MNTC P01 - Week #5 - Integrals - Techniques

Substitution Integrals

To practice computing integrals using substitutions, do as many of the problems from this section as you feel you need. The problems trend from simple to the more complex.

Note: In the solutions to these problems, we always show the substitution used. On a test, if you can compute the antiderivative in your head, you do *not* need to go through all the steps shown here. They are included in these solutions as learning & comprehension aid.

1. $\int te^{t^2}\ dt$

$$\text{Let } w = t^2, \text{ so } dw = 2t\ dt \text{ or } dt = \frac{1}{2t}\ dw$$

$$\int te^{t^2}\ dt = \int te^w\left(\frac{1}{2t}\ dw\right) = \int \frac{1}{2}e^w dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{t^2} + C$$

$$\text{Check: } \frac{d}{dt}\frac{1}{2}e^{t^2} + C = \frac{1}{2}(2t)e^{t^2} = te^{t^2}$$

$$= \text{ original function in integral. } \checkmark$$

2. $\int e^{3x}\ dx$

$$\text{Let } w = 3x, \text{ so } dw = 3\ dx \text{ or } dx = \frac{dw}{3}$$

$$\int e^{3x}\ dx = \int e^w\frac{dw}{3} = \frac{1}{3}e^w + C = \frac{1}{3}e^{3x} + C$$

$$\text{Check: } \frac{d}{dx}\frac{1}{3}e^{3x} + C = \frac{1}{3}e^{3x}(3) = e^{3x}$$

$$= \text{ original function in integral. } \checkmark$$

NOTE: we will not show the differentiation check for any later questions, as the process is always the same, and you should be comfortable enough with the derivative rules to do the checking independently. If you are uncertain about any problem, contact your instructor.

3. $\int e^{-x}\ dx$

$$\text{Let } w = -x, \text{ so } dw = -1\ dx \text{ or } dx = (-dw)$$

$$\int e^{-x}\ dx = \int e^w(-dw) = e^w + C = -e^{-x} + C$$

4. $\int 25e^{-0.2t}\ dt$

$$\text{Let } w = -0.2t, \text{ so } dw = -0.2\ dt$$

$$\text{or } dt = \frac{dw}{-0.2} = -5dw$$

$$\int 25e^{-0.2t}\ dt = \int 25e^w(-5dw) = -125e^w + C$$

$$= -125e^{-0.2t} + C$$

5. $\int t\cos(t^2)\ dt$

$$\text{Let } w = t^2, \text{ so } dw = 2t\ dt$$

$$\text{or } dt = \frac{dw}{2t}$$

$$\int t\cos(t^2)\ dt = \int t\cos(w)\ \frac{dw}{2t} = \frac{1}{2}\sin(w) + C$$

$$= \frac{1}{2}\sin(t^2) + C$$

6. $\int \sin(2x)\ dx$

$$\text{Let } w = 2x, \text{ so } dw = 2dx, \text{ or } dx = \frac{1}{2}dw$$

$$\int \sin(2x)dx = \frac{1}{2}\int \sin(w)dw = -\frac{1}{2}\cos(w) + C$$

$$= -\frac{1}{2}\cos(2x) + C$$

7. $\int \sin(3-t)\ dt$

$$\text{Let } w = 3 - t, \text{ so } dw = -1\ dt \text{ or } dt = -dw$$

$$\int \sin(3-t)\ dt = \int \sin(w)\ (-1)dw = -(-\cos(w)) + C$$

$$= \cos(3-t) + C$$

8. $\int xe^{-x^2}\ dx$

Let $w = -x^2$, so $dw = -2x\ dx$, or $dx = \frac{-1}{2x}dw$

$$\int xe^{-x^2}\ dx = \int xe^w \frac{-1}{2x}\ dw = \int \frac{-1}{2}e^w\ dw = -\frac{1}{2}e^w + C$$
$$= -\frac{1}{2}e^{-x^2} + C$$

9. $\int (r+1)^3\ dr$

Let $w = (r+1)$, so $dw = dr$

$$\int (r+1)^3\ dr = \int w^3\ dw = \frac{w^4}{4} + C = \frac{(r+1)^4}{4} + C$$

10. $\int y(y^2+5)^8\ dy$

Let $w = y^2+5$, so $dw = 2y\ dy$, or $dy = \frac{1}{2y}dw$

$$\int y(y^2+5)^8\ dy = \int yw^8 \frac{1}{2y}\ dw = \frac{1}{2}\int w^8\ dw = \frac{1}{2}\frac{w^9}{9} + C$$
$$= \frac{1}{18}(y^2+5)^9 + C$$

11. $\int t^2(t^3-3)^{10}\ dt$

Let $w = t^3-3$, so $dw = 3t^2\ dt$, or $dt = \frac{1}{3t^2}dw$

$$\int t^2(t^3-3)^{10}\ dt = \int t^2 w^{10}\frac{1}{3t^2}\ dw = \frac{1}{3}\int w^{10}\ dw$$
$$= \frac{1}{3}\frac{w^{11}}{11} + C = \frac{1}{33}(t^3-3)^{11} + C$$

12. $\int x^2(1+2x^3)^2\ dx$

Let $w = 1+2x^3$, so $dw = 6x^2\ dx$, or $dx = \frac{1}{6x^2}dw$

$$\int x^2(1+2x^3)^2\ dx = \int x^2w^2\frac{1}{6x^2}\ dw = \frac{1}{6}\frac{w^3}{3} + C$$
$$= \frac{1}{18}(1+2x^3)^3 + C$$

13. $\int x(x^2+3)^2\ dx$

Let $w = x^2+3$, so $dw = 2x\ dx$, or $dx = \frac{1}{2x}dw$

$$\int x(x^2+3)^2\ dx = \int xw^2\frac{1}{2x}\ dw = \frac{1}{2}\frac{w^3}{3} + C$$
$$= \frac{1}{6}(x^2+3)^3 + C$$

14. $\int x(x^2-4)^{7/2}\ dx$

Let $w = x^2-4$, so $dw = 2x\ dx$, or $dx = \frac{1}{2x}dw$

$$\int x(x^2-4)^{7/2}\ dx = \int xw^{7/2}\frac{1}{2x}\ dw = \frac{1}{2}\frac{w^{9/2}}{9/2} + C$$
$$= \frac{1}{9}(x^2-4)^{9/2} + C$$

15. $\int y^2(1+y)^2\ dy$

Trick (substitution) question: substitution seems not to work well here, because both factors have y^2 in them, so neither one is the derivative of the other. We're better off expanding the $(1+y)^2$ factor and then integrating each term separately:

$$\int y^2(1+y)^2 dy = \int y^2(1+2y+y^2)dy = \int y^2 + 2y^3 + y^4$$
$$= \frac{1}{3}y^3 + \frac{2}{4}y^4 + \frac{1}{5}y^5 + C$$

16. $\int (2t-7)^{73}\ dt$

Let $w = 2t-7$, so $dw = 2\ dt$, or $dt = \frac{1}{2}dw$

$$\int (2t-7)^{73}\ dt = \int w^{73}\frac{1}{2}\ dw = \frac{1}{2}\frac{w^{74}}{74} + C$$
$$= \frac{1}{148}(2t-7)^{74} + C$$

17. $\int \frac{1}{y+5}\ dy$

Let $w = y+5$, so $dw = dy$, making

$$\int \frac{dy}{y+5}dy = \int \frac{1}{w}dw = \ln|w| + C = \ln|y+5| + C$$

18. $\int \frac{1}{\sqrt{4-x}}\ dx$

Rewrite integral:

$$\int (4-x)^{-1/2}\ dx$$

Let $w = 4-x$, so $dw = -1\ dx$, or $dx = -dw$

$$\int (4-x)^{-1/2}\ dx = \int w^{-1/2}(-1)dw = -\frac{w^{1/2}}{1/2}$$
$$= -2(4-x)^{1/2} + C$$

19. $\int (x^2+3)^2\ dx$

Another non-substitution integral: since there is no x term outside the (x^2+3), it is easier to expand the square in this case and integrate term by term.

$$\int (x^2+3)^2\ dx = \int x^4+6x^2+9\ dx = \frac{x^5}{5}+\frac{6x^3}{3}+9x+C$$
$$= \frac{x^5}{5}+2x^3+9x+C$$

20. $\int x^2e^{x^3+1}\ dx$

Let $w = x^3+1$, so $dw = 3x^2\ dx$, or $dx = \frac{1}{3x^2}dw$

$$\int x^2e^{x^3+1}\ dx = \int x^2e^w\frac{1}{3x^2}\ dw = \frac{1}{3}e^w + C$$
$$= \frac{1}{3}e^{x^3+1}+C$$

21. $\int \sin\theta(\cos\theta+5)^7\ d\theta$

Let $w = \cos\theta+5$, so $dw = -\sin\theta d\theta$, making

$$\int \sin\theta(\cos\theta+5)^7 d\theta = -\int w^7 dw = -\frac{w^8}{8}+C$$
$$= -\frac{1}{8}(\cos\theta+5)^8+C$$

22. $\int \sqrt{\cos(3t)}\sin(3t)\ dt$

Let $w = \cos(3t)$ so $dw = -3\sin(3t)\ dt$,

or $dt = \frac{-1}{3\sin(3t)}dw$

$$\int \sqrt{\cos(3t)}\sin(3t)\ dt = \int w^{1/2}\sin(3t)\frac{-1}{3\sin(3t)}\ dt$$
$$= \frac{-1}{3}\frac{w^{3/2}}{3/2}+C$$
$$= \frac{-2}{9}(\cos(3t))^{3/2}+C$$

23. $\int \sin^6\theta\cos\theta\ d\theta$

Let $w = \sin(\theta)$ so $dw = \cos\theta\ d\theta$,

or $d\theta = \frac{1}{\cos\theta}dw$

$$\int (\sin\theta)^6\cos\theta\ d\theta = \int w^6\cos\theta\frac{1}{\cos\theta}dw = \frac{w^7}{7}+C$$
$$= \frac{1}{7}\sin^7\theta + C$$

