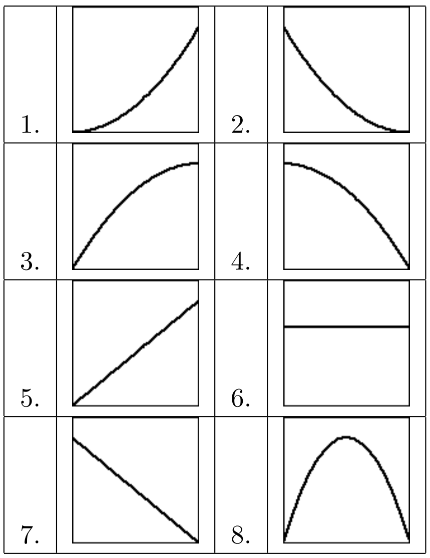
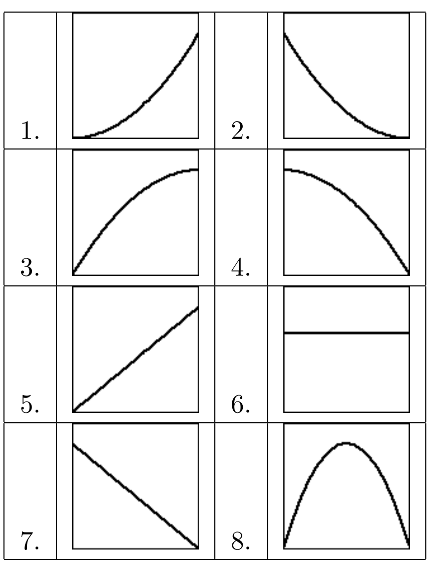
Week #1 - Derivatives - Foundations

Derivative Concept and Definition

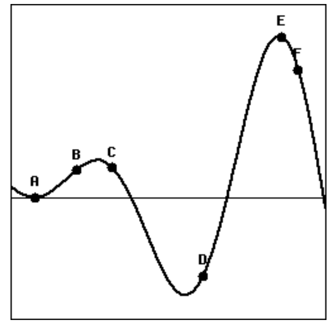
1. A car is driven at a speed that is initially high and then decreases, starting at noon. Which of the following could be a graph of the distance the car has traveled as a function of time past noon?



2. A car is driven at a constant speed, starting at noon. Which of the following could be a graph of the distance the car has traveled as a function of time past noon?

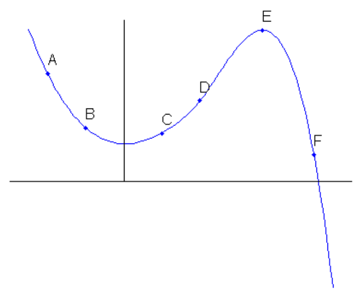


3. Match the points labeled on the curve below with the given slopes in the following table.



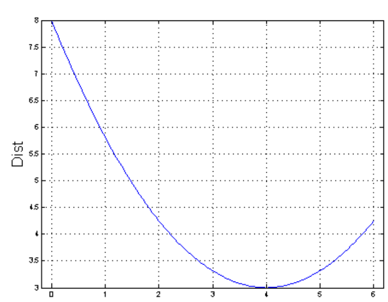
slope	-3	-1	-1/2	0	1	2
label						

4. Consider the function shown in the graph below.



(a)For each labelled point, is the slope of the graph positive, negative or zero?
(b) At which labeled point does the graph have the greatest (i. e., most positive) slope?
(c) At which labeled point does the graph have the **largest negative** slope?

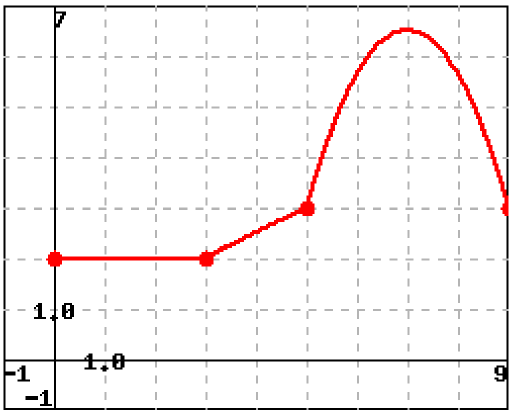
5. Consider the **distance vs time** graph shown below.



