

MNTC P01 - Week #6 - Integrals - Modeling

Numerical Integration

As part of this assignment, you should be able to reproduce the LHR rule calculations in MATLAB using a loop. You should know how to adapt it to handle either data from a file, or a function defined by a formula, as the case requires.

1. **Theory** Consider the problem of estimating the general form of the integral

$$\int_a^b f(x) dx$$

- (a) Assume $f(x)$ is a smooth and continuous function. For our the Left-Hand sum, $\text{LEFT}(n)$, by what factor do we reduce the error if we use 10 times the number of intervals?
- (b) Evaluate the integral $\int_0^6 \cos(x) dx$ exactly, using anti-derivatives and the Fundamental Theorem of Calculus.
- (c) Confirm your answers to part (a) by finding the change in the error for $\text{LEFT}(n)$ for the same integral, $\int_0^6 \cos(x) dx$ using $n = 20$, $n = 200$, and $n = 2000$. Find the error with each n value, and then compute the ratio of the errors each time you use $10\times$ as many intervals.

- (a) If $f(x)$ is a smooth function, if we use 10 times the intervals, the error for LHR's estimate will drop by a factor of $\frac{1}{10}$
- (b) We can evaluate the integral based on our earlier study of derivatives. The only thing to be careful of is that the input values will be in radians (not degrees).

$$\int_0^6 \cos(x) dx = \sin(x) \Big|_0^6 = \sin(6) - \sin(0) = -0.279415498$$

- (c) The linked script below computes $\text{LEFT}(n)$ for $n = 20, 200$ and 2000 . It also computes the error for each of those integral estimates.

W06CosineIntegral.m

Here are the results from the script (with the values for **I** being the integrals, and **Err** being the error in the estimates.

```
I1 =  
-0.271342274784534  
Err1 =  
0.008073223414391  
I2 =  
-0.278797096021967  
Err2 =  
6.184021769584658e-04  
I3 =  
-0.279355544067250  
Err3 =  
5.995413167569907e-05
```

We can see that as we increase the number of intervals by a factor of 10, the error in the $\text{LEFT}(n)$ estimate is scaled by a factor of (approximately) $\frac{1}{10}$.

- When going to $n = 200$ from $n = 20$, the error ratio is $\frac{6.184021769584658e-04}{0.008073223414391} \approx 0.07 \approx 0.1$ or $\frac{1}{10}$.
- When going to $n = 2000$ from $n = 200$, the error ratio is $\frac{5.995413167569907e-05}{6.184021769584658e-04} \approx 0.097 \approx 0.1$ or $\frac{1}{10}$.

2. For each of the following integrals,

- Evaluate the integral exactly,
- use MATLAB to compute the LEFT(1000) integral estimate, and
- comment on the agreement between the results.

(a) $\int_0^5 x^3 - 5 \, dx$

(b) $\int_1^{10} \log_{10}(x) \, dx$

(c) $\int_{-1}^1 x^2 e^{x^3} \, dx$

(a) Exact value: $\int_0^5 x^3 - 5 \, dx = \left. \frac{x^4}{4} - 5x \right|_0^5 = \left(\frac{5^4}{4} - 5(5) \right) - (0 - 0) = 131.25$

LEFT(1000) estimate (see link below to the script that computed this): 130.9377

The agreement is pretty good, with the LEFT(1000) within 1% of the exact value.

(b) $\int_1^{10} \log_{10}(x) \, dx$.

This requires integration by parts, and remembering the derivative rule for non-base- e logarithms: $\frac{d}{dx} \log_{10}(x) = \frac{1}{x \ln(10)}$ Computing the integral (without the $x = 1$ and $x = 10$ for now to keep the notation simpler):

$$\begin{aligned} \text{We choose } u &= \log_{10} x & \text{and } dv &= dx \\ \text{so } du &= \frac{1}{x \ln(10)} \, dx & \text{and } v &= x \end{aligned}$$

Using the integration by parts formula,

$$\begin{aligned} \int \log_{10} x \, dx &= x(\log_{10} x) - \int x \frac{1}{x \ln(10)} \, dx \\ &= x \log_{10} x - \int \frac{1}{\ln(10)} \, dx \\ &= x \log_{10} x - \frac{x}{\ln(10)} + C \end{aligned}$$

This gives the definite integral value of

$$\begin{aligned} \int_1^{10} \log_{10}(x) \, dx &= \left(x \log_{10} x - \frac{x}{\ln(10)} \right) \Big|_1^{10} \\ &= \left(10 \log_{10}(10) - \frac{10}{\ln(10)} \right) - \left(1 \log_{10}(1) - \frac{1}{\ln(10)} \right) \\ &= 10 - \frac{10}{\ln(10)} + \frac{1}{\ln(10)} \\ &\approx 6.0914 \end{aligned}$$

LEFT(1000) estimate (see link below to the script that computed this): 6.0868

The agreement is better for this function, with the LEFT(1000) within 0.1% of the exact value.

