Week #2 : Derivatives - Linearization

Goals:

- Describe the meaning and value of linearization
- Apply the technique of linearization to solve a variety of nonlinear equations
- Use MATLAB to graph and compare functions with their linearizations
- Use MATLAB to implement Newton's method

Linear Approximations

You should now feel comfortable in finding the derivative of a wide variety of functions with formulas.

In this section, we will explore how the derivatives you compute can be tied back to understanding the behaviour of the original function.

We will start by returning to the definition of the derivative, based on the $\left(\frac{\text{rise}}{\text{run}}\right)$ formula for slopes:

$\frac{\text{rise}}{\text{run}}$

$$\underbrace{f'(x) = \frac{d}{dx}f = \frac{df}{dx}}_{\text{notation for short form}} = \underbrace{\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{\text{definitions}}$$

For example, if $y = f(x)$, then

$$\cancel{\lim_{\Delta x \to 0}} \frac{\Delta y}{\Delta x} = f'(x)$$

Problem. What is the relationship between f' and Δy, Δx for merely *small* delta values?

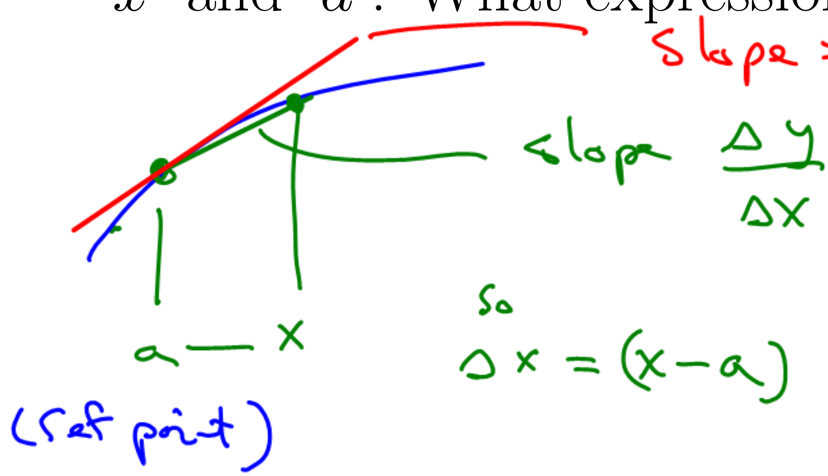
approximately equal to

Δx small, close to 0

$$\frac{\Delta y}{\Delta x} \simeq f'(x)$$

Now sketch a graph, and label the two points in the Δx difference 'x' and 'a'. What expression do we obtain for $f(x)$?

x – new point



So $\Delta x = (x-a)$

$$\frac{f(x) - f(a)}{(x-a)} \simeq f'(a)$$

linear

Δx

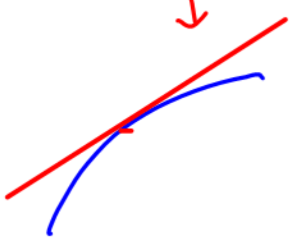
$$f(x) \approx f'(a)(x-a) + f(a)$$

curved funct. slope @ x=a value

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

Problem. What are some names for this linear function?

tangent line



linear approximation ($f(x) \cong \ldots$)

tangent line ($y = \ldots$

linearization ("to make $f(x)$ linear")

Problem. Consider the tangent line approximation to the graph of $f(x) = e^x$ at $(0, 1)$. Find the formula for the tangent line at that point.

