QUANTUM FIELD THEORY – Part I

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1 Introduction

Quantum Field Theory (abbreviated QFT) deals with the quantization of fields. A familiar example of a field is provided by the electromagnetic field. Classical electromagnetism describes the dynamics of electric charges and currents, as well as electro-magnetic waves, such as radio waves and light, in terms of Maxwell's equations. At the atomic level, however, the quantum nature of atoms as well as the quantum nature of electromagnetic radiation must be taken into account. Quantum mechanically, electromagnetic waves turn out to be composed of quanta of light, whose individual behavior is closer to that of a particle than to a wave. Remarkably, the quantization of the electromagnetic field is in terms of the quanta of this field, which are particles, also called *photons*. In QFT, this field particle correspondence is used both ways, and it is one of the key assumptions of QFT that to every elementary particle, there corresponds a field. Thus, the electron will have its own field, and so will every quark.

Quantum Field Theory provides an elaborate general formalism for the field–particle correspondence. The advantage of QFT will be that it can naturally account for the creation and annihilation of particles, which ordinary quantum mechanics of the Schrödinger equation could not describe. The fact that the number of particles in a system can change over time is a very important phenomenon, which takes place continuously in everyone's daily surroundings, but whose significance may not have been previously noticed.

In *classical mechanics*, the number of particles in a closed system is conserved, i.e. the total number of particles is unchanged in time. To each pointlike particle, one associates a set of position and momentum coordinates, the time evolution of which is governed by the dynamics of the system. *Quantum mechanics* may be formulated in two stages.

1. The *principles of quantum mechanics*, such as the definitions of states, observables, are general and do not make assumptions on whether the number of particles in the system is conserved during time evolution.

2. The *specific dynamics of the quantum system*, described by the Hamiltonian, may or may not assume particle number conservation. In introductory quantum mechanics, dynamics is usually associated with non-relativistic mechanical systems (augmented with spin degrees of freedom) and therefore assumes a fixed number of particles. In many important quantum systems, however, *the number of particles is not conserved*.

A familiar and ubiquitous example is that of *electromagnetic radiation*. An excited atom may decay into its ground state by emitting a single quantum of light or photon. The photon was not "inside" the excited atom prior to the emission; it was "created" by the excited atom during its transition to the grounds state. This is well illustrated as follows. An atom in a state of sufficiently high excitation may decay to its ground state in a single step by emitting a single photon. However, it may also emit a first photon to a lower excited state which in turn re-emits one or more photons in order to decay to the ground state (see Figure 1, (a) and (b)). Thus, given initial and final states, the number of photons emitted may vary, lending further support to the fact that no photons are "inside" the excited state to begin with.

Other systems where particle number is not conserved involve *phonons* and *spin waves* in condensed matter problems. Phonons are the quanta associated with vibrational modes of a crystal or fluid, while spin waves are associated with fluctuating spins. The number of particles is also not conserved in nuclear processes like fusion and fission.

1.1 Relativity and quantum mechanics

Special relativity invariably implies that the number of particles is not conserved. Indeed, one of the key results of special relativity is the fact that mass is a form of energy. A particle at rest with mass m has a rest energy given by the famous formula

$$E = mc^2 \tag{1.1}$$

The formula also implies that, given enough energy, one can *create particles* out of just energy – kinetic energy for example. This mechanism is at work in fire and light bulbs, where energy is being provided from chemical combustion or electrical input to excite atoms which then emit light in the form of photons. The mechanism is also being used in particle accelerators to produce new particles through the collision of two incoming particles. In Figure 1 the example of a photon scattering off an electron is illustrated. In (c), a photon of low energy ($\ll m_e c^2$) is being scattered elastically which results simply in a deflection of the photon and a recoil of the electron. In (d), a photon of high energy ($\gg m_e c^2$) is being scattered inelastically, resulting not only in a deflection of the photon and a recoil of the electron, but also in the *production of new particles*.

The particle data table also provides numerous examples of particles that are unstable and decay. In each of these processes, the number of particles is *not conserved*. To list just a few,

$$\begin{aligned}
n &\to p^+ + e^- + \bar{\nu}_e \\
\pi^0 &\to \gamma + \gamma \\
\pi^+ &\to \mu^+ + \nu_\mu \\
\mu^+ &\to e^+ + \nu_e + \bar{\nu}_\mu
\end{aligned}$$

As already mentioned, nuclear processes such as fusion and fission are further examples of systems in which the number of particles is not conserved.

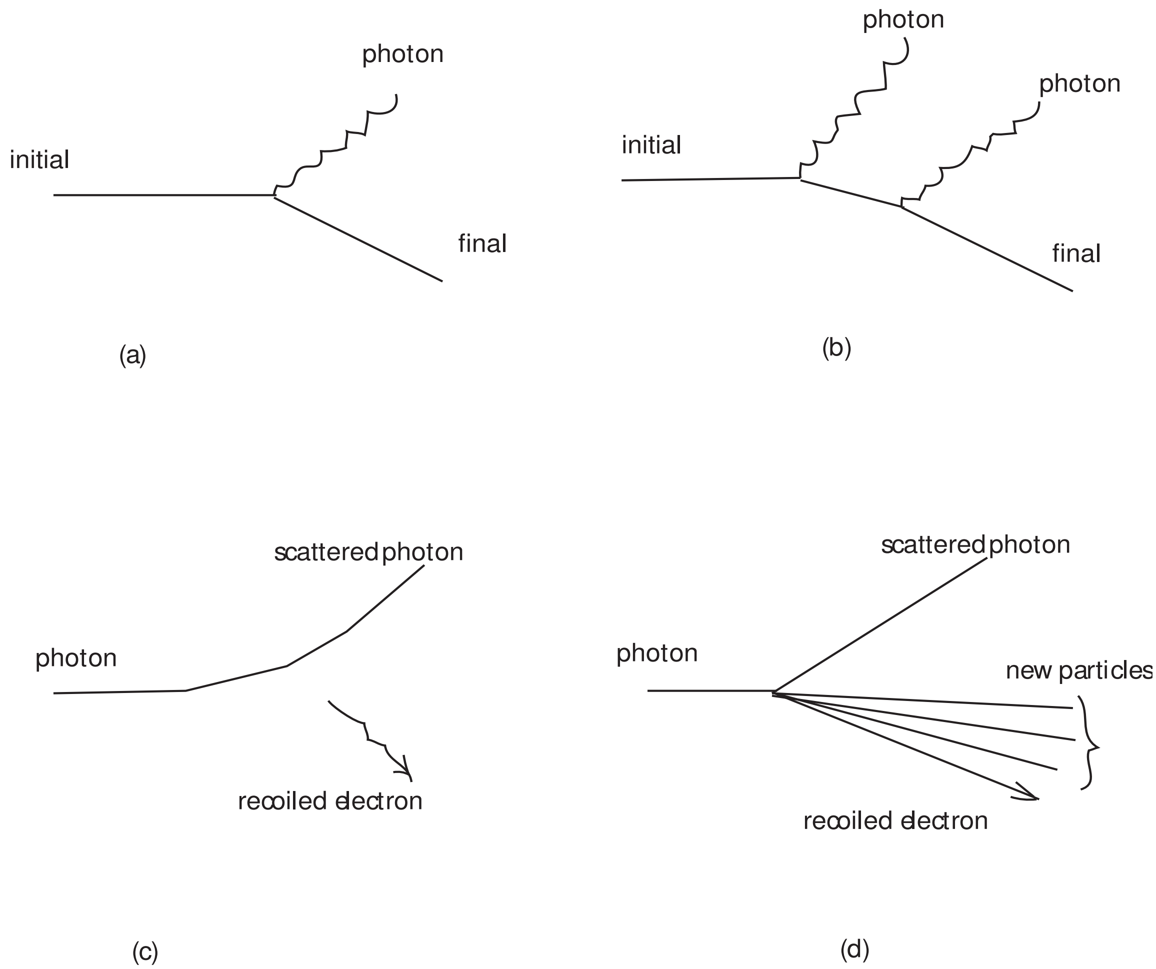

Figure 1: Production of particles : (a) Emission of one photon, (b) emission of two photons between the same initial and final states; (c) low energy elastic scattering, (d) high energy inelastic scattering of a photon off a recoiling electron.

1.2 Why Quantum Field Theory ?

Quantum Field Theory is a formulation of a quantum system in which the number of particles does not have to be conserved but may vary freely. QFT does not require a change in the principles of either quantum mechanics or relativity. QFT requires a different formulation of the dynamics of the particles involved in the system.

Clearly, such a description must go well beyond the usual Schrödinger equation, whose very formulation requires that the number of particles in a system be fixed. Quantum field theory may be formulated for non-relativistic systems in which the number of particles is not conserved, (recall spin waves, phonons, spinons etc). Here, however, we shall concentrate on *relativistic quantum field theory* because relativity forces the number of particles not to be conserved. In addition, relativity is one of the great fundamental principles of Nature, so that its incorporation into the theory is mandated at a fundamental level.

1.3 Further conceptual changes required by relativity

Relativity introduces some further fundamental changes in both the nature and the formalism of quantum mechanics. We shall just mention a few.

• **Space versus time**

In non-relativistic quantum mechanics, the position x, the momentum p and the energy E of free or interacting particles are all observables. This means that each of these quantities *separately* can be *measured* to arbitrary precision in an arbitrarily short time. By contrast, the accurarcy of the simultaneous measurement of x and p is limited by the Heisenberg uncertainty relations,

$$\Delta x \ \Delta p \ \sim \hbar$$

There is also an energy-time uncertainty relation $\Delta E \ \Delta t \sim \hbar$, but its interpretation is quite different from the relation $\Delta x \ \Delta p \ \sim \hbar$, because in ordinary quantum mechanics, time is viewed as a parameter and not as an observable. Instead the energy-time uncertainty relation governs the time evolution of an interacting system. In relativistic dynamics, particle-antiparticle pairs can always be created, which subjects an interacting particle always to a cloud of pairs, and thus inherently to an uncertainty as to which particle one is describing. Therefore, the momentum itself is no longer an instantaneous observable, but will be subject to the a momentum-time uncertainty relation $\Delta p \, \Delta t \sim \hbar/c$. As $c \to \infty$, this effect would disappear, but it is relevant for relativistic processes. Thus, momentum can only be observed with precision away from the interaction region.

Special relativity puts space and time on the same footing, so we have the choice of either treating space and time both as observables (a bad idea, even in quantum mechanics) or to treat them both as parameters, which is how QFT will be formulated.

• **"Negative energy" solutions and anti-particles**

The kinetic law for a relativistic particle of mass m is

$$E^2 = m^2c^4 + p^2c^2$$

Positive and negative square roots for E naturally arise. Classically of course one may just keep positive energy particles. Quantum mechanically, interactions induce transitions to negative energy states, which therefore cannot be excluded arbitrarily. Following Feynman, the correct interpretation is that these solutions correspond to negative frequencies, which describe physical anti-particles with positive energy traveling "backward in time".

• **Local Fields and Local Interactions**

Instantaneous forces acting at a distance, such as appear in Newton's gravitational force and Coulomb's electrostatic force, are incompatible with special relativity. No signal can travel faster than the speed of light. Instead, in a relativistic theory, the interaction must be *mediated* by another particle. That particle is the graviton for the gravitational force and the photon for the electromagnetic force. The true interaction then occurs at an instant in time and at a point in space, thus expressing the forces in terms of *local interactions* only. Thus, the Coulomb force is really a limit of relativistic "retarded" and "advanced" interactions, mediated by the exchange of photons. The exchange is pictorially represented in Figure 2.

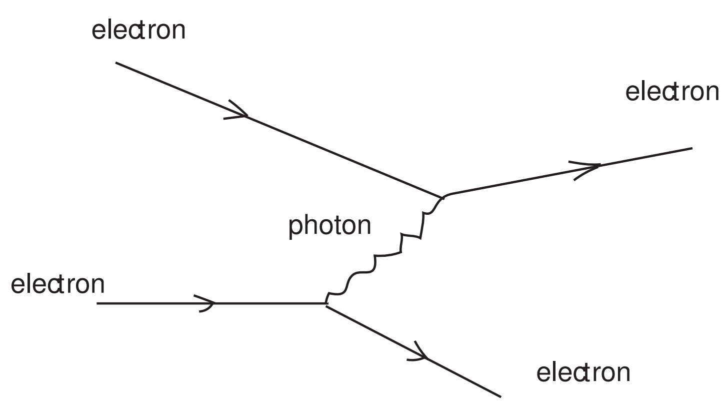

Figure 2: The Coulomb force results from the exchange of photons.

One may realize this picture concretely in terms of local fields, which have local couplings to one another. The most basic prototype for such a field is the electro-magnetic field. Once a field has been associated to the photon, then it will become natural to associate a field to every particle that is viewed as *elementary* in the theory. Thus, electrons and positrons will have their (Dirac) field. There was a time that a field was associated also to the proton, but we now know that the proton is composite and thus, instead, there are now fields for quarks and gluons, which are the elementary particles that make up a proton. The gluons are the analogs for the strong force of the photon for the electro-magnetic force, namely the gluons mediate the strong force. Analogously, the $W^\pm$ and Z^0 mediate the weak interactions.

1.4 Some History and present significance of QFT

The quantization of the elctro-magnetic field was initiated by Born, Heisenberg and Jordan in 1926, right after quantum mechanics had been given its definitive formulation by Heisenberg and Schrödinger in 1925. The complete formulation of the dynamics was given by Dirac Heiseberg and Pauli in 1927. Infinities in perturbative corrections due to high energy (or UV – ultraviolet) effects were first studied by Oppenheimer and Bethe in the 1930's. It took until 1945-49 until Tomonaga, Schwinger and Feynman gave a completely relativistic formulation of Quantum Electrodynamics (or QED) and evaluated the radiative corrections to the magnetic moment of the electron. In 1950, Dyson showed that the UV divergences of QED can be systematically dealt with by the process of renormalization.

In the 1960's Glashow, Weinberg and Salam formulated a renormalizable quantum field theory of the weak interactions in terms of a Yang-Mills theory. Yang-Mills theory was shown to be renormalizable by 't Hooft in 1971, a problem that was posed to him by his advisor Veltman. In 1973, Gross, Wilczek and Politzer discovered asymptotic freedom of certain Yang-Mills theories (the fact that the strong force between quarks becomes weak at high energies) and using this unique clue, they formulated (independently also Weinberg) the quantum field theory of the strong interactions. Thus, the elctro-magnetic, weak and strong forces are presently described – and very accurately so – by quantum field theory, specifically Yang-Mills theory, and this combined theory is usually referred to as *the STANDARD MODEL*. To give just one example of the power of the quantum field theory approach, one may quote the experimentally measured and theoretically calculated values of the muon magnetic dipole moment,

$$\begin{aligned} \frac{1}{2} g_\mu(\text{exp}) &= 1.001159652410(200) \\ \frac{1}{2} g_\mu(\text{thy}) &= 1.001159652359(282) \end{aligned} \tag{1.2}$$

revealing an astounding degree of agreement.

The gravitational force, described classically by Einstein's general relativity theory, does not seem to lend itself to a QFT description. String theory appears to provide a more appropriate description of the quantum theory of gravity. String theory is an extension of QFT, whose very formulation is built squarely on QFT and which reduces to QFT in the low energy limit.

The devlopment of quantum field theory has gone hand in hand with developments in Condensed Matter theory and Statistical Mechanics, especially critical phenomena and phase transitions.

A final remark is in order on QFT and mathematics. Contrarily to the situation with general relativity and quantum mechanics, there is no good "axiomatic formulation" of QFT, i.e. one is very hard pressed to lay down a set of simple postulates from which QFT may then be constructed in a deductive manner. For many years, physicists and mathematicians have attempted to formulate such a set of axioms, but the theories that could be fit into this framework almost always seem to miss the physically most relevant ones, such as Yang-Mills theory. Thus, to date, there is no satsifactory mathematical "definition" of a QFT.

Conversely, however, QFT has had a remarkably strong influence on mathematics over the past 25 years, with the development of Yang-Mills theory, instantons, monopoles, conformal field theory, Chern-Simons theory, topological field theory and superstring theory. Some developments is QFT have led to revolutions in mathematics, such as Seiberg-Witten theory. It is suspected by some that this is only the tip of the iceberg, and that we are only beginning to have a glimpse at the powerful applications of quantum field theory to mathematics.

2 Quantum Mechanics – A synopsis

The basis of QFT is quantum mechanics and therefore we shall begin by reviewing the foundations of this dsicipline. In fact, we begin with a brief summary of Lagrangian and Hamiltonian classical mechanics as this will also be of great value.

2.1 Classical Mechanics

Consider a system of N degrees of freedom $q_i(t)$, $i = 1, \cdots, N$. (For example n particles in $\mathbf{R}^3$ has $N = 3n$.) Dynamics is governed by the *Lagrangian function**

$$L(q_i, \dot{q}_i) \qquad\qquad \dot{q}_i(t) \equiv \frac{dq_i(t)}{dt} \tag{2.1}$$

or by the *action functional*

$$S[q_i] = \int_{t_1}^{t_2} dt\; L(q_i(t), \dot{q}_i(t)) \tag{2.2}$$

The classical time evolution of the system is determined as a solution to the variational problem as follows

$$\begin{cases} \delta S[q_i] = 0 \\ \delta q_i(t_1) = \delta q_i(t_1) = 0 \end{cases} \qquad \Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2.3}$$

and these equations are the *Euler-Lagrange equations*. In general, the differential equations are second order in t, and thus the Cauchy data are $q_i(t_1)$ and $\dot{q}_i(t_1)$ at initial time t_1 : namely positions and velocities.

• **Hamiltonian Formulation**

The *Hamiltonian formulation* of classical mechanics proceeds as follows. We introduce the *canonical momenta*,

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}(q_j, \dot{q}_j) \qquad \Rightarrow \qquad \dot{q}_i(q, p) \tag{2.4}$$

and introduce the Hamiltonian function

$$H(p, q) \equiv \sum_{i=1}^{N} p_i \dot{q}_i(p, q) - L(q_j, \dot{q}_j(p, q)) \tag{2.5}$$

The time evolution equation can now be re-expressed as the *Hamilton equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.6}$$

*For almost all the problems that we shall consider, L has no *explicit* time dependence and all time-dependence is implicit through $q_i(t)$ and $\dot{q}_i(t)$.

The time evolution of any function $F(p,q,t)$ along the classical trajectory may be expressed as follows

$$\begin{aligned} \frac{d}{dt}F(p,q,t) &= \{H,F\} + \frac{\partial F}{\partial t} \\ \{A,B\} &\equiv \sum_{i=1}^{N}\Big(\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i}\Big) \end{aligned} \tag{2.7}$$

The *Poisson bracket* $\{,\}$ is antisymmetric and satisfies the following relations

$$\begin{aligned} 0 &= \{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} \\ \{A,BC\} &= \{A,B\}C + \{A,C\}B \\ \{p_i,q_j\} &= \delta_{ij} \end{aligned} \tag{2.8}$$

The second line shows that the Poisson bracket acts as a derivation.

- **Symmetries**

A *symmetry* of a classical system is a transformation on the degrees of freedom q_i such that under its action, every solution to the corresponding Euler-Lagrange equations is transformed into a solution of these same equations. Symmetries form a *group* under successive composition. An important special class consists of *continuous symmetries*, where the transformation may be labeled by a parameter (or by several parameters) in a continuous way. Denoting a transformation depending on a single parameter $\alpha \in \mathbf{R}$ by $\sigma(\alpha)$, the transformation may be represented as follows,

$$\sigma(\alpha) \ : \ q_i(t) \longrightarrow q_i(t,\alpha) \qquad\qquad q_i(t,0) = q_i(t) \tag{2.9}$$

The transformation $\sigma(\alpha)$ is a *continuous symmetry* of the Lagrangian L <u>if and only if</u> the derivative with repect to α satisfies the following property

$$\frac{\partial}{\partial\alpha}L(q_i(t,\alpha),\dot q_i(t,\alpha)) = \frac{d}{dt}X(q_i,\dot q_i) \tag{2.10}$$

obtained <u>without using the Euler-Lagrange equations</u>.

- **Noether's Theorem**

The existence of a continuous symmetry implies the presence of a time independent charge Q (also sometimes called a first integral of motion) by Noether's theorem. An explicit formula is available for this charge

$$Q = \sum_{i=1}^{N} p_i \frac{\partial q_i(t,\alpha)}{\partial\alpha}\Big|_{\alpha=0} - X(p_i,q_i) \tag{2.11}$$

The relation $\dot{Q} = 0$ follows now by using the Euler-Lagrange equations. In the Hamiltonian formulation, the charge generates the transformation in turn by Poisson bracketting,

$$\frac{\partial q_i(t,\alpha)}{\partial \alpha} = \{Q, q_i(t,\alpha)\} \tag{2.12}$$

Continuous symmetries of a given L form a *Lie group*. Lie groups and algebras will be defined generally in the next section.

2.2 Principles of Quantum Mechanics

A quantum system may be defined in all generality in terms of a very small number of physical principles.

1. *Physical States* are represented by rays in a Hilbert space $\mathcal{H}$. Recall that a Hilbert space $\mathcal{H}$ is a complex (complete) vector space with a positive definite inner product, which we denote by $\langle \ | \ \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbf{C}$ à la Dirac, and we have

$$\begin{aligned} &\bullet \quad \langle \phi|\psi\rangle = \langle \psi|\phi\rangle^* \\ &\bullet \quad \langle \phi|\phi\rangle \geq 0 \\ &\bullet \quad \langle \phi|\phi\rangle = 0 \qquad \Rightarrow \qquad |\phi\rangle = 0 \end{aligned} \tag{2.13}$$

   A ray is an equivalence class in $\mathcal{H}$ by $|\phi\rangle \sim \lambda|\phi\rangle$, $\lambda \in \mathbf{C} - \{0\}$.

2. *Physical Observables* are represented by self-adjoint operators on $\mathcal{H}$. For the moment, we shall denote such operators with a hat, e.g. $\hat{A}$, but after this chapter, we drop hats. The operators are linear, so that $\hat{A}(a|\phi\rangle + b|\psi\rangle) = a\hat{A}|\phi\rangle + b\hat{A}|\psi\rangle$, for all $|\phi\rangle,\ |\psi\rangle \in \mathcal{H}$ and $a, b \in \mathbf{C}$. Recall that for finite-dimensional matrices, Hermiticity and self-adjointness are the same. For operators acting on an infinite dimensional Hilbert space, self-adjointness requires that the domains of the operator and its adjoint be the same as well.

3. *Measurements* : A state $|\psi\rangle$ in $\mathcal{H}$ has a definite measured value α for some observable $\hat{A}$ if the state is an eigenstate of $\hat{A}$ with eigenvalue α; $\hat{A}|\psi\rangle = \alpha|\psi\rangle$. (Since $\hat{A}$ is self-adjoint, states associated with different eigenvalues of $\hat{A}$ are orthogonal to one another.) In a quantum system, what you can measure in an experiment are the eigenvalues of various observables, e.g. the energy levels of atoms.

4. The *transition probability* for a given state $|\psi\rangle \in \mathcal{H}$ to be found in one of a set of orthogonal states $|\phi_n\rangle$ is given by

$$P(|\psi\rangle \to |\phi_n\rangle) = |\langle \psi|\phi_n\rangle|^2 \tag{2.14}$$

   for normalized states $\langle \psi|\psi\rangle = \langle \phi_n|\phi_n\rangle = 1$. The quantities $\langle \psi|\phi_n\rangle$ are referred to as the *transition amplitudes*. Notice that their dependence on the state $|\psi\rangle$ is linear, so that these amplitudes obey the superposition principle, while the probability amplitudes are quadratic and cannot be linearly superimposed.

Notice that amongst the principles of quantum mechanics, there is no reference to a Hamiltonian, a Schrödinger equation and the like. Those ingredients are dynamical and will depend upon the specific system under consideration.

• **Commuting observables – quantum numbers**

If two observables $\hat{A}$ and $\hat{B}$ commute, $[\hat{A}, \hat{B}] = 0$, then they can be diagonalized simultaneously. The associated physical observables can then be measured simultaneously, yielding eigenvalues α and β respectively. On the other hand, if $[\hat{A}, \hat{B}] \neq 0$, the associated physical observables cannot be measured simultaneously.

One defines a *maximal set of commuting observales* as a set of observables $\hat{A}_1, \hat{A}_2, \cdots, \hat{A}_n$ which all mutually commute, $[\hat{A}_i, \hat{A}_j] = 0$, for all $i, j = 1, \cdots, n$, and such that no other operator (besides linear combinations of the $\hat{A}_i$) commutes with all $\hat{A}_i$. It follows that the eigenspaces for each of the sets of eigenvalues is one-dimensional, and therefore label the states in $\mathcal{H}$ in a unique manner. Given the set $\{\hat{A}_i\}_{i=1,\cdots,n}$, the corresponding eigenvalues are the *quantum numbers*.

• **Symmetries in quantum systems**

Symmetries in quantum mechanics are realized by *unitary* transformations in Hilbert space $\mathcal{H}$. Unitarity guarantees that the transition amplitudes and thus probabilities will be invariant under the symmetry.

2.3 Quantum Systems from classical mechanics

A very large and important class of quantum systems derive from classical mechanics systems (for example the Hydrogen atom). Given a classical mechanics system, one can always construct an associated quantum system, using the *correspondence principle*. Consider a classical mechanics system with degrees of freedom $p_i(t), q_i(t) \in \mathbf{R}$, $i = 1, \cdots, N$ and with Hamiltonian $H(p, q)$. The associated quantum system is obtained by mapping the reals p_i and q_i into observables in a Hilbert space $\mathcal{H}$. The Hilbert space may be taken to be $L^2(\mathbf{R}^N)$, i.e. square integrable functions of q_i. The Poisson bracket is mapped into a commutator of operators. The detailed correspondence is

$$\begin{array}{rcl} p_i, \ q_i & \rightarrow & \hat{p}_i, \ \hat{q}_i \\ \{p_i, q_j\} = \delta_{ij} & \rightarrow & \frac{i}{\hbar}[\hat{p}_i, \hat{q}_j] = \delta_{ij} \\ \{p_i, p_j\} = \{q_i, q_j\} = 0 & \rightarrow & [\hat{p}_i, \hat{p}_j] = [\hat{q}_i, \hat{q}_j] = 0 \end{array}$$

Under the correspondence principle, the Hamiltonian $H(p, q)$ produces a self-adjoint operators $\hat{H}(\hat{p}, \hat{q})$. In general, however, this correspondence will not be unique, because in the classical system, p and q are commuting numbers, so their ordering in any expressing was immaterial, while in the quantum system, $\hat{p}$ and $\hat{q}$ are operators and their ordering will matter. If such ordering ambiguity arises in passing from the classical to the quantum system, further physical information will have to be supplied in order to lift the ambiguity.

If the classical system had a continuous symmetry, then by Noether's theorem, there is an associated time-independndt charge Q. Using the correspondence principle, one may construct an associated quantum operator $\hat{Q}$. However, the correspondence principle, in general, does not produce a unique $\hat{Q}$ and the question arises whether an operator $\hat{Q}$ can be constructed at all which is time independent. If yes, the classical symmetry extends to a quantum symmetry. If not, the classical symmetry is said to have an anomaly; this effect does not arise in quantum mechanics but only appears in quantum field theory. A particularly important symmetry of a system with time-independent Hamiltonian is time translation invariance. Time-evolution is governed by the Heisenberg equation

$$i\hbar \frac{d\hat{A}(t)}{dt} = [\hat{A}(t), \hat{H}] \tag{2.15}$$

The operators $\hat{p}$, $\hat{q}$ and $\hat{H}$ themselves are observable, and their eigenvalues are the *momentum, position and energy of the state.*

The prime example, and which will be extremely ubiquitous in the practice of quantum field theory is the Harmonic Oscillator. We consider here the simplest case with $N = 1$,

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2, \qquad \omega > 0 \tag{2.16}$$

Notice that since there are no terms involving both $\hat{p}$ and $\hat{q}$, the passage from classical to quantum was unambiguous here. Introducing the linear combinations,

$$\begin{aligned} \hat{a} &\equiv \frac{1}{\sqrt{2\omega\hbar}}(+i\hat{p} + \omega\hat{q}) \\ \hat{a}^\dagger &\equiv \frac{1}{\sqrt{2\omega\hbar}}(-i\hat{p} + \omega\hat{q}) \end{aligned} \tag{2.17}$$

The system may be recast in the following fashion,

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= 1 \\ \hat{H} &= \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \end{aligned} \tag{2.18}$$

The operators are referred to as *annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators* because their application on any state lowers and raises the energy of the state with precisely one quantum of energy $\hbar\omega$. This is shown as follows,

$$\begin{aligned} [\hat{H}, \hat{a}] &= -\hbar\omega\hat{a} \\ \{\hat{H}, \hat{a}^\dagger] &= +\hbar\omega\hat{a}^\dagger \end{aligned} \tag{2.19}$$

Thus, if $|E\rangle$ is an eigenstate with energy E, so that $\hat{H}|E\rangle = E|E\rangle$, then it follows that

$$\begin{aligned} \hat{H}(\hat{a}|E\rangle) &= (E - \hbar\omega)(\hat{a}|E\rangle) \\ \hat{H}(\hat{a}^\dagger|E\rangle) &= (E + \hbar\omega)(\hat{a}^\dagger|E\rangle) \end{aligned} \tag{2.20}$$

Using this algebraic formulation, the sytem may be solved very easily. Note that $\hat{H} > 0$, so there must be a state $|E_0\rangle \neq 0$, such that $\hat{a}|E_0\rangle = 0$, whence it follows that $E_0 = \frac{1}{2}\hbar\omega$ and the energies of the remaining states $(\hat{a}^\dagger)^n|E_0\rangle$ are $E_n = \hbar\omega(n + \frac{1}{2})$.

