Week #6 : Integrals - Modeling

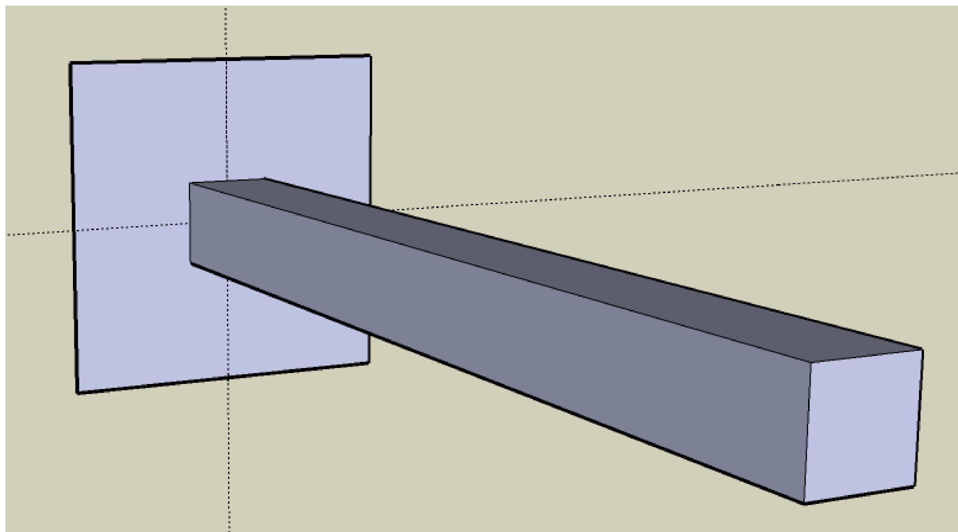
Goals:

- Use MATLAB to solve a variety of integration problems.
- Use integration to find the average value of a function.
- Use MATLAB to find the average value of a function.
- Use MATLAB to find the average value of a sequence of data.

Heat Transfer and Integrals

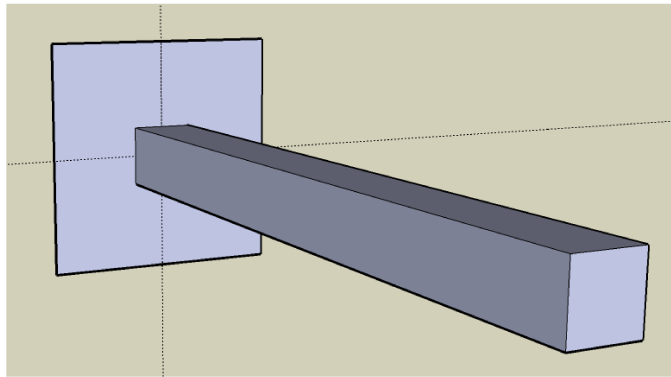
A common engineering challenge is to transfer heat generated by a motor or combustion into a nearby fluid.

This transfer is often made more effective by the use of **cooling fins**, which increase the surface area of contact with the fluid.

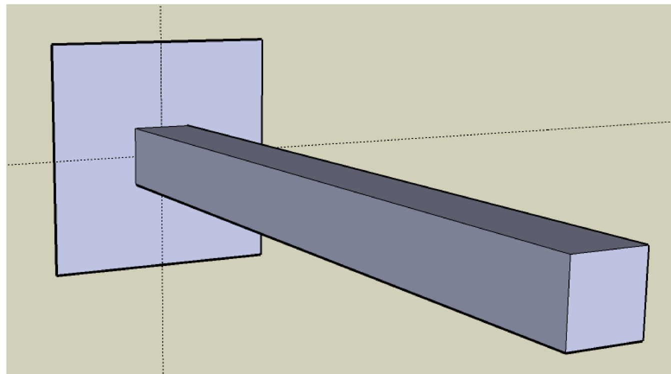


If we fix the temperature at the base, we can ask the question **“How quickly is heat radiated out of fin?”**

Problem. What factors affect the rate of heat transfer out of the fin?