$f(x) = e^x$

$\big\downarrow \ d/dx$

$f'(x) = e^x$

@ $x = 0 = a$

$f(0) = e^0 = 1$ value

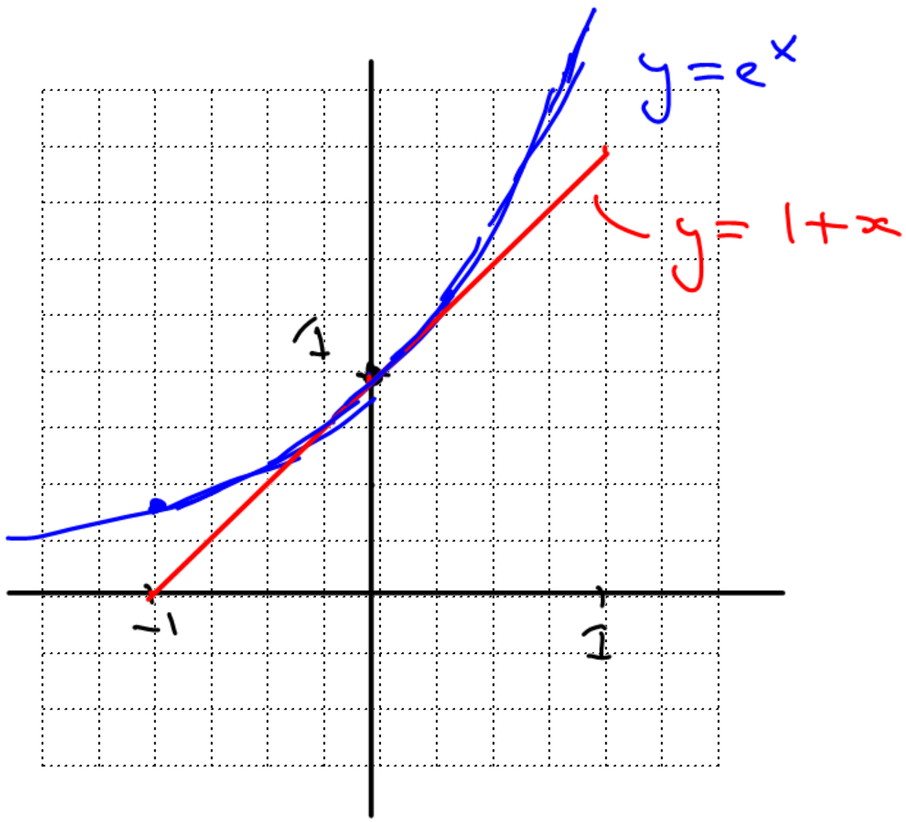
$f'(0) = e^0 = 1$ slope

Tgt line

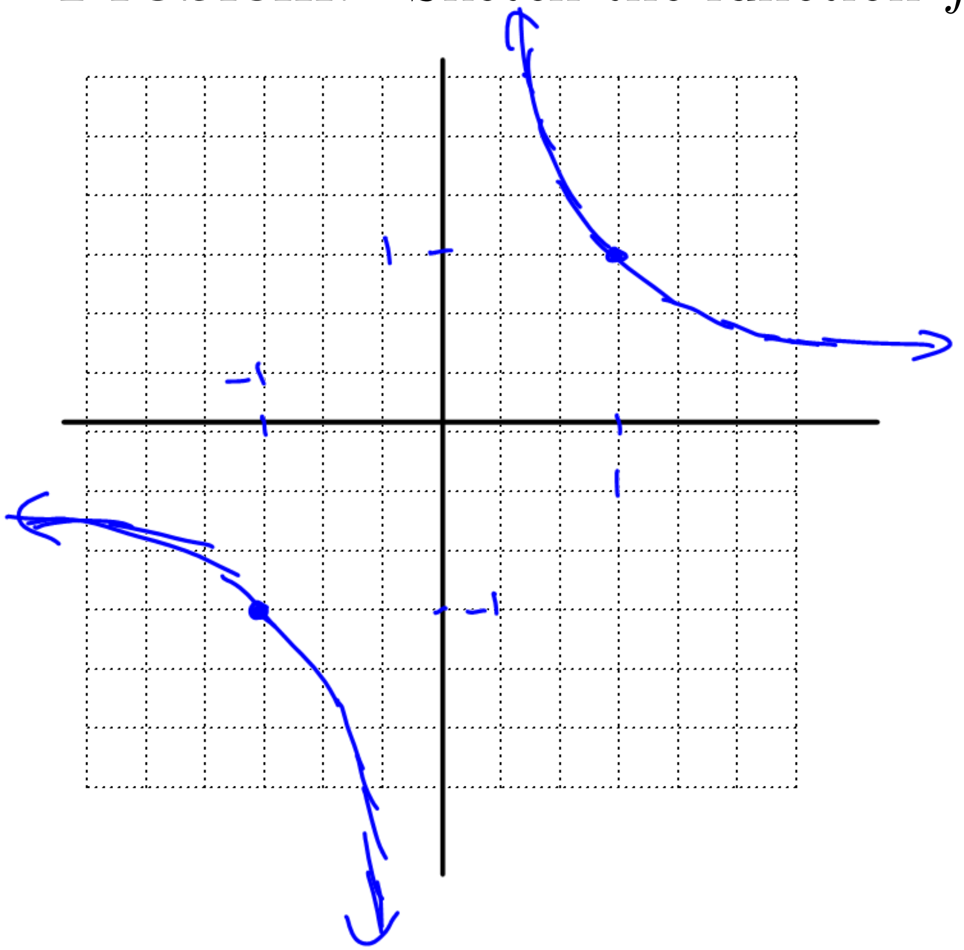
$y = 1 + 1(x - 0)$

$f(a) + f'(a)(x-a)$

Sketch the graph of $y = e^x$ and the linearization/tangent line.

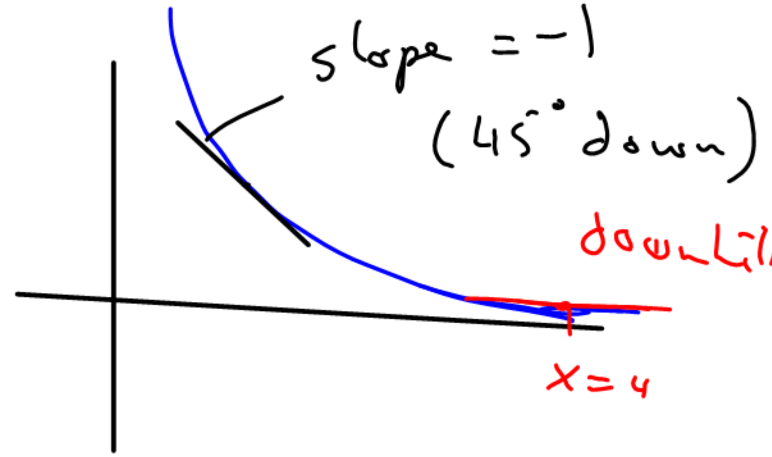


Problem. Sketch the function $f(x) = \dfrac{1}{x}$.



Problem. If we drew a tangent line to $f(x)$ at $x = 4$, what range we would expect for the slope there?

A. Slope above 1.



B. Slope between 0 and 1.

C. Slope between 0 and -1. ✓

=> negative slope

D. Slope below -1.

Problem. Find the linearization of $f(x) = \dfrac{1}{x}$ at $a = 4$.

$f(x) = 1/x = x^{-1}$

so $f'(x) = -1x^{-2} = -\dfrac{1}{x^2}$

at $a=4$

$f(4) = 1/4$ height/value

and

$f'(4) = \dfrac{-1}{16}$ slope

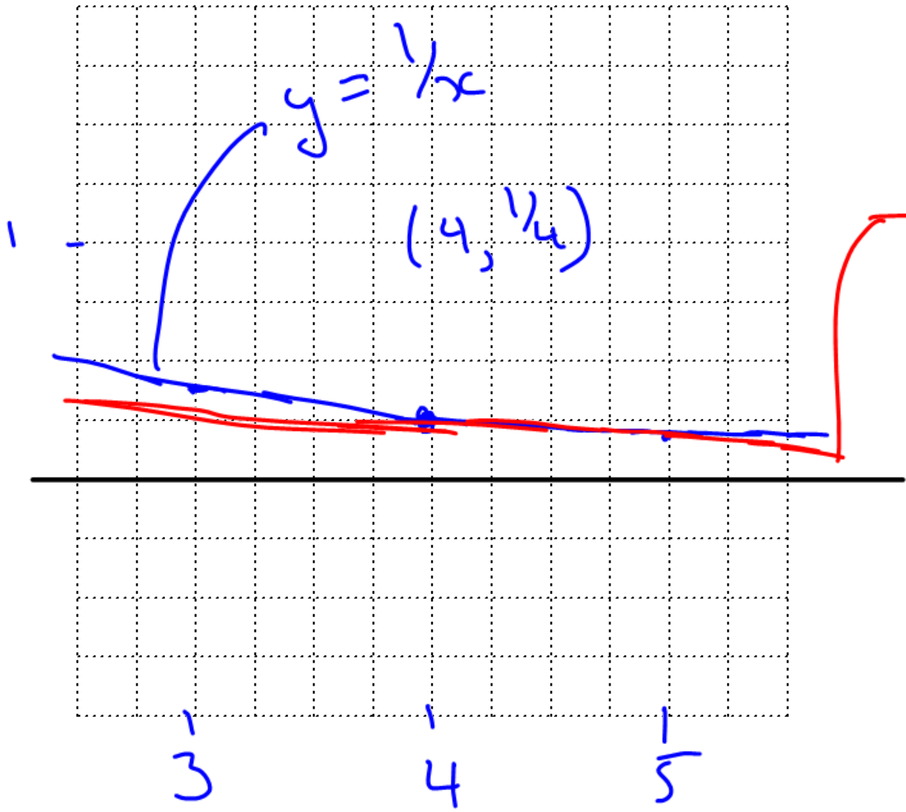
$$L(x) = \frac{1}{4} + \left(\frac{-1}{16}\right)(x-4)$$

$$f(a) + f'(a)(x-a)$$

Find the equation of the tangent line to the function $y = \dfrac{1}{x}$ at $x = 4$.

$$y = 1/4 + \left(\frac{-1}{16}\right)(x-4)$$

Problem. Sketch the graph of $y = \dfrac{1}{x}$, and its tangent line at $x = 4$.



$y = \frac{1}{4} + \left(-\frac{1}{16}\right)(x-4)$

point / slope

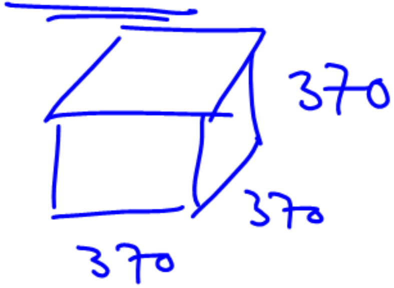
$(4, 1/4)$

$y = y_0 + m(x - x_0)$

Linear approximations can be useful for quick and (relatively) simple error calculations.

Problem. A rock formation with high density ores was identified using gravity measurements; the formation is roughly cubic in shape. The edge each side of the cube was found to be 370 m, with a possible error in measurement of 10 m.

Use the ideas of linear approximations to estimate the maximum possible error (positive or negative) in computing the **volume** of the formation.



