Week #2 : Derivatives - Linearization

Goals:

- Describe the meaning and value of linearization
- Apply the technique of linearization to solve a variety of nonlinear equations
- Use MATLAB to graph and compare functions with their linearizations
- Use MATLAB to implement Newton's method
- Calculate and interpret the first and second derivatives, as well as higher order derivatives

Linear Approximations

You should now feel comfortable in finding the derivative of a wide variety of functions with formulas.

In this section, we will explore how the derivatives you can compute can be tied back to understanding the behaviour of the original function.

We will start by returning to the definition of the derivative, based on the $\frac{\text{rise}}{\text{run}}$ formula for slopes:

$$f'(x) = \frac{d}{dx} f = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For example, if $y = f(x)$, then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

Problem. What is the relationship between f' and Δy, Δx for merely *small* delta values?

Now sketch a graph, and label the two points in the Δx difference 'x' and 'a'. What expression do we obtain for $f(x)$?

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

Problem. What are some names for this linear function?

Problem. Consider the tangent line approximation to the graph of $f(x) = e^x$ at $(0,1)$. Find the formula for the tangent line at that point.

Sketch the graph of $y = e^x$ and the linearization/tangent line.

Problem. Sketch the function $f(x) = \dfrac{1}{x}$.

Problem. If we drew a tangent line to $f(x)$ at $x = 4$, what range we would expect for the slope there?

A. Slope above 1.

B. Slope between 0 and 1.

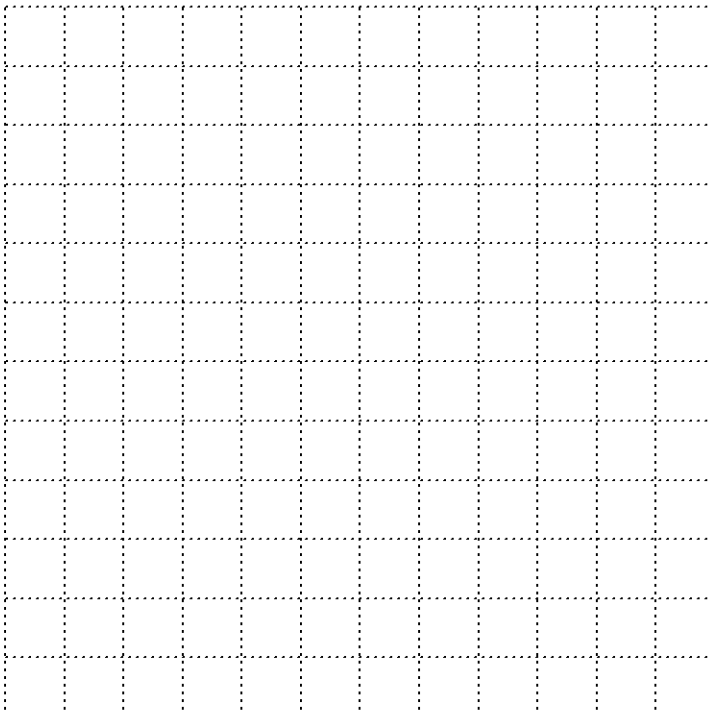
C. Slope between 0 and -1.

D. Slope below -1.

Problem. Find the linearization of $f(x) = \dfrac{1}{x}$ at $a = 4$.

Find the equation of the tangent line to the function $y = \dfrac{1}{x}$ at $x = 4$.

Problem. Sketch the graph of $y = \dfrac{1}{x}$, and its tangent line at $x = 4$.



Linear approximations can be useful for quick and simple error calculations.

Problem. A rock formation with high density ores was identified using gravity measurements; the formation is roughly cubic in shape. The edge each side of the cube was found to be 370 m, with a possible error in measurement of 10 m.

Use the ideas of linear approximations to estimate the maximum possible error (positive or negative) in computing the **volume** of the formation.

Problem. What are the trade-offs of using the linear approximation to obtain the above error estimate, compared to a direct calculation of the possible volumes with the error measurements?

