Week #4 : Integrals - Foundations

Goals:

- Use the definite integral to model and find a solution to a posed area- or accumulation-related problem.
- Scale and add definite integrals, describe the meaning of integral bounds and how to apply them.
- Recognize an anti-derivative of a function.
- Apply the theory of the Fundamental Theorem of Calculus to evaluate simple integrals.
- Distinguish between definite and indefinite integrals and their meaning.

Integration

If we had to summarize the first three weeks of the course, we would say that the focus was on **differentiation**.

All differentiation problems ask the same basic question: **At what rate does a process change**, and how does that rate of change relate to other characteristics of the process?

The key observation was that **at small scales, rates of change look linear**.

In the next three weeks of the course we will study **integration**. Again, the analysis will be made possible by the observation that on a very small scale all processes look linear. This time, though, we will use this fact to see how regarding a process as an **accumulation** of infinitely many small linear steps allows us to calculate the accumulated total even when the rate of accumulation is far from linear. **Integration is always in some way about finding the total at the end of a process of accumulation.**

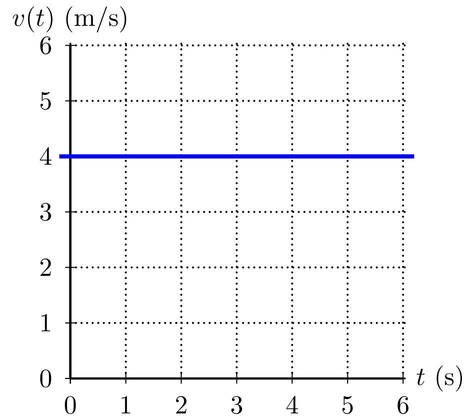
Distance and Velocity

Recall that if we measure distance x as a function of time t, the velocity is determined by differentiating $x(t)$:

Velocity is the **slope** on the **position graph**.

But now suppose we begin with a **graph of the velocity** with respect to time. How can we determine what **distance** will be traveled? Does distance also "appear" in the velocity graph somehow?

Problem. Consider the graph for the velocity of a particle shown below.



How far did the particle travel between $t = 0$ and $t = 5$ seconds?

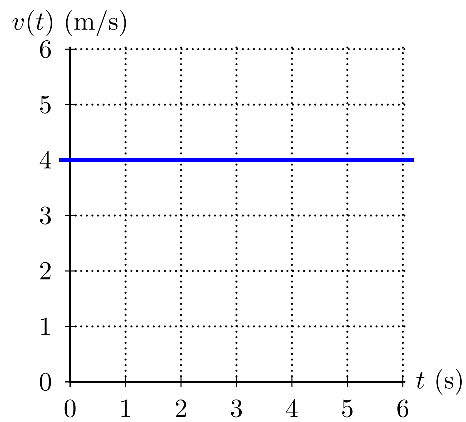
(a) 5 m

(b) 10 m

(c) 15 m

(d) 20 m

Problem.



Where does the distance travelled between $t = 0$ and $t = 5$ "appear" on this velocity graph?

(a) The distance travelled is the **slope** between $t = 0$ and $t = 5$

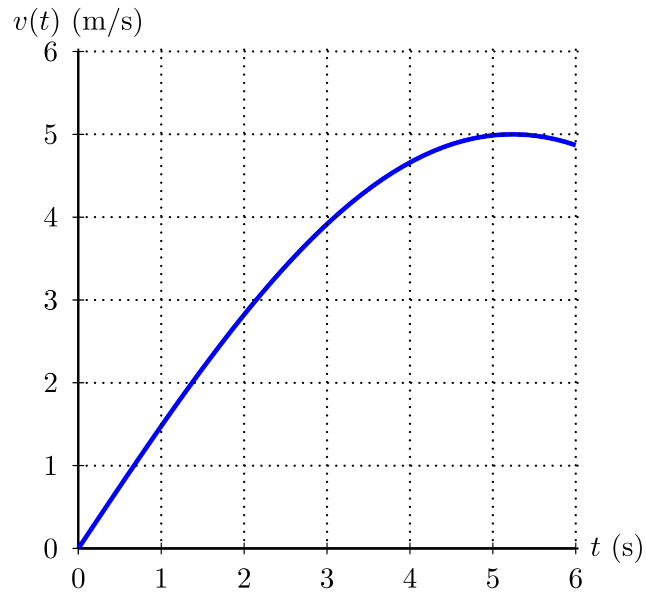
(b) The distance travelled is the **average height** of the graph between $t = 0$ and $t = 5$