$V = (370\,m)^3 = (L^3)$

change/error

$\frac{\Delta V}{\Delta L} \cong V'$

10 m → ΔL

$\Delta V \cong V' \cdot \Delta L$

$V = L^3$

so $V' = \frac{dV}{dL} = 3L^2$

at $L = 370$ m,

$V'(370) = 3(370)^2 = 410{,}700$

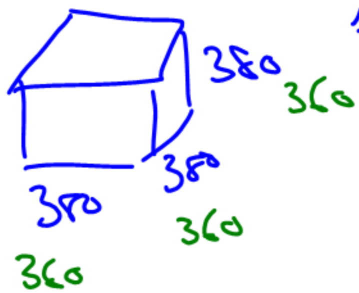
$\Delta V = (410700)(10)$

$\Delta V \cong 4 \text{ million } m^3$

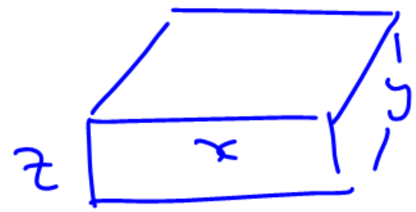
compare w/ $V = 370^3 = 50{,}653{,}000$

(so error $\approx 10\%$ of total volume)

Problem. What are the trade-offs of using the linear approximation to obtain the above error estimate, compared to a direct calculation of the possible volumes with the error measurements?



$\Delta L = 10$



$$\Delta V \cong V' \cdot \Delta L$$

The $\sin(x)$ Approximation

One of the most commonly-used approximation in physics is the relationship

$$\sin(x) \approx x$$

Problem. Derive this relationship using linearization.

at $a=0$

$f(x) = \sin(x)$

so $f'(x) = \cos(x)$

$f(0) = \sin(0) = 0$

and

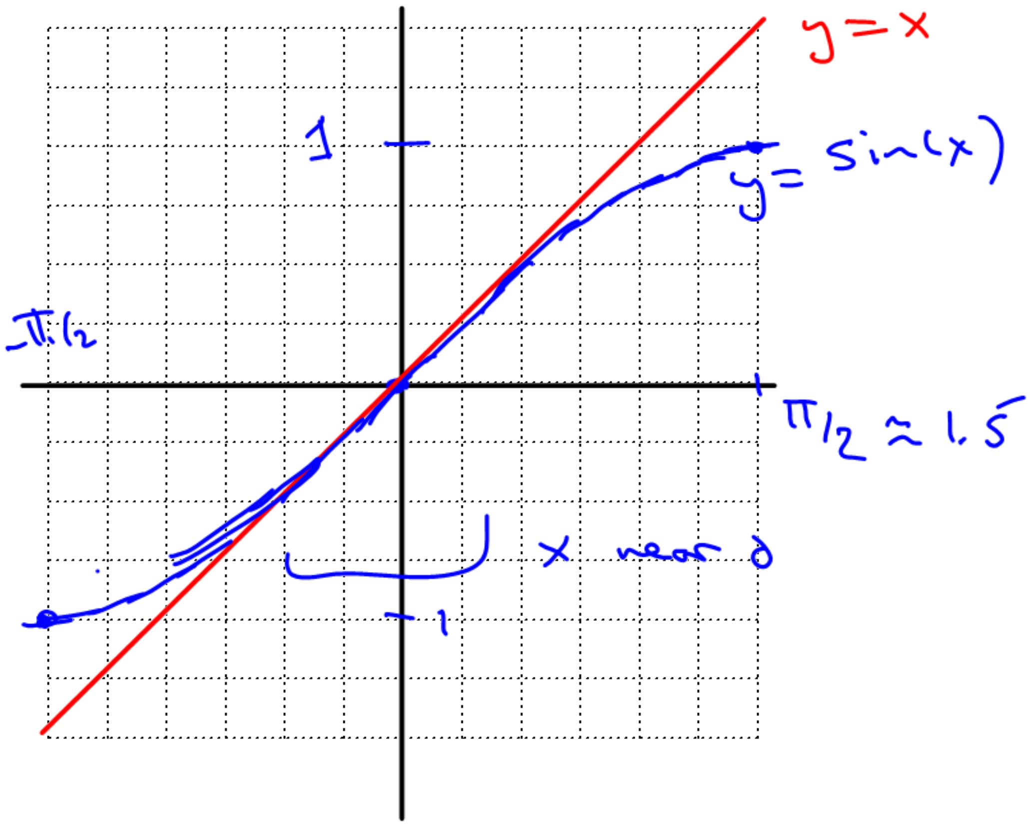
$f'(0) = \cos(0) = \underline{1}$

$$\sin(x) \approx \underbrace{0 + 1\,(x-0)}_{f(a) + f'(a)(x-a)} = x$$

What is the fine-print that should **always** be associated with this approximation?

for x near zero.

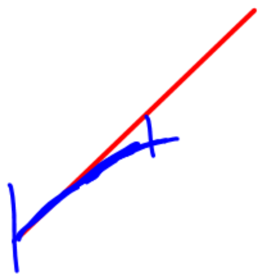
Problem. Sketch the graphs of $y = \sin(x)$ and $y = x$.
Focus on the domain $\dfrac{-\pi}{2} \leq x \leq \dfrac{\pi}{2}$.



Below are more detailed calculations relating $\sin(x)$ and x.

x (degrees)		5.7°		11.5°				23°
x (rad)	0.0500	0.1000	0.1500	0.2000	0.2500	0.3000	0.3500	0.4000
$\sin(x)$	0.0500	0.0998	0.1494	0.1987	0.2474	0.2955	0.3429	0.3894

Problem. Comment on the agreement between $y = x$ and $y = \sin(x)$ on the range shown.



$\sin(x) \cong x$

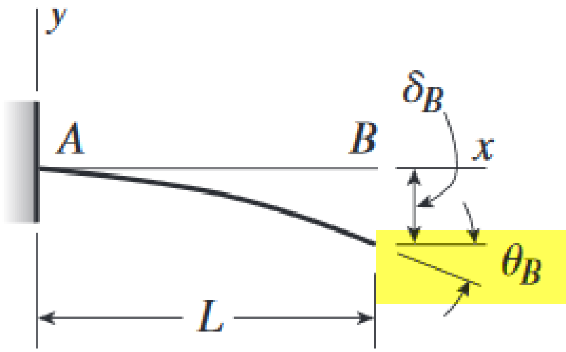
(x in radians)

$$(\text{degrees}) = \left(\frac{180 \text{ deg}}{\pi \text{ rad}}\right)(\text{rad})$$

Most people are more familiar with angles measured in degrees than radians: fill in the row indicating how many degrees are represented by the radian measures.

Linear Approximation In Beam Deflection

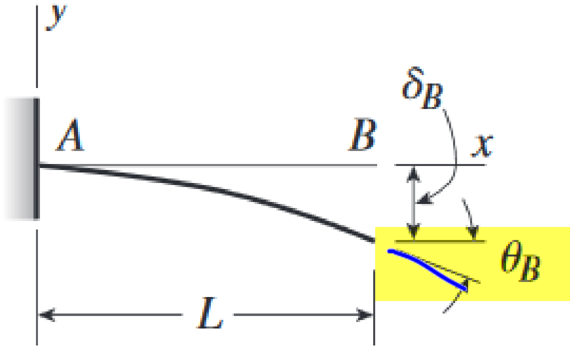
Sometimes engineering texts will use linear approximations without explaining the source. Here is a sample from a text on deflection of beams under load.



v = deflection in the y direction
$v' = dv/dx$ = slope of the deflection curve
$\delta_B = -v(L)$ = deflection at end B of the beam
$\theta_B = -v'(L)$ = angle of rotation at end B of the beam
EI = constant

Problem. Compare the two meanings of θ_B highlighted.

slope $\cong$ angle



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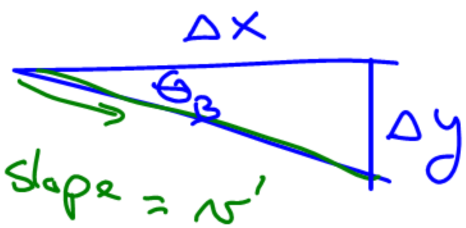
EI = constant