24. $\int \sin^3\alpha\cos\alpha d\alpha$

Let $w = \sin(\alpha)$ so $dw = \cos\alpha\ d\alpha$, or $d\alpha = \frac{1}{\cos\alpha}dw$

$$\int (\sin\alpha)^3\cos\alpha\ d\alpha = \int w^3\cos\alpha\frac{1}{\cos\alpha}dw = \frac{w^4}{4}+C$$
$$= \frac{1}{4}\sin^4\alpha + C$$

25. $\int \sin^6(5\theta)\cos(5\theta)\ d\theta$

Let $w = \sin(5\theta)$ so $dw = 5\cos(5\theta)\ d\theta$,

or $d\theta = \frac{1}{5\cos(5\theta)}dw$

$$\int (\sin(5\theta))^6\cos(5\theta)\ d\theta = \int w^6\cos(5\theta)\frac{1}{5\cos(5\theta)}dw$$
$$= \frac{1}{5}\frac{w^7}{7}+C = \frac{1}{35}\sin^7(5\theta)+C$$

26. $\int \tan(2x)\ dx$

Rewrite integral:

$$\int \frac{\sin(2x)}{\cos 2x}\ dx$$

Let $w = \cos(2x)$, so $dw = -2\sin(2x)\ dx$,

or $dx = \frac{-1}{2\sin(2x)}\ dw$

$$\int \frac{\sin(2x)}{\cos(2x)}\ dx = \int \frac{\sin(2x)}{w}\frac{-1}{2\sin(2x)}\ dw = \frac{-1}{2}\int w^{-1}\ dw$$
$$= \frac{-1}{2}\ln(|w|)+C = \frac{-1}{2}\ln(|\cos(2x)|)+C$$

27. $\int \frac{(\ln z)^2}{z}\ dz$

Let $w = \ln(z)$, so $dw = \frac{1}{z}\ dz$, or $dz = z\ dw$

$$\int \frac{(\ln z)^2}{z}\ dx = \int \frac{w^2}{z}(z\ dw) = \frac{w^3}{3}+C$$
$$= \frac{(\ln z)^3}{3}+C$$

28. $\displaystyle\int \frac{e^t+1}{e^t+t}\,dt$

Let $w = e^t + t$, so $dw = (e^t+1)\,dt$, or $dt = \dfrac{1}{e^t+1}\,dw$

$$\int \frac{e^t+1}{e^t+t}\,dt = \int \frac{e^t+1}{w}\frac{1}{e^t+1}\,dw = \int w^{-1}\,dw$$
$$= \ln(|w|) + C = \ln(|e^t+t|) + C$$

29. $\displaystyle\int \frac{y}{y^2+4}\,dy$

Let $w = y^2+4$, so $dw = 2y\,dy$, or $dy = \dfrac{1}{2y}\,dw$

$$\int \frac{y}{y^2+4}\,dy = \int \frac{y}{w}\frac{1}{2y}\,dw = \frac{1}{2}\int w^{-1}\,dw$$
$$= \frac{1}{2}\ln(|w|) + C = \frac{1}{2}\ln(|y^2+4|) + C$$

Note that y^2+4 is always positive, so we could remove the absolute values if we wished, as they are redundant in this case.

30. $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

Let $w = \sqrt{x}$, so $dw = \dfrac{1}{2}x^{-1/2}\,dx$, or $dx = 2\sqrt{x}\,dw$

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx = \int \frac{\cos(w)}{\sqrt{x}}(2\sqrt{x}\,dw) = 2\int \cos(w)\,dw$$
$$= 2\sin(w) + C = 2\sin(\sqrt{x}) + C$$

31. $\displaystyle\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy$

Let $w = \sqrt{y}$, so $dw = \dfrac{1}{2}y^{-1/2}\,dy$, or $dy = 2\sqrt{y}\,dw$

$$\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = \int \frac{e^w}{\sqrt{y}}(2\sqrt{y}\,dw) = 2\int e^w\,dw$$
$$= 2e^w + C = 2e^{\sqrt{y}} + C$$

32. $\displaystyle\int \frac{1+e^x}{\sqrt{x+e^x}}\,dx$

Let $w = x+e^x$, so $dw = (1+e^x)\,dx$, or $dx = \dfrac{1}{1+e^x}\,dw$

$$\int \frac{1+e^x}{\sqrt{x+e^x}}\,dx = \int \frac{1+e^x}{\sqrt{w}}\left(\frac{1}{1+e^x}\,dw\right) = \int w^{-1/2}\,dw$$
$$= \frac{w^{1/2}}{1/2} + C = 2\sqrt{x+e^x} + C$$

33. $\displaystyle\int \frac{e^x}{2+e^x}\,dx$

Let $w = 2+e^x$, so $dw = e^x\,dx$, or $dx = \dfrac{1}{e^x}\,dw$

$$\int \frac{e^x}{2+e^x}\,dx = \int \frac{e^x}{w}\left(\frac{1}{e^x}\,dw\right) = \int w^{-1}\,dw$$
$$= \ln(|w|) + C = \ln(|2+e^x|) + C$$

34. $\displaystyle\int \frac{x+1}{x^2+2x+19}\,dx$

Let $w = x^2+2x+19$, so $dw = (2x+2)\,dx$,

or $dx = \dfrac{1}{2(x+1)}\,dw$

$$\int \frac{x+1}{x^2+2x+19}\,dx = \int \frac{x+1}{w}\left(\frac{1}{2(x+1)}\,dw\right)$$
$$= \frac{1}{2}\int w^{-1}\,dw = \frac{1}{2}\ln(|w|) + C$$
$$= \frac{1}{2}\ln(|x^2+2x+19|) + C$$

35. $\displaystyle\int \frac{t}{1+3t^2}\,dt$

Let $w = 1+3t^2$, so $dw = 6t\,dt$, or $dt = \dfrac{1}{6t}\,dw$

$$\int \frac{t}{1+3t^2}\,dt = \int \frac{t}{w}\left(\frac{1}{6t}\,dw\right) = \frac{1}{6}\int w^{-1}\,dw$$
$$= \frac{1}{6}\ln(|w|) + C = \frac{1}{6}\ln(|1+3t^2|) + C$$

In this case we can remove the absolute values because $1+3t^2$ will always be positive, so the absolute values are redundant.

36. $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

Let $w = e^x+e^{-x}$, so $dw = (e^x-e^{-x})\,dx$,

or $dx = \dfrac{1}{e^x-e^{-x}}\,dw$

$$\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx = \int \frac{e^x-e^{-x}}{w}\left(\frac{1}{e^x-e^{-x}}\,dw\right)$$
$$= \int w^{-1}\,dw = \ln(|w|) + C$$
$$= \ln(|e^x+e^{-x}|) + C$$

In this case we can remove the absolute values because e^x+e^{-x} will always be positive, so the absolute values are redundant.

37. $\int \frac{(t+1)^2}{t^2}\ dt$

This question is probably more easily solved by expanding than by using substitution.

$$\begin{aligned}\int \frac{(t+1)^2}{t^2}\ dt = \int \frac{t^2+2t+1}{t^2}\ dt = \int 1 + \frac{2}{t} + \frac{1}{t^2}\ dt\\ = t + 2\ln(|t|) - t^{-1} + C\\ = t + 2\ln(|t|) - \frac{1}{t} + C\end{aligned}$$

38. $\int \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}\ dx$

$$\text{Let } w = \sin(x^2), \text{ so } dw = 2x\cos(x^2)\ dx \text{ or } dx = \frac{dw}{2x\cos(x^2)}$$

$$\begin{aligned}\int \frac{x\cos(x^2)}{\sqrt{\sin(x^w)}}\ dx = \int x\cos(x^2)(w^{-1/2}\frac{1}{2x\cos(x^2)}dw\\ = \frac{1}{2}\int w^{-1/2}\ dw = \frac{1}{2}\frac{w^{1/2}}{1/2} + C\\ = \sqrt{\sin(x^2)} + C\end{aligned}$$

39. $\int_0^{\pi} \cos(x+\pi)\ dx$

$$\begin{aligned}\int_0^{\pi} \cos(x+\pi)\ dx = \sin(x+\pi)\Big|_0^{\pi} = \sin(2\pi) - \sin(\pi)\\ = 0 - 0 = 0\end{aligned}$$

40. $\int_0^{1/2} \cos(\pi x)\ dx$

$$\begin{aligned}\int_0^{1/2} \cos(\pi x)\ dx = \frac{1}{\pi}\sin(\pi x)\Big|_0^{1/2}\\ = \frac{1}{\pi}[\sin(\pi/2) - \sin(0)] = \frac{1}{\pi}[1-0] = \frac{1}{\pi}\end{aligned}$$

41. $\int_0^{\pi/2} e^{-\cos(\theta)}\sin(\theta)\ d\theta$

In this solution, we will use the substitution just to find the antiderivative, but then we will switch back to the original integral variable θ to evaluate the limits.

$$\text{Let } w = -\cos(\theta), \text{ so } dw = \sin(\theta)\ d\theta \text{ or } d\theta = \frac{dw}{\sin(\theta)}$$

$$\begin{aligned}\int_0^{\pi/2} e^{-\cos(\theta)}\sin(\theta)\ d\theta = \int_{\theta=0}^{\theta=\pi/2} e^w\sin(\theta)\left(\frac{dw}{\sin(\theta)}\right)\\ = \int_{\theta=0}^{\theta=\pi/2} e^w\ dw = e^w\Big|_{\theta=0}^{\theta=\pi/2}\\ = e^{-\cos(\theta)}\Big|_{\theta=0}^{\theta=\pi/2}\\ = e^0 - e^{-1} = 1 - \frac{1}{e}\end{aligned}$$

42. $\int_1^2 2xe^{x^2}\ dx$

In this solution, we will convert the limits of integration to the substitution variable.

$$\begin{aligned}&\text{Let } w = x^2, \text{ so } dw = 2x\ dx \text{ or } dx = (1/2x)dw\\ &\text{then also } x = 1 \rightarrow w = 1^2 = 1,\\ &\text{and } x = 2 \rightarrow w = 2^2 = 4.\end{aligned}$$

$$\begin{aligned}\int_{x=1}^{x=2} 2xe^{x^2}\ dx = \int_{w=1}^{w=4} 2xe^w\left(\frac{1}{2x}\ dw\right)\\ = \int_{w=1}^{w=4} e^w\ dw = e^w\Big|_{w=1}^{w=4}\\ = e^4 - e^1\end{aligned}$$

