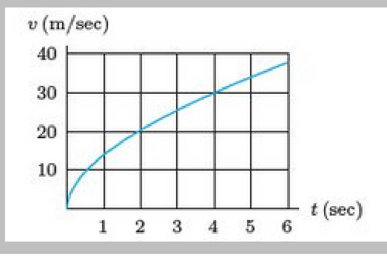
MNTC P01 - Week #4 - Integrals - Foundations

Distance And Velocity

1. The graph below shows the velocity, v, of an object (in meters/sec). Estimate the total distance the object traveled between $t = 0$ and $t = 6$.

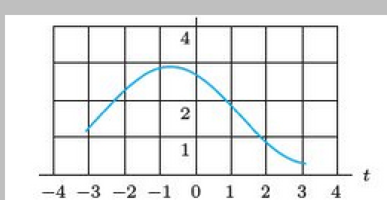


Just counting the squares (each of which has area representing 10 (m/s)·(s) = 10 m of distance), and allowing for the partial squares, we can see that the area under the curve from $t = 0$ to $t = 6$ is between 140 and 150 units. Therefore the distance traveled is between 140 and 150 meters.

2. The figure below shows the velocity of a particle, in cm/sec, along the t-axis for $-3 \leq t \leq 3$ (t in seconds).

   (a) Describe the motion in words. Is the particle changing direction or always moving in the same direction? Is the particle speeding up or slowing down?

   (b) Make over- and underestimates of the distance traveled for $-3 \leq t \leq 3$.



(a) The velocity is always positive, so the particle is moving in the same direction throughout. However, the particle is speeding up until shortly before $t = 0$, and slowing down thereafter.

(b) The distance traveled is represented by the area under the curve. Using whole grid squares, we can overestimate the area as 3+3+3+3+2+1 = 15 squares, and we can underestimate the area as 1+2+2+1+0+0= 6 squares. Each square represents 1 (cm/sec)·(s) = 1 cm, so the particle moved in one direction between 6 and 15 cm.

3. Consider the following table of values for $f(t)$.

t	15	17	19	21	23
$f(t)$	10	13	18	20	30

   (a) If we divide the time interval into $n = 4$ sub-intervals, what is Δt? What are t_0, t_1, t_2, t_3, t_4? What are $f(t_0), f(t_1), f(t_2), f(t_3), f(t_4)$?

   (b) Find the left and right sums using $n = 4$.

   (c) If we divide the time interval into $n = 2$ sub-intervals, what is Δt? What are t_0, t_1, t_2? What are $f(t_0), f(t_1), f(t_2)$?

   (d) Find the left and right sums using $n = 2$.

(a) With $n = 4$, we have $\Delta t = 2$. Then
$t_0 = 15, t_1 = 17, t_2 = 19, t_3 = 21, t_4 = 23$,
and
$f(t_0) = 10, f(t_1) = 13, f(t_2) = 18, f(t_3) = 20, f(t_4) = 30$.

(b)

$$\begin{aligned}\text{Left sum} &= (10)(2) + (13)(2) + (18)(2) + (20)(2)\\ &= 122\\ \text{Right sum} &= (13)(2) + (18)(2) + (20)(2) + (30)(2)\\ &= 162\end{aligned}$$

(c) With $n = 2$,we have $\Delta t = 4$. Then
$t_0 = 15; t_1 = 19; t_2 = 23$ and
$f(t_0) = 10; f(t_1) = 18; f(t_2) = 30$.

(d)

$$\begin{aligned}\text{Left sum} &= (10)(4) + (18)(4) = 112\\ \text{Right sum} &= (18)(4) + (30)(4) = 192\end{aligned}$$

4. Consider the following table of values for $f(t)$.

t	0	4	8	12	16
$f(t)$	25	23	22	20	17

   (a) If we divide the time interval into $n = 4$ subintervals, what is Δt? What are t_0, t_1, t_2, t_3, t_4? What are $f(t_0), f(t_1), f(t_2), f(t_3), f(t_4)$?
   (b) Find the left and right sums using $n = 4$.
   (c) If we divide the time interval into $n = 2$ subintervals, what is Δt? What are t_0, t_1, t_2? What are $f(t_0), f(t_1), f(t_2)$?
   (d) Find the left and right sums using $n = 2$.

(a) With $n = 4$, we have $\Delta t = 4$. Then $t_0 = 0; t_1 = 4; t_2 = 8; t_3 = 12; t_4 = 16$ and $f(t_0) = 25; f(t_1) = 23; f(t_2) = 22; f(t_3) = 20; f(t_4) = 17$

(b)

$$\begin{aligned}\text{Left sum} &= (25)(4) + (23)(4) + (22)(4) + (20)(4)\\ &= 360\\ \text{Right sum} &= (23)(4) + (22)(4) + (20)(4) + (17)(4)\\ &= 328\end{aligned}$$

(c) With $n = 2$,we have $\Delta t = 8$. Then $t_0 = 0; t_1 = 8; t_2 = 16$ and $f(t_0) = 25; f(t_1) = 22; f(t_2) = 17$

(d)

$$\begin{aligned}\text{Left sum} &= (25)(8) + (22)(8) = 376\\ \text{Right sum} &= (22)(8) + (17)(8) = 312\end{aligned}$$

5. At time t, in seconds, your velocity, v, in meters/ second, is given by

$$v(t) = 1 + t^2 \text{ for } 0 \leq t \leq 6.$$

Use $\Delta t = 2$ to estimate the distance traveled during this time. Find the upper and lower estimates, and then average the two.

Using $\Delta t = 2$,

$$\begin{aligned}\text{Lower estimate} &= v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2\\ &= 1(2) + 5(2) + 17(2)\\ &= 46\\ \text{Upper estimate} &= v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2\\ &= 5(2) + 17(2) + 37(2)\\ &= 118\\ \text{Average} &= \frac{46+118}{2}\\ &= 82\\ \text{Distance traveled} &\approx 82\left(\frac{\text{m}}{\text{s}}\right)(\text{s}) = 82 \text{ meters.}\end{aligned}$$

6. For time, t, in hours, $0 \leq t \leq 1$, a bug is crawling at a velocity, v, in meters/ hour given by

$$v = \frac{1}{1+t}.$$

Use $\Delta t = 0.2$ to estimate the distance that the bug crawls during this hour. Find an overestimate and an underestimate. Then average the two to get a new estimate.

Using $\Delta t = 0.2$, our upper estimate is comes from using the smallest possible t value on each interval, because that leads to the largest velocity. I.e. we use $t_0 = 0$, $t_1 = 0.2$, etc.

$$\underbrace{\frac{1}{1+0}}_{v(t_0)}(0.2) + \frac{1}{1+0.2}(0.2) + \frac{1}{1+0.4}(0.2) + \frac{1}{1+0.6}(0.2) + \frac{1}{1+0.8}(0.2) \approx 0.75$$

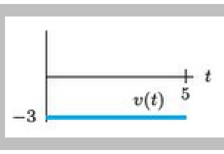
The lower estimate is

$$\frac{1}{1+0.2}(0.2) + \frac{1}{1+0.4}(0.2) + \frac{1}{1+0.6}(0.2) + \frac{1}{1+0.8}(0.2) + \frac{1}{1+1}(0.2) \approx 0.65$$

Since v is a decreasing function, the bug has crawled more than 0.65 meters, but less than 0.75 meters. We average the two to get a better estimate: 0.65 + 0.75 2 = 0.70 meters:

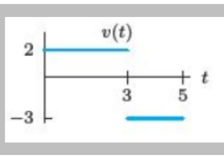
For questions 7 to 10, the graph shows the velocity, in cm/sec, of a particle moving along the x-axis. Compute the particle's change in position, left (negative) or right (positive), between times $t = 0$ and $t = 5$ seconds.

7.



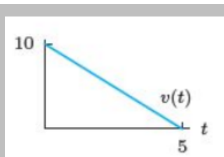
The velocity is constant and negative, so the change in position is $-3 \cdot 5$ cm, that is 15 cm to the left.

8.



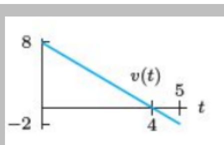
From $t = 0$ to $t = 3$, the velocity is constant and positive, so the change in position is $2 \cdot 3$ cm, that is 6 cm to the right.
From $t = 3$ to $t = 5$, the velocity is negative and constant, so the change in position is $3 \cdot 2$ cm, that is 6 cm to the left.
Thus the total change in position is 0. The particle moves 6 cm to the right, followed by 6 cm to the left, and returns to where it started.

9.



From $t = 0$ to $t = 5$ the velocity is positive so the change in position is to the right. The area under the velocity graph gives the distance traveled. The region is a triangle, and so has area $(1/2)bh = (1/2)5 \cdot 10 = 25$. Thus the change in position is 25 cm to the right.

10.



From $t = 0$ to $t = 4$ the velocity is positive so the change in position is to the right. The area under the velocity graph gives the distance traveled. The region is a triangle, and so has area $(1/2)bh = (1/2)4 \cdot 8 = 16$. Thus the change in position is 16 cm to the right for$t = 0$ to $t = 4$.

