Week #8 : Second Order Differential Equations

Goals:

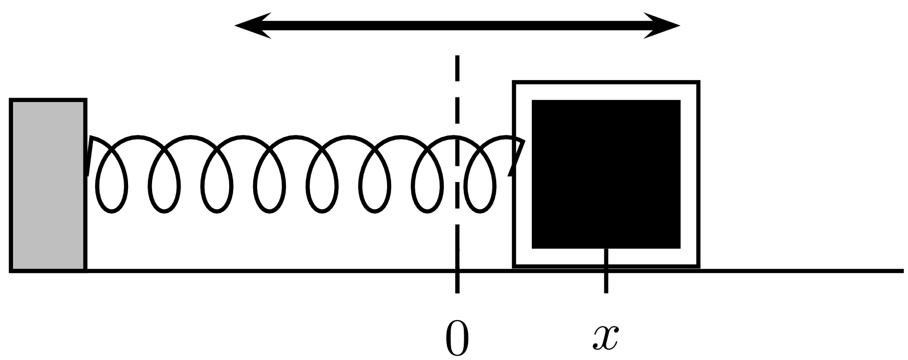
- Express real world situations in terms of second order linear differential equations.
- Describe the difference between homogeneous and nonhomogeneous second order linear differential equations.
- Use MATLAB to solve linear and nonlinear second order differential equations, both homogeneous and nonhomogeneous.

Second-Order Linear Equations - Spring System

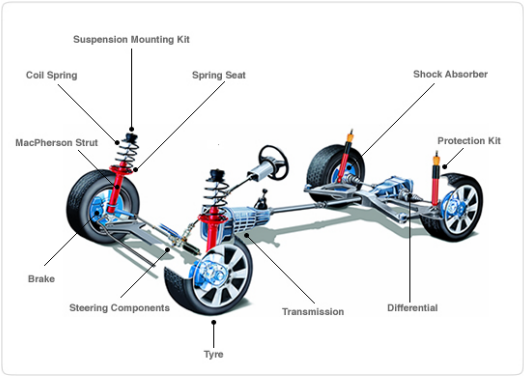
So far we have seen examples of **first-order DEs**, or equations with first derivatives of some unknown function.

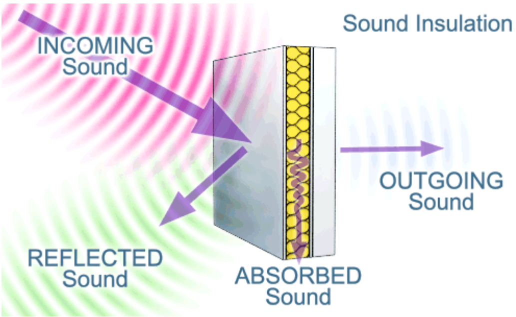
From here on in the course, we will study differential equations with **second or higher derivatives**.

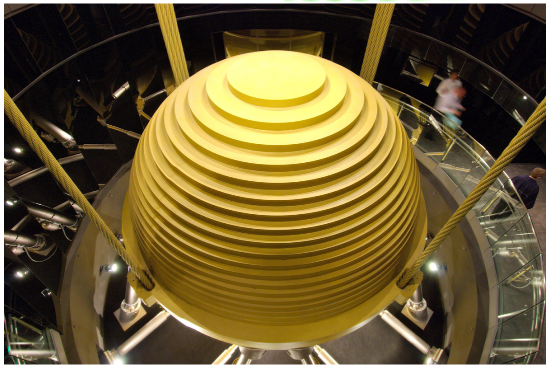
One classic source of differential equations of this type comes from analyzing the forces on a block at the end of a spring.