Problem. Use linearization to show that the slope at L is approximately equal to the angle of rotation.



$$\theta_B \cong \tan(\theta_B) = \frac{\Delta y}{\Delta x} = \text{slope} = v'(L)$$

angle in radians

slope

at $a = 0$

$$f(\theta) = \tan(\theta)$$

$$f'(\theta) = \sec^2(\theta)$$

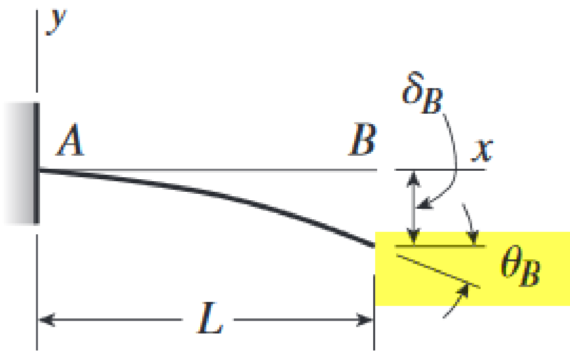
$$f(0) = \tan(0) = 0$$

and

$$f'(0) = \sec^2(0) = \frac{1}{\cos^2(0)} = \frac{1}{1^2} = 1$$

so $\tan(\theta) \cong 0 + 1(\theta - 0) = \theta$

$$f(a) + f'(a)(a - a)$$



v = deflection in the y direction
$v' = dv/dx$ = slope of the deflection curve
$\delta_B = -v(L)$ = deflection at end B of the beam
$\theta_B = -v'(L)$ = angle of rotation at end B of the beam
EI = constant

$$\text{slope} = \tan(\theta_B) \approx \theta_B$$

Problem. As with any linear approximation, what is the fine print on this relationship?

θ must be small (near zero)

okay because beams shouldn't deflect by much

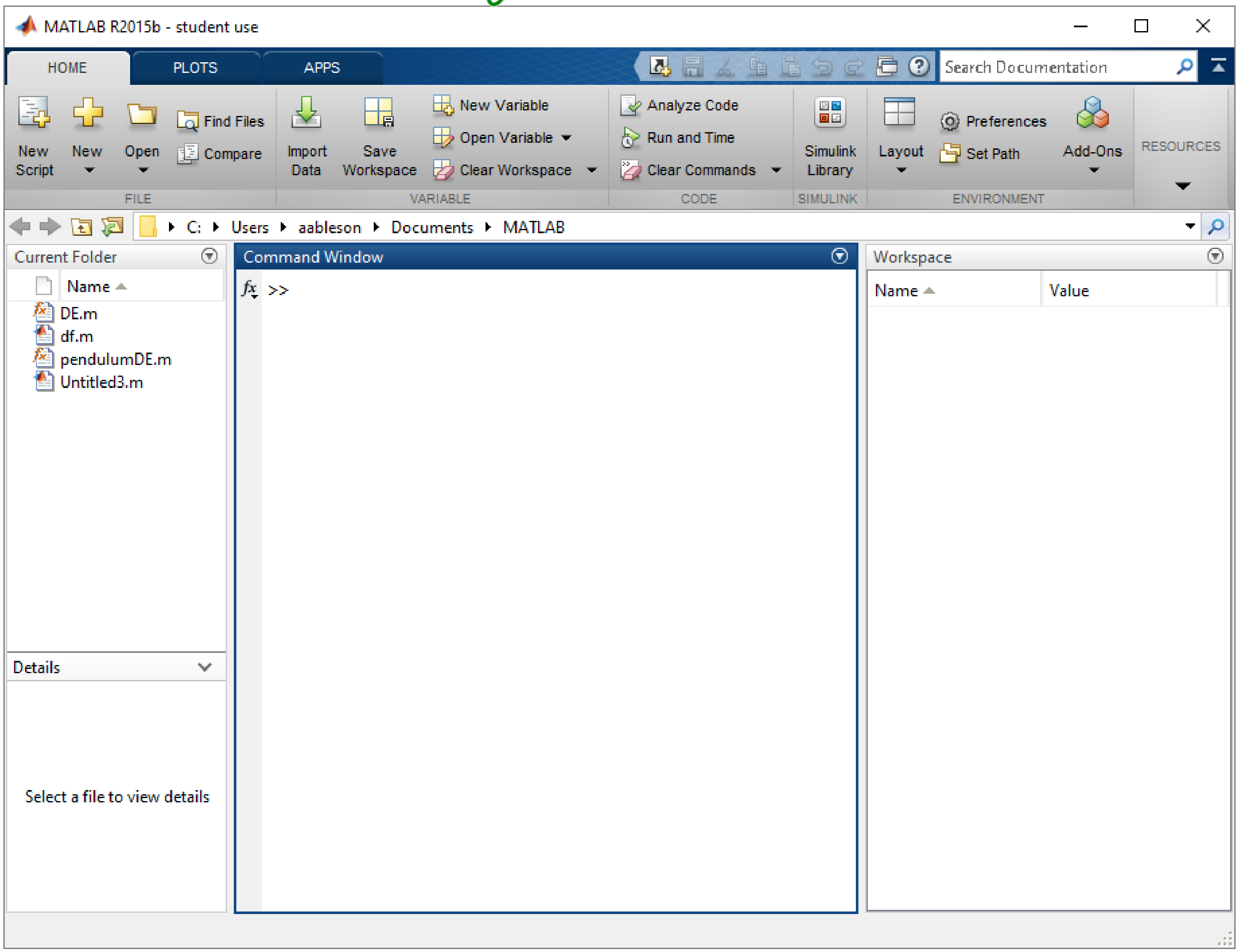
Extra caution: what units would the angle θ_B have to be in for this approximation to work?

θ measured in radians

MATLAB Orientation

Matrix Laboratory

$$\begin{bmatrix} 3 & 4 & 6 \\ -1 & 2 & 1 \end{bmatrix}$$



MATLAB - Basic Commands

Basic Usage:

- Enter simple calculations at the command prompt
- Define variables at the command prompt
- Use cursor keys to revisit previous commands
- Use first letters and cursor keys for more control

Scripts

Once you get past using single commands, you should store your program in a 'script', or M-file.

.m

- File/New/Blank M-File, or
- New Document button

You can run a script directly from the editor.

- F5, or
- Press the Play button

Further Commands

- Standard mathematical functions are built in:
  - `cos`, `sin`, `log`, `log10`, `exp`

    $\log_{10} = \log$

    ln

    e^x
- Suppress output with semi-colon
- User variable names should be different from built-in functions, variables
- File/Clear Workspace (or type `clear`) to clear all variables

Matrices and MATLAB

(Nearly) all MATLAB variables are matrices.

matrix

- scalars are `1x1` matrices
- vectors are `1xn` or `nx1` matrices
- `size` function gives dimensions of a matrix

$[2]$

$[1 \quad 2 \quad 3 \quad 4 \quad 5]$

Manually Creating Matrices

Use square brackets to create matrices or vectors from list of numbers.

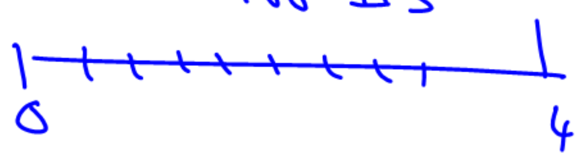
- `v = [1 3 10]`

Use colon notation to generate vectors by counting:

- `w = 0:10` — count by 1's
- `z = 10:0.1:12`

Use the `linspace` command to generate evenly-spaced values with over a range.

