

MNTC P01 - Week #9 - Differential Equations and Engineering

Pendulum

Consider the motion of a frictionless pendulum.

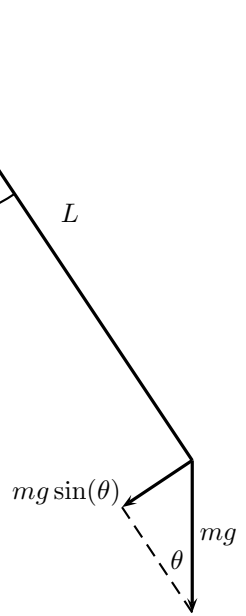
$$\begin{aligned}\text{Newton's Second Law: } mL^2\theta'' &= T_g \\ &= -mLg \sin(\theta)\end{aligned}$$

$$\text{Solving for } \theta'': \theta'' = -\frac{g}{L} \sin(\theta)$$

1. **Without** simulating the actual motion of the pendulum, we can compute the period, T , using the formula below:

$$T = 4\sqrt{L/g} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin(\frac{1}{2}\theta_0)$ and g is the acceleration due to gravity, 9.8 m/s.



For each set of values for L and θ_0 given below,

- (a) Use the MATLAB **integral** function to find the period of the pendulum oscillations by evaluating the integral given above, and
- (b) confirm the period by using **ode45** to simulate the motion of the pendulum for exactly that length of time, and plot a graph of the angular **velocity** against time. The velocity should just reach zero at the end of one cycle.

Do this with the following sets of L and θ_0 values.

- (i) $L = 2$ m, $\theta_0 = 40^\circ$,
- (ii) $L = 2.5$ m, $\theta_0 = 20^\circ$.
- (iii) $L = 5.0$ m, $\theta_0 = 90^\circ$.

In all the graphs, we see that the velocity returns to 0 at the end of the cycle, indicating that our integral calculation of the period matches the period from the **ode45** simulation.

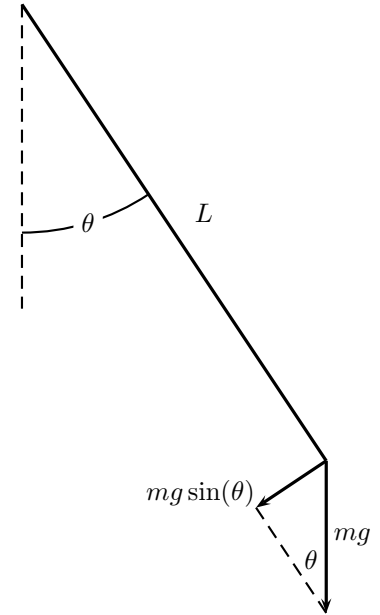
2. Consider the motion of a pendulum, this time **with** friction.

Newton's Second Law:

$$\begin{aligned} mL^2\theta'' &= T_g + T_f \\ &= -mLg\sin(\theta) - (\mu L^2 m)\theta' \end{aligned}$$

$$\text{Solving for } \theta'': \theta'' = -\frac{g}{L}\sin(\theta) - \mu\theta'$$

- (a) Write a MATLAB function for the differential equation, and a script that will simulate the scenario for $L = 1.5$ m, $g = 9.8$ m/s², and $\mu = 0.2$. Use an initial condition of $\theta_0 = \frac{7\pi}{8}$, which is close to vertical.
- (b) Experiment with the initial **angular velocity** of the pendulum and find the smallest **positive** initial velocity that will result in the pendulum passing over the top of the axis of rotation. Find the value to the nearest 0.1 rad/s.
Have MATLAB generate a plot of the angle vs time graph for both the initial velocity that achieves this result, and for the initial velocity 0.1 rad/s smaller, which does *not* go 'over the top'.
- (c) Repeat the analysis in part (b), but this time using a **negative** initial velocity.



Single Tank Problems

3. An aquarium pool has volume 2×10^6 liters. The pool initially contains pure fresh water. At $t = 0$ minutes, water containing 10 grams/liter of salt is poured into the pool at a rate of 60 liters/minute. The salt water instantly mixes with the fresh water, and the excess mixture is drained out of the pool at the same rate (60 liters/minute).
- (a) Write a differential equation for $S(t)$, the mass of salt in the pool at time t .
- (b) Use MATLAB solve the differential equation to predict $S(t)$ over time.
- (c) Based on the graph of the solution, what happens to $S(t)$ as $t \rightarrow \infty$?
- (d) Find this same value using only the information about the volume and the concentration of the incoming salt solution.
4. A 150 liter tank initially contains 60 liters of water with 0.5 kgs of salt dissolved in it. Water enters the tank at a rate of 0.9 liters/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + \cos(t))$ kgs/liter.
- (a) Draw a diagram of the inflow and outflow for this scenario.
- (b) Build a formula for the volume of water in the tank over time.
- (c) Find out how long it will be until the tank overflows.
- (d) Write a differential equation that describes the rate of change of the **amount of salt** in the tank.
- (e) Use MATLAB to generate a graph of the amount of salt in the tank over time, up until the tank overflows.
- (f) How much salt is in the tank when it overflows?

Other First Order Models

5. Differential equations are not only well-suited for physics applications: they are also widely used in biology, particularly in population models.