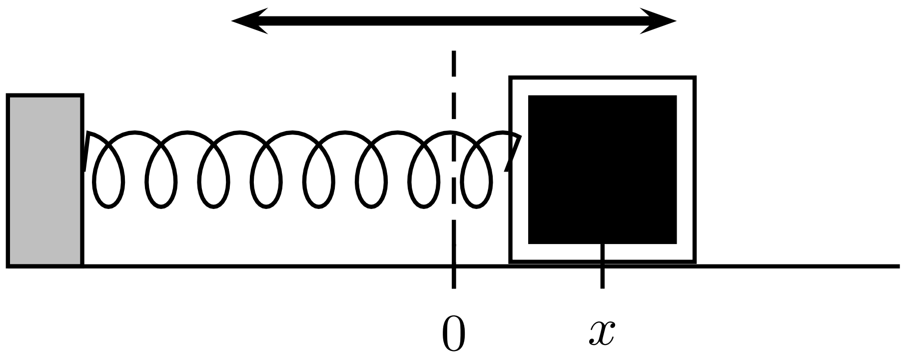
While the mathematics behind this simple system will be very interesting in their own right, we should also note at the outset that the simple spring/mass model can be applied to a wide variety of not-so-obviously related real-world problems.


Suspension Mounting Kit
Coil Spring
Spring Seat
Shock Absorber
MacPherson Strut
Protection Kit
Brake
Steering Components
Transmission
Differential
Tyre


Sound Insulation
INCOMING Sound
OUTGOING Sound
REFLECTED Sound
ABSORBED Sound



Spring System Analysis



In this system, how would you describe x in words?

Problem. Draw a free-body diagram for the mass. Indicate the magnitude of the forces, assuming

- the mass of the block is m kg, and
- the spring constant (in N/m) is given by the constant k.

Let us work with our intuition about this system before beginning the mathematics.
If the spring is very stiff, is k large or small?

Period: the length of time to complete one full cycle/oscillation.

If we increase the stiffness of the spring, do you expect the *period* of the oscillations to increase or decrease? Why?

If we increase the mass, do you expect the *period* of the oscillations to increase or decrease? Why?

If we know k and m, and assume that friction is negligible, should we be able to determine the exact period of the oscillations?

From the work so far, can we easily find the formula for the period?

The spring system is an excellent introduction to higher-order differential equations because

- we all have an intuition about how it *should* work physically,
- the mathematics and physics are simple, and
- there's no obvious way to predict critical features (e.g. the period) from the given information.

We clearly need some new tools!

Spring System as a DE

Use Newton's second law, $F = ma$, to construct an equation involving the position $x(t)$.

What order of differential equation does $F = ma$ produce for this spring/mass system?

To simplify matters temporarily, let us assume that both $k = 1$ N/m and $m = 1$ kg. Rewrite the previous differential equation.

This differential equation invites us to find a function $x(t)$ whose second derivative is its own negative. What function(s) would satisfy that?

Having found two (and more) solutions to the differential equation for the spring/mass system

Classifying Second-Order DEs

Classify the following DEs based on the terms *homogeneous*, *linear* and *constant coefficients*