The $\sin(x)$ Approximation

One of the most commonly-used approximation in physics is the relationship

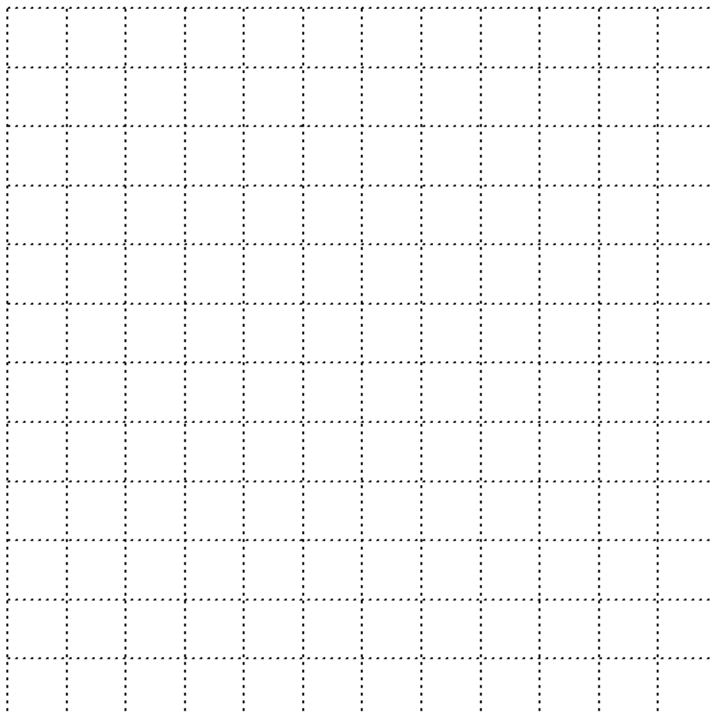
$$\sin(x) \approx x$$

Problem. Derive this relationship using linearization.

What is the fine-print that should **always** be associated with this approximation?

Problem. Sketch the graphs of $y = \sin(x)$ and $y = x$.

Focus on the domain $\dfrac{-\pi}{2} \leq x \leq \dfrac{\pi}{2}$.



Below are more detailed calculations relating $\sin(x)$ and x.

x (degrees)								
x (rad)	0.0500	0.1000	0.1500	0.2000	0.2500	0.3000	0.3500	0.4000
$\sin(x)$	0.0500	0.0998	0.1494	0.1987	0.2474	0.2955	0.3429	0.3894

Problem. Comment on the agreement between $y = x$ and $y = \sin(x)$ on the range shown.

Most people are more familiar with angles measured in degrees than radians: fill in the row indicating how many degrees are represented by the radian measures.

Just for fun, put your calculator into degree mode, and see whether $\sin(x) \approx x$ still holds for small x.

x (deg)	$\sin(x)$
2^o	
4^o	
6^o	

MATLAB Intro

Graphing in MATLAB

Summary:

```
x = linspace(-2, 2);
y = exp(x);
plot(x, y)
```

Tangent Lines in MATLAB

Solving Non-Linear Equations

One surprisingly challenging area of mathematics is in pre-calculus: simply solving equations with non-linear elements.

Problem. Compare the difficulty in solving these two single-variable equations.

Linear: $5x = 10$

Non-linear: $e^x = x + 5$

Some special types of non-linear equations **can** be solved algebraically.

Problem. Find, by hand or with the help of a calculator, the solutions to the following equations.

$$x^2 + 2x + 3 = 0$$

$$\log_{10}(x) = 3$$

$$\sin(3x) = \cos(3x)$$

Unfortunately, solving using algebra requires you understanding how to manipulate particular functions. Worse yet, equations can be simply too complex to solve algebraically.