2.4 Quantum systems associated with Lie algebras

Not all quantum systems have classical analogs. For example, quantum orbital angular momentum corresponds to the angular momentum of classical systems, but spin angular momentum has no classical counterpart. Another example would be the color assignment of quarks, and one may even view flavor assignments (namely whether they are *up, down, charm, strange, top* or *bottom*) as a quantum property without classical counterpart. It is natural to understand these quantum systems in terms of the theory of Lie groups and Lie algebras. After all, spin appeared as a special kind of representation of the rotation group that cannot be realized in terms of orbital angular momentum, and color will ultimately be associated with the gauge group of the strong interactions $SU(3)_c$. Even the quantum system of free particles will be thought of in terms of representations of the Poincaré group.

Consider a Lie group G and its associated Lie algebra $\mathcal{G}$. Let ρ be a unitary representation of $\mathcal{G}$ acting on a Hilbert space $\mathcal{H}$. (For representations of finite dimension N, we have $\mathcal{H} = \mathbf{C}^N$.) This set-up naturally defines a quantum system associated with the Lie algebra $\mathcal{G}$ and the representation ρ. The quantum observables are the linear self-adjoint operators $\rho(T^a)$. The maximal set of commuting observables is the largest set of commuting generators of $\mathcal{G}$. For semi-simple Lie algebras this is the Cartan sub-algebra. We conclude with a few important examples,

1. Angular momentum is associated with the group of space rotations $O(3)$; its unitary representations are labelled by half integers $j = 0,\ 1/2,\ 1,\ 3/2, \cdots$. Only for integer j is there a classical realization in terms of orbital angular momentum.

2. Special relativity states that the space-time symmetry group of Nature is the Poincaré group. Elementary particles will thus be associated with unitary representations of the Poincaré group.

2.5 Appendix I : Lie groups, Lie algebras, representations

The concept of a group was introduced in the context of polynomial equations by Evariste Galois, and in a more general context by Sophus Lie. Given an algebraic or differential equation, the set of transformations that map any solution into a solution of the same equation forms a *group*, where the group operation is composition of maps. As physics is formulated in terms of mathematical equations, groups naturally enter into the understanding and solution of these equations.

• Definition of a group

Generally, a set G forms a group provided it is endowed with an operation, which we denote $*$, and which satisfies the following conditions

1. Closure : $g_1, g_2 \in G$, then $g_1 * g_2 \in G$;

2. Associativity : $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$ for all $g_1, g_2, g_3 \in G$;

3. There exists a unit I such that $I * g = g * I = g$ for all $g \in G$;

4. Every $g \in G$ has an inverse g^{-1} such that $g^{-1} * g = g * g^{-1} = I$.

Additionally, the group $G_{,*}$ is called *commutative* or *Abelian* is $g_1 * g_2 = g_2 * g_1$ for all $g_1, g_2 \in G$. If on the other hand there is at least one pair g_1, g_2 such that $g_1 * g_2 \neq g_2 * g_1$, then the group is said to be *non-commutative* or *non-Abelian*.

Given a group G, a *subgroup* H is a subset of G that contains I such that $h_1 * h_2 \in H$ for all $h_1, h_2 \in H$. An *invariant subgroup* or *ideal* H of G is a subgroup such that $g * h * g^{-1} \in H$ for all $g \in G$ and all $h \in H$. For example, the Poincaré group has an invariant subgroup consisting of translations alone. A group whose only invariant subgroups are $\{e\}$ and G itself is called a *simple group*.

• Lie groups and Lie algebras

As we have encountered already many times in physics, group elements may depend on *parameters*, such as rotations depend on the angles, translations depend on the distance, and Lorentz transformations depend on the boost velocity). A *Lie group* is a parametric group where the dependence of the group elements on all its parameters is continuous (this by itself would make it a topological group) and differentiable. The number of independent parameters is called the *dimension* D of G. It is a powerful Theorem of Sophus Lie that if the parametric dependence is differentiable just once, the additional group property makes the dependence automatically infinitely differentiable, and thus real analytic.

A *Lie algebra* $\mathcal{G}$ associated with a Lie group G is obtained by expanding the group element g around the identity element I. Assuming that a set of local parameters x_i, $i = 1, \cdots, D$ has been made, we have

$$g(x_i) = I + \sum_{i=1}^{D} x_i T^i + \mathcal{O}(x^2) \tag{2.21}$$

The T^i are the *generators* of the Lie algebra $\mathcal{G}$ and may be thought of as the tangent vectors to the group manifold G at the identity. The fact that this structure forms a group allows us to compose and in particular form,

$$g(x_i)g(y_i)g(x_i)^{-1}g(y_i)^{-1} \ \in \ G \tag{2.22}$$

Retaining only the terms linear in x_i and linear in y_j, we have

$$g(x_i)g(y_i)g(x_i)^{-1}g(y_i)^{-1} = I + \sum_{i,j=1}^{D} x_i y_j [T^i, T^j] + \mathcal{O}(x^2, y^2) \tag{2.23}$$

Hence there must be a linear relation,

$$[T^i, T^j] = \sum_{k=1}^{D} f^{ijk}\ T^k \tag{2.24}$$

for a set of constants f^{ijk} which are referred to as the structure constants, which satisfy the *Jacobi identity*,

$$0 = [[T^i, T^j], T^k] + [[T^j, T^k], T^i] + [[T^k, T^i], T^j] \tag{2.25}$$

Lie's Theorem states that, conversely, if we have a Lie algebra $\mathcal{G}$, defined by generators T_i, $i = 1, \cdots, D$ satisfying both (2.24) and (2.25), then the Lie algebra may be integrated up to a Lie group (which is unique if connected and simply connected). Thus, except for some often subtle global issues, the study of Lie groups may be reduced to the study of Lie algebras, and the latter is much simpler in general ! Examples include,

1. $G = U(N)$, unitary $N \times N$ matrices (with complex entries); $g \in U(N)$ is defined by $g^\dagger g = I$; $D = N^2$. Its Lie algebra $\mathcal{G}$ (often also denoted by $U(N)$) consists of $N \times N$ Hermitian matrices, $(iT)^\dagger = (iT)$.

2. $G = O(N)$, orthogonal $N \times N$ matrices (with real entries); $g \in O(N)$ is defined by $g^t g = I$; $D = N(N-1)/2$. Its Lie algebra $\mathcal{G}$ (often also denoted by $O(N)$) consists of $N \times N$ anti-symmetric matrices, $T^t = -T$.

3. Any time S appears before the name, the group elements have also unit determinant. Thus $G = SU(N)$ consists of g such that $g^\dagger g = I$ and $\det g = 1$. And $G = SO(N)$ consists of g such that $g^t g = I$ and $\det g = 1$. Notice that $U(N)$ has two invariant subgroups : the diagonal $U(1)$ and $SU(N)$.

   Examples 1, 2 and 3 describe *compact Lie groups*, which are defined to be compact spaces, i.e. bounded and closed. A famous non-comapct group is described next.

4. $G = Gl(N, \mathbf{R})$ and $G = Gl(N, \mathbf{C})$ are the general linear groups of $N \times N$ matrices g with $\det g \neq 0$ (respectively with real and complex entries). $G = Sl(N, \mathbf{R})$ and $G = Sl(N, \mathbf{C})$ are the special linear groups defined by $\det g = 1$.

• **Representations of Lie groups and Lie algebras**

A (linear) *representation* R with dimension N of a group G is a map

$$R : G \quad \rightarrow \quad GL(N, \mathbf{C}) \text{ or } Gl(N, \mathbf{R}) \tag{2.26}$$

such that the group operation $*$ is mapped onto matrix multiplication in $Gl(N)$,

$$R(g_1 * g_2) = R(g_1)\ R(g_2) \qquad \& \qquad R(I) \ = \ I_N \tag{2.27}$$

In other words, the representation realizes the abstract group in terms of $N \times N$ matrices. A representation $R : G \rightarrow GL(N, \mathbf{C})$ is called a *complex representation*, while $R : G \rightarrow GL(N, \mathbf{R})$ is called a *real representation*. Two representations R and R' are *equivalent* if they have the same dimension and if there exists a constant invertible matrix S such that $R'(g) = S^{-1} R(g) S$ for all $g \in G$. A representation is said to be *reducible* if there exists a constant matrix D, which is not a scalar multiple of the identity matrix, and which commutes with $R(g)$ for all g. If no such matrix exists, the representation is *irreducible*. A special class of representations that is especially important in physics is that of *unitary representations*, for which $R : G \rightarrow U(N)$. It is a standard result that every finite-dimensional representation of a compact (or finite) group is equivalent to a unitary representation.

A representation of a Lie algebra $\mathcal{G}$ is a linear map $\mathcal{R} : \mathcal{G} \rightarrow \mathcal{G}l(N, \mathbf{C})$, such that

$$[\mathcal{R}(T^i), \mathcal{R}(T^j)] = \sum_{k=1}^{D} f^{ijk} \mathcal{R}(T^k) \tag{2.28}$$

As $\mathcal{R}(T^i)$ are matrices, the Jacobi identity is automatic.

A representation $R : G \rightarrow GL(N)$ naturally acts on an N-dimensional vector space, which in physics is a subspace of states or wave functions. The linear space is said to *transform under the representation* R, and the distiction between the representation and the vector space on which it acts is sometimes blurred.

2.6 Functional integral formulation

In preparation for the path integral formulation of quantum mechanics, for systems associated with classical mechanics systems with degrees of freedom $p(t)$, $q(t)$, we introduce two special adapted bases, one for position and one for momentum. Taking the first line in the table as a definition, the remaining lines follow.

The evolution operator

$$\hat{U}(t) = \exp\{-itH/\hbar\} \tag{2.29}$$

governs the transition amplitude $\langle\psi|\hat{U}(t)|\phi\rangle$, for an intital state $|\phi\rangle$ to evolve into a final state $|\psi\rangle$ after a time t. For systems whose only degrees of freedom are p and q, such amplitudes may be expressed in terms of the wave functions $\langle q|\phi\rangle$ and $\langle q|\psi\rangle$ of these states using the completeness relations,

$$\langle\psi|\hat{U}(t)|\phi\rangle = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \; \langle q|\phi\rangle\langle q'|\psi\rangle^* \langle q'|\hat{U}(t)|q\rangle \tag{2.30}$$

To evaluate the matrix elements of $\hat{U}$ in position basis, we make use of the group property of $\hat{U}$, $\hat{U}(t'+t'') = \hat{U}(t')\hat{U}(t'')$ and insert complete sets of position states,

$$\langle q'|\hat{U}(t'+t'')|q\rangle = \int_{-\infty}^{+\infty} dq'' \langle q'|\hat{U}(t'')|q''\rangle\langle q''|\hat{U}(t')|q\rangle \tag{2.31}$$

To evaluate $\langle q'|\hat{U}(t)|q\rangle$, we divide the time interval t into N equal segments of length $\epsilon = t/N$ and insert $N-1$ complete sets of position eigenstates (labelled by q_k, $k = 1, \cdots, N-1$) into the formula $\hat{U}(t) = \hat{U}(\epsilon)^N$. This yields

$$\langle q'|\hat{U}(t)|q\rangle = \prod_{k=1}^{N-1} \int_{-\infty}^{+\infty} dq_k \prod_{k=1}^{N} \langle q_k|\hat{U}(\epsilon)|q_{k-1}\rangle \tag{2.32}$$

where $q_N = q'$ and $q_0 = q$.

Position Basis	Momentum Basis
$\hat{q}\|q\rangle = q\|q\rangle$	$\hat{p}\|p\rangle = p\|p\rangle$
$\langle q'\|q\rangle = \delta(q'-q)$	$\langle p'\|p\rangle = \delta(p'-p)$
$\int_{-\infty}^{+\infty} dq \; \|q\rangle\langle q\| = I_{\mathcal{H}}$	$\int_{-\infty}^{+\infty} dp \; \|p\rangle\langle p\| = I_{\mathcal{H}}$
$e^{+ia\hat{p}}\hat{q}e^{-ia\hat{p}} = \hat{q} + a\hbar$	$e^{-ib\hat{q}}\hat{p}e^{+ib\hat{q}} = \hat{p} + b\hbar$
$e^{-ia\hat{p}}\|q\rangle = \|q + a\hbar\rangle$	$e^{+ib\hat{q}}\|p\rangle = \|p + b\hbar\rangle$

Table 1:

Each matrix element $\langle q_k|\hat{U}(\epsilon)|q_{k-1}\rangle$ may in turn be evaluated as follows. First, we insert a complete set of states in the momentum basis (labelled by p_k, $k = 1, \cdots, N$)

$$\langle q_k|\hat{U}(\epsilon)|q_{k-1}\rangle = \int_{-\infty}^{+\infty} dp_k \langle q_k|\hat{U}(\epsilon)|p_k\rangle\langle p_k|q_{k-1}\rangle \tag{2.33}$$

Using the overlaps of position and momentum eigenstates, we have

$$\langle q_k|p_k\rangle = e^{+ip_k q_k/\hbar} \qquad \langle p_k|q_{k-1}\rangle = e^{-ip_k q_{k-1}/\hbar} \tag{2.34}$$

The reason for introducing also the basis of momentum eigenstates is that now the needed matrix elements of the quantum Hamiltonian $\hat{H}(\hat{p}, \hat{q})$ can be evaluated.

In the limit $N \to \infty$, we have $\epsilon \to 0$ and thus the evolution operator over the infinitesimal time span ϵ may be evaluated by expanding the exponential to first order only,

$$\hat{U}(\epsilon) = I_{\mathcal{H}} - i\frac{\epsilon}{\hbar}\hat{H} + \mathcal{O}(\epsilon^2) \tag{2.35}$$

We assume that $\hat{H}$ is analytic in $\hat{p}$ and $\hat{q}$, so that upon using the commutation relation $[\hat{p}, \hat{q}] = -i\hbar$, we may rearrange $\hat{H}$ so as to put all $\hat{p}$ to the right and all $\hat{q}$ to the left in any given monomial. This allows us to define a classical function $H(p, q)$ by

$$H(p_k, q_k) \equiv \langle q_k|\hat{H}(\hat{p}, \hat{q})|p_k\rangle \tag{2.36}$$

and therefore, we have

$$\langle q_k|\hat{U}(\epsilon)|p_k\rangle = e^{-i\epsilon H(p_k,q_k)/\hbar}\langle q_k|p_k\rangle \tag{2.37}$$

Assembling all results, we now find the following expression,

$$\langle q'|\hat{U}(t)|q\rangle = \lim_{N\to\infty} \prod_{k=1}^{N-1} \int_{-\infty}^{+\infty} dq_k \prod_{l=1}^{N} \int_{-\infty}^{+\infty} dp_l \prod_{m=1}^{N} e^{iS_k(p,q)/\hbar} \tag{2.38}$$

where

$$S_k(p, q) \equiv p_k(q_k - q_{k-1}) - \epsilon H(p_k, q_k) \tag{2.39}$$

In the limit $N \to \infty$, for t fixed, the label k on p_k and q_k becomes continuous,

$$\begin{aligned}
p_k &\to p(t') \qquad\qquad t' = k\epsilon = kt/N, \quad \epsilon = dt' \\
q_k &\to q(t') \\
S_k(p, q) &\to dt'\Big(p(t')\dot{q}(t') - H(p(t'), q(t'))\Big)
\end{aligned} \tag{2.40}$$

As a result, the product of exponentials converges as follows

$$\lim_{N\to\infty}\prod_{m=1}^{N} e^{iS_k(p,q)/\hbar} = \exp\left\{\frac{i}{\hbar}\int_0^t dt'\Big(p\dot q - H(p,q)\Big)(t')\right\} \tag{2.41}$$

The measure "converges" to a random walk measure, which is denoted as follows,

$$\lim_{N\to\infty}\prod_{k=1}^{N-1}\int_{-\infty}^{+\infty} dq_k \prod_{l=1}^{N}\int_{-\infty}^{+\infty} dp_l = \int \mathcal{D}q\int\mathcal{D}p \tag{2.42}$$

The final result is the *path integral formulation of quantum mechanics*, which gives an expression for the position basis matrix elements of the evolution operator,

$$\langle q'|\hat U(t)|q\rangle = \int\mathcal{D}q\int\mathcal{D}p\exp\left\{\frac{i}{\hbar}\int_0^t dt'\Big(p\dot q - H(p,q)\Big)(t')\right\} \tag{2.43}$$

with the boundary conditions that $q(t'=0)=q$ and $q(t'=t)=q'$. This formulation was first proposed by Dirac and rederived by Feynman.

The quantity that enters the exponential in the path integral is really the classical action of the system,

$$S[p,q] \equiv \int dt'\Big(p\dot q - H(p,q)\Big)(t') \tag{2.44}$$

Therefore, in the *classical limit* where one formally takes the limit $\hbar\to 0$, the part integral will be dominated by the stationnary or saddle points of the integral, which are defined as those points at which the first order (functional derivatives of $S[p,q]$ with respect to p and q vanish. It is easy to work out what these equations are

$$\begin{aligned} 0 = \frac{\delta S}{\delta q} &= -\dot p - \frac{\partial H}{\partial q} \\ 0 = \frac{\delta S}{\delta p} &= +\dot q - \frac{\partial H}{\partial p} \end{aligned} \tag{2.45}$$

These are precisely the Hamilton equations of the classical Hamiltonian H. Thus, we recover very easily in the path integral formulation that in the classical limit, quantum mechanics receives its dominant contribution from classical solutions, as expected.

3 Principles of Relativistic Quantum Field Theory

The laws of quantum mechanics need to be supplemented with the principle of relativity. We shall ignore gravity and thus general relativity throughout and assume that space-time is flat Minkowski space-time.[†] The principle of relativity is then that of special relativity, which states that in flat Minkowski space-time, *the laws of physics are invariant under the Poincaré group*. Recall that the Poincaré group is the (semi-direct) product of the group of translations $\mathbf{R}^4$ and the Lorentz group $SO(1,3)$,

$$ISO(1,3) \equiv \mathbf{R}^4 \rtimes SO(1,3) \tag{3.1}$$

Quantum field theory is a quantum system, so we have the same definitions of states in Hilbert space and of self-adjoint operators corresponding to observables. In addition, the requirements that these states and operators transform consistently under the Poincaré group need to be implemented. In brief,

- The states and observables in Hilbert space will transform under unitary representations of the Poincaré group.
- The fields in QFT are local observables and transform under unitary representations of the Poincaré group, which induce finite dimensional representations of the Lorentz group on the components of the field.
- Micro-causality requires that any two local fields evaluated at points that are space-like separated are causally unrelated and thus must commute (or anti-commute for fermions) with one another.

Before making these principles quantitative, it is helpful to review the definitions and properties of the Poincaré and Lorentz groups and to discuss the unitary representations of the Poincaré group as well as the finite-dimensional representations of the Lorentz group.

3.1 The Lorentz and Poincaré groups and algebras

In special relativity the speed of light as observed from different inertial frames is the same, c. Inertial frames are related to one another by Poincaré transformations, which in turn may be defined as the transformations that leave the Minkowski distance between

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[†]Henceforth, we shall work in units where $c = \hbar = 1$.

two points invariant.[‡] This distance is defined by

$$s^2 \equiv (x^0 - y^0)^2 - (\vec{x} - \vec{y})^2 \equiv \eta_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) \tag{3.2}$$

where the Minkowski metric tensor is given by

$$\eta_{\mu\nu} \equiv \text{diag}\ [1\ \ -1\ \ -1\ \ -1] \tag{3.3}$$

Clearly, the Minkowski distance is invariant under *translations* by a^μ and *Lorentz transformations* by $\Lambda^\mu{}_\nu$,

$$R(\Lambda, a) \begin{cases} x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \\ y^\mu \to y'^\mu = \Lambda^\mu{}_\nu y^\nu + a^\mu \end{cases} \tag{3.4}$$

where the Lorentz transformation matrix must satisfy

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \qquad \Leftrightarrow \qquad \Lambda^T \eta \Lambda = \eta \qquad \Leftrightarrow \qquad (\Lambda^{-1})_\mu{}^\nu = \Lambda^\nu{}_\mu \tag{3.5}$$

The above transformation laws form the Poincaré group under the multiplication law,

$$R(\Lambda_1, a_1) R(\Lambda_2, a_2) = R(\Lambda_1 \Lambda_2, a_1 + \Lambda_1 a_2) \tag{3.6}$$

Special cases include,

1. $a = 0$ yields the Lorentz group, which is a subgroup of the Poincaré group. The definition (3.5) is analogous to that of an orthogonal matrix, $M^t I M = I$; for this reason the Lorentz group is denoted by $O(1,3)$.

2. A further special case of the Lorentz group is formed by the subgroup of rotations, characterized by $\Lambda^0{}_0 = 1, \Lambda^0{}_i = \Lambda^i{}_0 = 0$. Boosts do not by themselves form a subgroup; a boost in the direction 1 leaves $x'^{2,3} = x^{2,3}$ and

$$\begin{aligned} x'^0 &= +x^0 \text{ch}\tau - x^1 \text{sh}\tau \\ x'^1 &= -x^0 \text{sh}\tau + x^1 \text{ch}\tau \end{aligned} \qquad\qquad \text{ch}\tau = \sqrt{1 - v^2} \tag{3.7}$$

3. $\Lambda = I$ yields the translation group, which is an invariant subgroup of Poincaré.

[‡]Time and space coordinates are denoted by t and x^i, $i = 1, 2, 3$ respectively. Space coordinates are often regrouped in a 3-vector $\vec{x}$, while it will be convenient to measure time in distances, using the speed of light constant c via the relation $x^0 \equiv ct$. All four coordinates may then be conveniently regrouped into a 4-vector x^μ, $\mu = 0, 1, 2, 3$. Throughout, we adopt the Einstein convention in which repeated upper and lower indices are summed over and the summation symbol is implicit.

The *Poincaré algebra* is gotten by considering infintesimal translations by a^μ and infinitesimal Lorentz transformations,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \Rightarrow \eta_{\mu\rho}\omega^\mu{}_\rho + \eta_{\rho\nu}\omega^\nu{}_\sigma = 0 \qquad \Rightarrow \qquad \omega_{\mu\nu} + \omega_{\nu\mu} = 0 \tag{3.8}$$

so that $\omega_{\mu\nu}$ is antisymmetric, and has 6 independent components. It is customary to define the corresponding Lie algebra generators P_μ and $M_{\mu\nu}$ with a factor of i,

$$\begin{aligned} R(\Lambda, a) &= \exp\Big\{ -\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} + ia^\mu P_\mu \Big\} \\ &\sim I - \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} + ia^\mu P_\mu + \mathcal{O}(\omega^2, a^2, \omega a) \end{aligned} \tag{3.9}$$

so that the generators P_μ and $M_{\mu\nu}$ are self-adjoint in any unitary representation. The structure relations of the Poincaré algebra are given by

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [M_{\mu\nu}, P_\rho] &= +i\ \eta_{\nu\rho}P_\mu - i\ \eta_{\mu\rho}P_\nu \\ [M_{\mu\nu}, M_{\rho\sigma}] &= +i\ \eta_{\nu\rho}M_{\mu\sigma} - i\ \eta_{\mu\rho}M_{\nu\sigma} - i\ \eta_{\nu\sigma}M_{\mu\rho} + i\ \eta_{\mu\sigma}M_{\nu\rho} \end{aligned} \tag{3.10}$$

From these relations, it is clear that the generators $M_{\mu\nu}$ form the *Lorentz subalgebra*, while the generators P^μ form the *Abelian invariant subalgebra of translations*.

The Poincaré algebra has an immediate and important *quadratic Casimir operator*, namely a bilinear combination of the generators that commutes with all the generators of the Poincaré algebra. This Casimir is the *mass square operator*

$$m^2 \equiv P^\mu P_\mu \tag{3.11}$$

which is in a sense the classical definition of mass. m^2 is real, but despite its appearance, it can take positive or negative values.

3.2 Parity and Time reversal

Parity (or reversal of an odd number of space directions) and time reversal are elements of the Lorentz group with very special significance in physics. They are defined as follows,

$$\begin{aligned} P &: \ x'^0 = +x^0 \qquad x'^i = -x^i \qquad\qquad x'^\mu \equiv Px^\mu = x_\mu \\ T &: \ x'^0 = -x^0 \qquad x'^i = +x^i \qquad\qquad x'^\mu \equiv Tx^\mu = -x_\mu \end{aligned} \tag{3.12}$$

Notice that the product PT corresponds to $x'^\mu = -x^\mu$ or $\Lambda = -I$.

The role of P and T is important in the global structure (or topology) of the Lorentz group, as may be seen by studying the relation $\Lambda^t \eta \Lambda = \eta$ more closely. By taking the determinant on both sides, we find,

$$(\det\Lambda)^2 = 1 \tag{3.13}$$

while the $\mu = 0$, $\nu = 0$ component of the equation is

$$1 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)(\Lambda^i{}_0) \Rightarrow |\Lambda^0{}_0| \geq 1 \tag{3.14}$$

Transformations with $\det\Lambda = +1$ are *proper* while those with $\det\Lambda = -1$ are improper; those with $\Lambda^0{}_0 \geq 1$ are orthochronous, while those with $\Lambda^0{}_0 \leq -1$ are non-orthochronous. Hence the Lorentz group has four disconnected components.

$$\begin{array}{llll}
L_+^\uparrow & \det\Lambda = +1 & \Lambda^o{}_o \geq +1 & \text{proper orthochronous} \\
L_+^\downarrow & \det\Lambda = +1 & \Lambda^o{}_o \leq -1 & \text{proper non} - \text{orthochronous} \\
L_-^\uparrow & \det\Lambda = -1 & \Lambda^o{}_o \geq +1 & \text{improper orthochronous} \\
L_-^\downarrow & \det\Lambda = -1 & \Lambda^o{}_o \leq -1 & \text{improper non} - \text{orthochronous}
\end{array}$$

Since the identity is in $L_+^\uparrow$, this component is the only one that forms a group. Parity and time reversal interchange these components, for example

$$\begin{array}{rcccc}
P & : & L_+^\uparrow & \leftrightarrow & L_-^\uparrow \\
T & : & L_+^\uparrow & \leftrightarrow & L_-^\downarrow \\
PT & : & L_+^\uparrow & \leftrightarrow & L_+^\downarrow
\end{array} \tag{3.15}$$

Each component is itself connected. Any Lorentz transformation can be decomposed as a product of a rotation, a boost, and possibly T and P.

It will often be important to consider the complexified Lorentz group, which is still defined by $\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$, but now with Λ complex. We still have $\det\Lambda = \pm 1$, but it is not true that $|\Lambda^o{}_o| \geq 1$. Hence within the *complexified* Lorentz group, $L_+^\uparrow$ and $L_+^\downarrow$ are connected to one another.