The rate of heat flow from a metal fin to the environment is proportional to the temperature difference between the fin, T, and the environment, T_∞.

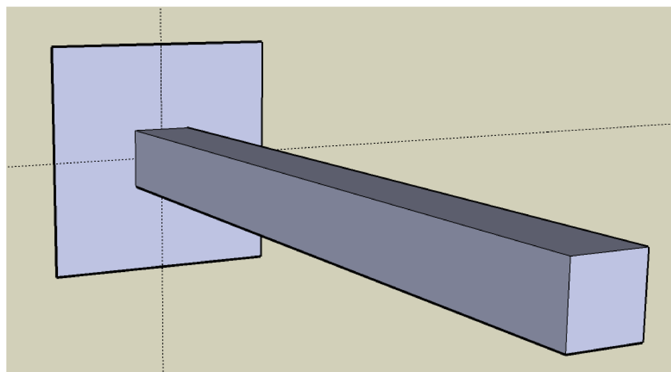


Problem. What issue is raised when trying to use this rule to compute the rate of heat flow out of the fin we are considering?

Numerical Integration - Motivation

Problem. Take a small slice of length Δx of the fin. What advantage is there to looking at a small slice, rather than the whole fin at once?

How much heat is lost through that slice?



Give an expression for the total amount of heat lost over the whole fin.

Integration

As soon as you see any sum of the form $\sum \ldots \Delta x$, you should be thinking "integral"!

For our fin example, net heat flow to environment is

If, for simplicity, we assume $T_\infty = 0$, our target integral becomes

Finding the Temperature Distribution

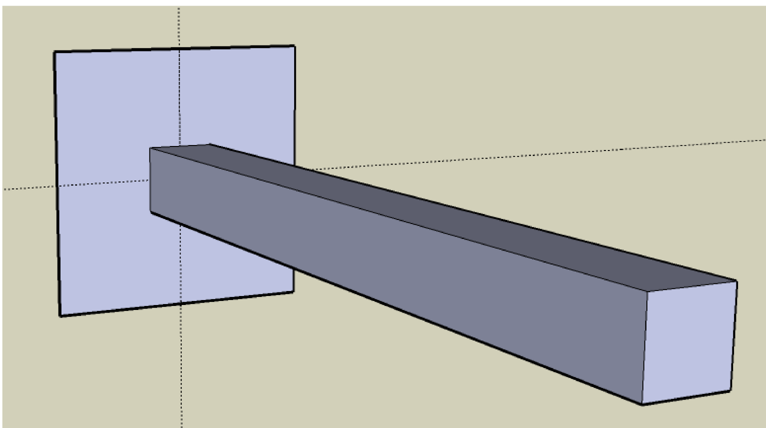
To evaluate the integral, we first need to find the temperature distribution along the fin. Without getting into all the gory details, tables or other methods will lead to the following formula for the temperature along the fin:

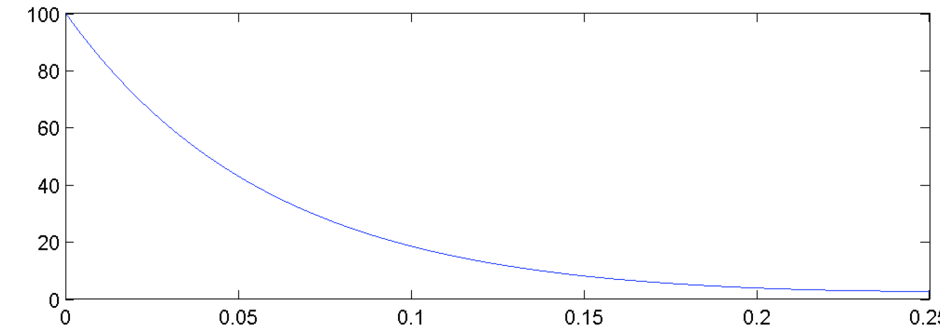
$$T(x) = \frac{T_b\left(\cosh(m(L-x)) + \frac{h}{mk}\right)(\sinh(m(L-x)))}{\cosh(mL) + \frac{h}{mk}\sinh(mL)}$$

