MNTC P01 - Week #3 - Derivatives - Applications

Taylor Polynomials

For reference, the general formula for the Taylor polynomial centered at x = a is:

$$P(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^n(a)}{n!}(x-a)^n$$

where $n!$ means "n factorial", or $n \times (n-1) \times (n-2) \times ...2 \times 1$. E.g. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

1. Suppose g is a function which has continuous derivatives, and that $g(5) = -3, g'(5) = -2$, $g''(5) = 1$, $g'''(5) = -5$.
   (a) What is the Taylor polynomial of degree 2 for g near 5?
   (b) What is the Taylor polynomial of degree 3 for g near 5?
   (c) Use the two polynomials that you found in parts (a) and (b) to approximate $g(5.1)$.

We have

$$g(x) = g(5)+g'(5)(x-5)+\frac{g''(5)}{2!}(x-5)^2+\frac{g'''(5)}{3!}(x-5)^3+\cdots$$

Substituting gives

$$g(x) = -3 - 2(x-5) + \frac{1}{2!}(x-5)^2 + \frac{-5}{3!}(x-5)^3 + \cdots$$

(a) The degree 2 Taylor polynomial, $P_2(x)$, is obtained by truncating after the $(x-5)^2$ term:

$$P_2(x) = -3 + (-2)(x-5) + \frac{1}{2!} \cdot 1(x-5)^2 .$$

(b) The degree 3 Taylor polynomial, $P_3(x)$, is obtained by truncating after the $(x-5)^3$ term:

$$P_3(x) = -3+(-2)(x-5)+\frac{1}{2!}\cdot 1(x-5)^2+\frac{1}{3!}(-5)(x-5)^3$$

(c) Substitute $x = 5.1$ into the Taylor polynomial of degree 2:

$$P_2(5.1) = -3+(-2)(5.1-5)+\frac{1}{2!}\cdot 1(5.1-5)^2 = -3.195.$$

From the Taylor polynomial of degree 3, we obtain

$$\begin{aligned} P_3(5.1) =& -3 + (-2)(5.1-5) + \frac{1}{2!} \cdot 1(5.1-5)^2 + \\ & \frac{1}{3!}(-5)(5.1-5)^3 \\ =& -3.19583. \end{aligned}$$

2. Find the Taylor polynomial of degree $n = 4$ for x near the point $a = \frac{\pi}{4}$ for the function $\cos(4x)$. Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around $x = \pi/4$.

Let $f(x) = \cos(4x)$. Then $f(\frac{\pi}{4}) = -1$, and

$f'(x) = -4\sin(4x)$	$f'(\frac{\pi}{4}) = 0$
$f''(x) = -16\cos(4x)$	$f'''(\frac{\pi}{4}) = 16$
$f'''(x) = 64\sin(4x)$	$f'''(\frac{\pi}{4}) = 0$
$f''''(x) = 256\cos(4x)$	$f''''(\frac{\pi}{4}) = -256$

So, $P_4(x) = -1 + \frac{16}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{-256}{24}\left(x - \frac{\pi}{4}\right)^4$.

MATLAB graphing:
W03TaylorTrig1.m

3. Find the fifth order Taylor polynomial for $\sin(x)$ near $x = \pi/4$.
   Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around $x = \pi/4$.

The fifth order Taylor polynomial is based on the first five derivatives of $\sin(x)$.

$$\begin{aligned} f(x) &= \sin(x) & f(\pi/4) &= \frac{1}{\sqrt{2}} \approx 0.7071 \\ f'(x) &= \cos(x) & f'(\pi/4) &= \frac{1}{\sqrt{2}} \approx 0.7071 \\ f''(x) &= -\sin(x) & f''(\pi/4) &\approx -0.7071 \\ f'''(x) &= -\cos(x) & f'''(\pi/4) &\approx -0.7071 \\ f^{(4)}(x) &= \sin(x) & f^{(4)}(\pi/4) &\approx 0.7071 \\ f^{(5)}(x) &= \cos(x) & f^{(5)}(\pi/4) &\approx 0.7071 \end{aligned}$$

Using the Taylor polynomial formula,

$$\begin{aligned} P_5(x) =& \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x-\pi/4) - \frac{1}{2}\frac{1}{\sqrt{2}}(x-\pi/4)^2 \\ & - \frac{1}{3!}\frac{1}{\sqrt{2}}(x-\pi/4)^3 + \frac{1}{4!}\frac{1}{\sqrt{2}}(x-\pi/4)^4 \\ & + \frac{1}{5!}\frac{1}{\sqrt{2}}(x-\pi/4)^5 \end{aligned}$$

MATLAB graphing:
W03TaylorTrig2.m

4. Find the Taylor polynomial of degree 3 around the point $x = -4$ of $f(x) = \sqrt{5+x}$.
Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around the reference point.

Let $f(x) = \sqrt{5+x} = (5+x)^{1/2}$.
Then

$$f'(x) = \frac{1}{2\sqrt{5+x}},$$

$$f''(x) = -\frac{1}{4(5+x)^{3/2}},$$

and

$$f'''(x) = \frac{3}{8(5+x)^{5/2}}.$$

The Taylor polynomial of degree three about $x = -4$ is thus

$$P_3(x) = \sqrt{5-4} + \frac{1}{2\sqrt{5-4}}(x+4) + (\frac{1}{2!})\left(-\frac{1}{4(5-4)^{3/2}}\right)(x+4)^2 + (\frac{1}{3!})\left(\frac{3}{8(5-4)^{5/2}}\right)(x+4)^3.$$

$$= 1 + \frac{1}{2}(x+4) + \frac{1}{2!}\frac{-1}{4}(x+4)^2 + \frac{1}{3!}\frac{3}{8}(x+4)^3$$

MATLAB graphing:
W03TaylorRoot1.m

5. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = 1$ for $f(x) = \ln(x+1)$.
Use MATLAB to graph the function and both the Taylor polynomials on a reasonable interval around the reference point.