3.3 Finite-dimensional Representations of the Lorentz Algebra

The representations of the Lorentz group that are relevant to physics may be *single-valued, i.e. tensorial* or *double valued, i.e. spinorial*, just as was the case for the representations of its rotation subgroup. In fact, more generally, we shall be interested in single and double valued representations of the full Poincaré group, which are defined by

$$R(\Lambda_1, a_1) R(\Lambda_2, a_2) = \pm R(\Lambda_1 \Lambda_2, a_1 + \Lambda_1 a_2) \tag{3.16}$$

The representations of the Lorentz group then correspond to having $a_1 = a_2 = 0$.

To study the representations of $O(1,3)$, it is convenient to notice that the *complexified* algebra factorizes, by taking linear combinations of rotations J_i and boosts K_i with complex coefficients. We define

$$J_i \equiv \frac{1}{2}\epsilon_{ijk}M_{jk} \qquad\qquad K_i \equiv M_{oi} \tag{3.17}$$

then the structure relations are

$$\begin{aligned} [J_i, J_j] &= +i\ \epsilon_{ijk}J_k \\ [J_i, K_j] &= +i\ \epsilon_{ijk}K_k \\ [K_i, K_j] &= -i\ \epsilon_{ijk}J_k \end{aligned} \tag{3.18}$$

Clearly, one should introduce

$$N_i^{\pm} \equiv \frac{1}{2}(J_i \pm i\ K_i) \tag{3.19}$$

so that the commutation relations decouple,

$$[N_i^{\pm}, N_j^{\pm}] = i\ \epsilon_{ijk}N_k^{\pm} \qquad\qquad [N_i^{+}, N_j^{-}] = 0 \tag{3.20}$$

Thus, each $\{N_i^{+}\}$ generates an $SU(2)$ algebra, considered here with complex coefficients.

The finite dimensional representations of $SU(2)$ are labelled by a half integer $j \geq 0$, and will be denoted by (j). It will be useful to have the tensor product formula,

$$(j) \otimes (j') = \bigoplus_{l=|j-j'|}^{j+j'} (l) \tag{3.21}$$

where the sum runs over integer spacings and the dimension is $\dim(j) = 2j+1$. A familiar example is from the addition of angular momentum representations. For example the addition of spin 1/2 to an orbital angular momentum l produces total momentum states with $j = l + 1/2$ and $j = l - 1/2$.

The representation theory of the complexified Lorentz algebra is obtained as the product of the representations of both $SU(2)$ factors, each of which may be described by the eigenvalue of the corresponding Casimir operator,

$$\begin{array}{lll} N_i^{+}N_i^{+} & j_+(j_+ + 1) & \\ N_i^{-}N_i^{-} & j_-(j_- + 1) & j_\pm = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots \end{array} \tag{3.22}$$

Thus, the finite-dimensional irreducible representations of $O(1,3)$ are labelled by (j_+, j_-). Since $J_3 = N_3^{+} + N_3^{-}$, the highest weight vector of the corresponding rotation group is just the sum $j_+ + j_-$, which should be thought of as the *spin* of the representation. Complex conjugation and parity have the same action on these representations,

$$\begin{aligned} (j_+, j_-)^* &= (j_-, j_+) \\ P\ (j_+, j_-)^* &= (j_-, j_+) \end{aligned} \tag{3.23}$$

and thus only the representations (j,j) or the reducible representations $(j_+, j_-) \oplus (j_-, j_+)$ can be real – the others are necessarily complex.

Some fundamental examples

1. $(0,0)$ with spin zero is the *scalar*; Its parity may be even orodd (scalar-pseudoscalar)

2. $(\frac{1}{2}, 0)$ is the spin $\frac{1}{2}$ *left-handed Weyl 2-component spinor* (and $(0, \frac{1}{2})$) is the *right-handed Weyl 2-component spinor*). Both are complex, and transform into one another under complex and parity conjugation.

3. $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ is the *Dirac spinor* when the left and right spinors are *independent*; it is the *Majorana spinor* when the left and right spinors are complex conjugates of one another, so that the Majorana spinor is *a real spinor*.

4. Tensor products of the Weyl spinors yield higher (j_+, j_-) representations.

5. $(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ with spin 1 and 4 components is a 4-*vector*, such as the E&M gauge potential and the momentum P_μ.

6. $(1,0) \oplus (0,1)$ with spin 1, is a real representation under which rank 2 antisymmetric tensors transform. Examples are the E& M field strength tensor $F_{\mu\nu} = -F_{\nu\mu}$, as well as $M_{\mu\nu}$ themselves.

7. $(1,0)$ (resp. $(0,1)$) with spin 1 and 3 complex components is the so-called self-dual (resp. anti self-dual) antisymmetric rank 2-tensor. To construct it in terms of tensors, one defines the *dual* by

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \qquad \epsilon^{0123} = -\epsilon_{0123} = 1 \tag{3.24}$$

Since $\tilde{\tilde{F}}_{\mu\nu} = -F_{\mu\nu}$, the eigenvalues of $\tilde{}$ are $\pm i$, and the eigenvectors are

$$F^\pm_{\mu\nu} \equiv \frac{1}{2}(F_{\mu\nu} \pm i\tilde{F}_{\mu\nu}) \qquad\qquad \tilde{F}^\pm_{\mu\nu} = \mp i F^\pm_{\mu\nu} \tag{3.25}$$

The self-dual F^+ corresponds to the representation $(1,0)$ (and the anti self-dual F^- to $(0,1)$. Notice that the presence of i in the definition of the dual, and therefore of

$F^{\pm}$ forces these representations to be complex. In particular N_i^+ and N_i^- correspond to the self-dual and anti self-dual part of $M_{\mu\nu}$. The precise relations are,

$$M_{0j}^{\pm} = \mp \frac{i}{2} \epsilon_{jkl} M_{kl}^{\pm} = \mp i N_j^{\pm} \tag{3.26}$$

8. $(1,1)$ of spin 2 corresponds to the symmetric traceless rank 2 tensor, i.e. the graviton.

9. Generalizing ordinary orbital angular momentum to include time yields another representation of the Lorentz algebra on functions of the coordinates x^μ,

$$L_{\mu\nu} \equiv i(x_\mu \partial_\nu - x_\nu \partial_\mu) \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \equiv \left(\frac{\partial}{\partial t} \, , \, \vec{\nabla} \right) \tag{3.27}$$

   A further generalization is to representations that include spin;

$$M_{\mu\nu} \equiv L_{\mu\nu} + S_{\mu\nu} \tag{3.28}$$

   where the spin representation $S_{\mu\nu}$ satisfies the Lorentz structure relations by itself, while $S_{\mu\nu}$ commutes with P_μ and $L_{\mu\nu}$. Supplementing this algebra with $P_\mu = i\partial_\mu$ yields a representation of the full Poincaré algebra.

3.4 Unitary Representations of the Poincaré Algebra

Next, the unitary representations of the Poincaré algebra are reviewed and constructed. Recall the structure relations of the algebra,

$$\begin{aligned}
[P_\mu, P_\nu] \quad &= 0 \\
[M_{\mu\nu}, P_\rho] \quad &= +i\eta_{\nu\rho} P_\mu - i\eta_{\mu\rho} P_\nu \\
[M_{\mu\nu}, M_{\rho\sigma}] &= +i\eta_{\nu\rho} M_{\mu\sigma} - i\eta_{\mu\rho} M_{\nu\sigma} - i\eta_{\nu\sigma} M_{\mu\rho} + i\eta_{\mu\sigma} M_{\nu\rho}
\end{aligned} \tag{3.29}$$

In any unitary representation, the generators M and P will have to be realized as self-adjoint operators. Constructing the representations of the Poincaré algebra is a bit more tricky than those of the Lorentz algebra, because the Lorentz algebra is semi-simple, but the Poincaré algebra has the *Abelian invariant subalgebra of translations*, and is therefore not semi-simple.

What are its representations? The Casimirs of the Lorentz algebra $N_i N_i$ and $N_i^+ N_i^+$ do not commutue with P_μ any longer. However, $P_\mu P^\mu$ is a relativistic invariant, and hence a Casimir of Poincaré. If we can find another 4-vector that commute with all P's, its square will also be a Casimir. The *Pauli-Lubanski* vector is defined by

$$W^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \tag{3.30}$$

It commutes with momentum, $[W^\mu, P^\nu] = 0$ and its square $W^\mu W_\mu$ commutes with the entire Poincaré algebra and is therefore a (quartic) Casimir operator for the Poincaré algebra. Now remember that the most general expression for $M_{\mu\nu}$ was

$$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu} \tag{3.31}$$

so that the angular momentum part cancels out and we are left with

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu S_{\rho\sigma} \tag{3.32}$$

The Casimirs of the Poincaré group are thus $P_\mu P^\mu$ and $W_\mu W^\mu$, and the states will be labelled by P_μ and W^3. W_μ generates an $SU(2)$ type algebra for fixed P_μ:

$$[W^\mu, W^\nu] = i\ \epsilon^{\mu\nu\alpha\beta} P_\alpha W_\beta \tag{3.33}$$

which is the so-called *little group*. The basic idea is to fix P^μ and then to obtain the representations of the little group.

• *One-particle state representations*

We apply the method above to the construction of the (unitary) *one-particle state representations*, which are the most basic building blocks of the Hilbert space of a QFT. We choose to diagonalize the components of the momentum operator P_μ since they mutualy commute and commute with the mass operator $m^2 = P^\mu P_\mu$. We denote the eigenstates by $|p, \imath\rangle$, where p_μ denotes the eigenvalues of the operator P_μ and the label i specifies whatever remaining quantum numbers,

$$P_\mu |p, i\rangle = p_\mu |p, i\rangle \tag{3.34}$$

One defines a *one particle state* as a representation in which the range of the label i is *discrete*. If one superimposes two one particle states, the index would include relative momenta, and this index would be continuous. At fixed M^2, the range of i is finite, both in QFT and in string theory.

To obtain the action of the Lorentz group, we proceed as follows. First, we list the transformations of the Poincaré generators under Poincaré transformations,

$$\begin{aligned} \mathcal{U}(\Lambda, a) M_{\mu\nu} \mathcal{U}(\Lambda, a)^{-1} &= (\Lambda^{-1})_\mu{}^\alpha (\Lambda^{-1})_\nu{}^\beta (M_{\alpha\beta} - a_\alpha P_\beta + a_\beta P_\alpha) \\ \mathcal{U}(\Lambda, a) P_\mu \mathcal{U}(\Lambda, a)^{-1} &= (\Lambda^{-1})_\mu{}^\alpha P_\alpha \end{aligned} \tag{3.35}$$

Applying these relations to one particle states, one establishes that the state $\mathcal{U}(\Lambda, 0)|p, i\rangle$ indeed has momentum Λp, as expected, $P^\mu \mathcal{U}(\Lambda, 0)|p, i\rangle = \Lambda^\mu{}_\nu p^\nu \mathcal{U}(\Lambda, 0)|p, i\rangle$, and therefore may be decomposed onto such states,

$$\mathcal{U}(\Lambda, 0)|p, i\rangle = \sum_j C_{i,j}(\Lambda, p)|\Lambda p, j\rangle \tag{3.36}$$

To find this action and understand it, we specify a reference momentum k^μ, for given M^2, so that p^μ is a Lorentz transform of k^μ,

$$p^\mu = L^\mu{}_\rho(p)k^\rho \qquad\qquad k^2 = m^2 \tag{3.37}$$

Thus, the states $|p, i\rangle$ may then be reconstructed as follows,

$$|p, i\rangle = \mathcal{U}(L(p), 0)|k, i\rangle \tag{3.38}$$

Once this construction has been effected, the action of general Lorentz transformations may be obtained as follows,

$$\begin{aligned} \mathcal{U}(\Lambda, 0)|p, i\rangle &= \mathcal{U}(\Lambda, 0)\mathcal{U}(L(p), 0)|k, i\rangle \\ &= \mathcal{U}(L(\Lambda p), 0)\mathcal{U}(W, 0)|k, i\rangle \end{aligned} \tag{3.39}$$

Defining now the composite transformation

$$\mathcal{U}(W, 0) = \mathcal{U}(L(\Lambda p), 0)^{-1}\mathcal{U}(\lambda, 0)\mathcal{U}(L(p), 0) \tag{3.40}$$

which represents the composite Lorentz transformation W,

$$\begin{aligned} W &= L(\lambda(p))^{-1}\Lambda L(p) \\ Wk &= L(\Lambda p)^{-1}\Lambda L(p)k = L(\Lambda p)^{-1}\Lambda p = k \end{aligned} \tag{3.41}$$

Therefore, W leaves the reference momentum k^μ invariant. The group of all such transformations is the little group. Thus, given the reference momentum k, the unitary representation of the one particle state is classified by the representation of the little group.

1. $P^\mu P_\mu = m^2 > 0$: These are the *massive particle states*; $k^\mu = (m, 0, 0, 0)$.
   It follows from the explicit form of W^μ that $W^\mu W_\mu = -m^2\vec{S}^2 = -m^2 s(s+1)$, where s is the *spin* of the representation of the little group, and it takes the usual values
   $$s = 0, \frac{1}{2}, 1, \dots \tag{3.42}$$

2. $P_\mu P^\mu = 0$: These are the *massless particle states*; $k^\mu = (\kappa, \kappa, 0, 0)$.
   It follows from the explicit form of W^μ that $W_\mu W^\mu = 0$. Since we also have $P_\mu W^\mu = 0$, it follows that $\vec{W} \cdot \vec{P} = \pm|\vec{W}||\vec{P}|$ and hence the vectors $\vec{W}$ and $\vec{P}$ must be colinear, $\vec{W} = \lambda\vec{P}$ and thus
   $$W^\mu = \lambda P^\mu \tag{3.43}$$
   Again, using the explicit form of W^μ, we have $W^0 = P_i S_i = \lambda P^0$ and we see that λ is the projection of spin onto momentum, i.e. *helicity* which takes values,
   $$\lambda = 0, \pm\frac{1}{2}, \pm 1, \dots \tag{3.44}$$
   Hence any massless particles of $s \neq 0$ has 2 degrees of freedom, photon ± 1, neutrino $\pm\frac{1}{2}$, graviton ± 2.

3. $P^\mu P_\mu < 0$: These are the *tachyonic particle states*; $k^\mu = (0, m, 0, 0)$.
   These representations are unitary, and are acceptable as free particles. As soon as interactions are turned on, however, they cause problems with micro-causality and it is generally believed that tachyons are unacceptable in physical theories.

Note that all these unitary representations are infinite dimensional (except for the trivial representation). Then there are of course also all the non-unitary representations, which we do not consider here. A standard reference is E.P. Wigner, "Unitary Representations of the Inhomogeneous Lorentz Group," Ann. Math. **40** (1939) 149.

3.5 The basic principles of quantum field theory

A summary is presented of *the fundamental principles of relativistic quantum field theory*. By analogy with quantum mechanics, these principles do not refer to any specific dynamics (except for the fact that the dynamics is invariant under Poincaré symmetry). Thus, the principles generalize those of quantum mechanics to include the principles of relativity.

1. States of QFT are vectors in a Hilbert space $\mathcal{H}$.

2. States transform under *unitary representations of the Poincaré group*, which will be denoted by $\mathcal{U}(\Lambda, a)$,

$$|\text{state}\rangle \;\rightarrow\; |\text{state}'\rangle = \mathcal{U}(\Lambda, a)|\text{state}\rangle \tag{3.45}$$

3. There exists a unique ground state (*the vacuum*), usually denoted by $|0\rangle$, which is the singlet representation of the Poincaré group,

$$\mathcal{U}(\Lambda, a)|0\rangle = |0\rangle \tag{3.46}$$

4. The fundamental observables in QFT are *local quantum fields*, which are space-time dependent self-adjoint operators on $\mathcal{H}$. They are generically denoted by $\phi_j(x)$; specific notations for the fields most relevant for physics will be introduced later.[§]

5. Throughout, the number n of independent fields $\phi_j(x)$, $j = 1, \cdots, n$ will be assumed to be FINITE. Generalizations with infinite numbers of fields can be constructed (for example string theory), but only at the cost of severe complications, which usually go outside the framework of standard QFT.

[§]Note that the operators or observables are thus time-dependent, and the states will be time-independent. This formulation is usually referred to as the *Heisenberg formulation of quantum mechanics*, as opposed to the Schrödinger formulation in which observables are time-independent but states are time-dependent.

6. Poincaré transformations on the fields are realized unitarily in $\mathcal{H}$, as follows,

$$\phi_j'(x) \equiv \mathcal{U}(\Lambda, a)\phi_j(x)\mathcal{U}(\Lambda, a)^\dagger = \sum_{k=1}^{n} S_{jk}(\Lambda^{-1})\phi_k(\Lambda x + a) \tag{3.47}$$

Since the number of fields is assumed finite, $S(\Lambda^{-1})$ must be a finite-dimensional representation of the Lorentz group. Except for the trivial one, such representations are always non-unitary. An important special case of this relation is when $\Lambda = 1$, which yields the behavior of observables under translations (this includes under time-translation, namely dynamics),

$$\phi_j'(x) \equiv e^{-ia^\mu P_\mu}\phi_j(x)e^{ia^\mu P_\mu} = \phi_j(x + a) \tag{3.48}$$

since $\mathcal{U}(I, a) = \exp\{-ia^\mu P_\mu\}$. The transformation law when $\Lambda \neq I$ will be examined in greater detail later.

7. *Microscopic Causality or local commutativity* states that any two observables ϕ_J and ϕ_k that have no mutual causal contact must be simultaneously observable and must therefore commute with one another,[¶]

$$[\phi_j(x^\mu), \phi_k(y^\mu)] = 0 \qquad \text{if} \qquad (x - y)^2 < 0 \tag{3.49}$$

Classically, as no relativistic signal can connect these two observables, it should better be possible to assign independent initial data. Quantum mechanically, it should be possible to measure them simultaneously to arbitrary precision and this independently at x^μ and y^μ.

8. There is a relation between states in $\mathcal{H}$ and fields. From the fields in QFT, it is possible to build up the Hilbert space. The simplest states (besides the vacuum) in Hilbert space are the *1-particle states*, denoted by $|p, i\rangle$. For every 1-particle species i, there must be a corresponding field $\phi_i(x)$ that produces this species from the vacuum state,

$$\langle 0|\phi_i(x)|p, i\rangle \neq 0 \tag{3.50}$$

9. Symmetries other than the Poincaré group will be realized by unitary transformations on Hilbert space which induce finite-dimensional transformations on the fields,

$$\phi_j'(x) \equiv \mathcal{U}(g)\phi_j(x)\mathcal{U}(g)^\dagger = \sum_k S(g^{-1})_j{}^k\phi_k(gx) \tag{3.51}$$

When g has no action on x, namely $gx = x$, the transformations generate *internal symmetries*; examples are flavor, color, isospin etc. On the other hand, if $gx \not{x}$, the

[¶]Fermionic fields will anti-commute.

transformations generate *space-time symmetries*; examples are Poincaré invariance, scale and conformal symmetries, parity, supersymmetry. Time-reversal is realized by an anti-unitary transformation.

3.6 Transformation of Local Fields under the Lorentz Group

Fields may be distinguished by the different finite-dimensional representations of the Lorentz group $S(\Lambda^{-1})$ under which they transform. It is convenient to decompose these representations into a direct sum of irreducible representations, whose study we now initiate. We apply the results of S3.3 to the general transformation rule of (3.47), specialized to Lorentz transformations. It is convenient to let $\Lambda \to \Lambda^{-1}$ and $x \to \Lambda x$, so that

$$\phi'_j(x') = \sum_{k=1}^{n} S_j{}^k(\Lambda)\phi_k(x) \qquad\qquad x' = \Lambda x \tag{3.52}$$

When the transformation is continuous (we shall treat Parity and Time-reversal separately) we may describe the transformation infinitesimally,

$$\begin{aligned} \Lambda^\mu{}_\nu &= \delta^\mu{}_\nu + \omega^\mu{}_\nu + \mathcal{O}(\omega^2) \\ \phi'_j(x) &= \phi_j(x) + \delta\phi_j(x) + \mathcal{O}(\omega^2) \\ S(I+\omega) &= I - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu} + \mathcal{O}(\omega^2) \end{aligned} \tag{3.53}$$

Combining these ingredients to compute $\delta\phi_j$, we find

$$\phi(x) + \delta x^\mu \partial_\mu \phi(x) + \delta\phi(x) = \phi(x) - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}\phi(x) + \mathcal{O}(\omega^2) \tag{3.54}$$

whence the transformation law naturally involves the total Lorentz generator $L+S$,

$$\delta\phi(x) = -\frac{i}{2}\omega^{\mu\nu}(L_{\mu\nu} + S_{\mu\nu})\phi(x) \tag{3.55}$$

1. The scalar field

The scalar has spin 0, corresponding to the representation $(0,0)$ in the notation of S3.3. Thus, $\phi'(\Lambda x) = \phi(x)$ for all Lorentz transformations in $L_+^\uparrow$. Infinitesimally, we have

$$\delta\phi(x) = -\frac{i}{2}\omega^{\mu\nu}L_{\mu\nu}\phi(x) \tag{3.56}$$

All irreducible representations are 1-dimensional; a single scalar field is denoted by ϕ.

2. The vector field

Vector fields are generally denoted by A_μ, such as for example the electro-magnetic potential. The transformation law of a vector field is

$$A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x) \tag{3.57}$$

which corresponds to the following infinitesimal transformation law,

$$\delta A_\mu(x) = A'_\mu(x) - A_\mu(x) = -\frac{i}{2}\omega^{\rho\sigma}L_{\rho\sigma}A_\mu(x) + \omega_\mu{}^\nu A_\nu(x) \tag{3.58}$$

(Using the transformation law for a scalar ϕ, it is easily checked that the derivative of a scalar $\partial_\mu\phi$ transforms as a vector with the above transformation law.) The explicit form of the vector representation matrices may be deduced from this formula,

$$(S_{\mu\nu})_\alpha{}^\beta = i\ \eta_{\mu\alpha}\delta_\nu{}^\beta - i\ \eta_{\nu\alpha}\delta_\mu{}^\beta \tag{3.59}$$

which is characteristic of the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation of the Lorentz group.

3. General tensor representations

These representations may be built up by taking the tensor product of vector representations using the tensor product formula,

$$(j_+, j_-) \otimes (j'_+, j'_-) = \sum_{s_\pm = |j_\pm - j'_\pm|}^{j_\pm + j'_\pm} (s_+, s_-) \tag{3.60}$$

The spins $s = s_+ + s_-$ are all integer. A rank n tensor $B_{\mu_1 \cdots \mu_n}$ transforms as

$$B'_{\mu_1\cdots\mu_n}(\Lambda x) = \Lambda_{\mu_1}{}^{\nu_1} \cdots \Lambda_{\mu_n}{}^{\nu_s} B_{\nu_1\cdots\nu_n}(x) \tag{3.61}$$

It is consistent with Lorentz transformations to symmetrize, anti-symmetrize or take the trace of tensor fields. For totally symmetric tensor, the spin and the rank coincide, $n = s$, but upon anti-symmetrization, $n > s$ in general. The representation $(1, 1)$ corresponds to the graviton. Fields of spin higher than 2 will never be needed in QFT, and in fact we shall always specialize to fields of spin ≤ 1.

4. The Dirac spinor field

The Dirac field, usually denoted $\psi(x)$, is a 4-component complex column matrix, whose transformation law is defined to be

$$\mathcal{U}(\Lambda, a)\psi(x)\mathcal{U}(\Lambda, a)^\dagger = S(\Lambda^{-1})\psi(\Lambda x + a) \tag{3.62}$$

where $S(\Lambda) = R(\Lambda, 0)$ is the spinor representation characterized by $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ in the classification of Lorentz representations. To obtain an explicit construction of $S(\Lambda)$, we proceed as follows.

First, recall that the 2-dimensional spinor representation of the rotation group is realized in terms of the 2×2 Pauli matrices, σ^i, $i = 1, 2, 3$, defined to satisfy the relation[‖] $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$. It is conventional to take the following basis,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.63}$$

The representation generators are $S^i = \sigma^i/2$, and satisfy the standard rotation algebra relations $[S^i, S^j] = i\epsilon^{ijk}S^k$.

The spinor representations of the Lorentz algebra are realized analogously in terms of the 4×4 Dirac matrices γ^μ, defined to satisfy the *Clifford-Dirac algebra*,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{3.64}$$

The 4-dimensional Dirac spinor representation is defined by the following generators,

$$S_{\mu\nu} \equiv \frac{i}{4}[\gamma_\mu, \gamma_\nu] \tag{3.65}$$

which satisfy the Lorentz algebra

$$[S_{\mu\nu}, S_{\rho\sigma}] = +i\ \eta_{\nu\rho}S_{\mu\sigma} - i\ \eta_{\mu\rho}S_{\nu\sigma} - i\ \eta_{\nu\sigma}S_{\mu\rho} + i\ \eta_{\mu\sigma}S_{\nu\rho} \tag{3.66}$$

This may be verified by making use of the Clifford-Dirac algebra relations only.

As for any representation of the Lorentz group, (3.35) holds, which in this case may be expressed as

$$S(\Lambda)S_{\mu\nu}S(\Lambda)^{-1} = (\Lambda^{-1})_\mu{}^\alpha(\Lambda^{-1})_\nu{}^\beta S_{\alpha\beta} \tag{3.67}$$

Furthermore, from the relation

$$[S_{\mu\nu}, \gamma_\rho] = i\ \eta_{\nu\rho}\gamma_\mu - i\ \eta_{\mu\rho}\gamma_\nu \tag{3.68}$$

it follows that $S(\Lambda)\gamma_\mu S(\Lambda)^{-1} = (\Lambda^{-1})_\mu{}^\nu\gamma_\nu$ or simply that the γ_μ matrices are *invariant*, provided both its "vector" and "spinor" indices are transformed,

$$\Lambda_\mu{}^\nu S(\Lambda)\gamma_\nu S(\Lambda)^{-1} = \gamma_\mu \tag{3.69}$$

The γ-matrices are the Clebsch-Gordon coefficients for the tensor product of two Dirac representations onto the vector representation.

[‖] $\{A, B\} \equiv AB + BA$ is defined to be the *anti-commutator* of A and B.

The Dirac spinor representation is *reducible*, as may be seen from the existence of the chirality matrix $\gamma^5 = \gamma_5$, which is defined by

$$\gamma^5 = \gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.70}$$

Clearly, we have $(\gamma^5)^2 = I$ and $\mathrm{tr}(\gamma^5) = 0$, so that γ^5 is not proportional to the identity matrix. The matrix γ^5 *anti-commutes with all* γ_μ and therefore γ^5 *commutes with* $S_{\mu\nu}$,

$$\{\gamma^5, \gamma_\mu\} = [\gamma^5, S_{\mu\nu}] = 0 \tag{3.71}$$

Thus, by Shur's lemma, the representation S is reducible, since its generators commute with a matrix γ^5 that is not proportional to the identity matrix. (Notice that the γ^μ *matrices do form an irreducible representation of the Clifford algebra.*)

A basis for all 4×4 matrices is given by the following set

$$I,\ \gamma^\mu,\ \gamma^{\mu\nu},\ \gamma^{\mu\nu\rho},\ \gamma^{\mu\nu\rho\sigma} \tag{3.72}$$

where $\gamma^{\mu_1\cdots\mu_n}$ is the product $\gamma^{\mu_1}\cdots\gamma^{\mu_n}$, completely antisymmetrized in its n indices. Some of these combinations may be reexpressed in terms of already familiar quantities,

$$\begin{aligned}
\gamma^{\mu\nu} &= -2i\ S^{\mu\nu} \\
\gamma^{\mu\nu\rho} &= -i\ \epsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\sigma \\
\gamma^{\mu\nu\rho\sigma} &= -i\ \epsilon^{\mu\nu\rho\sigma}\gamma^5
\end{aligned} \tag{3.73}$$

5. The Weyl spinor fields

The irreducible components of the Dirac representation are a - *chirality or left Weyl spinor* $(\frac{1}{2}, 0)$ and an *independent + chirality or right Weyl spinor* $(0, \frac{1}{2})$, corresponding to the -1 and $+1$ eigenvalues of the chirality matrix γ^5. In a *chiral basis* where γ^5 is diagonal, we have the following convenient representation of the γ-matrices,

$$\gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & +I_2 \end{pmatrix} \qquad\qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \tag{3.74}$$

where I_2 is the identity matrix in 2 dimensions, and

$$\sigma^\mu = (I_2, \sigma^i) \qquad\qquad \bar\sigma^\mu = (I_2, -\sigma^i) \tag{3.75}$$

The Dirac representation matrices $S(\Lambda)$ as well as the Dirac field $\psi(x)$ decompose into the Weyl spinor representation matrices $S_L(\Lambda)$ and $S_R(\Lambda)$ and the Weyl spinor fields $\psi_L(x)$

and $\psi_R(x)$ as follows,

$$S(\Lambda) = \begin{pmatrix} S_L(\Lambda) & 0 \\ 0 & S_R(\Lambda) \end{pmatrix} \qquad\qquad \psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \tag{3.76}$$

where the Weyl representation matrices are given by

$$\begin{aligned} S_L(\Lambda) &= \exp\Big\{ -\frac{i}{2}(\omega^+)^{\mu\nu} M^+_{\mu\nu} \Big\} = \exp\Big\{ \frac{i}{2}\vec\sigma \cdot (\vec\omega + i\vec\nu) \Big\} \\ S_R(\Lambda) &= \exp\Big\{ -\frac{i}{2}(\omega^-)^{\mu\nu} M^-_{\mu\nu} \Big\} = \exp\Big\{ \frac{i}{2}\vec\sigma \cdot (\vec\omega - i\vec\nu) \Big\} \end{aligned} \tag{3.77}$$

Here, $\omega^\pm$ and $M^\pm$ are the self-dual and anti self-dual parts of ω and M respectively, and $\vec\omega$ and $\vec\nu$ are real 3-vectors representing rotations and boosts respectively.

