Week #1 : Derivatives - Foundations

Goals:

- Interpret the derivative, and be able to discuss the difference between the secant line and the derivative
- Compute the derivative of polynomial, exponential, logarithmic, powers, trigonometric functions and their combinations, with the correct application of the product and quotient rules, and the chain rule
- Report the graphs, domain and range of the inverse trigonometric functions arcsin, arccos and arctan.
- Apply the derivative rules for arcsin, arccos and arctan.

Introduction

The two fundamental ideas in calculus are:

- The **derivative**, which gives the rate of change of a function, and

- The **integral**, which computes the total accumulation based on a rate.

Many fundamental quantities in physics are related through derivatives and integrals. For example:

$$\begin{tikzcd}
(m) & \textbf{Position} \arrow[d, bend right, "\text{derivative}"'] \\
(m/s) & \textbf{Velocity} \arrow[u, bend right, "\text{integral}"'] \arrow[d, bend right, "\text{derivative}"'] \\
(m/s^2) & \textbf{Acceleration} \arrow[u, bend right, "\text{integral}"']
\end{tikzcd}$$

We will spend the first half of this course examining differential calculus and integral calculus.

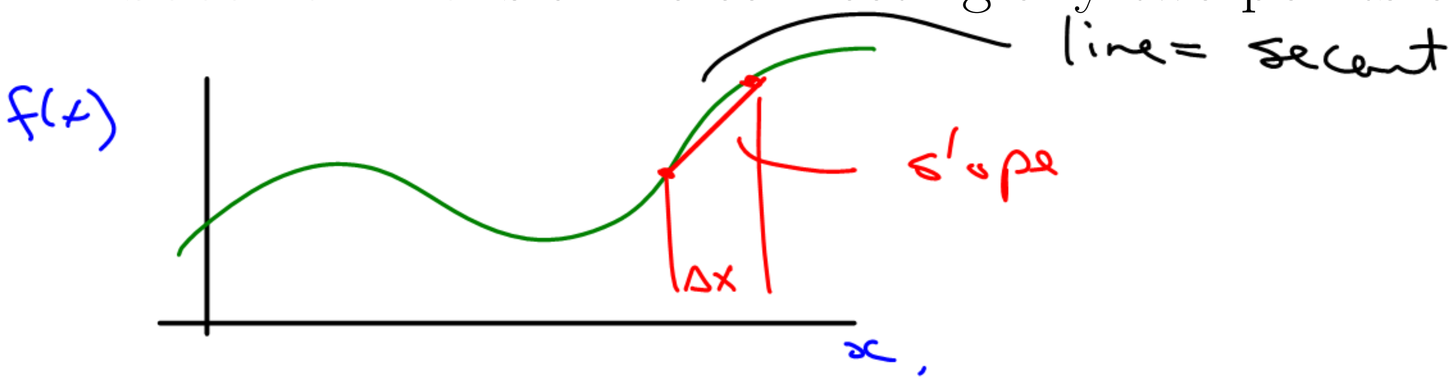
derivatives

The assumption for the derivative section is that the fundamental ideas will be review, so the pace will be fairly aggressive. We will also start including some calculations and graphical analysis using MATLAB.

Take advantage of the practice problems to practice your skills, and get in touch with the instructor if you need any help.

Slopes, Secants and the Derivative

A **secant line** is a line connecting any two points on a graph.



The **slope of a secant line** gives

Δ = change in

- the **average rate of change** of $f(x)$ over some interval Δx.
- the **average velocity** over an interval, if $f(t)$ represents **position**.
- the **average acceleration** over an interval, if $f(t)$ represents **velocity**.

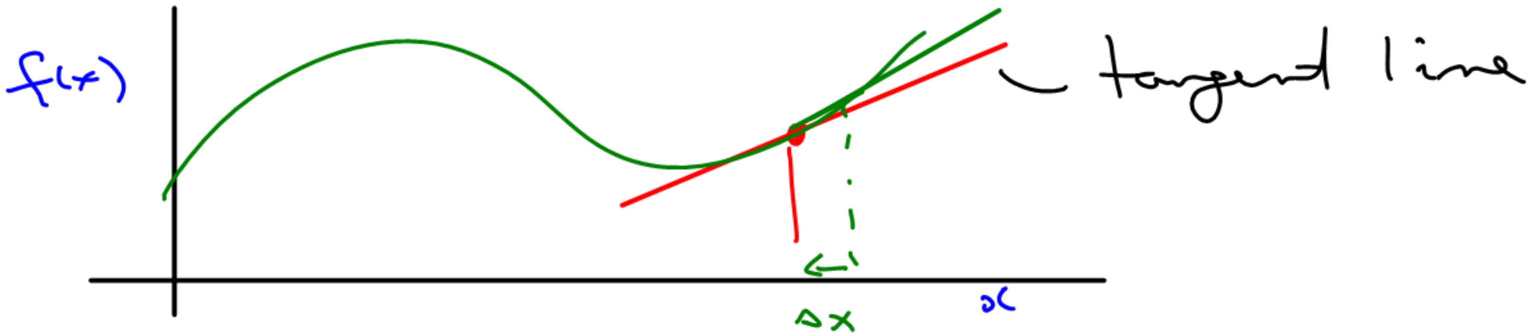
Problem. Give the units of the slope of a secant line.

$$\frac{\text{rise}}{\text{run}} = \frac{\text{units of } f(x)}{\text{" " } x}$$

Eg pos vs time

units $\frac{\text{pos}}{\text{time}}$

The **derivative** gives the **slope of a tangent line** at a **single point** on a graph.



2 points

The derivative gives:

- the **limit of the average slope** as the interval Δx approaches zero.
- the **instantaneous rate of change** of $f(x)$.
- the **velocity**, if $f(t)$ represents **position**.
- the **acceleration**, if $f(t)$ represents **velocity**.

Problem. Give the units of the derivative.

slope on $f(x)$ vs x

$$\frac{\text{rise}}{\text{run}} = \frac{\text{units of } f(x)}{\text{units of } x}$$

Eg. pos vs time

$$\frac{\text{units of pos}}{\text{units of time}} \quad (m/s)$$

Differentiability

The definition of the derivative is based on the classic $\left(\frac{\text{rise}}{\text{run}}\right)$ formula for slopes.

derivative operator
"take derivative of..."