From $t = 4$ to $t = 5$, the velocity is negative so the change in position is to the left. The distance traveled to the left is given by the area of the triangle, $(1/2)bh = (1/2)1 \cdot 2 = 1$.

Thus the total change in position is $16 - 1 = 15$ cm to the right.

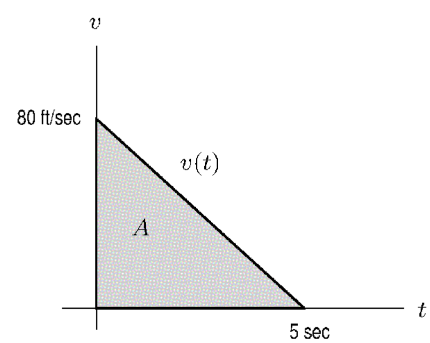
11. A car going 80 ft/s (about 90 km/h) brakes to a stop in five seconds. Assume the deceleration is constant.

    (a) Graph the velocity against time, t, for $0 \le t \le 5$ seconds.

    (b) Represent, as an area on the graph, the total distance traveled from the time the brakes are applied until the car comes to a stop.

    (c) Find this area and hence the distance traveled.

(a)



(b) The total distance is represented by the shaded region A, the area under the graph of $v(t)$.

(c) A is a triangle, so its area is

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(5\text{ sec})(80\text{ ft/s}) = 200\text{ ft}$$

12. A student is speeding down Route 11 in his fancy red Porsche when his radar system warns him of an obstacle 400 feet ahead. He immediately applies the brakes, starts to slow down, and spots a skunk in the road directly ahead of him. The "black box" in the Porsche records the car's speed every two seconds, producing the following table. The speed decreases throughout the 10 seconds it takes to stop, although not necessarily at a uniform rate.

Time since brakes applied (sec)	0	2	4	6	8	10
Speed (ft/sec)	100	80	50	25	10	0

(a) What is your best estimate of the total distance the student's car traveled before coming to rest?

(b) Which one of the following statements can you justify from the information given?

(i) The car stopped before getting to the skunk.

(ii) The "black box" data is inconclusive. The skunk may or may not have been hit.

(iii) The skunk was hit by the car.

To find the distance the car moved before stopping, we estimate the distance traveled for each two-second interval. Since speed decreases throughout, we know that the left-hand sum will be an overestimate to the distance traveled, and the right-hand sum an underestimate. Applying the formulas for these sums with $\Delta t = 2$ gives:

$$\text{LEFT} = 2(100 + 80 + 50 + 25 + 10) = 530 \text{ ft}$$
$$\text{RIGHT} = 2(80 + 50 + 25 + 10 + 0) = 330 \text{ ft}$$

(a) The best estimate of the distance traveled will be the average of these two estimates, or $\dfrac{530 + 330}{2} =$ 430 ft.

(b) All we can be sure of is that the distance traveled lies between the upper and lower estimates calculated above. In other words, all the black-box data tells us for sure is that the car traveled between 330 and 530 feet before stopping. So we can't be completely sure about whether it hit the skunk or not: answer (ii).

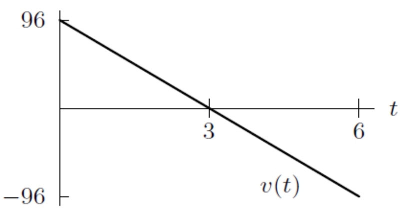
13. A baseball thrown directly upward at 96 ft/sec has velocity $v(t) = 96 - 32t$ ft/ sec at time t seconds.

(a) Graph the velocity from $t = 0$ to $t = 6$.

(b) When does the baseball reach the peak of its flight? How high does it go?

(c) How high is the baseball at time $t = 5$?

(a) See the figure below.



(b) The peak of the flight is when the velocity is 0, namely $t = 3$. The height at $t = 3$ is given by the area under the graph of the velocity from $t = 0$ to $t = 3$; see the figure above. The region is a triangle of base 3 seconds and altitude 96 ft/sec, so the height is $(1/2)3 \cdot 96 = 144$ feet.

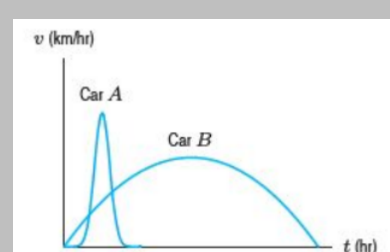
(c) The velocity is negative from $t = 3$ to $t = 5$, so the motion is downward then. The distance traveled downward can be calculated by the area of the triangular region which has base of 2 seconds and altitude of -64 ft/sec. Thus, the baseball travels $(1/2)2 \cdot 64 = 64$ feet downward from its peak height of 144 feet at $t = 3$. Thus, the height at time $t = 5$ is the total change in position, $144 - 64 = 80$ feet.

14. Two cars start at the same time and travel in the same direction along a straight road. The graph below gives the velocity, v, of each car as a function of time, t. Which car:

(a) Attains the larger maximum velocity?

(b) Stops first?

(c) Travels farther?



(a) Car A has the largest maximum velocity because the peak of car A's velocity curve is higher than the peak of B's.

(b) Car A stops first because the curve representing its velocity hits zero (on the t-axis) first.

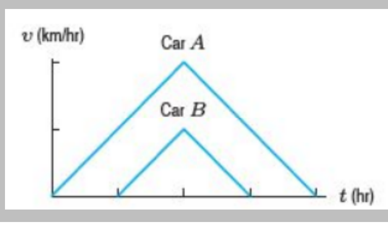
(c) Car B travels farther because the **area** under car B's velocity curve is the larger, and area under the velocity graph reprsents distance.

15. Two cars travel in the same direction along a straight road. The graph below shows the velocity, v, of each car at time t. Car B starts 2 hours after car A and car B reaches a maximum velocity of 50 km/ hr.

    (a) For approximately how long does each car travel?

    (b) Estimate car A's maximum velocity.

    (c) Approximately how far does each car travel?



(a) Since car B starts at $t = 2$, the tick marks on the horizontal axis (which we assume are equally spaced) are 2 hours apart. Thus car B stops at $t = 6$ and travels for 4 hours. Car A starts at $t = 0$ and stops at $t = 8$, so it travels for 8 hours.

(b) Car A's maximum velocity is approximately twice that of car B (50 km/h), so A's max velocity is 100 km/hr.

(c) The distance traveled is given by the area of under the velocity graph. Using the formula for the area of a triangle, the distances are given by

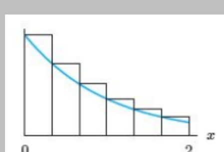
$$\begin{aligned}\text{Car A travels} &= \frac{1}{2}\cdot\text{Base }\cdot\text{Height}\\ &= \frac{1}{2}\cdot 8\cdot 100 = 400\text{ km}\\ \text{Car B travels} &= \frac{1}{2}\cdot\text{Base }\cdot\text{Height}\\ &= \frac{1}{2}\cdot 4\cdot 50 = 100\text{ km.}\end{aligned}$$

The Definite Integral

16. The figure below shows a Riemann sum approximation with n subdivisions to $\displaystyle\int_a^b f(x)dx$.

    (a) Is it a left- or right-hand approximation? Would the other one be larger or smaller?

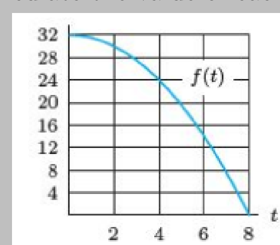
    (b) What are a, b, n and Δx?



(a) Left-hand sum. Right-hand sum would be smaller.

(b) We have $a = 0$, $b = 2$, $n = 6$, $\Delta x = \frac{2}{6} = \frac{1}{3}$.

17. Using the figure below, draw rectangles representing each of the following Riemann sums for the function f on the interval $0 \le t \le 8$. Calculate the value of each sum.

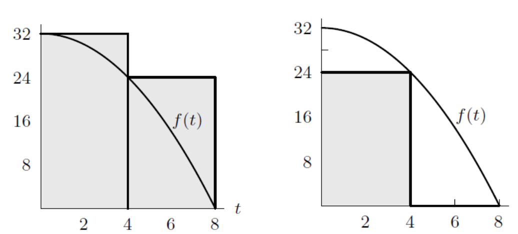


   (a) Left-hand sum with $\Delta t = 4$.

   (b) Right-hand sum with $\Delta t = 4$.

   (c) Left-hand sum with $\Delta t = 2$.

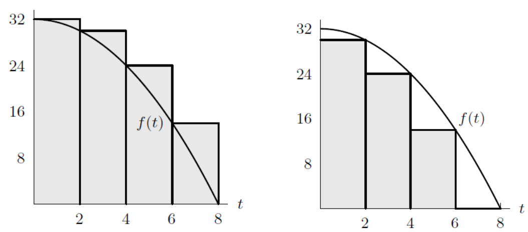
   (d) Right-hand sum with $\Delta t = 2$.



