Week #1 : Derivatives - Foundations

Goals:

- Interpret the derivative, and be able to discuss the difference between the secant line and the derivative
- Compute the derivative of polynomial, exponential, logarithmic, powers, trigonometric functions and their combinations, with the correct application of the product and quotient rules, and the chain rule
- Report the graphs, domain and range of the inverse trigonometric functions arcsin, arccos and arctan.
- Apply the derivative rules for arcsin, arccos and arctan.

Introduction

The two fundamental ideas in calculus are:

- The **derivative**, which gives the rate of change of a function, and

- The **integral**, which computes the total accumulation based on a rate.

Many fundamental quantities in physics are related through derivatives and integrals. For example:

Position

Velocity

Acceleration

We will spend the first half of this course examining differential calculus and integral calculus.

The assumption is that most of what is covered will be review so the pace will be fairly aggressive. We will also start including some calculations and graphical analysis using MATLAB.

Take advantage of the practice problems to practice your skills, and get in touch with the instructor if you need any help.

Slopes, Secants and the Derivative

The **slope of a secant line** gives

- the **average rate of change** of $f(x)$ over some interval Δx.
- the **average velocity** over an interval, if $f(t)$ represents **position**.
- the **average acceleration** over an interval, if $f(t)$ represents **velocity**.

Problem. Give the units of the slope of a secant line.

The **derivative** gives

- the **limit of the average slope** as the interval Δx approaches zero.
- a **formula for slopes** for the tangent lines to $f(x)$.
- the **instantaneous rate of change** of $f(x)$.
- the **velocity**, if $f(t)$ represents **position**.
- the **acceleration**, if $f(t)$ represents **velocity**.

Problem. Give the units of the derivative.

Differentiability

The definition of the derivative is based on the classic $\dfrac{\text{rise}}{\text{run}}$ formula for slopes.

$$f'(x) = \frac{d}{dx}f = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A function f is **differentiable** at a given point a if it has a derivative at a, or the limit above exists. There is also a graphical interpretation differentiability: **if the graph has a unique and finite slope at a point**. Since the slope in question is automatically the slope of the tangent line, we could also say that

> f IS DIFFERENTIABLE AT a IF ITS GRAPH HAS A (NON-VERTICAL) TANGENT AT $(a, f(a))$.

For functions of the form $y = f(x)$, we do not consider points with vertical tangent lines to have a real-valued derivative, because a vertical line does not have a finite slope.

Here are the ways in which a function can fail to be differentiable at a point a:

1. The function is not continuous at a.
2. The function has a corner (or a cusp) at a.
3. The function has a vertical tangent at $(a, f(a))$.

Problem. Sketch an example graph of each possible case.

Reminder: Differentiability is Common

You will notice that, despite our concern about some functions *not* being differentiable, most of our standard functions (polynomials, rationals, exponentials, logarithms, roots) **are** differentiable at most points. Therefore we should investigate what all these possible derivative/slope values could tell us.