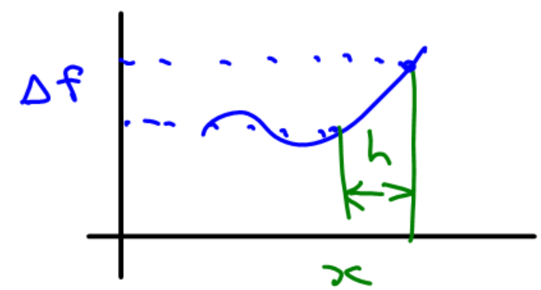
prime

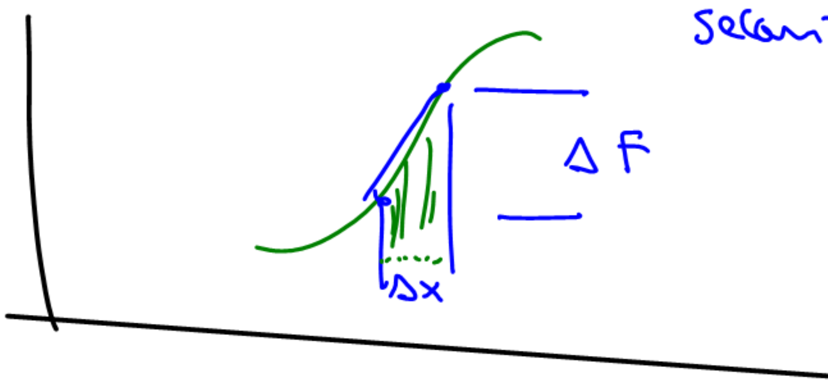
$h = \Delta x$ or Δt

$$\underbrace{f'(x) = \frac{d}{dx} f = \frac{df}{dx}}_{\text{derivative}} = \underbrace{\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{\text{definition}}$$

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta f}{\Delta x}$$

secant slope

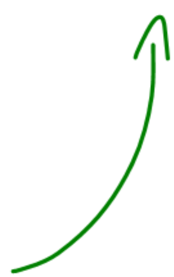


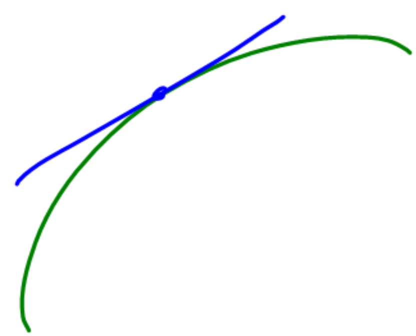


$x=a$

A function f is **differentiable** at a given point a if it has a derivative at a, or the limit above exists. There is also a graphical interpretation differentiability: **if the graph has a unique and finite slope at a point**. Since the slope in question is automatically the slope of the tangent line, we could also say that

> f IS DIFFERENTIABLE AT a IF ITS GRAPH HAS A (NON-VERTICAL) TANGENT AT $(a, f(a))$.





For functions of the form $y = f(x)$, we do not consider points with vertical tangent lines to have a real-valued derivative, because a vertical line does not have a finite slope.

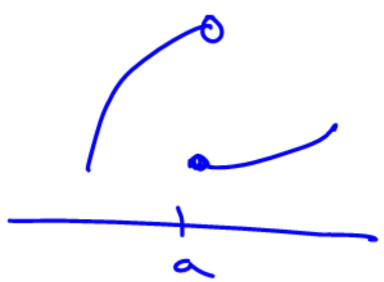
Here are the ways in which a function can fail to be differentiable at a point a:

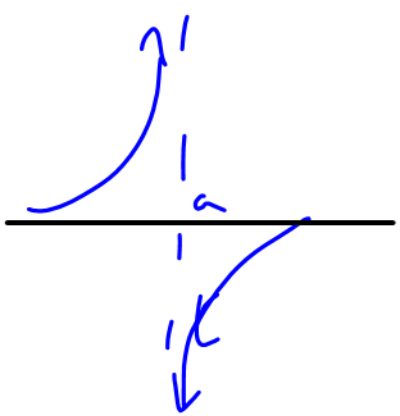
1. The function is not continuous at a.
2. The function has a corner (or a cusp) at a.
3. The function has a vertical tangent at $(a, f(a))$.

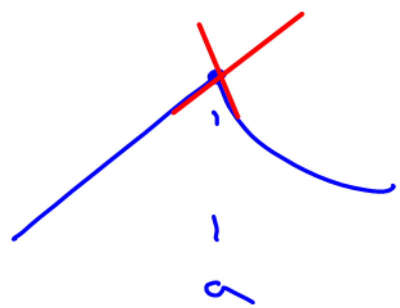
Problem. Sketch an example graph of each possible case.

$y = \sqrt[3]{x}$

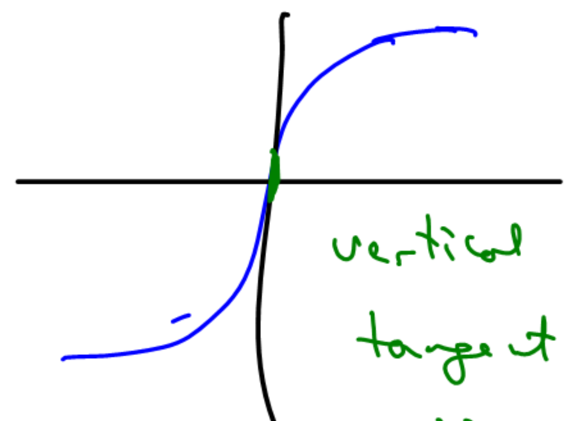
no derivative @ a







slope not unique



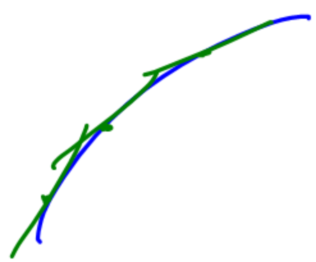
vertical tangent line

not finite slope

Reminder: Differentiability is Common

You will notice that, despite our concern about some functions *not* being differentiable, most of our standard functions (polynomials, rationals, exponentials, logarithms, roots) **are** differentiable at most points. Therefore we should investigate what all these possible derivative/slope values could tell us.

are diff'ble



Interpreting the Derivative — sign

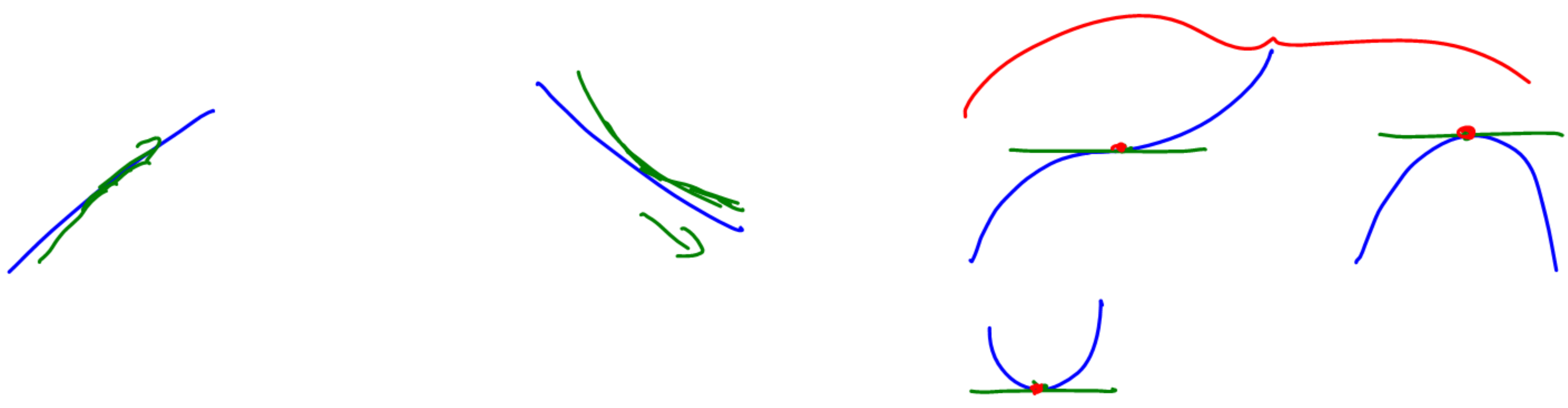
- Where $f'(x) > 0$, or the **derivative is positive**, $f(x)$ is **increasing**.

