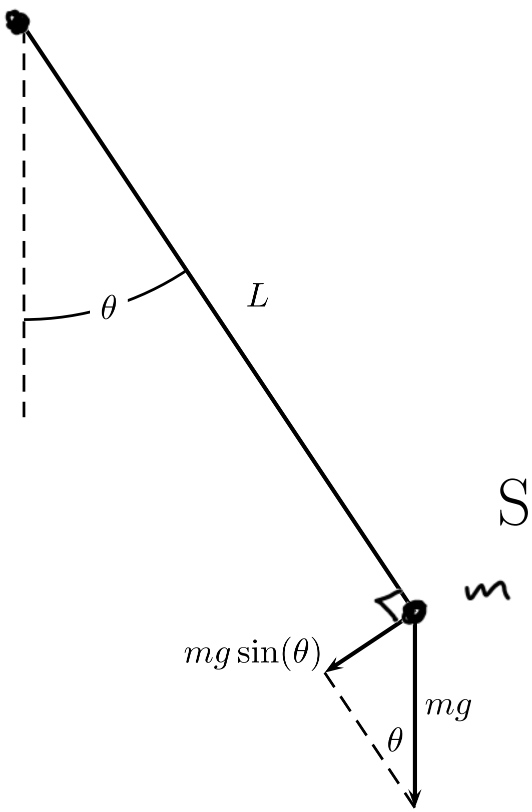
Week #9 : Differential Equations and Engineering

Goals:

- Take problems that can be modeled by differential equations, both first and second order, and give solutions both by hand and MATLAB
- Examine case studies of differential equations applied to engineering problems and reproduce those solutions

Application - Pendulum



Newton's Second Law:

$$mL^2\theta'' = T_g + T_f$$

Σ torques

angular accel

$$= -mLg\sin(\theta) - (\mu L^2 m)\theta'$$

angular velocity

Solving for θ'': $\theta'' = -\frac{g}{L}\sin(\theta) - \mu\theta'$ DE

ω_1 ω_2

2nd order DE

Problem. Turn this single second-order DE into a pair of first-order DEs.

let $\omega_1 = \theta$

and $\omega_2 = \theta'$

So

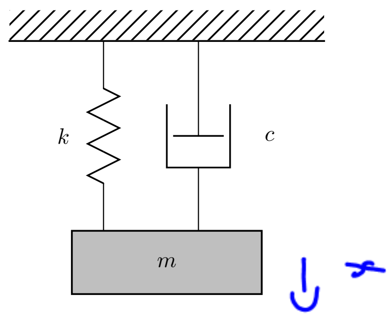
$$\frac{d\omega_1}{dt} = \theta' = \omega_2$$

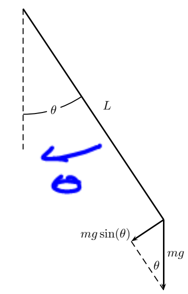
DE

$$\frac{d\omega_2}{dt} = (\theta')' = \theta'' = -\frac{g}{L}\sin(\omega_1) - \mu\,\omega_2$$

System of 1st order DEs

Problem. Compare the system of differential equations we obtained to the equations that define the motion of the damped spring/mass system.





$$\frac{dw_1}{dt} = w_2$$
$$\frac{dw_2}{dt} = \left(\frac{1}{m}\right)(-kw_1 - cw_2)$$

$$\frac{dw_2}{dt} = -d_1 w_1 - d_2 w_2$$

$$\frac{dw_1}{dt} = w_2$$
$$\frac{dw_2}{dt} = -\left(\frac{g}{L}\right)\sin(w_1) - \mu w_2$$

$$\frac{dw_2}{dt} = -a_1 \sin(w_1) - a_2 w_2$$

Problem. Create a new MATLAB function file called `pendulumDE.m`. Start with the first line

```
function dw_dt = pendulumDE(t, w, g, L, mu)
```

In the body of the function, implement the system of differential equations

$\vec{w} = [w_1, w_2]$

$$\frac{dw_1}{dt} = w_2$$

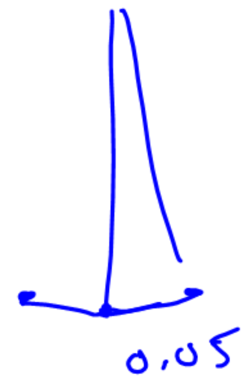
$$\frac{dw_2}{dt} = -\frac{g}{L}\sin(w_1) - \mu w_2$$

w(1), w(2)

Problem. Write a MATLAB script that simulates the motion of the pendulum using

ode45

$g = 9.8$ m/s^2, $L = 2$ m, $\mu = 0.1$, and
initial amplitude of 0.05 radians (≈ 2.9 degrees).
Generate a plot of the resulting angular position over time.



Pendulum - Period of Swings

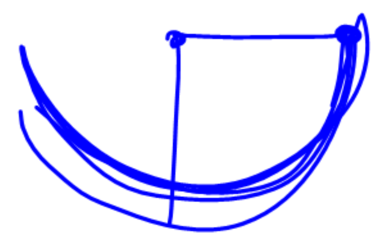
Galileo famously noticed the consistent period of pendulum swings, even if the amplitude of the swings was changed (so the actual distance travelled was different).