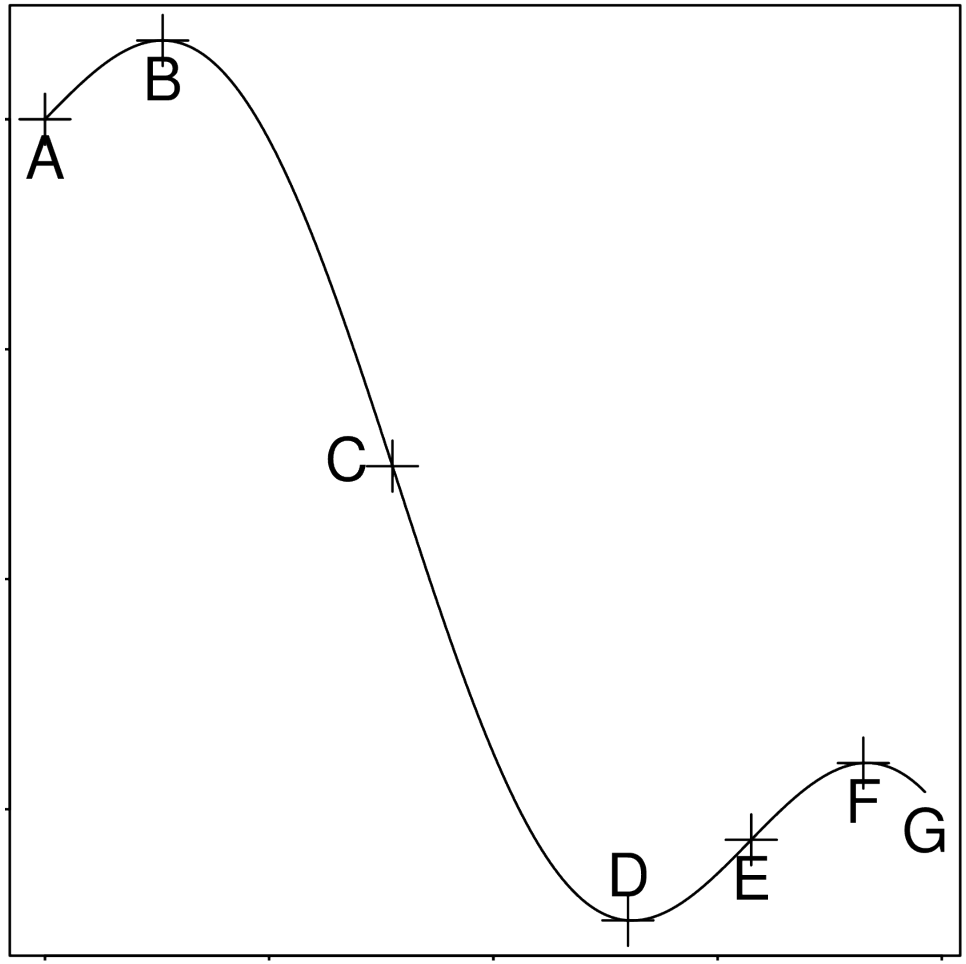
Interpreting the Derivative

- Where $f'(x) > 0$, or the **derivative is positive**, $f(x)$ is **increasing**.

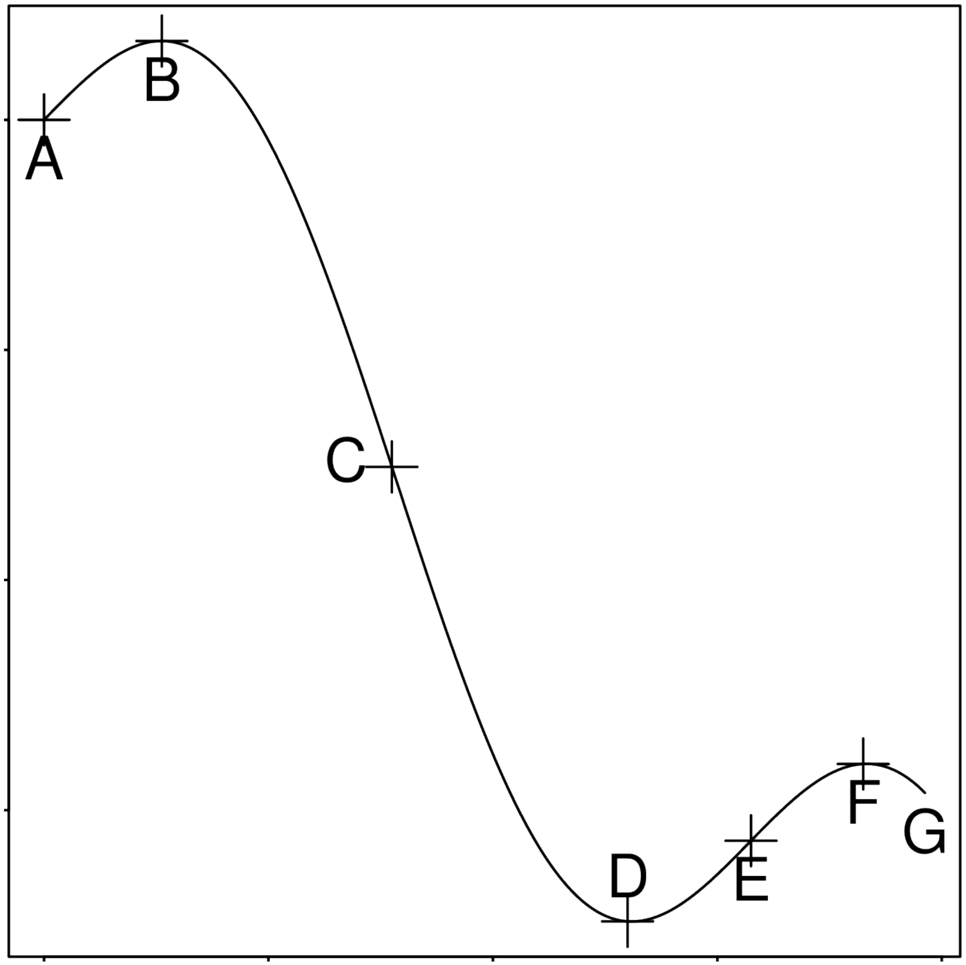
- Where $f'(x) < 0$, or the **derivative is negative**, $f(x)$ is **decreasing**.