6. Equivalent representations

If $S_{\mu\nu}$ in the Dirac representation satisfies the structure relations of the Lorentz algebra, then transposition and complex conjugation automatically also produce representations of the same dimensions, with matrices $-S^t_{\mu\nu}$ and $-S^*_{\mu\nu}$. These representations must be equivalent to the Dirac representation and hence related by conjugation. These conjugations reflect the discrete symmetries $\mathcal{C}$ and $\mathcal{P}$ (to be discussed shortly),

$$\begin{aligned} &\mathcal{C} \qquad\qquad & -S^t_{\mu\nu} &= C^{-1} S_{\mu\nu} C \\ &\mathcal{P} \qquad\qquad & -S^*_{\mu\nu} &= B^{-1} S_{\mu\nu} B \end{aligned} \tag{3.78}$$

7. The Majorana spinor field

A *Majorana spinor* corresponds to a real representation $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ where the two Weyl spinors are complex conjugates of one another. Complex conjugation by itself depends upon the basis chosen, but *charge conjugation*, to be defined in the subsequent subsection, is a Lorentz invariant operation. Thus, the proper requirement for a spinor to be "real" is provided by the Majorana spinor field condition,

$$\psi^c(x) = \psi(x) \qquad\qquad (\psi^c(x))^c = \psi(x) \tag{3.79}$$

A Majorana spinor is equivalent to its left field component (or its right field component).

8. The spin 3/2 field

Spin $\frac{3}{2}$ can be either $\left(\frac{3}{2},0\right)$ or $\left(\frac{1}{2},1\right)$ and their charge conjugates. Only the latter is physically significant. It can be gotten from

$$\left(\frac{1}{2},\frac{1}{2}\right) \otimes \left(\frac{1}{2},0\right) = \left(1,\frac{1}{2}\right) \oplus \left(0,\frac{1}{2}\right) \tag{3.80}$$

The Rarita-Schwinger field ψ_μ precisely corresponds to the above representation plus its complex conjugate. To project out the purely spin $(1,1/2) \oplus (1/2,1)$ part, the γ-trace must be removed which may be achieved by suplementary condition $\gamma^\mu \psi_\mu = 0$.

3.7 Discrete Symmetries

We group together here the transformations under the discrete symmetries $\mathcal{P}$, $\mathcal{T}$ and charge conjugation $\mathcal{C}$ of the basic fields.

3.7.1 Parity

One distinguishes two types of *scalars* under parity

$$\begin{aligned} \mathcal{P}\phi(x)\mathcal{P}^\dagger &= +\phi(Px) && \text{scalar} \\ \mathcal{P}\phi(x)\mathcal{P}^\dagger &= -\phi(Px) && \text{pseudo}-\text{scalar} \end{aligned} \tag{3.81}$$

This distinction is important as the particles $\pi^\pm$ and π^0 are for example pseudo-scalars, while the He_4 nucleus is a scalar. One also distinguished two types of *vectors* under parity,

$$\begin{aligned} \mathcal{P}A_\mu(x)\mathcal{P}^\dagger &= +A^\mu(Px) && \text{vector} \\ \mathcal{P}A_\mu(x)\mathcal{P}^\dagger &= -A^\mu(Px) && \text{axial vector} \end{aligned} \tag{3.82}$$

Again, this distinction is very important; the weak interactions are mediated by a superposition of a vector and an axial vector. Finally, under parity, the Dirac spinor transforms as follows,

$$\mathcal{P}\psi(x)\mathcal{P}^\dagger = e^{i\theta_P}\gamma^0\psi(x) \tag{3.83}$$

which amounts to an interchange of left and right spinor components, and θ_P is an arbitrary angle. (Note that a theory invariant under parity will have to contain both left and right spinors of each species.)

3.7.2 Time reversal

Time reversal symmetry is the only case which cannot be realized by a unitary transformation in Hilbert space; instead it is realized by an anti-unitary transformation $\mathcal{T}$, which has the property of changing the sign of the number i,

$$\mathcal{T}i = -i\mathcal{T} \tag{3.84}$$

Its action on the scalar, vector and Dirac sprinor is given by

$$\begin{aligned} \mathcal{T}\phi(x)\mathcal{T}^\dagger &= \phi(Tx) \\ \mathcal{T}A_\mu(x)\mathcal{T}^\dagger &= -A^\mu(Tx) \\ \mathcal{T}\psi(x)\mathcal{T}^\dagger &= iC\gamma^5\psi(Tx) \end{aligned} \tag{3.85}$$

3.7.3 Charge Conjugation

From the simple relation $(\sigma^i)^* = -\sigma^2 \sigma^i \sigma^2$, it follows that

$$S_L(\Lambda)^* = \sigma^2 S_R(\Lambda) \sigma^2 \tag{3.86}$$

As a result, the *charge conjugates* $\psi^c_{L,R}$ of $\psi_{L,R}$, defined by

$$\begin{aligned} \mathcal{C}\psi_L(x)\mathcal{C}^\dagger = \psi^c_L(x) &\equiv -\sigma^2 \psi^*_L(x) \\ \mathcal{C}\psi_R(x)\mathcal{C}^\dagger = \psi^c_R(x) &\equiv +\sigma^2 \psi^*_R(x) \end{aligned} \tag{3.87}$$

transforms under $S_R(\Lambda)$ and $S_L(\Lambda)$ respectively.

Charge conjugation as defined above, was formulated in a specific basis of σ-matrices, which is why σ^2 played a special role. It is possible to produce a basis independent definition of charge conjugation as follows. For any basis, the *charge conjugation matrix* C and the charge conjugate of a Dirac spinor are defined by

$$\gamma^t_\mu = -C^{-1} \gamma_\mu C \qquad\qquad \psi^c \equiv C \gamma^t_0 \psi^* = C \bar{\psi}^t \tag{3.88}$$

It follows that $[\gamma^5, C] = 0$. In the chiral basis, the transposition equations read

$$\begin{aligned} \gamma_{0,2} &= -C^{-1} \gamma_{0,2} C \\ \gamma_{1,3} &= +C^{-1} \gamma_{1,3} C \end{aligned}$$

which may be solved and we have

$$C = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & +\sigma^2 \end{pmatrix} \tag{3.89}$$

up to a multiplicative factor, which one may choose to be unity. With this factor, the preceding definition of charge conjugation on left and right spinors is recovered.

3.8 Significance of fields in terms of particle contents

spin 0		possible Higgs fields, pions etc.
spin 1	(vector field)	gauge fields, A_μ, the ρ-field
	(anti-symmetric tensor)	not used at present
spin 1/2		all the observed fermions
	Weyl	massless neutrinos
	Dirac	massive fermions $e^-, e^+, \ldots$
	Majorana	*possibly* massive neutrino
spin 3/2		*possibly* the gravitino (supergravity)
spion 2		graviton

3.9 Classical Poincaré invariant field theories

The point of departure for the construction of Poincaré invariant local quantum field theories will almost always be a classical action for a set of classical fields, which will have to be local as well. Maxwell theory is an example of such a classical field theory where the fundamental field is the gauge vector potential. Bosonic fields are ordinary functions of space-time; fermionic fields, however, will have to be represented by *anti-commuting or Grassmann valued functions*. These will be discussed in the next chapter.

Classical bosonic fields $\phi_i(x)$ (not necessarily scalars) are real or complex valued ordinary functions of x. We assume that the number of fields is finite, $i = 1, \cdots, n$. The fundamental actions in Nature appear to involve dependence on time-derivatives (and by Lorentz invariance also space-derivatives) only of first order. Therefore, the most general bosonic action of this type takes the form,

$$S[\phi_i] = \int d^4x \mathcal{L} \qquad\qquad \mathcal{L} = \mathcal{L}\Big(\phi_i(x), \partial_\mu \phi_i(x)\Big) \tag{3.90}$$

The *Euler-Lagrange equations* or *field equations* determine the stationnary field configurations of $S[\phi_i]$, and are given by

$$\frac{\delta S[\phi]}{\delta \phi_i(x)} = \frac{\partial \mathcal{L}}{\partial \phi_i}(x) - \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \Big)(x) = 0 \tag{3.91}$$

The canonical momentum π^μ_j conjugate to ϕ_j, and the Hamiltonian are defined by

$$\pi^\mu_j(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)}(x) \qquad\qquad H \equiv \int d^3x \Big[\sum_j \pi^0_j \partial_0 \phi_j - \mathcal{L} \Big] \tag{3.92}$$

As usual, we have Poisson brackets defined by

$$\{A, B\} = \sum_j \int d^3\vec{x} \left[\frac{\delta A}{\delta \pi^0_j(t,\vec{x})} \frac{\delta B}{\delta \phi_j(t,\vec{x})} - \frac{\delta A}{\delta \phi_j(t,\vec{x})} \frac{\delta B}{\delta \pi^0_j(t,\vec{x})} \right] \tag{3.93}$$

In particular, we have $\{\pi^0_j(t,\vec{x}), \phi_k(t,\vec{x}')\} = \delta_{jk}\delta^3(\vec{x} - \vec{x}')$.

A symmetry of the field equations is a transformation of the fields ϕ_j such that every solution is transformed into a solution of the same field equations. For continuous symmetries, the usual infinitesimal criterion may be applied. The infinitesimal transformation $\delta\phi_j(x) = \phi'_j(x) - \phi_j(x)$ is a symmetry provided

$$\delta \mathcal{L} = \partial_\mu X^\mu \tag{3.94}$$

without the use of the field equations. If so, Noether's theorem may be applied, which implies the existence of a *conserved current*,

$$j^\mu \equiv \sum_j \pi^\mu_j \delta\phi_j - X^\mu \qquad\qquad \partial_\mu j^\mu = 0 \tag{3.95}$$

and thus of a time independent charge

$$Q \equiv \int d^3x \ j^0(t,\vec{x}) \tag{3.96}$$

Furthermore, the transformation is recovered by Poisson bracketing with the charge,

$$\delta\phi_j = \{Q, \phi_j\} \tag{3.97}$$

Poincaré covariance of the field equations and Poincaré invariance of the action requires that the Lagrangian density $\mathcal{L}$ transform as a scalar. As a special case, translation invariance requires that $\mathcal{L}$ have no *explicit* x^μ dependence. Assuming Poincaré invariance of a Lagrangian $\mathcal{L}$, the conserved currents are as follows.

- Translations

$$\begin{aligned}
\delta\phi_j &= a^\mu \partial_\mu \phi_j \qquad \Rightarrow \qquad X^\mu = a^\mu \mathcal{L} \\
j^\mu &= \theta^\mu{}_\nu a^\nu
\end{aligned}$$

  where the conserved *canonical energy-momentum-stress tensor* – or simply the *stress tensor* – is defined by

$$\theta_{\mu\nu} \equiv \sum_j \pi_{j\mu} \partial_\nu \phi_j - \eta_{\mu\nu} \mathcal{L} \tag{3.98}$$

- Lorentz transformations

$$\begin{aligned}
\delta\phi_j &= \omega^{\kappa\lambda} \left(\frac{1}{2} x_\kappa \partial_\lambda \phi_j - \frac{1}{2} x_\lambda \partial_\kappa \phi_j - \frac{i}{2} S_{\kappa\lambda} \phi_j \right) \qquad \Rightarrow \qquad X^\mu = -\omega^{\mu\nu} x_\nu \mathcal{L} \\
j^\mu &= \sum_j \pi_j^\mu \omega^{\kappa\lambda} (x_\kappa \partial_\lambda \phi_j - \frac{i}{2} S_{\kappa\lambda} \phi_j) + \omega^{\mu\kappa} x_\kappa \mathcal{L} \\
&= \omega^{\kappa\lambda} x_\kappa \theta^\mu{}_\lambda - \frac{i}{2} \sum_j \pi_j{}^\mu \omega^{\kappa\lambda} S_{\kappa\lambda} \phi_j \\
&= \omega^{\kappa\lambda} x_\kappa \mathcal{T}^\mu{}_\lambda
\end{aligned} \tag{3.99}$$

  The conserved and symmetric *improved stress tensor* is given by

$$\begin{aligned}
T^{\mu\nu} &= \theta^{\mu\nu} + \partial_\rho B^{\mu\nu\rho} \\
B^{\mu\nu\rho} &= -B^{\nu\mu\rho} = -B^{\mu\rho\nu} = \frac{i}{2} \sum_j \pi_j^\mu S^{\nu\rho} \phi_j
\end{aligned} \tag{3.100}$$

A more universal interpretation of the stress tensor is that it is the quantity in a classical (and quantum) field theory that couples to variations in the space-time metric and thus to gravity (cfr Einstein's equations of gravity).

4 Free Field Theory

Free field theory corresponds to linear equations of motion, possibly with an inhomogeneous driving source, and thus to an action which is quadratic in the fields. In itself, free field theory is not so interesting, because no interactions take place at all. Free field theory offers, however, an excellent testing ground on which to realize the basic principles of QFT, and many of the methods needed to deal with interacting field theory may be reduced to problems in a theory that is effectively free. We shall deal successively with scalar, spin 1 and spin 1/2 free field theories. As stated from the outset, we restrict to Lagrangians with at most first time derivatives on the fields, and with Poincaré invariance.

4.1 The Scalar Field

The most general classical Poincaré invariant quadratic Lagrangian density and action of a single real scalar field $\phi(x)$ is given by

$$\mathcal{L}_0(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \qquad\qquad S_0[\phi] = \int d^4x \mathcal{L}_0(\phi, \partial_\mu \phi) \tag{4.1}$$

The Euler Lagrange equations are

$$\Box \phi + m^2 \phi = 0 \qquad\qquad \Box \equiv \partial_\mu \partial^\mu \tag{4.2}$$

The conjugate momentum $\pi(x)$ is the time component of the field $\pi^\mu(x)$,

$$\pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \qquad\qquad \pi = \pi^0 = \partial^0 \phi \tag{4.3}$$

The Hamiltonian is

$$H_0 \equiv \int d^3x [\pi \partial_0 \phi - \mathcal{L}_0] = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2\right] \tag{4.4}$$

The classical Poisson brackets are

$$\begin{aligned}
\{\phi(t,\vec x),\ \phi(t,\vec y)\} &= 0 \\
\{\pi(t,\vec x),\ \pi(t,\vec y)\} &= 0 \\
\{\pi(t,\vec x),\ \phi(t,\vec y)\} &= \delta^3(\vec x - \vec y)
\end{aligned} \tag{4.5}$$

It is straightforward to solve the classical field equations by using either Fourrier transformation techniques, or by solving the corresponding differential equations. We shall not do this here and directly pass to the quantization of the system; solving for the Heisenberg field equations instead.

• **Canonical quantization**

Canonical quantization associates with the fields ϕ and π operators that obey the canonical equal time commutation relations,

$$\begin{aligned}
[\phi(t,\vec{x}),\ \phi(t,\vec{y})] &= [\pi(t,\vec{x}),\ \pi(t,\vec{y})] = 0 \\
[\pi(t,\vec{x}),\ \phi(t,\vec{y})] &= -i\delta^{(3)}(\vec{x}-\vec{y})
\end{aligned} \tag{4.6}$$

The Heisenberg field equations for the ϕ-field are obtained as follows

$$\begin{aligned}
i\frac{\partial\phi}{\partial t}(x) = [\phi(x), H] &= \frac{1}{2}\int d^3\vec{y}\left[\phi(t,\vec{x}), \pi^2(t,\vec{y}) + (\vec{\nabla}\phi(t,\vec{y}))^2 + m^2\phi^2(y)\right] \\
&= \frac{1}{2}\int d^3\vec{y}\left[\phi(t,\vec{x}), \pi^2(t,\vec{y})\right] = i\pi(x)
\end{aligned} \tag{4.7}$$

while for the π-field they are derived analogously and found to be

$$i\frac{\partial\pi}{\partial t}(x) = [\pi(x), H] = i\Delta\phi(t,\vec{x}) - im^2\phi(t,\vec{x}) \tag{4.8}$$

Therefore, the Heisenberg field equations for the quantum field $\phi(x)$ are the same as the classical field equations, but now hold for the operator $\phi(x)$,

$$(\Box + m^2)\phi(x) = 0 \tag{4.9}$$

Although canonical quantization is not a manifestly Poincaré covariant procedure, the field equation are nonetheless Poincaré invariant.

The Heisenberg equations are solved by Fourrier transformation in $\vec{x}$ only,

$$\phi(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\ a(t,\vec{k})\ e^{i\vec{k}\cdot\vec{x}} \tag{4.10}$$

where the time-evolution of $a(t,\vec{k})$ is deduced from the field equations and is given by

$$\ddot{a}(t,\vec{k}) + \omega_k^2 a(t,\vec{k}) = 0 \qquad\qquad \omega_k \equiv \sqrt{\vec{k}^2 + m^2} \tag{4.11}$$

which can be solved,

$$a(t,\vec{k}) = \frac{1}{2\omega_k}\left(a(\vec{k})e^{-i\omega_k t} + \tilde{a}(\vec{k})e^{i\omega_k t}\right) \tag{4.12}$$

The reality condition on the classical field translates into the self-adjointness condition on the field $\phi^\dagger = \phi$, which in turn requires that $\tilde{a}(\vec{k}) = a^\dagger(-\vec{k})$. Once this condition has been satified, we may write the complete solution in a manifestly covariant manner,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right] \tag{4.13}$$

where $k^\mu = (k^0, \vec{k}),\ k^0 = \omega_k$, and henceforth, we shall us the notations $k \cdot x = k_\mu x^\mu$ and $k^2 = k_\mu k^\mu$. While this expression may not look Lorentz covariant, it actually is. This may be seen by recasting the measure in terms of a 4-dimensional integral,

$$\frac{d^3k}{(2\pi)^3 2\omega_k} = \frac{d^4k}{(2\pi)^4} 2\pi\ \delta(k^2 - m^2)\theta(k^0) \tag{4.14}$$

and noticing now that it is invariant under orthochronous Lorentz transformations.

Commutation relations on $\phi(x)$ and $\pi(x)$ imply the commutation relations between the a's, which are found to be given by a:

$$\begin{aligned} [a(\vec{k}),\ a(\vec{k}')] &= 0 \\ [a^\dagger(\vec{k}),\ a^\dagger(\vec{k}')] &= 0 \\ [a(\vec{k}),\ a^\dagger(\vec{k}')] &= (2\pi)^3\ 2\omega_k\ \delta^{(3)}(\vec{k} - \vec{k}') \end{aligned} \tag{4.15}$$

Using these results, an important consistency check of the basic principles announced earlier may be carried out. The commutator of two ϕ fields is,

$$[\phi(x),\ \phi(x')] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ik\cdot(x-x')} - e^{ik\cdot(x-x')} \right] \tag{4.16}$$

which is a relativistic invariant, and depends only on $(x-y)^2$. If $(x-y)^2 < 0$, then one may make $x^0 - y^0 = 0$ by a Lorentz transformation. By the equal time commutation relations, this commutator must thus vanish. This checks out because the integral also vanishes, since the integration measure is even under $\vec{k} \to -\vec{k}$, while the integrand is odd. Thus, we have the relativistic invariant equation,

$$[\phi(x),\ \phi(x')] = 0 \quad \text{when} \quad (x-x')^2 < 0 \tag{4.17}$$

which expresses the property of microcausality of the $\phi(x)$ field.

• A Comment on UV convergence

When the classical field equations are solved, the Fourrier coefficients $a(\vec{k})$ are determined by the initial conditions on the field. If the fields ϕ and π at initial time are well-behaved, then the integration over momenta $\vec{k}$ will be convergent.

When the Fourrier coefficients $a(\vec{k})$ are operators however, UV convergence of the $\vec{k}$ integral may be problematic, and indeed will be ! In those cases, the integral should be defined with a UV *regulator*. We shall encounter the need for regulators soon.

• **Fock space construction of the Hilbert space**

The states of the QFT associated with the ϕ field are vectors in a Hilbert space, which we shall now construct. To this end, we view the operator $a(\vec{k})$ as the annihilation operator for a ϕ-particle with momentum $\vec{k}$; similarly, its adjoint $a^\dagger(\vec{k})$ is the creation operator for a ϕ-particle with momentum $\vec{k}$. A basis of states is defined as follows,

- The ground state $|0\rangle$ or *vacuum* $\qquad a(\vec{k})|0\rangle = 0$ for all $\vec{k}$;
- The 1-particle state $|\vec{k}\rangle$ $\qquad |\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle$;
- The n-particle state $|\vec{k}_1, \cdots, \vec{k}_n\rangle$ $\qquad |\vec{k}_1, \cdots, \vec{k}_n\rangle = a^\dagger(\vec{k}_1) \cdots a^\dagger(\vec{k}_n)|0\rangle$.

The ground state is unique and may be normalized to 1, while particle states form a continuous spectrum. Their continuum normalization follows by the use of the canonical commutation relations, and we have for example,

$$\langle 0|0\rangle = 1 \qquad\qquad \langle \vec{k}|\vec{k}'\rangle = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k} - \vec{k}') \tag{4.18}$$

We can now check another basic principle : the one-particle state is obtained from the vacuum by applying the field ϕ,

$$\langle \vec{k}|\phi(x)|0\rangle = e^{ik_\mu x^\mu} \neq 0 \tag{4.19}$$

The *number operator*, N is defined as follows,

$$N \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} \, a^\dagger(\vec{k})a(\vec{k}) \tag{4.20}$$

and obeys the relations

$$\begin{cases} [N, a(\vec{k})] = -a(\vec{k}) \\ [N, a^\dagger(\vec{k})] = +a^\dagger(\vec{k}) \end{cases} \qquad\qquad \begin{cases} N|0\rangle = 0 \\ N|\vec{k}_1, \cdots, \vec{k}_n\rangle = n|\vec{k}_1, \cdots, \vec{k}_n\rangle \end{cases} \tag{4.21}$$

As a result of these commutation relations, N counts the number of particles in state.

• **Energy, Momentum and Angular Momentum**

The Hamiltonian, momentum and angular momentum may all be derived from the stress tensor. The classical stress tensor for a scalar is given by

$$T_c^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \tag{4.22}$$

and hence we have the translation and Lorentz generators

$$P^\mu = \int d^3x T_c^{0\mu} \qquad\qquad M^{\mu\nu} = \int d^3x (x^\mu T_c^{0\nu} - x^\nu T_c^{0\mu}) \tag{4.23}$$

Upon quantization, classical fields are replaced with quantum field operators. Carrying out this substitution for example on the Hamiltonian and momentum,

$$\begin{aligned} H &= \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right] \\ \vec{P} &= \int d^3x \pi \vec{\nabla}\phi \end{aligned} \tag{4.24}$$

Using the mode expansion for ϕ and π,

$$\pi(x) = \int \frac{d^3k}{2(2\pi)^3}\left[-ia(\vec{k})e^{-ik\cdot x} + ia^\dagger(\vec{k})e^{ik\cdot x}\right] \tag{4.25}$$

one finds

$$\begin{aligned} H &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\,\frac{\omega_k}{2}\left[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right] \\ \vec{P} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\,\frac{\vec{k}}{2}\left[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right] \end{aligned} \tag{4.26}$$

Both operators commute with the number operator. Therefore, in free field theory the number of particles in any state is conserved, as it was in non-relativistic quantum mechanics. In a fully interacting theory, particle number will of course no longer be conserved.

• The energy and momentum of the vacuum

To evaluate $H|0\rangle$ and $\vec{P}|0\rangle$, we use the defining property of the vacuum state, $a(\vec{k})|0\rangle = 0$ for the first terms of the integrands of H and $\vec{P}$. The second terms, however, require a re-ordering of the operators before use can be made of $a(\vec{k})|0\rangle = 0$,

$$\begin{aligned} a(\vec{k})a^\dagger(\vec{k}) &= a^\dagger(\vec{k})a(\vec{k}) + [a(\vec{k}),\ a^\dagger(\vec{k})] \\ &= 2\omega_k(2\pi)^3\delta^{(3)}(0) \end{aligned} \tag{4.27}$$

Here, a problem is encountered as $\delta^{(3)}(0) = \infty$. Actually, the quantity $(2\pi)^3\delta^{(3)}(0) \equiv V$ should be interpreted as the *volume of space*, as may be seen by considering a Fourrier transform of 1 in finite volume V, and then letting $\vec{k} \to 0$. The fact that the energy of the vacuum is proportional to the volume of space makes sense because the vacuum is homogeneous and any vacuum energy will be distributed uniformly throughout space. For the momentum, the integrand is now odd in $\vec{k}$ and therefore vanishes, $\vec{P}|0\rangle = 0$.

For the vacuum energy, however, a more serious problem is encountered, as may be seen by collecting the remaining contributions,

$$H|0\rangle = V\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2}\omega_k\ |0\rangle \tag{4.28}$$

This quantity is *quartically divergent*. The physical origin of this infinity is that there is one oscillator for each momentum mode, contributing more energy to the vacuum as momentum is increased.

Actually, we made a mistake in assuming that the quantum Hamiltonian was uniquely determinedby the classical Hamiltonian. Indeed, to the quantum Hamiltonian, terms of the form $[\pi(t,\vec{x}),\phi(t,\vec{x})]$ could be added which would vanish in the classical Hamiltonian. Such terms contribute a constant energy density and are immaterial for the time-evolution of the system, as they will commute with any field. Thus, it would have been more general to include a constant energy density ρ in the Hamiltonian,

$$\begin{aligned} H_\rho &= \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \rho\right] \\ H_\rho &= (2\pi)^3\delta^{(3)}(0)\rho + \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, \frac{\omega_k}{2}\left[a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right] \end{aligned} \tag{4.29}$$

and realizing that the constant ρ is a computable quantity that cannot be computed using the field quantization methods advocated here.

Progress is made by realizing that the energy of the vacuum (in the absence of gravity) is not physically observable. Instead, only differences in energy between states with different particle contents are observable. Therefore, the constant ρ may just as well be chosen so that the vacuum has vanishing energy, or alternatively,

$$H \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, \omega_k\ a^\dagger(\vec{k})a(\vec{k}) \tag{4.30}$$

The process of realizing that certain quantities (such as the vacuum energy) cannot be computed within the quantization procedure of the fields and subsequently fixing the arbitrariness of these quantities in terms of physical conditions goes under the name of *renormalization* (indicated with a subscript R on H_R). The physical conditions themselves are called *renormalization conditions*.

For renormalization in free field theory, it often suffices to use the operation of placing all creation operators to the left and all annihilation operators to the right, which is referred to as *normal ordering*, as was done here.

• Energy and momentum of particle states

Given the above results for the Hamiltonian and momentum operators, it is easy to evaluate them on particle states,

$$\begin{aligned} H|\vec{k}_1,\cdots,\vec{k}_n\rangle &= \sum_{i=1}^n \omega_{k_i}|\vec{k}_1,\cdots,\vec{k}_n\rangle \\ \vec{P}|\vec{k}_1,\cdots,\vec{k}_n\rangle &= \sum_{i=1}^n \vec{k}_i|\vec{k}_1,\cdots,\vec{k}_n\rangle \end{aligned} \tag{4.31}$$

Finally, it may be verified that all the generators of the Poincaré algebra, namely P^μ and $M^{\mu\nu}$ will have the proper commutation relations with ϕ, and reproduce the basic principle of Poincaré transformations of the scalar field $\mathcal{U}(\Lambda,a)\phi(x)\mathcal{U}(\Lambda,a)^\dagger = \phi(\Lambda x + a)$.

