Week #7: Differential Equations - Introduction

Goals:

- Understand and express real world situations in terms of first order differential equations
- Tell the difference between linear and nonlinear differential equations
- Solve basic first order separable and linear differential equations
- Use MATLAB to solve nonlinear first order differential equations

Differential Equations

An equation relating an unknown function and one or more of its derivatives is a **differential equation**.

We study differential equations because they are an incredibly powerful tool for modelling real world problems.

Using MATLAB to Solve Differential Equations

Since this course has a strong computational component, we will be using MATLAB to solve the majority of differential equations problems we come across. Be sure to watch the videos that accompany each problem to see how to enter code into MATLAB. This link also provides some help:
http://www.mathworks.com/help/matlab/ref/ode45.html
ode45 is going to be your new best friend when it comes to differential equations!

The **solution** is the *family* of all possible functions that satisfy a differential equation. It normally involves some parameters, i.e. arbitrary constants.

A **particular solution** is a specific function with satisfies the differential equation. We will be dealing with particular solutions in this course.

Problem. Use MATLAB to create a plot of the solution to the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x^2 - 1}$ which also satisfies $y(2) = 1$. Use the plot to describe the behaviour of this solution.

A first-order **initial value problem** is a differential equation

$$y'(x) = F\big(x, y(x)\big)$$

together with a point (x_0, y_0) in the domain of F called the **initial condition**.

A **solution** to an initial value problem is a function y that satisfies the differential equation and $y(x_0) = y_0$. This solution will be a **particular solution** to the differential equation.

Problem. Find the solution to the differential equation $\dfrac{dy}{dt} = 2t + 1$; $y(0) = 3$. Now use MATLAB to create a plot of the solution, and use the plot to describe the behaviour of this solution.

$$\frac{dy}{dt} = 2t + 1;\ y(0) = 3.$$

Problem. Use MATLAB to create a plot of the solution to the initial value problem: $\dfrac{dy}{dx} = x^2 e^x$; $y(0) = 2$. Describe the behaviour of the solution.

Separable Equations

We now introduce our first technique for solving first-order differential equations.

If the right side of the equation $\frac{dy}{dx} = f(x, y)$ is the product of a function in x and a function in y, then the equation is **separable**. By separating the variables and integrating, we obtain an implicit solution to the differential equation.

Problem. Solve $\frac{dy}{dx} = \frac{x-5}{y^2}$, with y(1) = 4. Use MATLAB to create a plot of the solution.

Problem. Use MATLAB to create a plot of the solution to the initial value problem $\frac{dx}{dt} = \frac{x-1}{t+3}$; $x(-1) = 0$.

Problem. Use MATLAB to plot the solution to the logistic equation: $\dfrac{dP}{dt} = (1-P)P$, where P(0) = 0.1. Try different values for the initial condition, what happens to the solution as the initial condition changes?

First-order Linear Equations

How does one integrate first-order linear equations?

A **linear first-order equation** is one that can be expressed in the form

$$a_1(x)y' + a_0(x)y = b(x)$$

Assuming that $a_1(x) \neq 0$, we can rewrite the equation in the **standard form**:

$$\frac{dy}{dx} + P(x)y = Q(x)\,.$$

Note: when $P(x)$ and $Q(x)$ are constants, this equation is also **separable** (seen earlier).

Problem. Use MATLAB to plot the solution to $\dfrac{dy}{dt} + 2y = 3$ with y$(3) = 4$.

Problem. Use MATLAB to plot the solution to $\dfrac{dy}{dx} + 2y = 3e^{x}$, with y(0) = 1

Note: this is a linear DE, but it is not separable.

Problem. Use MATLAB to plot the solution to the initial value problem where

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos(x) \text{ and } y\left(\frac{\pi}{2}\right) = 3$$