  slope of tangent line is positive

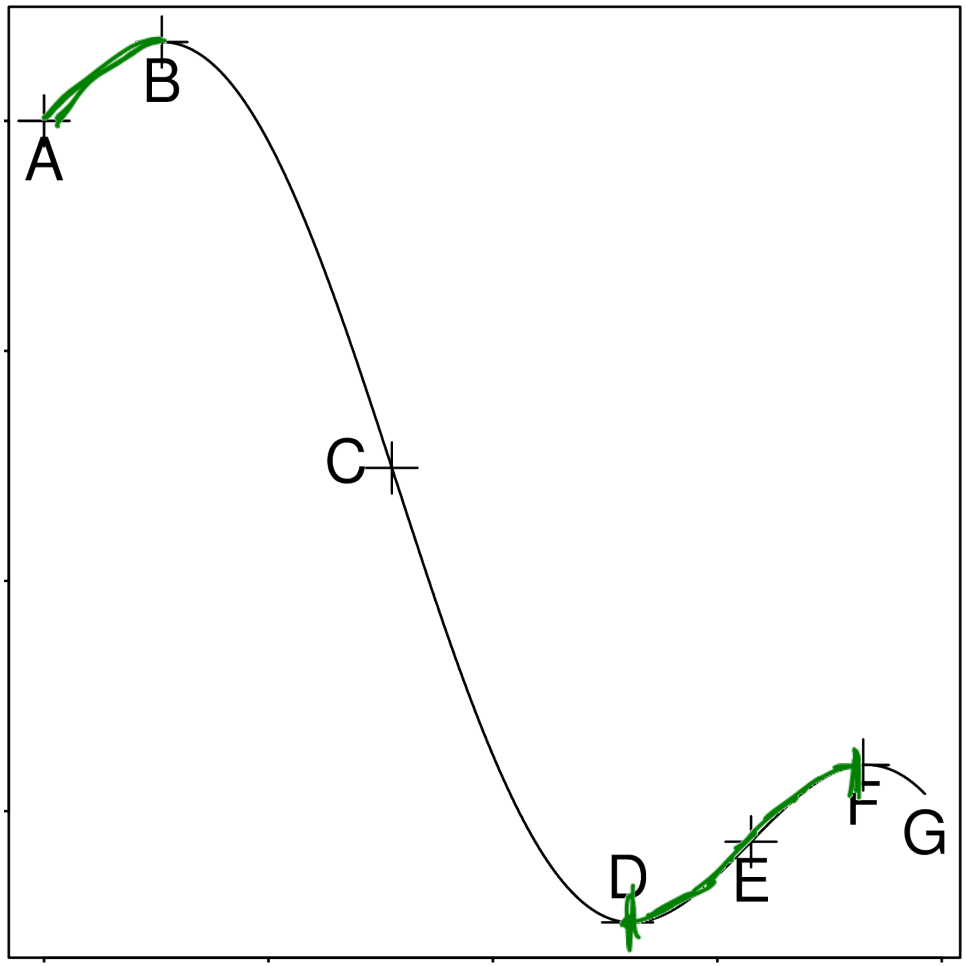
- Where $f'(x) < 0$, or the **derivative is negative**, $f(x)$ is **decreasing**.

- Where $f'(x) = 0$, or the **tangent line to the graph is horizontal**, $f(x)$ has a **critical point**.

  $f'(x) = 0$



Problem. Consider the graph of $f(x)$ shown below.



On what intervals is $f'(x) > 0$?

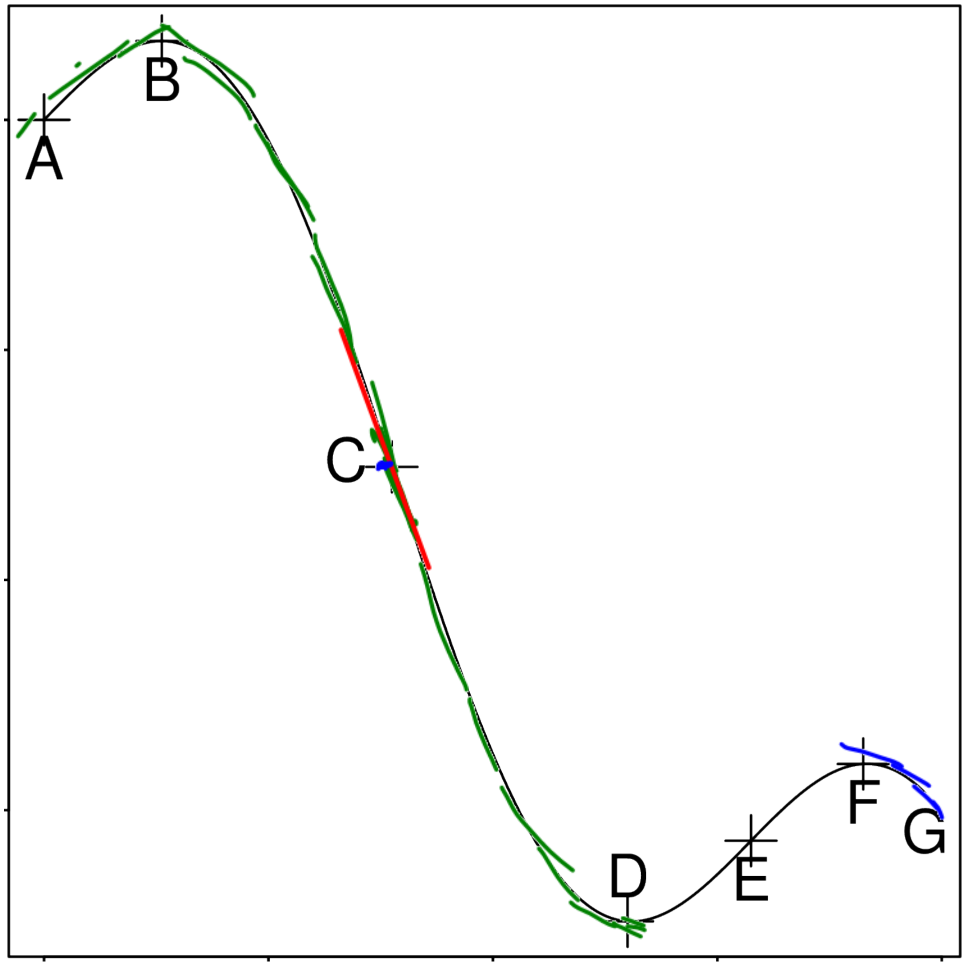
$\Updownarrow$

slopes positive

$\Updownarrow$

function increasing

$f'(x) > 0$ for x on (A, B)
and on (D, F)



At C.

Problem. Where does $f'(x)$ takes on its **largest negative value**?