= period

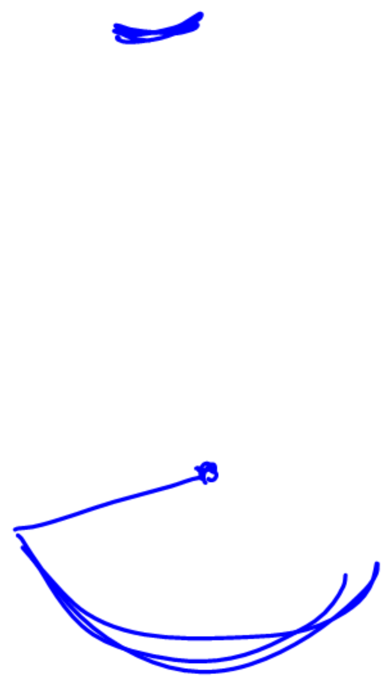
Problem. Compare the periods of the pendulum swings, using a range of initial angles from $\theta_0 = 0.05$ radians up to $\theta_0 = 0.25$ radians (≈ 14 degrees).

However, it turns out that pendulums are **not** perfectly consistent in their period, due to the non-linear term $-\frac{g}{L}\underbrace{\sin}_{\text{non-linear}}(\theta)$ in one of the forces: as the amplitudes get bigger, there is a gradual lengthening of the period.

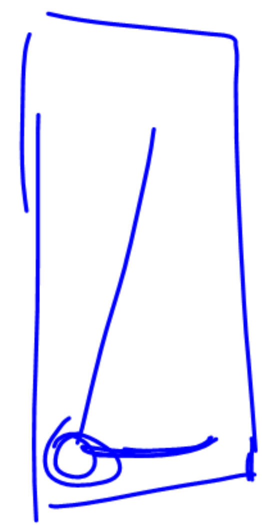
Problem. Compare the periods of the pendulum swings, using a range of initial angles from $\theta_0 = 0.25$ radians up to $\theta_0 = \frac{\pi}{2}$ radians ($= 90$ degrees).



Problem. Use these observations to explain the designs you see for pendulum-based clocks.