- Where $f'(x) = 0$, or the **tangent line to the graph is horizontal**, $f(x)$ has a **critical point**.

Problem. Consider the graph of $f(x)$ shown below.



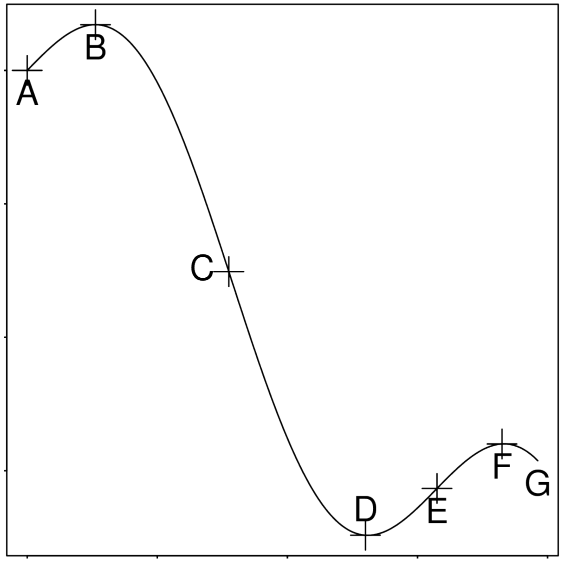
On what intervals is $f'(x) > 0$?

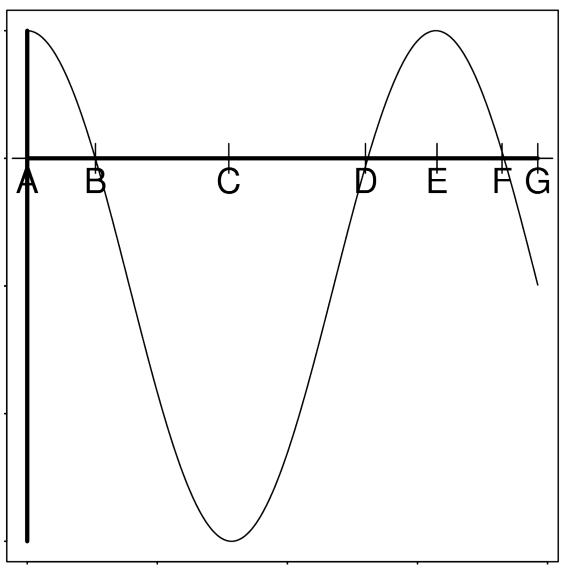


Problem. Where does $f'(x)$ takes on its **largest negative value**?

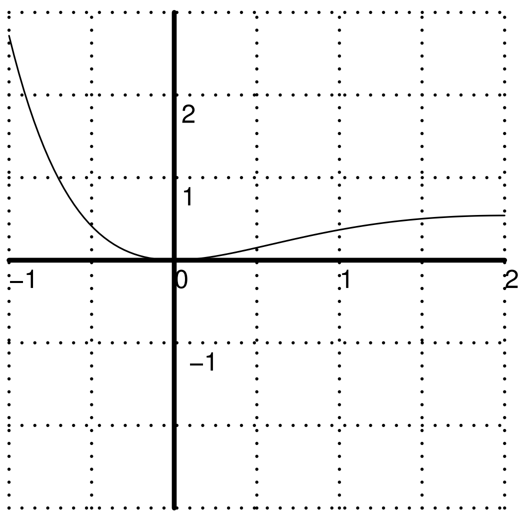
Graphs, and Graphs of their Derivatives

Problem. Consider the same graph again, and the graph of its derivative. Identify important features that associate the two.