$\Updownarrow$

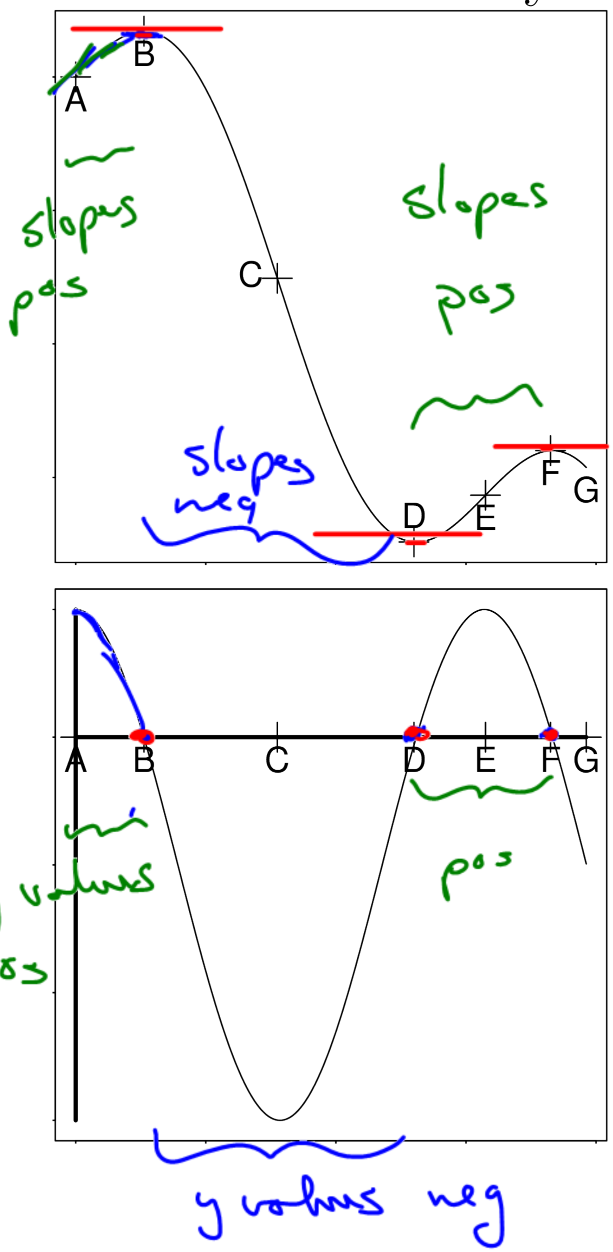
largest negative slope

$\Updownarrow$

steepest negative slope

Graphs, and Graphs of their Derivatives

Problem. Consider the same graph again, and the graph of its derivative. Identify important features that associate the two.



f(x)

slopes

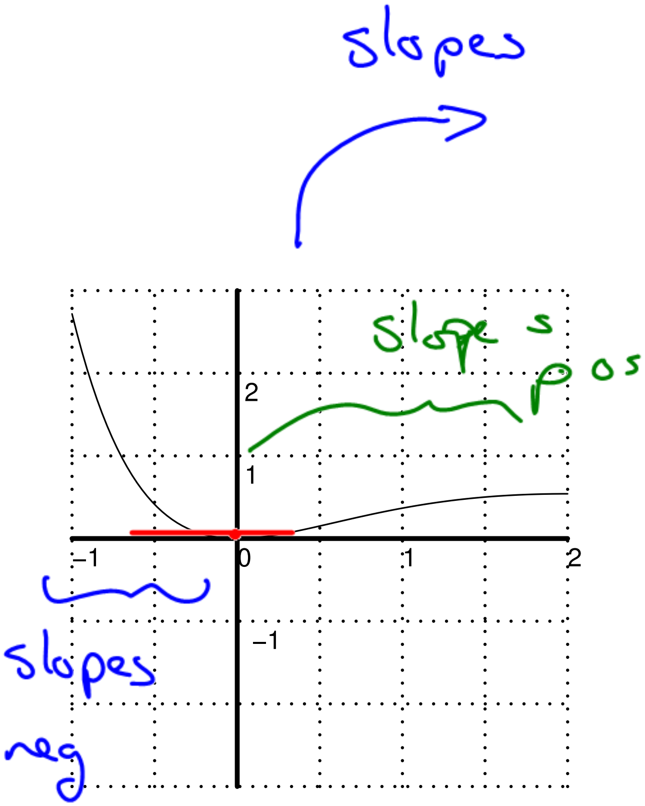
pos

f'(x)

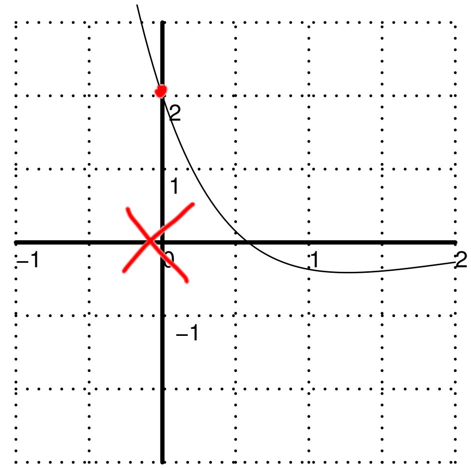
y-values

vel

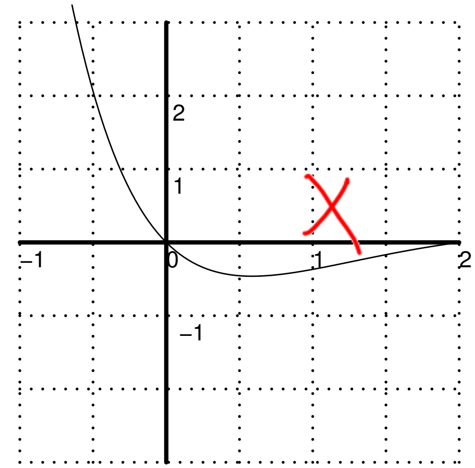
Problem. Consider the graph of $f(x)$ shown:



$f(x)$

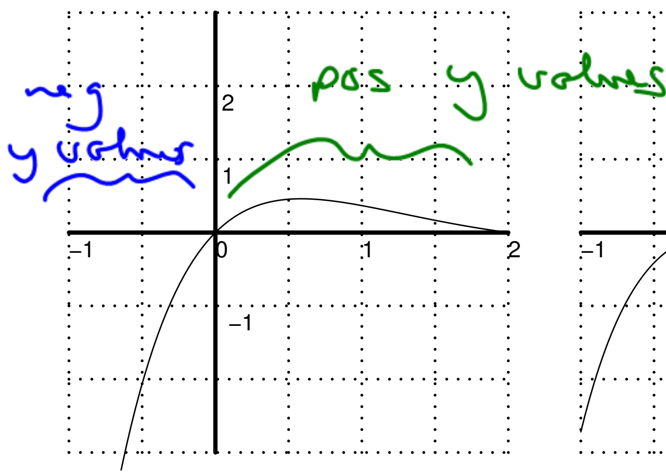


A

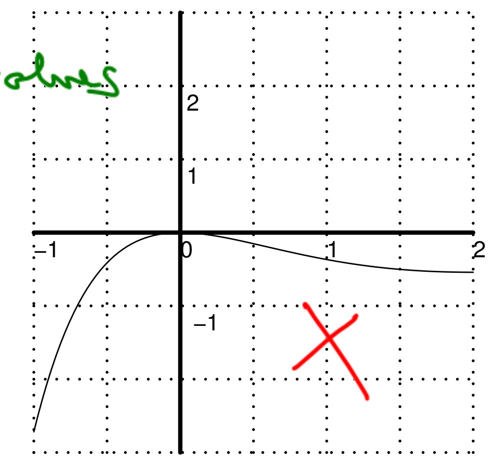


B

$f'(x)$



C

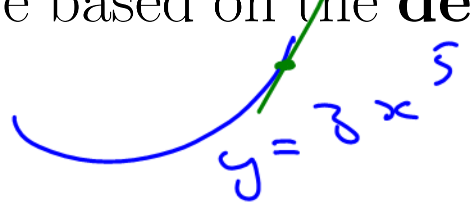


D

Which of the graphs on the right is the graph of the **derivative** of $f(x)$?