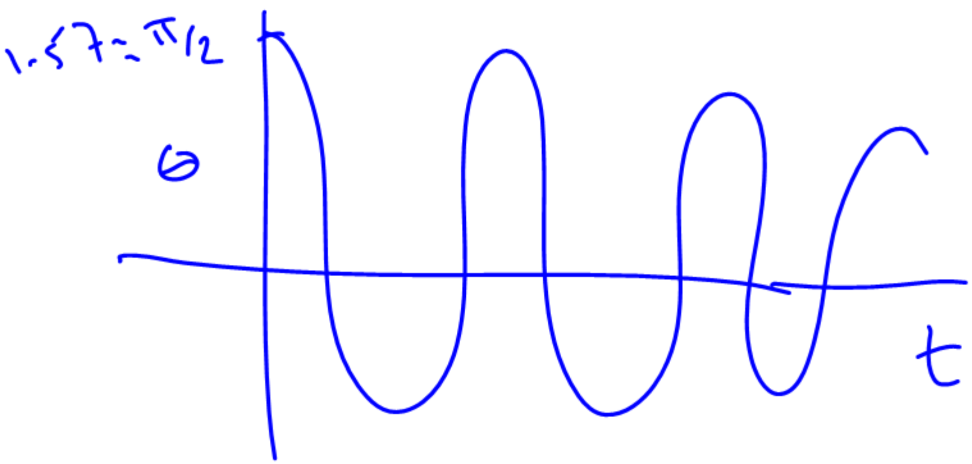


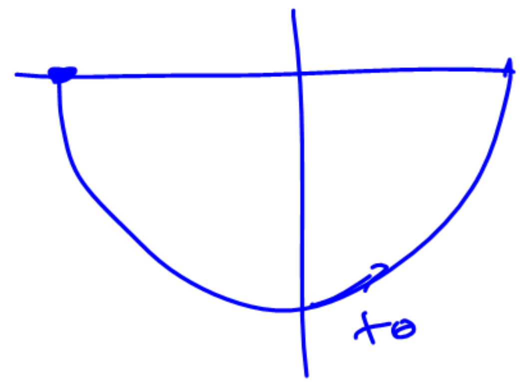
accurate clock



Pendulum - Including an Initial Velocity

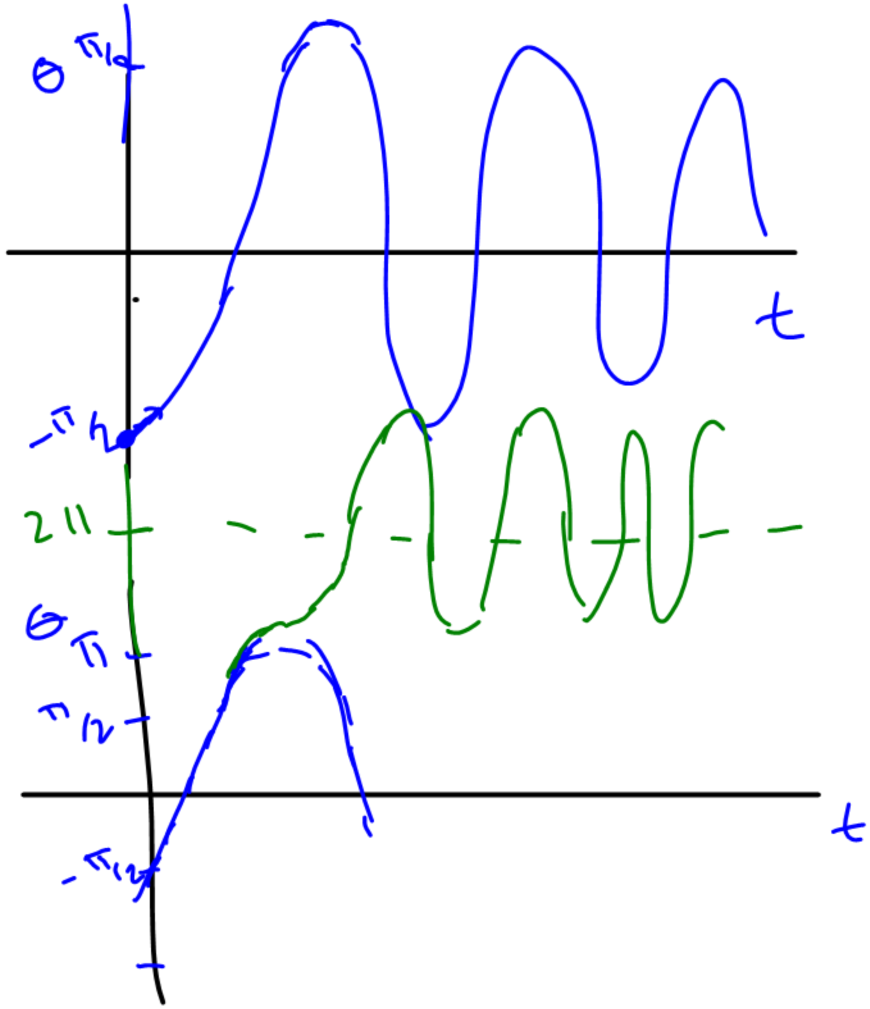
Problem. Write a new simulation script that starts the pendulum swinging from $\theta_0 = -\frac{\pi}{2}$, with no initial velocity. Simulate the motion for this scenario and generate a graph of the angle against time. Use the parameters $g = 9.8$ m/s^2, $L = 2$ m, and $\mu = 0.1$.



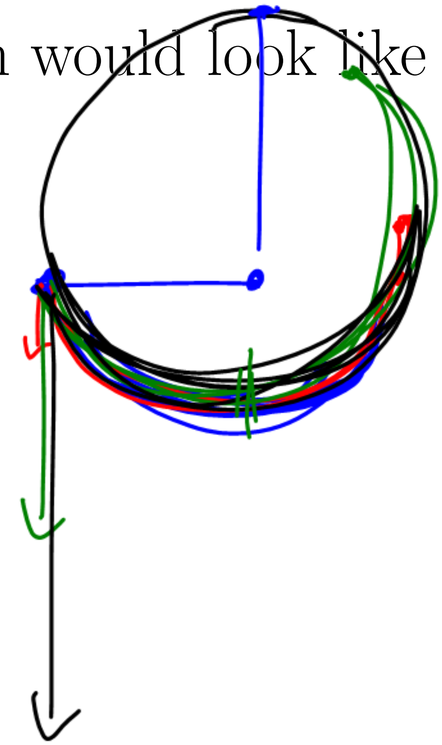


If we add a high enough initial ‘kick’, or initial velocity, it would be possible to make the mass of the pendulum go “over the top”, or above the point of rotation.

Problem. Sketch what the anglular position graph would look like for this scenario.



Small initial vel!



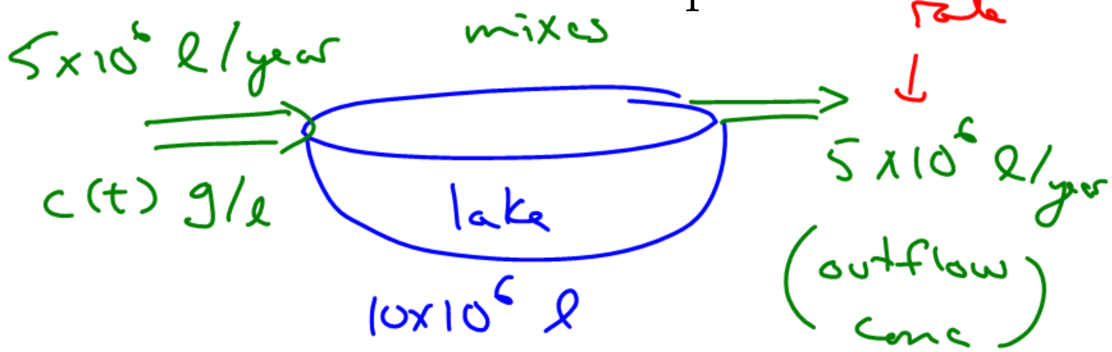
Problem. If we keep the initial angle at $-\frac{\pi}{2}$ (pendulum out horizontally), experiment with the MATLAB code to find the initial velocity that will push the pendulum "over the top".

Application - Lake Mixing Model

Consider a small lake that initially contains 10 million litres of fresh water. Water containing an undesirable chemical flows into the lake at the rate of 5 million litres per year; the mixture in the lake flows out at the same rate. The concentration $c(t)$ of chemical in the incoming water varies periodically with time according to the expression $c(t) = 2 + \sin(2t)\ \mathrm{g \cdot L^{-1}}$.