(c) The distance travelled is the **area** under the graph between $t = 0$ and $t = 5$.

When the velocity is **constant**, we have the equivalency:

$$\text{dist} = \text{vel} \times \text{time} \qquad \Longleftrightarrow \text{dist} = \text{area under the velocity graph}$$

Problem. What about when the velocity is **not** constant though?



Do the units of the "area" under this graph still make sense as a distance value?

Estimating Areas

It appears that

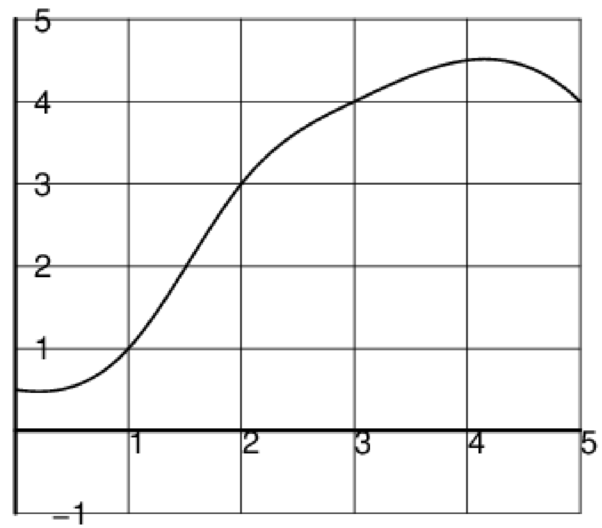
distance traveled = area under the graph of velocity

even when the velocity is changing. This means that we've found two equivalent problems, and can work with whichever version benefits us the most at a particular moment.

Unfortunately, many or most arbitrary areas are essentially impossible to find the area of when the shape isn't a simple composition of triangles, rectangles, or circles.

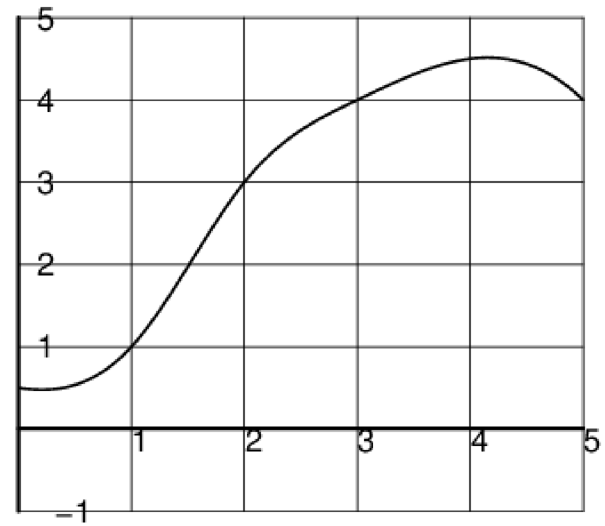
In these cases, we must use less direct methods. We will start by making an *estimate* of the area under the graph using shapes whose area is easier to calculate.

Problem. Suppose we are trying to find the area underneath the graph of the function $f(x)$ given below between $x = 1$ and $x = 4$. Shade in that region, and call that area A.



