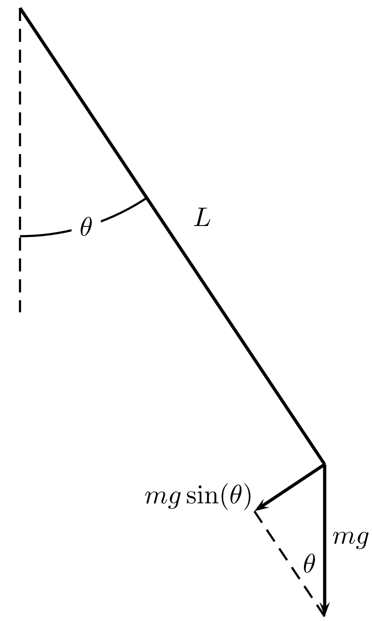
MNTC P01 - Week #9 - Differential Equations and Engineering

Pendulum



Consider the motion of a frictionless pendulum.

$$\begin{aligned} \text{Newton's Second Law: } mL^2\theta'' &= T_g \\ &= -mLg\sin(\theta) \\ \text{Solving for } \theta''\text{: } \theta'' &= -\frac{g}{L}\sin(\theta) \end{aligned}$$

1. **Without** simulating the actual motion of the pendulum, we can compute the period, T, using the formula below:

$$T = 4\sqrt{L/g}\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity, 9.8 m/s.

For each set of values for L and θ_0 given below,

(a) Use the MATLAB `integral` function to find the period of the pendulum oscillations by evaluating the integral given above, and

(b) confirm the period by using `ode45` to simulate the motion of the pendulum for exactly that length of time, and plot a graph of the angular **velocity** against time. The velocity should just reach zero at the end of one cycle.

Do this with the following sets of L and θ_0 values.

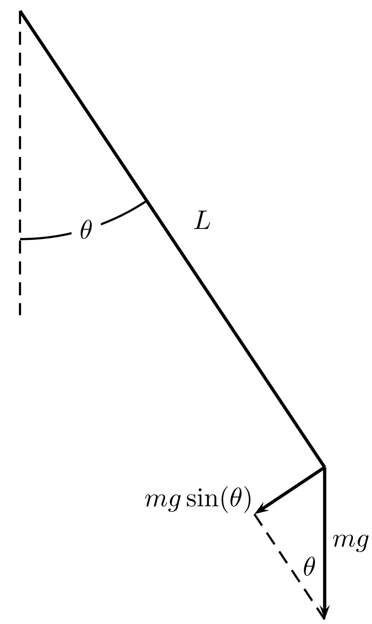
(i) $L = 2$ m, $\theta_0 = 40^o$,

(ii) $L = 2.5$ m, $\theta_0 = 20^o$.

(iii) $L = 5.0$ m, $\theta_0 = 90^o$.

In all the graphs, we see that that the velocity returns to 0 at the end of the cycle, indicating that our integral calculation of the period matches the period from the `ode45` simulation.

2. Consider the motion of a pendulum, this time **with** friction.



$$\begin{aligned} &\text{Newton's Second Law:} \\ mL^2\theta'' &= T_g + T_f \\ &= -mLg\sin(\theta) - (\mu L^2 m)\theta' \end{aligned}$$

$$\text{Solving for } \theta'' \text{: } \theta'' = -\frac{g}{L}\sin(\theta) - \mu\theta'$$

(a) Write a MATLAB function for the differential equation, and a script that will simulate the scenario for $L = 1.5$ m, $g = 9.8$ m/s^2, and $\mu = 0.2$. Use an initial condition of $\theta_0 = \frac{7\pi}{8}$, which is close to vertical.

(b) Experiment with the initial **angular velocity** of the pendulum and find the smallest **positive** initial velocity that will result in the pendulum passing over the top of the axis of rotation. Find the value to the nearest 0.1 rad/s.