Problem. Construct a mathematical model of this flow process.

rate →



let Q = amt (g) of chemical in lake

$$\left(\frac{g}{yr}\right)\ \underbrace{\frac{dQ}{dt}}_{\text{net rate of change}} = \underbrace{\overset{\cancel{l}/yr}{(5\times10^6)}\ \overset{g/\cancel{l}}{(c(t))}}_{\text{rate in}} - \underbrace{\overset{l/yr}{(5\times10^6)}\ \overset{g/l}{\left(\frac{Q(t)}{\underbrace{10\times10^6}_{\text{conc of chemical in lake}}}\right)}}_{\text{rate out}}$$

Problem. Use MATLAB and a differential equation solver to determine the amount of chemical in the lake over time, assuming that the lake started without any contamination.

$$\frac{dQ}{dt} = (5\times10^6)(2+\sin(2t)) - (5\times10^6)\left(\frac{Q}{10\times10^6}\right)$$

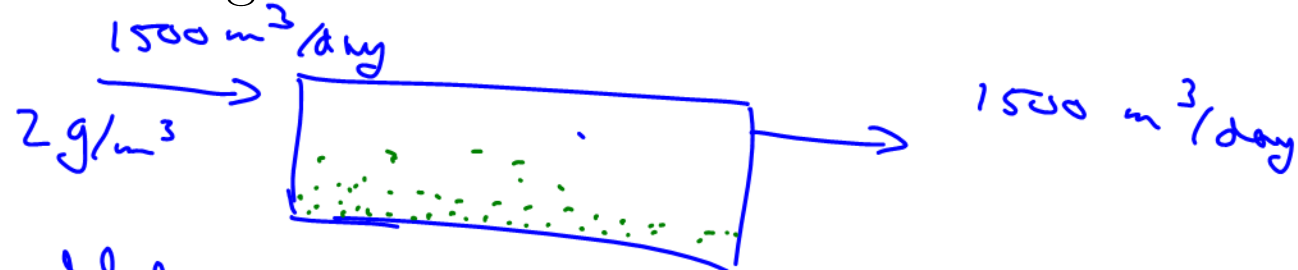
so $\quad Q(0) = 0$

Application - Tailings Pond With Sediment

Consider a tailings pond, where the the inflow contains both an environmentally sensitive chemical, and sediments that will settle out of the water.

- The volume of the pond is 40,000 cubic meters.
- Water is flowing in and out of the pond at a rate of 1,500 cubic meters per day.
- The water flowing into the pond contains 2 g of toxic chemical per cubic meter.
- The inflow water also contains 1% sediments

Problem. Sketch a diagram of this scenario.



→ sediment added at $15\ m^3/day$

pond

$40{,}000\ m^3$

→ $V(t)$ = water volume → $40000 - (15\ m^3/day)(t\ days)$

Problem. Write a differential equation that describes the rate of change of the concentration of the chemical in the water remaining in the tailings pond.

First look at amt, Q, g of chemical in water/pond.

$$\underbrace{\frac{dQ}{dt}}_{\text{g/day}} = \underbrace{(1500)}_{\text{m}^3\text{/day}}\underbrace{(2)}_{\text{g/m}^3} - \underbrace{(1500)}_{\text{m}^3\text{/day}}\left(\frac{Q(t)}{V(t)}\right)$$

net rate in pond — rate in — rate out

$$\boxed{V(t) = 40000 - 15t}$$

$$\underset{\text{g/m}^3}{C} = \frac{Q}{V} \quad \frac{\text{g}}{\text{m}^3} \xrightarrow{\text{take } d/dt \text{ of both sides}} \frac{d}{dt}(C) = \frac{d}{dt}\left(\frac{Q}{V}\right)$$

✓ quotient rule

$$\frac{dC}{dt} = \frac{\frac{dQ}{dt}\cdot V - Q\left(\frac{dV}{dt}\right)}{V^2}$$

$$\frac{dC}{dt} = \frac{\overbrace{\left((1500)(2) - 1500C\right)}^{dQ/dt}}{V} - \frac{\cancel{Q}^{\,C}}{\cancel{V}}\,\frac{1}{V}(-15)$$

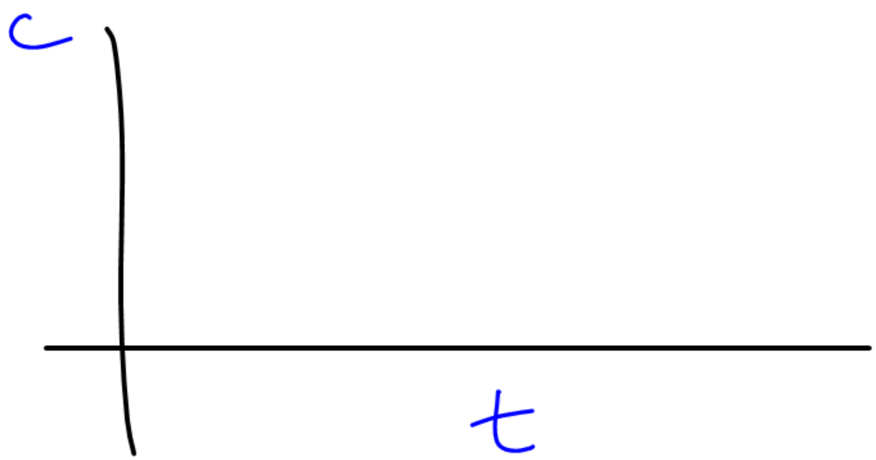
$$\frac{dC}{dt} = \frac{1}{V}\left(1500(2) - 1500C + 15C\right)$$

Problem. Use MATLAB and a differential equation solver to determine the concentration of chemical in the water part of the tailings pond, assuming that the pond started without any contamination.