Recall the general formula for the Taylor polynomial centered at $x =$ a:

$$P(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^n(a)}{n!}(x-a)^n$$

$$f(x) = \ln(x+1)$$

$$f'(x) = \frac{1}{x+1}$$

$$f''(x) = \frac{-1}{(x+1)^2}$$

$$f'''(x) = \frac{2}{(x+1)^3}$$

So, $f(x) = \ln(x+1)$ and $P_2(x) = A + B(x-1) + C(x-1)^2$
where
$A = f(1) = \ln(2) = 0.693147$
$B = f'(1) = \frac{1}{(2)} = 0.5$
$C = \frac{f''(1)}{2!} = -\frac{1}{(8)} = -0.125$
Thus,

$$P_2(x) = \ln(2) + \frac{1}{(2)}(x-1) - \frac{1}{(8)}(x-1)^2$$

or,

$$P_2(x) = 0.693147 + 0.5(x-1) - 0.125(x-1)^2$$

Similarly,
$P_3(x) = D + E(x-1) + F(x-1)^2 + G(x-1)^3$
where
$D = f(1) = \ln(2) = 0.693147$
$E = f'(1) = \frac{1}{(2)} = 0.5$
$F = \frac{f''(1)}{2!} = -\frac{1}{(8)} = -0.125$
$G = \frac{f'''(1)}{3!} = \frac{1}{(24)} = 0.0416667$
Thus,

$$P_3(x) = \ln(2) + \frac{1}{(2)}(x-1) - \frac{1}{(8)}(x-1)^2 + \frac{1}{(24)}(x-1)^3$$

or,

$$P_3(x) = P_2(x) + 0.0417(x-1)^3$$

MATLAB graphing:
W03TaylorLn1.m

6. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = \frac{\pi}{6}$ for $f(x) = \sin(x)$.
Use MATLAB to graph the function and both the Taylor polynomials on a reasonable interval around the reference point.

The 2nd and 3rd order Taylor polynomials are based on the first three derivatives of $\sin(x)$.

$$f(x) = \sin(x) \qquad f(\pi/6) = \frac{1}{2}$$

$$f'(x) = \cos(x) \qquad f'(\pi/6) = \frac{\sqrt{3}}{2} \approx 0.866$$

$$f''(x) = -\sin(x) \qquad f''(\pi/6) = \frac{-1}{2}$$

$$f'''(x) = -\cos(x) \qquad f'''(\pi/6) = \frac{-\sqrt{3}}{2} \approx -0.866$$

Using the Taylor polynomial formula,

$$P_2(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\pi/6) - \frac{1}{2!}\frac{1}{2}(x-\pi/6)^2$$

and P_3 will be the same, just with the cubic term added at the end.

$$P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\pi/6) - \frac{1}{2!}\frac{1}{2}(x-\pi/6)^2 - \frac{1}{3!}\frac{\sqrt{3}}{2}(x-\pi/6)^3$$

MATLAB graphing:
W03TaylorTrig3.m

7. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = 7$ for $f(x) = \frac{1}{1+x}$.
   Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around the reference point.

$$f(x) = \frac{1}{1+x}$$
$$f'(x) = \frac{-1}{(1+x)^2}$$
$$f''(x) = \frac{2}{(1+x)^3}$$
$$f'''(x) = \frac{-6}{(1+x)^4}$$

and $P_2(x) = A + B(x-7) + C(x-7)^2$
where
$A = f(7) = \frac{1}{(8)} = 0.125$
$B = f'(7) = -\frac{1}{(64)} \approx -0.0156$
$C = \frac{f''(7)}{2!} = \frac{1}{(512)} = 0.00195$
Thus,

$$P_2(x) = \frac{1}{(8)} - \frac{1}{(64)}(x-7) + \frac{1}{(512)}(x-7)^2$$

or,

$$P_2(x) = 0.125 - 0.0156(x-7) + 0.00195(x-7)^2$$

Similarly,
$P_3(x) = D + E(x-7) + F(x-7)^2 + G(x-7)^3$
where
$D = f(7) = \frac{1}{(8)} = 0.125$
$E = f'(7) = -\frac{1}{(64)} = -0.0156$
$F = \frac{f''(7)}{2!} = \frac{1}{(512)} = 0.00195$
$G = \frac{f'''(7)}{3!} = -\frac{1}{(4096)} = -0.000244$
Thus,

$$P_3(x) = \frac{1}{(8)} - \frac{1}{(64)}(x-7) + \frac{1}{(512)}(x-7)^2 - \frac{1}{(4096)}(x-7)^3$$

or,
$$P_3(x) = P_2(x) - 0.000244(x-7)^3$$

MATLAB graphing:
W03TaylorRecip1.m

8. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = 2$ for $f(x) = e^{-x} + e^{-2x}$.
   Use MATLAB to graph the function and both the Taylor polynomials on a reasonable interval around the reference point.

$$f(x) = e^{-x} + e^{-2x}$$
$$f'(x) = -e^{-x} - 2e^{-2x}$$
$$f''(x) = e^{-x} + 4e^{-2x}$$
$$f'''(x) = -e^{-x} - 8e^{-2x}$$

and $P_2(x) = A + B(x-2) + C(x-2)^2$
where
$A = f(2) = e^{-2} + e^{-4}$
$B = f'(2) = -\left(e^{-2} + 2e^{-4}\right)$
$C = \frac{f''(2)}{2!} = \frac{e^{-2}+4e^{-4}}{2}$
Thus,

$$P_2(x) = (e^{-2}+e^{-4}) - \left(e^{-2} + 2e^{-4}\right)(x-2) + \frac{e^{-2}+4e^{-4}}{2}(x-2)^2$$

Similarly,
$P_3(x) = D + E(x-2) + F(x-2)^2 + G(x-2)^3$
where
$D = f(2) = e^{-2} + e^{-4}$
$E = f'(2) = -\left(e^{-2} + 2e^{-4}\right)$
$F = \frac{f''(2)}{2!} = \frac{e^{-2}+4e^{-4}}{2}$
$G = \frac{f'''(2)}{3!} = -\frac{e^{-2}+8e^{-4}}{6}$
Thus, $P_3(x) =$

$$= e^{-2} + e^{-4} - \left(e^{-2} + 2e^{-4}\right)(x-2) + \frac{e^{-2}+4e^{-4}}{2}(x-2)^2 - \frac{e^{-2}+8e^{-4}}{6}(x-2)^3$$

