Week #2 - Derivatives - Linearization and Applications

Linear Approximations and Tangent Lines

1. Find the equation of the tangent line to the graph of f at (1,1), where f is given by $f(x) = 2x^3 - 2x^2 + 1$.

2. (a) Find the equation of the tangent line to $f(x) = x^3$ at $x = 2$.
   (b) Sketch the curve and the tangent line on the same axes, and decide whether using the tangent line to approximate $f(x) = x^3$ would produce *over-* or *under-*estimates of $f(x)$ near $x = 2$.

3. Find the equation of the line tangent to the graph of f at $(3, 57)$, where f is given by $f(x) = 4x^3 - 7x^2 + 12$.

4. Given a power function of the form $f(x) = ax^n$, with $f'(3) = 16$ and $f'(6) = 128$, find n and a.

5. Find the equation of the line tangent to the graph of f at $(2, 1)$, where f is given by $f(x) = 2x^3 - 5x^2 + 5$.

6. Find all values of x where the tangent lines to $y = x^8$ and $y = x^9$ are parallel.

7. Consider the function $f(x) = 9 - e^x$.
   (a) Find the slope of the graph of $f(x)$ at the point where the graph crosses the x-axis.
   (b) Find the equation of the tangent line to the curve at this point.
   (c) Find the equation of the line perpendicular to the tangent line at this point. (This is the *normal* line.)

8. Consider the function $y = 2^x$.
   (a) Find the tangent line based at $x = 1$, and find where the tangent line will intersect the x axis.
   (b) Find the point on the graph $x = a$ where the tangent line will pass through the origin.

9. (a) Find the tangent line approximation to $f(x) = e^x$ at $x = 0$.
   (b) Use a sketch of $f(x)$ and the tangent line to determine whether the tangent line produces over- or under-estimates of $f(x)$.
   (c) Use your answer from part (b) to decide whether the statement $e^x \geq 1 + x$ is always true or not.

10. The speed of sound in dry air is
$$f(T) = 331.3\sqrt{1 + \frac{T}{273.15}}\text{m/s}$$
where T is the temperature in degrees Celsius. Find a linear function that approximates the speed of sound for temperatures near 0^o C.

11. Find the equations of the tangent lines to the graph of $y = \sin(x)$ at $x = 0$, and at $x = \pi/3$.
   (a) Use each tangent line to approximate $\sin(\pi/6)$.
   (b) Would you expect these results to be equally accurate, given that they are taken at equal distances on either side of $\pi/6$? If there is a difference in accuracy, can you explain it?

12. Consider the graphs of $y = \sin(x)$ (regular sine graph), and $y = ke^{-x}$ (exponential decay, but scaled vertically by k).
   If $k \geq 1$, the two graphs will intersect. What is the smallest value of k for which two graphs will be *tangent* at that intersection point?

13. (a) Show that $1+kx$ is the local linearization of $(1+x)^k$ near $x = 0$.
   (b) Someone claims that the square root of 1.1 is about 1.05. Without using a calculator, is this estimate about right, and how can you decide using part (a)?

14. (a) Find the local linearlization of e^x near $x = 0$.
   (b) Square your answer to part (a) to find an approximation to e^{2x}.
   (c) Compare your answer in part (b) to the actual linearization to e^{2x} near $x = 0$, and discuss which is more accurate.

15. (a) Show that $1 - x$ is the local linearization of $\dfrac{1}{1+x}$ near $x = 0$.
   (b) From your answer to part (a), show that near $x = 0$,
$$\frac{1}{1+x^2} \approx 1 - x^2.$$
   (c) Without differentiating, what do you think the derivative of $\dfrac{1}{1+x^2}$ is at $x = 0$?

MATLAB Graphing

16. Create a smooth-looking graph of the function $y = \cos(x)$, x in radians, on the interval $[-\pi, 5\pi]$.

17. Create a smooth-looking graph of the function $y = e^{-x^2}$, over the interval $[-3, 3]$.

18. It is common in scientific plots to draw functions as lines, and plot data as distinct points.

    The following points mark the distance between Saturn and several of its moons:

Planet/Object	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
Mean Distance (AU), d	0.39	0.72	1	1.52	5.20	9.54	19.18	30.06	39.44
Period (Earth years), T	0.24	0.62	1	1.88	11.86	29.46	84.01	164.8	247.7

    The best-fit curve to this data is given by the formula $T = d^{3/2}$.

    Plot both the raw data (as points) and best fit curve (as a line) on a single graph. The best-fit graph should look smooth.

Newton's Method

19. Consider the equation $e^x + x = 2$. This equation has a solution near $x = 0$. By replacing the left side of the quation by its linearization near $x = 0$, find an approximate value for the solution.

    (In other words, perform one step of Newton's method, starting at $x = 0$, by hand.)

20. Use Newton's Method with the equation $x^2 = 2$ and initial value $x_0 = 3$ to calculate x_1, x_2, x_3 (the next three solution estimates generated by Newton's method). Do the calculations by hand.

21. Using MATLAB, write a script that applies Newton's Method to solve the equation $x^3 = 5$. Use 10 iterations of Newton's method.
    Compute the values of x^3 when you are done to confirm that it is close to 5.

22. Use Newton's Method to approximate $4^{\frac{1}{3}}$ and compare with the value obtained from a calculator.

    (Hint: write out a simple equation that $4^{\frac{1}{3}}$ would satisfy, and use Newton's method, with MATLAB, to solve that.)

23. Consider the equation $10xe^{-2x} = 0.4$.

    (a) On a single set of axes, draw both the graphs $y = 10xe^{-2x}$ and $y = 0.4$. The x locations of the intersections between these two graphs are the solutions.

    (b) Continue your MATLAB script so that you use Newton's Method to find **both** solutions to $10xe^{-2x} = 0.4$.

    (c) Confirm both solutions by subbing them into the original equation and verifying that the left and right hand sides of the equation are equal.