- **Time ordered product of two operators**

A key concept in QFT is that of *time ordering*. For any two scalar fields ϕ_1 and ϕ_2, one defines their *time-ordered product* as follows,**

$$T\phi_1(t,\vec{x})\phi_2(t',\vec{x}') = \theta(t-t')\phi_1(t,\vec{x})\phi_2(t',\vec{x}') + \theta(t'-t)\phi_2(t',\vec{x}')\phi_1(t,\vec{x}) \tag{4.32}$$

which can obviously be generalized to the case of the product of any number of fields. Time differentiation of the time ordered product follows a simple rule,

$$\partial_t T\phi_1(t,\vec{x})\phi_2(t',\vec{x}') = T\partial_t\phi_1(t,\vec{x})\phi_2(t',\vec{x}') + \delta(t-t')[\phi_1(t,\vec{x}),\phi_2(t,\vec{x}')] \tag{4.33}$$

For a single field, the time ordered product is a Klein-Gordon Green function,

$$\begin{aligned}
(\Box + m^2)_x T\phi(x)\phi(x') &= \partial_t^2 T\phi(x)\phi(x') + T(-\Delta + m^2)_x\phi_1(x)\phi(x') \\
&= \partial_t T\partial_t\phi(x)\phi(x') + T[-(\partial_t)^2\phi(x)\phi(x')] \\
&= \delta(t-t')[\pi(x),\ \phi(x')] = -i\delta^{(4)}(x-x')
\end{aligned} \tag{4.34}$$

The operator $\Box + m^2$ may be supplemented with a variety of boundary conditions, so care is needed when inverting it.

- **The Feynman propagator**

The *Feynman propagator* is defined by

$$\begin{aligned}
G_F(x-x') &\equiv \langle 0|T\phi(x)\phi(x')|0\rangle \\
(\Box + m^2)_x G_F(x-x') &= -i\delta^{(4)}(x-x')
\end{aligned} \tag{4.35}$$

so that G_F is the inverse of the differential operator $i(\Box + m^2)$. To evaluate the Feynman propagator, we first compute

$$\begin{aligned}
\langle 0|\phi(x)\phi(x')|0\rangle &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3l}{(2\pi)^3 2\omega_l} \langle 0| \left[a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}\right] \\
&\qquad \times \left[a(\vec{l})e^{-il\cdot x'} + a^\dagger(\vec{l})e^{il\cdot x'}\right]|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3l}{(2\pi)^3 2\omega_l} \langle 0|a(\vec{k})a^\dagger(\vec{l})|0\rangle\ e^{-ik\cdot x + il\cdot x'} \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\ e^{-ik\cdot(x-x')}
\end{aligned} \tag{4.36}$$

**The Heaviside θ-function, or *step-function*, is defined to be $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$, and $\theta(0) = 1/2$. Its derivative, defined in the sense of distributions, is $\partial_t\theta(t) = \delta(t)$.

Making the substitution $k^0 = \omega_k$, $l^0 = \omega_l$, and assembling both time orderings yields

$$G_F(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big[\theta(t-t') e^{-i\omega_k(t-t')+i\vec k(\vec x - \vec x')} + \theta(t'-t) e^{-i\omega_k(t'-t)+i\vec k(\vec x' - \vec x)} \Big]$$

Next, we make use of the following integral representation, valid for any $0 < \epsilon \ll 1$,

$$\frac{1}{2\omega_k} \Big[\theta(t-t') e^{-i\omega_k(t-t')} + \theta(t'-t) e^{+i\omega_k(t-t')} \Big] = -\int \frac{dk^0}{2\pi i} \, \frac{e^{-ik^0(t-t')}}{(k^0)^2 - \omega_k^2 + i\epsilon} \tag{4.37}$$

to recast the Feynman propagator as follows,

$$G_F(x-x') = \int \frac{d^4k}{(2\pi)^4} \, \frac{i}{k^2 - m^2 + i\epsilon} \, e^{ik\cdot(x-x')} \tag{4.38}$$

The Klein-Gordon equation with an extrenal (ϕ-independent) source $j(x)$ may be solved in terms of G_F,

$$(\Box + m^2)\phi(x) = j(x) \qquad \Leftrightarrow \qquad \phi(x) = i \int d^4x' G_F(x-x') j(x') \tag{4.39}$$

Note that for $m^2|(x-x')^2| \ll 1$, we have

$$G_F(x-x') = \frac{-i}{4\pi^2 (x-x')^2} \tag{4.40}$$

Feynman introduced a graphical notation for the propagator as a line joining the points x and x'. This is the simplest case of a Feynman diagran, and is depicted in Fig 3(a).

• **Time ordered product of n operators**

The time-ordered product of n canonical fields ϕ will play a fundamental role in scalar field theory. First, notice that the theory has a discrete symmetry

$$\phi'(x) = \mathcal{U}\phi(x)\mathcal{U}^\dagger = -\phi(x) \qquad \Leftrightarrow \qquad \begin{cases} \mathcal{U} a(\vec k) \mathcal{U}^\dagger = -a(\vec k) \\ \mathcal{U} a^\dagger(\vec k) \mathcal{U}^\dagger = -a^\dagger(\vec k) \end{cases} \tag{4.41}$$

Under this symmetry, the vacuum is invariant ($\mathcal{U}|0\rangle = |0\rangle$). By inserting $I = \mathcal{U}^\dagger \mathcal{U}$ between each pair of operators and the vacuum, we have

$$\begin{aligned} \langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle &= \langle 0|\mathcal{U}^\dagger T(\mathcal{U}\phi(x_1)\mathcal{U}^\dagger)\cdots(\mathcal{U}\phi(x_n)\mathcal{U}^\dagger)\mathcal{U}|0\rangle \\ &= (-)^n \langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle \end{aligned} \tag{4.42}$$

When n is odd, we find that $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = 0$. When $n = 2p$ is even, we proceed by applying the kinetic operator in one variable,

$$(\Box + m^2)_{x_n} \langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \sum_{k=1}^{n-1} -i\delta^{(4)}(x_n - x_k) \langle 0|T\phi(x_1)\cdots\hat\phi(x_k)\cdots\phi(x_{n-1})|0\rangle$$

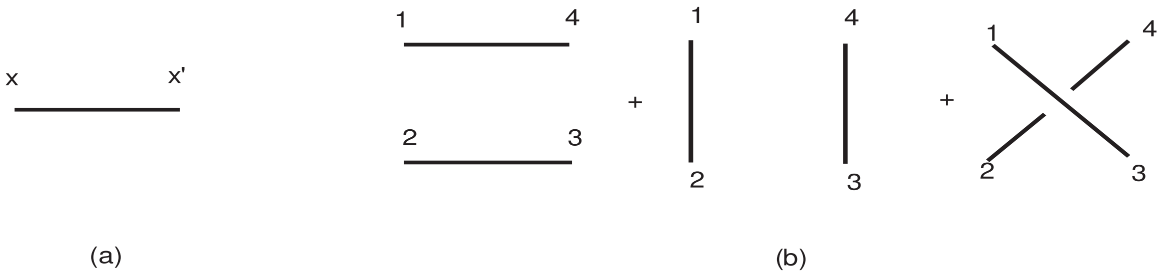


Figure 3: Free Scalar Correlation Functions; (a) The Feynman propagator $G_F(x-x')$, (b) the 4-point function $\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$.

Here, $\hat{\phi}$ stands for the fact that the operators $\phi(x_k)$ is to be omitted. The solution is again obtained in terms of the Feynman propagator $G_F(x-x')$ as follows,

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \sum_{k=1}^{n-1} G_F(x_n-x_k)\langle 0|T\phi(x_1)\cdots\hat{\phi}(x_k)\cdots\phi(x_{n-1})|0\rangle \tag{4.43}$$

The solution to this recursive equation is manifest,

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \sum_{\text{disjoint pairs}(k,l)} G_F(x_{k_1}-x_{l_1})\cdots G_F(x_{k_p}-x_{k_p}) \tag{4.44}$$

where the sum is over all possible pairs without common points. The simple example of the 4-point correlation function is given in Fig 3(b).

4.2 The Operator Product Expansion & Composite Operators

It is possible to re-analyze the issues of the vacuum energy of the free scalar field theory in terms of a more general problem. The canonical commutation relations provide expressions for the commutators of the canonical fields. In the construction of the Hamiltonian, momentum and Lorentz generators, *composite operators* enter, namely the operators $\phi(x)^2$, $\pi(x)^2$ and $(\vec{\nabla}\phi(x))^2$. These operators are obtained by putting two canonical fields at the same space-time point.

From the short distance behavior of the Feynman propagator, (4.40), it is clear that putting operators at the same point produces singularities, which ultimately result in the infinite expression for the energy density of the vacuum. So, one very important problem in QFT is to properly define composite operators. One of the most powerful ways is via the use of the Operator Product Expansion (or OPE), a method that was introduced by Ken Wilson in 1969. (Another way, which is mostly restricted to free field theory or at best perturbation theory, is normal ordering.)

If it were not for the fact that the product of local fields at the same space-time point is singular, one would naively be led to Taylor expand the product of two operators at nearly points,

$$? \qquad \phi(x)\phi(y) = \sum_{n=0}^{\infty} \frac{1}{n!}(x-y)^{\mu_1} \cdots (x-y)^{\mu_n} \left(\partial_{\mu_1} \cdots \partial_{\mu_n} \phi(y)\right) \phi(y) \tag{4.45}$$

Instead, the expansion is rather of the Laurent type, beginning with a singular term. In free field theory, the operators may be characterized by their standard dimension, and ordered according to increasing dimension. In the scalar free field case, reflection symmetry $\phi \to -\phi$ requires the operators on the rhs to be even in ϕ and of degree less or equal to 2. A list of possible operators for dimension less than 6 is given in table4.2.

Dimension	spin 0	spin 1	spin 2	spin 3	spin 4
0	I				
2	ϕ^2				
3		$\phi\partial_\mu\phi$			
4	$\phi\Box\phi$ $\partial_\mu\phi\partial^\mu\phi$		$\phi\partial_\mu\partial_\nu\phi$ $\partial_\mu\phi\partial_\nu\phi$		
5		$\phi\Box\partial_\mu\phi$ $\partial_\mu\phi\Box\phi$ $\partial_\mu\partial_\nu\phi\partial^\nu\phi$		$\phi\partial_\mu\partial_\nu\partial_\rho\phi$ $\partial_\mu\phi\partial_\nu\partial_\rho\phi$	
6	$\phi\Box^2\phi$ $\partial_\mu\phi\Box\partial^\mu\phi$ $\Box\phi\Box\phi$ $\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi$		$\phi\partial_\mu\partial_\nu\Box\phi$ $\partial_\mu\phi\partial_\nu\Box\phi$ $\partial_\mu\partial_\nu\phi\Box\phi$ $\partial_\mu\partial_\rho\phi\partial_\nu\partial^\rho\phi$		$\phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi$ $\partial_\mu\phi\partial_\nu\partial_\rho\partial_\sigma\phi$ $\partial_\mu\partial_\nu\phi\partial_\rho\partial_\sigma\phi$

Table 2:

If no further selection rules apply, the OPE of the two canonical fields will involve all of these operators and the coefficient functions may be calculated. Thus, we obtain an expansion of the type,

$$\phi(x)\phi(y) = \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}(x,y)\mathcal{O}(y) \tag{4.46}$$

The most singular part is proportional to the operator of lowest dimension, i.e. the identity,

$$\phi(x)\phi(y) = \frac{-i}{4\pi^2}\frac{I}{(x-y)^2} + [\phi^2](y) + \cdots \tag{4.47}$$

where the terms $\cdots$ vanish as $x \to y$. The coefficient function of the identity operator is the same as the limiting behavior as $(x-y)^2 \to 0$ in the Feynman propagator. The operator $[\phi^2]$ refers to the finite (or renormalized) operator associated with ϕ^2. It may be defined by the OPE in the following manner,

$$[\phi^2](y) \equiv \lim_{x \to y} \left(\phi(x)\phi(y) + \frac{i}{4\pi^2} \frac{I}{(x-y)^2} \right) \tag{4.48}$$

This limit is well-defined and unique.

More generally, but still in the scalar free field theory, the OPE of any two operators will be a Laurent series in terms of all the operators of the theory (not neccessarily of order 2 in the field ϕ), and we have

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_{\mathcal{O}} C_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}}(x,y)\mathcal{O}(y) \tag{4.49}$$

The coefficient functions may be calculated exactly as long as we are in free field theory, and they will have integer powers. For an interacting theory, a similar expansion exists, but fractional expansion powers will occur, refelecting the fact that in an interacting theory, the dimensions of the operators will received quantum corrections.

4.3 The Gauge or Spin 1 Field

The simplest free spin 1 field is the gauge vector potential A_μ in Maxwell theory of E&M. This field is real and its classical Lagrangian, in the absence of sources, is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad\qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.50}$$

The customary electric E_i and magnetic B_i fields are components of $F_{\mu\nu}$,

$$E_i \equiv F_{i0} \qquad\qquad B_i \equiv \frac{1}{2} \epsilon_{ijk} F_{jk} \tag{4.51}$$

By construction, the Lagrangian $\mathcal{L}$ is invariant under *gauge transformations* by any arbitary real scalar function $\theta(x)$, given by

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \theta(x) \tag{4.52}$$

Two gauge fields related by a gauge transformation have the same electric and magnetic fields and are thus physically equivalent.[††] The canonical momenta are defined as usual,

$$\pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} \tag{4.53}$$

and are found to be

$$\pi^0 = 0 \qquad \pi^i = E^i \tag{4.54}$$

The fact that $\pi^0 = 0$ implies that we are dealing with a *constrained system*, namely the canonical momenta and canonical fields are not independent of one another. The field A_0

[††]Except on non-simply connected domains when there can be glovbal effects.

has vanishing canonical momentum and is therefore *non-dynamical*. This is an immediate consequence of the gauge invariance of the Lagrangian and is also reflected in the structure of the field equations,

$$\partial_\mu F^{\mu\nu} = 0 \quad \Leftrightarrow \quad \begin{cases} \vec{\nabla} \cdot \vec{E} = 0 \\ \partial_t \vec{E} - \vec{\nabla} \times \vec{B} = 0 \end{cases} \tag{4.55}$$

Indeed, $\vec{\nabla} \cdot \vec{E} = 0$ is a non-dynamical equation since it gives a condition that the canonical momenta must satisfy at any time, including on the initial data. Yet a further manifestation of this phenomenon is that the second order differential operator, in terms of which the Lagrangian may be re-expressed (up to a total derivative term) is not invertible,

$$\mathcal{L} = \frac{1}{2} A_\mu (\Box \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu \tag{4.56}$$

The operator $\Box \eta^{\mu\nu} - \partial^\mu \partial^\nu$ is not invertible; it kernel are precisely gauge fields of the form $A_\mu = \partial_\mu \theta$, and gauge equivalent to 0. Clearly, the vanishing of one of the canonical momenta causes complications for the quantization of the gauge field.

Transverse gauge quantization (Lorentz non-covariant)

A first approach in dealing with the complications that ensue from gauge invariance and the constraints on the momenta is to make a gauge choice in which the constraint can be solved explicitly. This will leave only dynamical fields, on which canonical quantization may now be carried out. The drawback of this approach is that it cannot be Lorentz covariant, as a preferred direction is chosen in space-time.

With the help of a gauge transformation, any gauge field configuration may be transformed into a *transverse gauge* $\vec{\nabla} \vec{A} = 0$. As a result of the constraint $\vec{\nabla} \vec{E} = 0$, one automatically has $A_0 = 0$ and the Lagrangian and canonical momenta reduce to

$$\begin{aligned} \mathcal{L} &= +\frac{1}{2} (\dot{A}_i)^2 - \frac{1}{2} B_i^2 \\ \pi_i &= \frac{\partial \mathcal{L}}{\partial (\partial_0 A_i)} = \dot{A}_i \qquad\qquad \partial_i A_i = 0 \end{aligned} \tag{4.57}$$

The last equation reflects the gauge choice. The Euler-Lagrange equations are

$$(\Box \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\mu = 0 \tag{4.58}$$

and, in transverse gauge, reduce to

$$\Box A_i = 0 \qquad\qquad \partial_i A_i = 0 \tag{4.59}$$

The field equations are solved first by Fourier transform,

$$A_i(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big(a_i(\vec{k}) e^{-ik\cdot x} + a_i^\dagger(\vec{k}) e^{ik\cdot x} \Big) \tag{4.60}$$

after which we must further impose the radiation gauge conditions, which result in the following *transversality conditions* on the oscillators,

$$k_i a_i(\vec{k}) = k_i a_i^\dagger(\vec{k}) = 0 \tag{4.61}$$

It is convenient to introduce basis vectors for the two-dimensional space of *transverse polarization vectors* $\vec{\varepsilon}(\vec{k}, \alpha)$, $\alpha = 1, 2$, which satisfy

$$\left\{ \begin{array}{c} k_i \varepsilon_i(\vec{k}, \alpha) = 0 \\ \varepsilon_i(\vec{k}, \alpha)^* \ \varepsilon_i(\vec{k}, \beta) = \delta_{\alpha\beta} \\ \sum_\alpha \varepsilon_i(\vec{k}, \alpha) \varepsilon_j(\vec{k}, \alpha) = \delta_{ij} - \frac{k_i k_j}{k^2} \end{array} \right. \tag{4.62}$$

In terms of this basis, we may introduce independent field oscillators $a(\vec{k}, \alpha)$ in terms of which the old oscillators may be decomposed as

$$a_i(\vec{k}) = \sum_\alpha a(\vec{k}, \alpha) \varepsilon_i(\vec{k}, \alpha) \tag{4.63}$$

and the field takes the form

$$A_i(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\alpha=1,2} \Big(a(\vec{k}, \alpha) \varepsilon_i(\vec{k}, \alpha) e^{-ik\cdot x} + a^\dagger(\vec{k}, \alpha) \varepsilon_i^*(\vec{k}, \alpha) e^{ik\cdot x} \Big) \tag{4.64}$$

Clearly, the photon has two degrees of freedom, even though the field A_μ has 4 components.

The canonical commutation relations amongst the $a(\vec{k}, \alpha)$ and $a^\dagger(\vec{k}, \alpha)$ are just as for the scalar theory, but now with two independent components,

$$\begin{aligned} [a(\vec{k}, \alpha), \ a(\vec{k}', \beta)] &= [a^\dagger(\vec{k}, \alpha), \ a^\dagger(\vec{k}', \beta)] = 0 \\ [a(\vec{k}, \alpha), \ a^\dagger(\vec{k}', \beta)] &= \delta_{\alpha\beta} 2\omega_k (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \end{aligned} \tag{4.65}$$

The equal time commutators may be expressed in terms of the fields A_i as follows,

$$[\dot{A}_i(t, \vec{x}), \ A_j(t, \vec{y})] = -i P_{ij} \delta^{(3)}(\vec{x} - \vec{y}) \qquad\qquad \tilde{P}_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\Delta} \tag{4.66}$$

from which it is clear again that only two degrees of freedom are independent. The Hamiltonian is standard,

$$H = \int d^3x \Big(\frac{1}{2} \vec{E}^2 + \frac{1}{2} \vec{B}^2 \Big) \tag{4.67}$$

The Hilbert space is constructed very much as in the case of the single scalar field. The vacuum $|0\rangle$ is defined by

$$a(\vec{k},\alpha)|0\rangle = 0 \qquad \text{for all } \vec{k},\ \alpha \tag{4.68}$$

The n–particle states are found to be

$$|k_1,\alpha_1;k_2,\alpha_2;\cdots;k_n,\alpha_n\rangle \equiv a^\dagger(k_1,\alpha_1)a^\dagger(k_1,\alpha_1)\cdots a^\dagger(k_1,\alpha_1)|0\rangle \tag{4.69}$$

The (renormalized) Hamiltonian and momentum operators may be worked out,

$$\begin{aligned} H &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_\alpha \omega_k a^\dagger(\vec{k},\alpha)a(\vec{k},\alpha) \\ \vec{P} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_\alpha \vec{k} a^\dagger(\vec{k},\alpha)a(\vec{k},\alpha) \end{aligned} \tag{4.70}$$

Finally, the number operator may also be introduced,

$$N = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_\alpha a^\dagger(\vec{k},\alpha)a(\vec{k},\alpha) \tag{4.71}$$

Since this is a free theory, we have $[H,N]=[\vec{P},N]=0$, so the number of photons is conserved.

Gupta-Bleuler Quantization (Lorentz Covariant)

The second approach is to keep Lorentz invariance manifest at all times, possibly at the cost of giving up manifest gauge invariance. To see exactly what goes on, we introduce a conserved external source j^μ, which we assume is independent of A. The Lagrangian is,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 - A_\mu j^\mu \qquad \partial_\mu j^\mu = 0 \tag{4.72}$$

In this Lagrangian, for $\lambda \neq 0$, the time component A_0 does have a non-vanishing conjugate momentum, and the quadratic part of the Lagrangian involves an operator $\Box\eta_{\mu\nu} - (1-\lambda)\partial_\mu\partial_\nu$ which is invertible. The Lagrangian $\mathcal{L}$ is, however, *no longer gauge invariant*, so in the end we will have to deal with non-gauge invariant states. The canonical relations are

$$\begin{aligned} \pi^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_0)} &= -\lambda\partial_\nu A^\nu \qquad \pi^i = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i)} = F^{io} \\ [\pi^\mu(t,\vec{x}),\ A^\nu(t,\vec{x}')] &= i\ \eta^{\mu\nu}\delta^3(\vec{x}-\vec{x}') \end{aligned} \tag{4.73}$$

The sign in the canonical commutation relation is chosen such that the canonical quantization of the space components agree with the one derived previously. The sign for the time-components is then forced by Lorentz invariance.

Specializing to the case $\lambda = 1$ (Feynman gauge), the field equations reduce to $\Box A_\mu = 0$. The decomposition of $A_\mu(x)$ into modes then gives

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big(a_\mu(\vec{k}) e^{-ik\cdot x} + a^\dagger_\mu(\vec{k}) e^{ik\cdot x} \Big) \tag{4.74}$$

and the canonical commutation relations on the oscillators become,

$$\begin{aligned} [a_\mu(\vec{k}), a_\nu(\vec{k}')] &= [a^\dagger_\mu(\vec{k}), a^\dagger_\nu(\vec{k}')] = 0 \\ [a_\mu(\vec{k}), a^\dagger_\nu(\vec{k}')] &= -\eta_{\mu\nu} 2\omega_k (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \end{aligned} \tag{4.75}$$

We can still define the vacuum by the requirement that $a_\mu(\vec{k})|0\rangle = 0$, and one-particle states by

$$|\vec{k}, \mu\rangle = a^\dagger_\mu(\vec{k})|0\rangle \tag{4.76}$$

The normalization of these states may be worked out in terms of the canonical commutation relations as usual,

$$\langle \vec{k}, \mu | \vec{k}', \nu \rangle = -\eta_{\mu\nu} 2\omega_k (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \tag{4.77}$$

While the oscillators a_i create states with positive norm, as was the case in the scalar theory, the operators a_0 create states with *negative norm*, which cannot be physically acceptable in a quantum theory. These states are *unphysical*, and arise because we insisted on maintaining manifest Lorentz invariance, which forced us to carry along more degrees of freedom than physically needed or allowed. We now investigate how these unphysical degrees of freedom are purged from the theory.

To see how this can be done, we notice that there is a free field in the theory that is a pure gauge artifact. Look at the equations for A_μ:

$$\partial_\mu F^{\mu\nu} - \lambda \partial^\nu \partial_\mu A^\mu + j^\nu = 0 \tag{4.78}$$

Conservation of j^μ implies that $\lambda \Box \partial_\mu A^\mu = 0$; hence $\partial_\mu A^\mu$ is a free field with no coupling to the source j^μ. In view of the canonical quantization relations, we cannot simply set $\partial_\mu A^\mu = 0$. However, we had way too many states created by A^μ, so we may set it be zero *on the physical states*. In fact this is truly beneficial, because it implies that on the physical states, the Heisenberg equation of motion is gauge invariant:

$$\partial_\mu A^\mu |\text{phys}\rangle = 0 \tag{4.79}$$

It is very important to remark that we work with two types of spaces: (1) The Fock space for the full A_μ field, which contains states of negative norm; (2) The physical Hilbert space, obeying the restrictions above.

To solve for the field in terms of the source,

$$\Big(\Box\eta^{\mu\nu} + (\lambda - 1)\partial^\mu\partial^\nu\Big)A_\nu = j^\mu \tag{4.80}$$

we introduce the photon propagator,

$$\Big(\Box\eta^{\kappa\mu} + (\lambda - 1)\partial^\kappa\partial^\mu\Big)_x G_{\mu\nu}(x-y) = -i\delta^\kappa_\nu\delta^4(x-y) \tag{4.81}$$

The propagator is related to the time-ordered product of the quantum fields,

$$G_{\mu\nu}(x-y) = \langle 0|TA_\mu(x)A_\nu(y)|0\rangle \tag{4.82}$$

It may be obtained in terms of a Fourrier integral,

$$G_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\left[\eta_{\mu\nu} + \frac{1-\lambda}{\lambda}\frac{k_\mu k_\nu}{k^2+i\epsilon}\right]\frac{-i}{k^2+i\epsilon}\ e^{-ik\cdot(x-y)} \tag{4.83}$$

Clearly, the propagator in Feynman gauge ($\lambda = 1$) simplifies considerably; in Landau gauge ($\lambda = \infty$) it becomes transverse.

4.4 The Casimir Effect on parallel plates

Thus far, we have dealt with the quantization of the free electromagnetic fields $\vec{E}$ and $\vec{B}$, Already at this stage, it is possible to infer measurable predictions, such as the Casimir effect. The Casimir effect is a purely quantum field theory phenomenon in which the quantization of the electromagnetic fields in the presence of a electric conductors results in a net force between these conductors. The most famous example involves two conducting parallel plates, separated by a distance a, as represented in Fig4(a).

The assumptions entering the set-up are as follows.

1. For a physical conductor, the electric field $\vec{E}_p$ parallel to the plates will vanish for frequencies much lower than the typical atomic scale ω_c. At frequencies much higher than the atomic scale, the conductor effectively becomes transparent and no constraint is to be imposed on the electric field. In the frequency region comparable to the atomic scale ω_c, the conductivity is a complicated function of frequency, which we shall simply approximate by a step function $\theta(\omega_c - \omega)$.

2. The separation a should be taken much larger than interatomic distances, or $a\omega_c \gg 1$.

3. In order to regularize the problems associated with infinite volume, we study the problem in a square box in the form of a cube, with periodic boundary conditions, as depicted in Fig4(b).

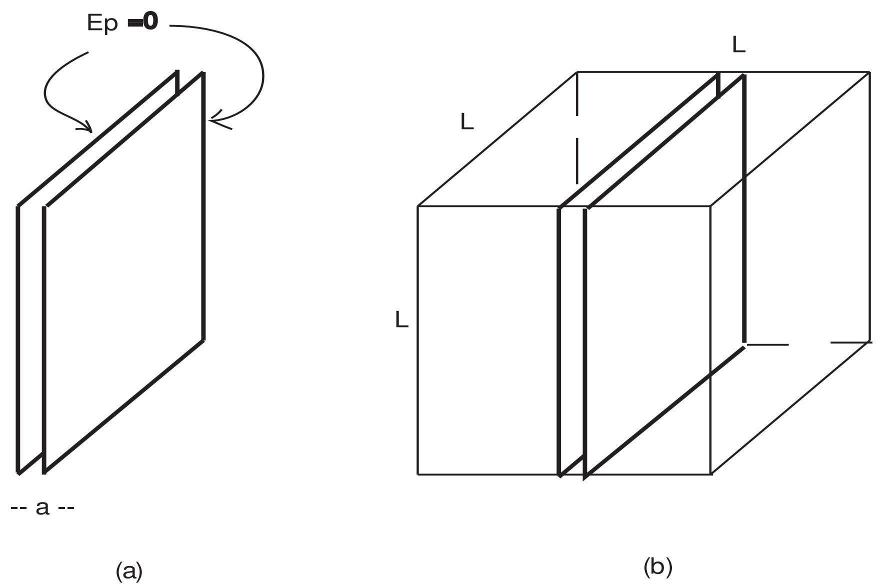


Figure 4: The Casimir effect : (a) electromagnetic vacuum fluctuations produce a force between two parallel plates; (b) the problem in a space box.