(c) $\int_{-1}^1 x^2 e^{x^3} \, dx$

This integral can be evaluated exactly using a substitution.

$$\begin{aligned} \text{Let } w = x^3, \text{ so } dw = 3x^2 dx \text{ or } dx = \frac{1}{3x^2} dw \\ \int x^2 e^{x^3} dx = \int x^2 e^w \left(\frac{1}{3x^2} dw \right) = \int \frac{1}{3} e^w dw = \frac{1}{3} e^w + C = \frac{1}{3} e^{x^3} + C \end{aligned}$$

This gives the definite integral value of

$$\begin{aligned} \int_{-1}^1 x^2 e^{x^3} dx &= \frac{1}{3} e^{x^3} \Big|_{-1}^1 \\ &= \frac{1}{3} (e^1 - e^{-1}) \\ &\approx 0.7834 \end{aligned}$$

LEFT(1000) estimate (see link below to the script that computed this): 0.7811

The agreement is good for this function, with the LEFT(1000) within 1% of the exact value.

Link to the MATLAB code:

W06LeftExamples.m

3. Use the **integral** function to estimate the following integrals, and print the estimates with at least 8 digits after the decimal. Use the default accuracy for the **integral** function. Note that the exact values for these integrals were computed in the previous question, so we can compare the **integral** estimate to the exact values.

- (a) $\int_0^5 x^3 - 5 dx$ (exact value: $\frac{5^4}{4} - 25$)
 (b) $\int_1^{10} 4 \log_{10}(x) dx$ (exact value: $\frac{(-36 + 40 \ln(2) + 40 \ln(5))}{\ln(2) + \ln(5)}$)
 (c) $\int_{-1}^1 x^2 e^{x^3} dx$

Code that does this would be e.g.

```
clc
format long
f = @(x) x.^3 - 5;
a = 0;
b = 5;
I = integral(f, a, b);
exact_value = (5^4/4 - 5*5);
[exact_value, I] % display both the exact value and the integral estimate
```

A link to the full MATLAB script for all the examples is included below.

The results are

- (a) function: `f = @(x) x.^3 - 5;`
 estimated integral: 131.25000000
 (b) function: `f = @(x) log10(x);`
 estimated integral: 6.091349662870734
 (c) function: `f = @(x) x.^2 .* exp(x.^3);`
 estimate integral: 0.783467462429201

Note that all these values are the same as the exact values for all or almost all the digits shown (12 digits of accuracy). This is a lot better performance than using the LEFT(*n*) approach, and is actually easier to code because the **integral** function is built into MATLAB.

Link to the MATLAB code:

W06IntegralExamples.m

4. When a pendulum oscillates, with maximum deviation angle θ_0 , the period of the pendulum is given by

$$T = 4\sqrt{L/g} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin(\frac{1}{2}\theta_0)$ and g is the acceleration due to gravity, 9.8 m/s.

Compute and compare the period of a pendulum with

- $L = 2$, $\theta_0 = 40^\circ$,
- $L = 2$, $\theta_0 = 20^\circ$.
- $L = 2.5$, $\theta_0 = 40^\circ$,
- $L = 2.5$, $\theta_0 = 20^\circ$.

Describe how significant the effect of maximum swing angle θ_0 is on the period of a pendulum, compared to the effect of the pendulum length.

Full solution is available in: W06PendulumPeriod.m

The periods computed are show below. `integral` was used for the integration.

$L = 2.0$ m, $\theta_0 = 40.0$ deg, period = 2.9274 s.

$L = 2.0$ m, $\theta_0 = 20.0$ deg, period = 2.8602 s.

$L = 2.5$ m, $\theta_0 = 40.0$ deg, period = 3.2729 s.

$L = 2.5$ m, $\theta_0 = 20.0$ deg, period = 3.1978 s.

From these results, it is clear that the angle has a minimal effect on the period of the oscillations, compared with the effect of the length. This insensitivity of the period to the oscillation angle explains why pendulum clocks and metronomes do not need a specific swing angle to be fairly accurate, but *do* need a specific length.