Computing Derivatives

Beyond the graphical interpretation of derivatives, there are all the algebraic rules. **All of these rules** are based on the **definition** of the derivative,



$$\underbrace{f'(x) = \overbrace{\frac{d}{dx} f = \frac{df}{dx}}^{\text{Liebniz}}}_{\text{notation}} = \underbrace{\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{\text{definition}} \qquad \frac{\text{rise}}{\text{run}}$$

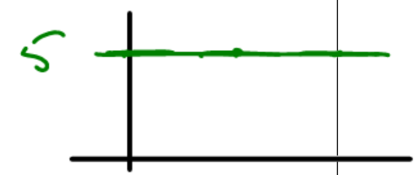
However, by finding common patterns in the derivatives of certain families of functions, we can compute derivatives much more quickly than by using the definition.

Sums, Powers, and Differences

constant by itself

Constant Functions: "what is the slope of..." $\longrightarrow$ $$\frac{d}{dx} k = 0$$

$$\frac{d}{dx}(5) = 0$$



Power rule: $\frac{d}{dx} x^{10} = 10x^9$ $$\frac{d}{dx} x^p = p x^{p-1}$$

bring power down front, subtract one from the power.

Sums : $$\frac{d}{dx}\left(f(x) + g(x)\right) = \left(\frac{d}{dx} f(x)\right) + \left(\frac{d}{dx} g(x)\right)$$

$$\frac{d}{dx}(x^{10} + 5) = 10x^9 + 0$$

Differences: $$\frac{d}{dx}\left(f(x) - g(x)\right) = \left(\frac{d}{dx} f(x)\right) - \left(\frac{d}{dx} g(x)\right)$$

<u>not</u> a constant by itself

Constant <u>Multiplier</u>: $\frac{d}{dx}[k f(x)] = k\left(\frac{d}{dx} f(x)\right)$, so long as k is a constant

$$\frac{d}{dx} 7x^{10} = 7 \frac{d}{dx} x^{10} = 7 \cdot (10x^9)$$
$$= 70x^9$$

Problem. Evaluate the following derivatives:

$$\frac{d}{dx}\left(x^5 - 2x\right) = 5x^4 - 2(1x^0)$$

(recall $x^0 = 1$)

$$= 5x^4 - 2$$

$$\frac{d}{dx}\left(4.1\sqrt{x} + \frac{4}{x}\right) = \frac{d}{dx}\left(4.1\, x^{1/2} + 4\, x^{-1}\right)$$

tip: write out as powers

$$= 4.1\left(\frac{1}{2}\, x^{-1/2}\right) + 4\left((-1)\, x^{-2}\right)$$

tidy

$$= \frac{4.1}{2}\,\frac{1}{\sqrt{x}} - \frac{4}{x^2}$$

Question: The derivative of $-3t^2 - \dfrac{1}{t^2}$ is

1. $-6t^3 + 2\dfrac{1}{t^3}$

2. $-6t + 2\dfrac{1}{t^3}$ ✓

3. $-6t - 2\dfrac{1}{t^3}$

4. $-t^3 + 2\dfrac{1}{t}$

$$\frac{d}{dt}\left(-3t^2 - t^{-2}\right)$$

$$=(-3)(2t) - (-2)(t^{-3})$$

tidy $$= -6t + \frac{2}{t^3}$$

Exponentials and Logs

$$\frac{d}{dx}e^x = e^x \underbrace{\ln(e)}_{=1}$$

e as a base: $$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}2^x = 2^x \ln(2)$$

Other bases: $$\frac{d}{dx}a^x = a^x(\ln(a))$$

base "e"

Natural Log: $$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Other Logs: $$\frac{d}{dx}\log_a(x) = \frac{1}{x}\frac{1}{\ln(a)}$$

Problem. Evaluate the following derivatives:

$$\frac{d}{dx}\left(3 \cdot 10^{x} + 10 \cdot x^{3}\right)$$

$$= 3\left(10^{x} \ln(10)\right) + 10 \cdot \left(3x^{2}\right)$$

$$\frac{d}{dx}\left(e^{x} + \log_{10}(x)\right)$$

$$= e^{x} + \frac{1}{x} \frac{1}{\ln(10)}$$

(Exponential and log derivatives are relatively straightforward, until we mix in the product, quotient, and chain rules.)

Product and Quotient Rules

(D 1st)·(2nd) + (1st)(D 2nd)

Products: $$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

(D top)(bottom) − (top)(D bottom)

Quotients: $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

bottom squared

Problem. Evaluate the following derivatives:

$$\frac{d}{dx}\left(\underset{f}{(2.1x^4)}\underset{g}{(\ln(x))}\right) = \underbrace{\left(2.1(4x^3)\right)}_{f'}\underbrace{\left(\ln(x)\right)}_{g} + \underbrace{\left(2.1x^4\right)}_{f}\underset{g'}{\left(\frac{1}{x}\right)}$$

tidy $= 8.4x^3\ln(x) + 2.1x^3$

$$\frac{d}{dx}\left(\frac{\overbrace{\sqrt{x}}^{\text{top}}}{\underbrace{e^x}_{\text{bottom}}}\right) = \frac{\overbrace{\left(\frac{1}{2\sqrt{x}}\right)}^{\text{(D top)}}\overbrace{(e^x)}^{\text{bott}} - \overbrace{(\sqrt{x})}^{\text{top}}\overbrace{(e^x)}^{\text{(D bottom)}}}{\underbrace{(e^x)^2}_{\text{(bottom squared)}}}$$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dt}\left(\underbrace{(7t^3)}_{f}\underbrace{(2^t)}_{g}\right) = (7)(3t^2)(2^t) + (7t^3)(2^t \cdot \ln(2))$$

$$= 21t^2 \cdot 2^t + 7t^3 2^t \ln(2)$$

Question: The derivative of $\dfrac{10^x}{x^3}$ is $\rightarrow \dfrac{d}{dx}\left(\dfrac{10^x}{x^3}\right)$

(a) $\dfrac{10^x}{\ln(10)}x^{-3} + 10^x(-3x^{-4})$

$$= \frac{(10^x \ln(10))(x^3) - (10^x)(3x^2)}{(x^3)^2}$$

(b) $\dfrac{10^x \ln(10)x^3 - 10^x(3x^2)}{x^6}$

$$\overset{\text{''} x^6}{} = 10^x \ln(10)\, x^{-3} - 3(10^x)\, x^{-4}$$

(c) $\dfrac{10^x \frac{1}{\ln(10)} x^3 - 10^x(3x^2)}{x^6}$

$$\frac{d}{dx}\left(10^x \cdot x^{-3}\right)$$

are equal.