$$x^2y'' + xy' + y = 10$$

$$100y'' + y = 4x^3$$

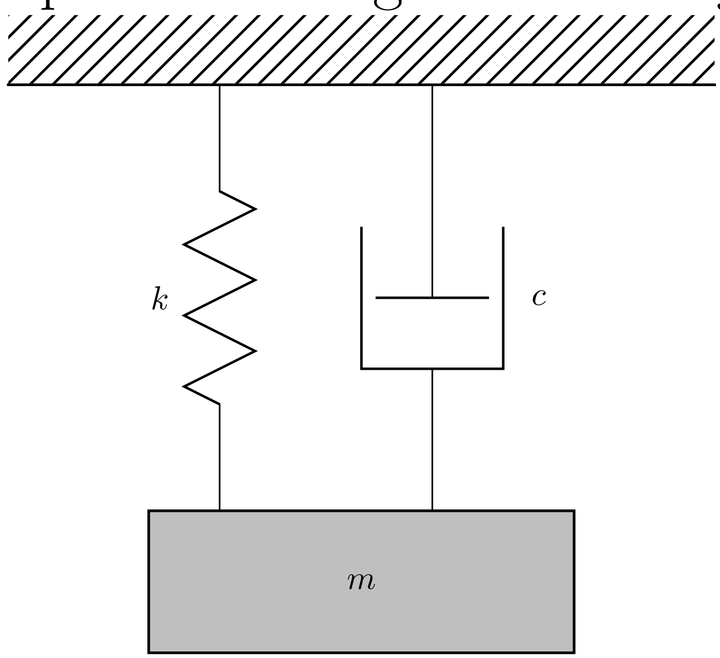
$$(y'')^2 + y = 4e^x$$

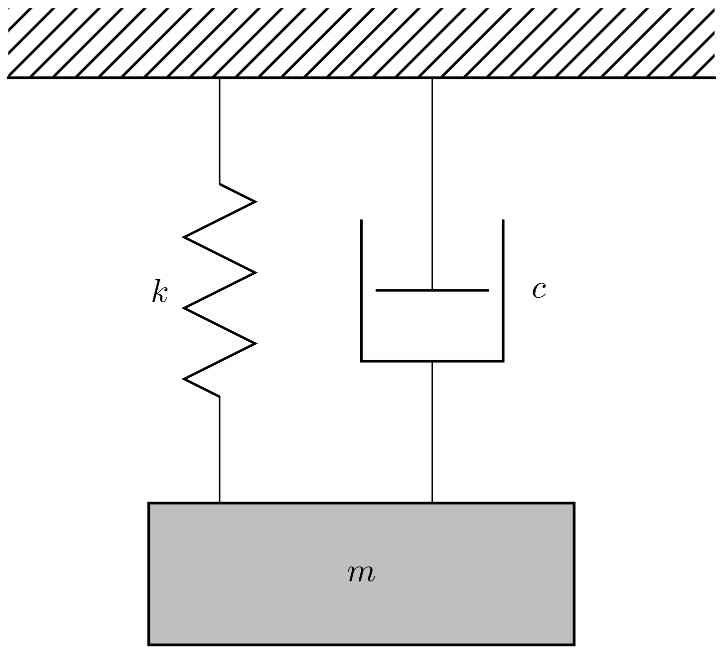
$$4y'' - 10y' + y = 0$$

Mechanical Vibrations - Spring-mass system

We now consider the spring/mass system seen earlier, but with more detail.

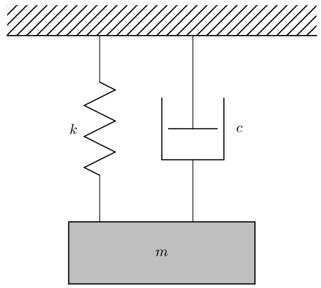
Consider a mass m hanging on the end of a vertical spring of unstretched length ℓ. Using Newton's second law, build a differential equation that governs the system.




k
c
m

Problem. Consider a mass of 0.5 kg with spring constant $k = 2\ \mathrm{N} \cdot \mathrm{m}^{-1}$ in an undamped unforced system. Assume the mass is displaced 0.4 m from equilibrium and released. Describe the long-term behaviour of the system.

Unforced Spring/Mass System - Patterns of Behaviour



$$my'' = -ky - cy'$$

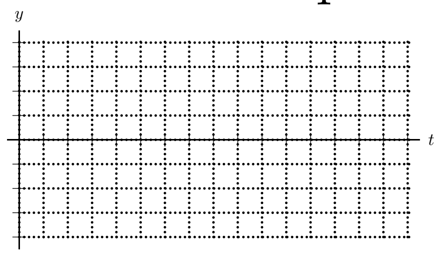
or

$$my'' + cy' + ky = 0$$

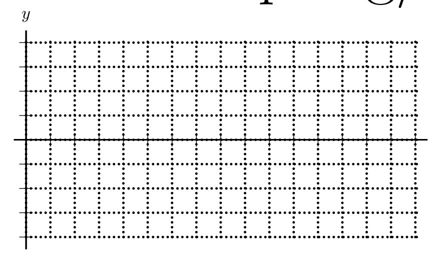
$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Damping	$c^2 - 4km$	r	Description
None $c = 0$			
Light $c^2 < 4km$			
Heavy $c^2 > 4km$			

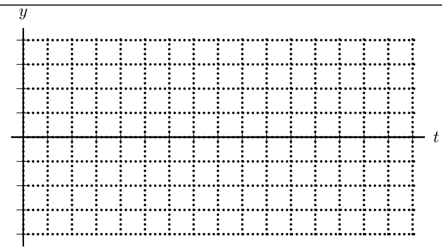
Problem. Sketch possible solutions for all four spring/mass cases.



