MNTC P01 - Week #8 - Second Order Differential Equations

1. Use `ode45` to generate a graph of the solution to the following DEs, over the specified interval, given the initial condition.

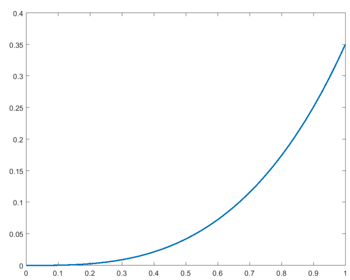
   (a) $\dfrac{dy}{dt} = t^2 + y^2$, $y(0) = 0$, and $0 \le t \le 1$.

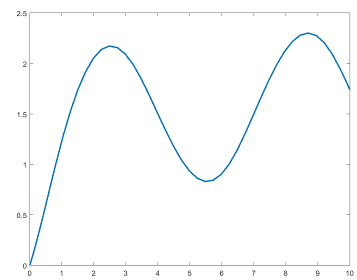
   (b) $\dfrac{dy}{dt} = \sin(t) + \cos(y)$, $y(0) = 0$, and $0 \le t \le 10$.

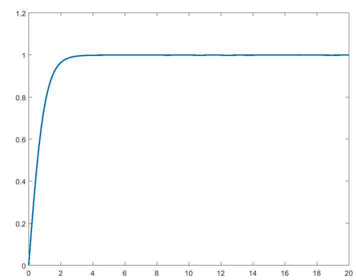
   (c) $\dfrac{dy}{dt} = (1 - y^2) + 0.2\sin(t)$, $y(0) = 0$, and $0 \le t \le 20$.

Link to the MATLAB code:
W08DE01.m

Here are the graphs of the solutions.







2. Newton's law of heating and cooling states that an object with temperature T in an environment at temperature T_{ext} will heat up or cool down according to the differential equation

$$\frac{dT}{dt} = -k(T - T_{ext})$$

Consider a garage used as a workshop. Its insulation and surface area give k a value of 0.1, if time t is measured in hours and the temperatures, T and T_{ext}, are in degrees Celsius.

The temperature outside changes during the day, as described by the formula

$$T_{ext} = 10 + 7\cos\left(\frac{\pi}{12}t\right)$$

We now imagine that the power goes out, with the garage at 23^o C at $t = 0$.

(a) Use ode45 and the DE to generate a numerical prediction of the garage's temperature T over time. Graph the solution over a time interval that shows both the initial and long-term behaviour of the temperature.

In your script, try to use the functions `title`, `xlabel`, `ylabel`, and `legend` to annotate the graph to make it easier for a reader to understand.

For the following questions, just use the graph or the numerical prediction of the temperature. You are *not* expected to solve the DE analytically.

(b) How many days does it take for the garage to get into a consistent temperature cycle? (You will need to estimate this by eye.)

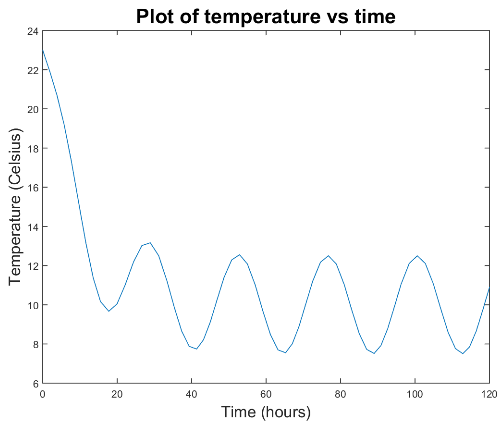
(c) How many degrees does the temperature in the building fluctuate by, once the temperature gets into a steady cycle?

(d) Suppose the building were better insulated, so that the rate of heat loss were cut in half. Should k be half as large, or twice as large?

(e) Generate a numerical prediction for the temperature over time in the better-insulated scenario, and produce a graph of the temperature vs time for both scenarios on the same axes.

(f) How large are the temperature fluctuations in the building, now that the extra insulation has been added? Does halving the net heat flow also halve the net temperature fluctuations?