43. $\int_1^8 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\ dx$

In this solution, we will convert the limits of integration to the substitution variable.

$$\begin{aligned}&\text{Let } w = x^{1/3}, \text{ so } dw = \frac{1}{3}x^{-2/3}\ dx \text{ or } dx = 3x^{2/3}dw\\ &\text{then also } x = 1 \rightarrow w = (1)^{1/3} = 1,\\ &\text{and } x = 8 \rightarrow w = 8^{1/3} = 2.\end{aligned}$$

$$\begin{aligned}\int_{x=1}^{x=8} \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\ dx = \int_{w=1}^{w=2} \frac{e^w}{x^{2/3}}\left(3x^{2/3}\ dw\right)\\ = \int_{w=1}^{w=2} 3e^w\ dw = 3e^w\Big|_{w=1}^{w=2} = 3e^2 - 3e^1\end{aligned}$$

44. $\displaystyle\int_{-1}^{e-2} \frac{1}{t+2}\, dt$

$$\begin{aligned}\text{Let } w &= t+2, \text{ so } dw = dt\\ \text{then also } t &= -1 \to w = (-1)+2 = 1\\ \text{and } t &= e-2 \to w = (e-2)+2 = e\end{aligned}$$

$$\begin{aligned}\int_{t=-1}^{t=e-2} \frac{1}{t+2}\, dt &= \int_{w=1}^{w=e} \frac{1}{w}\, dw = \ln(|w|)\Big|_{w=1}^{w=e}\\ &= \ln(e) - \ln(1) = 1 - 0 = 1\end{aligned}$$

45. $\displaystyle\int_{1}^{4} \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$

$$\begin{aligned}\text{Let } w = x^{1/2} \text{ so } dw &= \frac{1}{2}x^{-1/2}\, dx \text{ or } dx = 2x^{1/2}\, dw\\ \text{then also } x = 1 &\to w = (1)^{1/2} = 1\\ \text{and } x = 4 &\to w = (4)^{1/2} = 2\end{aligned}$$

$$\begin{aligned}\int_{x=1}^{x=4} \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx &= \int_{w=1}^{w=2} \frac{\cos(w)}{x^{1/2}}(2x^{1/2}\, dw)\\ &= \int_{w=1}^{w=2} 2\cos(w)\, dw = 2\sin(w)\Big|_{w=1}^{w=2}\\ &= 2\sin(2) - 2\sin(1)\end{aligned}$$

46. $\displaystyle\int_{0}^{2} \frac{x}{(1+x^2)^2}\, dx$

$$\begin{aligned}\text{Let } w = 1+x^2 \text{ so } dw &= 2x\, dx \text{ or } dx = \frac{1}{2x}\, dw\\ \text{then also } x = 0 &\to w = 1+0^2 = 1\\ \text{and } x = 2 &\to w = 1+2^2 = 5\end{aligned}$$

$$\begin{aligned}\int_{x=0}^{x=2} \frac{x}{(1+x^2)^2}\, dx &= \int_{w=1}^{w=5} \frac{x}{w^2}\left(\frac{1}{2x}\, dw\right)\\ &= \frac{1}{2}\int_{w=1}^{w=5} \frac{1}{w^2}\, dw = \frac{1}{2}\left(\frac{-1}{w}\right)\Big|_{w=1}^{w=5}\\ &= \frac{1}{2}\left[\frac{-1}{5} - \frac{-1}{1}\right] = \frac{1}{2}\frac{4}{5} = \frac{2}{5}\end{aligned}$$

47. If appropriate, evaluate the following integrals by substitution. If substitution is not appropriate, say so, and do not evaluate.

(a) $\displaystyle\int x\sin(x^2)\, dx$

(b) $\displaystyle\int x^2\sin(x)\, dx$

(c) $\displaystyle\int \frac{x^2}{1+x^2}\, dx$

(d) $\displaystyle\int \frac{x}{(1+x^2)^2}\, dx$

(e) $\displaystyle\int x^3 e^{x^2}\, dx$

(f) $\displaystyle\int \frac{\sin(x)}{2+\cos(x)}\, dx$

(a) This integral can be evaluated using integration by substitution. We use $w = x^2$, $dw = 2x\, dx$.

$$\begin{aligned}\int x\sin(x^2)\, dx &= \frac{1}{2}\int \sin(w)\, dw\\ &= -\frac{1}{2}\cos(w) + C = \frac{-1}{2}\cos(x^2) + C\end{aligned}$$

(b) This integral cannot be evaluated using a simple integration by substitution.

(c) This integral cannot be evaluated using a simple integration by substitution.

(d) This integral can be evaluated using integration by substitution. We use $w = 1+x^2$, $dw = 2x\, dx$.

$$\begin{aligned}\int \frac{x}{(1+x^2)^2}dx &= \frac{1}{2}\int \frac{1}{w^2}\, dw = \frac{1}{2}\left(\frac{-1}{w}\right) + C\\ &= \frac{-1}{2(1+x^2)} + C.\end{aligned}$$

(e) This integral cannot be evaluated using a simple integration by substitution.

(f) This integral can be evaluated using integration by substitution. We use $w = 2+\cos x, dw = -\sin x\, dx$.

$$\begin{aligned}\int \frac{\sin x}{2+\cos x}\, dx &= -\int \frac{1}{w}\, dw = -\ln|w| + C\\ &= -\ln|2+\cos x| + C:\end{aligned}$$

48. Find the exact area under the graph of $f(x) = xe^{x^2}$ between $x = 0$ and $x = 2$.

Since $f(x)$ is always positive, the area under the graph is equal to the integral of $f(x)$ on that interval. Thus we need to evaluate the definite integral

$$\int_0^2 xe^{x^2}\, dx.$$

This is done in Question 42, using the substitution $w = x^2$, the result being

$$\int_0^2 xe^{x^2}\,dx = \frac{1}{2}(e^4 - 1).$$

49. Find the exact area under the graph of $f(x) = 1/(x+1)$ between $x = 0$ and $x = 2$.

Since $f(x) = 1/(x+1)$ is positive on the interval $x = 0$ to $x = 2$, the area under the graph and the integral are equal to one another.

$$\int_0^2 \frac{1}{x+1}\,dx = \ln(x+1)\Big|_0^2 = \ln 3 - \ln 1 = \ln 3.$$

The area is $\ln 3 \approx 1.0986$.

50. Find $\int 4x(x^2+1)\,dx$ using two methods:

(a) Do the multiplication first, and then antidifferentiate.

(b) Use the substitution $w = x^2 + 1$.

(c) Explain how the expressions from parts (a) and (b) are different. Are they both correct?

(a) $\int 4x(x^2+1)\,dx = \int (4x^3+4x)\,dx = x^4+2x^2+C.$

(b) If $w = x^2+1$ then $dw = 2x\,dx$:

$$\int 4x(x^2+1)\,dx = \int 2w\,dw = w^2 + C = (x^2+1)^2 + C.$$

(c) The expressions from parts (a) and (b) look different, but they are both correct. Note that the answer from (b) can be expanded as

$$(x^2+1)^2 + C = x^4 + 2x^2 + \underbrace{1+C}_{\text{new const.}}.$$

In other words, the expressions from parts (a) and (b) differ only by a constant, so they are both correct antiderivatives.

51. (a) Find $\int \sin\theta\cos\theta\,d\theta$

(b) You probably solved part (a) by making the substitution $w = \sin\theta$ or $w = \cos\theta$. (If not, go back and do it that way.) Now find $\int \sin\theta\cos\theta\,d\theta$ by making the other substitution.

(c) There is yet another way of finding this integral which involves the trigonometric identities:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta.$$

Find $\int \sin\theta\cos\theta d\theta$ using one of these identities and then the substitution $w = 2\theta$.

(d) You should now have three different expressions for the indefinite integral $\int \sin\theta\cos\theta d\theta$. Are they really different? Are they all correct? Explain.

(a) We first try the substitution $w = \sin\theta$, so $dw = \cos\theta\,d\theta$. Then

$$\int \sin\theta\cos\theta\,d\theta = \int w\,dw = \frac{w^2}{2} + C = \frac{\sin^2\theta}{2} + C.$$

(b) If we instead try the substitution $w = \cos\theta$, $dw = -\sin\theta\,d\theta$, we get

$$\int \sin\theta\cos\theta\,d\theta = -\int w\,dw = -\frac{w^2}{2} + C = -\frac{\cos^2\theta}{2} + C.$$

(c) Once we note that $\sin(2\theta) = 2\sin\theta\cos\theta$ we can also say

$$\int \sin\theta\cos\theta\,d\theta = \frac{1}{2}\int \sin(2\theta)\,d\theta$$

Substituting $w = 2\theta$, $dw = 2\,d\theta$, the above equals

$$\frac{1}{4}\int \sin w\,dw = -\frac{\cos w}{4} + C = -\frac{\cos 2\theta}{4} + C.$$

(d) All these answers are correct. Although they have different forms, they differ from each other only in terms of a constant, and thus they are all acceptable antiderivatives.

For example,

$$1 - \cos^2\theta = \sin^2\theta$$

$$\text{so} \quad \underbrace{\frac{\sin^2\theta}{2}}_{\text{Answer (a)}} = -\frac{\cos^2\theta + 1}{2} = \underbrace{-\frac{\cos^2\theta}{2}}_{\text{Answer (b)}} - \frac{1}{2}$$

Thus the first two expressions differ only by a constant.
Similarly, $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$, so

$$\underbrace{-\frac{\cos(2\theta)}{4}}_{\text{Answer (c)}} = \underbrace{\frac{-\cos^2\theta}{2}}_{\text{Answer (b)}} + \frac{1}{4}$$

and thus the second and third expressions differ only by a constant. Of course, if the first two expressions and the last two expressions differ only in the constant C, then the first and last only differ in the constant as well, so they are all equally valid antiderivatives.

Substitution Integrals - Applications

52. Let $f(t)$ be the rate of flow, in cubic meters per hour, of a flooding river at time t in hours. Give an integral for the total flow of the river:

   (a) Over the 3-day period, $0 \le t \le 72$ (since t is measured in hours).

