

## MNTC P01 - Week #2 - Derivatives - Linearization and Applications

### Linear Approximations and Tangent Lines

- Find the equation of the tangent line to the graph of  $f$  at  $(1,1)$ , where  $f$  is given by  $f(x) = 2x^3 - 2x^2 + 1$ .
- (a) Find the equation of the tangent line to  $f(x) = x^3$  at  $x = 2$ .  
(b) Sketch the curve and the tangent line on the same axes, and decide whether using the tangent line to approximate  $f(x) = x^3$  would produce *over-* or *under-*estimates of  $f(x)$  near  $x = 2$ .
- Find the equation of the line tangent to the graph of  $f$  at  $(3,57)$ , where  $f$  is given by  $f(x) = 4x^3 - 7x^2 + 12$ .
- Given a power function of the form  $f(x) = ax^n$ , with  $f'(3) = 16$  and  $f'(6) = 128$ , find  $n$  and  $a$ .
- Find the equation of the line tangent to the graph of  $f$  at  $(2,1)$ , where  $f$  is given by  $f(x) = 2x^3 - 5x^2 + 5$ .
- Find all values of  $x$  where the tangent lines to  $y = x^8$  and  $y = x^9$  are parallel.
- Consider the function  $f(x) = 9 - e^x$ .
  - Find the slope of the graph of  $f(x)$  at the point where the graph crosses the  $x$ -axis.
  - Find the equation of the tangent line to the curve at this point.
  - Find the equation of the line perpendicular to the tangent line at this point. (This is the *normal* line.)
- Consider the function  $y = 2^x$ .
  - Find the tangent line based at  $x = 1$ , and find where the tangent line will intersect the  $x$  axis.
  - Find the point on the graph  $x = a$  where the tangent line will pass through the origin.
- (a) Find the tangent line approximation to  $f(x) = e^x$  at  $x = 0$ .  
(b) Use a sketch of  $f(x)$  and the tangent line to determine whether the tangent line produces over- or under-estimates of  $f(x)$ .  
(c) Use your answer from part (b) to decide whether the statement  $e^x \geq 1 + x$  is always true or not.

10. The speed of sound in dry air is

$$f(T) = 331.3\sqrt{1 + \frac{T}{273.15}} \text{ m/s}$$

where  $T$  is the temperature in degrees Celsius. Find a linear function that approximates the speed of sound for temperatures near  $0^\circ \text{ C}$ .

- Find the equations of the tangent lines to the graph of  $y = \sin(x)$  at  $x = 0$ , and at  $x = \pi/3$ .
  - Use each tangent line to approximate  $\sin(\pi/6)$ .
  - Would you expect these results to be equally accurate, given that they are taken at equal distances on either side of  $\pi/6$ ? If there is a difference in accuracy, can you explain it?
- Consider the graphs of  $y = \sin(x)$  (regular sine graph), and  $y = ke^{-x}$  (exponential decay, but scaled vertically by  $k$ ).  
If  $k \geq 1$ , the two graphs will intersect. What is the smallest value of  $k$  for which two graphs will be *tangent* at that intersection point?
- (a) Show that  $1+kx$  is the local linearization of  $(1+x)^k$  near  $x = 0$ .  
(b) Someone claims that the square root of 1.1 is about 1.05. Without using a calculator, is this estimate about right, and how can you decide using part (a)?
- (a) Find the local linearization of  $e^x$  near  $x = 0$ .  
(b) Square your answer to part (a) to find an approximation to  $e^{2x}$ .  
(c) Compare your answer in part (b) to the actual linearization to  $e^{2x}$  near  $x = 0$ , and discuss which is more accurate.
- (a) Show that  $1 - x$  is the local linearization of  $\frac{1}{1+x}$  near  $x = 0$ .  
(b) From your answer to part (a), show that near  $x = 0$ ,

$$\frac{1}{1+x^2} \approx 1 - x^2.$$

- Without differentiating, what do you think the derivative of  $\frac{1}{1+x^2}$  is at  $x = 0$ ?

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### MATLAB Graphing

16. Create a smooth-looking graph of the function  $y = \cos(x)$ ,  $x$  in radians, on the interval  $[-\pi, 5\pi]$ .

17. Create a smooth-looking graph of the function  $y = e^{-x^2}$ , over the interval  $[-3, 3]$ .
18. It is common in scientific plots to draw functions as lines, and plot data as distinct points.

The following points mark the distance between Saturn and several of its moons:

Planet/Object	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
Mean Distance (AU), $d$	0.39	0.72	1	1.52	5.20	9.54	19.18	30.06	39.44
Period (Earth years), $T$	0.24	0.62	1	1.88	11.86	29.46	84.01	164.8	247.7

The best-fit curve to this data is given by the formula  $T = d^{3/2}$ .

Plot both the raw data (as points) and best fit curve (as a line) on a single graph. The best-fit graph should look smooth.

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## Newton's Method

19. Consider the equation  $e^x + x = 2$ . This equation has a solution near  $x = 0$ . By replacing the left side of the equation by its linearization near  $x = 0$ , find an approximate value for the solution.
- (In other words, perform one step of Newton's method, starting at  $x = 0$ , by hand.)
20. Use Newton's Method with the equation  $x^2 = 2$  and initial value  $x_0 = 3$  to calculate  $x_1, x_2, x_3$  (the next three solution estimates generated by Newton's method). Do the calculations by hand.
21. Using MATLAB, write a script that applies Newton's Method to solve the equation  $x^3 = 5$ . Use 10 iterations of Newton's method.
- Compute the values of  $x^3$  when you are done to confirm that it is close to 5.
22. Use Newton's Method to approximate  $4^{1/3}$  and compare with the value obtained from a calculator.
- (Hint: write out a simple equation that  $4^{1/3}$  would satisfy, and use Newton's method, with MATLAB, to solve that.)
23. Consider the equation  $10xe^{-2x} = 0.4$ .
- (a) On a single set of axes, draw both the graphs  $y = 10xe^{-2x}$  and  $y = 0.4$ . The  $x$  locations of the intersections between these two graphs are the solutions.
- (b) Continue your MATLAB script so that you use Newton's Method to find **both** solutions to  $10xe^{-2x} = 0.4$ .
- (c) Confirm both solutions by subbing them into the original equation and verifying that the left and right hand sides of the equation are equal.