MATLAB graphing:
W03TaylorExp1.m

9. Calculate the Taylor polynomials $P_2(x)$ and $P_3(x)$ centered at $x = \frac{\pi}{4}$ for $f(x) = \tan(x)$.

   Use MATLAB to graph the function and both the Taylor polynomials on a reasonable interval around the reference point.

$$f(x) = \tan(x)$$
$$\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$$
$$\frac{d^2}{dx^2}\tan(x) = \frac{2\sin(x)}{\cos^3(x)}$$
$$\frac{d^3}{dx^3}\tan(x) = \frac{2\left(2\sin^2(x)+1\right)}{\cos^4(x)}$$

So, in this case,

$$P_2(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2$$

and

$$P_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}(x - \frac{\pi}{4})^3$$

MATLAB graphing:
W03TaylorTrig4.m

10. Find the second-degree Taylor polynomial $P_2(x)$ for the function $f(x) = \sqrt{15 + x^2}$ at the number $x = 1$.

    Use MATLAB to graph both the function and the Taylor polynomial on a reasonable interval around $x = \pi/4$.

$P_2(x) = 4 + (1/4) * (x - 1) + (15/(2(16)4))(x - 1)^2$
MATLAB graphing:
W03TaylorRoot2.m

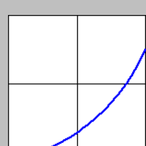
11. Find the second-degree Taylor polynomial for $f(x) = 2x^2 - 3x + 8$ about $x = 0$. What do you notice about your polynomial?

We note that $f(0) = 8$; $f'(x) = 4x - 3$, so that $f'(0) = -3$; and $f''(x) = 4$, so that $f''(0) = 4$.
Thus

$$P_2(x) = 8 - 3x + \frac{4}{2!}x^2 = 8 - 3x + 2x^2.$$

We notice that $f(x) = P_2(x)$ in this case, which makes sense because $f(x)$ is a polynomial.

12. Suppose that $P_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for the function f about $x = 0$. What can you say about the signs of a, b, c if f has the graph given below? Note that the central lines are the x and y axes.



Since $P_2(x)$ is the second degree Taylor polynomial for $f(x)$ about $x = 0$, $P_2(0) = f(0)$, which says $a = f(0)$. Since $\frac{d}{dx}P_2(x)\big|_{x=0} = f'(0)$, $b = f'(0)$; and since $\frac{d^2}{dx^2}P_2(x)\big|_{x=0} = f''(0)$, $2c = f''(0)$. In other words, a is the y-intercept of $f(x)$, b is the slope of the tangent line to $f(x)$ at $x = 0$ and c tells us the concavity of $f(x)$ near $x = 0$.

Thus $a < 0$; $b > 0$; and $c > 0$

13. The function $f(x)$ is approximated near $x = 0$ by the second degree Taylor polynomial $P_2(x) = 3x - 3 + 8x^2$.

    Give values the values of $f(0)$, $f'(0)$, and $f''(0)$.

Using the fact that

$$f(x) \approx P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

and identifying coefficients with those given for $P_2(x)$, we obtain:

$f(0) =$ the constant term, which equals -3, so $f(0) = -3$. $f'(0) =$ the coefficient of x, which equals 3, so $f'(0) = 3$. $\frac{f''(0)}{2!} =$ the coefficient of x^2, which equals 8, so $f''(0) = 16$.

Optimization Introduction

14. Let $f(x) = x^2 - 10x + 13$, and consider the interval $[0, 10]$.

    (a) Find the critical point c of $f(x)$ and compute $f(c)$.

    (b) Compute the value of $f(x)$ at the endpoints of the interval $[0, 10]$.

    (c) Determine the global min and max of $f(x)$ on $[0, 10]$.

    (d) Find the global min and max of $f(x)$ on $[0, 1]$. (Note: not the same interval as before)

(a) The critical point of $f(x)$ is the solution to $f'(x) = 0$. The derivative is$f'(x) = 2x - 10$. Setting this equal to zero and solving for x gives $x = 5$. Evaluating $f(5)$ yields -12.

(b) Evaluating $f(0)$ and $f(10)$, we find that each is equal to 13.

(c) The global min and max values must occur at critical points or at the endpoints of the interval. Since the value at the critical point is smaller than the value at the endpoints, the value of $f(5)$ is a minimum, and the value of $f(0)$ (or $f(10)$ since they are equal) is a maximum.

(d) Since there are no critical points in the interval $[0, 1]$, so the global min and max values lie at the endpoints of the interval. Computation yields $f(0) = 13$,$f(1) = 4$, so the minimum is 4 and the maximum is 13.

15. Find the maximum and minimum values of the function $f(x) = \dfrac{\ln(x)}{x}$ on the interval [1,3].

First we check for critical points. The critical point of $f(x)$ is the solution to $f'(x) = 0$.
The derivative is$f'(x) = x^{-2} - \dfrac{\ln(x)}{x^2}$.
Setting this equal to zero and solving for x gives $x = e^1 = e$.
Evaluating $f(e)$ yields the value $\frac{1}{e} = 0.3679$.
The values of the function at the endpoints of the interval are $f(1) = 0$, $f(3) = 0.3662$, so the minimum value is 0, and the maximum value is $\frac{1}{e} = 0.3679$.

16. Find the minimum and maximum values of $y = \sqrt{10}\theta - \sqrt{5}\sec\theta$ on the interval $[0, \frac{\pi}{3}]$.

Let $f(\theta) = \sqrt{10}\theta - \sqrt{5}\sec\theta$.
On the interval $[0, \frac{\pi}{3}]$, $f'(\theta) = \sqrt{10} - \sqrt{5}\sec\theta\tan\theta = 0$ at $\theta = \frac{\pi}{4}$.
(This can be found either by inspiration, guessing that the answer is a 'nice' angle like $\pi/3$, $\pi/4$ or $\pi/6$, or more mechanically by writing the whole equation in terms of $\sin(\theta)$, using $\cos^2(\theta) = 1 - \sin^2(\theta)$ and then solving a quadratic equation.)
The minimum value of f on this interval is at the endpoint $\theta = 0$, where $f(0) = -2.2361$,
whereas the maximum value over this interval is $f(\frac{\pi}{4}) = \sqrt{10}(\frac{\pi}{4} - 1) = -0.6786$.
At the second endpoint $\theta = \frac{\pi}{3}$,
$f(\frac{\pi}{3}) = \sqrt{10}\frac{\pi}{3} - 2\sqrt{5} = -1.1606$.