   (b) In terms of time T in **days** over the same 3-day period.

(a) A time period of Δt hours with flow rate of $f(t)$ cubic meters per hour has a flow of $f(t)\Delta t$ cubic meters. Summing the flows, we get total flow $\approx \sum f(t)\Delta t$, so

$$\text{Total flow } = \int_0^{72} f(t)\ dt \text{ cubic meters}$$

(b) Since 1 day is 24 hours, $t = 24T$. The constant 24 has units (hours/day), so $24T$ has units (hours/-day) $\times$ day $=$ hours. Applying the substitution $t = 24T$ to the integral in part (a), we get

$$\text{Total flow } = \int_0^{3} 24f(24T)\ dT \text{ cubic meters.}$$

53. Oil is leaking out of a ruptured tanker at the rate of $r(t) = 50e^{-0.02t}$ thousand liters per minute.

   (a) At what rate, in liters per minute, is oil leaking out at $t = 0$? At $t = 60$?

   (b) How many liters leak out during the first hour?

(a) At time $t = 0$, the rate of oil leakage $= r(0) = 50$ thousand liters/minute.
At $t = 60$, rate $= r(60) = 15.06$ thousand liters/minute.

(b) To find the amount of oil leaked during the first hour, we integrate the rate from $t = 0$ to $t = 60$:

$$\begin{aligned}\text{Oil leaked } &= \int_0^{60} 50e^{-0.02t}dt = \left(\frac{-50}{0.02}e^{-0.02t}\right)\Big|_0^{60} \\ &= -2500e^{-1.2} + 2500e^{0} \\ &\approx 1747 \text{ thousand liters.}\end{aligned}$$

54. If we assume that wind resistance is proportional to velocity, then the downward velocity, v, of a body of mass m falling vertically is given by

$$v = \frac{mg}{k}(1 - e^{-kt/m})$$

where g is the acceleration due to gravity and k is a constant. Find the height of the body, h, above the surface of the earth as a function of time. Assume the body starts at height h_0.

Since $v = dh/dt$, it follows that $h(t) = \int v(t)\ dt$ and $h(0) = h_0$. Since

$$\begin{aligned}v(t) &= \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \\ &= \frac{mg}{k} - \frac{mg}{k}e^{-\frac{k}{m}t}\end{aligned}$$

we have

$$\begin{aligned}h(t) &= \int v(t)\ dt \\ &= \frac{mg}{k}\int dt - \frac{mg}{k}\int e^{-k/m}t\ dt.\end{aligned}$$

The first integral is simply $\dfrac{mg}{k}t + C$.

To evaluate the second integral, make the substitution $w = -\dfrac{k}{m}t$. Then $dw = -\dfrac{k}{m}\ dt$, so

$$\begin{aligned}\int e^{-kt/m}\ dt &= \int e^{w}\left(\frac{-m}{k}\ dw\right) \\ &= -\frac{m}{k}e^{w} + C \\ &= -\frac{m}{k}e^{-kt/m} + C.\end{aligned}$$

Thus

$$\begin{aligned}h(t) &= \int v\ dt \\ &= \frac{mg}{k}t - \frac{mg}{k}\left(-\frac{m}{k}e^{-kt/m}\right) + C \\ &= \frac{mg}{k}t + \frac{m^2g}{k^2}e^{-kt/m} + C.\end{aligned}$$

Since $h(0) = h_0$,

$$h_0 = \frac{mg}{k} \cdot 0 + \frac{m^2 g}{k^2} e^0 + C;$$
$$C = h_0 - \frac{m^2 g}{k^2}.$$

Thus

$$h(t) = \frac{mg}{k} t + \frac{m^2 g}{k^2} e^{-kt/m} - \frac{m^2 g}{k^2} + h_0$$
$$h(t) = \frac{mg}{k} t - \frac{m^2 g}{k^2} \left(1 - e^{-kt/m}\right) + h_0.$$

This formula gives the height of the object above the surface of the earth as it falls.

55. The rate at which water is flowing into a tank is $r(t)$ gallons/minute, with t in minutes.

(a) Write an expression approximating the amount of water entering the tank during the interval from time t to time $t+\Delta t$, where Δt is small.

(b) Write a Riemann sum approximating the total amount of water entering the tank between $t = 0$ and $t = 5$. Then write an exact expression for this amount.

(c) By how much has the amount of water in the tank changed between $t = 0$ and $t = 5$ if $r(t) = 20e^{0.02t}$?

(d) If $r(t)$ is as in part (c), and if the tank contains 3000 gallons initially, find a formula for $Q(t)$, the amount of water in the tank at time t.

(a) Amount of water entering tank in a short period of time = Rate×Time = $r(t)\Delta t$.

(b) Summing the contribution from each of the small intervals Δt:

$$\text{Amount of water entering the tank} \approx \sum_i r(t_i)\Delta t_i$$

where $\Delta t = 5/n$. Taking a limit as $\Delta t \to 0$, we get the integral form instead of the sum:

$$\text{Exact amount of water entering the tank} = \int_0^5 r(t)\, dt.$$

(c) If $Q(t)$ is the amount of water in the tank at time t, then $Q'(t) = r(t)$. We want to calculate net change in volume between $t = 0$ and $t = 5$, or $Q(5) - Q(0)$.

By the Fundamental Theorem,

$$\begin{aligned}
&\text{Amount which has entered tank} \\
&= Q(5) - Q(0) \\
&= \int_0^5 r(t)\, dt \\
&= \int_0^5 20e^{0.02t}\, dt \\
&= \frac{20}{0.02} e^{0.02t} \Big|_0^5 \\
&= 1000(e^{(0.02)(5)} - 1) \approx 105.17 \text{ gallons.}
\end{aligned}$$

(d) By the Fundamental Theorem again,

$$\begin{aligned}
&\text{Amount which has entered tank} \\
&= Q(t) - Q(0) \\
&= \int_0^t r(t)\, dt \\
Q(t) - 3000 &= \int_0^t 20e^{0.02t}\, dt \\
\text{so } Q(t) &= 3000 + \int_0^t 20e^{0.02u}\, du
\end{aligned}$$

Note: t is already being used, so we put u inside the integral; since this is a definite integral, the variable inside the integral will disappear when we sub in the limits.

$$\begin{aligned}
Q(t) &= 3000 + \frac{20}{0.02} e^{0.02t} \Big|_0^t \\
&= 3000 + 1000(e^{0.02t} - e^0) \\
&= 1000e^{0.02t} + 2000.
\end{aligned}$$

This is the quantity of water in the tank at time t. Note that we can do a basic sanity check on the answer by verifying that $Q(0) = 3000$ (given), which is true for the formula we arrived at:

$$Q(0) = 1000e^{0.02 \cdot 0} + 2000 = 3000$$

56. After a spill of radioactive iodine, measurements at $t = 0$ showed the ambient radiation levels at the site of the spill to be four times the maximum acceptable limit. The level of radiation from an iodine source decreases according to the formula

$$R(t) = R_0 e^{-0.004t}$$

where R is the radiation level (in millirems/hour) at time t in hours and R_0 is the initial radiation level (at $t = 0$).

(a) How long will it take for the site to reach an acceptable level of radiation?

(b) Engineers look up the safe limit of radiation and find it to be 0.6 millirems/hour. How much total radiation (in millirems) will have been emitted by the time found in part (a)?

(a) If the level first becomes acceptable at time t_1, then $R_0 = 4R(t_1)$, and

$$\frac{1}{4}R_0 = R_0 e^{-0.004t_1}$$
$$\frac{1}{4} = e^{-.004t_1}$$

Taking natural logs on both sides yields

$$\ln\left(\frac{1}{4}\right) = -0.004t_1$$
$$t_1 = \frac{\ln\left(\frac{1}{4}\right)}{-0.004}$$
$$t_1 \approx 346.574 \text{ hours.}$$

(b) Since the initial radiation was four times the acceptable limit of 0.6 millirems/hour, we have $R_0 = 4(0.6) = 2.4$. The rate at which radiation is emitted is $R(t) = R_0 e^{-.004t}$, so

$$\text{Total radiation emitted} = \int_0^{346.574} 2.4e^{-0.004t}\, dt.$$

Finding by guess-and-check or substitution that an antiderivative of $e^{-0.004t}$ is $\dfrac{e^{-0.004t}}{-0.004}$,

$$\int_0^{346.574} 2.4e^{-0.004t}\, dt$$
$$= 2.4\frac{e^{-0.004t}}{-0.004}\Big|_0^{346.574}$$

$$= 2.4\left[\frac{e^{-0.004(346.574)}}{-0.004} - \frac{e^0}{-0.004}\right]$$
$$= 450.00$$

We find that 450 millirems were emitted during this time.

57. David is learning about catalysts in his Chemistry course. He has read the definition:

> Catalyst: A substance that helps a reaction to go faster without being used up in the reaction.

In today's Chemistry lab exercise, he has to add a catalyst to a chemical mixture that produces carbon dioxide. When there is no catalyst, the carbon dioxide is produced at a rate of 8.37×10^{-9} moles per second. When C moles of the catalyst are present, the carbon dioxide is produced at a rate of $(6.15 \times 10^{-8})C + 8.37 \times 10^{-9}$ moles per second.

The reaction begins at exactly 10:00 a.m. One minute later, at 10:01 sharp, David starts to add the catalyst at a constant rate of 0.5 moles per second.

How much carbon dioxide is produced between 10:00 (sharp) and 10:05?

In the first minute of the reaction no catalyst is present and the amount of carbon dioxide produced is

$$(8.37 \times 10^{-9}\text{mol}/s)(60s) = 5.02 \times 10^{-7}\text{mol}.$$

In the next four minutes, catalyst is being added. After t seconds of adding catalyst of $0.5\text{mol}/s$, there is $0.5t$mol of catalyst present. Thus, at time t carbon dioxide is being produced at a rate of

$$(6.15 \times 10^{-8})(0.5t) + (8.37 \times 10^{-9}\text{mol}/s)$$
$$= (3.075 \times 10^{-8})t + (8.37 \times 10^{-9}\text{mol}/s).$$

During the four minutes when catalyst is being added, the amount of carbon dioxide produced is:

$$\int_0^{240} (3.076 \times 10^{-8})t + (8.37 \times 10^{-9})dt$$
$$= (1.5375 \times 10^{-8})t^2 + (8.37 \times 10^{-9})t\Big|_0^{240}$$
$$= 8.88 \times 10^{-4}\text{mol}$$

The total amount of CO_2 produced in the first five minutes of the reaction is therefore

$$5.02 \times 10^{-7} + 8.88 \times 10^{-4} = 8.88 \times 10^{-4}\text{mol}.$$

Integration by Parts

58. For each of the following integrals, indicate whether integration by substitution or integration by parts is more appropriate. Do not evaluate the integrals.