$$= (10^x \ln(10))(x^{-3}) + 10^x(-3x^{-4})$$

(d) $\ln(10)10^x x^{-3} + 10^x(-3x^{-4})$

Chain Rule

D outside (inside same)

Nested Functions: $\dfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

outside

inside

(D inside)

Liebnitz form $\dfrac{d}{dx}\left(f(g(x))\right) = \dfrac{df}{dg}\dfrac{dg}{dx}$

Problem. Evaluate the following derivatives:

$$\frac{d}{dt}e^{(t^4)} = \underbrace{e^{(t^4)}}_{f'(g(t))} \cdot \underbrace{(4t^3)}_{g'(t)}$$

$f: \quad e^{\sim}$

$g: \quad t^4$

$$\frac{d}{dx}\ln(1+\sqrt{x}) \qquad \sqrt{x} = x^{1/2}$$

$$\frac{d}{dx}\ln(\sim) = \frac{1}{\sim}$$

$$= \underbrace{\left(\frac{1}{1+\sqrt{x}}\right)}_{f'(g(x))} \cdot \underbrace{\frac{d}{dx}\left(1+x^{1/2}\right)}_{g'(x)}$$

$$= \left(\frac{1}{1+\sqrt{x}}\right)\left(0+\frac{1}{2}x^{-1/2}\right)$$

$$= \left(\frac{1}{1+\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

const → don't need quotient

$$\frac{d}{dx}\left(\frac{1}{\ln(x)+x^3}\right) = \frac{d}{dx}\left(\ln(x)+x^3\right)^{-1}$$

$f: (\sim)^{-1}$

$g: (\ln(x)+x^3)$

$$= -1\left(\ln(x)+x^3\right)^{-2}\cdot\left(\frac{1}{x}+3x^2\right)$$

$$f'(g(x)) \quad \cdot \quad g'(x)$$

$$= \frac{-1}{\left(\ln(x)+x^3\right)^2}\left(\frac{1}{x}+3x^2\right)$$

$$\frac{d}{dt}\left(e^{(5t-1)} + 10^{(3t)}\right)$$

$$= \underbrace{e^{(5t-1)}}_{f'(g(t))} \cdot (5) + \underbrace{\left(10^{3t} \cdot \ln(10)\right)}_{f'(g(t))} \underbrace{(3)}_{g'(t)}$$

Question: The derivative of $e^{\sqrt{x}}$ is

(a) $\frac{1}{2}e^{\frac{1}{\sqrt{x}}}$

(b) $e^{\sqrt{x}}\left(\sqrt{x}\right)$

(c) $\frac{1}{2}e^{\sqrt{x}}\left(\frac{1}{\sqrt{x}}\right)$

(d) $\frac{1}{2}e^{\sqrt{x}}\left(\sqrt{x}\right)$

$$\frac{d}{dx} e^{\sqrt{x}} = \frac{d}{dx} e^{\left(x^{1/2}\right)}$$

$$= \left(e^{x^{1/2}}\right) \cdot \left(\frac{1}{2\sqrt{x}}\right)$$

$$= e^{\sqrt{x}} \cdot \left(\frac{1}{2\sqrt{x}}\right)$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

Problem. Prove the derivative rule for $\dfrac{d}{dx}\tan(x)$, using the definition $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and the other derivative rules.

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$= \frac{(\cos(x))[\cos(x)] - \sin(x)(-\sin(x))}{(\cos(x))^2}$$

quotient rule

$\cos^2(x) + \sin^2(x) = 1$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

def'n

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\boxed{\frac{d}{dx}\tan = \sec^2(x)}$$

Problem. Find the derivative of $4 + 6\cos(\pi t^2 + 1)$

(a) $4 - 6\sin(\pi t^2 + 1) \cdot (2\pi t)$

(b) $-6\cos(\pi t^2 + 1) \cdot (2\pi t)$

(c) $-6\sin(\pi t^2 + 1) \cdot (2\pi t)$ ✓

(d) $-6\sin(\pi t^2 + 1) \cdot (\pi t^2 + 1)$

(e) $6\sin(2\pi t)$

$$\frac{d}{dt}\left(4 + 6\cos(\pi t^2 + 1)\right)$$

$$= 0 + 6\left(\underbrace{-\sin(\pi t^2 + 1)}_{f'(g(t))}\right)\left(\pi(2t)\right)$$

$$= -6\sin(\pi t^2 + 1)(2\pi t)$$

Inverse Trig Functions

In addition to the 6 trig functions just seen, there are 6 inverse functions as well, though the inverses of sine, cosine, and tangent are the most commonly used.

inverse of sine

Notation: $\boxed{y = \arcsin(x)}$ and $\boxed{y = \sin^{-1}(x)}$ are both commonly used for the inverse of sine function.

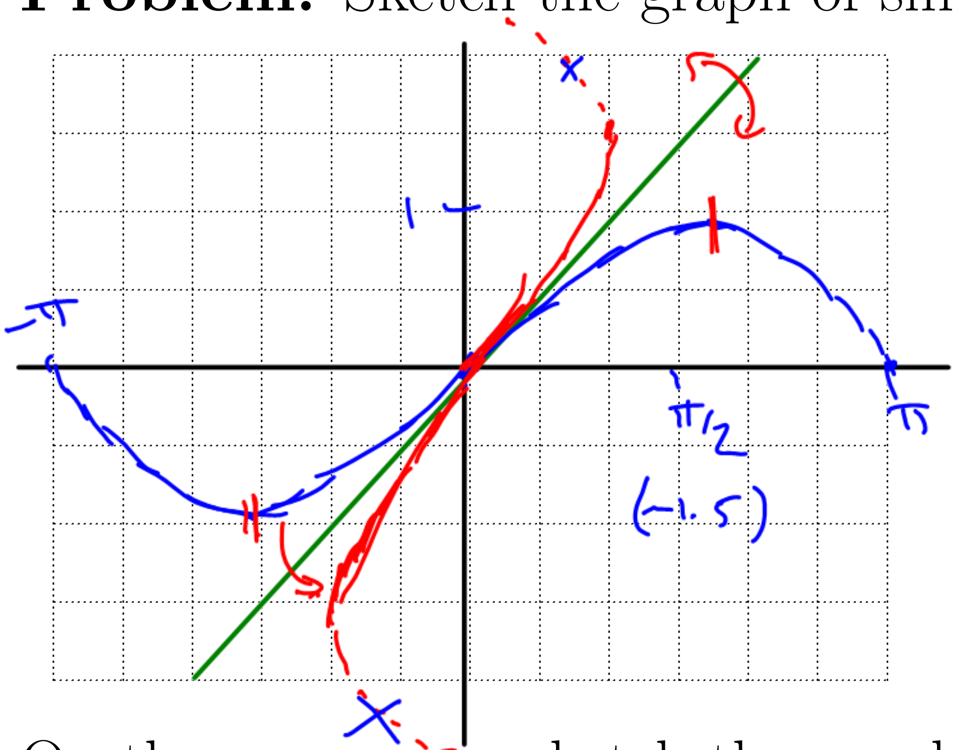
inverse of sin(x)

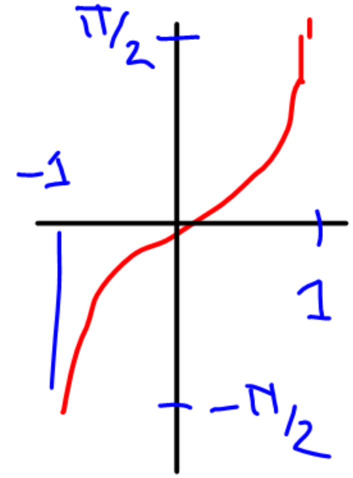
$\boxed{\frac{1}{\sin(x)} = \csc(x)}$

- We will use the arcsin notation as much as possible.