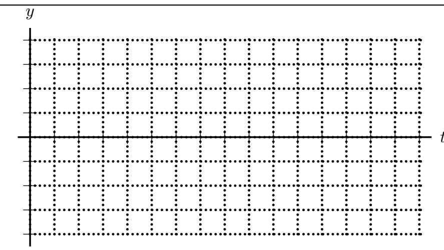
Undamped



Damped



Critically Damped



Overdamped

Demonstration - Spring/Mass

We will demonstrate how the solutions to the damped spring/mass DE change as the damping is gradually increased.

$$my'' + cy' + ky = 0$$

In this demonstration, we will use $m = 1$ kg, and $k = 25$ N/m.

$$y'' + cy' + 25y = 0$$

Problem. What damping level will produce *critical damping*?

What will the form of the solutions be when damping is **below** critical?

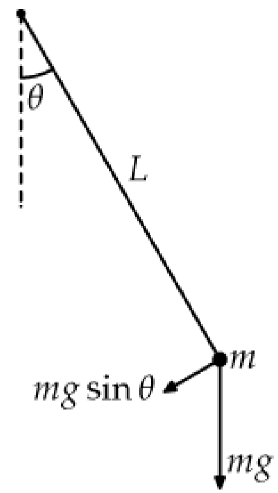
What will the form of the solutions be when damping is **above** critical?

During demonstration, try to ask yourself the following questions:

- As damping increases in general, does the **graph** of the solution change gradually or dramatically?
  What about the **mathematical form** of the solution?
- Near critical damping specifically, does the **graph** of the solution change gradually or dramatically?
  What about the **mathematical form** of the solution?

Applications - Pendulum

Problem. Consider the simple pendulum (mass at the end of a rod) shown below.



If we start from the rotational (torque) version of Newton's Second Law,

$$(\text{moment of inertia}) \cdot (\text{angular accel}) = \sum \text{torques}$$

we obtain

$$(mL^2) \cdot (\theta'') = -mgL\sin(\theta)$$

or, in (almost) standard form:

$$\theta'' + \frac{g}{L}\sin(\theta) = 0$$

Use a well-known approximation from calculus to simplify this to a linear DE:

What limitations does this put on our interpretation of the solution?

Find the general solution to the linearized differential equation

$$\theta'' + \frac{g}{L}\theta = 0$$

Use your solution to predict the period of the oscillations of a pendulum, given g and the length of the rod, L.

Comment: what does this mean about the swinging of a pendulum for larger angles?

Nonhomogeneous Linear Differential Equations

In the previous lectures, we have learned how to solve linear DEs with constant coefficients. Any degree was fine, but the equations had to be **homogeneous**:

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \ldots + a_1y' + a_0y = 0$$

This week, we explore how to deal with equations with more interesting right-hand sides.

The general solution of a nonhomogeneous n-th order linear differential equation has the form

$$y = \underbrace{c_1 y_1 + \cdots + c_n y_n}_{y_c} + y_{NH},$$

where

- $y_1, \ldots, y_n$ span the solution space to the corresponding **homogeneous** equation,
- $c_1, \ldots, c_n$ are arbitrary constants,
- y_c, the collection of $y_1, \ldots y_n$ and $c_1, \ldots c_N$, is called the "complementary solution" to the **homogeneous** equation, and
- y_{NH} is a specific solution to the **non**homogeneous equation.

Basic Strategy for Nonhomogeneous Linear Equations

1. Find a basis for the solution space of the **corresponding homogeneous equation**. Call those solutions $y_1, \ldots, y_n$.