Have MATLAB generate a plot of the angle vs time graph for both the initial velocity that achieves this result, and for the initial velocity 0.1 rad/s smaller, which does *not* go 'over the top'.

(c) Repeat the analysis in part (b), but this time using a **negative** initial velocity.

Single Tank Problems

3. An aquarium pool has volume 2×10^6 liters. The pool initially contains pure fresh water. At $t = 0$ minutes, water containing 10 grams/liter of salt is poured into the pool at a rate of 60 liters/minute. The salt water instantly mixes with the fresh water, and the excess mixture is drained out of the pool at the same rate (60 liters/minute).

   (a) Write a differential equation for $S(t)$, the mass of salt in the pool at time t.
   (b) Use MATLAB solve the differential equation to predict $S(t)$ over time.
   (c) Based on the graph of the solution, what happens to $S(t)$ as $t \to \infty$?
   (d) Find this same value using only the information about the volume and the concentration of the incoming salt solution.

4. A 150 liter tank initially contains 60 liters of water with 0.5 kgs of salt dissolved in it. Water enters the tank at a rate of 0.9 liters/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + \cos(t))$ kgs/liter.

   (a) Draw a diagram of the inflow and outflow for this scenario.
   (b) Build a formula for the volume of water in the tank over time.
   (c) Find out how long it will be until the tank overflows.
   (d) Write a differential equation that describes the rate of change of the **amount of salt** in the tank.
   (e) Use MATLAB to generate a graph of the amount of salt in the tank over time, up until the tank overflows.
   (f) How much salt is in the tank when it overflows?

Other First Order Models

5. Differential equations are not only well-suited for physics applications: they are are also widely used in biology, particularly in population models.

   Consider the fish population model below, based on a standard limited-resource population growth, minus a function of harvesting.

$$\frac{dP}{dt} = \underbrace{[(10 - P) \cdot P]}_{\text{natural population growth rate}} - \underbrace{h(t)}_{\text{harvesting rate}}$$

where

- P = population of fish (in thousands), and
- $\frac{dP}{dt}$ = rate of population change, in thousands per year
- $h(t)$ is the harvesting rate (in thousands of fish per year)

We want to study the impact of two harvesting models:

- $h_1 = k_1$; constant harvesting
- $h_2(t) = k_2(\sin(\pi t) + 1)$; seasonal model where the harvesting has a yearly cycle.

(a) Generate a prediction of the population over time, starting at initial populations of $P(0) = 15$ for each model. Use $k_1 = k_2 = 5$. Produce a graph showing the predicted population over time on the same graph, over a long enough time interval to show the long-term behaviour of both solutions.

One question that arises in such harvesting models is which fishing strategy permits a higher average harvesting rate can be maintained: seasonal harvesting, or constant harvesting? To decide this, we note that the average harvest rate for h_1 is k_1, and for h_2 is k_2, so whichever value of k_1 and k_2 is larger indicates the strategy with the greater average harvesting rate.

We will define the *maximum sustainable harvest rate* for both models as the *highest harvest rate for which the population is not driven to zero.*

(b) Find and report the maximum sustainable harvest level k_1 for the constant harvesting model (to the nearest integer). (Use trial and error if necessary, though more insightful DE-related ways are possible.) Indicate how you found the cut-off level.

NOTE: during this process, your model will predict a population of zero, which will then lead to large negative populations. This clearly makes no sense, so limit your plots with the command `ylim([0, P0])`. *This same problem will also trigger warnings in ode45 about error tolerances; you can safely ignore those warnings.*

(c) Generate a plot showing the population over time, using the same initial value used earlier, but using both the k_1 value just above, and just below the extinction level. (One line should remain positive, while the other should crash to zero at some point on the graph.)

(d) Use trial and error (theory isn't much help here) to find the maximum sustainable harvest level k_2 for the cyclic harvesting model (to the nearest integer). Include a plot showing the population over time with this harvesting level.

(e) Based on your experiments, can constant harvesting or cyclic harvesting sustain a greater average harvest in the long run? Explain your reasoning.