(a) A graph of the temperature over time is shown below:



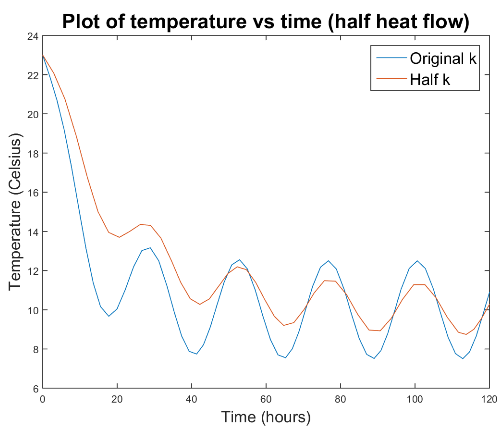
The file W08GarageTemp.m has the MATLAB code that generated the graph above.

(b) From the graph, it takes the building roughly 2 days (48 hours) to get into a repeating cycle of temperature variation.

(c) Careful zooming of the graph (or a look at the y values in the ode45 output) give a highest temperature of 12.5 (high) and 7.5 (low), for a net fluctuation of approximately 2.6 degrees per day.

(d) k represents the coefficient of heat flow between the building and the environment. The bigger k is, the *larger* the headflow between the two. Since we're adding insulation, this should *reduce* the heat flow, and so *lower* the value of k.

(e) A graph of the heat change over time, given better insulation, is shown below.



(f) Zooming in on the peaks of the **red line** graph (new insulation model), the temperature now fluctuates between approximately 11.3 and 8.8 degrees Celsius, for a range of 2.5 degrees. This *is* roughly half the magnitude of the fluctuations we saw earlier.

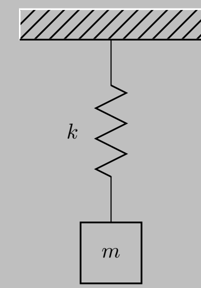
Modelling Spring Systems

3. Consider the single spring/mass system shown to the right, with no damper.

Newton's second law gives us the relationship:

$$ma = \sum F = F_{\text{spring}}$$
$$mx'' = -kx$$

where k is the spring constant.



(a) By hand, write this second order DE as a system of 1st order DEs, using the new variables $w_1 = x$ and $w_2 = x'$

(b) Write a MATLAB function file called `springDE1.m` starting with the first line
`function dw_dt = springDE1(t, w, m, k)`
that implements the system of differential equations from part (a).

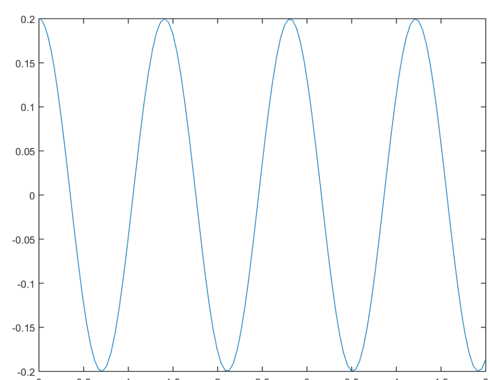
(c) Write a MATLAB script that simulates the motion of the mass using $m = 0.5$ kg and $k = 10$ N/m. Choose the time interval for the simulation so that 4-5 cycles of oscillation are shown.

(a) The first-order system would be:

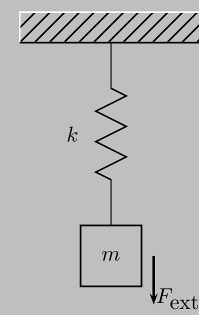
$$\frac{d}{dt}w_1 = x' = w_2$$
$$\frac{d}{dt}w_2 = x'' = \frac{1}{m}(-kx) = \frac{1}{m}(-kw_1)$$

(b) The function file springDE1.m implements the differential equation system, with the F_{ext} term left out.

(c) The main script W08SpringSimulation01.m has the code that will run this simulation.
In the resulting plot, we see a very nice example of simple harmonic motion.