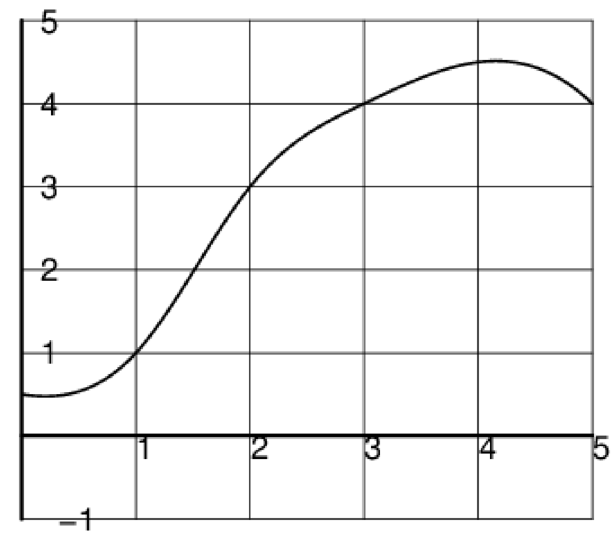
We can make a rough estimate of the area by drawing a rectangle that completely contains the area, or a rectangle that is completely contained by the area.

Problem. Calculate this overestimate and underestimate for the area A.



The next step is to use smaller rectangles to improve our estimate the area. We can divide the interval from $x = 1$ to $x = 4$ into 3 intervals of width 1, and use different size rectangles on each interval.

Problem. Estimate the area A by using 3 rectangles of width 1. Use the function value at the *left* edge of the interval as the height of each rectangle.



We can repeat this process for any number of rectangles, and we expect that our estimation of the area will get better the more rectangles we use. The method we used above, choosing for the height of the rectangles the function at the left edge, is called the **left hand sum**, and is denoted $\mathrm{LEFT}(n)$ if we use n rectangles.

Generalizing the Area Calculation

Suppose we are trying to estimate the area under the function $f(x)$ from $x = a$ to $x = b$ via the left hand sum with n rectangles.

- the **width** of each rectangle will be $\Delta x = \dfrac{b-a}{n}$.
- If we label the endpoints of the intervals to be $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, then the **heights** of the rectangles will be $f(x_i)$'s, and
- the formula for the left hand sum/area will be

$$\begin{aligned}\text{LEFT}(n) =& f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x \\ =& \sum_{i=1}^{n} f(x_{i-1})\Delta x.\end{aligned}$$

Aside: Summation Notation

The capital greek letter sigma, $\sum$, is used as a shortform notation for long sums. E.g.

$$\sum_{i=1}^{n} x_i$$

Problem. Translate the following into more traditional sums:

$$\sum_{i=1}^{100} i =$$

$$\sum_{n=1}^{10} n^2 + n =$$

$$\sum_{i=1}^{n} f(x_{i-1})\,\Delta x =$$

Summation notation lends itself very nicely to translations between hand-written sums and MATLAB computations. A `for` loop can be mapped easily on to a sum.

Problem. Implement each of the sums below as a MATLAB loop and compute the total.

$$\sum_{i=1}^{100} i =$$

$$\sum_{n=1}^{10} n^2 + n =$$

Riemann Sums

Area estimations like LEFT(n) are examples of **Riemann sums**, after the mathematician Bernahrd Riemann (1826-1866) who formalized many of the techniques of calculus. The general form for a Riemann Sum is

$$\begin{aligned} f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x \\ = \sum_{i=1}^{n} f(x_i^*)\Delta x \end{aligned}$$

where each x_i^* is some point in the interval $[x_{i-1}, x_i]$. For LEFT(n), we choose the left hand endpoint of the interval, so $x_i^* = x_{i-1}$.