2. Find a **single** solution, y_{NH}, to the **nonhomogeneous** equation.

3. The general solution to the nonhomogeneous DE will be

$$y = y_{NH} + \underbrace{(c_1 y_1 + \ldots c_n y_n)}_{y_c}$$

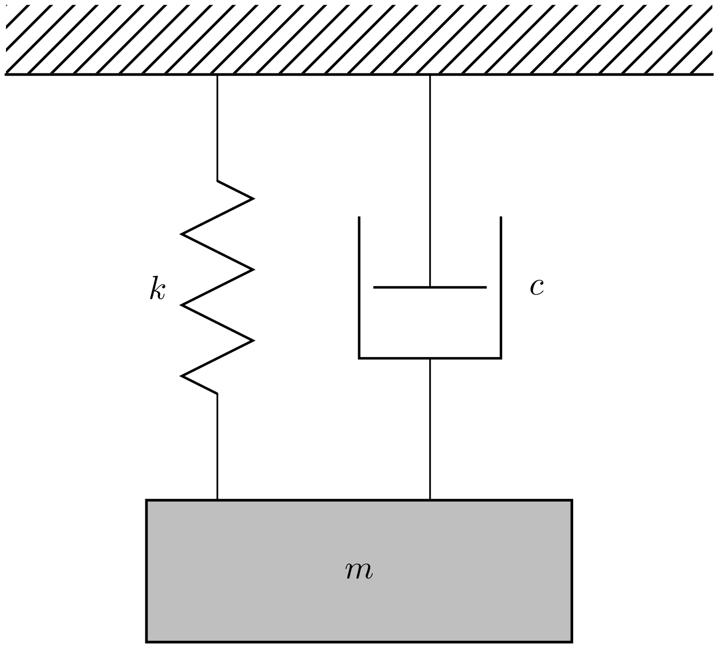
Periodic External Forces - Applications

We will next explore some of the ramifications of the nonhomogeneous solution form

$$y = \underbrace{c_1 y_1 + \cdots + c_n y_n}_{y_c} + y_{NH}$$

in several application areas.

Spring/Mass System



Problem.

Write out the DE for the position of the mass, given $F_{\text{ext}} = A\sin(Bt)$.

Problem. Sketch an RLC circuit, with the voltage supply (or 'electromotive force') of $E(t) = \dfrac{-A}{B}\cos(Bt)$, and write out the DE for the current in the circuit.

What might give rise to an EMF of this form?

Comment on the similarity between the mathematical model of the spring/mass, and of the RLC circuit.

We will explore the solutions of/predictions for the spring/mass system (and RLC circuits!) through changing parameters in a simulation. We will consider both

- graphs of the solutions, and
- mathematical forms of the solutions.

Your job during the demonstrations is to track:

- the amount of friction,
- the *natural* frequency (in y_c), and
- the *stimulating* frequency (in Fe and y_{NH})

Unforced Motion

Setting $F_{\text{ext}} = 0$, review the effect of increasing the friction coefficient c on the y_c solution.

What is the y_{NH} solution for all these simulations, and why?

Forced, but Undamped Motion

Now we turn off the friction (by setting $c = 0$) to zero, and start to apply the external force.
We set $F_{\text{ext}} = 10\sin(1.0t)$.
What is the form of y_c, and what is the form of y_{NH}?

Experiment with the frequency in F_{ext}, in the range 0.5 to 3, or well above 5. Does the motion of the mass over time make sense?

Resonance

Set $F_{\text{ext}} = 1\sin(5t)$. What is special about the frequency 5 rad/s?

Describe the graph of the solution.

Give the mathematical reasons for the graph of the simulation.

The condition of having the external force at *exactly* the same frequency as the natural vibrations is called **resonance.** Note: for true resonance, there must be *no friction.*

Near-Resonance

Admitedly, resonance is a unique event requiring **perfect** matching of the stimulating force with the natural frequency.

Explore frequencies in F_{ext} *close to* 5. Describe the resulting solutions, both for their amplitude and any other response.

Give the mathematical reasons for the shape of the graph for near-resonance response.

