MNTC P01 - Week #8 - Second Order Differential Equations

1. Use `ode45` to generate a graph of the solution to the following DEs, over the specified interval, given the initial condition.

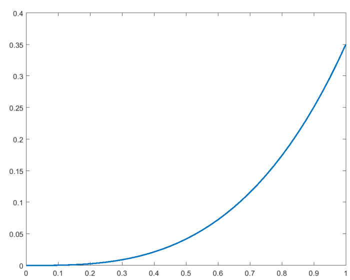
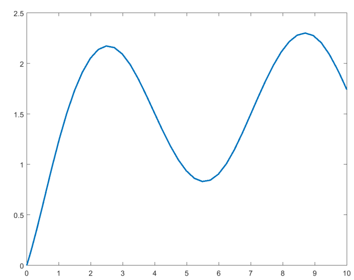
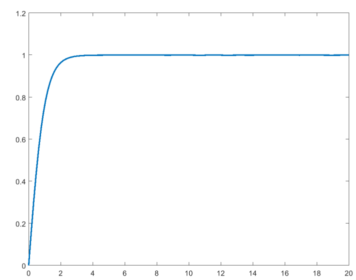
   (a) $\dfrac{dy}{dt} = t^2 + y^2$, $y(0) = 0$, and $0 \leq t \leq 1$.

   (b) $\dfrac{dy}{dt} = \sin(t) + \cos(y)$, $y(0) = 0$, and $0 \leq t \leq 10$.

   (c) $\dfrac{dy}{dt} = (1 - y^2) + 0.2\sin(t)$, $y(0) = 0$, and $0 \leq t \leq 20$.

Link to the MATLAB code:
W08DE01.m

Here are the graphs of the solutions.

2. Newton's law of heating and cooling states that an object with temperature T in an environment at temperature T_{ext} will heat up or cool down according to the differential equation

$$\frac{dT}{dt} = -k(T - T_{ext})$$

   Consider a garage used as a workshop. Its insulation and surface area give k a value of 0.1, if time t is measured in hours and the temperatures, T and T_{ext}, are in degrees Celsius.

   The temperature outside changes during the day, as described by the formula

$$T_{ext} = 10 + 7\cos\left(\frac{\pi}{12}t\right)$$

   We now imagine that the power goes out, with the garage at 23^o C at $t = 0$.

   (a) Use ode45 and the DE to generate a numerical prediction of the garage's temperature T over time. Graph the solution over a time interval that shows both the initial and long-term behaviour of the temperature.

   In your script, try to use the functions `title`, `xlabel`, `ylabel`, and `legend` to annotate the graph to make it easier for a reader to understand.

   For the following questions, just use the graph or the numerical prediction of the temperature. You are *not* expected to solve the DE analytically.

   (b) How many days does it take for the garage to get into a consistent temperature cycle? (You will need to estimate this by eye.)

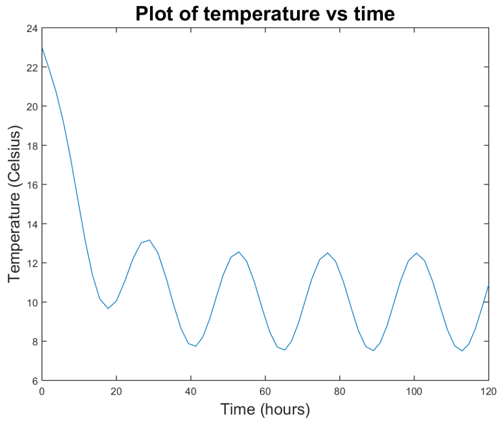
   (c) How many degrees does the temperature in the building fluctuate by, once the temperature gets into a steady cycle?

   (d) Suppose the building were better insulated, so that the rate of heat loss were cut in half. Should k be half as large, or twice as large?

   (e) Generate a numerical prediction for the temperature over time in the better-insulated scenario, and produce a graph of the temperature vs time for both scenarios on the same axes.

   (f) How large are the temperature fluctuations in the building, now that the extra insulation has been added? Does halving the net heat flow also halve the net temperature fluctuations?