Problem. Try to solve the following equations by hand:

$$\sin(3x) = x$$

$$xe^{-x} = 5$$

$$\sin(3x) = x$$
$$xe^{-x} = 5$$

In these more difficult cases, if we want a solution we must resort to **numerical methods**, which are all **fancy versions of guess and check**! This means numerical solutions are a poor second choice, compared to by-hand solving:

- Numerical solutions give no insight into the solution (existence, patterns).
- Numerical solving usually requires some amount of trial and error by the user.

Example - Trajectories

To generate a motivation for solving non-linear equations, we are going to simulate the launch a motorcycle off the end of a ramp. The launch parameters are v_0, y_0, and θ_0.

If you start with some basic physics equations, and break the trajectory into components, you can arrive at the following formula:

$$y = y_0 + \tan(\theta_0)\ x - \frac{1}{2}\frac{g}{\cos^2(\theta_0)(v_0)^2}\ x^2$$

Alex Harvill, age 19, lands 425 foot ramp-to-dirt record in May 2012.

Problem. Write a MATLAB script, `W2_1.m` that plots the trajectory of the motorcycle. Use

- $y_0 = 3$ m,
- $v_0 = 20$ m/s, and
- $\theta_0 = 30$ degrees.

and choose the x interval so it shows the impact point.

Problem. Add a horizontal line on the trajectory graph that shows the ground level, $y = 0$. Draw it in black.

Launch Angle - Ballistics

Now consider the ballistics targeting problem, with practical consequences on battlefields around the world.

Given the launch velocity, height, and a target x, what launch angle should be used?

Problem. Set $y_0 = 3$, and $v = 20$. Experiment with the launch angle θ in MATLAB to find an angle that lands the motorcycle at $x = 30$.

Comment on the simplicity and accuracy of this approach.

Root-Finding vs Equation Solving

In the ballistics example, it turns out to be very laborious, and fairly inaccurate, to find the launch angle that will launch a projectile on to a target.

Problem. Express the targeting problem as a solution to a non-linear equation.

We will return to that example, but first we will consider the more general challenge of solving non-linear equations, to give us a wider context.

Consider the following equations.

$$\text{Example 1: } \sin(x) = \frac{1}{10}\ln(x)$$

$$\text{Example 2: } x^5 - x^{10} = e^{-x}$$

Solving these equations as written involves balancing **two** functions, but life would be easier if we only had to deal with **one**.
Rewrite each equation so that only one function is required.

Any non-linear equation in the form $g(x) = h(x)$ can be written as $f(x) = 0$ simply by moving all terms to the left side; the solutions to both forms will be the same.

The $f(x) = 0$ form is preferable because:

- it only involves a single user-given function, and

- all *non-linear equation solving* problems become *root-finding* problems.