(a) (b)

(a) The left-hand sum with $n = 2$ intervals, or $\Delta t = 4$:

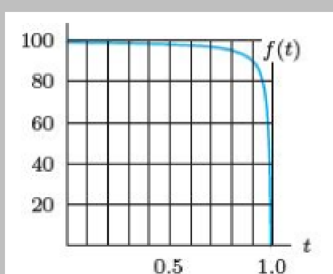
$32 \cdot 4 + 24 \cdot 4 = 224$.
(b) The right-hand sum with $n = 2$ intervals, or $\Delta t = 4$: $24 \cdot 4 + 0 \cdot 4 = 96$.



(c) (d)

(c) Left-hand sum with $n = 4$ intervals, or $\Delta t = 2$: $32 \cdot 2 + 30 \cdot 2 + 24 \cdot 2 + 14 \cdot 2 = 200$.
(d) Right-hand sum with $n = 4$ intervals, or $\Delta t = 2$: $30 \cdot 2 + 24 \cdot 2 + 14 \cdot 2 + 0 \cdot 2 = 136$.

18. The graph of a function $f(t)$ is given in the figure below.



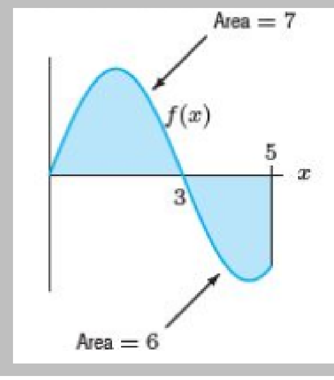
Which of the following four numbers could be an estimate of $\int_0^1 f(t)\, dt$, accurate to two decimal places? Explain how you chose your answer.
(a) -98.35 (b) 71.84 (c) 100.12 (d) 93.47

The graph given shows that f is positive for $0 \leq t \leq 1$, so the integral value must be positive: the answer cannot be -98.35.

Since the graph is contained within a rectangle of height 100 and length 1, the answer 100.12 is too large.

The graph of f is well above the horizontal line y = 80 for $0 \leq t \leq 0.95$, so the integral is likely much higher then $71.84 < 80$, so out of the choices given the best estimate is 93.47: answer (d).

19. (a) What is the area between the graph of $f(x)$ shown below and the x-axis, between $x = 0$ and $x = 5$?
 (b) What is $\int_0^5 f(x)\, dx$?

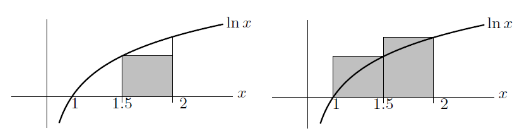


(a) The total area between $f(x)$ and the x-axis is the sum of the two given areas, so area = 7 + 6 = 13.
(b) To find the integral, we note that from $x = 3$ to $x = 5$, the function lies below the x-axis, and hence makes a negative contribution to the integral. So

$$\int_0^5 f(x)\, dx = \int_0^3 f(x)\, dx + \int_3^5 f(x)dx = 7 - 6 = 1$$

20. (a) On a sketch of $y = \ln(x)$, represent the left Riemann sum with $n = 2$ approximating $\int_1^2 \ln(x)\, dx$. Write out the terms in the sum, but do not evaluate it.
 (b) On another sketch, represent the *right* Riemann sum with $n = 2$ approximating $\int_1^2 \ln(x)\, dx$. Write out the terms in the sum, but do not evaluate it.
 (c) Which sum is an overestimate? Which sum is an underestimate?

(a) Below are the left and right sums respectively.



$$\begin{aligned} \text{Left sum} &= f(1)\Delta x + f(1.5)\Delta x \\ &= \underbrace{(\ln 1)}_{=0} 0.5 + \ln(1.5)0.5 = (\ln 1.5)0.5 \end{aligned}$$

(b)

$$\begin{aligned} \text{Right sum} &= f(1.5)\Delta x + f(2)\Delta x \\ &= (\ln 1.5)0.5 + (\ln 2)0.5 \end{aligned}$$

(c) Right sum is an overestimate, left sum is an underestimate.

21. Estimate $\int_1^2 x^2\,dx$ using left- and right-hand sums with four subdivisions, and then averaging them. How far from the true value of the integral could your final estimate be?

Left-hand sum gives:
$1^2(1/4) + (1.25)^2(1/4) + (1.5)^2(1/4) + (1.75)^2(1/4) = 1.96875$.

Right-hand sum gives:
$(1.25)^2(1/4) + (1.5)^2(1/4) + (1.75)^2(1/4) + (2)^2(1/4) = 2.71875$.

We improve our estimate the value of the integral by taking the average of these two sums, which is 2.34375.

Since x^2 is always increasing on $1 \le x \le 2$, the true value of the integral lies between 1.96875 and 2.71875. Thus the most our estimate could be off is 0.375 (half of the range from 2.34 to the lower and upper bounds 1.97 and 2.72). We expect our 2.34 estimate to be much closer than that possible error bound though. (And it is: the true value of the integral is $7/3 \approx 2.333$.)

22. Without computing the sums, find the difference between the right- and left-hand Riemann sums if we use $n = 500$ subintervals to approximate $\int_{-1}^1 (2x^3 + 4)\,dx$.

We have $\Delta x = 2/500 = 1/250$. The formulas for the left- and right-hand Riemann sums give us that

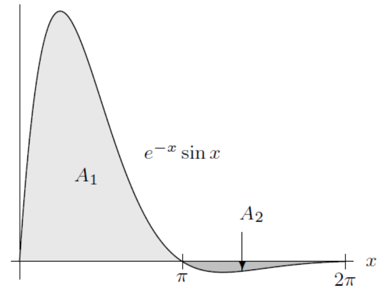
$$\begin{aligned} \text{Left} &= \Delta x[f(-1) + f(-1+\Delta x) + \dots \\ &\quad + f(1-2\Delta x) + f(1-\Delta x)] \\ \text{Right} &= \Delta x[f(-1+\Delta x) + f(-1+2\Delta x) + \dots \\ &\quad + f(1-\Delta x) + f(1)]: \end{aligned}$$

Subtracting these yields

$$\begin{aligned} \text{Right - Left} &= \Delta x[f(1) - f(-1)] \\ &= \frac{1}{250}[6-2] = \frac{4}{250} = \frac{2}{125}. \end{aligned}$$

The estimates from the Right and Left sums will differ by 2/125.

23. Without computation, decide if $\int_0^{2\pi} e^{-x}\sin x\,dx$ is positive or negative. [Hint: Sketch $e^{-x}\sin(x)$].



Looking at the graph of $y = e^{-x}\sin x$ shown above (remember the sketching of variable amplitudes earlier in the course!), for $0 \le x \le 2\pi$, we see that the area, A_1, which contributes a positive amount to the integral $\int_0^{2\pi} e^{-x}\sin(x)\,dx$, is much larger than the area A_2, which contributes a negative amount to the integral.

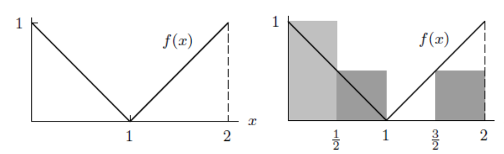
Since the positive integral contributions are larger than the negative, the overall integral will be positive.

24. (a) Graph $f(x) = \begin{cases} 1-x & \text{if } 0 \le x \le 1 \\ x-1 & \text{if } 1 < x \le 2 \end{cases}$

(b) Find the *exact* value of $\int_0^2 f(x)\,dx$ (hint: sketch and see what shapes you get).

(c) Calculate the 4-term left Riemann sum approximation to the definite integral. How does the approximation compare to the exact value?

(a)



(b) The area is made up of two triangles, with total area of $2 \times (1/2)(1)(1) = 1$ square unit.

(c) Using $\Delta x = 1/2$ in the 4-term Riemann sum shown in right-side graph above, we have

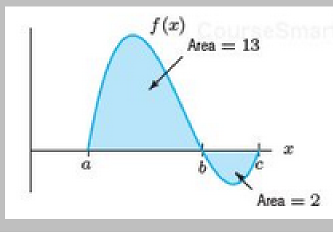
$$\begin{aligned} &\text{Left hand sum} \\ &= f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x \\ &= 1\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) + 0\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) \\ &= 1. \end{aligned}$$

We notice that in this case the approximation is exactly equal to the exact value of the integral. This is mostly coincidence due to the simple shape of $f(x)$. In general, approximations will not work out to be exactly the same as the value of the integral.

25. Using the figure below, find the values of

(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$

(c) $\int_a^c f(x)\,dx$ (d) $\int_a^c |f(x)|\,dx$

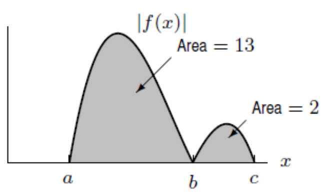


(a) The area between the graph of $f(x)$ and the x-axis between $x = a$ and $x = b$ is 13, so $\int_a^b f(x)\,dx = 13$.

(b) Since the graph of $f(x)$ is below the x-axis for $b < x < c$, $\int_b^c f(x)\,dx = -2$.