17. Find the maximum and minimum values of the function $f(x) = x - \dfrac{125x}{x+5}$ on the interval [0,21].

First we check for critical points. The critical point of $f(x)$ is the solution to $f'(x) = 0$.
The derivative is$f'(x) = 1 - \frac{625}{(x+5)^2}$.
Setting this equal to zero and solving for x gives $x = -5 \pm 25$, and of these two critical points only -5+25=20 lies in our interval.
Evaluating $f(-5 + 25) = f(20)$ yields the value -80.
The values of the function at the endpoints of the interval are $f(0) = 0$, $f(21) = -79.9615$, so the minimum value is -80, and the maximum value is 0.

18. The function $f(x) = -2x^3 + 21x^2 - 36x + 10$ has one local minimum and one local maximum. Find their (x, y) locations.

To identify any local extrema, we start by identifying critical points. We note that $f(x)$ is a polynomial, so its derivative is defined everywhere, so only points where $f'(x) = 0$ will be critical points.

$$\begin{aligned} & f'(x) = -6x^2 + 42x - 36 \\ \text{Setting } f'(x) = 0, \quad & 0 = -6x^2 + 42x - 36 \\ \text{Factoring,} \quad & 0 = -6(x^2 - 7x + 6) \\ & 0 = -6(x-1)(x-6) \end{aligned}$$

The two critical points are at $x = 1$ and $x = 6$. Subbing those x values back into the original function $f(x)$ gives us the points (1, -7) and $(6, 118)$.

Using test points and the first derivative test, or taking another derivative and using the second derivative test, you can find that:
there is a local minimum at (1,-7), and
there is a local maximum at (6, 118).

19. A University of Rochester student decided to depart from Earth after his graduation to find work on Mars. Before building a shuttle, he conducted careful calculations. A model for the velocity of the shuttle, from liftoff at t = 0 s until the solid rocket boosters were jettisoned at t = 80 s, is given by

$$v(t) = 0.001094333t^3 - 0.08215t^2 + 28.6t - 4.3$$

(in feet per second). Using this model, estimate the global maximum value and global minimum value of the **acceleration** of the shuttle between liftoff and the jettisoning of the boosters.

For simplicity of presentation, let $c_3 = 0.001094333$ and $c_2 = 0.08215$, so

$$v(t) = c_3t^3 - c_2t^2 + 28.6t - 4.3$$

Differentiating once gives the acceleration

$$a(t) = 3c_3t^2 - 2c_2t + 28.6$$

To find the critical points of the acceleration, we need to know when its rate of change is zero:

$$a'(t) = 6c_3t - 2c_2$$

This will have a zero value when

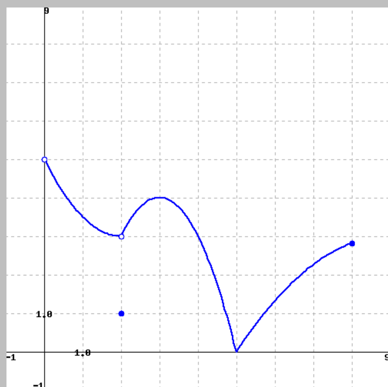
$$\begin{aligned} 0 &= 6c_3t - 2c_2 \\ t &= \frac{2c_2}{6c_3} = \frac{1}{3}\frac{0.08215}{0.001094333} \approx 25.022 \end{aligned}$$

Thus $t \approx 25.022$ is the only critical point.

We compute the acceleration at the end points of the interval ($t = 0$ and 80), and at the critical point:
$a(0) = 28.6$ ft/s^2
$a(25.022) = 26.5444$ ft/s^2
$a(80) = 36.4672$ ft/s^2

The global maximum acceleration is 36.4672 ft/s^2 and occurs at $t = 80$, while the global minimum acceleration is 26.5444 ft/s^2, and it occurs at $t = 25.022$ s.

20. Use the given graph of the function on the interval $(0, 8]$ to answer the following questions.



(a) Where does the function f have a local maximum?

(b) Where does the function f have a local minimum?

(c) What is the global maximum of f?

(d) What is the global minimum of f?

(a) x =3. ($x = 8$ is an end-point, and so is **not** considered a local max or min using our definitions.)

(b) x =2, 5

(c) none: at the left end, the interval is open, so the maximum is never reached.

(d) The global minimum of the function occurs at $x = 5$, and the value there is $f(5) = 0$.

21. Find the global maximum and minimum values of the following function on the given interval.
$f(t) = 4t\sqrt{4 - t^2},\ [-1, 2]$

The global maximum occurs at $x = 1.4142$ and $y = 8$.

The global minimum occurs at $x = -1$, and $y = -6.9282$

22. An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin(\theta) + \cos(\theta)}$$

where μ is a positive constant called the coefficient of friction and where $0 \le \theta \le \pi/2$. Find the value for $\tan\theta$ which minimizes the force. Your answer may depend on W and μ.

To minimize F, we differentiate with respect to θ:

$$F(\theta) = \mu W(\mu \sin(\theta) + \cos(\theta))^{-1}$$

$$\text{so} \quad F'(\theta) = -\frac{\mu W}{(\mu \sin(\theta) + \cos(\theta))^2}(\mu \cos(\theta) - \sin(\theta))$$

Setting the derivative equal to zero to identify critical points,

$$0 = -\frac{\mu W}{(\mu \sin(\theta) + \cos(\theta))^2}(\mu \cos(\theta) - \sin(\theta))$$

$$\begin{aligned} \text{requires } 0 &= (\mu \cos(\theta) - \sin(\theta)) \\ \sin(\theta) &= \mu \cos(\theta) \\ \frac{\sin(\theta)}{\cos(\theta)} &= \mu \\ \tan(\theta) &= \mu \end{aligned}$$

The question asked for the value of $\tan(\theta)$, so we have that now as μ.

The greater the coefficient of friction, μ, the more of our force should be directed upwards rather than forwards, to help minimize the friction effect.

23. Find the exact global maximum and minimum values of the function $f(t) = \frac{3t}{8+t^2}$ if its domain is all real numbers.