• Calculation of the frequencies

There are three distinct regimes in which the frequencies of the quantum oscillators need to be computed. The momenta parallel to the plates are always of the following form,

$$k_x = \frac{2\pi n_x}{L} \qquad k_y = \frac{2\pi n_y}{L} \qquad n_x, n_y \in \mathbf{Z}$$

Along the third direction, we distinguish three different regimes, so that the frequency has three different branches,

$$\omega^{(i)} = \sqrt{k_x^2 + k_y^2 + (k_z^{(i)})^2} \qquad i = 1, 2, 3 \tag{4.84}$$

The three regimes are as follows,

1. $\omega > \omega_c$: the frequencies are the same as in the absence of the plates, since the plates are acting as transparent objects,

$$k_z^{(1)} = \frac{2\pi n_z}{L} \qquad n_z \in \mathbf{Z}$$

2. $\omega < \omega_c$ & between the two plates.

$$k_z^{(2)} = \frac{\pi n_z}{a} \qquad 0 \le n_z \in \mathbf{Z}$$

3. $\omega < \omega_c$ & outside the two plates.

$$k_z^{(3)} = \frac{\pi n_z}{L-a} \qquad 0 \leq n_z \in \mathbf{Z}$$

• **Summing up the contributions of all frequencies**

Finally, we are not interested in the total energy, but rather in the enrgy in the presence of the plates minus the energy in the absence of the plates. Thus, when the frequencies exceed ω_c, all contributions to the enrgy cancel since the plates act as transparent bodies. Thus, we have

$$E(a) - E_0 = \sum_{n_x,n_y,n_z} \Big(\frac{1}{2}\omega^{(2)}(n_x,n_y,n_z) + \frac{1}{2}\omega^{(3)}(n_x,n_y,n_z) - \frac{1}{2}\omega^{(1)}(n_x,n_y,n_z) \Big) \tag{4.85}$$

When $n_z \neq 0$, two polarization modes have $\vec{E}$ with 2 components along the plates, while for $n_z = 0$, $n_x, n_y \neq 0$, there is only one. Therefore, we isolate $n_z = 0$,

$$\begin{aligned} E(a) - E_0 \ &= \ \frac{1}{2} \sum_{n_x,n_y} \omega(n_x,n_y,0) \qquad\qquad (4.86) \\ &+ \sum_{n_x,n_y} \sum_{n_z=1}^{\infty} \Big(\omega^{(2)}(n_x,n_y,n_z) + \omega^{(3)}(n_x,n_y,n_z) - 2\omega^{(1)}(n_x,n_y,n_z) \Big) \end{aligned}$$

Next, we are interested in taking the size of the box L very large compared to all other scales in the problem. The levels in n_x and n_y then become very closely spaced and the sums over n_x and n_y may be well-approximated by integrals, with the help of the following conversion process,

$$dn_x = \frac{L dk_x}{2\pi} \qquad dn_y = \frac{L k_y}{2\pi} \tag{4.87}$$

so that

$$\begin{aligned} E(a) - E_0 \ &= \ \frac{L^2}{4\pi^2} \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \Big[\frac{1}{2} \omega \theta(\omega_c - \omega) \\ &+ \sum_{n_z=1}^{\infty} \Big(\omega^{(2)} \theta(\omega_c - \omega^{(2)}) + \omega^{(3)} \theta(\omega_c - \omega^{(3)}) - 2\omega^{(1)} \theta(\omega_c - \omega^{(1)}) \Big) \Big] \end{aligned}$$

Now, for $\omega^{(1)}$ and $\omega^{(3)}$, the frequency levels are also close together and the sum over n_z may also be approximated by an integral,

$$\begin{aligned} \sum_{n_z=1}^{\infty} \omega^{(2)} \theta(\omega_c - \omega^{(2)}) \ &= \ \frac{L-a}{\pi} \int_0^{\infty} dk_z \sqrt{k_x^2 + k_y^2 + k_z^2}\ \theta(\omega_c - \sqrt{k_x^2 + k_y^2 + k_z^2}) \\ \sum_{n_z=1}^{\infty} 2\omega^{(1)}\ \theta(\omega_c - \omega^{(1)}) \ &= \ \frac{L}{2\pi}\ 2 \int_0^{\infty} dk_z \sqrt{k_x^2 + k_y^2 + k_z^2}\ \theta(\omega_c - \sqrt{k_x^2 + k_y^2 + k_z^2}) \end{aligned}$$

Introducing $k^2 = k_x^2 + k_y^2$, and in the last integral the continuous variable $n = ak_z/\pi$, we have

$$\begin{aligned} E(a) - E_0 &= \frac{L^2}{2\pi} \int_0^\infty dk\ k \Big[\frac{1}{2} k\theta(\omega_c - k) + \sum_{n=1}^{\infty} \sqrt{k^2 + \frac{\pi^2 n^2}{a^2}}\ \theta(\omega_c^2 - k^2 - \frac{\pi^2 n^2}{a^2}) \\ &\quad - \int_0^\infty dn \sqrt{k^2 + \frac{\pi^2 n^2}{a^2}}\ \theta(\omega_c^2 - k^2 - \frac{\pi^2 n^2}{a^2}) \Big] \end{aligned}$$

Introducing the function

$$f(n) \equiv \int_0^\infty \frac{dk}{2\pi} k \sqrt{k^2 + \frac{\pi^2 n^2}{a^2}}\ \theta(\omega_c^2 - k^2 - \frac{\pi^2 n^2}{a^2}) \tag{4.88}$$

we have

$$\frac{E(a) - E_0}{L^2} = \frac{1}{2} f(0) + \sum_{n=1}^{\infty} f(n) - \int_0^\infty dn f(n) \tag{4.89}$$

Now, there is a famous formula that relates a sum of the values of a function at integers to the integral of this function,

$$\int_0^\infty dn f(n) = \frac{1}{2} f(0) + \sum_{n=1}^{\infty} f(n) + \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} f^{(2p-1)}(0) \tag{4.90}$$

where the Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad B_2 = -\frac{1}{6}, \quad B_4 = -\frac{1}{30}, \cdots \tag{4.91}$$

Assuming a sharp cutoff, so that θ is a step function, we easily compute $f(n)$,

$$f(n) = \frac{1}{6\pi} \Big(\omega_c^3 - \frac{\pi^3 n^3}{a^3} \Big) \tag{4.92}$$

Thus, $f^{(2p-1)}(0) = 0$ as soon as $p > 2$, while $f^{(3)}(0) = -\pi^2/a^3$, and thus we have

$$\frac{E(a) - E_0}{L^2} = \frac{\pi^2}{a^3} \frac{B_4}{4!} = -\frac{\pi^2}{720} \hbar c \frac{L^2}{a^3} \tag{4.93}$$

This represents a universal, attractive force proportional to $1/a^4$. To make their dependence explicit, we have restored the factors of $\hbar$ and c.

4.5 Bose-Einstein and Fermi-Dirac statistics

In classical mechanics, particles are *distinguishable*, namely each particle may be tagged and this tagging survives throughout time evolution since particle identities are conserved.

In quantum mechanics, *particles of the same species are indistinguishable*, and cannot be tagged individually. They can only be characterized by the quantum numbers of the state of the system. Therefore, the the operation of interchange of any two particles of the same species must be a symmetry of the quantum system. The square of the interchange operation is the identity.[‡‡] As a result, the quantum states must have definite symmetry properties under the interchange of two particles.

All particles in Nature are either bosons or fermions;

- BOSONS : the quantum state is symmetric under the interchange of any pair of particles and obey *Bose-Einstein statistics*. Bosons have *integer spin*. For example, photons, $W^\pm$, Z^0, gluons and gravitons are bosonic elementary particles, while the Hydrogen atom, the He_4 and deuterium nuclei are composite bosons.

- FERMIONS : the quantum state is anti-symmetric under the interchange of any pair of particles and obey *Fermi-Dirac statistics*. Fermions have *integer plus half spin*. For example, all quarks and leptons are fermionic elementary particles, while the proton, neutron and He_3 nucleus are composite fermions.

Remarkably, the quantization of free scalars and free photons carried out in the preceeding subsections has Bose-Einstein statistics built in. The bosonic creation and annihilation operators are denoted by $a^\dagger_\sigma(\vec{k})$ and $a_\sigma(\vec{k})$ for each species σ. The canonical commutation relations inform us that all creation operators commute with one another $[a^\dagger_\sigma(\vec{k}), a^\dagger_{\sigma'}(\vec{k}')] = 0$. As a result, two states differening only by the interchange of two particle of the same species are identical quantum mechanically,

$$\begin{aligned} & a^\dagger_{\sigma_1}(\vec{k}_1)\cdots a^\dagger_{\sigma_i}(\vec{k}_i)\cdots a^\dagger_{\sigma_j}(\vec{k}_j)\cdots a^\dagger_{\sigma_n}(\vec{k}_n)|0\rangle \\ &= a^\dagger_{\sigma_1}(\vec{k}_1)\cdots a^\dagger_{\sigma_j}(\vec{k}_j)\cdots a^\dagger_{\sigma_i}(\vec{k}_i)\cdots a^\dagger_{\sigma_n}(\vec{k}_n)|0\rangle \end{aligned}$$

The famous *CPT Theorem* states that upon the quantization of a Poincaré and CPT invariant Lagrangian, integer spin fields will always produce particle states that obey Bose-Einstein statistics, while integer plus half fields always produce states that obey Fermi-Dirac statistics.

[‡‡]This statement holds in 4 space-time dimensions but it does not hold generally. In 3 space-time dimensions, the topology associated with the interchange of two particles allows for *braiding* and the square of this operation is not 1. The corresponding particles are *anyons*.

• Fermi-Dirac Statistics

It remains to establish how integer plus half spin fields are to be quantized. This will be the subject of the subsection on the Dirac equation. Here, we shall take a very simple approach whose point of departure is the fact that fermions obey Fermi-Dirac statistics.

First of all, a free fermion may be expected to be equivalent to a collection of oscillators, just as bosonic free fields were. But they cannot quite be the usual harmonic oscillators, because we have just shown above that harmonic operators produce Bose-Einstein statistics. Instead, the *Pauli exclusion principle* states that only a single fermion is allowed to occupy a given quantum state. This means that a fermion creation operator $\hat{b}^\dagger$ for given quantum numbers must square to 0. Then, its repeated application to any state (which would create a quantum state with more than one fermion particle with that species) will produce 0.

The smallest set of operators that can characterize a fermion state is given by the *fermionic oscillators* $\hat{b}$ and $\hat{b}^\dagger$, obeying the algebra

$$\{\hat{b}, \hat{b}\} = \{\hat{b}^\dagger, \hat{b}^\dagger\} = 0 \qquad\qquad \{\hat{b}, \hat{b}^\dagger\} = 1 \tag{4.94}$$

In analogy with the bosonic oscillator, we consider the following simple Hamiltonian,

$$\hat{H} = \frac{\omega}{2}\left(\hat{b}^\dagger\hat{b} - \hat{b}\hat{b}^\dagger\right) = \omega\left(\hat{b}^\dagger\hat{b} - \frac{1}{2}\right) \tag{4.95}$$

Naturally, the ground state is defined by $\hat{b}|0\rangle = 0$ and there are just two quantum states in this system, namely $|0\rangle$ and $\hat{b}^\dagger|0\rangle$ with energies $-\frac{1}{2}\omega$ and $+\frac{1}{2}\omega$ respectively. A simple representation of this algebra may be found by noticing that it is equivalent to the algebra of Pauli matrices or Clifford-Dirac algebra in 2 dimensions,

$$\sigma^1 = \hat{b} + \hat{b}^\dagger \qquad\qquad \sigma^2 = i\hat{b} - i\hat{b}^\dagger \qquad\qquad H = \frac{\omega}{2}\sigma^3 \tag{4.96}$$

Vice-versa, the γ-matrices in 4 dimensions are equivalent to two sets of fermionic oscillators $\hat{b}_\alpha$ and $\hat{b}_\alpha^\dagger$, $\alpha = 1, 2$. The above argumentation demsonstrates that its only irreducible representation is 4-dimensional, and spanned by the states $|0\rangle$, $\hat{b}_1^\dagger|0\rangle$, $\hat{b}_2^\dagger|0\rangle$, $\hat{b}_1^\dagger\hat{b}_2^\dagger|0\rangle$.

In terms of particles, we should affix the quantum number of momentum $\vec{k}$, so that we now have oscillators $b_\sigma(\vec{k})$ and $b_\sigma^\dagger(\vec{k})$, for some possible species index σ. Postulating anti-commutation relations, we have

$$\begin{aligned} \{b_\sigma(\vec{k}), b_{\sigma'}(\vec{k}')\} &= \{b_\sigma^\dagger(\vec{k}), b_{\sigma'}^\dagger(\vec{k}')\} = 0 \\ \{b_\sigma(\vec{k}), b_{\sigma'}^\dagger(\vec{k}')\} &= 2\omega_k\delta_{\sigma\sigma'}(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}') \end{aligned} \tag{4.97}$$

where again $\omega_k = \sqrt{\vec{k}^2 + m^2}$. The Hamiltonian and momentum operators are naturally

$$\begin{aligned} H &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, \omega_k \sum_\sigma b_\sigma^\dagger(\vec{k}) b_\sigma(\vec{k}) \\ \vec{P} &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\, \vec{k} \sum_\sigma b_\sigma^\dagger(\vec{k}) b_\sigma(\vec{k}) \end{aligned} \tag{4.98}$$

The vacuum state $|0\rangle$ is defined by $b_\sigma(\vec{k})|0\rangle = 0$ and multiparticle states are obtained by applying creation operators to the vacuum,

$$b_{\sigma_1}^\dagger(\vec{k}_1) \cdots b_{\sigma_n}^\dagger(\vec{k}_n)|0\rangle \tag{4.99}$$

Using the fact that creation operators always anti-commute with one another, it is straightforward to show that

$$\begin{aligned} & b_{\sigma_1}^\dagger(\vec{k}_1) \cdots b_{\sigma_i}^\dagger(\vec{k}_i) \cdots b_{\sigma_j}^\dagger(\vec{k}_j) \cdots b_{\sigma_n}^\dagger(\vec{k}_n)|0\rangle \\ & \quad = -\, b_{\sigma_1}^\dagger(\vec{k}_1) \cdots b_{\sigma_j}^\dagger(\vec{k}_j) \cdots b_{\sigma_i}^\dagger(\vec{k}_i) \cdots b_{\sigma_n}^\dagger(\vec{k}_n)|0\rangle \end{aligned}$$

so that the state obeys Fermi-Dirac statistics.

4.6 The Spin 1/2 Field

In view of the preceeding discussion on Fermi-Dirac statistics, it is clear that the fermionic field oscillators and thus the fermionic fields themselves should obey *canonical anti-commutation relations*, such as

$$\begin{aligned} \{\psi_\alpha(t,\vec{x}),\ \psi_\beta(t,\vec{y})\} &= 0 \\ \{\psi^{\dagger\alpha}(t,\vec{x}),\ \psi^{\dagger\beta}(t,\vec{y})\} &= 0 \\ \{\psi_\alpha(t,\vec{x}),\ \psi^{\dagger\beta}(t,\vec{y})\} &= i\hbar \delta_\alpha{}^\beta \delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \tag{4.100}$$

Here, α and β stand for spinor indices. The classical limit of commutation relations (without dividing by $\hbar$) confirms that classical bosonic fields are ordinary functions. The classical limit of anti-commutation relations (also without dividing by $\hbar$) for fermionic fields informs us that classical fermionic fields must be *Grassmann-valued functions*, and obey

$$\{\psi_\alpha(t,\vec{x}),\ \psi_\beta(t,\vec{y})\} = \{\psi_\alpha^*(t,\vec{x}),\ \psi_\beta^*(t,\vec{y})\} = \{\psi_\alpha(t,\vec{x}),\ \psi_\beta^*(t,\vec{y})\} = 0 \tag{4.101}$$

Grassmann numbers are well-defined by the above anticommutation relations. It is also useful, however, to obtain a concrete representation in terms of matrices. Let b_j, $j = 1, \cdots, N$ be N Grassmann numbers. These algebraic objects may be realized in terms of

γ-matrices in dimension $2N$, defined to obey the Clifford-Dirac algebra $\{\gamma_m, \gamma_n\} = 2\delta_{mn} I$, with $m, n, = 1, \cdots, 2N$. The following combinations

$$b_j \equiv \frac{1}{2} (\gamma_j + i\gamma_{j+N}) \qquad\qquad j = 1, \cdots, N \tag{4.102}$$

will form N Grassmann numbers. Note that $\{b_j, b_k^\dagger\} = \delta_{ij}$, so Grassmann numbers may be viewed as *half* of the γ-matrices.

• The Free Dirac equation

The field equations for the free Dirac field must be Poincaré covariant and linear in $\psi(x)$. Of course, one might postulate $(\Box + m^2)\psi(x) = 0$, but it turns out that this is neither the simplest nor the correct equation. The Dirac equation is a *first order partial differential equation*,

$$(i\gamma^\mu \partial_\mu - mI)\psi(x) = 0 \tag{4.103}$$

Here, γ^μ are the Dirac matrices in 4 dimensional Minkowski space-time and I is the identity matrix in the Dirac algebra. Often, I is not written explicitly. We begin by showing that this equation is Poincaré invariant. Translation invariance is manifest, and we recall the Lorentz transformation properties of the various ingredients,

$$\begin{aligned} \partial'_\mu &= \Lambda_\mu{}^\nu \partial_\nu \\ \gamma^\mu &= \Lambda^\mu{}_\nu S(\Lambda) \gamma^\nu S(\Lambda)^{-1} \\ \psi'(x') &= S(\Lambda)\psi(x) \end{aligned} \tag{4.104}$$

Combining all, we may establish the covariance of the equation,

$$\begin{aligned} (i\gamma^\mu \partial'_\mu - mI)\psi'(x') &= (i\Lambda^\mu{}_\nu S(\Lambda)\gamma^\nu S(\Lambda)^{-1} \Lambda_\mu{}^\rho \partial_\rho - mI) S(\Lambda)\psi(x) \\ &= (iS(\Lambda)\gamma^\mu S(\Lambda)^{-1} \partial_\mu - mI) S(\Lambda)\psi(x) \\ &= S(\Lambda)(i\gamma^\mu \partial_\mu - mI)\psi(x) \end{aligned} \tag{4.105}$$

Group-theoretically, the structure of the Dirac equation may be understood as follows,

$$\begin{aligned} \psi &\sim (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \qquad\qquad \partial \sim (\frac{1}{2}, \frac{1}{2}) \\ \partial\psi &\sim (\frac{1}{2}, \frac{1}{2}) \otimes \Big((\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \Big) = (0, \frac{1}{2}) \oplus (1, \frac{1}{2}) \oplus (\frac{1}{2}, 0) \oplus (\frac{1}{2}, 1) \end{aligned}$$

The role of the γ-matrices is to project $\partial\psi$ onto the spin 1/2 representations.

If $\psi(x)$ obeys the Dirac equation, then $\psi(x)$ automatically also obeys the Klein Gordon equation, as may be seen by multiplying the Dirac equation to the left by $(-i\gamma^\mu \partial_\mu - mI)$,

$$(-i\gamma^\mu \partial_\mu - mI)(i\gamma^\mu \partial_\mu - mI)\psi(x) = (\gamma^\mu \gamma^\nu \partial_\mu \partial_n u + m^2)\psi(x) = (\Box + m^2)\psi(x)$$

But the converse is manifestly not true.

4.6.1 The Weyl basis and Free Weyl spinors

In the Weyl basis, we have

$$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix} \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{4.106}$$

where $\sigma^\mu = (I, \sigma^i)$ and $\bar\sigma^\mu = (I, -\sigma^i)$. In this basis, the Dirac equation becomes

$$\begin{cases} i\bar\sigma^\mu \partial_\mu \psi_L - m\psi_R = 0 \\ i\sigma^\mu \partial_\mu \psi_R - m\psi_L = 0 \end{cases} \tag{4.107}$$

The mass term couples left and right spinors. When the mass vanishes, the two Weyl spinors ψ_L and ψ_R are decoupled. It now becomes possible to set one of the chiralities to zero and retain a consistent equation for the other. The left Weyl equation is

$$\bar\sigma^\mu \partial_\mu \psi_L = 0 \tag{4.108}$$

The spinors ψ_L and ψ_R transform under the representations $(1/2, 0)$ and $(0, 1/2)$ respectively. The Weyl equation is Poincaré invariant by itself.

- **The Free Weyl and Dirac actions**

The combination $\bar\psi \equiv \psi^\dagger \gamma^0$ has a simple Lorentz transformation property in view of the fact that $\gamma_\mu^\dagger = \gamma^0 \gamma_\mu \gamma^0$, and we have

$$\psi'(x') = S(\Lambda)\psi(x) \quad \Rightarrow \quad (\psi')^\dagger = \psi^\dagger(x) S(\Lambda)^\dagger \quad \Rightarrow \quad \bar\psi'(x) = \bar\psi(x) S(\Lambda)^{-1} \tag{4.109}$$

Therefore, we readily construct an invariant action,

$$S[\psi, \bar\psi] = \int d^4x \mathcal{L} \qquad \mathcal{L} = \bar\psi (i\gamma^\mu \partial_\mu - m)\psi \tag{4.110}$$

Viewing ψ and $\bar\psi$ as independent fields, it is clear that the Dirac equation follows from the variation of these fields. The canonical momenta are $\pi_\psi = i\bar\psi\gamma^0$ and $\pi_{\bar\psi} = 0$ and the Hamiltonian is given by

$$H = \int d^3x \, [\pi_\psi \partial_0 \psi - \mathcal{L}] = \int d^3x \bar\psi \left[-i\vec\gamma \cdot \vec\nabla + m \right] \psi \tag{4.111}$$

In the Weyl basis, the action becomes

$$S[\psi_L, \bar\psi_L, \psi_R, \bar\psi_R] = \int d^4x \left[\psi_L^\dagger i\bar\sigma^\mu \partial_\mu \psi_L + \psi_R^\dagger i\sigma^\mu \partial_\mu \psi_R - m\psi_L^\dagger \psi_R - m\psi_R^\dagger \psi_L \right] \tag{4.112}$$

- **Majorana spinors**

Recall that a Majorana spinor is a Dirac spinor which equals its own charge conjugate. In the Weyl basis, it takes the form,

$$\psi = \begin{pmatrix} \psi_L \\ -\sigma^2 \psi_L^* \end{pmatrix} \qquad\qquad \psi^c = \psi \tag{4.113}$$

Hence, a Majorana spinor in 4 space-time dimensions is equivalent to a Weyl spinor. All the properties can be deduced from this equivalence. Note that there exists an interesting *Majorana mass* term,

$$\mathcal{L} = \bar\psi_L i \not\partial \psi_L + \frac{1}{2} m \psi_L^T \sigma^2 \psi_L = \frac{1}{2}\bar\psi i \not\partial \psi - \frac{1}{2} m \bar\psi \psi \tag{4.114}$$

The Majorana mass term violates $U(1)$ symmetry and so a spinor with Majorana mass must be electrically neutral.

• Solutions to the Free Dirac equation

The free Dirac equation may be solved by a superposition of Fourrier modes with 4-momentum k^μ. Since any solution to the Dirac equation is also a solution to the Klein Gordon equation, we must have $k^2 = m^2$, and there will be solutions with $k^0 < 0$ as well as $k^0 > 0$. Just as we did in the case of bosonic fields, it is useful to separate these two branches, and we organize all solutions in the following manner, both with $k^0 > 0$,

$$\begin{aligned} \psi_+(x) &= u(k,r) e^{-ik\cdot x} \qquad\qquad & (k_\mu \gamma^\mu - m) u^s(k) &= 0 \\ \psi_-(x) &= v(k,r) e^{+ik\cdot x} \qquad\qquad & (k_\mu \gamma^\mu + m) v^s(k) &= 0 \end{aligned} \tag{4.115}$$

The superscript $r = 1, 2$ labels the linearly independent solutions. That there are precisely two solutions for each equation may be shown as follows.

When $m \neq 0$, we introduce the following projection operators

$$\Lambda_\pm(k) \equiv \frac{1}{2m}(\pm k_\mu \gamma^\mu + m) \tag{4.116}$$

which satisfy

$$(\Lambda_\pm)^2 = \Lambda_\pm \qquad \Lambda_+ + \Lambda_- = I \qquad \Lambda_+\Lambda_- = 0 \qquad \mathrm{tr}\Lambda_\pm = 2 \tag{4.117}$$

Hence the rank of both $\Lambda_\pm$ is 2, whence there are two independent solutions. The spinors u and v may be normalized as follows,

$$\begin{aligned} \sum_r u(\vec k, r)\bar u(\vec k, r) &= \not k + m \\ \sum_r v(\vec k, r)\bar v(\vec k, r) &= \not k - m \end{aligned} \tag{4.118}$$

Any bilinear in u and v may be calculated from this formula. For example, we have

$$\begin{cases} \bar{u}(\vec{k},r)u(\vec{k},s) = 2m\delta_{rs} \\ \bar{u}(\vec{k},r)v(\vec{k},s) = 0 \\ \bar{v}(\vec{k},r)v(\vec{k},s) = -2m\delta_{rs} \end{cases} \qquad \begin{cases} \bar{u}(\vec{k},r)\gamma^\mu u(\vec{k},s) = 2k^\mu\delta_{rs} \\ \bar{u}(\vec{k},r)\gamma^\mu v(\vec{k},s) = ? \\ \bar{v}(\vec{k},r)\gamma^\mu v(\vec{k},s) = 2k^\mu\delta_{rs} \end{cases} \tag{4.119}$$

Explicit forms of the solutions may be obtained as follows.

$$u(k,r) = \begin{pmatrix} +\sqrt{k\cdot\sigma}\ \xi(r) \\ +\sqrt{k\cdot\bar{\sigma}}\ \xi(r) \end{pmatrix} \qquad v(k,r) = \begin{pmatrix} +\sqrt{k\cdot\sigma}\ \eta(r) \\ -\sqrt{k\cdot\bar{\sigma}}\ \eta(r) \end{pmatrix} \tag{4.120}$$

with

$$\xi^\dagger(r)\xi(s) = \eta^\dagger(r)\eta(s) = \delta_{rs} \tag{4.121}$$

so that we may choose for example,

$$\xi(1) = \eta(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi(2) = \eta(2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.122}$$

When $m = 0$, there are still two solutions u and two solutions v, but the preceeding normalization becomes degenerate. Choose $k^\mu = (k^0, 0, 0, \pm k^0)$, so that the equations become $(\gamma^0 - \gamma^3)u(k,r) = 0$ and $(\gamma^0 + \gamma^3)v(k,r) = 0$, each of which has again two solutions. It suffices to take $k\cdot\bar{\sigma}\xi(r) = k\cdot\sigma\eta(r) = 0$ to obtain normalized left and right handed spinors. The above normalizations still hold good.