Consider the fish population model below, based on a standard limited-resource population growth, minus a function of harvesting.

$$\frac{dP}{dt} = \underbrace{[(10 - P) \cdot P]}_{\text{natural population growth rate}} - \underbrace{h(t)}_{\text{harvesting rate}}$$

where

- P = population of fish (in thousands), and
- $\frac{dP}{dt}$ = rate of population change, in thousands per year
- $h(t)$ is the harvesting rate (in thousands of fish per year)

We want to study the impact of two harvesting models:

- $h_1 = k_1$; constant harvesting
 - $h_2(t) = k_2(\sin(\pi t) + 1)$; seasonal model where the harvesting has a yearly cycle.
- (a) Generate a prediction of the population over time, starting at initial populations of $P(0) = 15$ for each model. Use $k_1 = k_2 = 5$. Produce a graph showing the predicted population over time on the same graph, over a long enough time interval to show the long-term behaviour of both solutions.

One question that arises in such harvesting models is which fishing strategy permits a higher average harvesting rate can be maintained: seasonal harvesting, or constant harvesting? To decide this, we note that the average harvest rate for h_1 is k_1 , and for h_2 is k_2 , so whichever value of k_1 and k_2 is larger indicates the strategy with the greater average harvesting rate.

We will define the *maximum sustainable harvest rate* for both models as the *highest harvest rate for which the population is not driven to zero*.

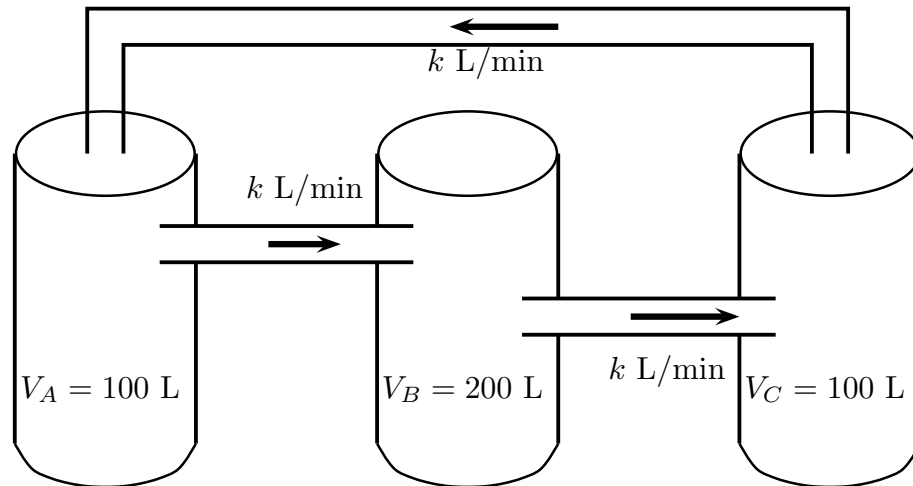
- (b) Find and report the maximum sustainable harvest level k_1 for the constant harvesting model (to the nearest integer). (Use trial and error if necessary, though more insightful DE-related ways are possible.) Indicate how you found the cut-off level.

NOTE: during this process, your model will predict a population of zero, which will then lead to large negative populations. This clearly makes no sense, so limit your plots with the command `ylim([0, P0])`. This same problem will also trigger warnings in `ode45` about error tolerances; you can safely ignore those warnings.

- (c) Generate a plot showing the population over time, using the same initial value used earlier, but using both the k_1 value just above, and just below the extinction level. (One line should remain positive, while the other should crash to zero at some point on the graph.)
- (d) Use trial and error (theory isn't much help here) to find the maximum sustainable harvest level k_2 for the cyclic harvesting model (to the nearest integer). Include a plot showing the population over time with this harvesting level.
- (e) Based on your experiments, can constant harvesting or cyclic harvesting sustain a greater average harvest in the long run? Explain your reasoning.

Multi-Tank Systems

6. Consider two interconnected tanks. Tank A initially contains 100 L of water and 200 g of salt, and tank B initially contains 200 L of water and 900 g of salt. The liquid inside each tank is kept well stirred.
- Liquid flows from tank A to tank B at a rate of $3 \text{ L} \cdot \text{min}^{-1}$;
 - liquid flows from tank B to tank A at rate of $2 \text{ L} \cdot \text{min}^{-1}$.
 - A salt brine with concentration $7 \text{ g} \cdot \text{L}^{-1}$ of salt flows into tank A at a rate of $5 \text{ L} \cdot \text{min}^{-1}$; an outflow pipe drains out of the Tank A at $4 \text{ L} \cdot \text{min}^{-1}$.
 - Moreover, a salt brine with concentration $3 \text{ g} \cdot \text{L}^{-1}$ of salt flows into tank B at a rate of $7 \text{ L} \cdot \text{min}^{-1}$; the solution is drained out at $8 \text{ L} \cdot \text{min}^{-1}$.
- (a) Draw a diagram for the flows and concentrations in this scenario.
- (b) Write a set of differential equations for the **amount** of salt in each tank.
- (c) Use MATLAB to simulate the concentration of salt in each tank over time. Generate two separate graphs, one for Tank A and one for Tank B, and use the `title` command to label each one.
- (d) Use the solution generated by MATLAB to estimate when Tank B reaches its lowest salt amount, and what that lowest amount is.
7. Consider the 3-tank system shown below.



- Write the set of differential equations that governs the **amount** of salt in each tank, S_A , S_B and S_C .
- Convert your answer from part (a) into a set of differential equations for the **concentrations** in each tank, C_A , C_B and C_C .
- Using the first line `function dw_dt = tankSystem2(t, w, k, VA, VB, VC)` and the definition that $\vec{w} = [C_A, C_B, C_C]$ to group the three dependent variables, write a MATLAB function file that implements the differential equation system from part (b).
- Write a script that simulates the changes in concentration over time, using the volumes shown in the diagram, a flow rate of $k = 2 \text{ L/min}$ for each connection, and a time span of 250 minutes.
- Use your knowledge of chemistry to explain the fact that all the tanks converge to a common concentration of 2.5 g/L .