The common property of all these approximations is that they involve

- a sum of rectangular areas, with

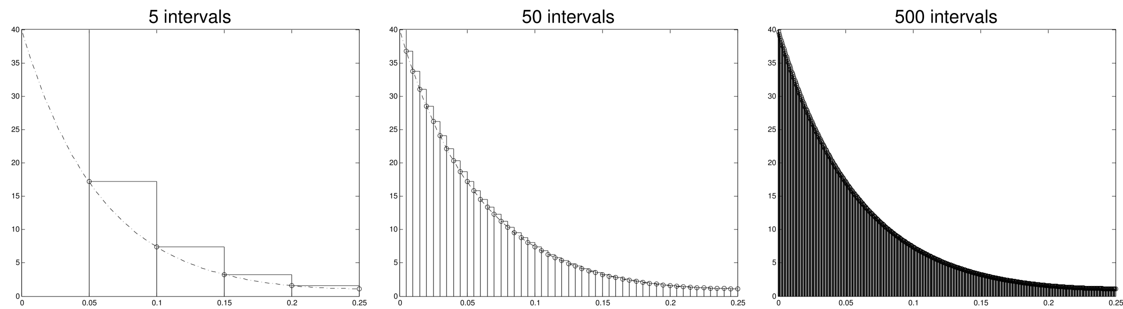
- widths (Δx), and

- heights $(f(x_i^*))$

There are other Riemann Sums that give slightly better estimates of the area underneath a graph, but they often require extra computation.

The Definite Integral

We observed that as we increase the number of rectangles used to approximate the area under a curve, our estimate of the area under the graph becomes more accurate. We can see that based on an example like the graph below:



This implies that to obtain the **exact area**, we should use a **limit** on our Riemann sums.

The area underneath the graph of $f(x)$ between $x = a$ and $x = b$ is

$$\text{equal to } \lim_{n\to\infty} \text{LEFT}(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x, \quad \text{where}$$

$$\Delta x = \frac{b-a}{n}.$$

This limit, which gives the **exact value of the accumulation of** $f(x)$ is called the **definite integral** of $f(x)$ from a to b, and is equal to the area under curve whenever $f(x)$ is a non-negative continuous function.

Notation for the Definite Integral

The definite integral of $f(x)$ between $x = a$ and $x = b$ is denoted by the symbol

$$\int_a^b f(x)\,dx$$

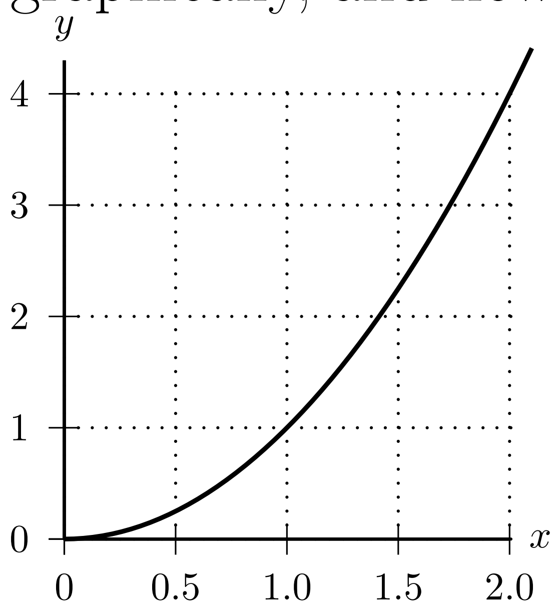
We call a and b the **limits of integration** and $f(x)$ the **integrand**. Note that this notation shares the same common structure with Riemann sums:

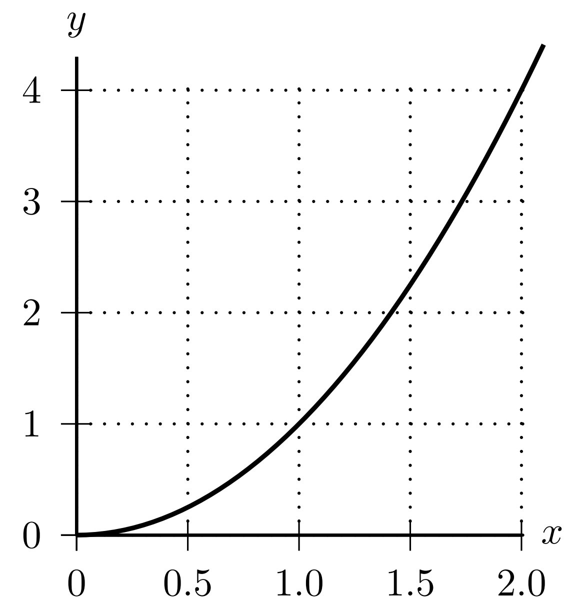
- a sum ($\displaystyle\int$ sign)
- widths (dx), and
- heights ($f(x)$)

Problem. Write the definite integral representing the area underneath the graph of $f(x) = x + \cos x$ between $x = 2$ and $x = 4$.