• Quantization of the Dirac Field

We are now ready to quantize the Dirac field. Since the field is complex valued, we must introduce two sets of operators in the decomposition of the field solution,

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_r \Big(u(k,r)\ b(k,r)e^{-ik\cdot x} + v(k,r)\ d^\dagger(k,r)e^{+ik\cdot x} \Big) \tag{4.123}$$

The anti-commutation relations for ψ and $\psi^\dagger$ translate into anti-commutation relations between the b and d oscillators,

$$\begin{aligned} \{b(\vec{k},r),\ b^\dagger(\vec{k}',r')\} &= 2\omega_k\delta_{rr'}(2\pi)^3\delta^2(\vec{k}-\vec{k}') \\ \{d(\vec{k},r),\ d^\dagger(\vec{k}',r')\} &= 2\omega_k\delta_{rr'}(2\pi)^3\delta^3(\vec{k}-\vec{k}') \end{aligned} \tag{4.124}$$

while all others vanish. The time-ordered product is defined by

$$T\psi_\alpha(x)\bar{\psi}_\beta(y) = \theta(x^o - y^o)\psi_\alpha(x)\bar{\psi}_\beta(y) - \theta(y^o - x^o)\bar{\psi}_\beta(y)\psi_\alpha(x) \tag{4.125}$$

It is straightforward to compute the Green function:

$$\begin{aligned} S_F(x-y) &= \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle \qquad (4.126) \\ &= (i\ \not\partial + m)_x G_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{\not k - m + i\epsilon}\ e^{-ik(x-y)} \end{aligned}$$

• Mode expansion of energy and momentum

These expansions are obtained as soon as the quantized fiels is available and we find,

$$P^\mu = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_r k^\mu [b^\dagger(\vec{k},r)b(\vec{k},r) + d^\dagger(\vec{k},r)d(\vec{k},r)] \tag{4.127}$$

where $k^0 = \omega_k$ is always positive. Using canonical anti-commutation relations, we find the commutators of P^μ with the oscillators,

$$\begin{aligned} [P^\mu,\ b(\vec{k},r)] &= -k^\mu b(\vec{k},r) \qquad & [P^\mu,\ b^\dagger(\vec{k},r)] = +k^\mu b^\dagger(k,r) \\ [P^\mu,\ d(\vec{k},r)] &= -k^\mu d(\vec{k},r) \qquad & [P^\mu,\ d^\dagger(\vec{k},r)] = +k^\mu d^\dagger(\vec{k},r) \end{aligned} \tag{4.128}$$

Note that the positive signs on the right hand side for creation operators confirms indeed that $b^\dagger$ and $d^\dagger$ create particles with positive energy only, while b and d annihilate particles with positive energy.

• **Internal symmetries**

The massive or massless Dirac action is invariant under phase rotations of the Dirac field, which form a $U(1)$ group of transformations,

$$U(1)_V \qquad \psi(x) \to \psi'(x) = e^{i\theta_V}\psi(x) \tag{4.129}$$

The associated conserved Noether current j^μ and the time independent charge Q are obtained as usual,

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \qquad\qquad Q = \int d^3x j^0 \tag{4.130}$$

In the quantum theory, the operator Q may be expressed in terms of the oscillators,

$$Q = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_r [b^\dagger(\vec{k},r)b(\vec{k},r) - d^\dagger(\vec{k},r)d(\vec{k},r)] \tag{4.131}$$

Thus, we have the following commutation relations of Q with the individual oscillators,

$$\begin{aligned} [Q,\ b(\vec{k},r)] &= -b(\vec{k},r) \qquad & [Q,\ b^\dagger(\vec{k},r)] = +b^\dagger(k,r) \\ [Q,\ d(\vec{k},r)] &= +d(\vec{k},r) \qquad & [Q,\ d^\dagger(\vec{k},r)] = -d^\dagger(\vec{k},r) \end{aligned} \tag{4.132}$$

Thus the particles created by $b^\dagger$ and by $d^\dagger$ have opposite charge. It will turn out that this quantity is electric charge and that b corresponds to electrons, while d corresponds to positrons.

The *massless Dirac action* has a *chiral symmetry*, which consists of a phase rotation independently on the left and right handed chirality fields,

$$\begin{aligned} &U(1)_L \qquad \begin{cases} \psi_L(x) \to \psi'_L(x) = e^{i\theta_L}\psi_L(x) \\ \psi_R(x) \to \psi'_R(x) = \psi_R(x) \end{cases} \\ &U(1)_R \qquad \begin{cases} \psi_L(x) \to \psi'_L(x) = \psi_L(x) \\ \psi_R(x) \to \psi'_R(x) = e^{i\theta_R}\psi_R(x) \end{cases} \end{aligned} \tag{4.133}$$

By taking linear combinations of the phases, $\theta_V = \theta_L + \theta_R$ and $\theta_A = \theta_L - \theta_R$, we may equivalently rewrite the transformations in terms of Dirac spinors,

$$
\begin{aligned}
&U(1)_V \qquad && \psi(x) \to \psi'(x) = e^{i\theta_V}\psi(x) \\
&U(1)_A \qquad && \psi(x) \to \psi'(x) = e^{i\theta_A\gamma^5}\psi(x)
\end{aligned}
\tag{4.134}
$$

The $U(1)_A$ is referred to as *axial symmetry*, and the associated conserved current is the *axial (vector) current* and the and time independent charge is the axial charge, given by

$$
j_5^\mu(x) = \bar\psi(x)\gamma^m u\gamma^5\psi(x) \qquad\qquad Q_5 = \int d^3x j_5^0 \tag{4.135}
$$

4.7 The spin statistics theorem

The spin statistics theorem states that in any QFT invariant under CPT symmetry, integer spin fields *must* be quantized with commutation relations, while half plus integer spin fields *must* be quantized with anti-commutation relations. This is in fact what we have assumed so far. We now show that this is necessary by studying some counterexamples.

4.7.1 Parastatistics ?

The first question to be answered is why we should have commutation or anticommutation relations at all, instead of some mixture of the two. To see this concretely, consider the case of Dirac spinors. Time translation invariance requires that $[H, b(\vec k, \alpha)] = -k^0 b(\vec k, \alpha)$ and thus for free field theory, we must have

$$
[H, b(\vec k, \alpha)] = \sum_{\alpha'} \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2}[b^+(\vec k', \alpha')b(\vec k', \alpha'),\ b(\vec k, \alpha)] \tag{4.136}
$$

The only way to realize this for all $\vec k$ is to have the following relation amongst the b's,

$$
\sum_{\alpha'} \frac{1}{2}[b^\dagger(\vec k', \alpha')b(\vec k', \alpha'),\ b(\vec k, \alpha)] = -2\omega_k b(\vec k, \alpha)(2\pi)^3\delta^3(\vec k - \vec k') \tag{4.137}
$$

Let us now postulate not the usual commutation or anti-commutation relations, but generalization thereof which allows for some mixture of the two. For simplicity, we shall retain a structure that is bilinear in the operators,

$$
\begin{aligned}
b(\vec k, \alpha)b^\dagger(\vec k', \alpha') + F_+\ b^\dagger(\vec k', \alpha')b(\vec k, \alpha) &= G_+(\vec k, \vec k') \\
b(\vec k, \alpha)b(\vec k', \alpha') + F_-\ b(\vec k', \alpha')b(\vec k, \alpha) &= G_-(\vec k, \vec k')
\end{aligned}
\tag{4.138}
$$

We always require these relations to be local in space-time so that $F_\pm$ are independent of $\vec k$ and $\vec k'$. When $F_\pm \neq \pm 1$, these relations are some-times referred to as *parastatistics*.

Working out the cubic equation now yields

$$G_- = 0 \qquad F_+ = F_- \qquad G_+ = (2\pi)^3 2\omega_k\ \delta^3(\vec{k} - \vec{k}') \tag{4.139}$$

Now look at the last equation and interchange $\vec{k}$ and $\vec{k}'$,

$$\begin{aligned} b(\vec{k})b(\vec{k}') + F_- b(\vec{k}')b(\vec{k}) &= 0 &(4.140)\\ b(\vec{k}')b(\vec{k}) + F_- b(\vec{k})b(\vec{k}') &= 0 &(4.141)\end{aligned}$$

Combining the two, we find $F_-^2 = +1$. Therefore, the only possibilities are commutation relations or anticommutation relations. This can be done similarly for any field.

4.7.2 Commutation versus anti-commutation relations

The next question to be addressed is how one decides between commutation relations and anticommutation relations. For a change, let's work with a complex scalar field and attempt quantization with anticommutation relations:

$$\begin{aligned} \phi(t,\vec{x}) &= \int \frac{d^3x}{(2\pi)^3 2\omega_k}\left[a(\vec{k})e^{-ikx} + c^\dagger(\vec{k})e^{ikx}\right] \\ \phi^+(t,\vec{x}) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[c(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{+ikx}\right] \end{aligned} \tag{4.142}$$

and postulate anticommutation relations between the fields ϕ and $\phi^d agger$,

$$\{\dot{\phi}^\dagger(t,\vec{x}),\ \phi(t,\vec{y})\} = -i\ \delta^3(\vec{x} - \vec{y}) \qquad \text{etc.} \tag{4.143}$$

Anticommutation relations also follow on the oscillators,

$$\begin{aligned} \{a^+(\vec{k}),\ a(\vec{k}')\} &= -(2\pi)^3\omega_k\delta^{(3)}(\vec{k} - \vec{k}') \\ \{c^\dagger(\vec{k}),\ c(\vec{k}')\} &= +(2\pi)^3 2\omega_k\delta^{(3)}(\vec{k} - \vec{k}') \end{aligned} \tag{4.144}$$

The fact that the two signs are opposite means that particle and antiparticle together cannot have both positive norm. If we attempt to quantize the fermion theory with equal time commutation relations, we get similar violations of known and well-established physical principles.

5 Interacting Field Theories – Gauge Theories

Free field theories were described by quadratic Lagrangians, so that the equations of motion are linear. Interactions will occur when the fields are coupled to external sources or to one another. As for free field theory, we shall insist on the interactions being Poincaré invariant (when the source is external, it should be also transformed under the Poincaré transformatiions) and local. In a fundamental theory of Nature, there is no room for external sources, since the descriptions of all phenomena is to be given in a single theory. Often, however, the effects of a certain sector of the theory may be summarized in the form of external sources, thereby simplifying the practical problems involved.

5.1 Interacting Lagrangians

We begin by presenting a brief list of some of the most important fundamental interacting field theories for elementary particles whose spin is less or equal to 1. These interacting field theories are described by local Poincaré invariant Lagrangians.

1. *Scalar field theory*

   The simplest case involves a free kinetic term plus an interaction potential.

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) \tag{5.1}$$

2. *The linear σ-model*

   This is a theory of N real scalar fields with quartic self-couplings,

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{N} \partial_\mu \sigma^a \partial^\mu \sigma^a - \frac{\lambda}{4}\left(\sum_{a=1}^{N} \sigma^a \sigma^a - v^2)^2\right) \tag{5.2}$$

3. *Electrodynamics*

   This theory governs most of the world that we see,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu} + \bar\psi(i \not\partial - m)\psi - eA_\mu \bar\psi \gamma^\mu \psi \tag{5.3}$$

4. *Yang-Mills theory*

   Yang-Mills theory describes the electro-weak and strong interactions. It is associated with a simple Lie algebra $\mathcal{G}$, with generators T^a, $a = 1, 2, \cdots, \dim\mathcal{G}$ and structure constants f^{abc}. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}\sum_a F^a_{\mu\nu} F^{a\mu\nu} \qquad\qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\sum_c f^{abc} A^b_\mu A^c_\nu \tag{5.4}$$

It is also possible to couple these various theories to one another.

5.2 Abelian Gauge Invariance

Consider a charged complex field $\psi(x)$, which transforms non-trivially under $U(1)$ gauge transformations:

$$\psi(x) \longrightarrow \psi'(x) = U(x)\ \psi(x) \qquad U(x) = e^{iq\theta(x)} \tag{5.5}$$

Bilinears, such as the current, are invariant under these transformations

$$\bar{\psi}\Gamma\psi(x) \longrightarrow \bar{\psi}\Gamma\psi(x) \qquad \Gamma = I, \Gamma^{\mu}, \Gamma^{\mu\nu} \ldots \Gamma^{5} \tag{5.6}$$

Derivatives of the field, however, do not transform well,

$$\partial_\mu\psi(x) \longrightarrow U(x)\ \partial_\mu\psi(x) + \partial_\mu U(x)\ \psi(x) \tag{5.7}$$

so that the standard kinetic term $\bar{\psi}\ \not{\partial}\psi$ is not invariant.

In QED, we have also a gauge field A_μ, whose transformation law involves an inhomogeneous piece in U, in such a way that the *covariant derivative* $D_\mu\psi$ transforms homogeneously,

$$\begin{cases} D_\mu & \equiv & \partial_\mu + iqA_\mu \\ & & \\ D'_\mu\psi'(x) & = & U(x)\ D_\mu\psi(x) \end{cases} \tag{5.8}$$

Working out the required transformation law of A_μ, we find

$$A'_\mu = A_\mu + \frac{i}{q}U^{-1}\partial_\mu U = A_\mu - \partial_\mu\theta \tag{5.9}$$

We are now guaranteed that $\bar{\psi}\ \not{D}\psi$ is invariant.

In additiion,

$$[D_\mu, D_\nu]\psi = iqF_{\mu\nu}\psi \tag{5.10}$$

which may be used to define the field strength $F_{\mu\nu}$. (Comparing with general relativity, one has $A_\mu \sim \Gamma^k_{\mu\nu}$ is a connection, while $F_{\mu\nu} \sim R_{\mu\nu\kappa\lambda}$ is a curvature).

5.3 Non-Abelian Gauge Invariance

On a multiplet of fields $\psi_i \quad i = 1, \ldots, N$, we are free to consider a larger set of transformations

$$\begin{array}{rclr} \psi(x) & \longrightarrow & \psi'(x) \equiv U(x)\psi(x) & \text{matrix form} \\ \psi_i(x) & \longrightarrow & \psi'_i(x) \equiv U_i{}^j(x)\psi_j(x) & \text{component form} \end{array} \tag{5.11}$$

where $U(x) \in G\ell(N,R)$ or $G\ell(N,C)$ depending on whether the fields ψ_i are real or complex.

While it is possible (and sometimes useful, such as in supergravity) to consider absolutely general gauge groups, for our purposes, we shall always be interested in groups with finite-dimensional unitary representations, i.e. compact groups. We denote these generically by G, and so $U(x) \in G$, for any x. This guarantees that bilinears of ψ are gauge-invariant

$$\bar{\psi}\Gamma\psi \to \bar{\psi}\Gamma\psi \qquad \Gamma = I, \gamma^\mu, \gamma^{\mu\nu}, \gamma^5 \dots \tag{5.12}$$

Just as in QED, covariant derivatives are again defined so that their transformation law is homogeneous. This requires introducing a *non-Abelian gauge field* $A_\mu(x)$, which is an $N \times N$ matrix so that

$$\begin{cases} D_\mu & \equiv & \partial_\mu + iA_\mu \\ D'_\mu \psi'(x) & \equiv & U(x)\ D_\mu \psi(x) \end{cases} \tag{5.13}$$

To work out the proper transformation law of A_μ, we need to be careful and take into account that the fields are now all (non-commuting) matrices.

$$(\partial_\mu + igA'_\mu)U(x)\psi(x) = U(x)(\partial_\mu + iA_\mu)\psi(x) \tag{5.14}$$

or

$$U^{-1}(x)(\partial_\mu + iA'_\mu(x))U(x)\psi(x) = (\partial_\mu + iA_\mu)\psi(x) \tag{5.15}$$

Working this out, and expressing A'_μ in terms of A_μ and U, we have

$$A'_\mu(x) = U(x)A_\mu(x)U^{-1}(x) + i(\partial_\mu U)(x)U^{-1}(x) \tag{5.16}$$

We proceed to compute the analogue of the field strength:

$$[D_\mu, D_\nu]\psi \;=\; [\partial_\mu + iA_\mu, \partial_\nu + iA_\nu]\psi = i(\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu])\psi \tag{5.17}$$

and therefore the object analogous to the field strength is

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \qquad\qquad F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^{-1}(x) \tag{5.18}$$

Notice that while for QED, the relation between field strength and field was linear, here it is non-linear. Thus, the requirement of non-Abelian gauge invariance has produced a novel interaction.

5.4 Component formulation

We assume that the gauge transformation belong to a Lie group G (compact), and that ψ transforms under a representation T of G, which is unitary. At the level of the Lie algebra, we have the expansion

$$U(\omega_a) = I + i\omega_a T^a + O(\omega^2) \qquad a = 1, \dots, \dim \mathcal{G} \tag{5.19}$$

where ω_a are the generators of the Lie group and T^a are the representation matrices of $\mathcal{G}$ in the representation T. They satisfy $(T^a)^\dagger = T^a$ and the structure relations

$$[T^a, T^b] = i\ f^{abc} T^c \tag{5.20}$$

where f^{abc} is totally antisymmetric and real.

The transformation laws in components now take the form

$$\begin{cases} \delta\psi(x) = \psi'(x) - \psi(x) = i\omega_a(x) T^a \psi(x) \\ \delta A_\mu(x) = A'_\mu(x) - A_\mu(x) = -T^a \partial_\mu \omega_a + i\omega_c [T^c, A_\mu] \end{cases} \tag{5.21}$$

Therefore A_μ transforms under the adjoint representation of $\mathcal{G}$. The matrix A_μ itself acts in the representations T on ψ. Thus, we may decompose A_μ as follows:

$$A_\mu = A_\mu^a\ T^a \qquad\qquad a = 1, \dots, \dim \mathcal{G} \tag{5.22}$$

The transformation law of the component is

$$\begin{aligned} \delta A_\mu^a &= -\partial_\mu \omega^a - f^{cba} \omega_c A_\mu^b & (5.23) \\ &= -(\partial_\mu \omega^a - f^{abc} A_\mu^b \omega^c) & (5.24) \end{aligned}$$

Recalling that f^{abc} are the representation matrices for the adjoint representation, and that ω^a itself transforms under the adjoint representation, we have

$$\delta A_\mu^a = -(D_\mu \omega)^a \qquad\qquad (D_\mu \omega)^a \equiv \partial_\mu \omega^a - f^{abc} A_\mu^n \omega^c \tag{5.25}$$

The field strength in components is

$$\begin{cases} F_{\mu\nu} \equiv F_{\mu\nu}^a T^a \\ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c \end{cases} \tag{5.26}$$

Of course, $F_{\mu\nu}$ also transforms under the adjoint rep. of $\mathcal{G}$.

5.5 Bianchi identities

D_μ is an operator on the space of fields and therefore must satisfy the Jacobi identity

$$[D_\mu,\ [D_\nu,\ D_\rho]] + [D_\nu,\ [D_\rho,\ D_\mu]] + [D_\rho,\ [D_\mu,\ D_\nu]] = 0 \tag{5.27}$$

In 4-dimension, this equality is equivalent to

$$\epsilon^{\mu\nu\rho\sigma}[D_\nu,\ [D_\rho,\ D_\sigma]] = 0 \tag{5.28}$$

Using $[D_\rho,\ D_\sigma] = iF_{\rho\sigma}$, we get the *Bianchi identity*,

$$\epsilon^{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = D_\nu \tilde{F}^{\mu\nu} = 0 \tag{5.29}$$

It generalizes the case of QED: $\partial_\nu \tilde{F}^{\mu\nu} = 0$.

5.6 Gauge invariant combinations

$$\begin{cases} \text{tr } F_{\mu\nu}F^{\mu\nu} & \\ \text{tr } F_{\mu\nu}\tilde{F}^{\mu\nu} & \qquad \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \end{cases} \tag{5.30}$$

To pass to components, use the fact that the Cartan-Killing form is positive-definite and use the normalization

$$\text{tr } T^aT^b = \frac{1}{2}\delta^{ab} \tag{5.31}$$

We then have

$$\begin{cases} \text{tr } F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}F^a_{\mu\nu}F^{a\mu\nu} \\ \text{tr } F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu \\ K^\mu \sim \epsilon^{\mu\alpha\beta\gamma}\left(\partial_\alpha F_{\beta\gamma} - \frac{1}{3}A_\alpha A_\beta A_\gamma\right) \qquad = \text{“Chern-Simons term”} \end{cases} \tag{5.32}$$

When G is a simple group, these invariants are the only ones possible. When G is not simple, it must be of the form of a product of simple groups times a product of $U(1)$ factors.

$$\begin{array}{cccccccc} G = & G_1\ x & \dots & x\ G_p\ x & U(1)_1\ x & \dots & x\ U(1)_q \\ & A_{1\mu} & \dots & A_{p\mu} & B_{1\mu} & \dots & B_{q\mu} \\ & F_{1\mu\nu} & \dots & F_{p\mu\nu} & G_{1\mu\nu} & \dots & G_{q\mu\nu} \\ & g_1 & \dots & g_p & e_1 & \dots & e_q \end{array} \tag{5.33}$$

and we have independent invariants for each:

$$\begin{cases} \text{tr } F_{i\mu\nu}F^{i\mu\nu} \\ \text{tr } F_{i\mu\nu}\tilde{F}^{i\mu\nu} \end{cases} \tag{5.34}$$

The Abelian factors in general may mix; they can however be diagolized,

$$\sum_{i,j=1}^{q} C_{ij} G_{i\mu\nu} G_j^{\mu\nu} \qquad \longrightarrow \qquad \sum_{i=1}^{q} \frac{1}{e_i^2} G_{i\mu\nu} G_i^{\mu\nu} \tag{5.35}$$

and for the simple parts

$$\sum_{j=1}^{p} \frac{1}{g_j^2} \mathrm{tr}\ F_{j\mu\nu} F_j^{\mu\nu} \tag{5.36}$$

5.7 Classical Action & Field equations

A suitable action for the Yang-Mills field is given by the expression (dimension 4), for each simple group component

$$S[A,J] \equiv -\frac{1}{2g^2} \int d^4x\ \mathrm{tr}\ F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{8\pi^2 g^2} \int d^4x\ \mathrm{tr}\ F_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{2}{g} \int d^4x (J A_\mu) \tag{5.37}$$

where $J^\mu \equiv J^{\mu a} T^a$ is a gauge current coupling to the Yang-Mills field A_μ and g is a coupling constant. It is often convenient to change normalization of the field by factoring out the coupling constant g.

$$\begin{aligned} A_\mu &\longrightarrow gA_\mu \\ \frac{1}{g} F_{\mu\nu} &\longrightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \end{aligned} \tag{5.38}$$

The action is then

$$S[A,J] = -\frac{1}{2} \int d^4x\ \mathrm{tr}\ F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{8\pi^2} \int d^4x\ \mathrm{tr}\ F_{\mu\nu} \tilde{F}^{\mu\nu} - 2 \int d^4x\ \mathrm{tr}\ J^\mu A_\mu \tag{5.39}$$

and its variation is given by

$$\delta S = -\int d^4x\ \mathrm{tr}(F^{\mu\nu} \delta F_{\mu\nu} + 2J^\mu \delta A_\mu) \tag{5.40}$$

To evaluate this variation, one makes use of the following relations,

$$\begin{aligned} igF_{\mu\nu} &= [D_\mu, D_\nu] \\ \delta D_\mu &= ig\delta A_\mu \\ ig\delta F_\mu &= ig[\delta A_\mu, D_\nu] + ig[D_\mu, \delta A_\nu] \end{aligned} \tag{5.41}$$

which imply that the variation of the field strength is very simple,

$$\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu \tag{5.42}$$

$$\delta S = -2 \int d^4x \ \mathrm{tr}(F^{\mu\nu} D_\mu \delta A_\nu + J^\nu \delta A_\nu) \tag{5.43}$$

We'd like to integrate by parts, but D_μ is not quite a derivative in the usual sense. Nonetheless, all proceeds as if it were.

$$\begin{aligned}
\mathrm{tr}\ F^{\mu\nu} D_\mu \delta A_\nu &= \mathrm{tr}\ F^{\mu\nu}(\partial_\mu \delta A_\nu + ig[A_\mu, \delta A_\nu]) \\
&= \partial_\mu(\mathrm{tr}\ F^{\mu\nu}\delta A_\nu) - \mathrm{tr}\ \partial_\mu F^{\mu\nu}\delta A_\nu + ig\ \mathrm{tr}(F^{\mu\nu} A_\mu \delta A_\nu - F^{\mu\nu}\delta A_\nu A_\mu) \\
&= \partial_\mu(\mathrm{tr}\ F^{\mu\nu}\delta A_\nu) - \mathrm{tr}\ \delta A_\nu D_\mu F^{\mu\nu}
\end{aligned} \tag{5.44}$$

Hence, we have

$$\delta S = -2 \int d^4x \ \mathrm{tr}(-D_\mu F^{\mu\nu} + J^\nu)\delta A_\nu \tag{5.45}$$

Thus, the field equations are

$$D_\mu F^{\mu\nu} = J^\nu \tag{5.46}$$

As a consequence, the current J^ν is *covariantly conserved.*

$$D_\nu D_\mu F^{\mu\nu} = \frac{1}{2}[D_\nu, D_\mu]F^{\mu\nu} = \frac{1}{2}[F_{\nu\mu}, F^{\mu\nu}] = 0 \tag{5.47}$$

$$D_\nu J^\nu = \partial_\nu J^\nu + ig[A_\nu, J^\nu] = 0 \tag{5.48}$$

5.8 Lagrangians including fermions and scalars

The basic Lagrangian for a gauge field A_μ associated with a simple gauge group G, in the presence of "matter fields"

- fermions in representations T_L and T_R of G;
- scalars in representation T_S of G.

Then the Lagrangians of relevance are of the form

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2g^2}\mathrm{tr}\ F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_L \gamma^\mu (i\partial_\mu - gA_\mu^a T_L^a)\psi_L + \bar{\psi}_R \gamma^\mu (i\partial_\mu - gA_\mu^a T_R^a)\psi_R \\
&\quad + D_\mu \phi^+ D^\mu \phi - V(\phi)
\end{aligned} \tag{5.49}$$

where $V(\phi)$ is invariant under G and

$$D_\mu \phi \equiv \partial_\mu \phi + igA_\mu^a T_s^a \phi \tag{5.50}$$

Actually, when $T_L^\star \otimes T_R \otimes T_S$ contains one or more singlets, one may also have Yukawa terms

$$\lambda(\bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^+ \psi_L) \tag{5.51}$$

In addition, if $T_L^\star \otimes T_R$ contains singlets, one may also have a "Dirac mass term"

$$M(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \tag{5.52}$$

Finally, if $T_L = T_R$, then the theory is parity conserving, while it is parity violating when $T_L \neq T_R$.

5.9 Examples

1. QCD (ignoring the weak interactions); $G = SU(3)_c$

$$\psi^t = (u\ d\ c\ s\ t\ b)^c \tag{5.53}$$

- c = color $SU(3)$
- each flavor is in fundamental representation $SU(3)$
- $T_L = T_R = \underbrace{3 \oplus \ldots \oplus 3}_{\text{6 times}} = T$

$$\mathcal{L} = -\frac{1}{2g_3^2} \mathrm{tr}\ F_{\mu\nu} F^{\mu\nu} + \sum_{f=1}^{6} \bar{q}_f \gamma^\mu (i\partial_\mu - g_3 A_\mu^a T^a) q_f - \sum_{f=1}^{6} m_f \bar{q}_f q_f \tag{5.54}$$

2. $SU(2)_L$ weak interactions $U(1)_Y$ (no QCD);

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \qquad \ell_L = \begin{pmatrix} e^- \\ \nu_e \end{pmatrix}_L \qquad \ell_R = e_R^- \tag{5.55}$$

$$\begin{aligned} \mathcal{L} \ \ = \ \ & -\frac{1}{2g_2^2} \mathrm{tr}\ F_{\mu\nu} F^{\mu\nu} + \bar{q}_L \gamma^\mu (i\partial_\mu - g_2 A_\mu^a T_L^a - g_1 B_\mu Y_L^q) q_L + \bar{q}_R \gamma^\mu (i\partial_\mu - g_1 B_\mu Y_R) q_R \\ & + D_\mu \phi^+ D_\mu \phi + \bar{\ell}_L \gamma^\mu (i\partial_\mu - g_2 A_\mu^a T^a - g_1 Y_L^\ell B_\mu) \ell_L + \bar{\ell}_R \gamma^\mu (i\partial_\mu - g_1 B_\mu) \ell_R \\ & - V(\phi^+ \phi) - \bar{q}_L \phi q_R - \bar{\ell}_L \phi \ell_R - \bar{q}_L \tilde{\phi} q_R \end{aligned} \tag{5.56}$$

6 The S-matrix and LSZ Reduction formulas

So far we have only discussed *fields*, but we are interested in the interaction of *particles*. However remember from the introduction that the positions and the momenta of the particles are not directly observable. In particular the *position* can never be observed to arbitrary precision due to the occurrence of anti-particles. On the other hand, momenta of particles are observable provided one observes the particles asymptotically as $\Delta t \to \infty$, then $\Delta p \to 0$.

Hence we will be interested in observing the dependence on the momenta of incoming and outgoing particles. To do this we idealize the interaction.