Differentiating using the quotient rule gives

$$f'(t) = \frac{3(8 + t^2) - 3t(28t)}{(8 + t^2)^2} = \frac{3(8 - t^2)}{(8 + t^2)^2}.$$

The critical points are the solutions to $\frac{3(8-t^2)}{(8+t^2)^2} = 0$, which are $t = \pm\sqrt{8}$.

Since $f'(t) > 0$ for $-\sqrt{8} < t < \sqrt{8}$ and $f'(t) < 0$ otherwise, there is a local minimum at $t = -\sqrt{8}$ and a local maximum at $t = \sqrt{8}$.

As $t \to \pm\infty$, we have $f(t) \to 0$. Thus, the local maximum at $t = \sqrt{8}$ is a global maximum of $f(\sqrt{8}) = \frac{3\sqrt{8}}{8+8}$, and the local minimum at $t = -\sqrt{8}$ is a global minimum of $f(-\sqrt{8}) = \frac{-3\sqrt{8}}{2(8)}$.

24. A ball is thrown up on the surface of a moon. Its height above the lunar surface (in feet) after t seconds is given by the formula

$$h = 217t - \frac{7}{4}t^2.$$

(a) Find the time that the ball reaches its maximum height.

(b) Find the maximal height attained by the ball.

(a) When the ball reaches its maximum, the velocity will be zero, so we can solve for when velocity $= h'(t) = 0$.

$$h'(t) = 217 - \frac{7}{2}t$$

setting h'=0, $$0 = 217 - \frac{7}{2}t$$

$$t = \frac{2}{7}217 = 62 \text{ s}$$

(b) At the time of zero velocity, the height will be $h(62) = 6727$ m.

25. In a certain chemical reaction, substance A combines with substance B to form substance Y. At the start of the reaction, the quantity of A present is a grams, and the quantity of B present is b grams. Assume $a < b$ and $y \leq a$. At time t seconds after the start of the reaction, the quantity of Y present is y grams. For certain types of reactions, the rate of the reaction, in grams/sec, is given by

$$\text{Rate} = k(a-y)(b-y),$$

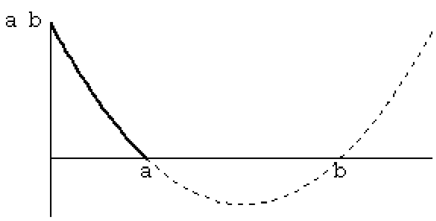
where k is a positive constant.

(a) Sketch a graph of the rate against y.

(b) For what values of y is the rate non-negative?

(c) Use your graph to find the value of y at which the rate of the reaction is fastest.

(a)



(b) If we expect the rate to be non-negative, we must have $0 \leq y \leq a$ or $b \leq y$. Since we assume $a < b$, we restrict y to $0 \leq y \leq a$.

In fact, the expression for the rate is non-negative for y greater than a but these values of y are not meaningful for the reaction. See the figure above (which shows the rate with $k = 1$).

(c) From the graph, we see that the maximum rate occurs when $y = 0$; that is, at the start of the reaction.

26. At what value(s) of x on the curve $y = 1 + 250x^3 - 3x^5$ does the tangent line have the largest slope?

The slope of the tangent line is given by $y' = 750x^2 - 15x^4$.

Consider this to be a new function, $g(x)$, that we want to maximize (to get the *largest* slope). To maximize $g(x)$, we differentiate to find critical points:

$$\begin{aligned} g(x) &= 750x^2 - 15x^4 \\ \text{so } g'(x) &= 1500x - 60x^3 \\ \text{set } g' = 0\text{: } 0 &= 1500x - 60x^3 \\ 0 &= 60x(25 - x^2) \\ 0 &= 60x(5-x)(5+x) \\ x &= 0, 5, -5 \end{aligned}$$

These are the critical points of the slope function. To determine which is a max, and which is a min, we can use either the first or second derivative tests. Let's use the 2nd here because differentiation of g' will be easy:
$g''(x) = 1500 - 180x^2$
$g''(-5) = -3000 < 0$: concave down; $x = -5$ is a local max.
$g''(0) = 1500 > 0$: concave up; $x = 0$ is a local min.
$g''(5) = -3000 < 0$: concave down; $x = 5$ is a local max.

The values of $g(-5) = 9375$ and $g(5) = 9375$ are the slopes of the original function at $x = -5$ and $x = 5$. They are equal, so they are both the common global maximum slope of 9375.

Optimization With MATLAB

For Questions 27-32, use MATLAB to:

- generate a graph of the given function on the domain shown, and
- use the `fminbnd` function to find the global maximum and global minimum of the function on that interval.

27. $f(x) = 7e^{7x^3-7x}$, on $-1 \leq x \leq 0$

All the examples will be solved with the same basic architecture.

In the MATLAB plots,

- the global **min** will be shown as a **red** dot, and
- the global **max** will be shown as a **green** dot.

If there is anything special that is required in the solving, it will be mentioned in text, as well as a commetn in the MATLAB code.

W03OptExp1.m

```
% Optimize 7 e^{7x^3-7x} on [-1, 0]
close all;
f = @(x) 7 * exp(7*x.^3 - 7*x);
x = linspace(-1, 0);
plot(x, f(x));
hold on;

% Basic minimum search
xmin = fminbnd(f, -1, 0);
plot(xmin, f(xmin), '.r', 'markersize', 20);
disp('Global min (x, y) ');
disp([xmin, f(xmin)])

% Max search: look for mins of -f(x)
fn = @(x) -1*f(x);
xmax = fminbnd(fn, -1, 0);
plot(xmax, f(xmax), '.g', 'markersize', 20);
disp('Global max (x, y) ');
disp([xmax, f(xmax)])
```

Final answer:

```
Global min (x, y)
   -0.0001    7.0032

Global max (x, y)
   -0.5774  103.5662
```

28. $f(x) = 7x - 21\ln(x)$, on $[1, 4]$

For all the following problems, we will just provide a link to the solution MATLAB file, and the final answer, rather than including it in the PDF.

Looking at the graph, this function has a clear global max at the left boundary of the interval, and a global min at the critical point at $x = 1$.