Newton's Method

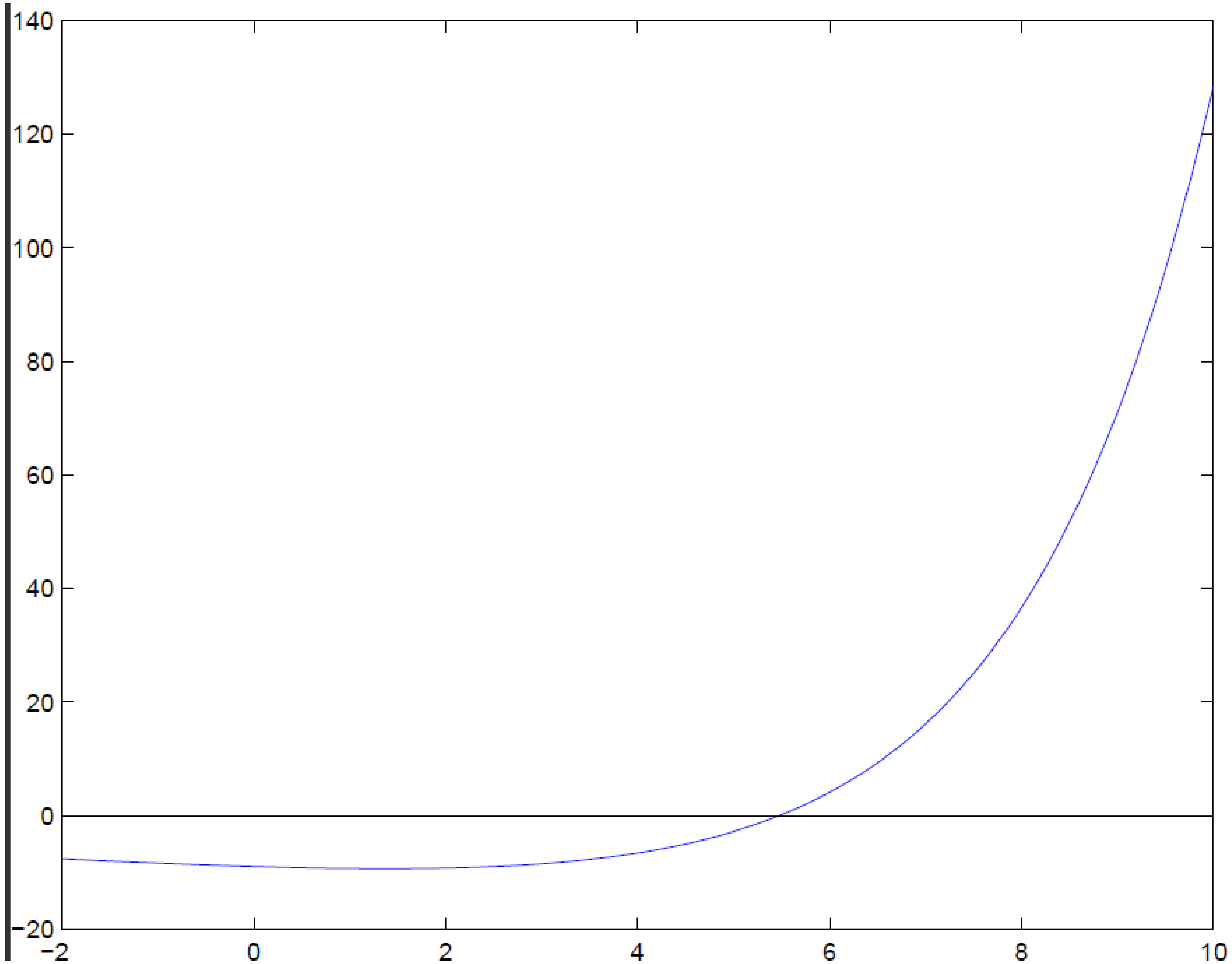
Concept: If we are looking for the **root of a non-linear function**,
We can use that information to speed up our search.

- Pretend that $f(x)$ is linear, through some starting point $x = a, y = f(a))$.
- Find the tangent line at $x = a$
- Find c, the root of the tangent line. This should be closer to the real root of $f(x)$ than our original guess.

Graphical Example

We want to find the x values where the graph below crosses $y = 0$.