Rank the following quantities as if along the real line (i.e. largest negative, through zero, up to largest positive).
A - Instantaneous velocity at $t = 1$.
B - Instantaneous velocity at $t = 3$.
C - Instantaneous velocity at $t = 4$.

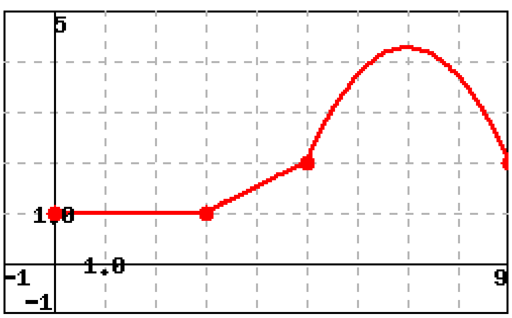
D - Instantaneous velocity at $t = 5$.
E - Average velocity over $t = 1 \ldots 3$.
F - Average velocity over $t = 4 \ldots 5$.

6. Let $f(x)$ be the function whose graph is shown below.



Determine the derivative of $f(a)$ at the points $a = 1, 2, 4, 7$.

7. Let $f(x)$ be the function whose graph is shown below.



Which is larger?

- A. $f'(6.5)$
- B. $f'(5.5)$

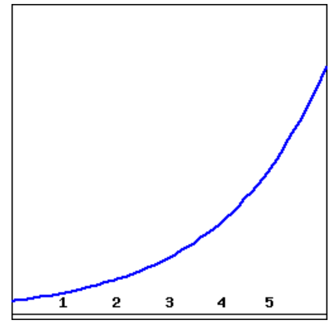
8. Use algebra to evaluate the following limit.

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h}$$

9. Estimate the following limit by substituting smaller and smaller values of h, and by using algebra (the two answers should be very similar!).

$$\lim_{h \to 0} \frac{(4+h)^3 - 64}{h}$$

10. Suppose $y = f(x)$ graphed in the figure below represents the cost of manufacturing x kilograms of a chemical.

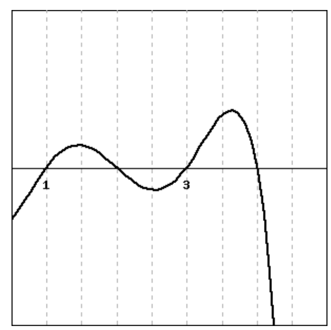


$f(x)/x$ represents the average cost of producing 1 kilogram of the chemical when x kilograms are made. This problem asks you to visualize these averages graphically.

(a) We can represent $f(4)/4$ as the slope of a line. Through what points does this line extend?

(b) Which is larger, $f(3)/3$ or $f(4)/4$?

11. Consider the function $y = f(x)$ graphed below.



Give the x-coordinate of a point where:

(a) The derivative of the function is positive.

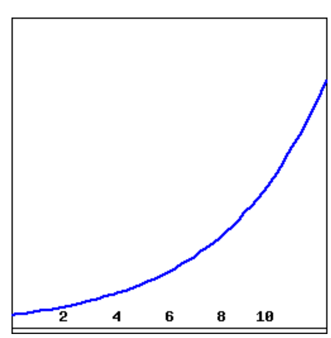
(b) The value of the function is positive.

(c) The derivative of the function is largest.

(d) The derivative of the function is zero.

(e) The derivative of the function is approximately the same as the derivative at $x = 3.25$.

12. Consider the graph of the function $f(x)$ shown below.



Using this graph, for each of the following pairs of numbers decide which is larger. *Be sure that you can explain your answer.*

(a) $f(8)$ vs. $f(10)$

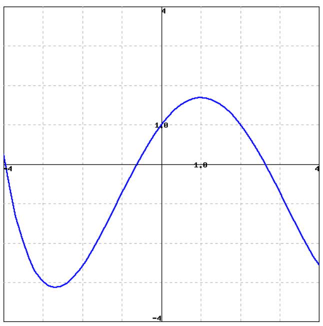
(b) $f(8) - f(6)$ vs. $f(6) - f(4)$

(c) $\dfrac{f(6) - f(4)}{6 - 4}$ vs. $\dfrac{f(8) - f(4)}{8 - 4}$

(d) $f'(4)$ vs. $f'(10)$

The Derivative as a Function

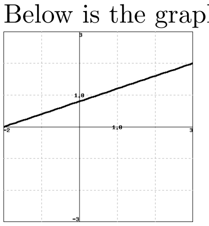
13. Consider the function $f(x)$ shown in the graph below.



