Week #5 : Integrals - Techniques

Goals:

- Recognize the family of functions that can be solved with the technique of integration by substitution.
- Solve integration problems using the technique of substitution.
- Recognize the family of functions that can be solved with the technique of integration by parts.
- Solve integration problems using the technique of integration by parts.

We now return to the challenge of finding a *formula* for an anti-derivative function. We saw simple cases last week, and now we will extend our methods to handle more complex integrals.

Anti-differentiation by Inspection: The Guess-and-Check Method

Often, even if we do not see an anti-derivative immediately, we can make an educated guess and eventually arrive at the correct answer.

Problem. Based on your knowledge of derivatives, what should the anti-derivative of $\cos(3x)$, $\displaystyle\int \cos(3x)\ dx$, look like?

Problem. Find $\displaystyle\int e^{3x-2}\,dx$.

Both of our previous examples had *linear* ‘inside’ functions. Here is an integral with a *quadratic* ‘inside’ function:

$$\int xe^{-x^2}\,dx$$

Problem. Evaluate the integral.

Why was it important that there be a factor x in front of e^{-x^2} in this integral?

Integration by Substitution

We can formalize the guess-and-check method by defining an *intermediate variable* the represents the "inside" function.

Problem. Show that $\int x^3\sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2}+C$.

$$\int x^3\sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2} + C$$

Problem. Relate this result to the **chain rule**.

Problem. Now use the **method of substitution** to evaluate $\displaystyle\int x^3\sqrt{x^4+5}\,dx$

Steps in the Method Of Substitution

1. Select a simple function $w(x)$ that appears in the integral.
   - Typically, you will also see w' as a **factor** or **multiplier** in the integrand as well.
2. Find $\dfrac{dw}{dx}$ by differentiating. Re-write it in the form $\ldots dw = \ dx$
3. Rewrite the integral using only w and dw (no x nor dx).
   - If you can now evaluate the integral, the substitution was effective.
   - If you cannot remove all the x's, or the integral became harder instead of easier, then either try a different substitution, or a different integration method.

Problem. Find $\int \tan(x)\; dx$.

Though it is not required unless specifically requested, it can be re-assuring to check the answer.

Problem. Verify that the anti-derivative you found is correct.

Problem. Find $\displaystyle\int x^3 e^{x^4-3}\,dx$.

Problem. For the integral,

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2} dx$$

both $w = e^x - e^{-x}$ and $w = e^x + e^{-x}$ are seemingly reasonable substitutions.

Question: Which substitution will change the integral into the simpler form?

1. $w = e^x - e^{-x}$
2. $w = e^x + e^{-x}$

Problem. Compare both substitutions in practice.

$$\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2}\, dx$$

with $w = e^x - e^{-x}$	with $w = e^x + e^{-x}$

Problem. Find $\displaystyle\int \frac{\sin(x)}{1+\cos^2(x)}dx.$

Using the Method of Substitution for Definite Integrals

If we are asked to evaluate a **definite** integral such as

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin x} dx \quad ,$$

where a substitution will ease the integration, we have two methods for handling the limits of integration ($x = 0$ and $x = \pi/2$).

a) When we make our substitution, convert both the *variables* x and the *limits* (in x) to the new variable; or

b) do the integration while keeping the limits explicitly in terms of x, writing the final integral back in terms of the original x variable as well, and *then* evaluating.

Problem. Use method (a) (converting both the integral and the limits to the new variable) to evaluate the integral

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin x} dx$$

Problem. Use method (b) (converting back to x's to evaluate at the end points) to evaluate

$$\int_9^{64} \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} dx \quad .$$

Integration by Parts

So far in studying integrals we have used

- direct anti-differentiation, for relatively simple functions, and
- integration by substitution, for some more complex integrals.

However, there are many integrals that can't be evaluated with these techniques.

Problem. Try to find $\displaystyle\int xe^{4x}\,dx$.

This particular integral can be evaluated with a different integration technique, **integration by parts.** This rule is related to the **product rule** for derivatives.

Problem. Expand

$$\frac{d}{dx}(uv) =$$

Integrate both sides with respect to x and simplify.

Express $\displaystyle\int u\frac{dv}{dx}\,dx$ relative to the other terms.

Integration by Parts

For short, we can remember this formula as

$$\int udv = uv - \int vdu$$

Integration by parts:

- Choose a part of the integral to be u, and the remaining part to be dv.
- **Differentiate** u to get du.
- **Integrate** dv to get v.
- Replace $\displaystyle\int u\,dv$ with $\displaystyle uv - \int vdu$.
- Hope/check that the new integral is easier to evaluate.

Problem. Use integration by parts to evaluate $\int xe^{4x}\,dx$.

Problem. Verify that your anti-derivative is correct.

Integration By Parts - Examples

Guidelines for selecting u and dv

- Ensure you can actually integrate the dv part by itself, then
- Try to select u and dv so that either
  - u' is simpler than u or
  - $\int dv$ is simpler than dv

Problem. Find $\displaystyle\int x \cos x \; dx$.

Problem. Now evaluate the slightly more challenging integral

$$\int x^2 \cos x \, dx$$

$$\int x^2 \cos x \, dx$$

Integration By Parts - Definite Integrals

When using integration by parts to evaluate *definite* integrals, you need to apply the limits of integration to the **entire** anti-derivative that you find.

Problem. Evaluate $\displaystyle\int_0^{\pi} x \sin 4x \; dx$

Don't forget that dv does not require any other factors besides dx. That can help when there is only a single factor in the integrand.

Problem. Find the area under the graph of $\ln x$ between $x = 1$ and $x = 2$.

General integration advice:

- Look for a substitution in your integral first - they are the simplest method to use, and usually the most obvious.
- Only try integration by parts if substitution fails.
- With all methods, you may need to **experiment** with your choice of u, dv, or your substitution.