- On your calculator, you will typically see the $\sin^{-1}$ notation.

Problem. Sketch the graph of $\sin(x)$ on the axes below.



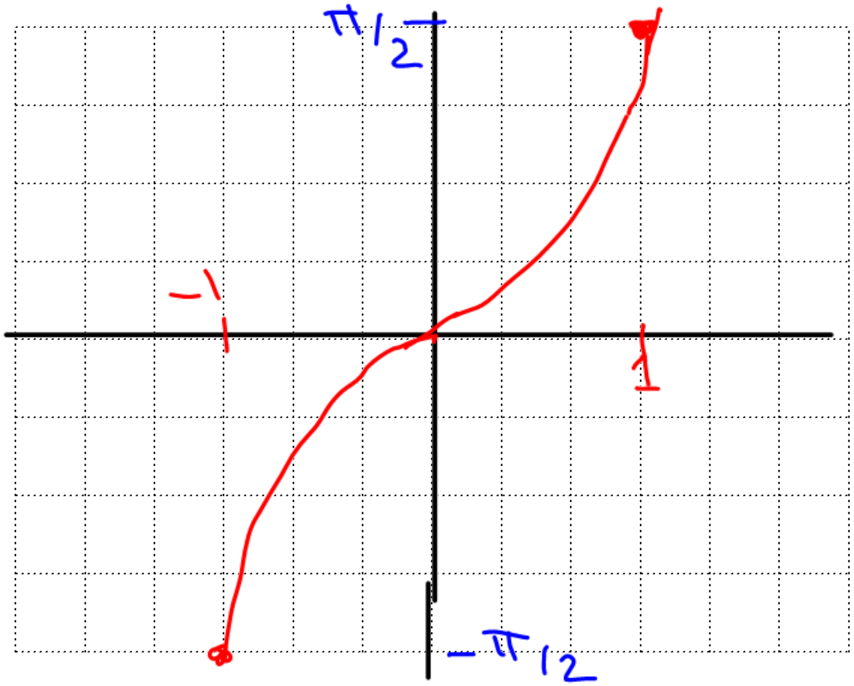


On the same axes, sketch the graph of arcsin(x), or $\sin^{-1}(x)$.

inverse "undoes" the original function

$\sin(\theta) = \text{ratio} \quad \left(\frac{\text{opp}}{\text{hyp}}\right)$

$\arcsin(\text{ratio}) = \theta$



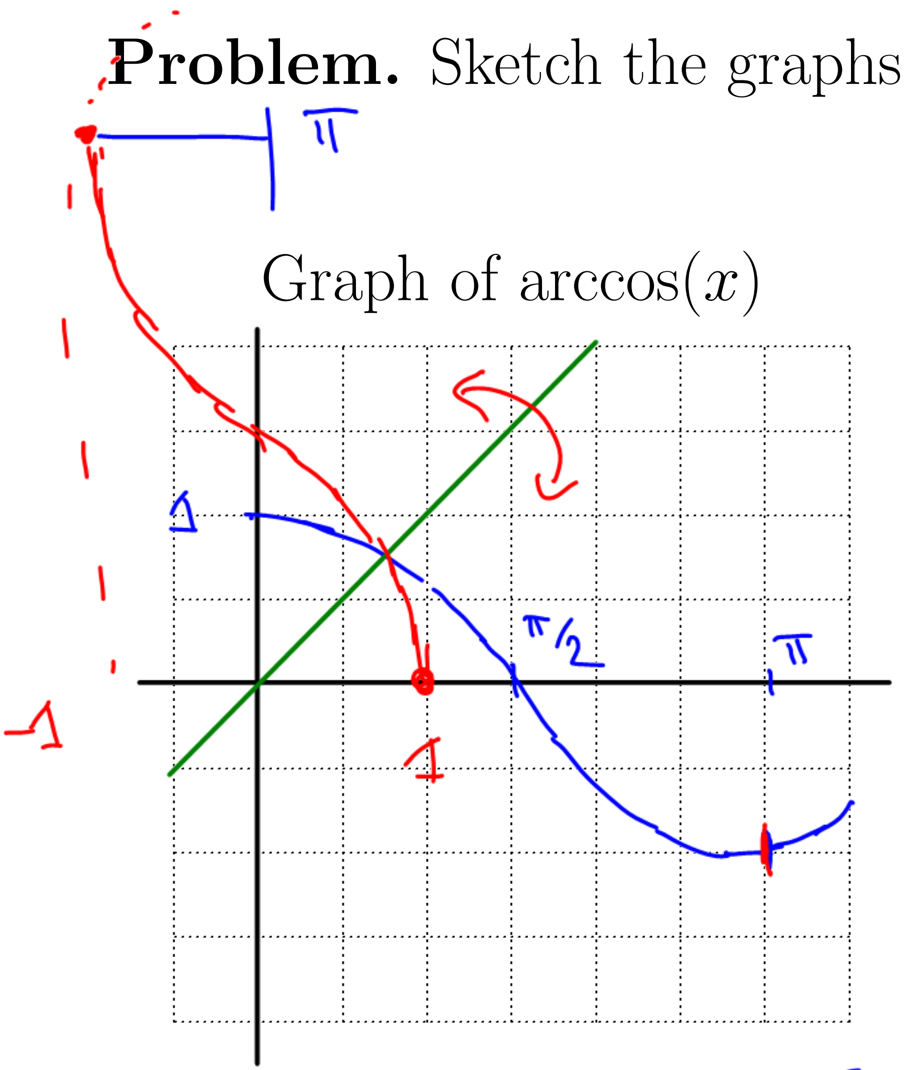
Problem. What is the domain of $\arcsin(x)$?