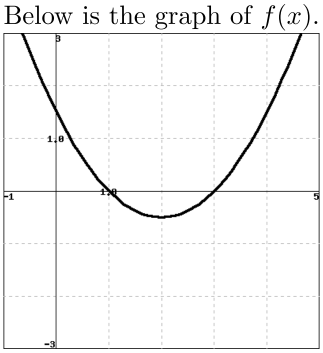
Carefully sketch the derivative of the given function (you will want to estimate values on the derivative function at different x values as you do this). Use your derivative function graph to estimate the following values on the derivative function.

at $x =$	-3	-1	1	3
the derivative is	__	__	__	__

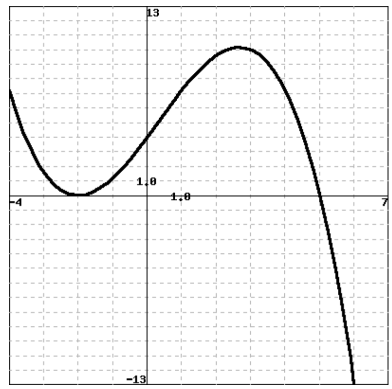
14. Below is the graph of $f(x)$. Sketch the graph of $f'(x)$.



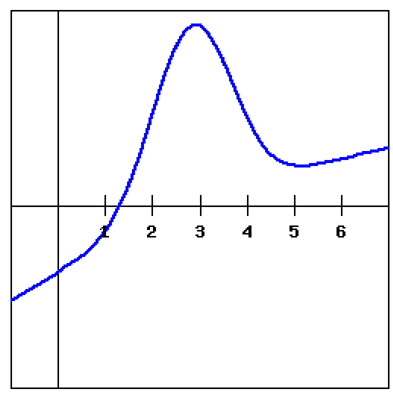
15. Below is the graph of $f(x)$. Sketch the graph of $f'(x)$.



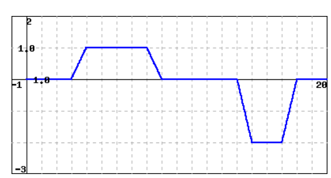
16. Below is the graph of $f(x)$. Sketch the graph of $f'(x)$.



17. For the function $f(x)$ shown in the graph below, sketch a graph of the derivative.



18. A child fills a pail by using a water hose. After finishing, the child plays in a sandbox for a while before tipping the pail over to empty it. If $V(t)$ gives the volume of the water in the pail at time t, then the figure below shows $V'(t)$ as a function of t.



At what time does the child:

(a) Begin to fill the pail?

(b) Finish filling the pail?

(c) Tip the pail over?

Computing Derivatives

Below are a **small sample of problems** involving the computation of derivatives. They are **not enough** to properly learn and memorize how to apply all the derivative rules. You should practice with as many problems as you need to become proficient at computing derivatives.

Further practice problems can be found in any calculus textbook. For example,

From Hughes-Hallett 5th edition,
Section 3.1 - 7-47 (odd)†
Section 3.2 - 1-33 (odd)
Section 3.3 - 3-29 (odd)
Section 3.4 - 1-49 (odd)
Section 3.5 - 3-39 (odd)
Section 3.6 - 1-33 (odd)

From Hughes-Hallett 6th edition,
Section 3.1 - 7-49 (odd)†
Section 3.2 - 1-25 (odd)
Section 3.3 - 3-29 (odd)
Section 3.4 - 1-55 (odd)
Section 3.5 - 3-47 (odd)
Section 3.6 - 1-41 (odd)

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19. Let $f(x) = 4e^x - 9x^2 + 5$. Compute $f'(x)$.

20. Let $f(x) = 2x^6\sqrt{x} + \dfrac{-5}{x^3\sqrt{x}}$. Compute $f'(x)$.

21. Let $f(x) = \dfrac{7x^2 + 7x + 5}{\sqrt{x}}$.

    (a) Compute $f'(x)$. (b) Find $f'(3)$.

22. Let $f(t) = 7t^{-7}$.

    (a) Compute $f'(t)$. (b) Find $f'(3)$.