4. Consider the single spring/mass system shown to the right, with no damper.
This is the same as in Question 3, except with the addition of the F_{ext} shown as an external applied force.
Newton's second law gives us the relationship:



$$ma = \sum F = F_{\text{spring}} + F_{\text{ext}}$$
$$mx'' = -kx + F_{\text{ext}}$$

where k is the spring constant.

(a) By hand, write this second order DE as a system of 1st order DEs, using the new variables $w_1 = x$ and $w_2 = x'$

(b) We will now incorporate an external force of the form $F_{\text{ext}} = a\sin(bt)$. Write a MATLAB function file called `springDE2.m` starting with the first line
`function dw_dt = springDE2(t, w, m, k, a, b)`
that implements the system of differential equations from part (a).

(c) Create a new MATLAB script. In the script, set $m = 0.5$ kg, $k = 10$ N/m, and use $a = 5$ and $b = 1$ in $F_{\text{ext}} = a\sin(bt)$. Use `ode45` to simulate the motion of the spring for 30 seconds (`tspan = [0, 30]`), given an initial displacement of $x(0) = 0.2$ m, and initial velocity of zero: $x'(0) = 0$.

(d) Explain why the motion looks so disorganized.

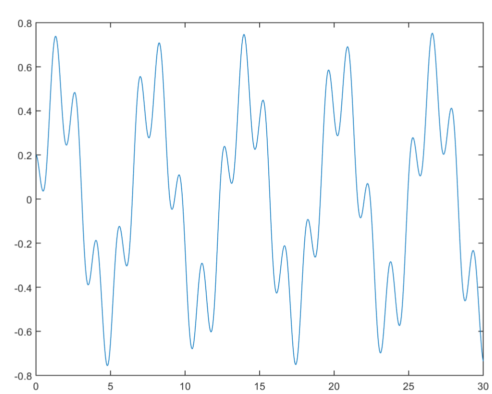
(e) Repeat Question (4c), but with an external force of $F_{\text{ext}} = \sin(4t)$. Explain why the motion in this case has cyclic waves in its amplitude.

(a) The first-order system would be:

$$\frac{d}{dt}w_1 = x' = w_2$$
$$\frac{d}{dt}w_2 = x'' = \frac{1}{m}(-kx + F_{\text{ext}}) = \frac{1}{m}(-kw_1 + F_{\text{ext}})$$

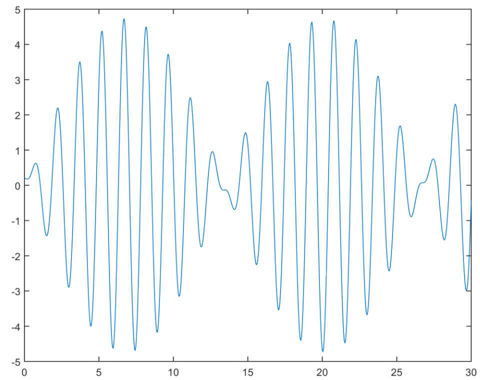
(b) The file springDE2.m implements the differential equation system, with new external force $F_{\text{ext}} = a\sin(bt)$.

(c) The file W08SpringSimulation02.m has the code that will run this simulation.
In the resulting plot, see some wildly varying and irregular oscillations.



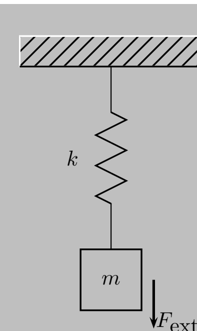
(d) The motion of the mass looks very disorganized because the natural frequency (the frequency at which the mass would oscillate if you just let swing on its own) is different from the frequency that we are pushing and pulling on it with through F_{ext}.
Recall: the natural frequency of a spring/mass system is given by $\omega = \sqrt{k/m}$, which for this scenario gives $\omega = \sqrt{\frac{25}{4}}$

(e) The file W08SpringSimulation02.m has the code that will run this simulation. Here is a graph of the resulting mass motion.



