# MNTC P01 - Week #8 - Second Order Differential Equations

1. Use `ode45` to generate a graph of the solution to the following DEs, over the specified interval, given the initial condition.

   (a) $\dfrac{dy}{dt} = t^2 + y^2$, $y(0) = 0$, and $0 \leq t \leq 1$.

   (b) $\dfrac{dy}{dt} = \sin(t) + \cos(y)$, $y(0) = 0$, and $0 \leq t \leq 10$.

   (c) $\dfrac{dy}{dt} = (1 - y^2) + 0.2\sin(t)$, $y(0) = 0$, and $0 \leq t \leq 20$.

2. Newton's law of heating and cooling states that an object with temperature $T$ in an environment at temperature $T_{ext}$ will heat up or cool down according to the differential equation

$$\frac{dT}{dt} = -k(T - T_{ext})$$

   Consider a garage used as a workshop. Its insulation and surface area give $k$ a value of 0.1, if time $t$ is measured in hours and the temperatures, $T$ and $T_{ext}$, are in degrees Celsius.

   The temperature outside changes during the day, as described by the formula

$$T_{ext} = 10 + 7\cos\left(\frac{\pi}{12}t\right)$$

   We now imagine that the power goes out, with the garage at $23^o$ C at $t = 0$.

   (a) Use ode45 and the DE to generate a numerical prediction of the garage's temperature $T$ over time. Graph the solution over a time interval that shows both the initial and long-term behaviour of the temperature.

   In your script, try to use the functions `title`, `xlabel`, `ylabel`, and `legend` to annotate the graph to make it easier for a reader to understand.

   For the following questions, just use the graph or the numerical prediction of the temperature. You are *not* expected to solve the DE analytically.

   (b) How many days does it take for the garage to get into a consistent temperature cycle? (You will need to estimate this by eye.)

   (c) How many degrees does the temperature in the building fluctuate by, once the temperature gets into a steady cycle?

   (d) Suppose the building were better insulated, so that the rate of heat loss were cut in half. Should $k$ be half as large, or twice as large?

   (e) Generate a numerical prediction for the temperature over time in the better-insulated scenario, and produce a graph of the temperature vs time for both scenarios on the same axes.

   (f) How large are the temperature fluctuations in the building, now that the extra insulation has been added? Does halving the net heat flow also halve the net temperature fluctuations?

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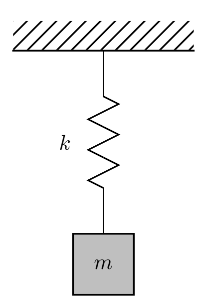
## Modelling Spring Systems

3. Consider the single spring/mass system shown to the right, with no damper.
   Newton's second law gives us the relationship:

$$ma = \sum F = F_{\text{spring}}$$
$$mx'' = -kx$$

   where $k$ is the spring constant.

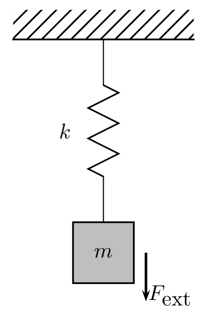


(a) By hand, write this second order DE as a system of 1st order DEs, using the new variables $w_1 = x$ and $w_2 = x'$

(b) Write a MATLAB function file called `springDE1.m` starting with the first line
`function dw_dt = springDE1(t, w, m, k)`
that implements the system of differential equations from part (a).

(c) Write a MATLAB script that simulates the motion of the mass using $m = 0.5$ kg and $k = 10$ N/m. Choose the time interval for the simulation so that 4-5 cycles of oscillation are shown.

4. Consider the single spring/mass system shown to the right, with no damper.
This is the same as in Question 3, except with the addition of the $F_{\text{ext}}$ shown as an external applied force.
Newton's second law gives us the relationship:




$$\begin{aligned} ma &= \sum F = F_{\text{spring}} + F_{\text{ext}} \\ mx'' &= -kx + F_{\text{ext}} \end{aligned}$$

where $k$ is the spring constant.