Problem. Write an integral that represents the area under the graph of $f(x) = x^2$ on the interval $x \in [0, 2]$.

Problem. Sketch how the LEFT(4) sum would be represented graphically, and how it differs from the integral value.





Problem. Use a LEFT sum with $n = 4$ to estimate $\displaystyle\int_0^2 x^2\,dx$.

Repeat your LEFT estimate, but using MATLAB.

Repeat the calculation again in MATLAB, but now with 100 intervals.

The Role of Riemann sums

1. They are needed to say what we mean by an integral.

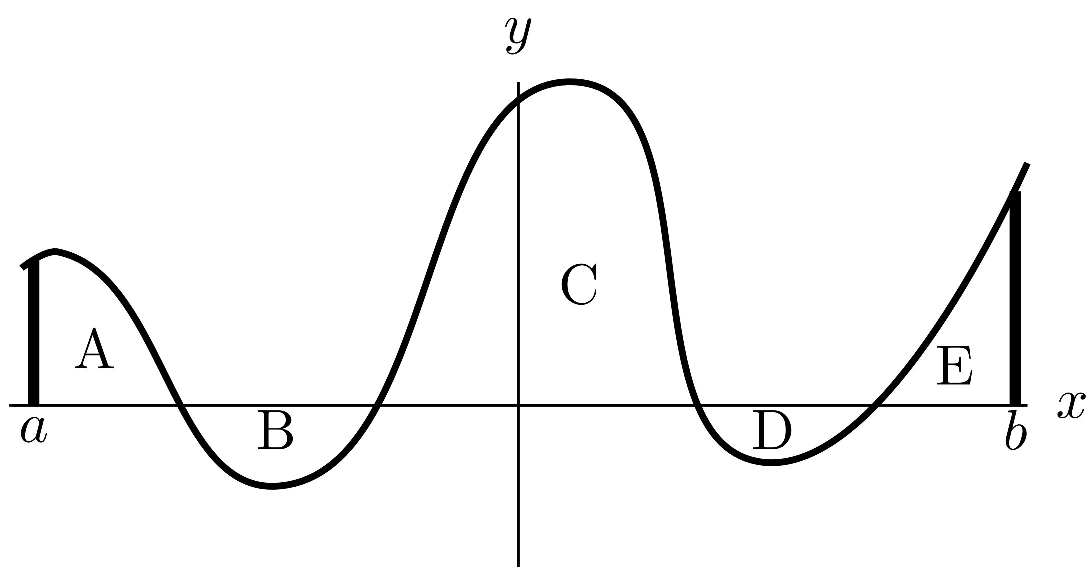
2. They enable us to decide which integral is appropriate in a word problem.

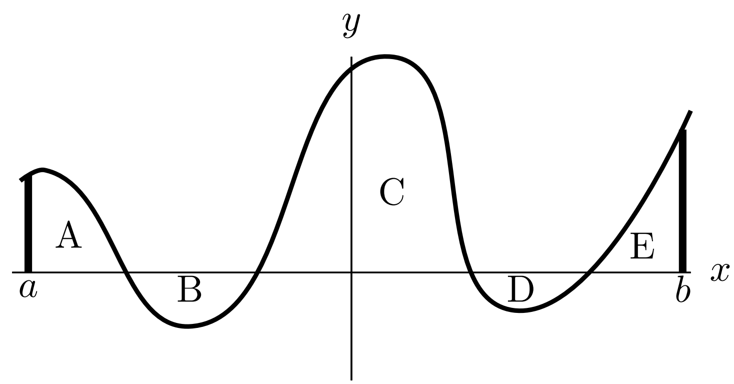
3. They can also be used, with a finite number of intervals, to give an approximate value of the integral.