In the plot, we see that the natural frequency and the regular stimulation by the outside force are close to each other: the natural frequency is $\omega = \sqrt{\frac{10}{0.5}} \approx 4.5$, rad/s, and the stimulating frequency is at $\omega = 4$ rad/s. This close match of the frequencies leads to the phenonenon called *beats*, or *near resonance.*

5. We return to the same single spring/mass system from Question 4, shown to the right, with no damper.
The F_{ext} shown is an external applied force.



(a) For a mass of $m = 5$ kg , and a spring constant of $k = 2$ N/m, what is the natural frequency of the system?

(b) Define an external force of the form $F_{\text{ext}} = \sin(bt)$ that will produce **resonance** in the system.

(c) Use MATLAB to simulate the motion of the spring, with your selected external force, using an initial condition where the mass starts at its equilibrium and at rest.

(d) The system will break if the oscillations become too large, specifically if $x(t)$ exceeds 2 m (in the positive or negative directions). Does the system break, and if so, how long does it take for the system to break?

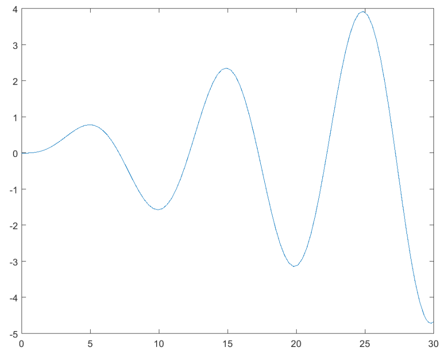
(a) The natural frequency of a spring/mass system is given by $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{5}} \approx 0.6325$ rad/s.

(b) If we want to produce resonance in the oscillations, our applied force's frequency must be exactly at the same frequency as the natural frequency. That way, the natural oscillations and the applied force are perfectly synchronized, leading to a build-up of the energy in the system.
This means we should select $F_{\text{ext}} = sin(0.6325t)$.

(c) For this problem, we can recycle the differential equation code in springDE2.m from Question 4, because the forces (spring and F_{ext}) are in the same form as in that problem.
In our new main script, W08Resonance1.m we will set up all the constants needed for the simulation:

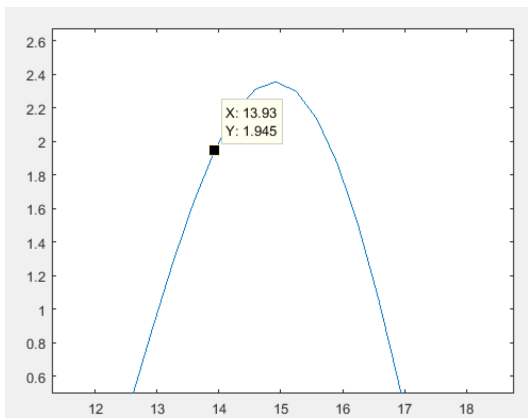
- $m = 5$, $k = 2$,
- $a = 1$ and $b = 0.6325$ to define $F_{\text{ext}} = a\sin(bt) = 1 \cdot \sin(0.6325t)$.

Below is a graph of the simulated motion of the mass over 30 seconds:



(d) From the graph, we see that the amplitude of the oscillations are growing with every cycle, as expected when we induce resonance. This means that the system *will* break at some point.

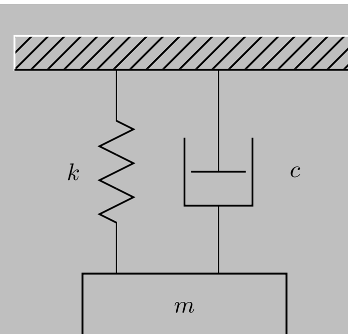
The specific limit we were given was that breakage will occur if $x(t)$ exceeds 2 m. Zooming in, we can see that the system will reach $x(t) \approx 2$ m around $t = 14$ seconds.



6. Consider the damped system shown at right.
The damping force exerted by the dashpot/damper is proportional to the velocity of the mass.
Newton's second law gives us the relationship:

$$ma = \sum F = F_{\text{spring}} + F_{\text{damp}}$$
$$mx'' = -kx - cx'$$

where k is the spring constant in N/m, and c is the damping coefficient in N/(m/s).