(c) Since the graph of $f(x)$ is above the x-axis for $a < x < b$ and below for $b < x < c$, $\int_a^c f(x)\,dx = 13 - 2 = 11$.

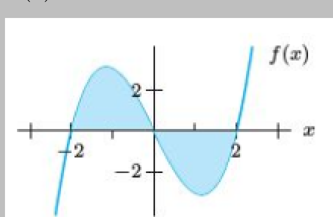
(d) The graph of $|f(x)|$ is the same as the graph of $f(x)$, except that the part *below* the x-axis is reflected to be *above* it (see graph of $|f(x)|$ below). Thus $\int_a^c |f(x)|\,dx = 13 + 2 = 15$.



26. Given the figure below, and the statement that $\int_{-2}^0 f(x)\,dx = 4$, estimate

(a) $\int_0^2 f(x)\,dx$ (b) $\int_{-2}^2 f(x)\,dx$

(c) The total shaded area.



The region shaded between $x = 0$ and $x = 2$ appears to have approximately the same area as the region shaded between $x = -2$ and $x = 0$, but it lies below the axis. Since $\int_{-2}^0 f(x)\,dx = 4$, we have the following results:

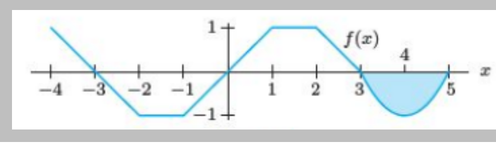
(a) $\int_0^2 f(x)\,dx \approx$ negative of $\int_{-2}^0 f(x)\,dx = -4$

(b) $\int_{-2}^2 f(x)\,dx \approx 4 - 4 = 0$.

(c) The total area shaded is approximately $4 + 4 = 8$.

27. (a) Using the graph below, find $\int_{-3}^0 f(x)\,dx$.

(b) If the area of the shaded region is A, estimate $\int_{-3}^4 f(x)\,dx$.



(a) $\int_{-3}^0 f(x)\,dx = -2$: counting squares or computing areas of the triangles and rectangles in this region.

(b) We break the integral over the interval $x = -3 \ldots 5$ into pieces, each of which we can find the areas of.

$$\begin{aligned}\int_{-3}^4 f(x)\,dx &= \int_{-3}^0 f(x)\,dx + \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx \\ &= -2 + 2 - \frac{A}{2} \\ &= \frac{-A}{2}\end{aligned}$$

28. Calculate the following approximations to $\int_0^6 x^2\,dx$. (a) LEFT(2); (b) RIGHT(2); (c) TRAP(2); (d) MID(2)

(a) Since two rectangles are being used, the width of each rectangle is $\Delta x = (6 - 0)/2 = 3$. The height is given by the left-hand endpoint so we have

$$\begin{aligned}\text{LEFT}(2) &= f(0)\cdot 3 + f(3)\cdot 3 \\ &= 0^2\cdot 3 + 3^2\cdot 3 = 27\end{aligned}$$

(b) Again, $\Delta x = 3$. The height of each rectangle is given by the right-hand endpoint so we have

$$\begin{aligned}\text{RIGHT}(2) &= f(3)\cdot 3 + f(6)\cdot 3 \\ &= 32\cdot 3 + 62\cdot 3 = 135\end{aligned}$$

(c) We know that TRAP is the average of LEFT and RIGHT and so

$$TRAP(2) = \frac{27 + 135}{2} = 81$$

(d) $\Delta x = 3$ again. The height of each rectangle is given by the height at the midpoint so we have

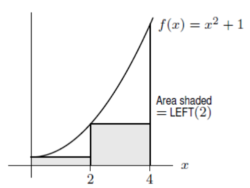
$$\begin{aligned} MID(2) &= f(1.5)\cdot 3 + f(4.5)\cdot 3 \\ &= (1.5)^2\cdot 3 + (4.5)^2 \cdot 3 = 67.5 \end{aligned}$$

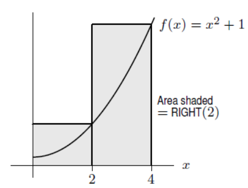
29. (a) Find LEFT(2) and RIGHT(2) for $\int_0^4 (x^2 + 1)\, dx$.
    (b) Illustrate your answers to part (a) graphically. Is each approximation an underestimate or overestimate?

(a)

$$\begin{aligned} \text{LEFT}(2) &= 2\cdot f(0) + 2\cdot f(2) \\ &= 2\cdot 1 + 2\cdot 5 \\ &= 12 \\ \text{RIGHT}(2) &= 2\cdot f(2) + 2\cdot f(4) \\ &= 2\cdot 5 + 2\cdot 17 \\ &= 44 \end{aligned}$$

(b)





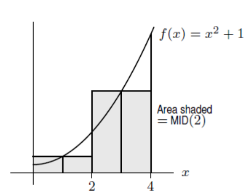
LEFT(2) will be an under-estimate, while RIGHT(2) will be an over-estimate.

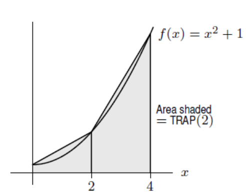
30. (a) Find MID(2) and TRAP(2) for $\int_0^4 (x^2 + 1)\, dx$.
    (b) Illustrate your answers to part (a) graphically. Is each approximation an underestimate or overestimate?

(a)

$$\begin{aligned} \text{MID}(2) &= 2\cdot f(1) + 2\cdot f(3) \\ &= 2\cdot 2 + 2\cdot 10 \\ &= 24 \\ \text{TRAP}(2) &= \frac{\text{LEFT}(2) + \text{RIGHT}(2)}{2} \\ &= \frac{12+44}{2} \\ &= 28 \end{aligned}$$

(b)





MID(2) is an underestimate, since $f(x) = x^2 + 1$ is concave up; this means that the new growth of the function after the midpoint (missed by MID) will be greater than extra area before (captured by MID).

Since $f(x)$ is concave up, TRAP(2) is an overestimate, because the trapezoidal secant lines in the TRAP rule will always lie above the original curve, leading to more area in the trapezoids than under the curve.

31. Calculate the following approximations to $\int_0^{\pi} \sin(\theta) d\theta$. (a) LEFT(2); (b) RIGHT(2); (c) TRAP(2); (d) MID(2)

(a) Since two rectangles are being used, the width of each rectangle is $\Delta x = (\pi - 0)/2 = \pi/2$. The height is given by the left-hand endpoint so we have

$$\begin{aligned} \text{LEFT}(2) &= f(0)\cdot\frac{\pi}{2} + f(\pi/2)\cdot\frac{\pi}{2} \\ &= \sin 0\cdot\frac{\pi}{2} + \sin(\pi/2)\cdot\frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

(b) Again $\Delta x = \pi/2$. The height of each rectangle is given by the right-hand endpoint so we have

$$\begin{aligned} RIGHT(2) &= f(\pi/2)\cdot\frac{\pi}{2} + f(\pi)\cdot\frac{\pi}{2} \\ &= \sin(\pi/2)\cdot\frac{\pi}{2} + \sin(\pi)\cdot\frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

(c) We know that TRAP is the average of LEFT and RIGHT and so

$$\text{TRAP}(2) = \frac{\text{LEFT}(2) + \text{RIGHT}(2)}{2} = \frac{\pi/2 + \pi/2}{2} = \frac{\pi}{2}$$

(d) Again, $\Delta x = \pi/2$. The height of each rectangle is given by the height at the midpoint so we have

$$\begin{aligned} \text{MID}(2) &= f(\pi/4)\cdot\frac{\pi}{2} + f(3\pi/4)\cdot\frac{\pi}{2} \\ &= \sin(\pi/4)\cdot\frac{\pi}{2} + \sin(3\pi/4)\cdot\frac{\pi}{2} \\ &= \frac{\sqrt{2}\pi}{2} \end{aligned}$$

32. Using the table below, estimate the total distance traveled from time $t = 0$ to time $t = 6$ using LEFT, RIGHT, and TRAP.

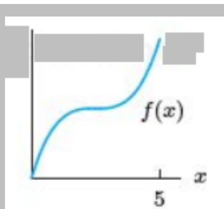
Time, t (s)	0	1	2	3	4	5	6
Velocity, v (m/s)	3	4	5	4	7	8	11

Let $s(t)$ be the distance traveled at time t, and $v(t)$ be the velocity at time t. Then the distance traveled during the interval $0 \le t \le 6$ is given by $\int_0^6 v(t)\ dt$.

We estimate the distance by estimating this integral. From the table, we find: LEFT(6) = 31 m; RIGHT(6) = 39 m, TRAP(6) = 35 m.

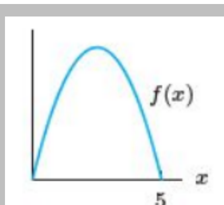
For the functions in Problems 33 – 36, pick which approximation- left, right, trapezoid, or midpoint- is guaranteed to give an *overestimate* for $\int_0^5 f(x)\ dx$, and which is guaranteed to give an *underestimate*. (There may be more than one.)

