Week #3 : Derivatives - Applications

Goals:

- Calculate and interpret the first and second derivatives, as well as higher order derivatives.
- Define and calculate Taylor Polynomials.
- Use MATLAB to graph and compare functions with their Taylor polynomial approximations.
- Find and use critical points for global and local optimization problems.
- Use MATLAB optimizers and equation solvers to identify optimal values and critical points.

Second and Higher Derivatives

The information about the graph of a function f provided by the sign of $f'(x)$ and $f''(x)$ on an interval (a, b) is expressed in the following table. (a and b are assumed to be finite.)

$f'(x) > 0$ on (a, b)	f increasing on $[a, b]$
$f'(x) < 0$ on (a, b)	f decreasing on $[a, b]$
$f''(x) > 0$ on (a, b)	f concave up on $[a, b]$
$f''(x) < 0$ on (a, b)	f concave down on $[a, b]$

Problem. Sketch the possible graphs combining different signs of positive and negative first and second derivatives.

Problem. Sketch graphs where the **first** derivative has a zero value.

Problem. Sketch graphs where the **second** derivative has a zero value.

Aside from their graphical interpretation, second derivatives frequently have important physical interpretations in kinematics problems.

Problem. If $x(t) = 4\sin(2t)$ gives the position of a particle at time t, what is particle's **speed** at $t = \dfrac{\pi}{6}$?

For the same particle, what is its **acceleration** at $t = \dfrac{\pi}{6}$?

While their interpretations are not as immediately obvious, it is possible to compute 3rd, 4th, or higher derivatives of function if we want.

Problem. Find the first four derivatives of the function

$$f(x) = 7(2^x) + \ln(x).$$

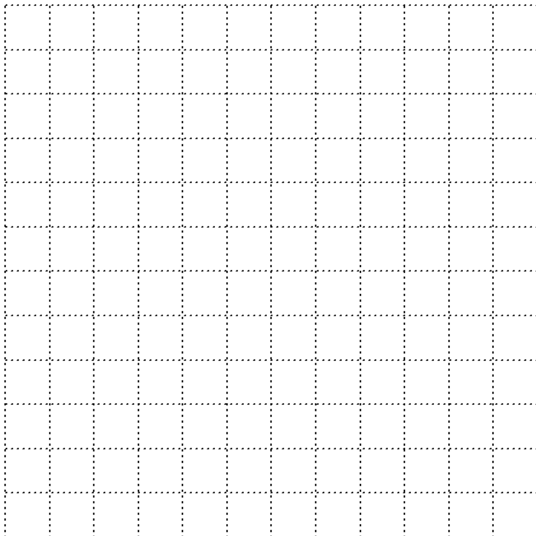
Taylor Polynomials

One application of higher derivative information is to help us build **polynomial approximations** to complicated functions.

Previously we found a formula for linear approximations to functions $f(x)$ around a point $x = a$:

This linear approximation, or tangent line formula, can also be called the **Taylor polynomial of degree 1 approximating** $f(x)$ **near** $x = a$**.**

Problem. Sketch the graph of $\cos(x)$ around $x = 0$, and add its tangent line based at $x = 0$.



The linearization or tangent line is clearly a very limited approximation to this function. What might be a *slightly* more complex form of function that would work better in this case?

Taylor Polynomial of Degree 2

$$f(x) \approx \quad f(a) \quad + f'(a)(x-a) \quad + \frac{f''(a)}{2}(x-a)^2$$

is a *quadratic* approximation to $f(x)$ near $x = a$.

Problem. For values of x close to a do you think this quadratic approximation will be a better or worse approximation than the tangent line? Why?

Problem. Find the quadratic Taylor approximation to $f(x) = \cos(x)$ near $x = 0$.

Problem. Use MATLAB to draw the graph of $\cos(x)$ around $x = 0$, and add both its 1st and 2nd degree Taylor polynomial approximations for x near 0.

There is a very good reason for the particular form of the Taylor polynomial.

Problem. What mathematical features will the original $f(x)$ share with its 2nd degree Taylor approximation at the point $x = a$?

Taylor Polynomials of Higher Degree

Problem. If we wanted a still-better approximation for a function $f(x)$ near a specific point $x = a$, how could we generalize our earlier 1st and 2nd degree Taylor polynomials?

Below is the general formula for the terms in a Taylor polynomial, up to degree n.

$$f(x) \approx \; f(a) \; + f'(a)(x-a) \; + \frac{f''(a)}{2}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- $f^{(n)}$ means "the n-th derivative of f".

- $n!$ means "n factorial"

Higher Degree Taylor Polynomials - Example

Consider the function $f(x) = \sin(x)$.

Problem. Find the first five derivatives of $f(x)$, and evaluate them at $x = 0$.

Problem. Write out the Taylor poylnomial of degree 5 for $f(x) = \sin(x)$.

Problem. Write out the general form of the Taylor poylnomial of degree n for $f(x) = \sin(x)$.

Problem. Use MATLAB to plot the graph of $f(x) = \sin(x)$ and the Taylor polynomial approximations up to degree 5.

MATLAB Demo of increasing higher degrees.