W03OptLn1.m

```
Global min (x, y)
    3.0000   -2.0709

Global max (x, y)
    1.0000    6.9994
```

29. $f(x) = 4e^{-x} - 4e^{-2x}$, on $[0, 1]$

Looking at the graph, this function has a clear global max at the left boundary of the interval, and a global min at the critical point at $x \approx 0.6931$ (exact value is $x = \ln(2)$).

W03OptExp2.m

Notice that MATLAB doesn't return *exactly* $x = 0$ for the global minimum, but just the very small $x = 0.0661 \times 10^{-3} = 0.0000661$. This kind of slight deviation from the exact value is common with numerical methods.

```
Global min (x, y)
   1.0e-03 *

    0.0661    0.2644

Global max (x, y)
    0.6931    1.0000
```

30. $f(x) = 7x - 14\cos(x)$, on $[-\pi, \pi]$

Looking at the graph, this function has

- a clear global max at the right boundary of the interval, and
- a global min at the critical point at $x \approx -0.5236$.

There is a local max around $x = -2.8$ that isn't found by `fminbnd` when the interval is $[-\pi, pi]$.

W03OptTrig1.m

```
Global min (x, y)
   -0.5236  -15.7895

Global max (x, y)
    3.1415   35.9907
```

31. $f(x) = (3\cos x)/(20 + 10\sin x)$, $0 \leq x \leq 2\pi$

This example requires some care, as the default search with `fminbnd` does **not** find the correct global maximum.

W03OptTrig2.m

```
Global min (x, y)
    3.6652   -0.1732

Endpoint (x, y)
    0.0001    0.1500

Global max (x, y)
    5.7596    0.1732
```

32. $f(t) = \frac{10}{t} + 4, 0 < t \leq 1$
Note the open end due to the $0 < t$ instead of $0 \leq t$.

Variables: note that you can choose to change the x variables in your script to `t`, or just note that the function $f(x) = \dfrac{10}{x} + 4$ has all the same properties as $f(t) = \dfrac{10}{t} + 4$.

Open Interval: Even though we are working on an open interval, we can still try to use `fminbnd`. We just need to be careful when running it, and interpreting its results.

W03OptRecip1.m

If you run your `fminbnd` with $t = 0$ as a boundary, you will get an error:

```
Error using fminbnd (line 219)
User supplied objective function must return a scalar value.
```

This points to the fact that $f(0)$ is actually undefined (dividing by zero). As a result, the best you can do is try to use a small but non-zero left limit, but then check your results against the graph to be sure.

```
Global min (x, y)
    0.9999   14.0005
```

```
Global max (x, y)
   1.0e+04 *

    0.0000    6.0209
```

However, the global max is actually incorrect. This function won't have a global max, because there is a vertical asymptote at $t = 0$, so there is no single highest point for this function.

Optimization Word Problems

33. Some airlines have restrictions on the size of items of luggage that passengers are allowed to take with them. Suppose that one has a rule that the sum of the length, width and height of any piece of luggage must be less than or equal to 192 cm. A passenger wants to take a box of the maximum allowable volume.

    (a) If the length and width are to be equal, what should the dimensions be?

    (b) In this case, what is the volume?

    (c) If the length is be twice the width, what should the dimensions be?

    (d) In this case, what is the volume?

    Include units in all your answers.

Let the length, width and height of the box be L, w and h, respectively. Then the volume of the box is $V = Lwh$. The sum $L + w + h = 192$, and, for the first part, we know that $L = w$. Thus $2w + h = 192$, so $h = 192 - 2w$, and the volume equation becomes $V = Lwh = w \cdot w \cdot (192 - 2w) = 192 \cdot w^2 - 2w^3$. Since we need $h \geq 0$ and $h = 192 - 2w$, the domain for w is $0 \leq w \leq 96$.

Critical points are where $\frac{dV}{dw} = 2 \cdot 192 \cdot w - 6 \cdot w^2 = 0$, so $w = 0$ or $w = 64$. The global maximum must occur either at this point or at the end points. $V(64) > 0$ while $V(0) = V(96) = 0$, so the global maximum is at $L = w = 64$, in which case $h = 64$ as well. The volume is then $V = 64^3 = 262144\text{cm}^3$.

If $L = 2w$, $2w + w + h = 192$, so $V = (2w)(w)(192 - 3w) = 384w^2 - 6w^3$. Proceeding as before, we find $w = \frac{128}{3}$, $L = \frac{256}{3}$ and $h = 64$, so that $V = \frac{2097152}{9}$.

34. A wire 3 meters long is cut into two pieces. One piece is bent into a square for a frame for a stained glass ornament, while the other piece is bent into a circle for a TV antenna.

    (a) To reduce storage space, where should the wire be cut to **minimize** the total area of both figures?

    (b) Where should the wire be cut to **maximize** the total area?

(a) Note that we are interested in *the total area enclosed by the two figures* . Our first task is therefore to find an equation for this area, which will be the sum of the areas of the two figures.

    Suppose we cut x meters of wire to make the circular antenna. Then there are $3 - x$ meters left for the square. To find the area of the circle we need its radius. The circumference of a circle of radius r is $C = 2\pi r$, so the radius of the circle is given by $2\pi r = x$, and so $r = \frac{x}{2\pi}$. The area of the circular antenna is therefore $A_c = \pi r^2 = \frac{1}{4\pi}x^2$.
    Then the perimeter of a square with side length s is $P = 4s = 3-x$, so the side length is $s = \frac{1}{4}(3-x)$. Then the area of a square is $A_s = s^2$, so the area of the square is $A_s = (\frac{1}{16})(3 - x)^2$.
    The total area is therefore

    $$A = \frac{1}{4\pi}x^2+(\frac{1}{16})(3-x)^2 = \frac{1}{4\pi}x^2+\frac{1}{(16)}(9-6x+x^2).$$

    The domain for x is $0 \leq x \leq 3$.
    The maximum and minimum of A will occur at critical or end points. Critical points are where $dA/dx = 0$, or, where

    $$\frac{1}{2\pi}x + \frac{1}{(16)}(2x - 6) = 0.$$

    Collecting all terms in x we have

    $$\left(\frac{1}{2\pi} + \frac{2}{(16)}\right) x = \frac{6}{16},$$

so, after simplifying,

$$x = \frac{3\pi}{4+\pi}.$$

To determine if this is a local maximum or minimum, we use the second derivative test.

$$A'' = \left(\frac{1}{2\pi} + \frac{2}{(16)}\right) > 0,$$

so the function is concave up everywhere and this is a local minimum. Also, because this is the *only* critical point, this is also a *global* minimum for the area.