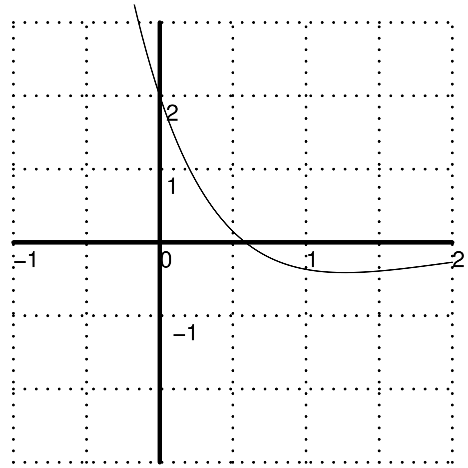




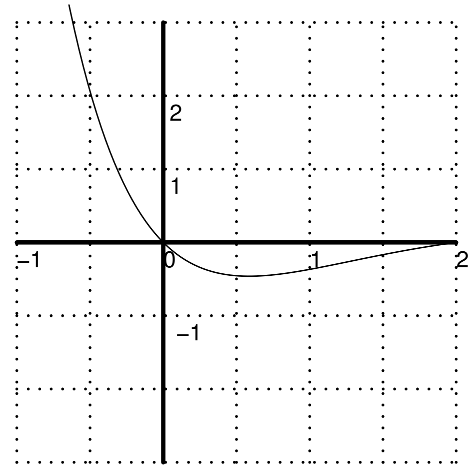
Problem. Consider the graph of $f(x)$ shown:



$f(x)$



A



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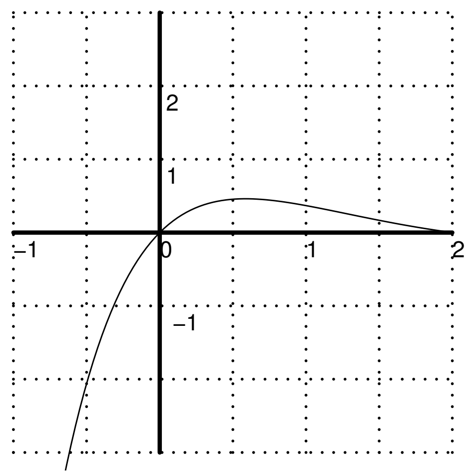

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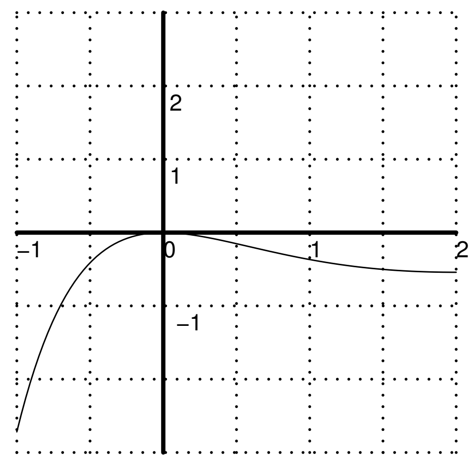

D

Which of the graphs on the right is the graph of the **derivative** of $f(x)$?

Computing Derivatives

Beyond the graphical interpretation of derivatives, there are all the algebraic rules. **All of these rules** are based on the **definition** of the derivative,

$$f'(x) = \frac{d}{dx}f = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

However, by finding common patterns in the derivatives of certain families of functions, we can compute derivatives much more quickly than by using the definition.

Sums, Powers, and Differences

Constant Functions: $$\frac{d}{dx}\,k = 0$$

Power rule: $$\frac{d}{dx}\,x^p = px^{p-1}$$

Sums : $$\frac{d}{dx}\,f(x) + g(x) = \left(\frac{d}{dx}\,f(x)\right) + \left(\frac{d}{dx}\,g(x)\right)$$

Differences: $$\frac{d}{dx}\,f(x) - g(x) = \left(\frac{d}{dx}\,f(x)\right) - \left(\frac{d}{dx}\,g(x)\right)$$

Constant Multiplier: $\displaystyle\frac{d}{dx}\,[kf(x)] = k\left(\frac{d}{dx}\,f(x)\right)$, so long as k is a constant

Problem. Evaluate the following derivatives:

$$\frac{d}{dx}\left(x^5 - 2x\right)$$

$$\frac{d}{dx}\left(4.1\sqrt{x} + \frac{4}{x}\right)$$

Question: The derivative of $-3t^2 - \dfrac{1}{t^2}$ is

1. $-6t^3 + 2\dfrac{1}{t^3}$

2. $-6t + 2\dfrac{1}{t^3}$

3. $-6t - 2\dfrac{1}{t^3}$

4. $-t^3 + 2\dfrac{1}{t}$

Exponentials and Logs

e as a base: $$\frac{d}{dx}\,e^x = e^x$$

Other bases: $$\frac{d}{dx}\,a^x = a^x(\ln(a))$$

Natural Log: $$\frac{d}{dx}\,\ln(x) = \frac{1}{x}$$

Other Logs: $$\frac{d}{dx}\,\log_a(x) = \frac{1}{x}\frac{1}{\ln(a)}$$

Problem. Evaluate the following derivatives:

$$\frac{d}{dx}\left(3 \cdot 10^x + 10 \cdot x^3\right)$$

$$\frac{d}{dx}\left(e^x + \log_{10}(x)\right)$$

(Exponential and log derivatives are relatively straightforward, until we mix in the product, quotient, and chain rules.)