    (a) $\int x\sin(x)\ dx$

    (b) $\int \frac{x^2}{1+x^3}\ dx$

    (c) $\int xe^{x^2}\ dx$

    (d) $\int x^2\cos(x^3)\ dx$

    (e) $\int \frac{1}{\sqrt{3x+1}}\ dx$

    (f) $\int x^2\sin x\ dx$

    (g) $\int \ln x\ dx$

(a) This integral can be evaluated using integration by parts with $u = x$, $dv = \sin x\ dx$.

(b) We evaluate this integral using the substitution $w = 1 + x^3$.

(c) We evaluate this integral using the substitution $w = x^2$.

(d) We evaluate this integral using the substitution $w = x^3$.

(e) We evaluate this integral using the substitution $w = 3x + 1$.

(f) This integral can be evaluated using integration by parts with $u = x^2$, $dv = \sin x\ dx$.

(g) This integral can be evaluated using integration by parts with $u = \ln x$, $dv = dx$.

To practice computing integrals by parts, do as many of the problems from this section as you feel you need. The problems trend from simple to the more complex.

For Questions #59 to #82, evaluate the integral.

59. $\int t\sin t\ dt$

$$\begin{aligned} &\text{We choose } u = t && \text{and } dv = \sin t\ dt \\ &\text{so } \frac{du}{dt} = 1 && \text{and } v = \int \sin t\ dt \\ &\text{or } du = 1\ dt && v = -\cos(t) \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int \underbrace{t}_{u}\underbrace{\sin t\ dt}_{dv} &= \underbrace{t}_{u}\underbrace{(-\cos t)}_{v} - \int \underbrace{(-\cos t)}_{v}\underbrace{dt}_{du} \\ &= -t\cos t + \int \cos t\ dt \\ &= -t\cos t + \sin t + C \end{aligned}$$

As always, we can check our integral is correct by differentiating:

$$\begin{aligned} \frac{d}{dt}\left(-t\cos t + \sin t + C\right) &= -\cos t - t(-\sin t) + \cos t \\ &= t\sin t \\ &= \text{ original integrand. } \checkmark \end{aligned}$$

60. $\int te^{5t}\ dt$

$$\begin{aligned} &\text{We choose } u = t && \text{and } dv = e^{5t}\ dt \\ &\text{so } \frac{du}{dt} = 1 && \text{and } v = \int e^{5t}\ dt \\ &\text{or } du = 1\ dt && v = e^{5t}/5 \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int \underbrace{t}_{u}\underbrace{e^{5t}\ dt}_{dv} &= \underbrace{t}_{u}\underbrace{(e^{5t}/5)}_{v} - \int \underbrace{(e^{5t}/5)}_{v}\underbrace{dt}_{du} \\ &= \frac{te^{5t}}{5} - \frac{1}{5}\int e^{5t}\ dt \\ &= \frac{te^{5t}}{5} - \frac{1}{5}\left(e^{5t}/5\right) + C \\ &= \frac{te^{5t}}{5} - \frac{1}{25}e^{5t} + C \end{aligned}$$

As always, we can check our integral is correct by differentiating:

$$\begin{aligned} \frac{d}{dt}\left(\frac{te^{5t}}{5} - \frac{e^{5t}}{25} + C\right) &= \frac{1}{5}(e^{5t} + t(5e^{5t})) - \frac{5e^{5t}}{25} \\ &= te^{5t} + \frac{e^{5t}}{5} - \frac{e^{5t}}{5} \\ &= te^{5t} \\ &= \text{ original integrand. } \checkmark \end{aligned}$$

From now on, for brevity, we won't show quite as many steps in the solution, nor will we check the answer by differentiating. However, you should always remember that you **can** check your antiderivatives/integrals by differentiation.

61. $\int pe^{-0.1p}\ dp$

$$\begin{aligned} &\text{We choose } u = p &&\text{and } dv = e^{-0.1p}\ dp\\ &\text{so } du = dp &&\text{and } v = e^{-0.1p}/(-0.1)\end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned}\int \underbrace{p}_{u}\underbrace{e^{-0.1p}\ dp}_{dv} &= \underbrace{p}_{u}\underbrace{e^{-0.1p}/(-0.1)}_{v} - \int \underbrace{e^{-0.1p}/(-0.1)}_{v}\underbrace{dp}_{du}\\ &= \frac{pe^{-0.1p}}{-0.1} + 10\int e^{-0.1p}\ dp\\ &= -10pe^{-0.1p} + 10\left(e^{-0.1p}/(-0.1)\right) + C\\ &= -10pe^{-0.1p} - 100e^{-0.1p} + C\end{aligned}$$

62. $\int (z+1)e^{2z}\ dz$

$$\begin{aligned} &\text{We choose } u = (z+1) &&\text{and } dv = e^{2z}\ dz\\ &\text{so } du = dz &&\text{and } v = e^{2z}/2\end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned}\int (z+1)e^{2z}\ dz &= (z+1)e^{2z}/2 - \int e^{2z}/2dz\\ &= \frac{(z+1)e^{2z}}{2} - \frac{1}{2}\int e^{2z}\ dz\\ &= \frac{(z+1)e^{2z}}{2} - \frac{1}{2}\left(e^{2z}/2\right) + C\\ &= \frac{(z+1)e^{2z}}{2} - \frac{1}{4}e^{2z} + C\end{aligned}$$

There is no need for further simplifications.

63. $\int \ln x\ dx$

In this problem, we recall that the only simple calculus formula related to $\ln x$ is that its **derivative** is known: $\frac{d}{dx}\ln(x) = 1/x$. This means that we have to select $u = \ln x$ so that it will be differentiated.

$$\begin{aligned} &\text{We choose } u = \ln x &&\text{and } dv = dx\\ &\text{so } du = \frac{1}{x}\ dx &&\text{and } v = x\end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned}\int \ln x\ dx &= x(\ln x) - \int x\frac{1}{x}\ dx\\ &= x\ln x - \int 1\ dx\\ &= x\ln x - x + C\end{aligned}$$

64. $\int y\ln y\ dy$

In this problem, we recall that the only simple calculus formula related to $\ln y$ is that its **derivative** is known: $\frac{d}{dy}\ln(y) = 1/y$. While it might be tempting to keep with our earlier pattern of choosing $u = y$ and $dv = \ln y\ dy$, that won't work because we won't be able to integrate dv. As a result,

$$\begin{aligned} &\text{we choose } u = \ln y &&\text{and } dv = y\ dy\\ &\text{so } du = \frac{1}{y}\ dy &&\text{and } v = y^2/2\end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned}\int y\ln y\ dy &= (\ln y)(y^2/2) - \int (y^2/2)\frac{1}{y}\ dy\\ &= \frac{y^2\ln y}{2} - \frac{1}{2}\int y\ dy\\ &= \frac{y^2\ln y}{2} - \frac{1}{2}(y^2/2) + C\\ &= \frac{y^2\ln y}{2} - \frac{y^2}{4} + C\end{aligned}$$

65. $\int x^3\ln x\ dx$

$$\begin{aligned} &\text{We choose } u = \ln x &&\text{and } dv = x^3\ dx\\ &\text{so } du = \frac{1}{x}\ dx &&\text{and } v = x^4/4\end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned}\int x^3\ln x\ dx &= (\ln x)x^4/4 - \int (x^4/4)\left(\frac{1}{x}\ dx\right)\\ &= \frac{x^4\ln x}{4} - \frac{1}{4}\int x^3\ dx\\ &= \frac{x^4\ln x}{4} - \frac{1}{4}(x^4/4) + C\\ &= \frac{x^4\ln x}{4} - \frac{x^4}{16} + C\end{aligned}$$

66. $\int q^5\ln(5q)\ dq$

$$\begin{aligned} \text{We choose } u &= \ln(5q) & \text{and } dv &= q^5\ dq \\ \text{so } du &= \frac{1}{5q}(5)\ dq & \text{and } v &= q^6/6 \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int q^5 \ln(5q)\ dq &= (\ln(5q))(q^6/6) - \int (q^6/6)\left(\frac{1}{q}\ dq\right) \\ &= \frac{q^6\ln(5q)}{6} - \frac{1}{6}\int q^5\ dq \\ &= \frac{q^6\ln(5q)}{6} - \frac{1}{6}(q^6/6) + C \\ &= \frac{q^6\ln(5q)}{6} - \frac{1}{36}q^6 + C \end{aligned}$$

67. $\int t^2 \sin t\ dt$

$$\begin{aligned} \text{We choose } u &= t^2 & \text{and } dv &= \sin t\ dt \\ \text{so } du &= 2t\ dt & \text{and } v &= -\cos t \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int t^2 \sin t\ dt &= -t^2\cos t - \int (-\cos t)\ (2t\ dt) \\ &= -t^2\cos t + 2\int t\cos t\ dt \end{aligned}$$

While we have traded our original integral for a slightly simpler one, it is still not simple enough to evaluate by finding an obvious antiderivative. In fact, it is of the form of one of our earlier examples of integration by parts, so here we must apply integration by parts *again* to finally evaluate the original integral.

We focus on just the new integral part, $\int t\cos t\ dt$:

$$\begin{aligned} \text{We choose } u &= t & \text{and } dv &= \cos t\ dt \\ \text{so } du &= \ dt & \text{and } v &= \sin t \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int t\cos t\ dt &= t\sin t - \int (\sin t)\ dt \\ &= t\sin t - \int \sin t\ dt \\ &= t\sin t - (-\cos t) + C \\ &= t\sin t + \cos t + C \end{aligned}$$

Subbing this result back into the original integral,

$$\begin{aligned} &\int t^2\sin t\ dt = \underbrace{-t^2\cos t + 2\underbrace{\int t\cos t\ dt}_{\text{Second by parts step}}}_{\text{First by parts step}} \\ &= -t^2\cos t + 2\left(t\sin t + \cos t\right) + C \end{aligned}$$

No further simplification is necessary for a pure 'evaluate the integral' question.