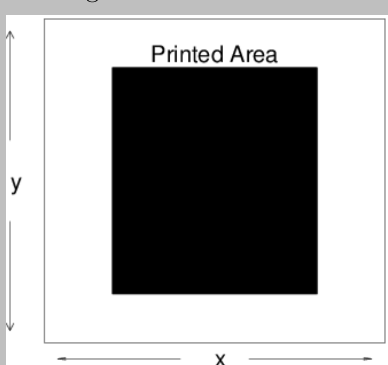
Thus to minimize area we use $\frac{3\pi}{4+\pi}$ meters of wire for the circle and $3 - \frac{3\pi}{4+\pi}$ meters for the square.

(b) To *maximize* the area, we can't use our critical point, which was a minimum; instead we must use the endpoints. The areas at the endpoints are

$$A(0) = \frac{9}{16} \approx 0.56 \qquad \text{and} \qquad A(3) = \frac{9}{4\pi} \approx 0.72,$$

the larger of which is $A(3)$, so the maximum area occurs when all of the wire is used for the circle and none for the square.

35. A printed poster is to have a total area of 799 square inches with top and bottom margins of 6 inches and side margins of 4 inches. What should be the dimensions of the poster so that the printed area be as large as possible? Let x denote the width of the poster and let y denote the length.



(a) Write the function of x and y that you need to maximize.

(b) Express that function in terms of x alone.

(c) Find the critical points of the function.

(d) Use the second derivative test to verify that $f(x)$ has a maximum at this critical point

(e) Find the optimal dimensions of the poster, and the resulting area. Include units.

(a) Area $= A = (x - 2\cdot 4)(y - 2\cdot 6) = (x-8)(y-12)$

(b) By using the requirement that $799 = xy$, we get $A = (x-8)(799/x - 12)$

(c) Differentiating and setting the derivative equal to zero, we obtain $x = 23.08$.

(d) The second derivative of A will be negative at $x = 23.08$, so A is concave down there, indicating $x = 23.08$ is a local maximum for the printed area.

(e) The dimensions of the poster with the largest printed area will be 23.08×34.62, with a net printed area of 341.09 cm.

36. A box with an open top has vertical sides, a square bottom, and a volume of 32 cubic meters. If the box has the least possible surface area, find its dimensions.

This is the same question studied in the course videos. Let the dimensions of the box be w and h; the bottom is square so w can represent the length of both sides of the bottom. These combine to produce

$$A = w^2 + 2(wh) + 2(wh), \qquad V = w^2 h = 32$$

Solving for h in the V equation, $h = 32/w^2$, we can write A as just a function of w:

$$A = w^2 + 4w(32/w^2)$$
$$A = w^2 + 128/w$$

Differentiating and setting $A' = 0$, you will find $w = 4$, and consequently $h = 32/w^2 = 2$.

37. Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle with sides of length 2 if one side of the rectangle lies on the base of the triangle.

The optimal rectangle will be 1 unit on the base, and have height $\frac{\sqrt{3}}{2}$.

38. Find the minimum distance from the parabola

$$x - y^2 = 0$$

to the point (0,3).

The point of closest approach will occur at $y = 1$, and that will give a distance of $\sqrt{1^2 + (3-1)^2} = \sqrt{5}$.

39. I have enough pure silver to coat 2 square meters of surface area. I plan to coat a sphere and a cube.

    (a) Allowing for the possibility of all the silver going onto one of the solids, what dimensions should they be if the total volume of the silvered solids is to be a maximum?

    (b) Now allowing for the possibility of all the silver going onto one of the solids, what dimensions should they be if the total volume of the silvered solids is to be a minimum?

Let s be the length of the cube, A the total area, and V the total volume. Then

$$A = 4\pi r^2 + 6s^2$$

and

$$V = \frac{4}{3}\pi r^3 + s^3.$$

Obviously, the volume is maximized if we put all of our stock in the sphere. In that case,

$$r = \sqrt{\frac{A}{4\pi}} \approx 0.3989 \text{ meters}$$

(and $s = 0$ meters). To minimize the volume we eliminate one of the variables and find a stationary point as usual. Solving the area equation for s gives

$$s = \sqrt{\frac{A - 4\pi r^2}{6}}.$$

Substituting this value in the volume equation gives

$$V = \frac{4}{3}\pi r^3 + \left(\frac{A - 4\pi r^2}{6}\right)^{\frac{3}{2}}.$$

Differentiating with respect to r and setting to zero gives:

$$V' = 4\pi r^2 - \frac{3}{2} \times \frac{8\pi r}{6}\left(\frac{A - 4\pi r^2}{6}\right)^{\frac{1}{2}} = 0.$$

This simplifies to

$$2r = \sqrt{\frac{A - 4\pi r^2}{6}}.$$

Squaring gives

$$4r^2 = \frac{A - 4\pi r^2}{6}$$

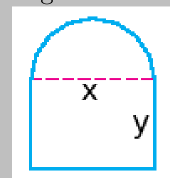
which gives

$$r = \sqrt{\frac{A}{24 + 4\pi}} \approx 0.2339 \text{ meters}.$$

The corresponding value of s is

$$s = 2r = \sqrt{\frac{A}{6 + \pi}} \approx 0.4677 \text{ meters}.$$

40. Suppose that 241 ft of fencing are used to enclose a corral in the shape of a rectangle with a semicircle whose diameter is a side of the rectangle as the following figure:



Find the dimensions of the corral with maximum area.