Product and Quotient Rules

Products: $$\frac{d}{dx} f(x) \cdot g(x) = f'(x)g(x) + f(x)g'(x)$$

Quotients: $$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

Problem. Evaluate the following derivatives:

$$\frac{d}{dx} \left(2.1x^4 \ln(x)\right)$$

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{e^x}\right)$$

$$\frac{d}{dt}\left(4t^3 2^t\right)$$

Question: The derivative of $\dfrac{10^x}{x^3}$ is

(a) $\dfrac{10^x}{\ln(10)}x^{-3} + 10^x(-3x^{-4})$

(b) $\dfrac{10^x \ln(10)x^3 - 10^x(3x^2)}{x^6}$

(c) $\dfrac{10^x \frac{1}{\ln(10)}x^3 - 10^x(3x^2)}{x^6}$

(d) $\ln(10)10^x x^{-3} + 10^x(-3x^{-4})$

Chain Rule

Nested Functions: $\dfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

Liebnitz form $\dfrac{d}{dx} f(g(x)) = \dfrac{df}{dg}\dfrac{dg}{dx}$

Problem. Evaluate the following derivatives:

$$\frac{d}{dt}e^{t^4}$$

$$\frac{d}{dx}\ln(1+\sqrt{x})$$

$$\frac{d}{dx}\left(\frac{1}{\ln(x)+x^3}\right)$$

$$\frac{d}{dt}\left(e^{5t-1} + 10^{3t}\right)$$

Question: The derivative of $e^{\sqrt{x}}$ is

(a) $\frac{1}{2}e^{\frac{1}{\sqrt{x}}}$

(b) $e^{\sqrt{x}}\left(\sqrt{x}\right)$

(c) $\frac{1}{2}e^{\sqrt{x}}\left(\frac{1}{\sqrt{x}}\right)$

(d) $\frac{1}{2}e^{\sqrt{x}}\left(\sqrt{x}\right)$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

Problem. Prove the derivative rule for $\dfrac{d}{dx}\tan(x)$, using the definition $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and the other derivative rules.

Problem. Find the derivative of $4 + 6\cos(\pi t^2 + 1)$

(a) $4 - 6\sin(\pi t^2 + 1) \cdot (2\pi t)$

(b) $-6\cos(\pi t^2 + 1) \cdot (2\pi t)$

(c) $-6\sin(\pi t^2 + 1) \cdot (2\pi t)$

(d) $-6\sin(\pi t^2 + 1) \cdot (\pi t^2 + 1)$

(e) $6\sin(2\pi t)$

Inverse Trig Functions

In addition to the 6 trig functions just seen, there are 6 inverse functions as well, though the inverses of sine, cosine, and tangent are the most commonly used.

Problem. Sketch the graph of $\sin(x)$ on the axes below.

On the same axes, sketch the graph of $\arcsin(x)$, or $\sin^{-1} x$, or the inverse of $\sin(x)$.

Problem. What is the domain of $\arcsin(x)$?

What is the range of $\arcsin(x)$?

Problem. Sketch the graphs of $\arccos(x)$ and $\arctan(x)$.

Graph of $\arccos(x)$

Graph of $\arctan(x)$

Derivative of Inverse Trig Functions

Problem. Simplify $\sin(\arcsin x)$.

Differentiate both sides of this equation, using the chain rule on the left. You should end up with an equation involving $\dfrac{d}{dx} \arcsin x$.

Problem. Solve for $\frac{d}{dx}\arcsin x$, and simplify the resulting expression by means of the formula

$$\cos\theta = \sqrt{1-\sin^2\theta},$$

which is valid if $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

From this, we see that

$$\frac{d}{dx} \arcsin x =$$

Graph of $\arcsin(x)$

Graph of $\dfrac{d}{dx} \arcsin(x)$

Through a similar process, we can also show that:

$$\boxed{\begin{gathered}\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \\[2em] \frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}} \\[2em] \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}\end{gathered}}$$

Problem. Note any common themes or differences, related to the earlier trigonometric derivatives.

Problem. Evaluate the following derivatives.

$$\frac{d}{dt}\arcsin\left(\frac{x}{4}\right)$$

$$\frac{d}{dx}\arctan\left(e^{3x}\right)$$