MNTC P01 - Week #6 - Integrals - Modeling

Numerical Integration

As part of this assignment, you should be able to reproduce the LHR rule calculations in MATLAB using a loop. You should know how to adapt it to handle either data from a file, or a formula, as the case requires.

1. **Theory** Consider the problem of estimating the general form of the integral

$$\int_a^b f(x)\ dx$$

   (a) Assume $f(x)$ is a smooth and continuous function. For our the Left-Hand sum, LEFT(n), by what factor do we reduce the error if we use 10 times the number of intervals?

   (b) Confirm your earlier answers by finding the change in the error for LEFT(n) using $N = 20$ and $N = 200$ on the following integrals. Find the error with both N values, and then compute the ratio of the errors.

   - $\displaystyle\int_0^6 \cos(x)\ dx$ - exact value is $(\sin(6) - \sin(0)) = \sin(6)$

   (a) If $f(x)$ is a smooth function, if we use 10 times the intervals, the error for
   - LHR's estimate will drop by a factor of $\frac{1}{10}$,
   - Trapezoidal rule's estimate will drop by a factor of $\frac{1}{10^2} = \frac{1}{100}$, and
   - Simpson's rule's estimate will drop by a factor of $\frac{1}{10^4} = \frac{1}{10{,}000}$,

   (b) Any function with a discontinuity or even a discontinuous derivative is at risk of having poorer-than-expected convergence. See the sawtooth example.

   (c) Here are the errors and ratios for both examples.

   - $\displaystyle\int_0^6 \cos(x)\ dx$

	$N = 20$ error	$N = 200$ error	Ratio
LHR	0.0080732234	0.00061840218	$0.0766 \approx \frac{1}{10}$
TRAP	0.0020987664	2.0956477e-005	$0.00999 \approx \frac{1}{100}$
SIMP	-1.2709701e-005	-1.2575047e-009	$9.89\text{e-}005 \approx \frac{1}{10{,}000}$

   - $\displaystyle\int_0^6$ `mod(x, pi)` dx

	$N = 20$ error	$N = 200$ error	Ratio
LHR	-0.40234864N	-0.063587542	$0.158 \approx \frac{1}{10}$
TRAP	0.026412458N	-0.020711432	$0.784 \not\approx \frac{1}{100}$
SIMP	-0.13066717N	-0.036419395	$0.279 \not\approx \frac{1}{10{,}000}$

Write code that will find the LHR, Trapezoidal Rule, and Simpson's Rule integral estimate of the following integrals: **Assignment Note:** On a test, you may be asked to implement any of the three fixed-interval techniques we saw in class (LHR, TRAP and SIMP).

1. $\displaystyle\int_0^5 x^3 - 5\ dx$ (exact value: $\displaystyle\frac{5^4}{4} - 25$)
2. $\displaystyle\int_1^{10} 4\log_{10}(x)\ dx$ (exact value: $\displaystyle\frac{(-36 + 40\ln(2) + 40\ln(5))}{\ln(2) + \ln(5)}$
3. $\displaystyle\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\ dx$ (exact value not known; approximately 0.95450)

   For all integral estimates, use $N = 100$ intervals.
   Using the 'lhr.m', 'trap.m', and 'simp.m' built in class, you should obtain the following results.

1. $\displaystyle\int_0^5 x^3 - 5\ dx$

```
**** f =     @(x)x^3-5
Integral from 0.0 to 5.0
Exact value of integral = 131.25
LHR estimate = 128.14063
Trap estimate = 131.26563
Simp estimate = 131.25
```

2. $\int_1^{10} 4\log_{10}(x)\, dx$

```
**** f =     @(x)4*log10(x)

Integral from 1.0 to 10.0
Exact value of integral = 24.365399
LHR estimate = 24.184344
Trap estimate = 24.364344
Simp estimate = 24.365397
```

3. $\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\, dx$

```
**** f =     @(x)1/sqrt(2*pi)*exp(-x.^2/2)

Integral from -2.0 to 2.0
Exact value of integral = 0.9545
LHR estimate = 0.95447094
Trap estimate = 0.95447094
Simp estimate = 0.95449973
```

In this case, our approximation generated using Simpson's rule is likely more accurate than the "exact value" listed, as the "exact value" was given to only 5 significant digits.

Use the `quad` function to estimate the following integrals, and print the estimates with 8 digits after the decimal. Use the default accuracy for the `quad` function.