(a) A graph of the temperature over time is shown below:



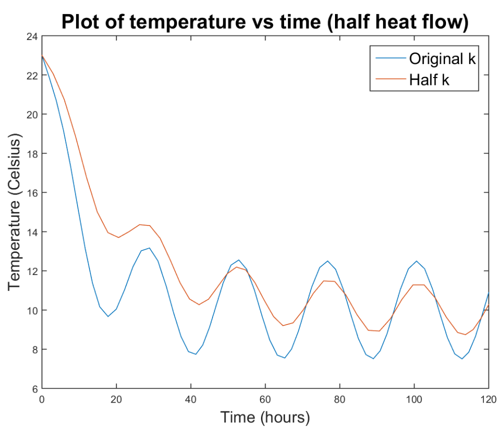
The file W08GarageTemp.m has the MATLAB code that generated the graph above.

(b) From the graph, it takes the building roughly 2 days (48 hours) to get into a repeating cycle of temperature variation.

(c) Careful zooming of the graph (or a look at the y values in the ode45 output) give a highest temperature of 12.5 (high) and 7.5 (low), for a net fluctuation of approximately 2.6 degrees per day.

(d) k represents the coefficient of heat flow between the building and the environment. The bigger k is, the *larger* the headflow between the two. Since we're adding insulation, this should *reduce* the heat flow, and so *lower* the value of k.

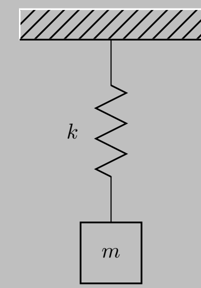
(e) A graph of the heat change over time, given better insulation, is shown below.



(f) Zooming in on the peaks of the graph, the temperature now fluctuates between approximately 11.3 and 8.8 degrees Celsius, for a range of 2.5 degrees. This *is* roughly half the magnitude of the fluctuations we saw earlier.

Modelling Spring Systems

3. Consider the single spring/mass system shown to the right, with no damper.
Newton's second law gives us the relationship:



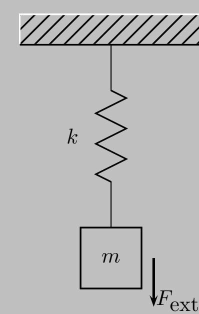
$$ma = \sum F = F_{\text{spring}}$$
$$mx'' = -kx$$

where k is the spring constant.

(a) By hand, write this second order DE as a system of 1st order DEs, using the new variables $w_1 = x$ and $w_2 = x'$

(b) Write a MATLAB function file called `springDE1.m` starting with the first line
`function dw_dt = springDE1(t, w, m, k)`
that implements the system of differential equations from part (a).

4. Consider the single spring/mass system shown to the right, with no damper.
The F_{ext} shown is an external applied force.
Newton's second law gives us the relationship:



$$ma = \sum F = F_{\text{spring}} + F_{\text{ext}}$$
$$mx'' = -kx + F_{\text{ext}}$$

where k is the spring constant.

(a) By hand, write this second order DE as a system of 1st order DEs, using the new variables $w_1 = x$ and $w_2 = x'$

(b) Write a MATLAB function file called `springDE1.m` starting with the first line
`function dw_dt = springDE1(t, w, m, k)`
that implements the system of differential equations from part (a).
For this function, **only include the spring force**, leaving the $F_{\text{ext}} = 0$ or absent.

(c) Create a new MATLAB script. In the script, set $m = 0.5$ kg, $k = 10$ N/m. Use `ode45` to simulate the motion of the spring, given an initial displacement of $x(0) = 0.2$ m, and initial velocity of zero: $x'(0) = 0$. Generate a plot with

- position against time (do *not* show the velocity), and
- choosing the time interval used for the `ode45` simulation to show the first 4 to 5 cycles only.

(d) We will now incorporate an external force of the form $F_{\text{ext}} = a\sin(bt)$. Write a MATLAB function file called `springDE2.m` starting with the first line
`function dw_dt = springDE2(t, w, m, k, a, b)`
that implements the system of differential equations from part (a), now with the external force included.

(e) Create a new MATLAB script. In the script, again set $m = 0.5$ kg, $k = 10$ N/m, and use $a = 5$ and $b = 1$ in $F_{\text{ext}} = a\sin(bt)$. Use `ode45` to simulate the motion of the spring for 30 seconds (`tspan = [0, 30]`), given an initial displacement of $x(0) = 0.2$ m, and initial velocity of zero: $x'(0) = 0$.

(f) Explain why the motion looks so disorganized.