Multi-Tank Systems

6. Consider two interconnected tanks. Tank A initially contains 100 L of water and 200 g of salt, and tank B initially contains 200 L of water and 900 g of salt. The liquid inside each tank is kept well stirred.

- Liquid flows from tank A to tank B at a rate of $3\ \mathrm{L} \cdot \mathrm{min}^{-1}$;
- liquid flows from tank B to tank A at rate of $2\ \mathrm{L} \cdot \mathrm{min}^{-1}$.
- A salt brine with concentration $7\ \mathrm{g} \cdot \mathrm{L}^{-1}$ of salt flows into tank A at a rate of $5\ \mathrm{L} \cdot \mathrm{min}^{-1}$; an outflow pipe drains out of the Tank A at $4\ \mathrm{L} \cdot \mathrm{min}^{-1}$.
- Moreover, a salt brine with concentration $3\ \mathrm{g} \cdot \mathrm{L}^{-1}$ of salt flows into tank B at a rate of $7\ \mathrm{L} \cdot \mathrm{min}^{-1}$; the solution is drained out at $8\ \mathrm{L} \cdot \mathrm{min}^{-1}$.

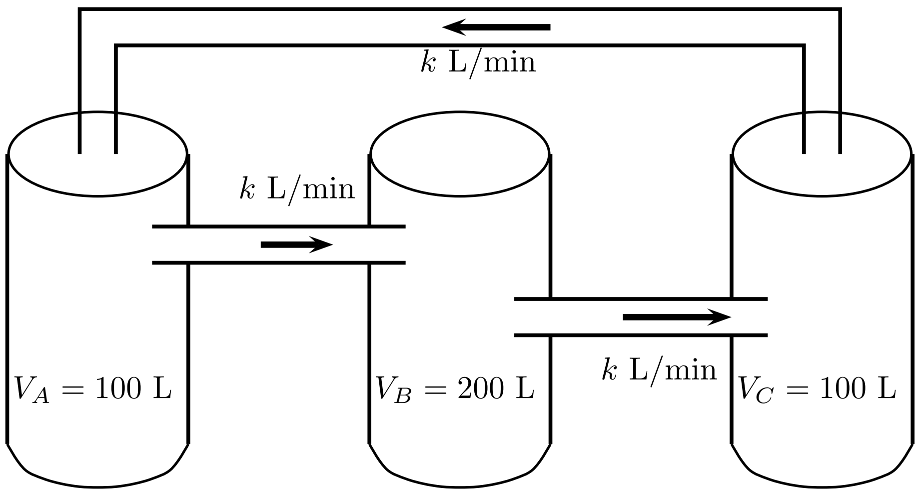
(a) Draw a diagram for the flows and concentrations in this scenario.

(b) Write a set of differential equations for the **amount** of salt in each tank.

(c) Use MATLAB to simulate the concentration of salt in each tank over time. Generate two separate graphs, one for Tank A and one for Tank B, and use the `title` command to label each one.

(d) Use the solution generated by MATLAB to estimate when Tank B reaches its lowest salt amount, and what that lowest amount is.

7. Consider the 3-tank system shown below.



(a) Write the set of differential equations that governs the **amount** of salt in each tank, S_A, S_B and S_C.

(b) Convert your answer from part (a) into a set of differential equations for the **concentrations** in each tank, C_A, C_B and C_C.

(c) Using the first line
`function dw_dt = tankSystem2(t, w, k, VA, VB, VC)` and the definition that $\vec{w} = [C_A, C_B, C_C]$ to group the three dependent variables, write a MATLAB function file that implements the differential equation system from part (b).

(d) Write a script that simulates the changes in concentration over time, using the volumes shown in the diagram, a flow rate of $k = 2$ L/min for each connection, and a time span of 250 minutes.

(e) Use your knowledge of chemistry to explain the fact that all the tanks converge to a common concentration of 2.5 g/L.