33.



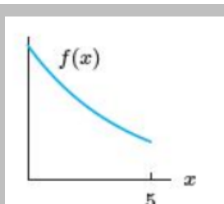
$f(x)$ is increasing, so RIGHT gives an overestimate and LEFT gives an underestimate.

34.



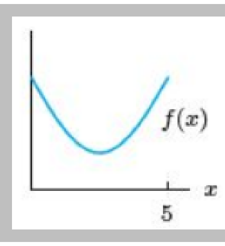
$f(x)$ is concave down, so MID gives an overestimate and TRAP gives an underestimate.

35.



$f(x)$ is decreasing and concave up, so LEFT and TRAP give overestimates and RIGHT and MID give underestimates.

36.



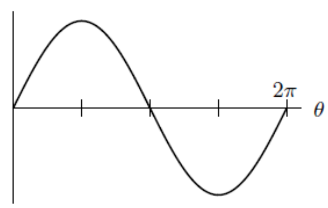
$f(x)$ is concave up, so TRAP gives an overestimate and MID gives an underestimate.

37. (a) Find the exact value of $\int_0^{2\pi} \sin\theta d\theta$ without calculation (i.e. from a sketch).

(b) Explain, using pictures, why the MID(1) and MID(2) approximations to this integral give the exact value.

(c) Does MID(3) give the exact value of this integral? How about MID(n)? Explain.

(a)



From the graph of sine above, and the fact that the integral includes a complete $[0, 2\pi]$ cycle, it is clear from symmetry that the positive and negative contributions to the integral must be equal, so

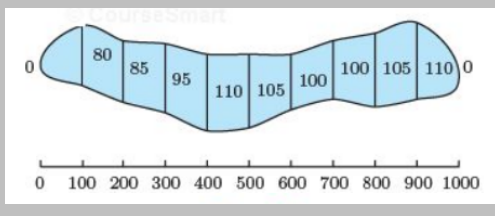
$$\int_0^{2\pi} \sin(\theta)\ d\theta = 0$$

(b) Again, refer to the graph above. MID(1) is 0 since the midpoint of 0 and 2π is π and $\sin\pi = 0$. Thus MID(1) $= 2\pi \sin\pi = 0$.

The midpoints we use for MID(2) are $\pi/2$ and $3\pi/2$, and $\sin(\pi/2) = -\sin(3\pi/2)$. Thus MID(2) $= \pi \sin(\pi/2) + \pi\sin(3\pi/2) = 0$.

(c) MID(3) = 0. In general, MID(n) = 0 for all n, even though your calculator (because of round-off error) might not return it as such. The reason is that $\sin(x) = -sin(2\pi - x)$. If we use MID(n), we will always take sums where we are adding pairs of the form $\sin(x)$ and $\sin(2\pi - x)$, so the sum will cancel to 0. (If n is odd, we will get a $\sin\pi$ in the sum which does not pair up with anything, but $\sin\pi$ is already 0.)

38. The width, in feet, at various points along the fairway of a hole on a golf course is given in the figure below. If one pound of fertilizer covers 200 square feet, estimate the amount of fertilizer needed to fertilize the fairway.



We approximate the area of the playing field by using Riemann sums. From the data provided, LEFT(10) = RIGHT(10) = TRAP(10) = 89,000 square feet.

Thus approximately $\dfrac{89,000 \text{ sq. ft.}}{200\text{sq. ft./lb}} = 445$ lbs. of fertilizer should be necessary.

Note that the fact that LEFT=RIGHT=TRAP doesn't mean there is no error in the estimate; it actually means that the error is completely unknown. You will always have this relationship when the first and last heights are equal because that makes LEFT = RIGHT, and then TRAP (= average of LEFT and RIGHT) will equal the same value as well).

Definite Integrals in Modeling

39. The rate at which the world's oil is being consumed is continuously increasing. Suppose the rate of oil consumption (in billions of barrels per year) is given by the function $r = f(t)$, where t is measured in years and $t = 0$ is the start of 2004.

    (a) Write a definite integral which represents the total quantity of oil used between the start of 2004 and the start of 2009.

    (b) Suppose $r = 32e^{0.05t}$. Using a left-hand sum with five subdivisions, find an approximate value for the total quantity of oil used between the start of 2004 and the start of 2009.

    (c) Interpret each of the five terms in the sum from part (b) in terms of oil consumption.

(a) Quantity used $= \displaystyle\int_0^5 f(t)\, dt$.

(b) Using a left sum, our approximation is

$$32e^{0.05(0)} + 32e^{0.05(1)} + 32e^{0.05(2)} + 32e^{0.05(3)} + 32e^{0.05(4)}$$
$$= 177.270.$$

Since f is an increasing function, the left endpoint values of f are lower than anywhere else on the subintervals, so this represents an underestimate.

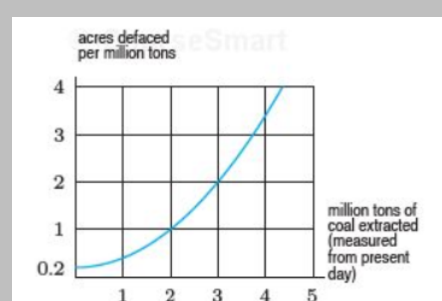
(c) Each term is a lower estimate of one year's consumption of oil.

40. As coal deposits are depleted, it becomes necessary to strip-mine larger areas for each ton of coal. The graph below shows the number of acres of land per million tons of coal that will be defaced during strip-mining as a function of the number of million tons removed, starting from the present day.

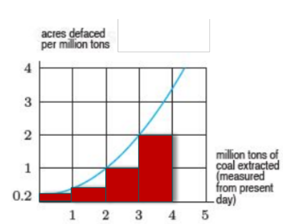
    (a) Estimate the total number of acres defaced in extracting the next 4 million tons of coal (measured from the present day). Draw four rectangles under the curve, and compute their area.

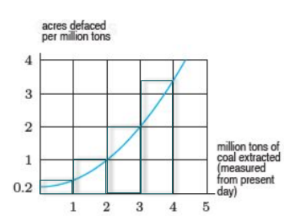
    (b) Re-estimate the number of acres defaced using rectangles above the curve.

    (c) Combine your answers to parts (a) and (b) to get a better estimate of the actual number of acres defaced.



(a)





Using rectangles under the curve, we get

Acres defaced $\approx (1)(0.2 + 0.4 + 1 + 2) = 3.6$ acres.

(b) Using rectangles above the curve, we get

Acres defaced $\approx (1)(0.4 + 1 + 2 + 3.5) = 6.9$ acres.

(c) The number of acres defaced is between 3.6 and 6.9, so we estimate the average, 5.25 acres.

41. The following table gives the emissions, E, of nitrogen oxides, in millions of metric tons per year, in the US. Let t be the number of years since 1970 and $E = f(t)$.

    (a) What are the units and meaning of $\int_0^{30} f(t)\,dt$?

    (b) Estimate $\int_0^{30} f(t)\,dt$.

Year	1970	1975	1980	1985	1990	1995	2000
E	26.9	26.4	27.1	25.8	25.5	25.0	22.6

(a) The integral $\int_0^{30} f(t)\,dt$ represents the total emissions of nitrogen oxides, in millions of metric tons, during the period 1970 to 2000.

(b) We estimate the integral using left- and right-hand sums:

$$\begin{aligned}\text{Left sum} &= (26.9)5 + (26.4)5 + (27.1)5 + (25.8)5 \\ &\quad + (25.5)5 + (25.0)5 \\ &= 783.5. \\ \text{Right sum} &= (26.4)5 + (27.1)5 + (25.8)5 + (25.5)5 \\ &\quad + (25.0)5 + (22.6)5 \\ &= 762.0.\end{aligned}$$

We average the left- and right-hand sums to find an improved estimate of the integral:

$$\int_0^{30} f(t)\,dt \approx \frac{783.5 + 762.0}{2} = 772.8 \text{ million metric tons.}$$

Between 1970 and 2000, about 772.8 million metric tons of nitrogen oxides were emitted.

42. Coal gas is produced at a gasworks. Pollutants in the gas are removed by scrubbers, which become less and less efficient as time goes on. The following measurements, made at the start of each month, show the rate at which pollutants are escaping (in tons/ month) in the gas:

Time (months)	0	1	2	3	4	5	6
Rate pollutants escape (tons/month)	5	7	8	10	13	16	20

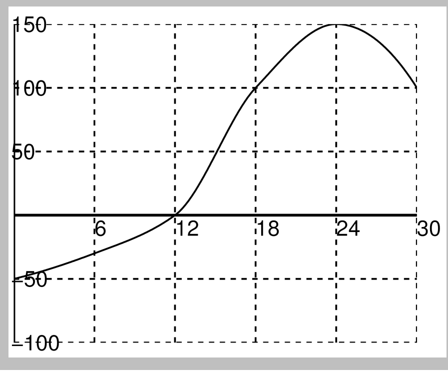
    (a) Make an overestimate and an underestimate of the total quantity of pollutants that escape during the **first month**.