100 #'s



- `x = linspace(0, 4)`
- `x = linspace(0, 4, 10000)`

Graphing in MATLAB

Basic syntax for generating a graph in MATLAB is:

```
close all
x = linspace(-2, 2);
y = exp(x);
plot(x, y)
```

Matrices and Graphing

Try the following code to plot of a parabola

```
x = linspace(-2, 2)
y = x^2
plot(x, y)
```

This is a very common issue when using MATLAB.

Matrix

`x^2` means:

x * x matrix version
of multiplication

`x.^2` means:

.*

./

x=[0, 1, 2, 3 4]

each
element of x, square it separately $y=[0^2, 1^2, 2^2, 3^2, 4^2]$

Tangent Lines in MATLAB

Problem. Use MATLAB to graph the function $y = \frac{1}{x}$, and the tangent line to that graph at $x = 4$.

$y = \frac{1}{4} - \frac{1}{16}(x-4)$ tangent line

Useful new commands:

- `hold on`
- `xlim([2, 5])`
- `ylim[[0, 1])`

Problem. Use MATLAB to graphically show the relationship between $y = \tan(x)$ and its linear approximation $y = x$ for small x values.

Solving Non-Linear Equations

One surprisingly challenging area of mathematics is back in pre-calculus: simply solving equations with non-linear elements.

Problem. Compare the difficulty in solving these two single-variable equations.

Linear: $5x = 10$

$$x = 10/5 = 2$$

Non-linear: $e^x = x + 5$??

$$\ln(e^x) = \ln(x+5)$$

$$x = \ln(x+5) \;??$$

by hand

Some special types of non-linear equations **can** be solved algebraically.

Problem. Find, by hand or with the help of a calculator, the solutions to the following equations.

poly'ls

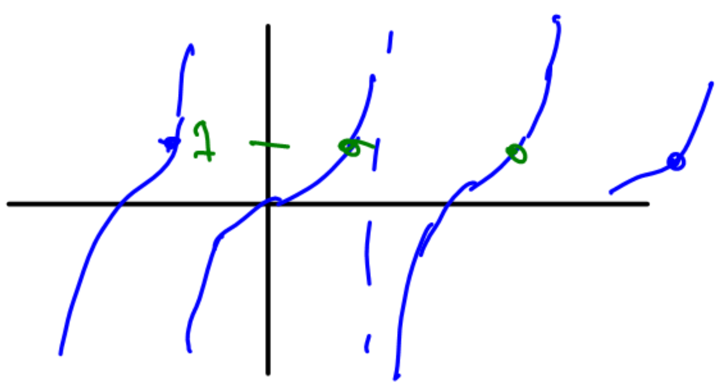
$$x^2 + 2x + 3 = 0$$

$$(x+3)(x-1) = 0$$

$x = -3, 1$ are solutions

$$10^{\log_{10}(x)} = 10^{3}$$

$$x = 10^3 = 1000$$

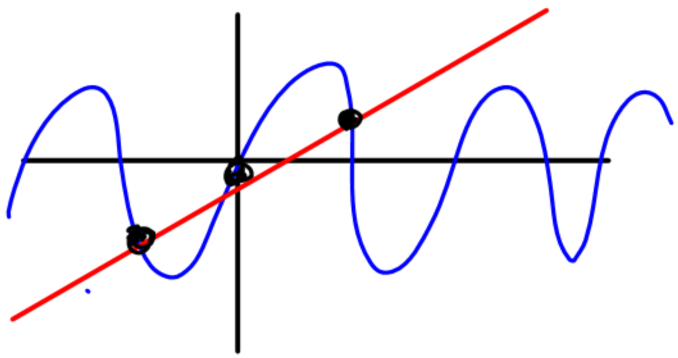


$$\frac{\sin(3x)}{\cos(3x)} = \cancel{\cos(3x)}\ 1$$

$$\tan(3x) = 1$$

Unfortunately, solving using algebra requires understanding how to manipulate specific functions. Worse yet, some equations can be simply too complex to solve algebraically.

Problem. Try to solve the following equations by hand:



y= y=

$$\sin(3x) = x$$

can't isolate x...

$$xe^{-x} = 5$$

can't isolate x = ...

$$\sin(3x) = x$$
$$xe^{-x} = 5$$

In these more difficult cases, if we want a solution we must resort to **numerical methods**, which are all **fancy versions of guess and check**! This means numerical solutions are a poor second choice, compared to by-hand solving:

- Numerical solutions give no insight into the solution (existence, patterns).
- Numerical solving usually requires some amount of trial and error by the user.

Example - Trajectories

To generate a motivation for solving non-linear equations, we are going to simulate the launch a motorcycle off the end of a ramp.

Alex Harvill, age 19, lands 425 foot ramp-to-dirt record in May 2012

Speed angle

The launch parameters are v, θ, and y_0.
If you start with some basic physics equations, and break the trajectory into components, you can arrive at the following formula:

parabola/quadratic in x

$$y = y_0 + \tan(\theta)\, x - \frac{1}{2}\frac{g}{\cos^2(\theta)(v)^2}\, x^2$$

Problem. Write a MATLAB script that plots the trajectory of the motorcycle. Use

- $y_0 = 3$ m,
- $v = 20$ m/s, and
- $\theta = 30$ degrees.

$= \frac{30}{180}\pi$

Choose the x interval so it shows the impact point.

Add a horizontal line on the trajectory graph that shows the ground level, $y = 0$. Draw it in black.

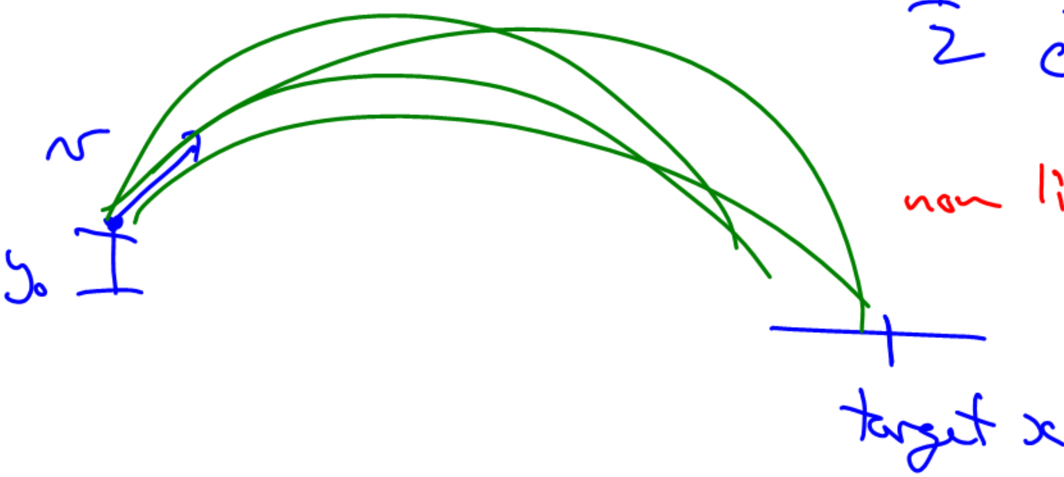
Launch Angle - Ballistics

Now consider the ballistics targeting problem, with practical consequences on battlefields around the world.

Given the launch velocity, height, and a target x, what launch angle should be used?

$$0 = y_0 + \tan(\theta)\cdot 30 - \frac{1}{2}\frac{g}{\cos^2(\theta)\, v^2}(30)^2$$

non linear equation in θ



$$x e^{-x} = 3$$

Problem. Set $y_0 = 3$, and $v = 20$. Experiment with the launch angle θ in MATLAB to find an angle that lands the motorcycle at $x = 30$.

$\theta \cong 16.75^\circ \longrightarrow$ lands at $x \cong 30.01$

not simple/ manual

okay but not great

Comment on the simplicity and accuracy of this approach.

