MNTC P01 - Week #7 - Differential Equations - Introduction

Verifying Solutions

1. Show that $y = \frac{2}{3}e^x + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^x$.

$$y = \frac{2}{3}e^x + e^{-2x} \Rightarrow y' = \frac{2}{3}e^x - 2e^{-2x}$$

To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left-hand side of the equation and show that the left-hand side of the equation and show that the left-hand side is equal to the right-hand side.

$$\begin{aligned} \text{LHS} = y' + 2y &= \frac{2}{3}e^x - 2e^{-2x} + 2(\frac{2}{3}e^x + e^{-2x}) \\ &= \frac{2}{3}e^x - 2e^{-2x} + \frac{4}{3}e^x + 2e^{-2x} = \frac{6}{3}e^x = 2e^x \\ &= \text{RHS} \end{aligned}$$

2. (a) For what values of r does the function $y = e^{rx}$ satisfy the differential equation $2y'' + y' - y = 0$?
   (b) If r_1 and r_2 are the values of r that you found in part (a), show that every member of the family of functions $y = ae^{r_1 x} + be^{r_2 x}$ is also a solution.

(a)

$$y = e^{rx} \Rightarrow y' = re^{rx} \Rightarrow y'' = r^2 e^{rx}$$

Substituting these expressions into the differential equation $2y'' + y' - y = 0$, we get

$$\begin{aligned} & 2r^2 e^{rx} + re^{rx} - e^{rx} = 0 \\ \Rightarrow \quad & (2r^2 + r - 1)e^{rx} = 0 \\ \Rightarrow \quad & (2r - 1)(r + 1) = 0 \end{aligned}$$

(since e^{rx} is never zero) $r = \frac{1}{2}$ or -1.

(b) Let $r_1 = \frac{1}{2}$ and $r_2 = -1$, so we need to show that every member of the family of functions $y = ae^{x/2} + be^{-x}$ is a solution of the differential equation $2y'' + y' - y = 0$.

$$\begin{aligned} & y = ae^{x/2} + be^{-x} \\ \Rightarrow \quad & y' = \frac{1}{2}ae^{x/2} - be^{-x} \\ \Rightarrow \quad & y'' = \frac{1}{4}ae^{x/2} + be^{-x} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= 2y'' + y' - y \\ &= 2(\frac{1}{4}ae^{x/2} + be^{-x}) + (\frac{1}{2}ae^{x/2} - be^{-x}) \\ &\quad - (ae^{x/2} + be^{-x}) \\ &= \frac{1}{2}ae^{x/2} + 2be^{-x} + \frac{1}{2}ae^{x/2} - be^{-x} \\ &\quad - ae^{x/2} - be^{-x} \\ &= (\frac{1}{2}a + \frac{1}{2}a - a)e^{x/2} + (2b - b - b)e^{-x} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

3. (a) For what values of k does the function $y = \cos(kt)$ satisfy the differential equation $4y'' = -25y$?
   (b) For those values of k, verify that every member of the vamily of functions $y = A \sin kt + B \cos kt$ is also a solution.

(a)

$$y = \cos kt \;\Rightarrow\; y' = -k \sin kt \;\Rightarrow\; y'' = -k^2 \cos kt$$

Substituting expressions into the differential equation $4y'' = -25y$, we get

$$\begin{aligned} & 4(-k^2 \cos kt) = -25(\cos kt) \\ \Rightarrow\; & (25 - 4k^2) \cos kt = 0 (\text{for all } t) \\ \Rightarrow\; & 25 - 4k^2 = 0 \\ \Rightarrow\; & k^2 = \frac{25}{4} \;\Rightarrow\; k = \pm\frac{5}{2} \end{aligned}$$

(b)

$$\begin{aligned} & y = A \sin kt + B \cos kt \\ \Rightarrow\; & y' = Ak \cos kt - Bk \sin kt \\ \Rightarrow\; & y'' = -Ak^2 \sin kt - Bk^2 \cos kt \end{aligned}$$

The given differential equation $4y'' = -25y$ is equivalent to $4y'' + 25y = 0$. Thus,

$$\begin{aligned} \text{LHS} &= 4y'' + 25y \\ &= 4(-Ak^2 \sin kt - Bk^2 \cos kt) \\ &\quad + 25(A \sin kt + B \cos kt) \\ &= -4Ak^2 \sin kt - 4Bk^2 \cos kt \\ &\quad + 25A \sin kt + 25B \cos kt \\ &= (25 - 4k^2) A \sin kt + (25 - 4k^2) B \cos kt \\ &= 0 \quad \text{since } k^2 = \frac{25}{4} \end{aligned}$$

4. Consider the differential equation $\dfrac{dy}{dx} = -y^2$.
   (a) If you were asked whether the solutions to this equation would *increase* or *decrease* as x increased, what could you say based on only the equation itself?
   (b) Verify that all members of the family $y = 1/(x + C)$ are solutions of the equation in part (a).
   (c) Can you think of a (very simple) solution of the differential equation $y' = -y^2$ that is *not* a member of the family in part (b)?
   (d) Find the solution to the initial-value problem

$$y' = -y^2 \qquad y(0) = 0.5$$

(a) Since the derivative of $y' = -y^2$ is always negative (or 0 if $y = 0$), the function y must be **decreasing** (or maybe horizontal) on any interval on which it is defined.