Note again that there is *no friction* with this response.

Adding in Friction - Practical Resonance

Set $F_{\text{ext}} = 1\sin(5t)$ again, but set friction c to 0.1 Compare this solution to the one without friction, both graphically and mathematically.

Gradually increase the amount of friction. Describe how the solution changes.

How is this more realistic than the undamped case?

How do the mathematical forms of the pure-resonance (undamped) and practical resonance (some damping) compare?

Transient and Steady-State Solutions

More generally, when there is friction in the spring/mass system and an oscillatory external force, we can break the solution into two parts: *transient* and *steady state*.

We set $F_{\text{ext}} = 1\sin(t)$, and $c = 1$ for friction, which is fairly high for an oscillating system. Describe the resulting behaviour.

Specifically, what are the two natural frequencies in the solution?

Which frequency is present in the **long run**?

In a damped physical system with forced oscillations, is y_{NH} **transient** or **steady-state**, and why?

In a damped physical system, is y_c **transient** or **steady-state**, and why?

Reminder: we study the simple damped spring/mass system in such depth because

- the system displays all the interesting mathematical solution forms as we vary just a few parameters,
- we hope that the simplicity of the system, and its resemblance to familiar systems like swings or bouncing a basket ball, help to you associate the mathematics with the real-world behaviour.
- the DE for the spring/mass system is either identical or similar to those for a number of other real-world scenarios, like electronic circuits.

Analytic solutions to DEs

To compute an analytic solution, we must

- Recognize form of DE
- Properly execute solution technique
- Include "+C" at appropriate moment
- Solve for C using initial conditions

Finally get $y(t) =$

Unfortunately, that can all be a lot of work. Sometimes, we just want the prediction of y, position, velocity, etc. over time.

- In *principle*, the pair (DE + initial condition) is enough to define prediction for all time.
- The difficulties in finding analytic solutions come from the necessary integration/solution step.
- MATLAB can generate an *approximate numerical solution* **without** need to integrate!

In MATLAB

There are two main ingredients to generating a numerical solution/prediction from a differential equation:

- Define the DE as a MATLAB function.
- In a separate script, pass the DE to a DE solver, along with the initial conditions and the desired time span for the simulation.

Problem. Create a file called `tempDE.m`, which computes the derivative based on the DE

$$\frac{dy}{dt} = -k(y - T_{\mathrm{ext}})$$

Note that the MATLAB function is a **direct implementation** of the DE formula: you do **not** do any solving at this stage!

Generating Numerical Solution

Problem. Search for "first order ODEs" or "initial value problem" in MATLAB help.

Problem. What form of differential equation does MATLAB assume we have?

Note that differential equation solvers are a big component of MATLAB: you may have to do some reading to sort out the kind of equation you have, and what the appropriate solving tool is.

Problem. Which of the solvers is recommended as a "first try" solver?

ode45

The first solver to reach for in MATLAB is `ode45`. To run it, we need

- the DE function **in form** $\dfrac{dy}{dt} = f(t, y)$;
- the time span for the solution/simulation, $[t_0, t_{\text{end}}]$; and
- the initial condition $(y(t_0))$

Example - Step 1

The following comes from the script `W8_1.m`, on the course web site.

```
% Define temp DE in the form dy/dt = f(t, y)
% and set other constants
k = 0.7;  % /min

T_ext = 20; % external/environment temp

DE = @(t, y) tempDE(y, k, T_ext);
```

Example - Step 2

```
% Solve based on initial condition
y0 = 100;  % initial temperature

time_span = [0, 30];  % interval for solution

[t, y] = ode45(DE, time_span, y0);

plot(t, y);
```

To decipher and work with the output, it is critical that you understand what MATLAB provides.

- The final values of `t` and `y` are

- `t` starts at

- `t` ends at

- `y` starts at

Basic Process

How does MATLAB do it?

