Week #10: Linear Algebra - An Introduction

Goals:

- Express vectors, linear combinations, and compute dot products.
- Write information in matrix form in the context of engineering applications.
- Understand the definition of the transpose, and use MATLAB to compute it.
- Use MATLAB to compute the inverse of a matrix.

Introduction to Vectors

In a physical context we often need to describe measurable quantities such as pressure, mass, and speed, and these objects can be completely described by a single number known as the **magnitude**. However there are other quantities which help us describe the world around us such as force, velocity and acceleration that require more than magnitude to describe them. They also need a **direction**. A **vector** is a magnitude (a number describing how much, how fast, etc) in combination with a direction. Vectors are denoted by boldface letters, such as $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{x}$, $\overrightarrow{y}$, $\overrightarrow{z}$.

When discussing direction, we need to decide on a frame of reference. We will be using the Cartesian coordinate system. A *point* in the two-dimensional Cartesian plane, denoted as $\mathbb{R}^2$ is denoted by an ordered pair (x,y) of real numbers, which we call *coordinates*. So for example, (2,1) can be represented as a point on a grid like so:

Consider the point (x,y) in $\mathbb{R}^2$. If we draw a directed line segment from the **origin** (the point (0,0)) to (x,y), we get the following picture: "insert picture here"
Notice there is an arrow pointing at (x,y). We call this the **head** of the vector and the **tail** is at the origin, indicating a direction. The **magnitude** of the vector is its length from (0,0) to (x,y).

To find the magnitude of a vector, we first find its **components**. If the head of the vector is at the point (x', y') and the tail of the vector is at the point (x,y), then the components of the vector are $x' - x$ and $y' - y$, and we represent them in the following way:

$$\begin{bmatrix} x' - x \\ y' - y \end{bmatrix}$$

Once you have found the components, the formula for finding the magnitude of the vector is:

$$\text{magnitude} = \sqrt{(x' - x)^2 + (y' - y)^2}$$

Draw and find the magnitude of the following vectors: $\overrightarrow{a}$ has the tail at (-1,1) and the head at (4,3), $\overrightarrow{b}$ has the tail at (0,4) and the head at (-2,-2), and $\overrightarrow{c}$ has the tail at (3,0) and the head at (0,3). We use the notation $||\overrightarrow{a}||$ to represent the magnitude of $\overrightarrow{a}$.
"Insert picture here"

An interesting fact to note is that two vectors are **equal** if their components are equal. Consider the vector $\overrightarrow{A}$ going from (-1,4) to (-3,1) (tail to head), $\overrightarrow{B}$ going from (2,3) to (0,0) and $\overrightarrow{C}$ going from (5,5) to (3,2). All of these vectors occupy the different regions of $\mathbb{R}^2$, yet we can define them as equal since they have the same components. A **position vector** is a vector whose tail starts at the origin, and if a vector is given with just its components, you may assume it is a position vector.

Vector Addition and Scalar Multiplication

If we have two vectors in $\mathbb{R}^2$ with components

$$\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \overrightarrow{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

then the **sum** of the vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\overrightarrow{x} + \overrightarrow{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

So if we have vectors

$$\overrightarrow{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \overrightarrow{y} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, \text{ and } \overrightarrow{z} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

Then

$$\overrightarrow{x} + \overrightarrow{y} = \begin{bmatrix} 1 + (-2) \\ 1 + 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Or we could have

$$\overrightarrow{y} + \overrightarrow{z} = \begin{bmatrix} -2 + 4 \\ 5 + (-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

We can interpret vector addition geometrically.
“Insert picture here”

If $\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a vector and c is a scalar (meaning a real number), then the **scalar** multiple $c\overrightarrow{x}$, meaning every component of $\overrightarrow{x}$ is multiplied by c, is $c\overrightarrow{x} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$. If $c > 0$, then $c\overrightarrow{x}$ is in the same direction as $\overrightarrow{x}$. If $c < 0$, then $c\overrightarrow{x}$ is in the opposite direction as $\overrightarrow{x}$.
“Insert picture here”

For example, if $c = -1$, $d = 2$ and $\overrightarrow{y} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ (Recall that $\overrightarrow{y}$ is a position vector, whose tail starts at the origin), then $c\overrightarrow{y} = \begin{bmatrix} (-1)\cdot(4) \\ (-1)\cdot(-1) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $d\overrightarrow{y} = \begin{bmatrix} (2)\cdot(4) \\ (2)\cdot(-1) \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$
"insert picture here"

3-Dimensional Vectors

3-dimensional (also known as 3-space) vectors exist in $\mathbb{R}^3$, meaning that we are now dealing another axis, the z-axis. Just like in $\mathbb{R}^2$, there is an origin where all of the axes meet, (0,0,0). Points in $\mathbb{R}^3$ are represented by an ordered triplet (x, y, z). The points (2,1,3), (4, -2, -2), and (1,0,5) would be drawn like so:
"Insert Picture Here"

Components for 3-dimensional vectors are defined in the same way as 2-dimensional vectors, except that now there are three of them. So for a vector $\overrightarrow{u}$ whose tail starts at the point (x,y,z) and has its head at the point (x',y',z'), the components of $\overrightarrow{u}$ are:

$$\begin{bmatrix} x' - x \\ y' - y \\ z' - z \end{bmatrix}$$

A 3-dimensional vector whose components are given without information about the head or the tail is a position vector, and you can assume its tail starts at the origin.

All of the rules for vector addition and scalar multiplication we presented for 2-dimensional vectors are the same for the 3-dimensional versions. If we have position vector $\overrightarrow{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ then the magnitude of $\overrightarrow{a}$ is

$$||\overrightarrow{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Keep in mind that we get this formula from the fact that the tail of $\overrightarrow{a}$ is at the origin (0,0,0).