5. When underground mining operations are in progress, one concern the monitoring or predicting the subsidence of the rock between the mine and the surface, as “the differential settlement and horizontal strain developed during subsidence tend to be critical in terms of structural damage”^a

If the rock is considered isomorphic in character, then complete subsidence s at a point on the surface is given by an integral of the form

$$s = \int_0^R p(r) dr$$

where R is the effective radius of influence, and $p(r)$ is an empirically derived “influence function”.

Consider the influence function $p(r) = \frac{1}{R^2} e^{-\pi r^2/R^2}$ and use an appropriate technique to evaluate the integral

$$s = \int_0^R p(r) dr$$

with $R = 100$.

^aSME Mining Engineering Handbook, 3rd edition, page 632.

The resulting integral is $\int_0^{100} \frac{1}{100^2} e^{-\pi r^2/(100^2)} dr$.

Since r is the variable, we have an integral of the form essentially $\int e^{-x^2} dx$, which **cannot** be evaluated by hand. Thus we will use a numerical method in MATLAB.

Of the two methods we have in MATLAB, `LEFT(n)` and the `integral` function, the `integral` function is both easier to use and more accurate.

The actual script required to just evaluate this integral is remarkably short.

```
R = 100;
p = @(r) (1/R^2) * exp(-pi * r.^2/R^2);
I = integral(p, 0, R)
```

and the output gives an integral value of $I = 0.0049$.

6. The x coordinate of the center of mass of a object can be computed using the formula

$$\bar{x} = \frac{\int_a^b x \cdot f(x) dx}{\int_a^b f(x) dx}$$

where $f(x)$ is the height (or mass) at the point x .

For the following functions, use MATLAB, with the `integral` function and other tools, to

- plot the graph of $f(x)$,
- compute the x center of mass of the object, and
- draw the center of mass as a vertical line on the same graph.

(a) $f(x) = x^3$ on $x = [1, 4]$.

(b) $f(x) = xe^{-x}$ on $x = [0, 10]$.

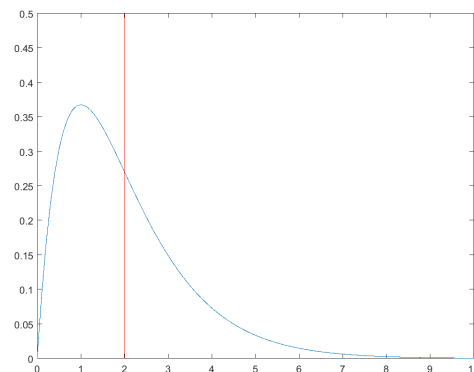
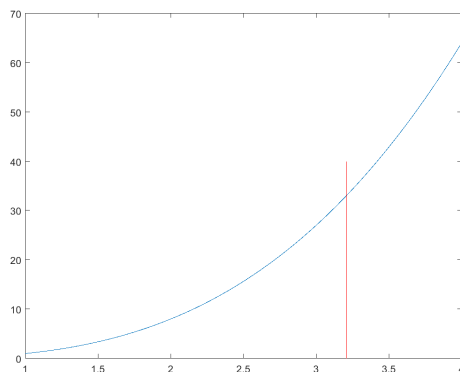
In this problem, the challenge in coding the MATLAB script is to remember that when using the `integral` function, you need to specify separate (MATLAB) function for each integral. In the listings below, we did this using the syntax

```
f = @(x) x.^3;
% Note: we also need a _new_ function for the integral in the numerator
fn = @(x) x .* f(x);
xcenter = integral(fn, 1, 4) / integral(f, 1, 4)
```

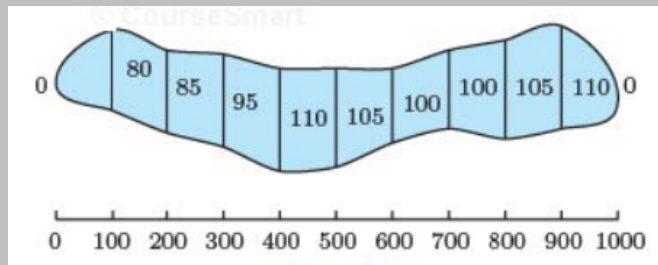
Full solutions:

W06CenterOfMass.m

- (a) The x center of mass for $f(x) = x^3$ on $[1, 4]$ is $\bar{x} = 3.2094$, or closer to the right end of the interval, because the function has higher values there.
- (b) The x center of mass for $f(x) = xe^{-x}$ on $[0, 10]$ is $\bar{x} = 1.9955$, or closer to the left end of the interval, because the function has a peak there before declining to lower values as x approaches 10.