Critical Points

If $f(x)$ is defined on the interval (a, b), then we call a point c in the interval a **critical point** if:

- $f'(c) = 0$, or
- $f'(c)$ does not exist.

We will also refer to the point $(c, f(c))$ on the graph of $f(x)$ as a critical point. We call the function value $f(c)$ at a critical point c a **critical value**.

Technical Notes:

1. By this definition, $f(c)$ must be **defined** for c to be a critical point.

   Sketch $f(x) = 1/x$, and decide whether $x = 0$ is a critical point.

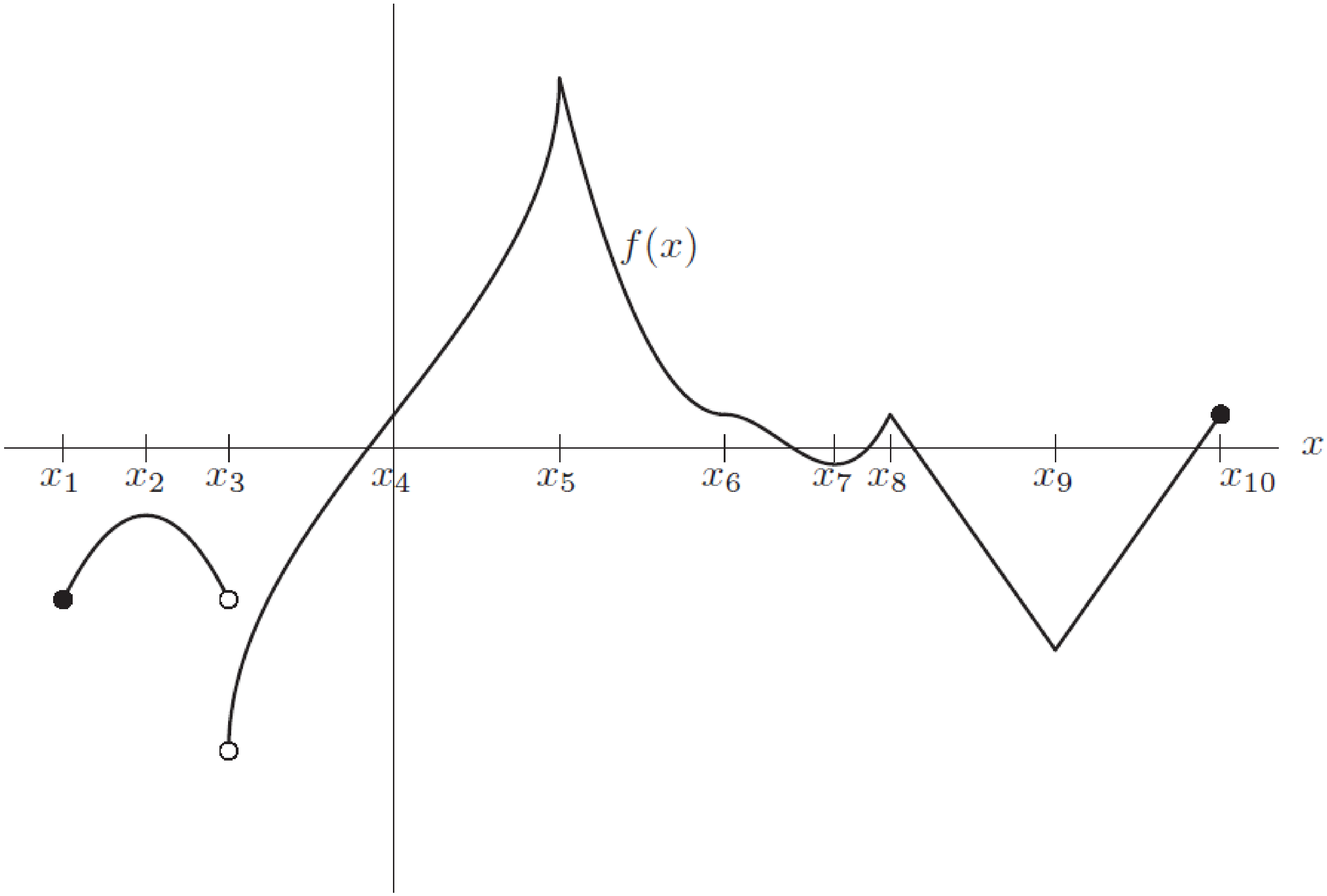
   Sketch $g(x) = |x|$, and decide whether $x = 0$ is a critical point.

2. By the definition, if a function is defined on a closed interval, the endpoints of interval **cannot** be critical points.

   Sketch the graph of $f(x) = \sqrt{x}$ and decide whether $x = 0$ is a critical point.

Sketch the graph of $g(x) = \sqrt[3]{x}$ and decide whether $x = 0$ is a critical point.

Problem. Identify all the critical points on the graph below, and characterize any other interesting points by continuity, limits, or other properties.



Problem. Consider the function

$$f(x) = \frac{x}{x^2+1}$$

Construct a sign chart for both f' and f'', and use this information to sketch $f(x)$.

Optimization - Introduction

Optimization - Critical Points