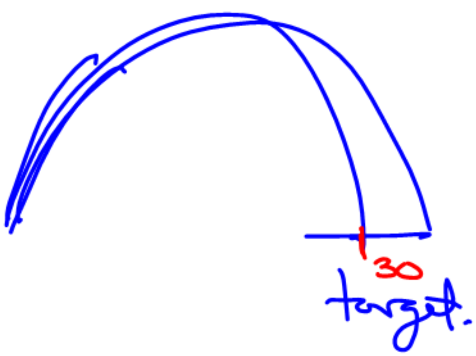
Root-Finding vs Equation Solving

In the ballistics example, it turns out to be very laborious, and fairly inaccurate, to manually experiment with angles to find the launch angle that will launch a projectile on to a target. We would like to find a more systematic and robust approach.

Problem. Express the targeting problem as a solution to a non-linear equation.

$$0 = 3 + \tan(\theta)\cdot 30 - \frac{1}{2}\frac{g}{\cos^2\theta \cdot 20^2}30^2$$

equation in θ



We will return to that example, but first we will consider the more general challenge of solving non-linear equations, to give us a wider context.

Consider the following equations.

$0 = \ldots$

$$\text{Example 1: } \sin(x) = \frac{1}{10}\ln(x)$$

$$\text{Example 2: } x^5 - x^{10} = e^{-x}$$

Solving these equations as written involves balancing **two** functions, but life would be easier if we only had to deal with **one**.
Rewrite each equation so that only one function is required.

$$\underbrace{\sin(x) - \frac{1}{10}\ln(x)}_{f(x)} = 0$$

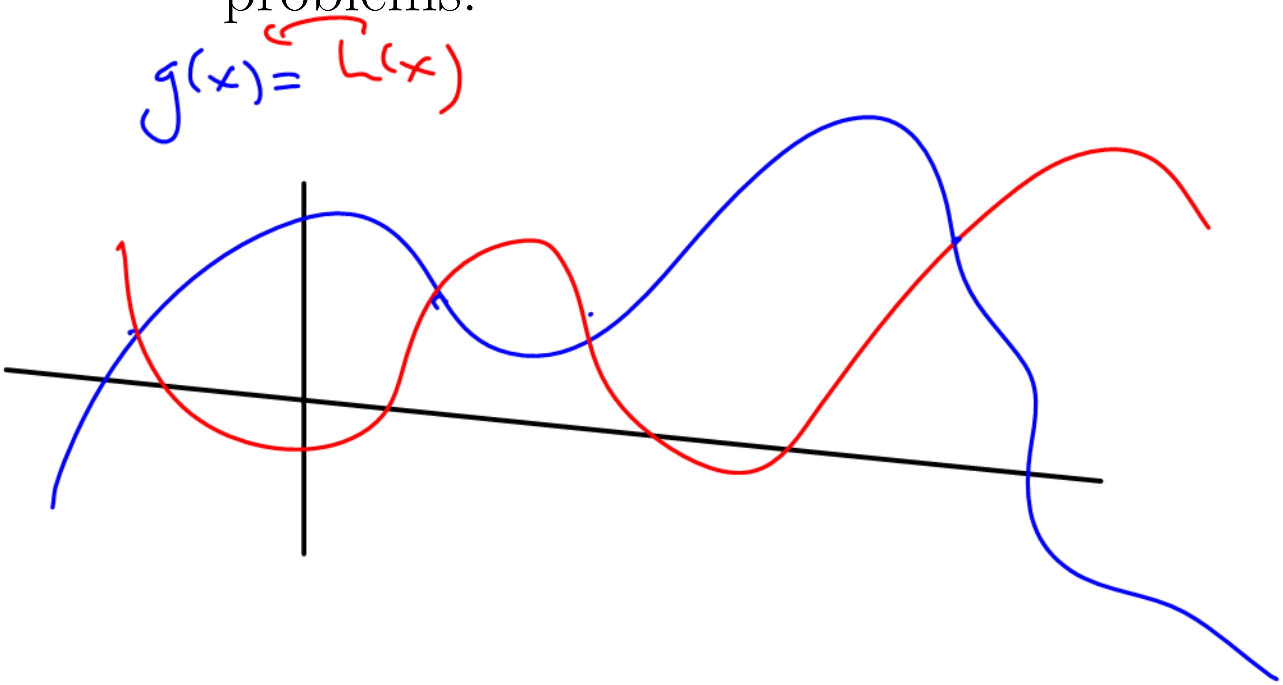
$$= 0$$

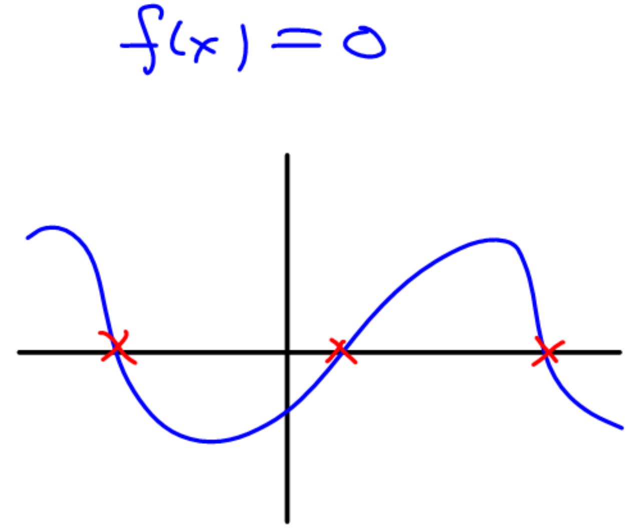
$$x^5 - x^{10} - e^{-x} = 0$$

Any non-linear equation in the form $g(x) = h(x)$ can be written as $f(x) = 0$ simply by moving all terms to the left side; the solutions to both forms will be the same.

The $f(x) = 0$ form is preferable because:

- it only involves a single user-given function, and
- all *non-linear equation solving* problems become *root-finding* problems.



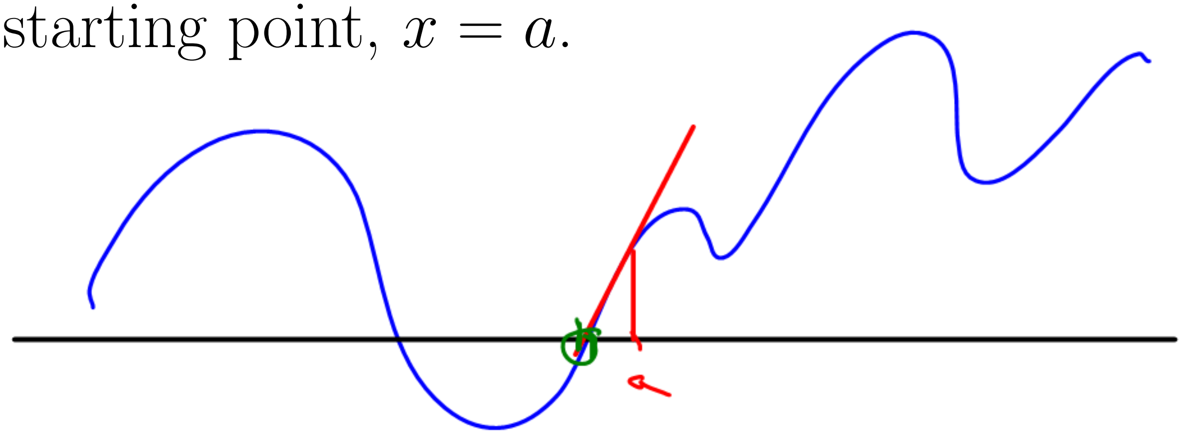


Newton's Method - Concept

Concept: If we are looking for the **root of a non-linear function**, here is a reasonable search strategy.