7. The width (in feet) of the golf course fairway was measured at 100-foot intervals as indicated on the figure. Estimate the square footage of the fairway, using any appropriate means taught in the course.



NOTE: In this problem, because we are only given data and not a function, the only technique we have that would work is $\text{LEFT}(n)$. We **can't** use the Fundamental Theorem or the **integral** function because both of those require a formula for the function that we don't have.

The key steps are:

1. Create a vector of the widths, *including* the zero widths at the start and end of the fairway.
2. Note that $\Delta x = 100$ feet for each left-to-right distance.
3. Note that there are 10 intervals (11 widths), so we will be performing a $\text{LEFT}(10)$ calculation.
4. Approximate the area on each left-to-right interval as width at the left $\times \Delta x$ (this is the LEFT rule on each interval).
5. Use a loop to accumulate the total area estimated.

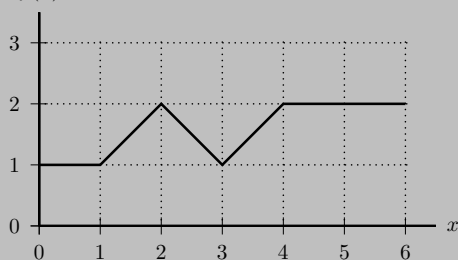
Link to the MATLAB code:
W06Golf.m

The final estimated area is 88,000 square feet. We don't have a formula for the widths, so this is simply our best estimate of the area.

Average Value

8. (a) Using the graph shown below, find

$$\int_1^6 f(x) dx.$$



- (b) What is the average value of f on $[1, 6]$?

The integral represents the area below the graph of $f(x)$ but above the x -axis.

- (a) Since each square has area 1, by counting squares and half-squares we find

$$\int_1^6 f(x) dx = 8.5.$$

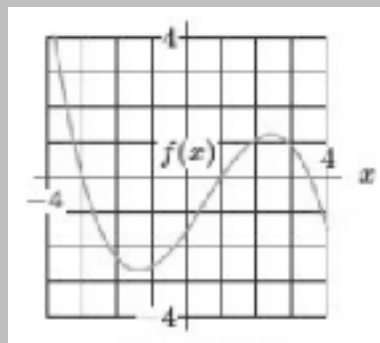
- (b) The average value is

$$\frac{1}{6-1} \int_1^6 f(x) dx = \frac{8.5}{5} = 1.7$$

9. (a) Using the graph below, estimate

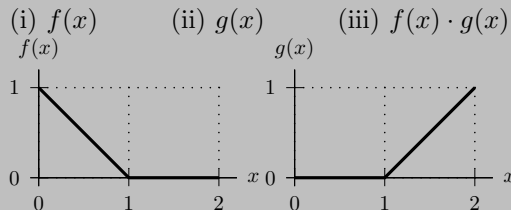
$$\int_{-3}^3 f(x) dx.$$

- (b) Which of the following average values of $f(x)$ is larger?
- (i) Between $x = -3$ and $x = 3$, or
 - (ii) Between $x = 0$ and $x = 3$?



- (a) The integral is the area above the x -axis minus the area below the x -axis. Thus, we can see that $\int_{-3}^3 f(x) dx$ is about $-6 + 2 = -4$, based on the negative of the area from $t = -3$ to $t = 1$ plus the area from $t = 1$ to $t = 3$. Note that your estimate may differ from this, depending on how you counted squares under the curved shape.
- (b) Since the integral in part (a) is negative, the average value of $f(x)$ between $x = -3$ and $x = 3$ is negative. From the graph, however, it appears that the integral of $f(x)$ from $x = 0$ to $x = 3$ is positive overall, meaning that the average value will also be positive. Hence (ii) is the larger quantity.

10. (a) Using the graphs of $f(x)$ and $g(x)$ shown below, find the average value on $0 \leq x \leq 2$ of



- (b) Is the statement that

$$\text{Average}(f) \cdot \text{Average}(g) = \text{Average}(f \cdot g)$$

true or not? Explain your answer.