From the picture, we see that x is the width of the corral, and therefore the diameter of the semicircle, and that y is the height of the rectangular section. Thus the perimeter of the corral can be expressed by the equation $2y + x + \frac{\pi}{2}x = 2y + (1 + \frac{\pi}{2})x = 241$ ft or equivalently, $y = \frac{1}{2}(241 - (1 + \frac{\pi}{2})x)$. Since x and y must both be non-negative, it follows that x must be restricted to the interval $[0, \frac{241}{1+\pi/2}]$. The area of the corral is the sum of the area of the rectangle and semicircle, $A = xy + \frac{\pi}{8}x^2$. Making the substitution for y from the constraint equation,

$A(x) = \frac{1}{2}x(241 - (1 + \frac{\pi}{2})x) + \frac{\pi}{8}x^2 = 120.5x - \frac{1}{2}(1 + \frac{\pi}{2})x^2 + \frac{\pi}{8}x^2$.
Now, $A'(x) = 120.5 - (1 + \frac{\pi}{2})x + \frac{\pi}{4}x = 0$ implies $x = \frac{120.5}{(1+\frac{\pi}{4})} \approx 67.4919$.
With $A(0) = 0$,

$A(\frac{120.5}{(1+\frac{\pi}{4})}) \approx 4066.39$ and $A(\frac{241}{1+\frac{\pi}{2}}) \approx 3451.11$,
it follows that the corral of maximum area has dimensions

$x = \frac{120.5}{1+\frac{\pi}{4}}$ and
$y = \frac{1}{2}(241 - (1 + \frac{\pi}{2})\frac{120.5}{1+\frac{\pi}{4}}) \approx 33.746$.

41. A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$4 per square foot and the metal for the sides costs \$6 per square foot. Find the dimensions that minimize cost if the box has a volume of 35 cubic feet.

Let $x > 0$ be the length of a side of the square base and $z > 0$ the height of the box. With volume $x^2z = 35$, we have $z = 35/x^2$ and cost

$$C(x) = 4 \cdot 2 \cdot x^2 + 6 \cdot 4 \cdot xz = 8x^2 + 840\frac{1}{x}.$$

Solve $C'(x) = 8 \cdot 2x - 840x^{-2} = 0$ to obtain $x = \left(\frac{35\cdot 6}{4}\right)^{1/3}$. Since $C(x) \to \infty$ as $x \to 0+$ and as $x \to \infty$, the minimum cost is $C\left(\left(\frac{35\cdot 6}{4}\right)^{1/3}\right) \approx \$336.499$ when $x \approx 3.74444$ ft and $z \approx 2.49629$ ft.

42. Centerville is the headquarters of Greedy Cablevision Inc. The cable company is about to expand service to two nearby towns, Springfield and Shelbyville. There needs to be cable connecting Centerville to both towns. The idea is to save on the cost of cable by arranging the cable in a Y-shaped configuration. Centerville is located at $(8, 0)$ in the xy-plane, Springfield is at $(0, 5)$, and Shelbyville is at $(0, -5)$. The cable runs from Centerville to some point $(x, 0)$ on the x-axis where it splits into two branches going to Springfield and Shelbyville. Find the location $(x, 0)$ that will minimize the amount of cable between the 3 towns and compute the amount of cable needed. Justify your answer.

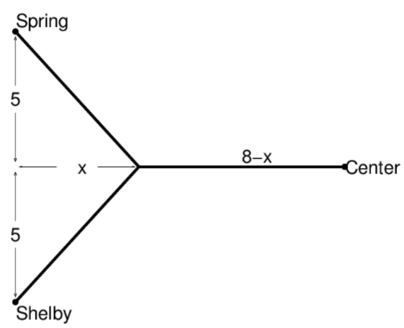
    (a) What function of x needs to be minimized to solve this problem?

    (b) Find the critical points of $f(x)$.

    (c) Use the second derivative test to verify that $f(x)$ has a minimum at this critical point.

    (d) Compute the minimum amount of wire needed.

(a) Draw a sketch.



With x being the horizontal component of the diagonal lines, the total length of the cable will be $L(x) = 2\sqrt{x^2 + 5^2} + (8 - x)$.

(b) Taking the derivative and finding critical points of $L(x)$ yields $x = 2.89$.

(c) The second derivative of $L(x)$ will be positive at $x = 2.89$, indicating that the critical point is a local minimum for the length of cable.

(d) $L(2.89) = 2\sqrt{(2.89)^2 + 25} + (8 - 2.89) = 16.66$ units of cable.

43. A cylinder is inscribed in a right circular cone of height 4 m and radius (at the base) equal to 3.5 m. What are the dimensions of such a cylinder which has maximum volume?

As we are attempting to maximize the volume of the inscribed cylinder, we must first come up with a formula for the volume of this cylinder. Let x be the radius of the cylinder, $v(x)$ the volume. We know from basic geometry that the formula for volume is give by $\pi x^2 h$ where x is the radius and h is the height of the cylinder. So in order to come up with a formula for volume in terms of x only, we need to relate x to h.

This is where the information about the cone comes in handy. The cone is a right circular cone. Thus, inscribing the cylinder will fill up some of the base of the cone, and just touch the slanted side, leaving a similar right circular cone at the top. This new cone will have a radius of x and a height of $4-h$ where x and h are as in the formula for the volume of our cylinder. As this cone is similar to the original, we can use ratios to get:

$$\frac{x}{3.5} = \frac{4-h}{4}$$

Simplifying this, we get $h = 4 - \frac{4}{3.5}x$. Therefore, our formula for volume in terms of x becomes $v(x) = \pi x^2(4 - \frac{4}{3.5}x) = (4\pi)x^2 - (\frac{4}{3.5}\pi)x^3$

Now, we want to maximize this. So we will first take the derivative. Using the rules for differentiation of polynomials, the derivative is $v'(x) = 2(4\pi)x - 3(\frac{4}{3.5}\pi)x^2$. Solving for zero, we get, as we don't want $x = 0$, the following.

$$\begin{aligned} v'(x) &= 0 \\ 2(4\pi)x - 3(\frac{4}{3.5}\pi)x^2 &= 0 \\ \pi x(2(4) - 3\frac{4}{3.5}x) &= 0 \\ 2(4) - 3\frac{4}{3.5}x &= 0 \\ 3\frac{4}{3.5}x &= 2(4) \\ x &= \frac{2}{3}(3.5) = 2.333 \text{ m} \end{aligned}$$

Then, using the formula for height we came up with before, the height can be determined by:

$$h = 4 - \frac{4}{3.5}(2.333) = 1.333 \text{ m}$$