68. $\int x^2\cos(3x)\ dx$

$$\begin{aligned} \text{We choose } u &= x^2 & \text{and } dv &= \cos(3x)\ dx \\ \text{so } du &= 2x\ dx & \text{and } v &= \sin(3x)/3 \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int x^2\cos(3x)\ dx &= x^2\sin(3x)/3 - \int (\sin(3x)/3)\ (2x\ dx) \\ &= \frac{x^2\sin(3x)}{3} - \frac{2}{3}\int x\sin(3x)\ dx \end{aligned}$$

While we have traded our original integral for a slightly simpler one, it is still not simple enough to evaluate by finding an obvious antiderivative. In fact, it is of the form of one of our earlier examples of integration by parts, so here we must apply integration by parts *again* to finally evaluate the original integral.

We focus on just the new integral part, $\int x\sin(3x)\ dx$:

$$\begin{aligned} \text{We choose } u &= x & \text{and } dv &= \sin(3x)\ dx \\ \text{so } du &= \ dx & \text{and } v &= -\cos(3x)/3 \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int x\sin(3x)\ dx &= -x\cos(3x)/3 - \int (-\cos(3x)/3)\ dx \\ &= -\frac{x\cos(3x)}{3} + \frac{1}{3}\int \cos(3x)\ dx \\ &= -\frac{x\cos(3x)}{3} + \frac{1}{3}(\sin(3x)/3) + C \\ &= -\frac{x\cos(3x)}{3} + \frac{1}{9}\sin(3x) + C \end{aligned}$$

Subbing this result back into the original integral,

$$\begin{aligned} &\int x^2\cos(3x)\ dx \\ &= \underbrace{\frac{x^2\sin(3x)}{3} - \frac{2}{3}\underbrace{\int x\sin(3x)\ dx}_{\text{Second by parts step}}}_{\text{First by parts step}} \\ &= \frac{x^2\sin(3x)}{3} - \frac{2}{3}\left(-\frac{x\cos(3x)}{3} + \frac{\sin(3x)}{9} + C\right) \\ &= \frac{x^2\sin(3x)}{3} + \frac{2}{9}x\cos(3x) - \frac{2\sin(3x)}{27} + C_2 \end{aligned}$$

where $C_2 = -(2/3)C$ is a new integration constant.

69. $\int (\ln t)^2 \, dt$

We only know how to differentiate $(\ln t)^2$, so we have to choose it as u.

$$\begin{aligned} &\text{We choose } u = (\ln t)^2 && \text{and } dv = dt \\ &\text{so } du = \frac{2\ln t}{t}\, dt && \text{and } v = t \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int (\ln t)^2 \, dt &= (\ln t)^2 t - \int (t)\left(\frac{2\ln t}{t}\right) \, dt \\ &= t(\ln t)^2 - 2\int \ln t \, dt \end{aligned}$$

We are left with a simpler integral, but not an easy one (unless you look at the examples from the course notes!)

We focus on $I_2 = \displaystyle\int \ln t \, dt$, and apply integration by parts once more.

$$\begin{aligned} &\text{We choose } u = \ln t && \text{and } dv = dt \\ &\text{so } du = \frac{1}{t}\, dt && \text{and } v = t \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int \ln t \, dt &= (\ln t)t - \int (t)\left(\frac{1}{t}\right) \, dt \\ &= t(\ln t) - \int 1 \, dt \\ &= t\ln t - t + C \end{aligned}$$

Thus, going back to our original integral,

$$\begin{aligned} \int (\ln t)^2 \, dt &= t(\ln t)^2 - 2\underbrace{\int \ln t \, dt}_{I_2} \\ &= t(\ln t)^2 - 2(t\ln t - t + C) \end{aligned}$$

70. $\int t^2 e^{5t} \, dt$

$$\begin{aligned} &\text{We choose } u = t^2 && \text{and } dv = e^{5t}\, dt \\ &\text{so } du = 2t\, dt && \text{and } v = e^{5t}/5 \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int t^2 e^{5t} \, dt &= t^2 e^{5t}/5 - \int (e^{5t}/5)(2t\, dt) \\ &= \frac{t^2 e^{5t}}{5} - \frac{2}{5}\underbrace{\int te^{5t} \, dt}_{I_2} \end{aligned}$$

Applying integration by parts again to the integral marked I_2 will lead to

$$I_2 = \frac{te^{5t}}{5} - \frac{e^{5t}}{25} + C$$

so the overall integral will be

$$\begin{aligned} \int t^2 e^{5t} \, dt &= \frac{t^2 e^{5t}}{5} - \frac{2}{5}\underbrace{\int te^{5t} \, dt}_{I_2} \\ &= \frac{t^2 e^{5t}}{5} - \frac{2}{5}\left(\frac{te^{5t}}{5} - \frac{e^{5t}}{25} + C\right) \\ &= \frac{t^2 e^{5t}}{5} - \frac{2te^{5t}}{25} + 2\frac{e^{5t}}{125} + C_2 \end{aligned}$$

where C_2 is a multiple of the original C.

71. $\int y\sqrt{y+3} \, dy$

We choose $u = y$ and $dv = (y+3)^{1/2} \, dx$,
so $du = dx$ and $v = \dfrac{2}{3}(y+3)^{3/2}$:

$$\begin{aligned} \int y\sqrt{y+3} \, dy &= \frac{2}{3}y(y+3)^{3/2} - \int \frac{2}{3}(y+3)^{3/2} \, dy \\ &= \frac{2}{3}y(y+3)^{3/2} - \frac{2}{3}\frac{(y+3)^{5/2}}{5/2} + C \\ &= \frac{2}{3}y(y+3)^{3/2} - \frac{4}{15}(y+3)^{5/2} + C. \end{aligned}$$

72. $\int (t+2)\sqrt{2+3t} \, dt$

Let $u = t+2$ and $dv = \sqrt{2+3t}$,
so $du = dt$ and $v = \frac{2}{9}(2+3t)^{3/2}$.

Then

$$\begin{aligned} \int (t+2)\sqrt{1+3t} \, dt &= \frac{2}{9}(t+2)(2+3t)^{3/2} - \frac{2}{9}\int (2+3t)^{3/2} \, dt \\ &= \frac{2}{9}(t+2)(2+3t)^{3/2} \\ &\quad - \frac{4}{135}(2+3t)^{5/2} + C. \end{aligned}$$

73. $\int (p+1)\sin(p+1) \, dp$

Let $u = p+1$ and $dv = \sin(p+1)$,
so $du = dx$ and $v = -\cos(p+1)$.

$$\begin{aligned} &\int (p+1)\sin(p+1) \, dp \\ &= -(p+1)\cos(p+1) + \int \cos(p+1) \, dp \\ &= -(p+1)\cos(p+1) + \sin(p+1) + C \end{aligned}$$

74. $\int \frac{z}{e^z}\ dz$

Rewriting the integral,

$$\int \frac{z}{e^z}\ dz = \int ze^{-z}\ dz$$

Let $u = z$, $dv = e^{-z}\ dx$.
Thus $du = dz$ and $v = -e^{-z}$. Integration by parts gives:

$$\begin{aligned}\int ze^{-z}\ dz &= -ze^{-z} - \int (-e^{-z})\ dz \\ &= -ze^{-z} + \int e^{-z}\ dz \\ &= -ze^{-z} - e^{-z} + C\end{aligned}$$

75. $\int \frac{\ln x}{x^2}\ dx$

Let $u = \ln x$, $dv = \frac{1}{x^2}\ dx$.
Then $du = \frac{1}{x}\ dx$ and $v = -\frac{1}{x}$.
Integrating by parts, we get:

$$\begin{aligned}\int \frac{\ln x}{x^2}\ dx &= -\frac{1}{x}\ln x - \int \left(-\frac{1}{x}\right)\frac{1}{x}\ dx \\ &= -\frac{1}{x}\ln x + \int \frac{1}{x^2}\ dx \\ &= -\frac{1}{x}\ln x - \frac{1}{x} + C\end{aligned}$$

76. $\int \frac{y}{\sqrt{5-y}}\ dy$

Let $u = y$ and $dv = \frac{1}{\sqrt{5-y}}$,
so $du = dy$ and $v = -2(5-y)^{1/2}$

$$\begin{aligned}\int y\sqrt{5-y}\ dy &= -2y(5-y)^{1/2} - \int (-2)(5-y)^{1/2}dy \\ &= -2y(5-y)^{1/2} + 2\int (5-y)^{1/2}dy \\ &= -2y(5-y)^{1/2} + 2\frac{2}{3}(5-y)^{3/2}(-1) + C \\ &= -2y(5-y)^{1/2} - \frac{4}{3}(5-y)^{3/2} + C\end{aligned}$$

77. $\int \frac{t+7}{\sqrt{5-t}}\ dt$

Since we have a fraction in the numerator, we can split the integral into a sum, and then evaluate each term separately.

$$\int \frac{t+7}{\sqrt{5-t}}\ dt = \underbrace{\int \frac{t}{\sqrt{5-t}}\ dt}_{I_1} + \underbrace{\int \frac{7}{\sqrt{5-t}}\ dt}_{I_2}$$

I_1 can be evaluated using integration by parts.
Let $u = t$ and $dv = \frac{1}{\sqrt{5-t}}\ dt$,
so $du = dx$ and $v = -2(5-t)^{1/2}$.

$$\begin{aligned}&\int \frac{t}{\sqrt{5-t}}\ dt \\ &\qquad = -2t(5-t)^{1/2} + 2\int (5-t)^{1/2}\ dt \\ &\qquad = -2t(5-t)^{1/2} - \frac{4}{3}(5-t)^{3/2} + C.\end{aligned}$$

I_2 can be integrated directly:

$$\begin{aligned}\int \frac{7}{\sqrt{5-t}}\ dt &= 7(2)(5-t)^{1/2}(-1) + C_1 \\ &= -14(5-t)^{1/2} + C_1\end{aligned}$$