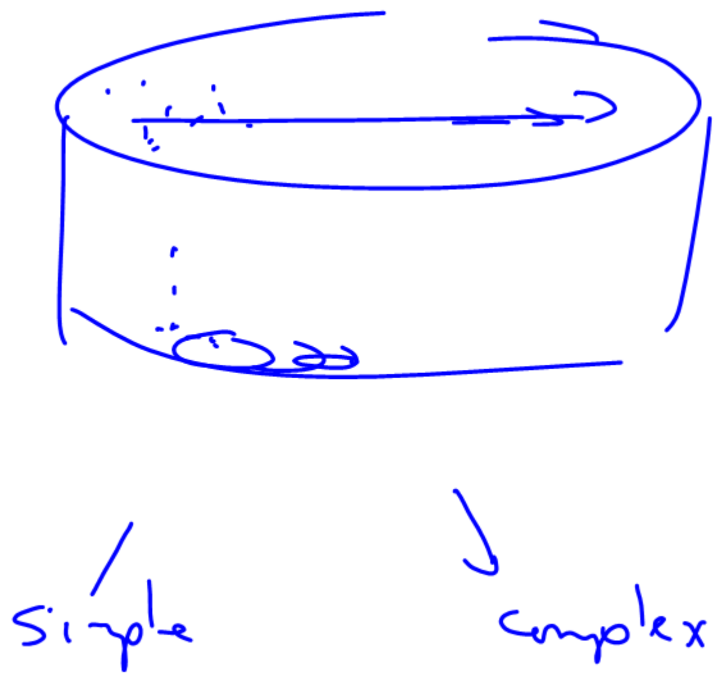
$$\frac{dc}{dt} = \frac{1}{V}\left(1500(\overset{g/m^3}{2}) - 1500c + 15c\right)$$

$$V = 40000 - 15t$$

$$c(0) = 0.$$

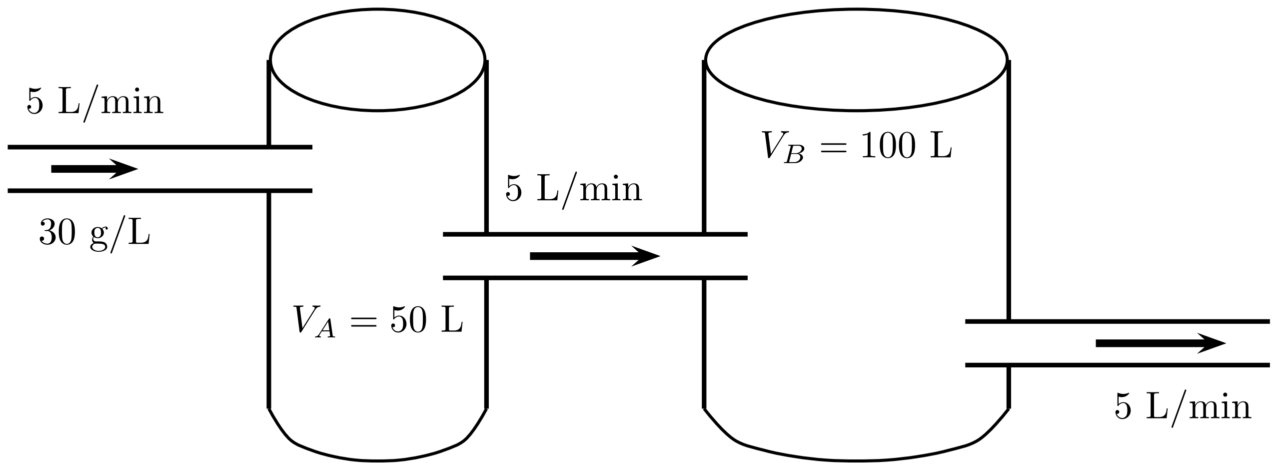


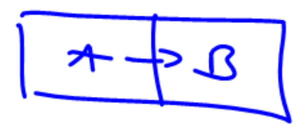
Problem. Comment on any mismatch between the model and the reality that should be addressed to make the model more accurate.



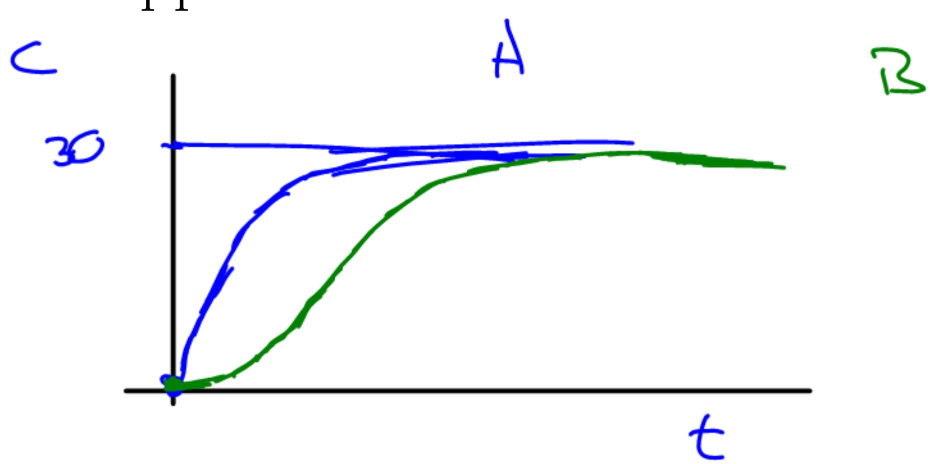
Application- Interconnected Tanks

Consider the tanks shown below, which shows water flowing between the tanks, and the concentration of a salt solution coming in. Within each tank, the water/salt solution is kept well mixed.