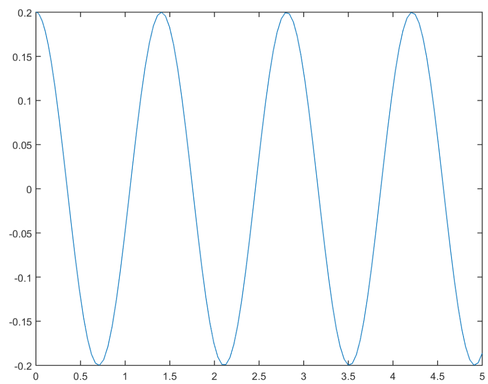
(g) Repeat Question (4e), but with an external force of $F_{\text{ext}} = \sin(4t)$. Explain why the motion in this case has cyclic waves in its amplitude.

(a) The first-order system would be:

$$\frac{d}{dt}w_1 = x' = w_2$$
$$\frac{d}{dt}w_2 = x'' = \frac{1}{m}\left(-kx + F_{\text{ext}}\right) = \frac{1}{m}\left(-kw_1 + F_{\text{ext}}\right)$$

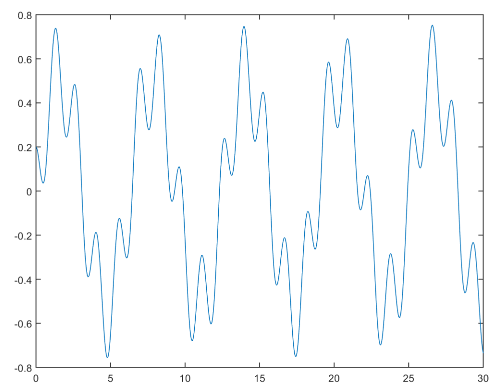
(b) The function file springDE1.m implements the differential equation system, with the F_{ext} term left out.

(c) The main script W08SpringSimulation01.m has the code that will run this simulation.
In the resulting plot, we see a very nice example of simple harmonic motion.



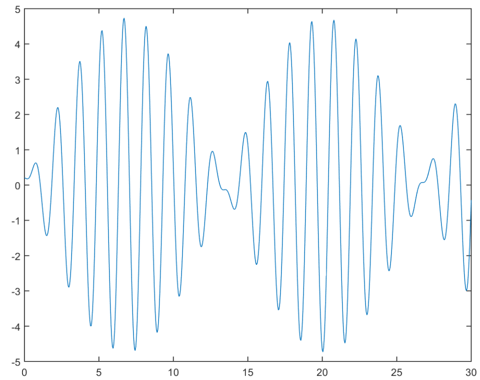
(d) The file springDE2.m implements the differential equation system, with new external force $F_{\text{ext}} = a\sin(bt)$.

(e) The file W08SpringSimulation02.m has the code that will run this simulation.
In the resulting plot, see some wildly varying and irregular oscillations.



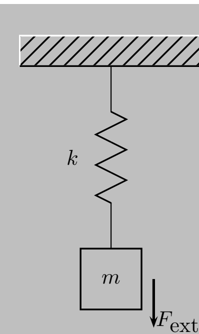
(f) The motion of the mass looks very disorganized because the natural frequency (the frequency at which the mass would oscillate if you just let swing on its own) is different from the frequency that we are pushing and pulling on it with through F_{ext}.
Recall: the natural frequency of a spring/mass system is given by $\omega = \sqrt{k/m}$, which for this scenario gives $\omega = \sqrt{10/0.5} \approx 4.47$ rad/s.

(g) The file W08SpringSimulation02.m has the code that will run this simulation. Here is a graph of the resulting mass motion.



In the plot, we see that the natural frequency and the regular stimulation by the outside force are close to each other: the natural frequency is $\omega = \sqrt{\frac{10}{0.5}} \approx 4.5$, rad/s, and the stimulating frequency is at $\omega = 4$ rad/s. This close match of the frequencies leads to the phenonenon called *beats*, or *near resonance*.

5. We return to the same single spring/mass system from Question 4, shown to the right, with no damper.
The F_{ext} shown is an external applied force.



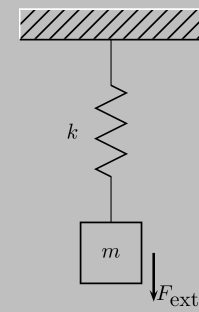
(a) For a mass of $m = 5$ kg , and a spring constant of $k = 2$ N/m, what is the natural frequency of the system?

(b) Define an external force of the form $F_{\text{ext}} = \sin(bt)$ that will produce **resonance** in the system.

(c) Use MATLAB to simulate the motion of the spring, with your selected external force, using an initial condition where the mass starts at its equilibrium and at rest.

(d) The system will break if the oscillations become too large, specifically if $x(t)$ exceeds 0.4 m. Does the system break, and if so, how long does it take for the system to break?