Adding the two integrals back together, we obtain

$$\begin{aligned}\int \frac{t+7}{\sqrt{5-t}}\ dt &= \underbrace{-2t(5-t)^{1/2} - \frac{4}{3}(5-t)^{3/2} + C}_{I_1} \\ &\quad + \underbrace{-14(5-t)^{1/2} + C_1}_{I_2}\end{aligned}$$

78. $\int x(\ln x)^2\ dx$

Select $u = (\ln x)^2$ and $dv = x\ dx$,
so $du = 2\ln x\left(\frac{1}{x}\right)\ dx$ and $v = x^2/2$.
Using the integration by parts formula,

$$\begin{aligned}\int x(\ln x)^2\ dx &= (\ln x)^2\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{2\ln x}{x}\right)\ dx \\ &= \frac{1}{2}x^2(\ln x)^2 - \underbrace{\int x\ln x\ dx}_{I_2}\end{aligned}$$

This second integral, I_2, can be evaluated with a second application of integration by parts. This was done earlier in Question #64 (using y instead of x though):

$$\underbrace{\int x\ln x\ dx}_{I_2} = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$$

Going back to the original integral

$$\begin{aligned}&\int x(\ln x)^2\ dx \\ &= \frac{1}{2}x^2(\ln x)^2 - \underbrace{\left(\frac{x^2\ln x}{2} - \frac{x^2}{4}\right)}_{I_2} + C \\ &= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{x^2}{4} + C\end{aligned}$$

79. $\int \arcsin(w)\ dw$

We don't know the integral of arcsin, but we do know its derivative. Therefore we pick
$u = \arcsin(w)$ and $dv = dw$,
so $du = \dfrac{1}{\sqrt{1-w^2}}\ dw$ and $v = w$.
Using the integration by parts formula,

$$\int \arcsin(w)\ dw = w\ \arcsin(w) - \underbrace{\int \frac{w}{\sqrt{1-w^2}}\ dw}_{I_2}$$

The new integral I_2 can be evaluate using a substitution.
Let $z = 1 - w^2$, so $\frac{dz}{dw} = -2w$ or $\dfrac{-1}{2w}\ dz = dw$:

$$\begin{aligned}\underbrace{\int \frac{w}{\sqrt{1-w^2}}\ dw}_{I_2} &= \int \frac{w}{\sqrt{z}}\left(\frac{-1}{2w}\ dz\right) \\ &= \frac{-1}{2}\frac{1}{\sqrt{z}}\ dz \\ &= \frac{-1}{2}(2z^{1/2}) + C \\ &= -\sqrt{1-w^2} + C\end{aligned}$$

Going back to the original integral,

$$\begin{aligned}\int \arcsin(w)\ dw &= w\ \arcsin(w) - \underbrace{\int \frac{w}{\sqrt{1-w^2}}\ dw}_{I_2} \\ &= w\ \arcsin(w) - \left(-\sqrt{1-w^2} + C\right) \\ &= w\ \arcsin(w) + \sqrt{1-w^2} + C_2\end{aligned}$$

80. $\int \arctan(7x)\ dx$

We don't know the integral of arctan, but we do know its derivative. Therefore we pick
$u = \arctan(7x)$ and $dv = dx$,
so $du = \dfrac{7}{1+(7x)^2}\ dx$ and $v = x$.
Using the integration by parts formula,

$$\int \arctan(7x)\ dx = x\ \arctan(7x) - \underbrace{\int \frac{7x}{1-49x^2}\ dx}_{I_2}$$

The new integral I_2 can be evaluate using a substitution.

Let $w = 1 + 49x^2$, so $\frac{dw}{dx} = 98x$ or $\dfrac{1}{98x}\ dw = dx$:

$$\begin{aligned}\underbrace{\int \frac{7x}{1+49x^2}\ dx}_{I_2} &= \int \frac{7x}{w}\left(\frac{1}{98x}\ dw\right) \\ &= \frac{1}{14}\frac{1}{w}\ dw \\ &= \frac{1}{14}\ln|w| + C \\ &= \frac{1}{14}\ln|1+49x^2| + C\end{aligned}$$

Going back to the original integral,

$$\begin{aligned}\int \arctan(x)\ dx &= x\ \arctan(x) - \underbrace{\int \frac{7x}{1-49x^2}\ dx}_{I_2} \\ &= x\ \arctan(x) - \frac{1}{14}\ln|1+49x^2| + C_2\end{aligned}$$

81. $\int x\arctan(x^2)\ dx$

This question starts off best with a substitution, due to the x^2 inside the arctan, and the x outside:
Let $w = x^2$, so $\frac{1}{2x}\ dw = dx$

$$\begin{aligned}\int x\arctan(x^2)\ dx &= \int x\arctan(w)\left(\frac{1}{2x}\ dw\right) \\ &= \frac{1}{2}\underbrace{\int \arctan(w)\ dw}_{I_2}\end{aligned}$$

Evaluating I_2, we use by parts, following the same approach as Question #80, but without the '7' factor,

$$\underbrace{\int \arctan(w)\ dw}_{I_2} = w\ \arctan(w) - \frac{1}{2}\ln|1+w^2| + C$$

Going back to the original integral, and using $w = x^2$,

$$\begin{aligned}&\int x\arctan(x^2)\ dx \\ &= \frac{1}{2}\underbrace{\int \arctan(w)\ dw}_{I_2} \\ &= \frac{1}{2}\left(w\ \arctan(w) - \frac{1}{2}\ln|1+w^2| + C\right) \\ &= \frac{1}{2}\left(x^2\ \arctan(x^2) - \frac{1}{2}\ln|1+(x^2)^2| + C\right) \\ &= \frac{1}{2}x^2\ \arctan(x^2) - \frac{1}{4}\ln|1+x^4| + C_2\end{aligned}$$

82. $\int x^3 e^{x^2}\ dx$

In this problem, we note that we can't integrate e^{x^2} by itself (no closed-form anti-derivative). However, if we package it with one of the x's from the x^3, we'll get xe^{x^2}, and that *can* be integrated, using substitution ($w = x^2$).

Let $u = x^2$ and $dv = xe^{x^2}$,
so $du = 2x\ dx$ and $v = \frac{1}{2}e^{x^2}$. Using that,

$$\int x^3 e^{x^2}\ dx = \frac{1}{2}x^2 e^{x^2} - \underbrace{\int xe^{x^2}\ dx}_{I_2} = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C$$

where I_2 is evaluated using the same substitution $w = x^2$.

83. $\int_1^5 \ln t\ dt$

We integrate by parts, as done in Question #63:

$$\begin{aligned}\int_1^5 \ln t\ dt &= \underbrace{t\ln t - t}_{\text{from \#63}}\Big|_1^5\\ &= (5\ln 5 - 5) - (1\ln 1 - 1)\\ &= 5\ln 5 - 4.\end{aligned}$$

84. $\int_3^5 x\cos x\ dx$

Integrating by parts with $u = x$ and $dv = \cos x\ dx$ gives

$$\begin{aligned}\int_3^5 x\cos x\ dx &= x\sin(x) + \cos(x)\Big|_3^5\\ &= 5\sin(5) + cos(5) - (3\sin(3) + \cos(3))\\ &\approx -3.944\end{aligned}$$

85. $\int_0^{10} ze^{-z}\ dz$

The integral is the same as in Question #74. We use by parts with $u = z$ and $dv = e^{-z}\ dz$, giving

$$\begin{aligned}\int_0^{10} ze^{-z}\ dz &= \underbrace{-ze^{-z} - e^{-z}}_{\text{from \#74}}\Big|_0^{10}\\ &= -10e^{-10} - e^{-10} - (0 - e^0)\\ &= -11e^{-10} + 1 \approx 0.9995\end{aligned}$$

86. $\int_1^3 t\ln(t)\ dt$

The integral is the same as in Question #64. We use by parts, with $u = \ln(t)$ and $dv = t\ dt$.

$$\begin{aligned}\int_1^3 t\ln(t)\ dt &= \underbrace{\frac{t^2\ln t}{2} - \frac{t^2}{4}}_{\text{from \#64}}\Big|_1^3\\ &= \left(\frac{9\ln 3}{2} - \frac{9}{4}\right) - \left(\frac{1\ln 1}{2} - \frac{1}{4}\right)\\ &= \frac{9\ln 3}{2} - 2 \approx 2.944\end{aligned}$$

87. $\int_0^1 \arctan(y)\ dy$

We follow the work from Question #80, but without the '7' factor, using $u = \arctan(y)$ and $dv = dy$.

$$\begin{aligned}\int_0^1 \arctan(y)\ dy &= y\ \arctan(y) - \frac{1}{2}\ln|1+y^2|\Big|_0^1\\ &= \left(1\cdot\arctan(1) - \frac{1}{2}\ln(2)\right) - \left(0\cdot\arctan(0)\right)\\ &= \frac{\pi}{4} - \frac{1}{2}\ln(2) \approx 0.439.\end{aligned}$$

88. $\int_0^5 \ln(1+t)\ dt$

We use the solution to Question #63, or applying by parts with $u = \ln(1+t)$ and $dv = \ dt$.

$$\begin{aligned}\int_0^5 \ln(1+t)\ dt &= \underbrace{(1+t)\ln(1+t) - (1+t)}_{\text{from \#63, with } x=1+t}\Big|_0^5\\ &= (6\cdot\ln 6 - 6) - (1\cdot\ln 1 - 1)\\ &= 6\ln 6 - 5 \approx 5.751\end{aligned}$$

89. $\int_0^1 \arcsin z\ dz$

Using the solution from Question #79, or $u = \arcsin z$ and $dv = dz$,

$$\begin{aligned}\int_0^1 \arcsin z\ dz &= z\ \arcsin(z) + \sqrt{1-z^2}\Big|_0^1\\ &= \left(\arcsin(1) + \sqrt{0}\right) - \left(0\cdot\arcsin(0) + \sqrt{1}\right)\\ &= \frac{\pi}{2} - 1 \approx 0.571\end{aligned}$$

90. $\displaystyle\int_0^1 x \arcsin(x^2)\ dx$

We first simplify the integral with the substitution $w = x^2$, which leads to the new limits
$x = 0 \to w = 0^2 = 0$ and
$x = 1 \to w = 1^2 = 1$.