    (b) Make an overestimate and an underestimate of the total quantity of pollutants that escape during the **six months** shown in the table.

(a) An overestimate is (7 tons/month)$\cdot$ (1 month) = 7 tons. An underestimate is (5 tons/month)$\cdot$(1 month) = 5 tons.

(b) An overestimate is 7+8+10+13+16+20 = 74 tons. An underestimate is 5+7+8+10+13+16 = 59 tons.

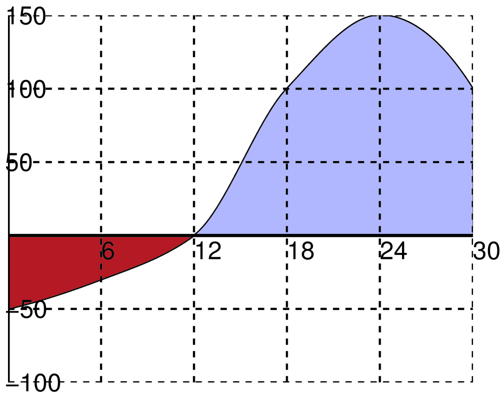
43. The graph below shows the rate of change of the quantity of water in a water tower, in liters per day, during the month of April. If the tower had 12,000 liters of water in it on April 1, estimate the quantity of water in the tower on April 30.



Suppose $V(t)$ represents the total quantity of water in the water tower at time t, where t is in days since April 1. Then the graph shown in the problem is a graph of the volume's rate of change, or $\frac{dV}{dt}$ or $V'(t)$. By the Fundamental Theorem, the change in the volume over the 30 days of April is given by:

$$V(30) - V(0) = \int_0^{30} V'(t)\,dt.$$

This relationship means we can calculate the change in the volume of water by calculating the area under the curve. Each box represents about (50 l/d)$\cdot$(6 d) = 300 liters.



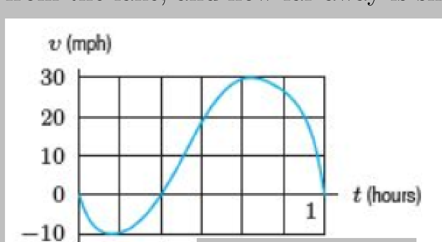
In the red region, from $t = 0$ to $t = 12$, the graph is below the t axis, indicating that water is being lost from the tower. There is a little more than one box, but for simplicity we'll call it one square, so roughly ≈ 300 liters were lost.
From $t = 12$ to $t = 30$ (blue region), there are between 6 and 7 boxes in total (let's say 6 for simplicity), or roughly +1800 liters.
Thus,

$$\int_0^{30} V'(t)\ dt \approx 1800 - 300 = 1500$$

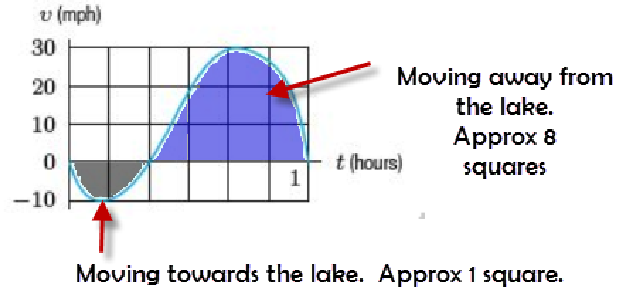
This is the net *change* in the volume of water. Solving for $V(30)$, the *amount* of water on April 30th,

$$\begin{aligned} V(30) &= V(0) + \int_0^{30} V'(t)\ dt \\ &= 12,000 + 1500 = 13,500 \text{ liters.} \end{aligned}$$

44. A bicyclist pedals along a straight road with velocity v given in the graph below. She starts 5 miles from a lake; positive velocities take her away from the lake and negative velocities take her toward the lake. When is the cyclist farthest from the lake, and how far away is she then?



Notice that the area of a square on the graph represents (10 mph)$\cdot$ (1/6) h = (5/3) miles.



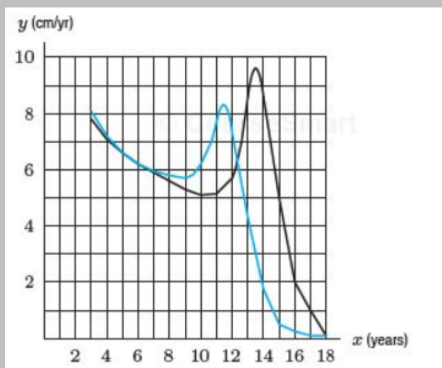
At $t = 1/3$ hours, $v = 0$. The area between the curve v and the t-axis over the interval $0 \le t \le 1/3$ is $-\int_0^{1/3} v\ dt \approx$ one square $= 5/3$ miles. Since v is negative here, she is moving toward the lake. Since she starts 5 miles from the lake at time $t = 0$, at $t = 1/3$ she is about 5 - 5/3 = 10/3 miles from the lake (so closer than when she started).

For $1/3 \le t \le 1$, v is positive, so she is moving away from the lake. Her *change* in distance from the lake between the times $t = 0$ and the end point $t = 1$ is given by

$$\begin{aligned} \int_0^1 v\ dt &= \int_0^{1/3} v\ dt + \int_{1/3}^1 v\ dt \\ &\approx -\frac{5}{3} + 8 \cdot \frac{5}{3} = \frac{35}{3} = 11.667 \text{ miles,} \end{aligned}$$

Relative to her starting point of 5 miles at $t = 0$, the cyclist is about $5 + 35/3 = 50/3 = 16.667$ miles from the lake at $t = 1$. Since, starting from the moment $t = 1/3$, she is always moving away from the lake, the cyclist will be farthest from the lake at the latest possible time, $t = 1$. The maximal distance at that time equals $50/3 = 16.667$ miles.

45. Height velocity graphs are used by endocrinologists (doctors specializing in the study of hormones) to follow the progress of children with growth deficiencies. The graph below shows the height velocity curves of an average boy and an average girl between ages 3 and 18.



(a) Which curve is for girls and which is for boys? Explain how you can tell.

(b) About how much does the average boy grow between ages 3 and 10?

(c) The growth spurt associated with adolescence and the onset of puberty occurs between ages 12 and 15 for the average boy and between ages 10 and 12.5 for the average girl. Estimate the height gained by each average child during this growth spurt.

(d) When fully grown, about how much taller is the average man than the average woman? (The average boy and girl are about the same height at age 3.)

(a) Since men are generally taller than women, the curve with the larger area under it is the height velocity of the boys. The area under each curve represents the change in growth in centimeters. Therefore, the black curve is for boys, the lighter grey one for girls.

(b) Each square below the height velocity curve has area (1 cm/yr) $\cdot$ 1 yr = 1 cm. Counting squares lying below the black curve gives about 43 cm. Thus, on average, boys grow about 43 cm between ages 3 and 10.

(c) Counting squares lying below the black curve gives about 23 cm growth for boys during their growth spurt. Counting squares lying below the colored curve gives about 18 cm for girls during their growth spurt.

(d) We can measure the difference in growth by counting squares that lie between the two curves. Between ages 2 and 12.5, the average girl grows faster than the average boy. Counting squares yields about 5 cm between the colored and black curves for $2 \le x \le 12.5$. Counting squares between the curves for $12.5 \le x \le 18$ gives about 18 squares. Thus, there is a net increase of boys over girls by about 18 - 5 = 13 cm.

Properties of Definite Integrals

For questions 46 to 49 , find $\int_2^5 f(x)\ dx$.

46. $f(x)$ is odd and $\int_{-2}^5 f(x)\ dx = 8$.

To use the fact that f is odd, we split the interval up into the symmetric interval, [-2, 2], and the non-symmetric interval, [2, 5].

We have

$$8 = \int_{-2}^5 f(x)\ dx = \int_{-2}^2 f(x)\ dx + \int_2^5 f(x)\ dx$$

Since f is odd, $\int_{-2}^2 f(x)\ dx = 0$, so $\int_{-2}^5 f(x)\ dx = 8$.

47. $f(x)$ is even, $\int_{-2}^2 f(x)\ dx = 6$, and $\int_{-5}^5 f(x)\ dx = 14$.

Since f is even, $\int_0^2 f(x)\ dx = (1/2)\cdot 6 = 3$ and $\int_0^5 f(x)\ dx = (1/2)\cdot 14 = 7$.