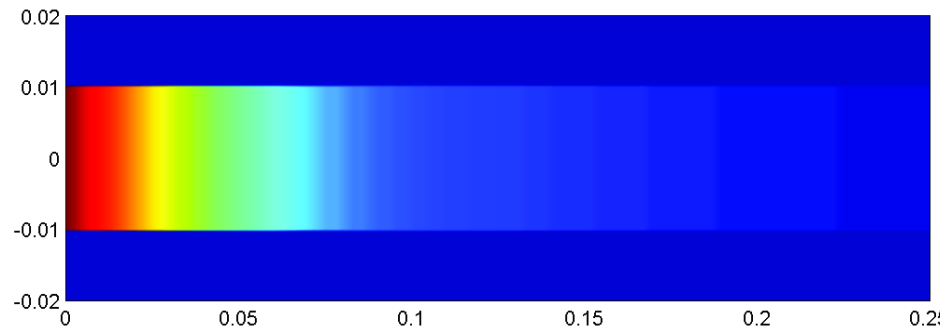
Problem. What are these new $\cosh(x)$ and $\sinh(x)$ functions?

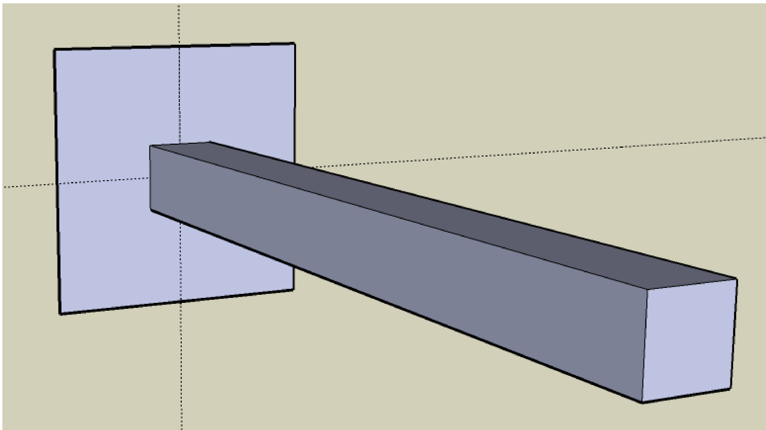
Graphically

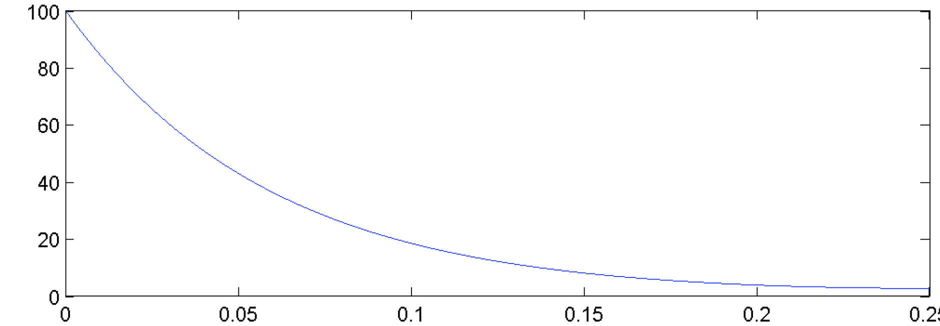
Once we have the temperature distribution, we can graph it along the length of the fin to see if it makes sense.

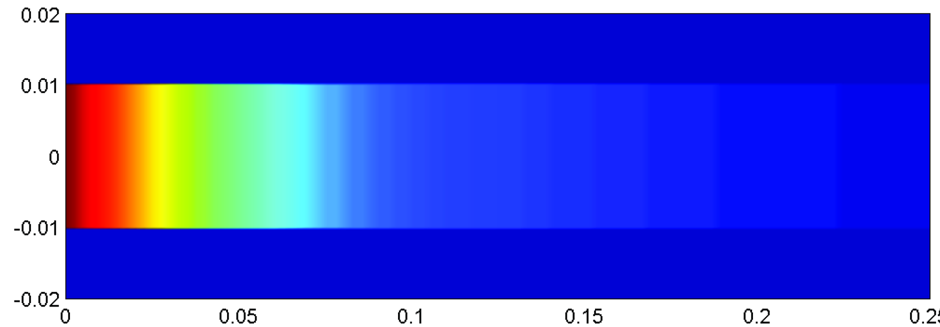












Our next step is to see if we can use this temperature distribution, $T(x)$, to compute the rate of heat transfer by integration:

$$Q = \int_0^L h\ P\ T(x)\ dx \qquad\qquad (T_\infty \text{ assumed} = 0)$$

Computing Total Heat Loss

Now, we have addressed one challenge in our problem: we know the steady-state temperature along the fin. Next, we want to compute the net rate of heat flow out, or the cooling ability of the fin.

The heat flow out of the fin is given by

$$Q = \int_0^L h\ P\ T(x)\ dx \qquad\qquad (T_\infty \text{ assumed} = 0)$$

Our first approach, if it is possible, should be direct anti-differentiation (think $\int x^2\ dx = \frac{1}{3}x^3$).

$$Q = \int_0^L h\ P\ T(x)\ dx \qquad\qquad (T_\infty \text{ assumed} = 0)$$

For this problem, given the earlier temperature we found, $T(x)$, we **can** evaluate the integral exactly:

$$Q = \frac{1}{2}\frac{T_b\,P\,h\,(m\,k\,exp(2\,m\,L) - m\,k + h\,exp(2\,m\,L) + h - 2\,h\,exp(m\,L)}{m(\cosh(m\,L)\,m\,k + h\ \sinh(m\,L))\,exp(-m\,L)}$$

Evaluated with appropriate constants for the material, base temp, etc. we would obtain the final value

$$Q = 2.363\ \mathrm{J/s}$$

Comments on Anti-Derivatives

Through this last step, we reached what would be the important engineering goal: obtaining the **numerical value** for the integral. When we compute integrals analytically, by using anti-derivatives, we are doing the best possible thing.

- Integrals give exact values.
- Integrals can be re-used immediately with different constants.

Unfortunately, actually computing the numerical value of an integral using antiderivatives isn't always an option:

- Some functions don't have antiderivatives.
- Sometimes we don't have a function, but only data.
- Sometimes we forget how to find the anti-derivative!

Numerical Quadrature

The word *quadrature* comes from the Greek challenge of trying to *square the circle*, or finding the **area** (in square units) of the round circle.

When you hear **quadrature** think **numerical integration**.

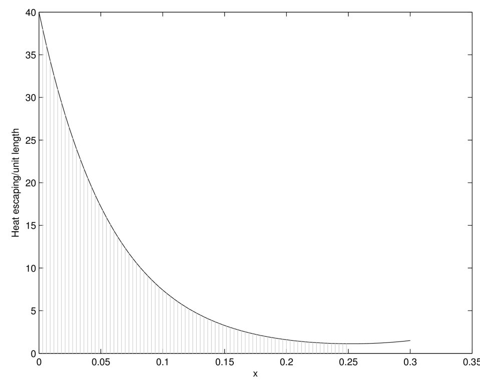
The two common scenarios where we need *numerical* integration will be:

- **Formula** for $f(x)$ known, want $\displaystyle\int_a^b f(x)dx$
- **Data** for $f(x)$ collected at $f(x_i)$, want $\displaystyle\int_a^b f(x)dx$

We will study the formula case initially, because it is easier to experiment with. We will continue to use our cooling fin example, where

$$Q = \int_0^L h\ P\ T(x)\ dx = 2.36269950112023 \text{ J/s}$$

Graphically, $\displaystyle\int_0^L \underbrace{h\ P\ T(x)}_{f(x)}\ dx$ is the area shown below:



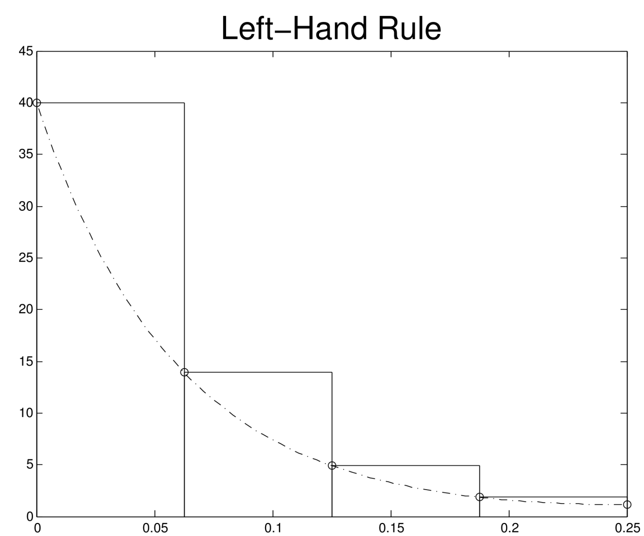
Numerical integration is performed by

- separating the desired interval into *panels*
- on each *panel*, evaluating the integrand, $f(x)$ one or more times and those values are combined in some way to *estimate* the area of the panel

Left-Hand Sum

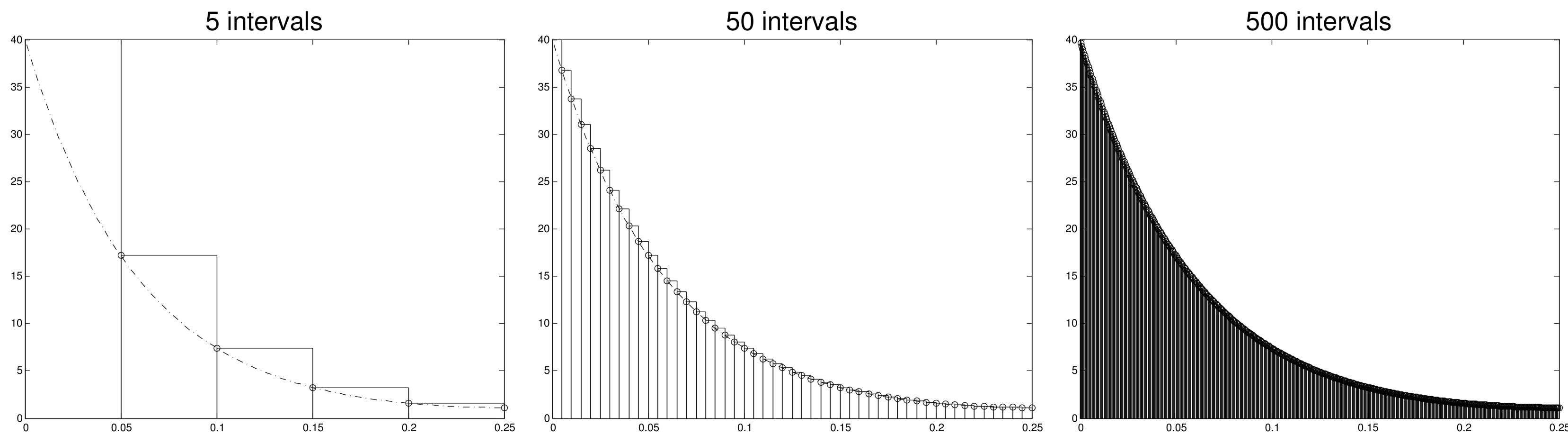
The simplest quadrature rule is one we have already seen: the LEFT(n) sum.

- Divide the interval into n panels, width $\Delta x = (b-a)/n$
- Evaluate function at the left end point, $f(x_{i-1})$ on each panel.
- Compute the area of rectangles, $\displaystyle\sum_{i=1}^{N} f(x_{i-1}) \cdot \Delta x$ or



Quadrature Principles

We are approximating a complex shape with simpler shapes for which we can compute the area. The **more panels** we use, the **more accurate** the area estimate will be:



By using enough panels, we can reduce the error to any level we like, but then it takes longer to compute.

Problem. Download the file `Week06CoolingFin.m`, and extend it so it plots the graph of the temperature along the fin.