1. $\int_0^5 x^3 - 5\, dx$ (exact value: $\frac{5^4}{4} - 25$)
2. $\int_1^{10} 4\log_{10}(x)\, dx$ (exact value: $\frac{(-36 + 40\ln(2) + 40\ln(5))}{\ln(2) + \ln(5)}$
3. $\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\, dx$ (exact value not known; approximately 0.95450)

Code that does this would be e.g.

```
f = @(x) x.^3 - 5; % note: quad will pass in _vectors_ of x values
% so we need to use '.^' instead of just '^' for the exponent/power
I = quad(f, 0, 5);
fprintf('quad estimate: %.8f\n', I);
```

The results are

1. function: `f = @(x) x.^3 - 5;`
   estimated integral: 131.25000000
2. function: `f = @(x) 4*log10(x);`
   estimated integral: 24.36539860

3. function: `f = @(x) (1/sqrt(2*pi)) * exp(-x.^2/2);`
   estimate integral: 0.95449979

Note that all these values are within 10^{-7} of the exact values (except the last, which we only know to 10^{-5}).

The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated on the figure. Estimate the area of the pool. The integration technique you choose should be the one that provides the most accurate answer, out of the techniques covered in class.

NOTE: In this problem, because we are only given data and not a function, we can only use LHR, Trapezoidal or Simpson's rules. We cannot use "quad", the adaptive integrator, because it would try to evaluate the function (width of the pool) in between the measured data points, and we don't have that information.

Since Simpson's rule generally gives the most accurate integral estimate, compared with the LHR and Trapezoidal rule, we should use it here.

A simple vector-based version of Simpson's rule is shown below, applied to these pool dimensions.

```
% Compute the area of the pool

% Store the widths of the pool, using zeros at the ends!
% we need to count the areas in the last intervals as well
w = [0 6.2 7.2 6.8 5.6 5.0 4.8 4.8 0];

% Use a loop to compute the Simpson's rule estimate of the area
% dx = 2 m, from the question
dx = 2;
n = length(w) -1;  % # intervals = # points - 1
I = 0;
for (i = 1:2:n)
    I = I + (w(i) + 4*w(i+1) + w(i+2))/6 * 2*dx;
end
fprintf('Surface area of pool is approximately %.5g m^2\n', I);
```

This script produces the estimate

"Surface area of pool is approximately 84.267 m^2"

This is reasonable, as the median width is 5.6 m, and the total length of the pool is 16 m, giving approximately 89.6 m^2 if it were rectangular. This is safely in the same ballpark as the more accurate result above. Without more information, though, we can't improve our integral estimate.

A steel scribing tip is designed to have a circular cross-section. The shape can be generated by rotating the function $y = \dfrac{0.0007}{10x + 0.1}$ around the x axis, starting at $x = 0$ and ending at $x = 0.02$ (all dimensions in meters). Compute the volume of steel required to make the scribing tip. Use an appropriate integration technique and parameters, and report your answer to five significant digits. Be prepared to defend your choice of technique.

You will need to review the volume of rotation to answer this question.

Volumes of revolution can be computed by taking slices through the shape. Each slice has a volume $(\Delta x) \cdot \pi y^2$. Adding up the slices using an integral gives the total volume.

For this shape, the volume will be given by the integral

$$\int_0^{0.02} \pi \left(\frac{0.0007}{10x + 0.1} \right)^2 dx$$

The analytic value of this integral is approximately 1.0262536×10^{-6}.

The script "q_3Dshape.m" computes this integral. In the solution, the I used "quad", the adaptive integrator. The rationale for this was, knowing that the integral had a value of approximately 10^{-6}, I could set the absolute tolerance to 10^{-12}, to give 6 figures of accuracy. Note that the "quad" error is only an estimate of the true error; however, it is usually correct within an order of magnitude (e.g. setting the tolerance to 10^{-12} usually ensures that the exact error is no worse that 10^{-11}.

When a pendulum oscillates, with maximum deviation angle θ_0, the period of the pendulum is given by

$$T = 4\sqrt{L/g} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity, 9.8 m/s.

Compute and compare the period of a pendulum with

- $L = 2$, $\theta_0 = 40^o$,
- $L = 2$, $\theta_0 = 20^o$.

- $L = 2.5$, $\theta_0 = 40^o$,
- $L = 2.5$, $\theta_0 = 20^o$.

Describe how significant the effect of maximum swing angle θ_0 is on the period of a pendulum, compared to the effect of the pendulum length.

Full solution is available in `q_pendulumPeriod.m`.

The periods computed are show below. "quad" was used for the integration.

L = 2.0 m, theta0 = 40.0 deg, period = 2.9274 s.
L = 2.0 m, theta0 = 20.0 deg, period = 2.8602 s.
L = 2.5 m, theta0 = 40.0 deg, period = 3.2729 s.
L = 2.5 m, theta0 = 20.0 deg, period = 3.1978 s.

From these results, it is clear that the angle has a minimal effect on the period of the oscillations, compared with the effect of the length. This insensitivity of the period to the oscillation angle explains why pendulum clocks and metronomes do not need a specific swing angle to be fairly accurate, but *do* need a specific length.