Example - Pendulum

$$\begin{aligned}
\text{Newton's Second Law: } mL^2\theta'' &= T_g + T_f \\
&= -mLg\sin(\theta) - (\mu L^2 m)\theta' \\
\text{Solving for } \theta''\text{: } \theta'' &= -\frac{g}{L}\sin(\theta) - \mu\theta'
\end{aligned}$$

Problem. Recalling your work from your differential equations class, turn this single second-order DE into a pair of first-order DEs.

Generality of 1st Order Systems

Out of all the techniques you learned in your DE class, this transformation of higher-order DEs to systems of first-order ones will be essential for this class:

MATLAB solvers like `ode45` **only** handle 1st order systems of DEs

In practice, this is not a limitation because:

Problem. Download the representation of this system of DEs, `pendulumDE.m`. Compare it to our DE system,

$$\frac{dy_1}{dt} = y_2$$
$$\frac{dy_2}{dt} = -\frac{g}{L}\sin(y_1) - \mu y_2$$

Problem. Download and run the script `W8_2.m`.

Problem. What were the initial conditions of the pendulum?

Problem. Based on that information, what do the two curves on the graph represent?

Problem. Change the script so the pendulum also has an initial velocity.

Problem. If we keep the initial angle at $-\frac{\pi}{2}$ (pendulum out horizontally), experiment and find the initial velocity that will push the pendulum "over the top".

Vector-based Pendulum Model

We are now going to model the pendulum's motion as if it were free to move anywhere in space (two full degrees of freedom). This will require a more complicated model, but will be more general (i.e. more like the roller coaster scenario!)

Free Body Diagram The obvious way to determine how the pendulum would move is to consider all the forces involved. For now we will just use gravity and the rod (no friction).

Problem. Draw the free body diagram for the pendulum's mass.

Problem. Write out as a force vector.

The force is more complicated.

Problem. If the pendulum is at the position (x, y), what is the **direction** of ?

Problem. What is the **magnitude** of ?

Problem. Write out a formula or calculation to determine , given the position (x, y).

Creating the Differential Equations

Problem. Use Newton's Second Law, $ma = \sum F$, to obtain a differential equation for x and another for y.

Problem. Using the vector $\vec{w} = [x, x', y, y'] = [w_1, w_2, w_3, w_4]$ rewrite our second-order system of DEs for x and y as a first-order system for $\vec{w} = [w_1, w_2, w_3, w_4]$.

In MATLAB

Problem. Download the file `pendulumXYDE.m`. Complete it by computing the values of $\dfrac{d\vec{w}}{dt} = \vec{w}'$, based on the current state, $\vec{w}$. You are literally transcribing the 1st order DE system we just found into MATLAB.

Problem. Download and run the script `W8_3.m`. This script simply sets the initial conditions, the time span for the simulation, then calls `ode45` to do the simulating.

Problem. Fix any errors to get the simulation to run.

Problem. The output of the simulation is a vector `t` and a matrix `w`. Sketch them and what their contents mean.

Problem. Add the command we used last class, `plot(t, w)`, to the script and re-run the script.

Problem. Interpret the graphs produced.

Problem. Modify the script so that MATLAB shows the path or trajectory of the pendulum over the entire simulation. Comment on the quality of the simulation.

Accuracy

`ode45` generates a *numerical approximation* to the real trajectory. As soon as we say "approximation", we mean "there is error". Clearly, the default level of error is unacceptable!

Problem. Look up the function `odeset` in MATLAB Help. Look for the key words like 'tolerance' and 'error'.

Problem. In `W8_3.m`, replace

```
[t, w] = ode45(DE, [0, t_end], w0);
```

with

```
options = odeset('RelTol', 1e-10)
[t, w] = ode45(DE, [0, t_end], w0, options);
```

Problem. Comment on the change in the quality of the simulation.

Extension Exercises

Problem. Change the static plot into an animation, showing the location of the pendulum, as well as the connecting rod and the axis of rotation.

Problem. Change the initial condition to $(x, y) = (-1, 1)$, above the axis of rotation and show the resulting trajectory.

Problem. Add an initial **velocity** of 1 m/s to the model. Hint: in what direction must the initial velocity be for a pendulum?

Problem. Add a friction force to the differential equations.