Problem. Using MATLAB, estimate the integral $Q = \displaystyle\int_0^L hP\,T(x)\,dx$ with the left-hand rule,

$$Q \approx \sum_{i=1}^{n} hP\;T(x_{i-1})\;\Delta x$$

Problem. Add a print statement that shows the number of intervals used, and the resulting error. Experiment by doubling the number of intervals and seeing the resultant reduction in error.

When you double the number of intervals, what happens to the error?

Note: If you are using computer software for modeling in your career, it would be a good idea to get familiar with *numerical methods* and *numerical analysis* concepts.

Built-In Integration in MATLAB

Previously, we have used the LEFT(n) approximation to estimate the value of an integral, both by hand and using MATLAB to speed up the computation.

As you might expect, numerical integration is something we can do using built-in MATLAB functions, instead of writing our own LEFT(n) rule.

Problem. Look up the functions `quad` and `integral` in MATLAB help.

Problem. Starting with the `Week06CoolingFin.m`, use `quad` or `integral` to evaluate the rate of heat transfer from the cooling fin. Compare it with the exact integral value,

$$Q = \int_0^L h\ P\ T(x)\ dx = 2.36269950112023\ \mathrm{J/s}$$

Look for options that could be used to increase the accuracy of the numerical integral estimate.

Numerical Integration in MATLAB - Examples

We now have two techniques we can try when evaluating definite integrals like $\displaystyle\int_a^b f(x)\,dx$ where the formula for $f(x)$ is given:

- The Fundamental Theorem of Calculus: finding an anti-derivative $F(x)$ then evaluating $F(b) - F(a)$; or
- Numerical integration tools.

Problem. Use the Fundamental Theorem of Calculus to evaluate the integral

$$\int_1^3 \ln(x)\; dx.$$

Use the built-in numerical integration tools in MATLAB to estimate the value of

$$\int_1^3 \ln(x)\; dx.$$

Problem. Use the Fundamental Theorem of Calculus to evaluate the integral

$$\int_0^{\pi} x^2 \cos(x^3)\, dx.$$

Use the built-in numerical integration tools in MATLAB to estimate the value of

$$\int_0^{\pi} x^2 \cos(x^3)\, dx.$$

Problem. Try to use the Fundamental Theorem of Calculus to evaluate the integral

$$\int_{-2}^{2} e^{-x^2}\, dx.$$

Problem. Use MATLAB to plot the graph of $f(x) = e^{-x^2}$ on the interval $x \in [-2, 2]$.