Therefore

$$\begin{aligned}\int_2^5 f(x)\ dx &= \int_0^5 f(x)\ dx - \int_0^2 f(x)\ dx \\ &= 7 - 3 = 4.\end{aligned}$$

48. $\int_2^5 (3f(x)+4)\ dx) = 18.$

We have

$$\begin{aligned} 18 &= \int_2^5 (3f(x)+4)\ dx \\ &= 3\int_2^5 f(x)\ dx + \int_2^5 4\ dx \end{aligned}$$

Thus, since $\int_2^5 4\ dx = 4(5-2) = 12$, we have

$$3\int_2^5 f(x)\ dx = 18 - \underbrace{\int_2^5 4\ dx}_{12} = 6$$

so

$$\int_2^5 f(x)\ dx = \frac{6}{3} = 2.$$

49. $\int_2^4 2f(x)\ dx = 8$ and $\int_5^4 f(x)\ dx = 1.$

We have $\int_2^4 f(x)\ dx = 8/2 = 4$ and $\int_4^5 f(x)\ dx = -\int_5^4 f(x)\ dx = -1$. Thus

$$\begin{aligned} \int_2^5 f(x)\ dx &= \int_2^4 f(x)\ dx + \int_4^5 f(x)\ dx \\ &= 4 - 1 = 3 \end{aligned}$$

50. Without any computation, find the values of

(a) $\int_{-2}^{2} \sin x\ dx.$

(b) $\int_{-\pi}^{\pi} x^{113}\ dx.$

(a) 0, since the integrand is an odd function and the limits are symmetric around 0.

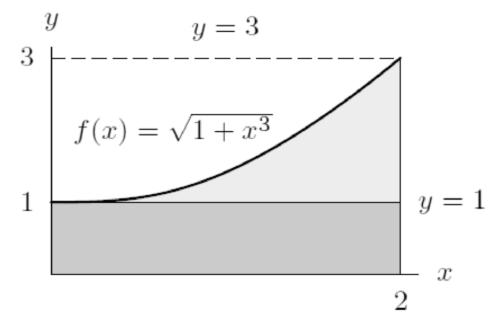
(b) 0, since the integrand is an odd function and the limits are symmetric around 0.

51. Without integrating, show that

$$2 \le \int_0^2 \sqrt{1+x^3}\ dx \le 6.$$

Notice that $f(x) = \sqrt{1+x^3}$ is increasing for $0 \le x \le 2$, since x^3 gets bigger as x increases. This means that, looking at the interval [0, 2] over which we are integrating, $f(0) \le f(x) \le f(2)$.

For this function, $f(0) = \sqrt{1+0} = 1$ and $f(2) = \sqrt{1+8} = 3$. Thus, the area under $f(x)$ lies between the area under the line $y = 1$ and the area under the line $y = 3$ on the interval $0 \le x \le 2$. See the diagram below.



That is,

$$\underbrace{1}_{h}\underbrace{(2-0)}_{w} \le \int_0^2 \sqrt{1+x^3}\ dx \le \underbrace{3}_{h}\underbrace{(2-0)}_{w}$$

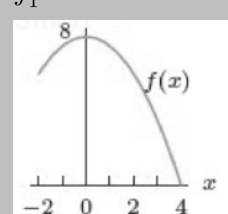
52. Without calculating the integral, explain why the following statements are false.

(a) $\int_{-2}^{-1} e^{x^2}\ dx = -3$

(b) $\int_{-1}^{1} \left|\frac{\cos(x+2)}{1+\tan^2 x}\right|\ dx = 0.$

(a) The integrand is always positive, so the integral (sum of positive $f(x)$ times positive Δx values) cannot be negative.
(b) The integrand is always ≥ 0 because of the absolute value sign. If the integral $= 0$, then with a non-negative integrand, the integrand must be exactly 0 at every x value on the interval, which is definitely not the case here.

53. Based on the graph of $f(x)$ shown below, write $\int_0^3 f(x)\ dx$ in terms of $\int_{-1}^{1} f(x)\ dx$ and $\int_1^3 f(x)\ dx.$



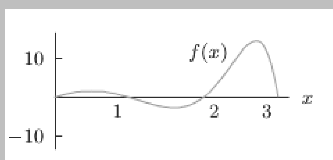
We know that we can divide the integral up as follows:

$$\int_0^3 f(x)\ dx = \int_0^1 f(x)\ dx + \int_1^3 f(x)\ dx.$$

The graph suggests that f is an even function for $-1 \le x \le 1$, so $\int_{-1}^{1} f(x)\ dx = 2\int_0^1 f(x)\ dx$. Substituting this into the preceding equation, we have

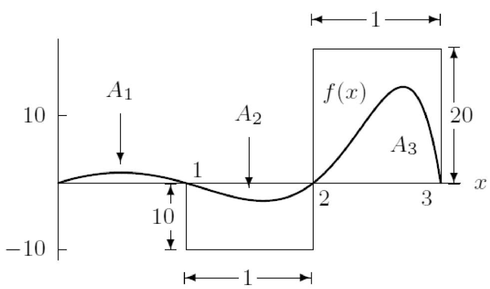
$$\int_0^3 f(x)\ dx = \frac{1}{2}\int_{-1}^{1} f(x)\ dx + \int_1^3 f(x)\ dx.$$

54. Using the graph of $f(x)$ shown below, arrange the following quantities in increasing order, from least to greatest.



(i) $\int_0^1 f(x)\ dx$ (ii) $\int_1^2 f(x)\ dx$
(iii) $\int_0^2 f(x)\ dx$ (iv) $\int_2^3 f(x)\ dx$
(v) $-\int_1^2 f(x)\ dx$ (vi) The number 0
(vii) The number 20 (viii) The number -10

See the figure below.



Since $\int_0^1 f(x)\ dx = A_1$ and $\int_1^2 f(x)\ dx = -A_2$ and $\int_2^3 f(x)\ dx = A_3$, we know that

$$0 < \int_0^1 f(x)\ dx < -\int_1^2 f(x)\ dx < \int_2^3 f(x)\ dx.$$

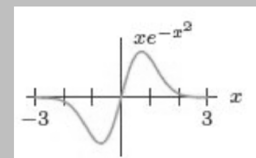
In addition, $\int_0^2 f(x)\ dx = A_1 - A_2$, which is negative, but smaller in magnitude than $\int_1^2 f(x)\ dx$. Thus

$$\int_1^2 f(x)\ dx < \int_0^2 f(x)\ dx < 0.$$

The area A_3 lies inside a rectangle of height 20 and base 1, so $A_3 < 20$.
The area A_2 lies inside a rectangle below the x-axis of height 10 and width 1, so $-10 < A_2$. Thus:

(viii) < (ii) < (iii) < (vi) < (i) < (v) < (iv) < (vii).

55. (a) Use the graph of $y = xe^{-x^2}$ shown below to explain why $\int_{-3}^{3} xe^{-x^2}\ dx = 0$.



(b) Find the left-hand sum approximation with $n = 3$ to $\int_0^3 xe^{-x^2}\ dx$. Give your answer to four decimal places.

(c) Repeat part (b) for $\int_{-3}^{0} xe^{-x^2}\ dx$.

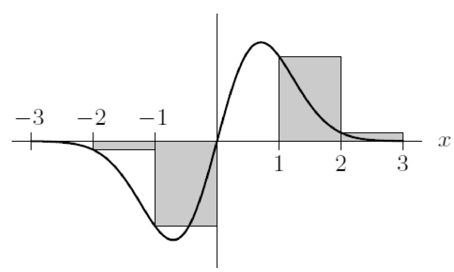
(d) Do your answers to parts (b) and (c) add to 0? Explain.

(a) Since the function is odd, and the interval used is symmetric across $x = 0$, the areas above and below the x-axis cancel. Thus,

$$\int_{-3}^{0} xe^{-x^2} dx = -\int_0^3 xe^{-x^2} dx,$$

so $$\int_{-3}^{3} xe^{-x^2} dx = \underbrace{\int_{-3}^{0} xe^{-x^2} dx + \int_0^3 xe^{-x^2} dx}_{\text{equal magnitude, opposite signs}} = 0.$$

(b) For $0 \le x \le 3$ with $n = 3$, we have $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$, and $\Delta x = 1$. See the figure below.



$$\begin{aligned}\text{Left sum} &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x \\ &= 0e^{-0^2}\cdot 1 + 1e^{-(1^2)}\cdot 1 + 2e^{-(2^2)}\cdot 1 \\ &= 0.4045.\end{aligned}$$

(c) For $-3 \le x \le 0$, with $n = 3$, we have $x_0 = -3, x_1 = -2, x_2 = -1, x_3 = 0$, and $\Delta x = 1$. (See the diagram again.)

$$\begin{aligned}\text{Left sum} &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x \\ &= -3e^{-(-3)^2}\cdot 1 - 2e^{-(-2)^2}\cdot 1 - 1e^{-(-1)^2}\cdot 1 \\ &= -0.4049.\end{aligned}$$

(d) No. The rectangles between $x = -3$ and 0 are not the same size as those between $x = 0$ and 3. See the diagram. There are three rectangles with nonzero height on [-3, 0] and only two on [0, 3], so the estimates computed must be different, so we won't get the exact result of zero using left sums.

Integration By Anti-Derivatives

To practice computing anti-derivatives, do as many of the problems from the following section as you need.

From Hughes-Hallett 5th edition,
Section 6.2 - 1-63 (odd)

From Hughes-Hallett 6th edition,
Section 6.2 - 1-60 (odd)

Note: In questions 53-63 (5th Ed.) and 51-60 (6th Ed.), 'evaluate numerically' means 'plug the limit values into the anti-derivative using your calculator to get a decimal number'. It does *not* mean use rectangles/trapezoids/Riemann sums to approximate the integral values.