6. Damped spring
Give the differential equation.



(a) We define a system with a mass of $m = 10$ kg, spring constant $k = 2$ N/m, and a damping coefficient of $c = 0.2$ N/(m/s). It the system is displaced by 0.5 m and then let go with zero initial velocity, how long (in both seconds and cycles) does it take for the oscillations to reach approximately 10% of their original amplitude?

(b) What damping coefficient would be needed for the oscillations to be reduced to 10% of their original amplitude within 3 cycles? You will need to estimate your answer based on guessing and checking against the graph.

(c) Double the mass to 20 kg. What damping coefficient would be needed now for the oscillations to be reduced to 10% of their original amplitude within 3 cycles? You will need to estimate your answer based on guessing and checking against the graph.

7. For a spring/mass system with $m = 1$ kg, $c = 6$ N/(m/s), and $k = 45$, approximately what frequency of external forcing would produce the largest amplitude steady-state vibration? You will need to estimate the answer based on guessing and checking against the graph.

The natural frequency is found using the auxiliary equation $r^2 + 6r + 45 = 0$, giving $r = -3 \pm 6i$, so $x_c = c_1 e^{-3t}\cos(6t) + c_2 e^{-3t}\sin(6t)$

Since this system's natural or intrinsic oscillations are at 6 rad/s, the spring/mass will show the largest amplitude response to an outside force if that outside force also has a frequency close to 6 rad/s.

8. For a spring/mass system with $m = 1$ kg, $c = 10$ N/(m/s), and $k = 650$, approximately what frequency of external forcing would produce the largest amplitude steady-state vibration? You will need to estimate the answer based on guessing and checking against the graph.

$m = 1, c = 10, k = 650, F_0 = 100$, so

$$x'' + 10x' + 650x = \cos(\omega t)$$

The natural frequency is found using the auxiliary equation $r^2 + 10r + 650 = 0$, giving $r = -5 \pm 25i$, so $x_c = c_1 e^{-5t}\cos(25t) + c_2 e^{-5t}\sin(25t)$

It will show its largest amplitude response to stimuli with frequencies close to 25 rad/s.

9. Consider the equation for the spring/mass system with $m = 1$ kg, $c = 4$ N/(m/s) and $k = 4$ N/m, and which is being forced by an external periodic force of $10\cos(3t)$ N:

$$x'' + 4x' + 4x = 10\cos(3t)$$

(a) Use a MATLAB simulation to estimate the amplitude of the steady-state oscillations in this system.

(b) If you modify the frequency of the external force, so $F_{\text{ext}} = 10\cos(bt)$, use trial and error to find a frequency that produces steady-state oscillations with amplitude at least twice as large as you found in part (a).

The system is damped, so the x_c solution will have negative exponentials in it. i.e. that part of the solution will be **transient** because its contribution $\to 0$ over time. That is why x_c is referred to as $x_{\text{transient}}$ or x_{tr} throughout the problems in this section.

Since the RHS is pure cos/sine, its differential family will not overlap with the exp sin/cos in x_c. This means x_p will be a simple linear combination of $\cos(3t)$ and $\sin(3t)$.

Let $x_p = A\cos(3t) + B\sin(3t)$.
$x_p' = -3A\sin(3t) + 3B\cos(3t)$
$x_p'' = -9A\cos(3t) - 9B\sin(3t)$

Substituting into the DE,

$$(-9A\cos(3t) - 9B\sin(3t)) + 4(-3A\sin(3t) + 3B\cos(3t)) + 4(A\cos(3t) + B\sin(3t)) = 10\cos(3t)$$

$$\begin{aligned} \text{sine coeffs:} \quad & -9B - 12A + 4B = 0 & \qquad & -12A - 5B = 0 \\ \text{cos coeffs:} \quad & -9A + 12B + 4A = 10 & \qquad & -5A + 12B = 10 \end{aligned}$$

$$\begin{aligned} -60A - 25B &= 0 \\ -60A + 144B &= 120 \end{aligned}$$

$$-169B = -120$$

Solving gives $A = \frac{-50}{169}, B = \frac{120}{169}$, so

$$x_p = x_{sp} = \frac{-50}{169}\cos(3t) + \frac{120}{169}\sin(3t)$$

The amplitude of the sum $A\cos(bt)+B\sin(bt)$ is given by $C = \sqrt{A^2 + B^2}$, so the amplitude here is $\sqrt{\left(\frac{-50}{169}\right)^2 + \left(\frac{120}{169}\right)^2} \approx$ 0.77 meters or 77 cm.