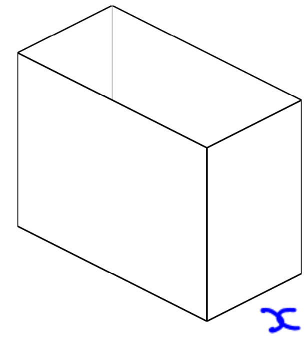
$$18 = 4x^3$$

$$x^3 = \frac{18}{4} = \frac{9}{2}$$

so $x = \sqrt[3]{9/2}$ is the only crit point.

$x = \sqrt[3]{9/2}$





$C(x) = 20x^2 + \frac{180}{x}$

$C'(x) = 40x - \frac{180}{x^2}$

$\downarrow$

$C''(x) = 40 - 180(-2x^{-3})$

$= 40 + \frac{360}{x^3}$ (+) for $x > 0$

$C(x)$ concave up $\Rightarrow x = \sqrt[3]{9/2}$ is global min for cost

$x \cong 1.65$ m $\qquad z = \frac{5}{(1.65)^2} = 1.84$ m

$C(1.65) = \$163.54$

Optimization in MATLAB

MATLAB has built-in numerical methods tools that can also perform optimization, given the formula for a function. However, those tools have the same limitations as our earlier work with numerical methods for solving equations.

These **numerical methods** are all **fancy versions of guess and check**! This means numerical solutions are a poor second choice, compared to by-hand solving:

- Numerical optimums give no insight into the optimal value (existence, patterns).
- Finding numerical optimums usually requires some amount of trial and error by the user.

Despite these limitations though, numerical methods are great in that they can be applied to any function, and don't require any by-hand solving for critical points.

Defining New MATLAB Functions

$f(\theta) = e^{\theta} + 2$

User-written functions are an important tool in MATLAB. For purposes of our optimization problems, we will use the simplest implementation, called **anonymous functions**.

```
f = @(x)  x.^2;
```

function that takes x as input, returns x^2

```
x = linspace(-2, 2);
plot(x, f(x));
```

Optimizing MATLAB functions

The single-variable optimization function in MATLAB is called `fminbnd`.

Problem. Look at the help menu for `fminbnd`.

f – minimize

– with bounds

Problem. Use `fminbnd` to find a minimum of $f(x) = e^{-x} + x$.

Graph $f(x) = e^{-x} + x$ and add a point at the minimum that was found.

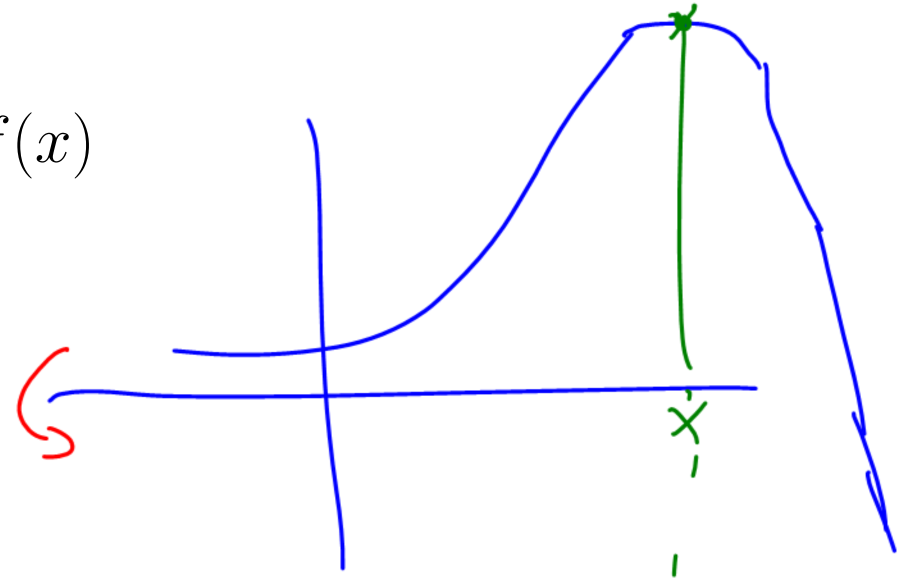
Problem. Use `fminbnd` to find the global minimum of $g(x) = x\sin(x)$ on the interval $x \in [0, 4\pi]$.

Graph $g(x)$ and comment on steps you need to take to find the global minimum.

MATLAB Optimization - Further Examples

You'll notice that command **`fminbnd`** does **not** have a sister function **`fmaxbnd`**. Instead, MATLAB requires users to turn any maximization problem they have into a minimization problem themselves.

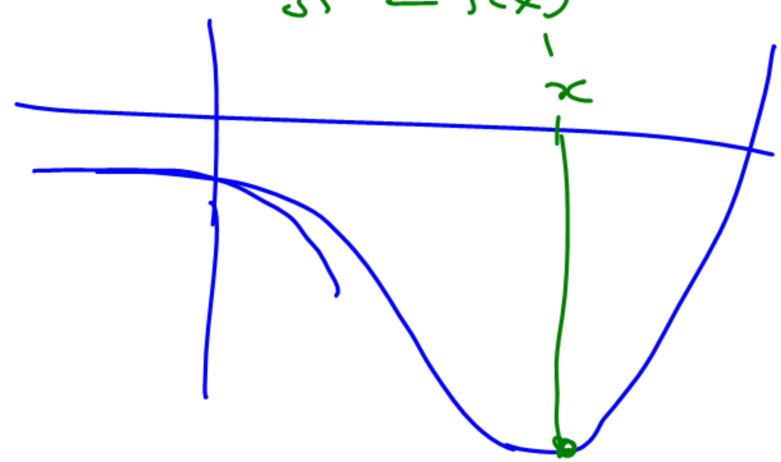
Problem. Sketch a graph $y = f(x)$ which has a local maximum.



max

$\Rightarrow$ find location of min(s) of $-f(x)$

Now sketch the graph of $y = -f(x)$.
What do you notice about the x locations of the minima on this function?



Problem. Use `fminbnd` to find the global **maximum** of $g(x) = x\sin(x)$ on the interval $x \in [0, 4\pi]$.

Problem. Use **`fminbnd`** to find the length of side fence x that maximizes an enclosed rectangular area, if one side is against the wall. The total length of fence is 480 m.

$$A(x) = x\frac{(480 - x)}{2}$$

Also find the total area enclosed with the optimal design.

Notation Confusion

There is an unfortunate coincidence in MATLAB notation that often catches new users.

Problem. Contrast the meaning of the `y(1.5)` in these two MATLAB examples.

```
x = linspace(-2, 2);
y = x.^2;

y(1.5)
```

y vector

↳ y(1.5) = find the 1.5th element of the vector

```
x = linspace(-2, 2);
y = @(x)  x.^2;

y(1.5)
```

$= 1.5^2 = 2.25$

↳ y as a function

put x=1.5 into formula