$$\int_{x=0}^{x=1} x \arcsin(x^2)\ dx = \underbrace{\frac{1}{2}\int_{w=0}^{w=1} \arcsin(w)\ dw}_{\text{after substitution}}$$

At this point, we have returned to the integral in Question #79, which can be evaluated using by parts, with $u = \arcsin(w)$ and $dv = dw$.

$$\begin{aligned}
&\int_{x=0}^{x=1} x \arcsin(x^2)\ dx \\
&= \frac{1}{2}\int_{w=0}^{w=1} \arcsin(w)\ dw \\
&= \underbrace{\frac{1}{2}\left(w\ \ \arcsin(w) + \sqrt{1-w^2}\right)\Big|_0^1}_{\text{by parts}} \\
&= \frac{1}{2}\left[\left(\arcsin(1) + \sqrt{0}\right) - \left(0 \cdot \arcsin(0) + \sqrt{1}\right)\right] \\
&= \frac{\pi}{4} - \frac{1}{2} \approx 0.285
\end{aligned}$$

91. Find the area under the curve $y = te^{-t}$ on the interval $0 \le t \le 2$.

The function te^{-t} is always positive on the interval $0 \le t \le 2$ so the area under the curve is equal to the integral

$$\int_0^2 te^{-t}\ dt$$

Proceeding in the same way as Question #74, using $u = t$ and $dv = e^{-t}\ dt$,

$$\begin{aligned}
\int_0^2 te^{-t}\ dt &= \underbrace{-te^{-t} - e^{-t}}_{\text{from \#74}}\Big|_0^2 \\
&= \left(-2e^{-2} - e^{-2}\right) - \left(0e^0 - e^0\right) \\
&= -3e^{-2} + 1
\end{aligned}$$

92. Find the area under the curve $f(z) = \arctan z$ on the interval $[0, 2]$.

On the interval $t \in [0, 2]$, the function $\arctan(z)$ is always positive, so the area equals the integral $\displaystyle\int_0^2 \arctan(z)\ dz$.

To evaluate the integral, we follow the work from Question #80, but without the '7' factor, using $u = \arctan(z)$ and $dv = dz$.

$$\begin{aligned}
\int_0^2 \arctan(z)\ dz &= z\ \ \arctan(z) - \frac{1}{2}\ln|1+z^2|\Big|_0^2 \\
&= \left(2\arctan(2) - \frac{1}{2}\ln 5\right) - \left(0\arctan(0) - \frac{1}{2}\right. \\
&= 2\arctan(2) - \frac{\ln 5}{2}
\end{aligned}$$

93. Use integration by parts twice to find $\displaystyle\int e^x \sin(x)\ dx$.

There are several ways to evaluate this integral; we'll show just one here.
Let $u = \sin(x)$ and $dv = e^x\ dx$,
so $du = \cos(x)\ dx$ and $v = e^x$.

$$\underbrace{\int e^x \sin(x)\ dx}_{\text{our goal, } I} = \sin(x)e^x - \underbrace{\int \cos(x)e^x\ dx}_{I_2}$$

To evaluate I_2, we select the trig function again as u and the exponential as dv: Let $u = \cos(x)$ and $dv = e^x\ dx$,
so $du = -\sin(x)\ dx$ and $v = e^x$.

$$\begin{aligned}
&\underbrace{\int e^x \sin(x)\ dx}_{\text{our goal, } I} \\
&= \sin(x)e^x - \underbrace{\int \cos(x)e^x\ dx}_{I_2} \\
&= \sin(x)e^x - \left(\cos(x)e^x - \int (-\sin(x))e^x\ dx\right) \\
&= \sin(x)e^x - \left(\cos(x)e^x - \int (-\sin(x))e^x\ dx\right)
\end{aligned}$$

Tidying, we obtain

$$\underbrace{\int e^x \sin(x)\ dx}_{I} = \sin(x)e^x - \cos(x)e^x - \underbrace{\int \sin(x)e^x\ dx}_{I}$$

Grouping the integrals I, which are what we are looking for,

$$\begin{aligned}
2\int e^x \sin(x)\ dx &= \sin(x)e^x - \cos(x)e^x \\
\text{or } \int e^x \sin(x)\ dx &= \frac{1}{2}\left(\sin(x)e^x - \cos(x)e^x\right)
\end{aligned}$$

94. Use integration by parts twice to find $\int e^y \cos(y)\, dy$.

This question is done in the same manner as the previous one. For variety, and to show it works as well, we will select the exponential function as u and the trig functions as dv. (Both choices work, so long as you are consistent in both integration by parts steps.)

Let $u = e^y$ and $dv = \cos(y)\, dy$,
so $du = e^y\, dy$ and $v = \sin(y)$.

$$\underbrace{\int e^y \cos(y)\, dy}_{\text{our goal, } I} = \sin(y)e^y - \underbrace{\int \sin(y)e^y\, dy}_{I_2}$$

To evaluate I_2, we select the exponential function again as u and the trig as dv:
Let $u = e^y$ and $dv = \sin(y)\, dy$,
so $du = e^y\, dy$ and $v = -\cos(y)$.

$$\begin{aligned}
&\underbrace{\int e^y \cos(y)\, dy}_{\text{our goal, } I} \\
&= \sin(y)e^y - \underbrace{\int \sin(y)e^y\, dy}_{I_2} \\
&= \sin(y)e^y - \left((-\cos(y))e^y - \int (-\cos(y))e^y\, dy\right) \\
&= \sin(y)e^y + \cos(y)e^y - \int \cos(y)e^y\, dy
\end{aligned}$$

Tidying, we obtain

$$\underbrace{\int e^y \cos(y)\, dy}_{I} = \sin(y)e^y + \cos(y)e^y - \underbrace{\int \cos(y)e^y\, dy}_{I}$$

Grouping the integrals I, which are what we are looking for,

$$\begin{aligned}
2\int e^y \cos(y)\, dy &= \sin(y)e^y + \cos(y)e^y \\
\text{or} \int e^y \cos(y)\, dy &= \frac{1}{2}\left(\sin(y)e^y + \cos(y)e^y\right)
\end{aligned}$$

95. Use integration by parts to show that

$$\int x^n \cos(ax)\, dx = \frac{1}{a}x^n \sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\, dx$$

We are simply asked to change one integral into another, which can be done here directly with integration by parts.
Let $u = x^n$ and $dv = \cos(ax)\, dx$,
so $du = nx^{n-1}\, dx$ and $v = \frac{1}{a}\sin(ax)$
Applying the by parts formula,

$$\begin{aligned}
&x^n \cos(ax)\, dx \\
&= x^n\left(\frac{1}{a}\sin(ax)\right) - \int \frac{1}{a}\sin(ax)\cdot n \cdot x^{n-1}\, dx \\
&= x^n\left(\frac{1}{a}\sin(ax)\right) - \frac{n}{a}\int x^{n-1}\sin(ax)\, dx
\end{aligned}$$

which is the desired formula.

96. The concentration, C, in ng/ml, of a drug in the blood as a function of the time, t, in hours since the drug was administered is given by

$$C = 15te^{-0.2t}.$$

The area under the concentration curve is a measure of the overall effect of the drug on the body, called the *bioavailability*. Find the bioavailability of the drug between $t = 0$ and $t = 3$.

We have

$$Bioavailability = \int_0^3 15te^{-0.2t}\, dt.$$

We first use integration by parts to evaluate the indefinite integral of this function.
Let $u = 15t$ and $dv = e^{-0.2t}\, dt$,
so $du = 15\, dt$ and $v = -5e^{-0.2t}$. Then,

$$\begin{aligned}
\int 15te^{-0.2t}\, dt &= (15t)(-5e^{-0.2t}) - \int (-5e^{-0.2t})(15\, dt) \\
&= -75te^{-0.2t} + 75\int e^{-0.2t}\, dt \\
&= -75te^{-0.2t} - 375e^{-0.2t} + C.
\end{aligned}$$

$$\begin{aligned}
\text{Thus, } \int_0^3 15te^{-0.2t}\, dt &= \left(-75te^{-0.2t} - 375e^{-0.2t}\right)\Big|_0^3 \\
&= -329.29 + 375 = 45.71.
\end{aligned}$$

The bioavailability of the drug over this time interval is 45.71 (ng/ml)-hours

97. During a surge in the demand for electricity, the rate, r, at which energy is used can be approximated by

$$r = te^{-at}$$

where t is the time in hours and a is a positive constant.

(a) Find the total energy, E, used in the first T hours. Give your answer as a function of a.

(b) What happens to E as $T \to \infty$?

We know that $\frac{dE}{dt} = r$, so the total energy E used in the first T hours is given by

$$E = \int_0^T te^{-at}dt.$$

We use integration by parts.
Let $u = t$, $dv = e^{-at}\ dt$.
Then $du = dt$, and $v = (-1/a)e^{-at}$.

$$\begin{aligned} E &= \int_0^T te^{-at}dt \\ &= -(t/a)e^{-at}\Big|_0^T - \int_0^T -(1/a)e^{-at}\ dt \\ &= -(1/a)Te^{-aT} + (1/a)\int_0^T e^{-at}dt \\ &= -(1/a)Te^{-aT} + (1/a^2)(1 - e^{-aT}). \end{aligned}$$

(b)

$$\lim_{T\to\infty} E = -(1/a)\lim_{T\to\infty}\left(\frac{T}{e^{aT}}\right) + (1/a^2)\left(1 - \lim_{T\to\infty}\frac{1}{e^{aT}}\right)$$

Since $a > 0$, the second limit on the right hand side in the above expression is 0. In the first limit, although both the numerator and the denominator go to infinity, the denominator e^{aT} goes to infinity more quickly than T does (can verify with l'Hopital's rule). So in the end the denominator e^{aT} is much greater than the numerator T. Hence $\lim\limits_{T\to\infty}\dfrac{T}{e^{aT}} = 0$.

Thus $\lim_{T\to\infty} E = \frac{1}{a^2}$.

In words this means that the total amount of energy in the surge, accounting for over all time ($T \to \infty$) is $\dfrac{1}{a^2}$ Joules.