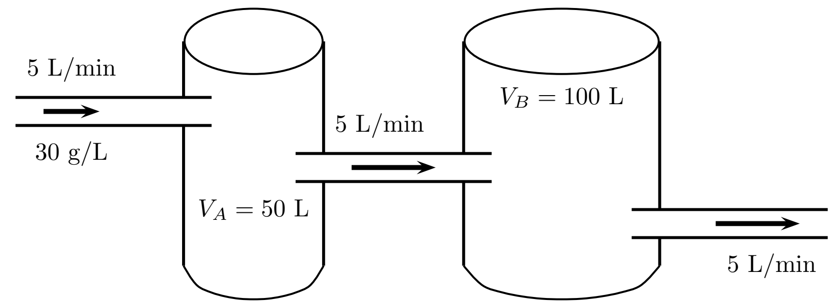




Problem. If both tanks start with no salt, sketch what you expect will happen to the concentration within each tank over time.



Problem. Create a system of differential equations that dictate how the two tank concentrations will evolve over time.



let Q_A, Q_B = g salt in Tank A, B

$$\underbrace{\frac{dQ_A}{dt}}_{\text{net rate for Tank A}} = \underbrace{(5)(30)}_{\text{rate in}} - \underbrace{(5)\left(\frac{Q_A}{V_A}\right)}_{\text{rate out}}$$

$\frac{g}{min}$; (L/min)(g/L) (L/min)(g/L)

$$\frac{dQ_B}{dt} = \underbrace{(5)\left(\frac{Q_A}{V_A}\right)}_{\text{rate in}} - \underbrace{(5)\left(\frac{Q_B}{V_B}\right)}_{\text{rate out}}$$

$$C = \frac{Q}{V} \Rightarrow \frac{dC}{dt} = \frac{1}{V}\frac{dQ}{dt}$$

const

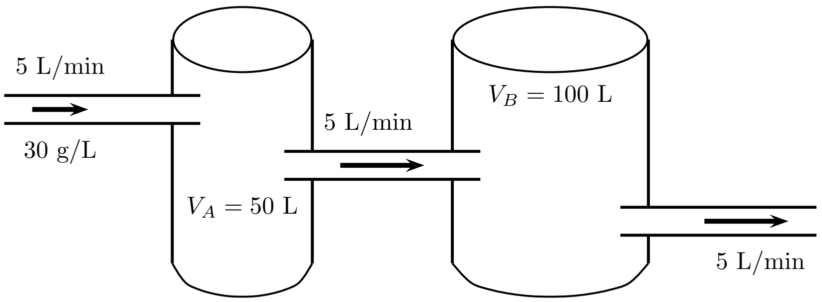
$$\frac{dC_A}{dt} = \frac{1}{V_A}\left(5 \cdot 30 - 5C_A\right)$$

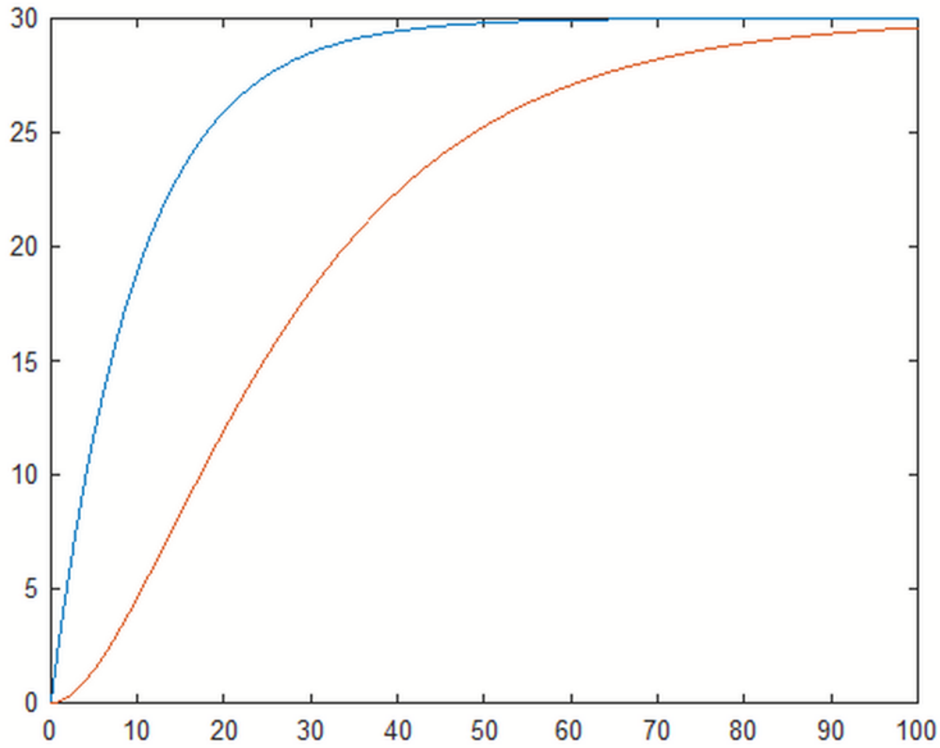
$$\frac{dC_B}{dt} = \frac{1}{V_B}\left(5C_A - 5C_B\right)$$

$$\vec{C} = [C_A, C_B]$$

c(1) c(2)