(b) We sub in the proposed solution into the original equation. To do this, we will need the derivative of y: $y = \dfrac{1}{x + C} \Rightarrow y' = -\dfrac{1}{(x + C)^2}$.
LHS $= y' = -\frac{1}{(x+C)^2} = -\left(\frac{1}{x+C}\right)^2 = -y^2 =$ RHS Therefore, any function of the form $y(x) = \frac{1}{x+C}$ **is** a solution to $y' = -y^2$.

(c) $y = 0$ is a simple solution to $y' = -y^2$ that is not a member of the family in part (b). We can confirm this by subbing $y = 0$ into the DE and checking the LHS equals the RHS. If $y = 0$, then $y' = 0$ as well, so LHS $= y' = 0 = -y^2 = RHS$.

(d) We already know that the solutions will be of the form $y(x) = \frac{1}{x+C}$; we just need to sub in the initial value to solve for C.

If $y(x) = \frac{1}{x+C}$, then $y(0) = \frac{1}{0+C} = \frac{1}{C}$. Since $y(0) = 0.5$, $\frac{1}{C}\frac{1}{2} \Rightarrow C = 2$, so $y = \frac{1}{x+2}$

Numerical ODE Solutions With MATLAB

5. Create a plot for the solution to the differential equation $y' - \frac{y^2}{x^3} = 0$ if y(2) = 1. Include a large enough `xspan` to see the long-term behaviour.

For this first example of use MATLAB to build a numerical solution to a DE, we will show the full listing of a script that generates a solution to the given differential equation. In later solutions, we will only include the key lines for the MATLAB script.

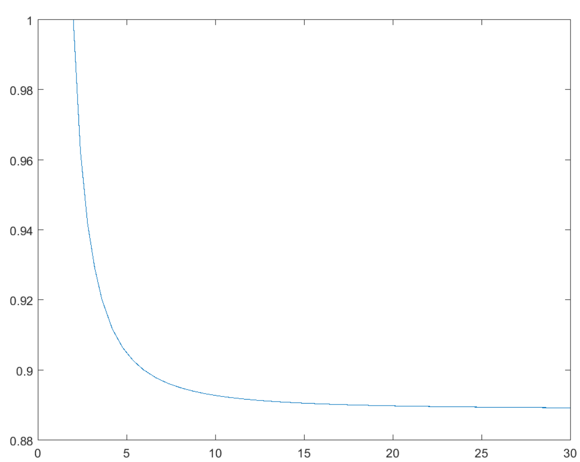
Notes:

- We set `xspan` to start at 2 in the line `xspan = [2, 30]`. This is used because the solution MATLAB is generating will start at the coordinates x_0 = first element of `xspan`, and y_0 = `y0` in the code, and our initial condition is $x = 2$, $y = 1$.
- We find the second value in the time span with some trial and error. Any value larger than 15 or 20 would be sufficient to show the long-term trend in the solution.

```
% ode45 solution to y' = - y^2/x^3, y(1) = 1
close all;
xspan = [2, 30]; % must start at x=2, from y(2) = 1
y0 = 1; % = y value at the start of xspan; y(2) = 1
[x, y] = ode45( @(x, y) -y.^2./x.^3, xspan, y0); % have MATLAB solve the DE
plot(x, y);
```

Link to the MATLAB code:
W07DE01.m

Here is the graph of the solution.



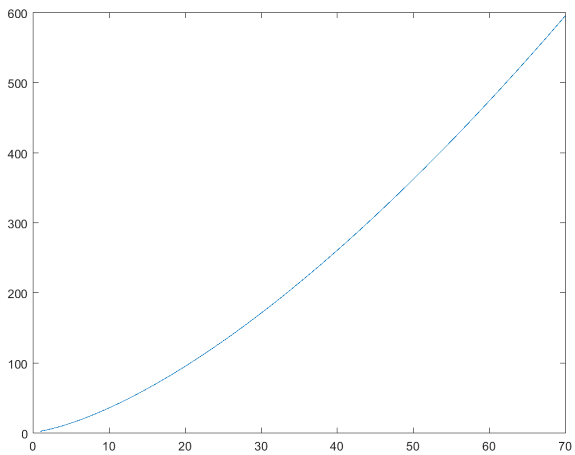
6. Create a plot for the solution to the differential equation $(2y - 4)y' - 3x^2 = 4x - 4$, if y(1) = 3.

To generate a first-order DE solution in MATLAB, the differential equation must be written first in the form $y' = \ldots$.

$$\begin{aligned}(2y-4)y' - 3x^2 &= 4x - 4\\ (2y-4)y' &= 3x^2 + 4x - 4\\ y' &= \frac{(3x^2+4x-4)}{(2y-4)}\end{aligned}$$

Link to the MATLAB code:
W07DE02.m

Here is the graph of the solution.

