MNTC P01 - Week #6 - Integrals - Modeling

Numerical Integration

As part of this assignment, you should be able to reproduce the LHR rule calculations in MATLAB using a loop. You should know how to adapt it to handle either data from a file, or a function defined by a formula, as the case requires.

1. **Theory** Consider the problem of estimating the general form of the integral

$$\int_a^b f(x)\ dx$$

   (a) Assume $f(x)$ is a smooth and continuous function. For our the Left-Hand sum, LEFT(n), by what factor do we reduce the error if we use 10 times the number of intervals?

   (b) Evaluate the integral $\int_0^6 \cos(x)\ dx$ exactly, using anti-derivatives and the Fundamental Theorem of Calculus.

   (c) Confirm your answers to part (a) by finding the change in the error for LEFT(n) for the same integral, $\int_0^6 \cos(x)\ dx$ using $n = 20$, $n = 200$, and $n = 2000$. Find the error with each n value, and then compute the ratio of the errors each time you use $10\times$ as many intervals.

(a) If $f(x)$ is a smooth function, if we use 10 times the intervals, the error for LHR's estimate will drop by a factor of $\frac{1}{10}$

(b) We can evaluate the integral based on our earlier study of derivatives. The only thing to be careful of is that the input values will be in radians (not degrees).

$$\int_0^6 \cos(x)\ dx = \sin(x)\Big|_0^6 = \sin(6) - \sin(0) = -0.279415498$$

(c) The linked script below computes LEFT(n) for $n = 20, 200$ and 2000. It also computes the error for each of those integral estimates.

W06CosineIntegral.m

Here are the results from the script (with the values for `I` being the integrals, and `Err` being the error in the estimates.

```
I1 =
  -0.271342274784534
Err1 =
   0.008073223414391
I2 =
  -0.278797096021967
Err2 =
       6.184021769584658e-04
I3 =
  -0.279355544067250
Err3 =
       5.995413167569907e-05
```

We can see that as we increase the number of intervals by a factor of 10, the error in the LEFT(n) estimate is scaled by a factor of (approximately) $\frac{1}{10}$.

- When going to $n = 200$ from $n = 20$, the error ratio is $\dfrac{6.184021769584658e-04}{0.008073223414391} \approx 0.07 \approx 0.1$ or $\dfrac{1}{10}$.
- When going to $n = 2000$ from $n = 200$, the error ratio is $\dfrac{5.995413167569907e-05}{6.184021769584658e-04} \approx 0.097 \approx 0.1$ or $\dfrac{1}{10}$.

2. For each of the following integrals,

   - Evaluate the integral exactly,
   - use MATLAB to compute the LEFT(1000) integral estimate, and
   - comment on the agreement between the results.

   (a) $\int_0^5 x^3 - 5\ dx$

   (b) $\int_1^{10} \log_{10}(x)\ dx$

   (c) $\int_{-1}^{1} x^2 e^{x^3}\ dx$

(a) Exact value: $\int_0^5 x^3 - 5\ dx = \frac{x^4}{4} - 5x\Big|_0^5 = \left(\frac{5^4}{4} - 5(5)\right) - (0 - 0) = 131.25$
LEFT(1000) estimate (see link below to the script that computed this): 130.9377
The agreement is pretty good, with the LEFT(1000) within 1% of the exact value.

(b) $\int_1^{10} \log_{10}(x)\ dx.$

This requires integration by parts, and remembering the derivative rule for non-base-e logarithms: $\frac{d}{dx}\log_{10}(x) = \frac{1}{x\ln(10)}$ Computing the integral (without the $x = 1$ and $x = 10$ for now to keep the notation simpler):

$$\text{We choose } u = \log_{10} x \qquad\qquad \text{and } dv = dx$$
$$\text{so } du = \frac{1}{x\ln(10)}\ dx \qquad\qquad \text{and } v = x$$

Using the integration by parts formula,

$$\begin{aligned}
\int \log_{10} x\ dx &= x(\log_{10} x) - \int x\frac{1}{x\ln(10)}\ dx \\
&= x\log_{10} x - \int \frac{1}{\ln(10)}\ dx \\
&= x\log_{10} x - \frac{x}{\ln(10)} + C
\end{aligned}$$

This gives the definite integral value of

$$\begin{aligned}
\int_1^{10} \log_{10}(x)\ dx &= \left(x\log_{10} x - \frac{x}{\ln(10)}\right)\Big|_1^{10} \\
&= \left(10\log_{10}(10) - \frac{10}{\ln(10)}\right) - \left(1\log_{10}(1) - \frac{1}{\ln(10)}\right) \\
&= 10 - \frac{10}{\ln(10)} + \frac{1}{\ln(10)} \\
&\approx 6.0914
\end{aligned}$$

LEFT(1000) estimate (see link below to the script that computed this): 6.0868
The agreement is better for this function, with the LEFT(1000) within 0.1% of the exact value.