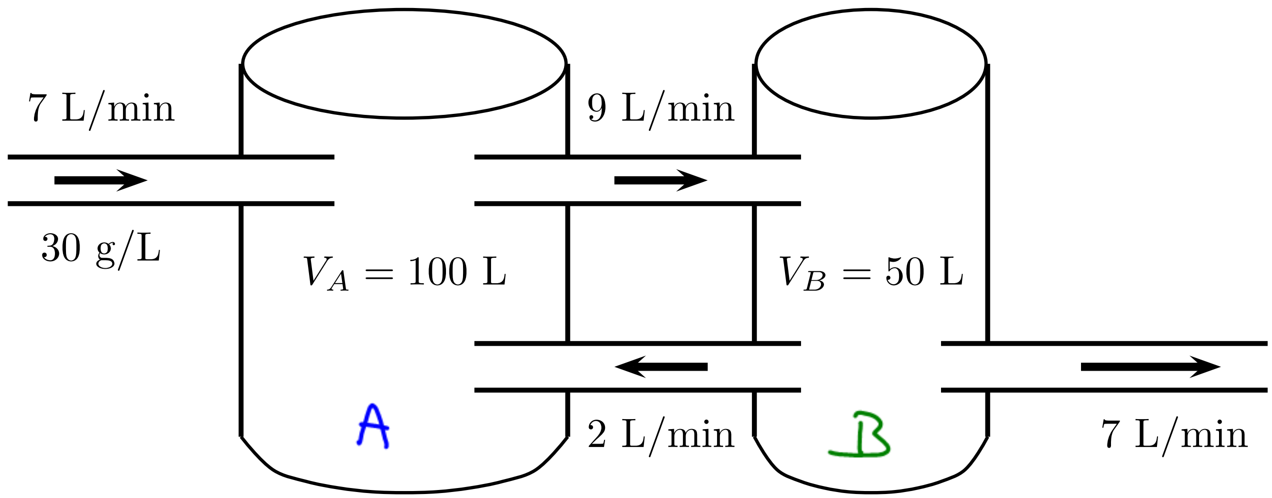
Problem. Use MATLAB and a differential equation solver to predict the exact salt concentrations over time in **both tanks**.



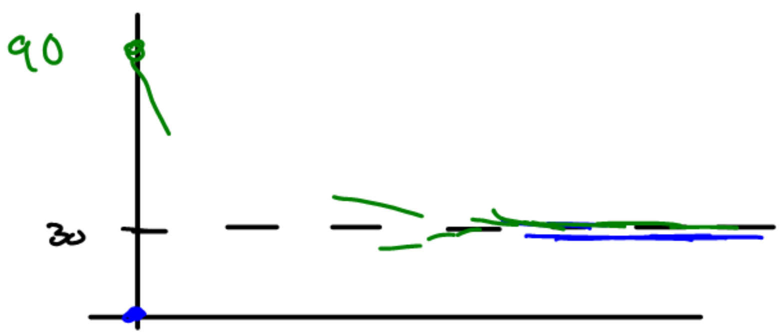


Tank Model - Example 2

Consider the more complicated tank arrangement shown below.

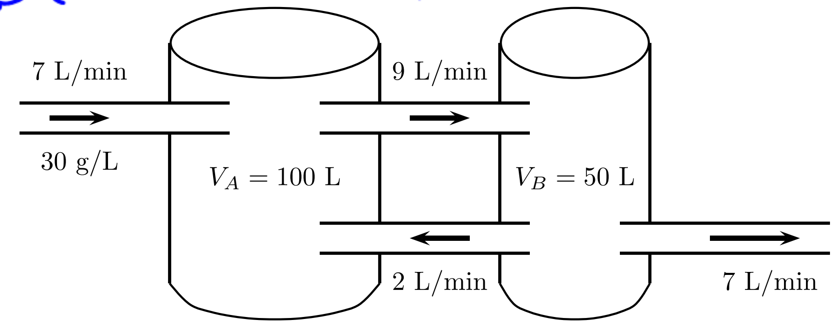


Problem. Given that the initial concentrations are
$c_A(0) = 0$ g/L and $c_B(0) = 90$ g/L,
sketch what you would predict for the concentration in each tank over time.