23. Let $f(x) = 4e^x + e^1$. Compute $f'(x)$.

24. Let $f(x) = 4e^x + 4x$. Compute $f'(x)$.

25. $f(x) = (3x^2 - 2)(6x + 3)$.

    (a) Compute $f'(x)$. (b) Find $f'(4)$.

26. Let $f(x) = \dfrac{\sqrt{x} - 4}{\sqrt{x} + 4}$. Compute $f'(9)$.

27. Consider $f(x) = \dfrac{4x + 3}{3x + 2}$.

    (a) Compute $f'(x)$. (b) Find $f'(5)$.

28. Consider $f(x) = \dfrac{7 - x^2}{7 + x^2}$

    (a) Compute $f'(x)$. (b) Find $f'(1)$.

29. Let $f(x) = -2x(x - 3)$.

    (a) Compute $f'(x)$. (b) Find $f'(-5)$.

30. $f(x) = \dfrac{4x^3 - 3}{x^4}$

    (a) Compute $f'(x)$. (b) Find $f'(2)$.

31. $g(x) = \dfrac{e^x}{5 + 4x}$. Compute $g'(x)$.

32. $f(x) = \dfrac{4x^2 \tan x}{\sec x}$.

    (a) Find $f'(x)$. (b) Find $f'(3)$.

33. $f(x) = 7\sin x + 12\cos x$

    (a)Compute $f'(x)$. (b) Find $f'(1)$.

34. Let $f(x) = \cos x - 2\tan x$. Compute $f'(x)$.

35. $f(x) = \dfrac{5\sin x}{3 + \cos x}$

    (a) Compute $f'(x)$. (b) Find $f'(2)$.

36. $f(x) = 7x(\sin x + \cos x)$

    (a) Compute $f'(x)$. (a) Find $f'(3)$.

37. Let $f(x) = \cos(\sin(x^2))$. Compute $f'(x)$.

38. Let $f(x) = 2\sin^3 x$. Compute $f'(x)$.

39. Let $y = (8 + \cos^2 x)^6$. Compute $\dfrac{dy}{dx}$.

40. Let $f(x) = -3\ln[\sin(x)]$. Compute $f'(x)$.

41. Let $f(x) = 2\ln(4 + x)$. Compute $f'(x)$.

42. Consider the function $\cos(\arccos(x))$.

    (a) Simplify the expression to get a simple equation.

    (b) Differentiate both sides of the equation, and then solve for $\dfrac{d}{dx}\arccos(x)$.

    (c) Use the trig identity $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$ to simplify your expression for $\dfrac{d\arccos}{dx}(x)$.

43. Consider the function $\tan(\arctan(x))$.

†For this section, simplify products and fractions *before* you differentiate, rather than using the product rule and quotient rule.
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(a) Simplify the expression to get a simple equation.

(b) Differentiate both sides of the equation, and then solve for $\frac{d}{dx}$ arctan(x).

(c) Use the trig identity $\sec^2(\theta) = 1 + tan^2(\theta)$ to simplify your expression for $\frac{d\text{arctan}}{dx}(x)$.

44. Take the derivative of $y = \arcsin(4x)$.

45. If $f(x) = x \arctan(x)$, find $f'(x)$.

46. Find the slope of $y = \frac{3x}{\arccos(x)}$ at the point $x = 0$.

47. Let $f(x) = \arcsin(x)$.

    (a) Compute $f'(x)$ (b) Find $f'(0.4)$.

48. Let $f(x) = \frac{\arccos(14x)}{\arcsin(14x)}$. Compute $f'(x)$.

49. Let
$$P = \frac{V^2 R}{(R + r)^2}.$$
Calculate $\frac{dP}{dr}$, assuming that r is variable and R and V are constant.

Inverse Trig Functions

50. (a) Sketch the graph of $y = \arccos(x)$.

    (b) Find the exact values of

    (i) $\arccos(0)$

    (ii) $\arccos(0.5)$

    (iii) $\arccos(1)$

    (iv) $\arccos(-1)$

51. (a) Sketch the graph of $y = \arctan(x)$.

    (b) Find the exact values of

    (i) $\arctan(0)$

    (ii) $\arctan(1)$

    (iii) $\arctan(-1)$

    (iv) $\arctan(x)$ as x gets very large.