- Find a reasonable starting point for our solution search, $x = a, y = f(a)$.
- Imagine that $f(x)$ is approximately linear.
- Find the tangent line to the function at $x = a$.
- Find c, the root of the tangent line.
  - The new $x = c$ point should be closer to the real root of $f(x)$ than our original starting point, $x = a$.

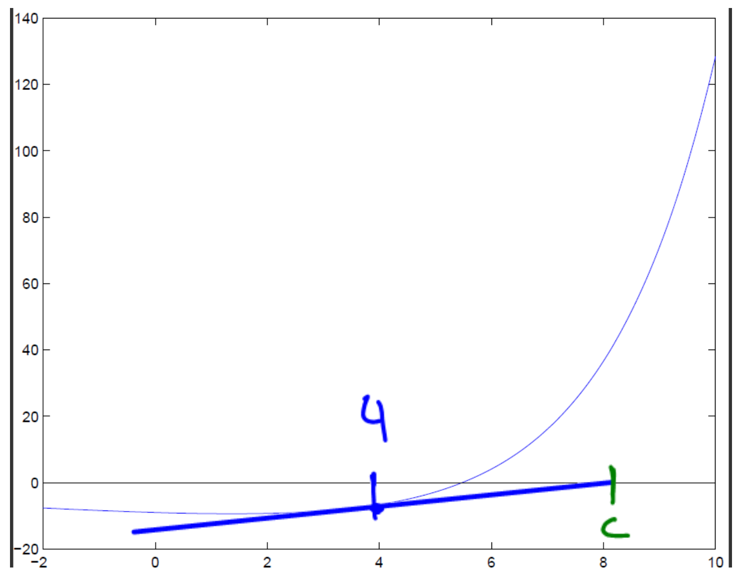
$f(x) = 0$

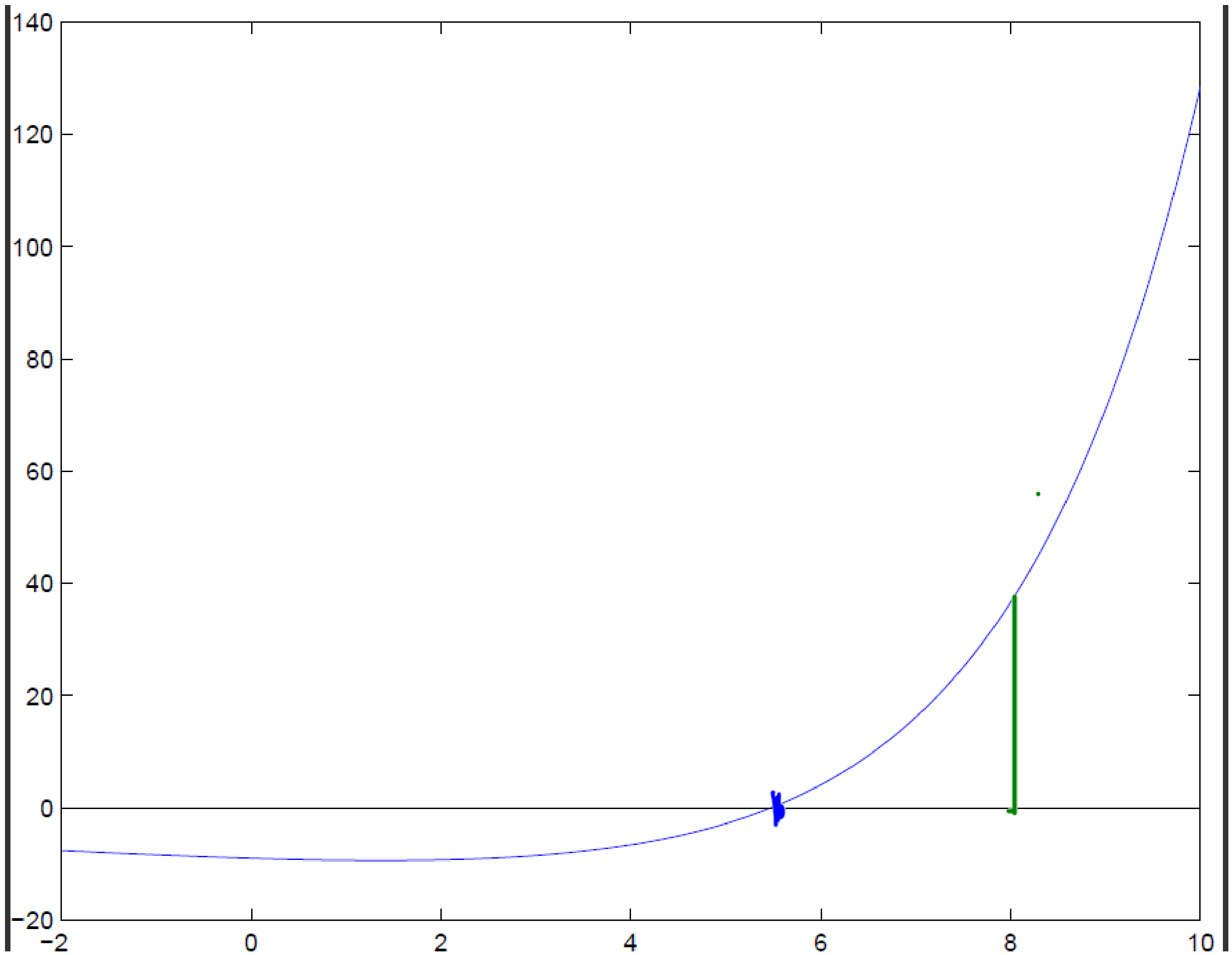


Graphical Example

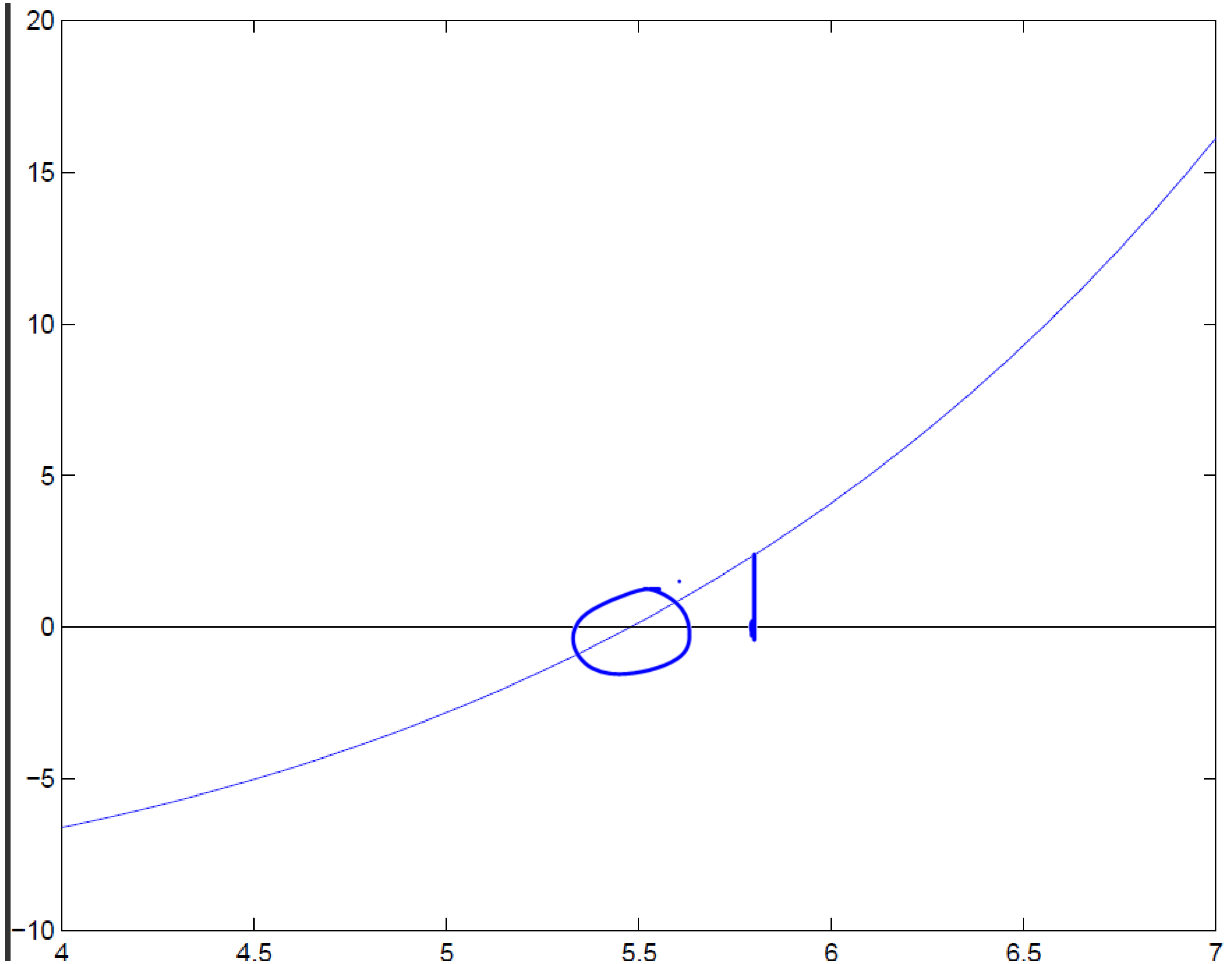
We want to find the x values where the graph below crosses $y = 0$.