- Suppose we start at initial guess of $a = 3$.
- Draw a line through the pair of points, and follow it to its root, $x = c$.
- Replace a with the new point c.
- Repeat the last two steps until a and c are almost the same, or don't change between iterations.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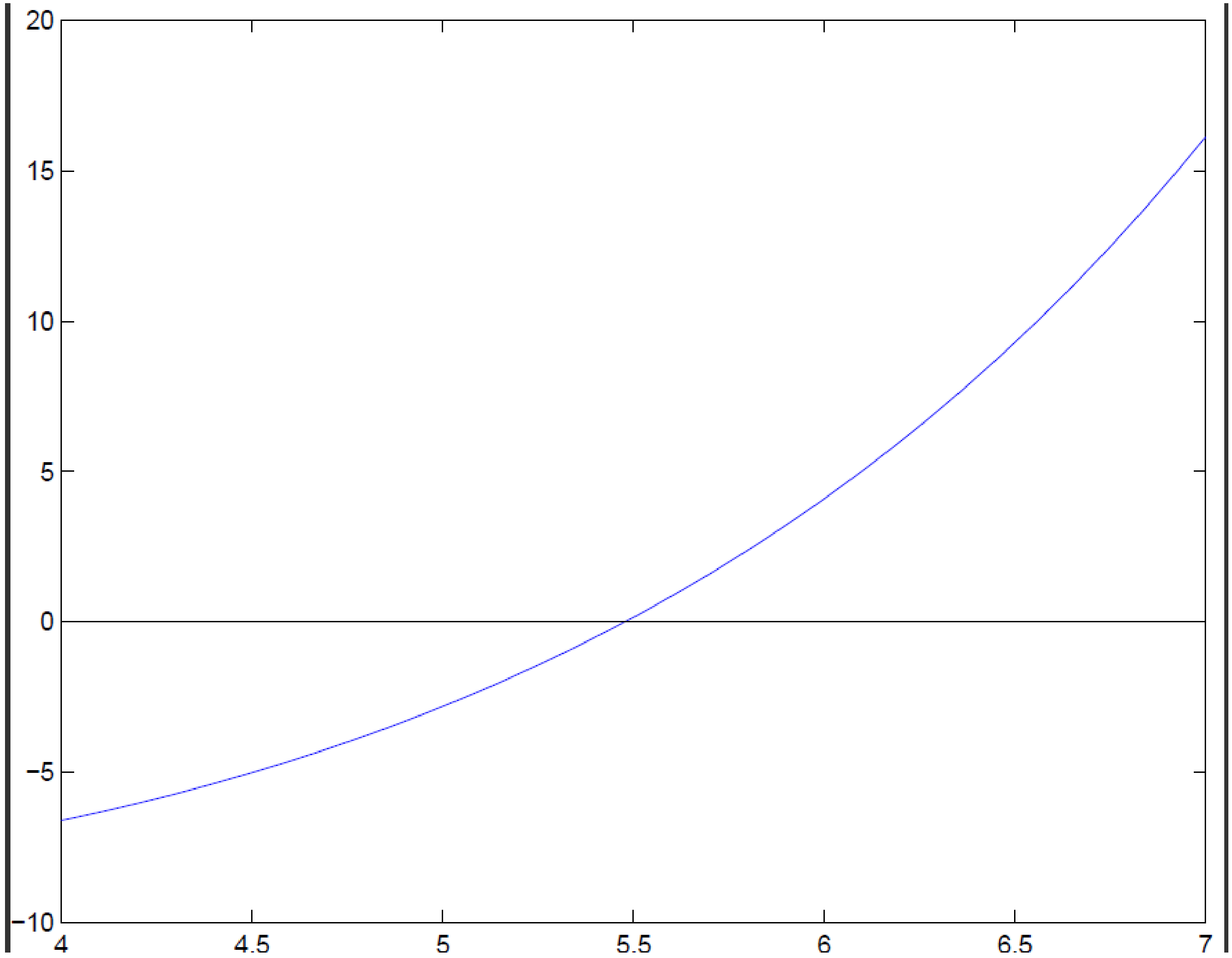
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7

Computational Algorithm

The previous graph was the graph of

$f(x) = e^{x/2} - x - 10$, which came from the equation $e^{x/2} = x + 10$

Problem. Use an anonymous function to define this function in MATLAB.

How do we calculate the root of the tangent line to $f(x)$, if the line is based on the point $x = a$?

Comments

Newton's method is a sophisticated method for solving non-linear equations. Some quick notes:

- The starting value of $x = a$ must be **close to** the root (see practice problems).
  - In practice, people will graph the function $f(x)$ to get a rough idea for where the roots might be.
- Newton's method is vastly superior to using guess-and-check, or using loops, to find high-accuracy approximate solutions.

Newton's Method - Practice

Problem. Use Newton's method to find an approximate solution to

$$xe^{-x} = 5$$

Problem. Use Newton's method to find an approximate solution to

$$sin(x) = \frac{1}{10}\ln(x)$$