Sketch the graphical interpretation of the integral

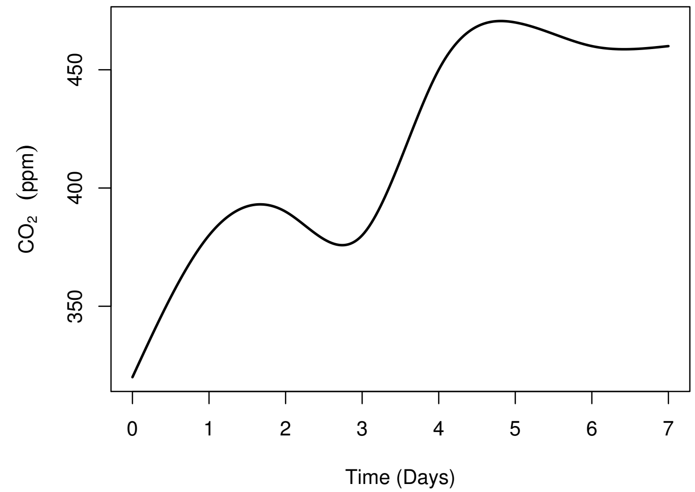
$$\int_{-2}^{2} e^{-x^2}\,dx.$$

Use the built-in numerical integration tools in MATLAB to estimate the value of

$$\int_{-2}^{2} e^{-x^2}\,dx.$$

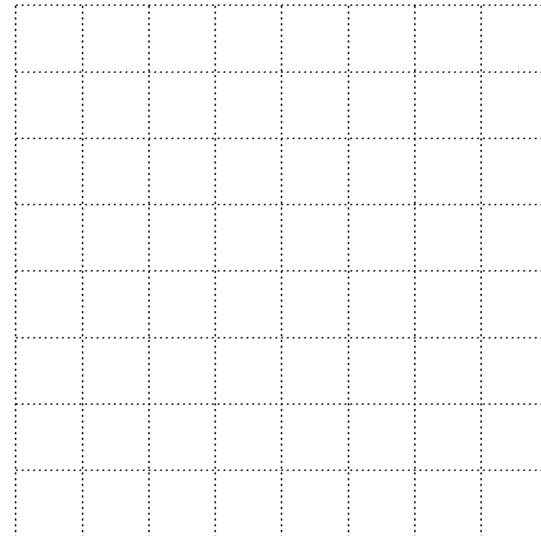
Applications of Integrals - Average Value

Problem. If the following graph describes the level of CO_2 in the air in a mine over a week, estimate the *average level* of CO_2 over that period.

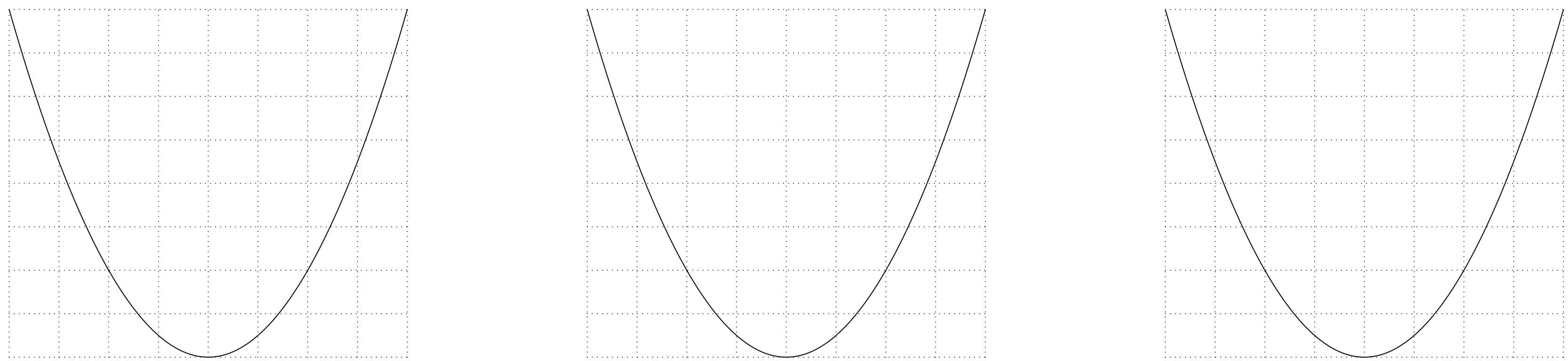


Problem. Give the units of the average CO_2 level.

Problem. Sketch the graph of $f(x) = x^2$ from $x = -2$ to 2, and estimate the average value of f on that interval.



Problem. What makes the single "average" f value different or distinct from other possible f values?



Problem. Use this property to find a general expression for the **average value of** $f(x)$ **on the interval** $x \in [a, b]$.

Problem. Find the exact average of $f(x) = x^2$ on the interval $x \in [-2, 2]$ using this formula.

Use MATLAB to generate a graph of $f(x) = x^2$, compute the average value using the built-in integration tools, and then add a line at the average value. Use the same $x \in [-2, 2]$ interval.

Average Value of a Function on $[a, b]$

The average value of a function $f(x)$ on the interval $[a, b]$ is given by

$$A = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Problem. The temperature in a house is given by $H(t) = 18 + 4\sin(\pi t/12)$, where t is in hours and H is degrees C. Sketch the graph of $H(t)$ from $t = 0$ to $t = 12$, then find the average temperature between $t = 0$ and $t = 12$.

Problem. You are told that the average value of $f(x)$ over the interval $x \in [0,3]$ is 5. What is the value of $\int_0^3 f(x)\,dx$?