Additional problems. Evaluate the following integrals.

56. $\int_{-1}^{2} (x^3 - 2x)dx$

$$\int_{-1}^{2} (x^3 - 2x)dx = \left[\frac{x^4}{4} - x^2\right]_{-1}^{2} = \left(\frac{2^4}{4} - 2^2\right) - \left(\frac{(-1)^4}{4} - (-1)^2\right) = (4-4) - (\frac{1}{4} - 1) = 0 - (-\frac{3}{4}) = \frac{3}{4}$$

57. $\int_{-1}^{1} x^{100}dx$

$$\int_{-1}^{1} x^{100}dx = \left[\frac{1}{101}x^{101}\right]_{-1}^{1} = \frac{1}{101} - (-\frac{1}{101}) = \frac{2}{101}$$

58. $\int_{1}^{4} (5 - 2t + 3t^2)dt$

$$\int_{1}^{4} (5 - 2t - 3t^2)dt = [5t - t^2 + t^3]_1^4 = (20 - 16 + 64) - (5 - 1 + 1) = 68 - 5 = 63$$

59. $\int_{0}^{1} (1 + \frac{1}{2}u^4 - \frac{2}{5}u^9)du$

$\int_0^1 (1 + \frac{1}{2}u^4 - \frac{2}{5}u^9)du = [u + \frac{1}{10}u^5 - \frac{1}{25}u^{1}0]_0^1 = (1 + \frac{1}{10} - \frac{1}{25}) - 0 = \frac{50+5-2}{50} = \frac{53}{50}$

60. $\int_{1}^{9} \sqrt{x}dx$

$$\int_{1}^{9} \sqrt{x}dx = \int_{1}^{9} x^{1/2}dx = \left[\frac{x^{3/2}}{3/2}\right]_1^9 = \frac{2}{3}\left[x^{2/3}\right]_1^9 = \frac{2}{3}(9^{3/2} - 1^{3/2}) = \frac{2}{3}(27 - 1) = \frac{52}{3}$$

61. $\int_{1}^{8} x^{-2/3}dx$

$$\int_{1}^{8} x^{-2/3}dx = \left[\frac{x^{1/3}}{1/3}\right]_1^8 = 3\left[x^{1/3}\right]_1^8 = 3(8^{1/3} - 1^{1/3}) = 3(2-1) = 3$$

62. $\displaystyle\int_{\pi/6}^{\pi} \sin\theta d\theta$

$$\int_{\pi/6}^{\pi} \sin\theta d\theta = [-\cos\theta]_{\pi/6}^{\pi} = -\cos\pi - (-\cos\frac{\pi}{6}) = -(-1) - (-\sqrt{3}/2) = 1 + \sqrt{3}/2$$

63. $\displaystyle\int_{-5}^{5} e\ dx$

The number e is just a constant.

$$\int_{-5}^{5} e\ dx = [ex]_{-5}^{5} = 5e - (-5e) = 10e$$

64. $\displaystyle\int_{0}^{1} (u+2)(u-3)du$

You need to expand the product before you can integrate.

$$\int_0^1 (u+2)(u-3)du = \int_0^1 (u^2 - u - 6)du = \left[\frac{1}{3}u^3 - \frac{1}{2}u^2 - 6u\right]_0^1 = \left(\frac{1}{3} - \frac{1}{2} - 6\right) - 0 = -\frac{37}{6}$$

65. $\displaystyle\int_{0}^{4} (4-t)\sqrt{t}\ dt$

To evaluate this integral, you need to expand the product first; you can't integrate the product $(4-t)\sqrt{t}$ as it is originally written.

$$\int_0^4 (4-t)\sqrt{t}dt = \int_0^4 (4-t)t^{1/2}dt = \int_0^4 (4t^{1/2} - t^{3/2})dt = \left[\frac{8}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right]_0^4 = \frac{8}{3}(8) - \frac{2}{5}(32) = \frac{320 - 192}{15} = \frac{128}{15}$$

66. $\displaystyle\int_{1}^{9} \frac{x-1}{\sqrt{x}}dx$

$$\int_1^9 \frac{x-1}{\sqrt{x}}dx = \int_1^9 \left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right) dx = \int_1^9 (x^{1/2} - x^{-1/2})dx = \left[\frac{2}{3}x^{3/2} - 2x^{1/2}\right]_1^9 = (\frac{2}{3}\cdot 27 - 2\cdot 3) = (\frac{2}{3} - 2) = 12 - (-\frac{4}{3}) = \frac{40}{3}$$

67. $\displaystyle\int_{0}^{2} (y-1)(2y+1)dy$

$$\int_0^2 (y-1)(2y+1)dy = \int_0^2 (2y^2 - y - 1)dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^2 - y\right]_0^2 = (\frac{16}{3} - 2 - 2) - 0 = \frac{4}{3}$$

68. $\displaystyle\int_{0}^{\pi/4} \sec^2 t\ dt$

$$\int_0^{\pi/4} \sec^2 t\ dt = [\tan t]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$$

69. $\displaystyle\int_{0}^{\pi/4} \sec\theta\tan\theta d\theta$

$$\int_0^{\pi/4} \sec\theta\tan\theta d\theta = [\sec\theta]_0^{\pi/4} = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$$

70. $\int_1^2 (1+2y)^2 dy$

$$\int_1^2 (1+2y)^2 dy = \int_1^2 (1+4y+4y^2) dy = [y+2y^2+\frac{4}{3}y^3]_1^2 = (2+8+\frac{32}{3}) - (1+2+\frac{4}{3}) = \frac{62}{3} - \frac{13}{3} = \frac{49}{3}$$

71. $\int_0^3 (2\sin x - e^x) dx$

$$\int_0^3 (2\sin x - e^x) dx = [-2\cos 3 - e^3) - (-2-1) = 3 - 2\cos 3 - e^3$$

72. $\int_1^2 \frac{v^3+3v^6}{v^4} dv$

$$\int_1^2 \frac{v^3+3v^6}{v^4} = \int_1^2 \left(\frac{1}{v} + 3v^2\right) dv = [\ln|v| + v^3]_1^2 = (\ln 2 + 8) - (\ln 1 + 1) = \ln 2 + 7$$

73. $\int_1^{18} \sqrt{\frac{3}{z}} dz$

$$\int_1^{18} \sqrt{\frac{3}{z}} dz = \int_1^{18} \sqrt{3} z^{-1/2} dz = \sqrt{3}[2z^{1/2}]_1^{18} = 2\sqrt{3}(18^{1/2} - 1^{1/2}) = 2\sqrt{3}(3\sqrt{2} - 1)$$

74. $\int_0^1 (x^e + e^x) dx$

$$\int_0^1 (x^e e^x) dx = \left[\frac{x^{e+1}}{e+1} + e^x\right]_0^1 = \left(\frac{1}{e+1} + e\right) - (0+1) = \frac{1}{e+1} + e - 1$$

75. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} dx$

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2} dx = [8\arctan x]_{1/\sqrt{3}}^{\sqrt{3}} = 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 8\left(\frac{\pi}{6}\right) = \frac{4\pi}{3}$$

76. $\int_1^2 \frac{4+u^2}{u^3} du$

$$\int_1^2 \frac{4+u^2}{u^3} du = \int_1^2 (4u^{-3} + u^{-1}) du = \left[\frac{4}{-2}u^{-2} + \ln|u|\right]_1^2 = \left[\frac{-2}{u^2} + \ln u\right]_1^2 = (-\frac{1}{2} + \ln 2) - (-2 + \ln 1) = \frac{3}{2} + \ln 2$$

77. $\int_{-1}^1 e^{u+1} du$

$$\int_{-1}^1 e^{u+1} du = [e^{u+1}]_{-1}^1 = e^2 - e^0 = e^2 - 1 \text{ [or start with } e^{u+1} = e^u e^1]$$

78. $\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$

$$\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx = [4\arcsin x]_{1/2}^{1/\sqrt{2}} = 4\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = 4(\frac{\pi}{12}) = \frac{\pi}{3}$$

79. $\displaystyle\int_0^{\pi} f(x)dx$ where $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le \pi \end{cases}$

If $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le \pi \end{cases}$ then

$$\int_0^{\pi} f(x)dx = \int_0^{\pi/2} \sin x dx + \int_{\pi/2}^{\pi} \cos x dx = [-\cos x]_0^{\pi/2} + [\sin x]_{\pi/2}^{\pi} = -\cos\frac{\pi}{2} + \cos 0 + \sin\pi - \sin\frac{\pi}{2} = -0 + 1 + 0 - 1 = 0$$

80. $\displaystyle\int_{-2}^{2} f(x)dx$ where $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \end{cases}$

If $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \end{cases}$ then

$$\int_{-2}^{2} f(x)dx = \int_{-2}^{0} 2dx + \int_0^2 (4 - x^2)dx = [2x]_{-2}^{0} + [4x - \frac{1}{3}x^3]_0^2 = [0 - (-4)] + (\frac{16}{3} - 0) = \frac{28}{3}$$