(a) By hand, write this second order DE as a system of 1st order DEs, using the new variables $w_1 = x$ and $w_2 = x'$

(b) We define a system with a mass of $m = 10$ kg, spring constant $k = 2$ N/m, and a damping coefficient of $c = 0.4$ N/(m/s). If the system is displaced by 0.5 m and then let go with zero initial velocity, use MATLAB to find out how long (in both seconds and cycles) it takes for the oscillations to reach approximately 10% of their original amplitude. Note: you will need to write both a MATLAB function for the differential equation, and a main script to run the simulation.

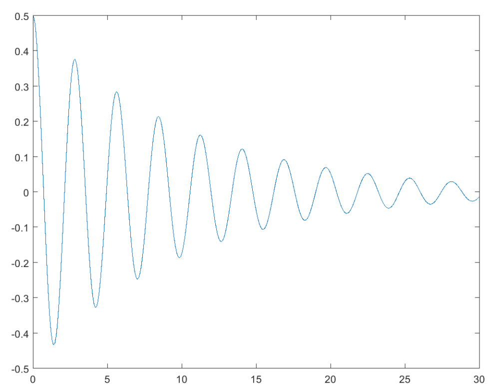
(c) What damping coefficient would be needed for the oscillations to be reduced to 10% of their original amplitude within 3 cycles? You will need to estimate your answer based on guessing and checking against the graph. Hint: add horizontal lines to the solution plot at $x = 0.05$ and $x = -0.05$ to see easily whether the oscillations are reduced to that level.

(a) The first-order system would be:

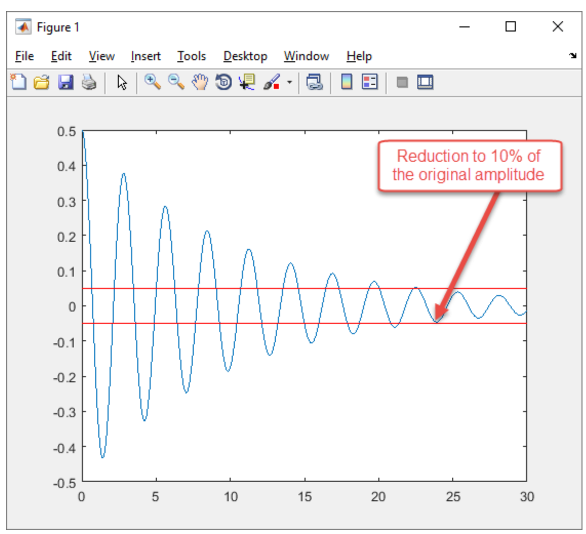
$$\begin{aligned}
\frac{d}{dt}w_1 &= x' = w_2 \\
\frac{d}{dt}w_2 &= x'' = \frac{1}{m}\left(-kx - cx'\right) = \frac{1}{m}\left(-kw_1 - cw_2\right)
\end{aligned}$$

(b) To simulate the motion of the spring/mass system, we need a function file with the differential equation, which we called springDEDamped.m, and a main script, which we called W08DampedSpringSystem.m.

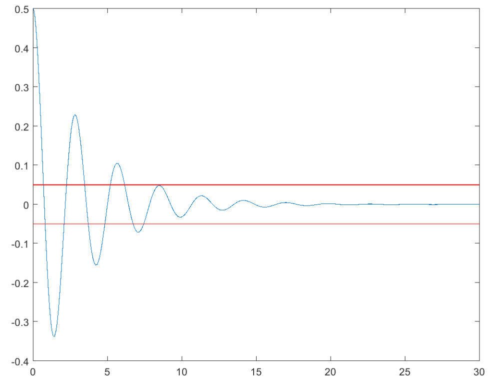
Here is the graph of the resulting motion, showing a nice oscillatory pattern, but the amplitude diminishing over time due to the damping.



seconds, or between 8 and 9 cycles.



(c) Using $c = 1.1$ N/(m/s) gives the required damping, putting the amplitude of the oscillations below 10% of their original magnitude after 3 cycles.