(c) $\int_{-1}^{1} x^2 e^{x^3}\ dx$
This integral can be evaluated exactly using a substitution.

$$\text{Let } w = x^3, \text{ so } dw = 3x^2\ dx \text{ or } dx = \frac{1}{3x^2}\ dw$$
$$\int x^2 e^{x^3}\ dx = \int x^2 e^w \left(\frac{1}{3x^2}\ dw\right) = \int \frac{1}{3} e^w dw = \frac{1}{3} e^w + C = \frac{1}{3} e^{x^3} + C$$

This gives the definite integral value of

$$\begin{aligned}\int_{-1}^{1} x^2 e^{x^3}\ dx &= \frac{1}{3} e^{x^3}\Big|_{-1}^{1} \\ &= \frac{1}{3}\left(e^1 - e^{-1}\right) \\ &\approx 0.7834\end{aligned}$$

LEFT(1000) estimate (see link below to the script that computed this): 0.7811
The agreement is good for this function, with the LEFT(1000) within 1% of the exact value.

Link to the MATLAB code:
W06LeftExamples.m

3. Use the `integral` function to estimate the following integrals, and print the estimates with at least 8 digits after the decimal. Use the default accuracy for the `integral` function. Note that the exact values for these integrals were computed in the previous question, so we can compare the `integral` estimate to the exact values.

   (a) $\int_0^5 x^3 - 5\ dx$ (exact value: $\frac{5^4}{4} - 25$)

   (b) $\int_1^{10} 4\log_{10}(x)\ dx$ (exact value: $\frac{(-36 + 40\ln(2) + 40\ln(5))}{\ln(2) + \ln(5)}$

   (c) $\int_{-1}^{1} x^2 e^{x^3}\ dx$

Code that does this would be e.g.

```
clc
format long
f = @(x) x.^3 - 5;
a = 0;
b = 5;
I = integral(f, a, b);
exact_value = (5^4/4 - 5*5);
[exact_value, I]  % display both the exact value and the integral estimate
```

A link to the full MATLAB script for all the examples is included below.

The results are

(a) function: `f = @(x) x.^3 - 5;`
    estimated integral: 131.25000000

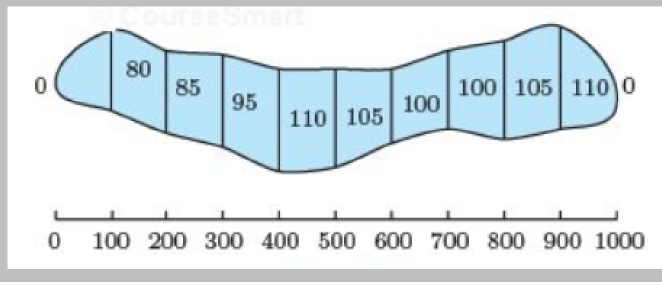
(b) function: `f = @(x) log10(x);`
    estimated integral: 6.091349662870734

(c) function: `f = @(x) x.^2 .* exp(x.^3);`
    estimate integral: 0.783467462429201

Note that all these values are the same as the exact values for all or almost all the digits shown (12 digits of accuracy). This is a lot better performance than using the LEFT(n) approach, and is actually easier to code because the `integral` function is built into MATLAB.

Link to the MATLAB code:
W06IntegralExamples.m

4. The width (in feet) of the golf course fairway was measured at 100-foot intervals as indicated on the figure. Estimate the square footage of of the fairway.



NOTE: In this problem, because we are only given data and not a function, the only technique we have that would work is LEFT(n). We **can't** use the Fundamental Theorem or the `integral` function because both of those require a formula for the function that we don't have.

The key steps are:

1. Create a vector of the widths, *including* the zero widths at the start and end of the fairway.
2. Note that $\Delta x = 100$ feet for each left-to-right distance.
3. Note that there are 10 intervals (11 widths), so we will be performing a LEFT(10) calculation.
4. Approximate the area on each left-to-right interval as width at the left $\times$ Δx (this is the LEFT rule on each interval).
5. Use a loop to accumulate the total area estimated.

Link to the MATLAB code:
W06Golf.m

The final estimated area is 88,000 square feet. We don't have a formula for the widths, so this is simply our best estimate of the area.

5. When a pendulum oscillates, with maximum deviation angle θ_0, the period of the pendulum is given by

$$T = 4\sqrt{L/g}\int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin\left(\frac{1}{2}\theta_0\right)$ and g is the acceleration due to gravity, 9.8 m/s.

Compute and compare the period of a pendulum with

- $L = 2$, $\theta_0 = 40^o$,
- $L = 2$, $\theta_0 = 20^o$.
- $L = 2.5$, $\theta_0 = 40^o$,
- $L = 2.5$, $\theta_0 = 20^o$.

Describe how significant the effect of maximum swing angle θ_0 is on the period of a pendulum, compared to the effect of the pendulum length.

Full solution is available in `q_pendulumPeriod.m`.

The periods computed are show below. "quad" was used for the integration.

L = 2.0 m, theta0 = 40.0 deg, period = 2.9274 s.
L = 2.0 m, theta0 = 20.0 deg, period = 2.8602 s.
L = 2.5 m, theta0 = 40.0 deg, period = 3.2729 s.
L = 2.5 m, theta0 = 20.0 deg, period = 3.1978 s.

From these results, it is clear that the angle has a minimal effect on the period of the oscillations, compared with the effect of the length. This insensitivity of the period to the oscillation angle explains why pendulum clocks and metronomes do not need a specific swing angle to be fairly accurate, but *do* need a specific length.

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