Negative Integral Values

So far we have only dealt with the areas under/intergrals of **positive** functions. Will the definite integral still be equal to the area underneath the graph if $f(x)$ is always negative? What happens if $f(x)$ crosses the x-axis several times?

Problem. Suppose that $f(x)$ has the graph shown below, and that A, B, C, D, and E are the areas of the regions shown.

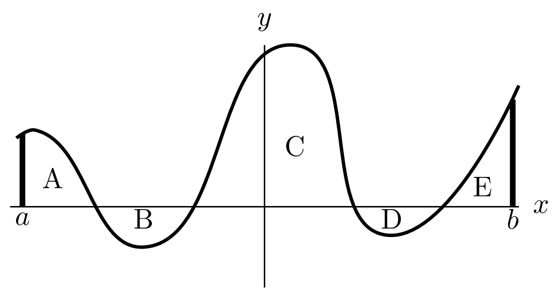




If we were to partition $[a, b]$ into small subintervals and construct a corresponding Riemann sum, then the first few terms in the Riemann sum would correspond to the region with area A, the next few to B, etc.

Problem. Which of these sets of terms have positive values?

Which of these sets have negative values?



Problem. Express the integral $\int_a^b f(x)dx$ in terms of the (positive) areas A, B, C, D, and E.

If f were to represent velocity over time, what would the "negative areas" represent?

The Fundamental Theorem of Calculus

We started our discussion of integration by stressing the fact that an integral problem is always at heart a problem in which something **accumulates**.

We then converted that accumulation problem into the problem of computing the **area under a graph**, and representing both using the notation

$$\int_a^b f(x)\,dx.$$

So far, we don't know how to evaluate that exactly: we can only use our approximation based on a Riemann Sum,

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i*)\,dx$$

If we want to evaluate this integral exactly, we need a new tool.

The new idea is one of the most important discoveries in calculus: the connection between integrals and the **inverse of differentiation**.

Fundamental Theorem of Calculus

If f is continuous and F is an **anti-derivative** of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Problem. Explain in words how we would use this to evaluate an integral over the interval $x = a \dots b$.

Using the Fundamental Theorem of Calculus

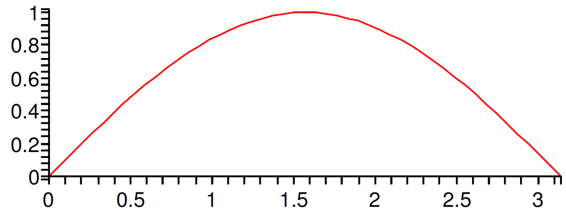
To use the Fundamental Theorem of Calculus, we need to know formulas for the anti-derivatives of functions. We already know quite a few.

Problem. Complete the following table of basic anti-derivatives by asking yourself the question, "$f(x)$ is the derivative of what function, $F(x)$?".

Function $f(x)$	**Anti-derivative** $F(x)$
$x^n \ (n \neq -1)$	
$\dfrac{1}{x}$	
e^x	
$\cos x$	
$\sin x$	
$\sec^2 x$	
$\dfrac{1}{\sqrt{1-x^2}}$	
$\dfrac{1}{1+x^2}$	

Fundamental Theorem - Simple Examples

Problem. Use the Fundamental Theorem to calculate the area under one section of the graph of $\sin x$, the part from 0 to π.



Problem. Calculate $\displaystyle\int_0^4 \sqrt{x}\, dx$.

Notation

The expression $F(b) - F(a)$ comes up so often that there is a special notation for it. It is written as

$$F(x)\Big|_a^b \quad \text{or} \quad [F(x)]_a^b$$

Problem. Calculate $\displaystyle\int_{\pi/4}^{\pi/3} \sec^2(\theta)\, d\theta$.

Properties of Integrals

Before we go on to refine our skill at calculating integrals, we should first reflect on some basic properties of integrals that derive from their origins as limits of more and more accurate Riemann sums.