- Suppose we start at initial starting point of $a = 4$.
- Draw the tangent line at $x = a$, and follow it to its root, $x = c$.
- Replace a with the new point c.
- Repeat the last two steps until a and c are almost the same, or don't change between iterations.





140
120
100
80
60
40
20
0
−20
−2
0
2
4
6
8
10


20
15
10
5
0
−5
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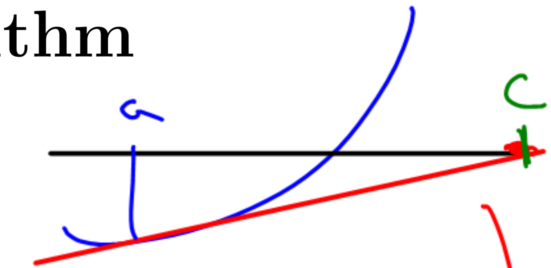
5.5
6
6.5
7

MATLAB Simulation

Newton's Method - Computational Algorithm



The two key steps of Newton's method are:

- Find the tangent line to the function at $x = a$.
- Find c, the root of the tangent line.

Problem. How do we calculate the root of the tangent line to $f(x)$, if the line is based on the point $x = a$?

tangent line: $y = f(a) + f'(a)(x-a)$

Solve by hand for x when $y=0$

$$0 = f(a) + f'(a)(x-a)$$

$$f'(a)(x-a) = -f(a)$$

$$(x-a) = -\frac{f(a)}{f'(a)}$$

$$\boxed{x = a - \frac{f(a)}{f'(a)}}$$

c ''

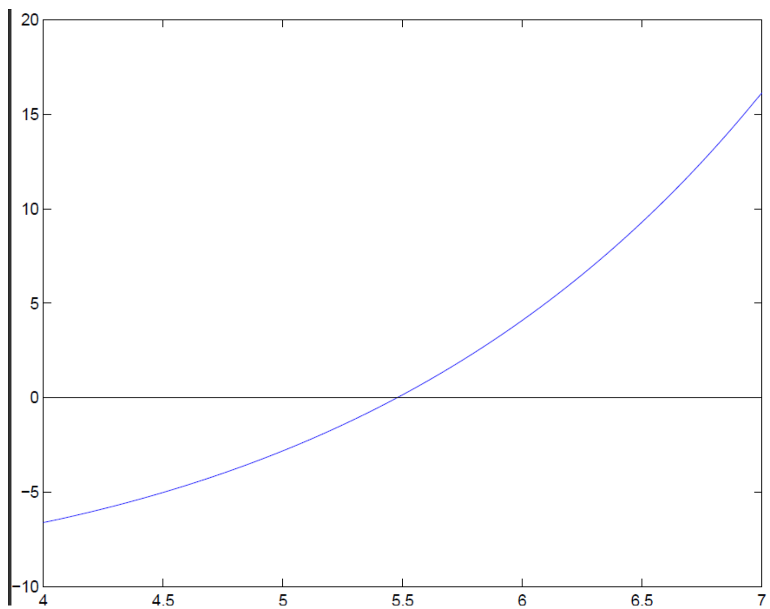
Problem. In MATLAB, use Newton's method to find an approximation solution to

$$15.47.. = 15.47...$$

$$e^{x/2} = x + 10 \qquad x \approx 5.4790$$

$$\underbrace{e^{x/2} - x - 10}_{f(x)} = 0$$

Need $f'(x) = e^{x/2}\left(\frac{1}{2}\right) - 1 = (0.5)e^{x/2} - 1$



Graph of $f(x) = e^{x/2} - x - 10$

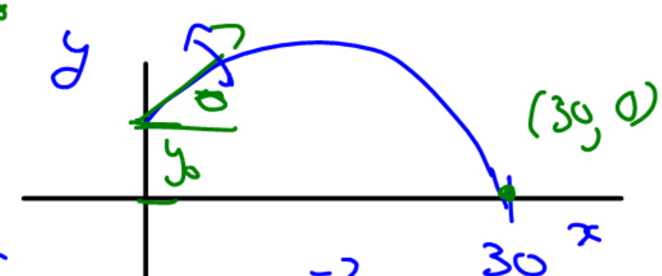
Newton's Method - Trajectory Solution

Problem. In MATLAB, use Newton's method to find the launch angle that will land the a projectile at $x = 30$, using the formula

$$\overset{0}{\cancel{y}} = \underbrace{\overset{3}{\cancel{y_0}} + \tan(\theta)\, \overset{30}{\cancel{x}} - \frac{1}{2}\frac{g}{\cos^2(\theta)(\underset{20}{\cancel{v}})^2}\, \overset{30}{\cancel{x}}^2}_{f(\theta)}$$

v

with $y_0 = 3$ m and $v = 20$ m/s.



$$f(\theta) = y_0 + x\tan(\theta) - \frac{1}{2}\frac{g x^2}{v^2}(\cos(\theta))^{-2}$$

so $\Big\downarrow \, d/d\theta$

$$f'(\theta) = 0 + x\sec^2(\theta) - \frac{1}{\cancel{2}}\frac{g x^2}{v^2}\left(-\cancel{2}(\cos(\theta))^{-3}\cdot(-\sin(\theta))\right)$$

$$f'(\theta) = x\sec^2(\theta) - \frac{g x^2}{v^2}\frac{\sin\theta}{(\cos(\theta))^3}$$

θ is <u>radians</u>

$$c = a - \frac{f(a)}{f'(a)}$$

Comments

Newton's method is a sophisticated method for solving non-linear equations. Some quick notes:

- The starting value of $x = a$ must be **close to** the root (see practice problems).
  - In practice, people will graph the function $f(x)$ to get a rough idea for where the roots might be.
- Newton's method is vastly superior to using guess-and-check, or using loops, to find high-accuracy approximate solutions.