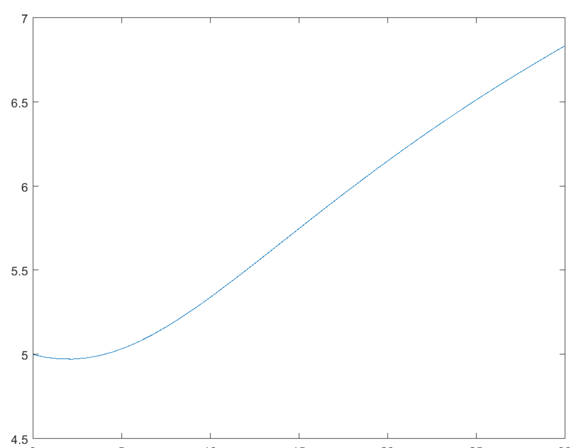


> 7. Create a plot for the solution to the differential equation $y' = e^{-y}(2t-4)$ if $y(0) = 5$

This DE is already in the form $y' = \ldots$, so we can input it into MATLAB as-is. Note that the independent variable in this example is t, so we will use that in MATLAB instead of the variable x.

Link to the MATLAB code:
W07DE03.m

Here is the graph of the solution.



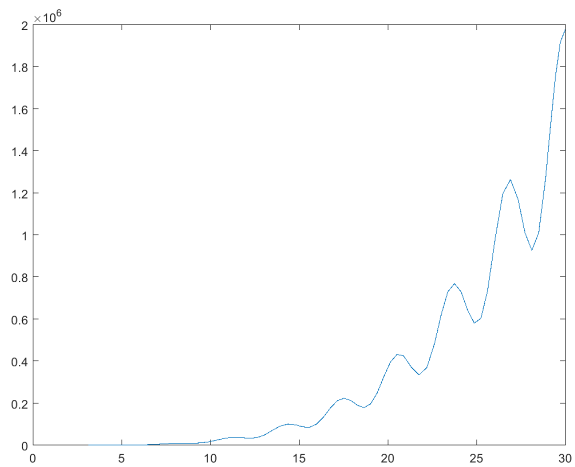
> 8. Create a plot for the solution to the differential equation $ty' - 2y = t^5\sin(2t) - t^3 + 4t^4$, if $y(\pi) = \frac{3}{2}\pi^4$

To generate a first-order DE solution in MATLAB, the differential equation must be written first in the form $y' = \ldots$.

$$
\begin{aligned}
ty' - 2y &= t^5 \sin(2t) - t^3 + 4t^4 \\
ty' &= 2y + t^5 \sin(2t) - t^3 + 4t^4 \\
y' &= \frac{1}{t}(2y + t^5 \sin(2t) - t^3 + 4t^4)
\end{aligned}
$$

Link to the MATLAB code:
W07DE04.m

Here is the graph of the solution.



Note that in this example, because of the $\sin(2t)$ introducing an oscillation in the system, the solution won't look at simple as some of the other examples.

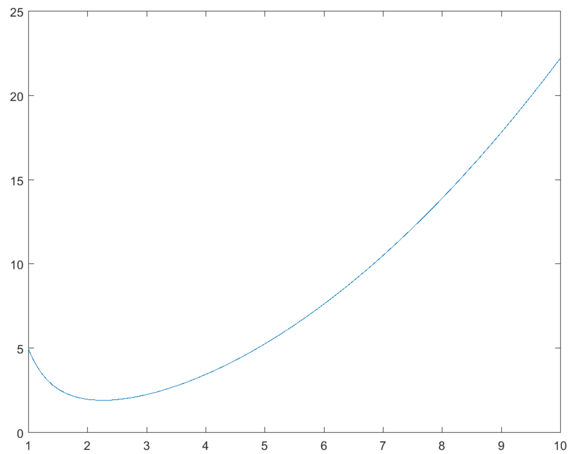
9. Create a plot for the solution to the differential equation $ty' + 2y = t^2 - t + 1$, if $y(1) = 0.5$.

To generate a first-order DE solution in MATLAB, the differential equation must be written first in the form $y' = \ldots$.

$$
\begin{aligned}
ty' + 2y &= t^2 - t + 1 \\
ty' &= -2y + t^2 - t + 1 \\
y' &= \frac{1}{t}(-2y + t^2 - t + 1)
\end{aligned}
$$

Link to the MATLAB code:
W07DE05.m

Here is the graph of the solution.



10. Create a plot for the solution to the differential equation $2xy^2 + 4 = 2(3 - x^2y)y'$ if $y(5) = 8$.

To generate a first-order DE solution in MATLAB, the differential equation must be written first in the form $y' = \ldots$. We start by switching both sides of the equation to put y' on the left.

$$2(3 - x^2y)y' = 2xy^2 + 4$$
$$y' = \frac{2xy^2 + 4}{2(3 - x^2y)}$$

Link to the MATLAB code:
W07DE06.m

Here is the graph of the solution.