1. If $a > b$ then $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

2. If $a = b$ then $\displaystyle\int_a^b f(x)\,dx = 0.$

3. $\displaystyle\int_a^b c\,dx = c\,(b - a).$

4. $\displaystyle\int_a^b \left[f(x) \pm g(x)\right]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

5. $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx.$

6. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

Problem. Find the value of the integral $\int_{-2}^{2} (4x^2 - 3e^x)\, dx$

Problem. Evaluate $\displaystyle\int_3^3 \sin(x^3)\,dx$

Problem. Compute both

$$\int_{-1}^{3} x^2\ dx, \text{ and}$$

$$\int_{-1}^{0} x^2\ dx + \int_{0}^{3} x^2\ dx.$$

Explain the relationship between the two answers with a sketch.

Net Change Theorem

Note that we create an anti-derivative $F(x)$, we are building it such that $f(x) = F'(x)$. This means that f gives the rate of change of F. Notice that this observation was made much earlier, when we started our discussion of integration: when an integral is associated with a process of **accumulation** then the **rate of accumulation** is always precisely the integrand.

Consider $F(x)$ as the quantity we are tracking, so F' is its rate of change. Another statement of the Fundamental Theorem of Calculus Part 2 would then be

$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a).}$$

"The integral of a rate of change is the total change". Textbooks refer to this the **Net Change Theorem**.

Problem. State what this means if $F(t)$ represents the position of an object at time t.

Problem. Suppose water is flowing into/out of a tank at a rate given by $r(t) = 200 - 10t$ L/min, where *positive* rates indicate flow *in*. By how much does the water level in the tank change during the first 45 minutes after $t = 0$?

Problem. What is an assumption you would have to make about the initial amount of water in the tank for this to make sense?

The Indefinite Integral

The second part of the Fundamental Theorem of Calculus,

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

reduces the problem of evaluating an integral to that of finding the anti-derivative F. For that reason the anti-derivative is often called the **indefinite integral** and written as

$$\int f(x)\,dx \left(= F(x)\right).$$

- A **definite integral** has the form $\int_a^b f(x)\,dx$.
  - It represents
  - Its value is
- An **indefinite integral** has the form $\int f(x)\,dx$.
  - It represents
  - Its value is

Problem. Find $\displaystyle\int \left(\sqrt{t} - \frac{3}{\sqrt{t}}\right)\,dt$.

Note: When you are asked to find an indefinite integral, as in the previous question, it is important to add the "$+C$" to indicate that there is more than one anti-derivative, and that all of them differ from each other by a constant.

Finding Anti-Derivatives - Guess and Check

Finding antiderivatives is surprisingly difficult. You would think that if we know how to find derivatives then we should know how to find antiderivatives. In fact, however, the latter is much more difficult, as illustrated next.

Problem. Calculate $\displaystyle\int \frac{1}{x^3+1}\,dx$

Guess and Check

Next week we will study the "Substitution Rule", or "Method of Substitution", the first of a list of techniques for finding anti-derivatives. To prepare us for the Method of Substitution, let's explain an informal method often called "guess and check".

Problem. Find $\displaystyle\int \cos(5x)\,dx$.

Problem. Find $\displaystyle\int \cos(x^2)\,dx$.

Problem. Now what if we were given the problem $\int x\cos(x^2)\,dx$?

So why does "guess and check" work for $\int x\cos(x^2)\,dx$ but not for $\int \cos(x^2)\,dx$?

Problem. Which of the following anti-differentiations would you predict can be evaluated by the guess-and-check method?

1. $\displaystyle\int x^2 e^{x^3}\,dx$ 2. $\displaystyle\int x^2 e^{x^2}\,dx$ 3. $\displaystyle\int x e^{x^2}\,dx$

A. 1

B. 2

C. 3

D. 1 and 3

E. 1 and 2

Problem. Calculate $\int \cos(x)\, e^{\sin(x)}\, dx$.

Next week we will turn these ideas into a formal procedure called "Substitution".