Q_A, Q_B = amt of salt in Tank A, B

Problem. Construct the differential equation for the salt concentration in each tank.



$$\frac{g}{\min} \qquad \frac{dQ_A}{dt} = \Big[\overbrace{(7)(30) + 2\,\frac{Q_B}{V_B}}^{\text{rate in}}\Big] - \underbrace{(9)\left(\frac{Q_A}{V_A}\right)}_{\text{rate out}}$$

$$C_A = \frac{Q_A}{V_A} \qquad \text{const} \rightarrow V_A$$

$$\frac{dQ_B}{dt} = \overbrace{(9)\left(\frac{Q_A}{V_A}\right)}^{\text{rate in}} - (2)\left(\frac{Q_B}{V_B}\right) - 7\left(\frac{Q_B}{V_B}\right)$$

$$= (9)\left(\frac{Q_A}{V_A}\right) - 9\left(\frac{Q_B}{V_B}\right)$$

$$C_B = \frac{Q_B}{V_B} \qquad \text{const} \rightarrow V_B$$

and $\dfrac{dC}{dt} = \dfrac{1}{V}\dfrac{dQ}{dt}$

$$\frac{dC_A}{dt} = \frac{1}{V_A}\left((7)(30) + 2C_B - 9C_A\right)$$

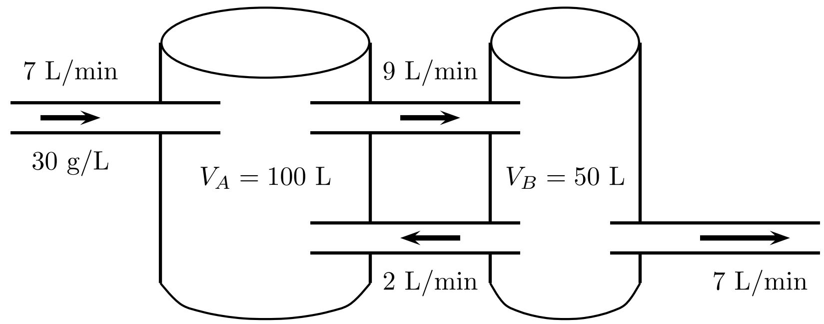
$$\frac{dC_B}{dt} = \frac{1}{V_B}\left(9C_A - 9C_B\right)$$

let $\vec{C} = [C_A, C_B]$

$\qquad C(1) \qquad C(2)$

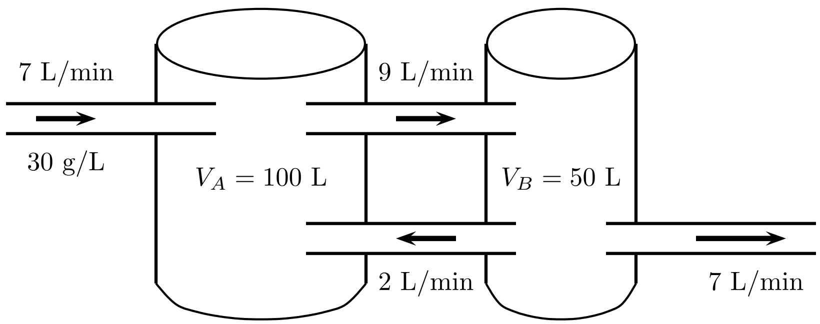
Problem. Use MATLAB and a differential equation solver to predict the salt concentrations over time by solving the system of differential equations

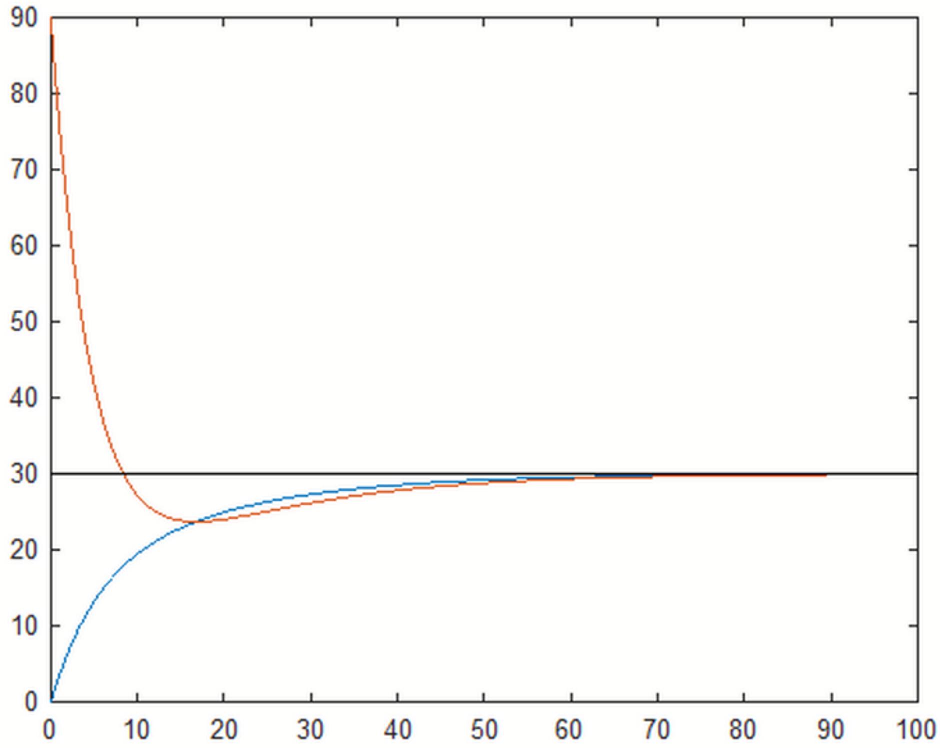
$$\frac{dc_A}{dt} = -0.09c_A + 0.02c_B + 2.1$$
$$\frac{dc_B}{dt} = 0.18c_A - 0.18c_B$$



$c_A \rightarrow c(1)$

$c_B \rightarrow c(2)$


7 L/min
30 g/L
$V_A = 100$ L
9 L/min
2 L/min
$V_B = 50$ L
7 L/min


90
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