(a) By hand, write this second order DE as a system of 1st order DEs, using the new variables $w_1 = x$ and $w_2 = x'$

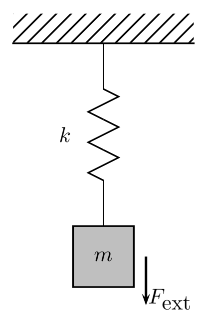
(b) We will now incorporate an external force of the form $F_{\text{ext}} = a\sin(bt)$. Write a MATLAB function file called `springDE2.m` starting with the first line
`function dw_dt = springDE2(t, w, m, k, a, b)`
that implements the system of differential equations from part (a).

(c) Create a new MATLAB script. In the script, set $m = 0.5$ kg, $k = 10$ N/m, and use $a = 5$ and $b = 1$ in $F_{\text{ext}} = a\sin(bt)$. Use `ode45` to simulate the motion of the spring for 30 seconds (`tspan = [0, 30]`), given an initial displacement of $x(0) = 0.2$ m, and initial velocity of zero: $x'(0) = 0$.

(d) Explain why the motion looks so disorganized.

(e) Repeat Question (4c), but with an external force of $F_{\text{ext}} = \sin(4t)$. Explain why the motion in this case has cyclic waves in its amplitude.

5. We return to the same single spring/mass system from Question 4, shown to the right, with no damper.
The $F_{\text{ext}}$ shown is an external applied force.




(a) For a mass of $m = 5$ kg , and a spring constant of $k = 2$ N/m, what is the natural frequency of the system?

(b) Define an external force of the form $F_{\text{ext}} = \sin(bt)$ that will produce **resonance** in the system.

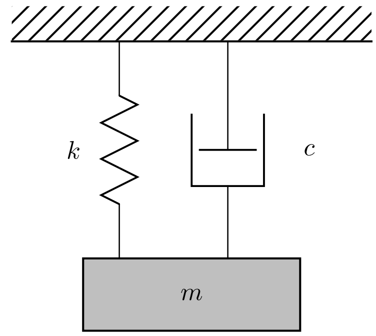
(c) Use MATLAB to simulate the motion of the spring, with your selected external force, using an initial condition where the mass starts at its equilibrium and at rest.

(d) The system will break if the oscillations become too large, specifically if $x(t)$ exceeds 2 m (in the positive or negative directions). Does the system break, and if so, how long does it take for the system to break?

6. Consider the damped system shown at right.
The damping force exerted by the dashpot/damper is proportional to the velocity of the mass.
Newton's second law gives us the relationship:

$$\begin{aligned} ma &= \sum F = F_{\text{spring}} + F_{\text{damp}} \\ mx'' &= -kx - cx' \end{aligned}$$

where $k$ is the spring constant in N/m, and $c$ is the damping coefficient in N/(m/s).




(a) By hand, write this second order DE as a system of 1st order DEs, using the new variables $w_1 = x$ and $w_2 = x'$

(b) We define a system with a mass of $m = 10$ kg, spring constant $k = 2$ N/m, and a damping coefficient of $c = 0.4$ N/(m/s). If the system is displaced by 0.5 m and then let go with zero initial velocity, use MATLAB to find out how long (in both seconds and cycles) it takes for the oscillations to reach approximately 10% of their original amplitude. Note: you will need to write both a MATLAB function for the differential equation, and a main script to run the simulation.

(c) What damping coefficient would be needed for the oscillations to be reduced to 10% of their original amplitude within 3 cycles? You will need to estimate your answer based on guessing and checking against the graph. Hint: add horizontal lines to the solution plot at $x = 0.05$ and $x = -0.05$ to see easily whether the oscillations are reduced to that level.

7. For a spring/mass system with $m = 1$ kg, $c = 0.5$ N/(m/s), and $k = 45$, approximately what frequency of external forcing would produce the largest amplitude steady-state vibration? Give your answer to the nearest 0.5 rad/s.

You will need to estimate the answer based on guessing and checking against the graph.