- (i) Since the triangular region under the graph of $f(x)$ has area $\frac{1}{2}$, we have

$$\text{Average}(f) = \frac{1}{2-0} \int_0^2 f(x) dx = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

- (ii) Similarly,

$$\text{Average}(g) = \frac{1}{2-0} \int_0^2 g(x) dx = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

- (iii) Since $f(x)$ is nonzero only for $0 \leq x < 1$ and $g(x)$ is nonzero only for $1 < x \leq 2$, the product $f(x)g(x) = 0$ for all x . (To confirm this, try picking any x value between 0 and 2, and subbing it into the expression $f(x) \cdot g(x)$.)

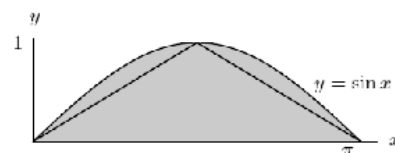
Thus

$$\text{Average}(f \cdot g) = \frac{1}{2-0} \int_0^2 f(x)g(x) dx = \frac{1}{2} \int_0^2 0 dx = 0.$$

- (b) Since the average values of $f(x)$ and $g(x)$ are nonzero, their product is nonzero. Thus the left side of the statement, $\text{Avg}(f) \cdot \text{Avg}(g)$, is nonzero. However, the average of the product $f(x)g(x)$ is zero. Thus, the right side of the statement is zero, so the statement is not true.

11. (a) Without computing any integrals, explain why the average value of $f(x) = \sin x$ on $[0, \pi]$ must be between 0.5 and 1.
- (b) Compute the exact average of $\sin x$ on $[0, \pi]$.
- (c) Use MATLAB to plot the graph of $\sin(x)$ on $[0, \pi]$, and draw the average value on the graph as well.

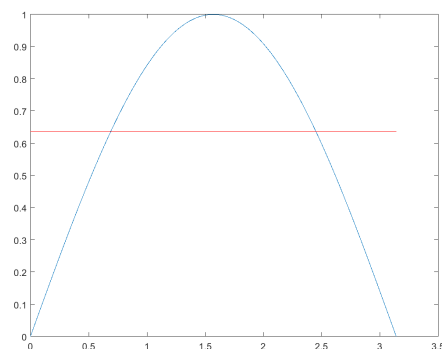
- (a) Since $f(x) = \sin x$ over $[0, \pi]$ is between 0 and 1, the average of $f(x)$ must itself be between 0 and 1. Furthermore, since the graph of $f(x)$ is concave down on this interval, the average value must be greater than the average height of the triangle shown in the figure below, namely $y = 0.5$.



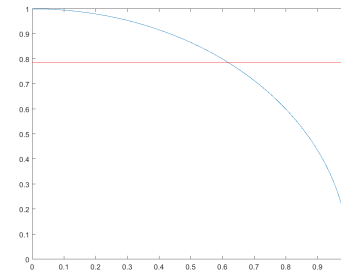
- (b) $\text{Average} = \frac{1}{\pi - 0} \int_0^\pi \sin x dx \approx 0.64.$

- (c)

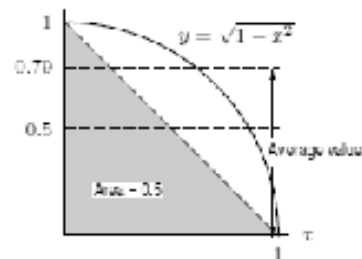
```
% Plot of sin(x) on the interval [0, pi],
close all;
x = linspace(0, pi);
y = sin(x);
plot(x, y);
hold on;
% we will re-calculate the average value h
f = @(x) sin(x);
avg = (1/pi) * integral(f, 0, pi);
plot([0, pi], [avg, avg], 'r');
```



12. (a) What is the average value of $f(x) = \sqrt{1-x^2}$ over the interval $0 \leq x \leq 1$? (Hint: we don't have the integration tools to evaluate the integral of $f(x)$ from first principles, but can use MATLAB to evaluate it.)
- (b) Use MATLAB to plot the graph of $f(x)$ on $[0, 1]$, and draw the average value on the graph as well.
- (c) How can you tell whether this average value is more or less than 0.5 without doing any calculations?



- (c) The area between the graph of $y = 1 - x$ and the x -axis is 0.5. Because the graph of $y = \sqrt{1-x^2}$ is concave down, it lies above the straight line $y = 1 - x$, so its average value is above 0.5. See the figure below.



- (a) Solution is computed in the script below, with the average value being 0.7854.
- (b) Graphing is also included in the script below.

W06RootAverage.m