7. For a spring/mass system with $m = 1$ kg, $c = 0.5$ N/(m/s), and $k = 45$, approximately what frequency of external forcing would produce the largest amplitude steady-state vibration? Give your answer to the nearest 0.5 rad/s.

   You will need to estimate the answer based on guessing and checking against the graph.

The natural frequency is going to be near $\omega = \sqrt{k/m} \approx 6.7$ rad/s.

To run the simulation in MATLAB, we know we need to build the differential equation based on the sum of the forces in MATLAB. $ma = \sum F$ gives us

$$mx'' = -kx - cx' + \sin(bt)$$

where b is the frequency we can experiment with to maximize the steady-state oscillations.

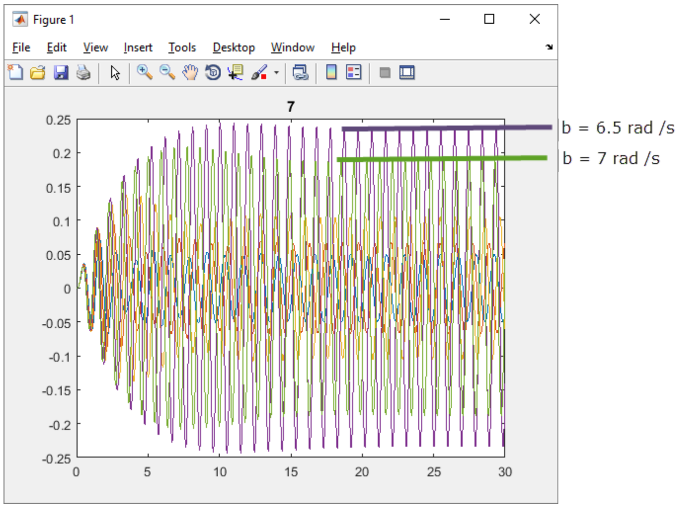
The first-order system version of this is:

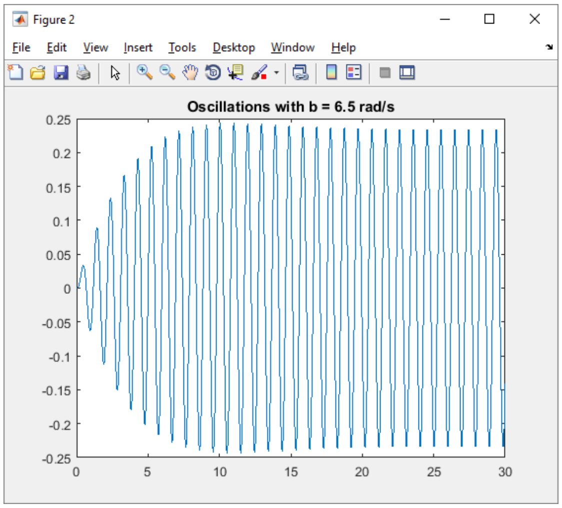
$$\begin{aligned}
\frac{d}{dt}w_1 &= x' = w_2 \\
\frac{d}{dt}w_2 &= x'' = \frac{1}{m}\left(-kx - cx' + \sin(bt)\right) \\
&= \frac{1}{m}\left(-kw_1 - cw_2 + \sin(bt)\right)
\end{aligned}$$

We built this system of differential equations into the file springDEDampedAndForced.m and the main script W08DampedSpring

If we are only interested in the steady-state behaviour, then the initial conditions won't matter (their influence fades in the long run). We make the easiest choice, which is the mass starting at equilibrium at rest, $x(0) = 0$ and $x'(0) = 0$.

Below is a graph showing the response of the system to $b = 5, 5.5, 6, 6.5$ and 7 rad/s, and then the graph for $b = 6.5$ rad/s only.





Looking for the steady-state oscillations means looking at the amplitudes once the graph steadies into a regular repeating pattern. The graph corresponding to $F_{\text{ext}} = \sin(6.5\ t)$ is the graph with the highest amplitude once that steady-state oscillation pattern is reached.