$[-1, 1]$

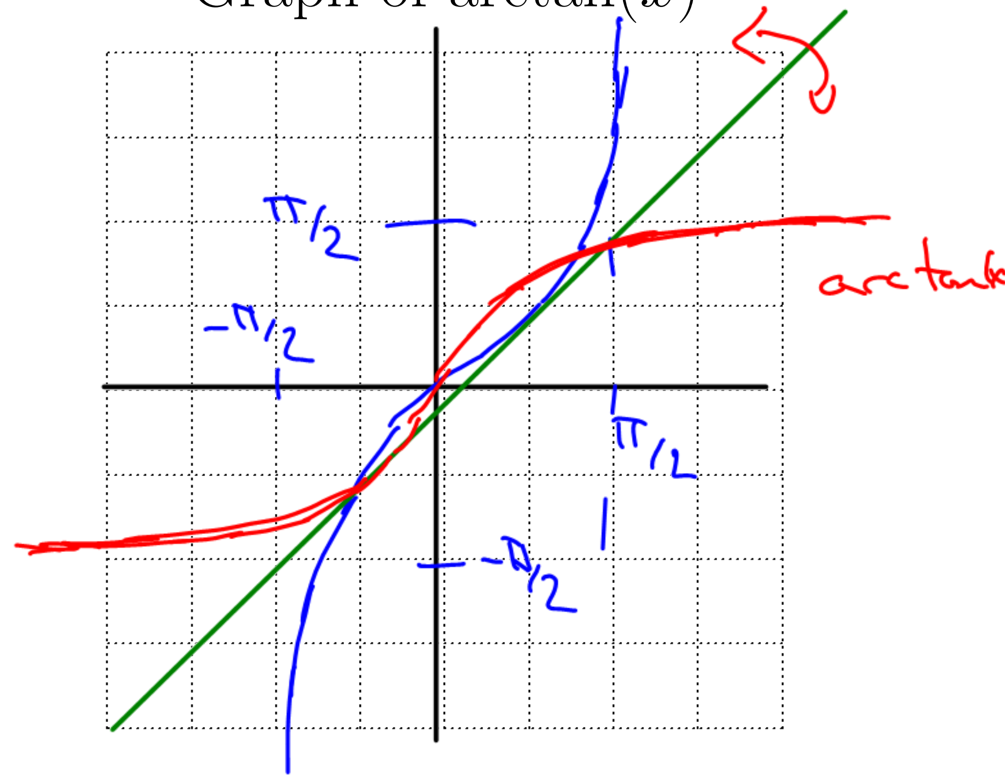
What is the range of $\arcsin(x)$?

$[-\pi/2, \pi/2]$

Problem. Sketch the graphs of arccos(x) and arctan(x).



Graph of arccos(x)



Graph of arctan(x)

domain arccos(x) [-1, 1]

ratio

range arccos(x) [0, π]

domain arctan(x)
all real values

range: $(-\pi/2, \pi/2)$

Derivative of Inverse Trig Functions

$$e^{\ln(x)} = x$$

$$(\sqrt{x})^2 = x$$

Problem. Simplify the expression $\sin(\arcsin x)$.

$$\sin(\underbrace{\arcsin}_{\text{inverse sin}}(x)) = x$$

Differentiate both sides of this equation, using the chain rule on the left. You should end up with an equation involving $\dfrac{d}{dx} \arcsin x$.

$$\frac{d}{dx}\left(\sin(\arcsin(x))\right) = \frac{d}{dx} x$$

$$\cos(\arcsin(x)) \cdot \underbrace{\frac{d}{dx}(\arcsin(x))}_{\text{want!}} = 1$$

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\cos(\arcsin(x))}$$

True, but not useful

Problem. Solve for $\frac{d}{dx}\arcsin x$, and simplify the resulting expression by means of the formula

$$\sin^2\theta + \cos^2\theta = 1 \longrightarrow \cos\theta = \sqrt{1-\sin^2(\theta)},$$

which is valid if $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))}$$

$$x = \sin(\arcsin(x))$$

$$= \frac{1}{\sqrt{1-\sin^2(\arcsin(x))}}$$

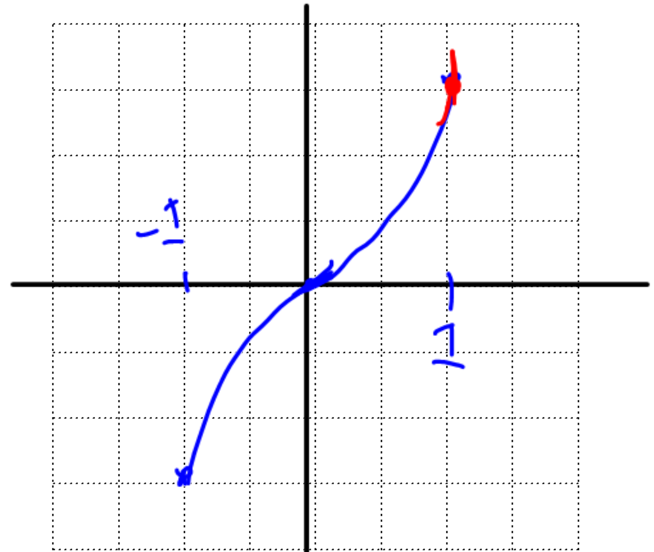
$$(\sin(\arcsin(x)))^2 = x^2$$

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

From this, we see that

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

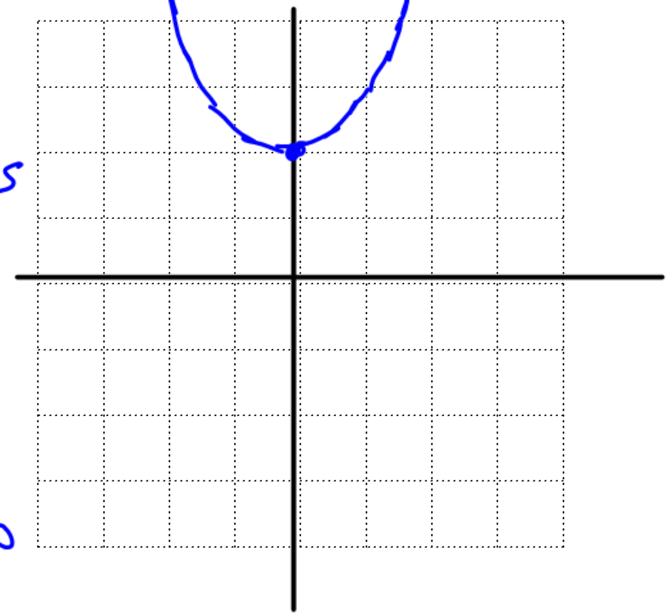
Graph of $\arcsin(x)$



Slopes

y values

Graph of $\dfrac{d}{dx}\arcsin(x)$



or $\dfrac{1}{\sqrt{1-x^2}}$

as $x \to 1$, $1-x^2 \to 0$

$1/\sqrt{1-x^2} \to \infty$

Through a similar process, we can find the derivatives of all the commonly used inverse trig functions.

sin(arcsin(x)) = x
$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

cos(arccos(x)) = x
$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

tan(arctan(x)) = x
$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Problem. Note any common themes or differences, related to the earlier trigonometric derivatives.

- derivs are not trig/inverse trig.
- $\frac{d}{dx}$ co... = $-$

Problem. Evaluate the following derivatives.

$$\frac{d}{dt}\arcsin\left(\frac{\cancel{x}\,t}{4}\right)$$

$$\frac{d}{dt}\arcsin(u) = \frac{1}{\sqrt{1-u^2}}$$

$$= \underbrace{\frac{1}{\sqrt{1-(t/4)^2}}}_{f'(g(t))}\ \underbrace{\left(\frac{1}{4}\right)}_{g'(t)}$$

$$\frac{d}{dx}\arctan\left(e^{3x}\right) \qquad 3x = h$$

$$\frac{d}{dx}\arctan(u) = \frac{1}{1+u^2}$$

$$= \underbrace{\frac{1}{1+(e^{3x})^2}}_{f'(g(\cancel{x}\,h(x)))}\ \left(\underbrace{e^{3x}}_{g'(h(x))}\cdot\underbrace{3}_{h'(x)}\right)$$