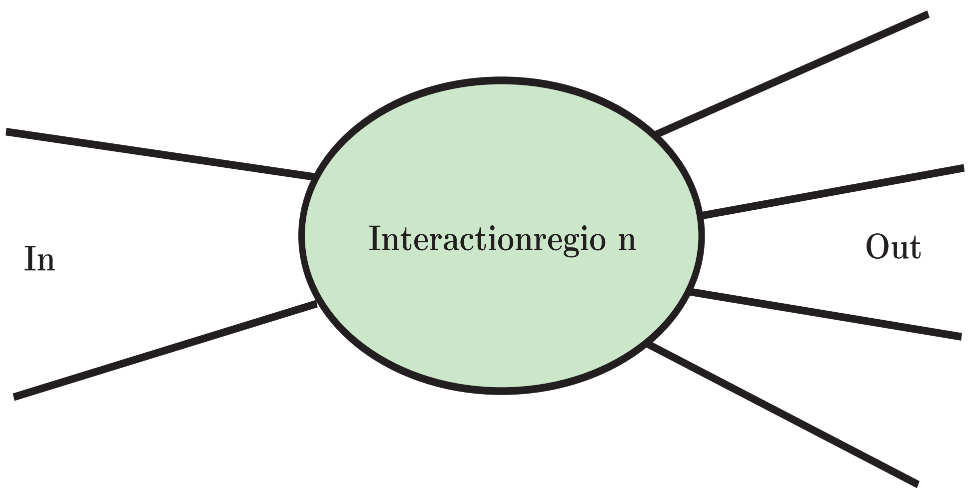


Figure 5: Idealized view of the process of free initial, interacting and free final particles

The basic assumption is that as $t \to \pm\infty$, the particles behave as free particles. Is this true for the interactions that we know of ?

1. Strong nuclear forces at internuclear distances are governed by the Yukawa potential,

$$\sim \frac{1}{r} \exp\{-r\ m_\pi\} \tag{6.1}$$

   These are so-called *short-ranged* and effectively vanish beyond the size of the nucleus. This is why effectively, in day to day life, we do not see their effects. For short ranged forces, the idealized view should hold.

2. The weak forces are also short ranged,

$$\sim \frac{1}{r} \exp\{-r\ M_{W^\pm}\} \qquad \text{or} \qquad \sim \frac{1}{r} \exp\{-r\ M_{Z^0}\} \tag{6.2}$$

3. The electro-magnetic force falls off at infinity as well,

$$\sim \frac{1}{r^2} \tag{6.3}$$

   By contrast with the weak and strong forces however, this fall-off is only *power-like* and such forces are referred to as *long-ranged*. This means that electro-magnetic fields can extend over long distances and will not be confined to some small interaction region. For example, the galactic magnetic fields extend over many parsecs. The idealized picture may or may not hold. We should foresee difficulties at low energies and low momenta (so-called IR propblems) when we apply the idealized view here.

4. The force of gravity behaves in the same manner as the electro-magnetic force and is also long-ranged. Fortunately, the force of gravity is not on the menu for this course.

5. The most problematic of all – and the most interesting – is QCD which describes the strong forces at a fundamental level. The force between two quarks is constant, even for large distances (ignoring quark pair creation) and the quarks are *confined*, as it would require an infinite amount of energy to separate two quarks that are subject to a law of attraction that is constant. This effect is so dramatic that the spectrum is altered to a hadronic spectrum.

6.0.1 In and Out States and Fields

If at $t \to -\infty$ the particles are free, then we better introduce a free field for them, which are the so-called *in-field*, denoted generically by $\Phi^I_{\text{in}}(x)$. The Fock space associated with Φ_{in} is $\mathcal{H}_{\text{in}}$, and the states are labelled exactly as the free field states were. If at $t \to +\infty$ the particles are free, we introduce free *out-fields* $\Phi^J_{\text{out}}(x)$ and the Fock space is $\mathcal{H}_{\text{out}}$.

Of course we also have the Hilbert space of the full quantum field theory, which we denote by $\mathcal{H}$. Axiomatic field theory requires $\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} = \mathcal{H}$. Here, however, we shall attempt to define these different theories, to link them and then to check the nature of the various Hilber spaces. Obviously, we should have the property $\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}}$, which is called *asymptotic completeness*.

The n-particle *in-states* created by *in fields* are denoted by

$$|\text{in},\ p_1, \cdots, p_n;\ I_1, \cdots, I_n\rangle \tag{6.4}$$

with particle momenta $p_1, \cdots, p_n$ and internal indices $I_1, \cdots, I_n$. The *out-states* created by the *out fiels* are of the form

$$|\text{out},\ q_1 \ldots q_m,\ J_1 \cdots J_m\rangle \tag{6.5}$$

where the m-particle *out state* has particle momenta $q_1 \ldots q_m$, and internal indices $J_1 \ldots J_m$. Note that it makes sense to label states in this way since momenta are observable to arbitrary precision in the $t = to \pm \infty$ limit.

6.0.2 The S-matrix

The fundamental question in scattering theory and in quantum field theory is going to be as follows. Given an initial in-state (the state of the system at $t \to -\infty$),

$$|\text{in } p_1, \ldots p_n;\ I_1, \ldots I_n\rangle \tag{6.6}$$

what is the probability (given the dynamics of the theory) for the final out-state (the state at $t \to +\infty$) to be

$$|\text{out } q_1 \ldots q_m;\ J_1, \cdots, J_m\rangle \tag{6.7}$$

The answer is given by the inner product of the two states, as always

$$S_{f \leftarrow i} = \langle \text{out } q_1 \ldots q_m;\ J_1 \cdots J_m | \text{in } p_1, \ldots p_n,\ I_1 \cdots I_n\rangle \tag{6.8}$$

and the probability is

$$W_{f \leftarrow i} = |S_{f \leftarrow i}|^2 \tag{6.9}$$

Clearly, this question can be asked for every state in $\mathcal{H}_{\text{in}}$ onto every state in $\mathcal{H}_{\text{out}}$. Assuming asymptotic completeness, $\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}}$, then there should exist a matrix S that relates the in-states to the out-states,

$$|\text{out}\rangle = S|\text{in}\rangle \tag{6.10}$$

The equivalence of the two Hilbert spaces requires that S be unitary. This may also be seen from the fact that for a given initial state, a summation over all possible final states must give unit probability. Thus, the S-matrix must be unitary ! Consequently, we may describe all these overlap probabilities in terms of $|\text{in}\rangle$ or $|\text{out}\rangle$ states and the S-matrix.

6.0.3 Scattering cross sections

From the probability, one can compute the thing the experimental high energy physicist talks about all day and dreams about: the scattering cross section. We shall limit ourselves to an initial state with only 2 particles.

The incoming state is now in general a *wave packet* with distribution ρ, so

$$|\text{in}\rangle = \int \frac{d^3p_1}{(2\pi)^3 2p_1^o} \int \frac{d^3p_2}{(2\pi)^3 2p_2^o} \rho_1(p_1)\rho_2(p_2)|\text{in } p_1, p_2\rangle \tag{6.11}$$

This wave packet is associated with a distribution $\tilde{\rho}(x)$ (for a free particle) and positive energy

$$\tilde{\rho}(x) = \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-ip\cdot x}\rho(p) \tag{6.12}$$

The flux is given by

$$\Phi_\rho \equiv \int \frac{d^3p}{(2\pi)^3 2p^o}|\rho(p)|^2 = \frac{\#\text{ of particles}}{\text{unit of time}} \tag{6.13}$$

To isolate the interaction part from the free part, it is useful to decompose the S-matrix into an identity part and a T matrix part, describing interaction

$$S = I + iT \tag{6.14}$$

Energy momentum conservation then implies that

$$\langle f|T|p_1, p_2\rangle = (2\pi)^4\delta^4(P_f - p_1 - p_2)\langle f|\mathcal{T}|p_1, p_2\rangle \tag{6.15}$$

In general, one will be interested in scattering cross sections for which $\langle f|i\rangle = 0$. (If $|i\rangle = |f\rangle$, one deals with so-called *forward* scattering, and we shall not deal with it here.) In the above case

$$\langle f|S|i\rangle = \langle f|i\rangle - i\langle f|T|i\rangle = -i\langle f|T|i\rangle \tag{6.16}$$

and

$$W_{f\leftarrow i} = |\langle f|S|i\rangle|^2 = |\langle f|T|i\rangle|^2 \tag{6.17}$$

$$\begin{aligned} W_{f\leftarrow i} &= \int dp_1 \int dp_2 \int dp_1' \int dp_2'\ \rho_1^*(p_1)\rho_1(p_1')\rho_2^*(p_2)\rho_2(p_2')(2\pi)^4\delta(p_1 + p_2 - P_f) \\ &\quad \times (2\pi)^4\delta(p_1 + p_2 - p_1' - p_2')\langle f|\mathcal{T}|p_1, p_2\rangle^*\langle f|\mathcal{T}|p_1', p_2'\rangle \end{aligned} \tag{6.18}$$

Using now the fact that

$$(2\pi)^4\delta^4(k) = \int d^4x e^{-ikx} \tag{6.19}$$

we may represent the second momentum conservation δ-function as a Fourrier integral,

$$\begin{aligned} W_{f\leftarrow i} &= \int d^4x \int dp_1 \int dp_2 dp_1' \int dp_2' \rho_1^*(p_1)\rho_1(p_1')\rho_2^*(p_2)\rho_2(p_2') \\ &\quad \times (2\pi)^4\delta^4(p_1 + p_2 - P_f)e^{-ip_1+p_2-p_1'-p_2')\cdot x}\langle f|\mathcal{T}|p_1, p_2\rangle^*\langle f|\tau|p_1', p_2'\rangle \end{aligned} \tag{6.20}$$

Now we assume that the distributions ρ_1 and ρ_2 are *very narrow*, and sharply peaked about momenta $\bar{p}_1$ and $\bar{p}_2$. We assume that the variation of the ρ's is so small over this distribution that the matrix elements of $\mathcal{T}$ are essentially constant over that variation. We then have

$$\langle f|\mathcal{T}|p_1,p_2\rangle \sim \langle f|\mathcal{T}|p_1',p_1'\rangle \simeq \langle f|\mathcal{T}|\bar{p}_1,\bar{p}_2\rangle \tag{6.21}$$

Hence

$$W_{f\leftarrow i} = \int d^4x|\tilde{\rho}_1(x)|^2|\tilde{\rho}_2(x)|^2(2\pi)^4\delta^4(\bar{p}_1+\bar{p}_2-P_f)|\langle f|\mathcal{T}|\bar{p}_1,\bar{p}_2\rangle|^2 \tag{6.22}$$

Now to get the cross section, we want the transition probability per unit volume and per unit time:

$$\frac{dW_{p\leftarrow i}}{dtdV} = |\tilde{\rho}_1(x)|^2|\tilde{\rho}_2(x)|^2(2\pi)^4\delta^4(\bar{p}_1+\bar{p}_2-P_f)|\langle f|\mathcal{T}|\bar{p}_1,\bar{p}_2\rangle|^2 \tag{6.23}$$

Now the relative flux between distributions ρ_1 and ρ_2 is given by

$$4|m_2\gamma_2p_1-m_1\gamma_1p_2||\tilde{f}_1(x)|^2|\tilde{f}_2(x)|^2 \tag{6.24}$$

and therefore, the differential cross section is given by

$$\boxed{d\sigma = (2\pi)^4\delta^4(P_f-\bar{p}_1-\bar{p}_2)\frac{1}{2E_12E_2|\vec{v}_1-\vec{v}_2|}|\langle f|\mathcal{T}|\bar{p}_1,\bar{p}_2\rangle|^2} \tag{6.25}$$

The normalization holds for scalar, photons and spin 1/2 particles all the same!

6.1 Relating the interacting and in- and out-fields

The quantum field theory is formulated in terms of interacting fields $\phi(x)$ instead of the free fields $\phi_{\rm in}(x)$ and $\phi_{\rm out}(x)$. To compute elements of the S matrix, we need to connect the states $|{\rm in}\rangle$ and $|{\rm out}\rangle$ to the interacting fields in the theory. We carry this out here for the case of a single scalar field ϕ. Other cases are completely analogous.

The in and out-fields satify the free field equations

$$\begin{aligned}(\Box+m^2)\phi_{\rm in}(x) &= 0\\ (\Box+m^2)\phi_{\rm out}(x) &= 0\end{aligned} \tag{6.26}$$

as well as the canonical commutation relations

$$[\dot{\phi}_{\rm in}(x),\ \phi_{\rm in}(y)]\delta(x^0-y^0) = -i\ \delta(x-y)$$

$$[\dot\phi_{\rm out}(x),\ \phi_{\rm out}(y)]\delta(x^0-y^0) \ = \ -i\ \delta(x-y) \tag{6.27}$$

The in and out states are now built with the help of the Fourier mode operators:

$$\begin{aligned} |{\rm in}\ p_1,\cdots,p_n\rangle &= a^\dagger_{\rm in}(\vec p_1)a^\dagger_{\rm in}(\vec p_2)\cdots a^\dagger_{\rm in}(p_n)|0\rangle \\ |{\rm out}\ q_1,\cdots,q_m\rangle &= a^\dagger_{\rm out}(\vec q_1)a^\dagger_{\rm out}(\vec q_2)\cdots a^\dagger_{\rm out}(q_m)|0\rangle \end{aligned} \tag{6.28}$$

Recall the normalizations of the free in and out-fields with the 1-particle states,

$$\begin{aligned} \langle{\rm in}\ p|\phi_{\rm in}(x)|0\rangle &= e^{ip\cdot x} \\ \langle{\rm out}\ q|\phi_{\rm out}(x)|0\rangle &= e^{iq\cdot x} \end{aligned} \tag{6.29}$$

Note that the normalization on the rhs is 1.

As remarked above, the dynammics of the theory is described by neither $\phi_{\rm in}$ nor $\phi_{\rm out}$, but rather by the interacting field ϕ. In some sense, the field ϕ should behave like $\phi_{\rm in}$ or $\phi_{\rm out}$ according to whether $t\to-\infty$ or $t\to+\infty$.

To see concretely how this happens, consider the overlap of the field ϕ with the 1-particle states. Accoding to general principles, the field ϕ should have non-zero overlap with $|{\rm in}\ p\rangle$. The x- and p-dependence of this overlap is determined by Poincaré symmetry in the following manner,

$$\begin{aligned} \langle{\rm in}\ p|\phi(x)|0\rangle = \langle{\rm in}\ p|e^{ix\cdot P}\phi(0)e^{-ix\cdot P}|0\rangle &= e^{ip\cdot x}\langle{\rm in}\ p|\phi(0)|0\rangle \\ &= \sqrt{Z}e^{ip\cdot x} \end{aligned} \tag{6.30}$$

Z is indepedent of p. We therefore conclude that

$$\begin{aligned} \langle{\rm in}\ p|\phi(x)|0\rangle &= \sqrt{Z}\langle{\rm in}\ p|\phi_{\rm in}(x)|0\rangle \\ \langle{\rm out}\ p|\phi(x)|0\rangle &= \sqrt{Z}\langle{\rm out}\ p|\phi_{\rm in}(x)|0\rangle \end{aligned} \tag{6.31}$$

Therefore, we have – in a somewhat formal sense – the following limits

$$\begin{aligned} \lim_{t\to-\infty}\phi(x) &= \sqrt{Z}\phi_{\rm in}(x) \\ \lim_{t\to+\infty}\phi(x) &= \sqrt{Z}\phi_{\rm out}(x) \end{aligned} \tag{6.32}$$

This limit cannot really be understood in the sense of a limit of operators, but rather must be viewed as a limit that will hold when matrix elements are taken with states containing a finite number of particles.

6.2 The Källen-Lehmann spectral representation

To evaluate Z, we construct the Källen-Lehmann representation for the interacting theory. The starting point is the commutator,

$$i\ \Delta(x,x') = \langle 0|[\phi(x),\ \phi(x')]|0\rangle \tag{6.33}$$

Let us collectively denote all the states of the theory $|n\rangle$. Then, using completeness of the states $|n\rangle$, we insert a complete set of normalized states,

$$i\ \Delta(x,x') = \sum_n \Big(\langle 0|\phi(x)|n\rangle\langle n|\phi(x')|0\rangle - \langle 0|\phi(x')|n\rangle\langle n|\phi(x)|0\rangle \Big) \tag{6.34}$$

Translation symmetry of the theory allows us to pull out the x-dependence using $\phi(x) = e^{ix\cdot P}\phi(0)e^{-ix\cdot P}$ so that

$$i\ \Delta(x,x') = \sum_n |\langle 0|\phi(0)|n\rangle|^2 \Big(e^{-ip_n\cdot(x-x')} - e^{ip_n\cdot(x-x')} \Big) \tag{6.35}$$

Using $1 = \int d^4q\delta(q-p_n)$, we have

$$i\ \Delta'(x,x') = \int \frac{d^4q}{(2\pi)^3}\rho(q)\Big(e^{-iq(x-x')} - e^{iq(x-x')} \Big) \tag{6.36}$$

where the *spectral density* is defined to be

$$\rho(q) \equiv (2\pi)^3 \sum_n \delta^4(q-p_n)|\langle 0|\phi(0)|n\rangle|^2 \tag{6.37}$$

Notice that ρ is positive, and vanishes when either $q^2 < 0$ or $q^0 < 0$. Also it is Lorentz invariant, and therefore can only depend upon q^2. Thus, it is convenient to introduce the function $\sigma(q^2)$ as follows,

$$\rho(q) = \theta(q^0)\sigma(q^2) \qquad \sigma(q^2) = 0 \quad \text{if} \quad q^2 < 0 \tag{6.38}$$

Recall the free commutator,

$$i\ \Delta_0(x-y,m^2) = \int \frac{d^4q}{(2\pi)^3}\theta(q^0)\delta(q^2-m^2)\Big(e^{-iq(x-y)} - e^{iq(x-y)} \Big) \tag{6.39}$$

Hence we may express the commutator of the fully interacting fields as,

$$\Delta(x-x') = \int_0^\infty d\mu^2\sigma(\mu^2)\Delta_0(x-x';\mu^2) \tag{6.40}$$

This is the Källén-Lehmann representation for the commutator.

Figure 6: Support of the spectral function $\sigma(\mu^2)$

The contribution to the spectral density due to the one particle state is given in terms of the constant Z. The remaining contributions come from states with more than 1 particle, and may be summarized as follows,

$$\Delta(x-x') = Z\Delta_0(x-x', m^2) + \int_{m_t^2}^{\infty} d\mu^2 \sigma(\mu^2)\Delta_0(x-x';\mu^2) \tag{6.41}$$

Here, m_t^2 is the mass-squared at which the remaining spectrum starts.

If the interactions in the theory are very weak, the particles will behave almost as free particles. If we further assume that no bound states occur in the full spectrum (but even at weak coupling this assumption may be false, see the hydrogen atom), then we can give a general characterization of the spectral function and the nature of the thresholds of $\sigma(\mu^2)$. A state with n (nearly) free particles starts contributing to $\sigma(\mu^2)$ provided

$$\begin{aligned} \mu^2 &= (p_1 + p_2 \ldots + p_n)^2 \\ &= (p_1^o + p_2^o + \ldots + p_n^o)^2 - (\vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n)^2 \geq n^2 m^2 \end{aligned} \tag{6.42}$$

Thus, under the assumptions made here, there will be successive thresholds starting at $4m^2$, $9m^2$, $16m^2$ etc. In most theories however, the spectrum may be much more complicated.

To obtain Z, it sufficies to take the x^0 derivative at both sides and set $x^0 = x'^0$. This produces the canonical commutators, as follows,

$$\delta^3(\vec{x}-\vec{x}') = Z\ \delta^3(\vec{x}-\vec{x}') + \int_{m_t^2}^{\infty} d\mu^2 \sigma(\mu^2)\delta^3(\vec{x}-\vec{x}') \tag{6.43}$$

and identifying the coefficients of $\delta(\vec{x}-\vec{x}')$, we get

$$1 = Z + \int_{m_t^2}^{\infty} d\mu^2 \sigma(\mu^2) \tag{6.44}$$

Since $\sigma(\mu^2) \geq 0$, any interacting theory will have

$$0 < Z < 1 \tag{6.45}$$

In particular, $Z = 1$ is possible only when $\phi = \phi_{\rm in}$ is a free field. On the other hand, $Z = 0$ would mean that the field ϕ does not create the 1-particle state, which is in contradiction with one of our basic assumptions.

One may also obtain a Källen-Lehmann representation for the two-point function in terms of similar methods,

$$\langle 0|T\phi(x)\phi(x')|0\rangle = \int_0^\infty d\mu^2 \sigma(\mu^2) G_F(x-y,\mu^2) \tag{6.46}$$

Notice that the field ϕ which governs the dynammics and which obeys canonical commutation relations is **not** the field that obeys a canonical normalization on one-particle states. There is a multiplicative factor between the two fields. This is the first non-trivial example of the phenomenon of renormalization: the parameter that govern the dynamics in the theory like the field, the mass, the coupling constants are *not* the parameters that are observable in the asymptotic regime. Later on it will be convenient to work directly with this renormalized field

$$\phi_R(x) \equiv \frac{1}{\sqrt{Z}}\, \phi(x) \tag{6.47}$$

Of course the field ϕ_R will no longer satisfy canonical commutation relations.

6.3 The Dirac Field

A similar treatment is available for the Dirac field $\psi(x)$. We restrict to quoting the results here. The normalizations of the in and out fields and of the interacting field are as follows,

$$\begin{aligned} \langle 0|\psi_{\rm in}(x)|\text{in } p,\lambda\rangle &= u(p,\lambda)e^{-ip\cdot x} \\ \langle 0|\psi(x)|\text{in}|p,\lambda\rangle &= \sqrt{Z_2}\langle 0|\psi_{\rm in}(x)\text{in } p,\lambda\rangle \\ &= \sqrt{Z_2}\langle 0|\psi_{\rm out}(x)|\text{out } p,\lambda\rangle \end{aligned} \tag{6.48}$$

The Källen-Lehmann representation is obtained by introducing the spectral density

$$\rho_\alpha^\beta(q) = (2\pi)^3 \sum_n \delta^4(p_n - q)\langle 0|\psi_\alpha(0)|n\rangle\langle\bar\psi^\beta(0)|0\rangle \tag{6.49}$$

so that, using Lorentz invariance, one gets

$$\rho_\alpha^\beta(q) = \theta(q^0)\Big(\sigma_1(q^2)(\not q)_\alpha{}^\beta + \sigma_2(q^2)\delta_\alpha{}^\beta\Big) \tag{6.50}$$

The result is most easily expressed in terms of the free scalar propagator,

$$\langle 0\{\psi_\alpha(x)\bar\psi^\beta(x')\}|0\rangle = \int_0^\infty d\mu^2\Big(i\sigma_1(\mu^2)\ \not\partial_x + \sigma_2(\mu^2)\Big)_\alpha{}^\beta G_F(x-x';\mu^2) \tag{6.51}$$

Properties of σ_1 and σ_2 are as follows, $\sigma_1(\mu^2)\geq 0$, $\sigma_2(\mu^2)\geq 0$, and $\mu\sigma_1(\mu^2)-\sigma_2(\mu^2)\geq 0$. The spectral representation for Z_2 is obtained again via the equal time commutators,

$$i\langle 0|\{\psi_\alpha(x)\bar\psi^\beta(x')\}|0\rangle = Z_2 S(x-x') + \int_{m_t^2}^\infty d\mu^2\Big(i\sigma_1(\mu^2)\ \not\partial_x + \sigma_2(\mu^2)\Big)_\alpha{}^\beta G_F(x-x';\mu^2)$$

and we get

$$1 = Z_2 + \int_{4m^2}^\infty d\mu^2 \sigma_1(\mu^2) \tag{6.52}$$

6.4 The LSZ Reduction Formalism

It is now our task to evaluate the S-matrix elements in terms of quantities which can be expressed in terms of the dynammics of the QFT. We shall treat in detail here the case of the real scalar field; the other cases may be handled analogously. The key object of interest is the transition amplitude

$$\langle \text{out } q_1 \cdots q_m | \text{in } p_1 \cdots p_n \rangle \tag{6.53}$$

We shall work out a recursion formula which decreases the number of particles in the *in* and *out* states and replaces the overlap with expectation values of the fully interacting quantum field ϕ. To this end, we isolate the in-particle with momentum p_n, and write

$$\begin{aligned}
\langle \text{out } q_1 \cdots q_m | \text{in } p_1 \cdots p_n \rangle &= \langle \text{out } \beta | \text{in } \alpha \; p_n \rangle \\
|\text{in } \alpha\rangle &= |\text{in } p_1 \cdots p_{n-1}\rangle \\
|\text{out } \beta\rangle &= |\text{out } q_1 \cdots q_m\rangle \\
|\text{in } p_1 \cdots p_n\rangle &= a^\dagger_{\text{in}}(p_n) |\text{in } \alpha\rangle
\end{aligned} \tag{6.54}$$

6.4.1 In- and Out-Operators in terms of the free field

The decomposition into creation and annihilation operators of a free scalar field $\varphi(x)$ with mass m and its canonical momentum $\partial_0 \phi(t, \vec{x})$ are given in terms by

$$\begin{aligned}
\varphi(t, \vec{x}) &= \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \Big(a(\vec{k}) e^{-k \cdot x} + a^\dagger(\vec{k}) e^{ik \cdot x} \Big) \\
\partial_0 \varphi(t, \vec{x}) &= \int \frac{d^3 k}{(2\pi)^3 2} \Big(-ia(\vec{k}) e^{-ik \cdot x} + ia^\dagger(\vec{k}) e^{ik \cdot x} \Big)
\end{aligned} \tag{6.55}$$

The creation and annihilation operators may be recovered by by taking the space Fourier transforms

$$\begin{aligned}
a^\dagger(\vec{k}) &= -i \int d^3 x \Big(e^{-ik \cdot x} \overleftrightarrow{\partial_0} \varphi(x) \Big) \\
a(\vec{k}) &= +i \int d^3 x \Big(e^{+ik \cdot x} \overleftrightarrow{\partial_0} \varphi(x) \Big)
\end{aligned} \tag{6.56}$$

Here, the symbol $\overleftrightarrow{\partial}$ stands for the antisymmetrixed derivative $u \overleftrightarrow{\partial_0} v \equiv u \partial_0 v - \partial_0 u v$. The left hand side is independent of the time coordinate x^0 at which the field $\varphi(x)$ is evaluated on the left hand side as long as the momentum satisfies $k^2 = m^2$.

6.4.2 Asymptotics of the interacting field

Next, we make use of the fact that, when evaluated between states, where it is to play the role of creation operator, the field tends towards free field expectation values,

$$\begin{aligned}
\lim_{x^0\to+\infty} \langle \text{out}|\phi(x)|\text{in}\rangle &\sim Z^{-\frac{1}{2}}\langle \text{out}|\phi_{\text{out}}(x)|\text{in}\rangle \\
\lim_{x^0\to-\infty} \langle \text{out}|\phi(x)|\text{in}\rangle &\sim Z^{-\frac{1}{2}}\langle \text{out}|\phi_{\text{in}}(x)|\text{in}\rangle
\end{aligned} \tag{6.57}$$

Combining this result with the free field formulas for the creation operators for in- and out-operators, we obtain (omitting now the in- and out-states matrix elements),

$$\begin{aligned}
a_{\text{in}}^\dagger(\vec{k}) &= -\frac{i}{\sqrt{Z}} \lim_{x^0\to-\infty} \int d^3x \Big(e^{-ik\cdot x} \overleftrightarrow{\partial_0}\, \phi(x)\Big) \\
a_{\text{out}}^\dagger(\vec{k}) &= -\frac{i}{\sqrt{Z}} \lim_{x^0\to+\infty} \int d^3x \Big(e^{-ik\cdot x} \overleftrightarrow{\partial_0}\, \phi(x)\Big) \\
a_{\text{in}}(\vec{k}) &= +\frac{i}{\sqrt{Z}} \lim_{x^0\to-\infty} \int d^3x \Big(e^{+ik\cdot x} \overleftrightarrow{\partial_0}\, \phi(x)\Big) \\
a_{\text{out}}(\vec{k}) &= +\frac{i}{\sqrt{Z}} \lim_{x^0\to+\infty} \int d^3x \Big(e^{+ik\cdot x} \overleftrightarrow{\partial_0}\, \phi(x)\Big)
\end{aligned} \tag{6.58}$$

Using the fact that the difference between the same quantity evaluated at $x^0 \to \pm\infty$ may be expressed as an integral,

$$\lim_{x^0\to+\infty} \psi(x) - \lim_{x^0\to-\infty} \psi(x) = \int_{-\infty}^{+\infty} dx^0 \partial_0 \psi(x) \tag{6.59}$$

we obtain the following difference formulas between in- and out-operators,

$$\begin{aligned}
a_{\text{in}}^\dagger(\vec{k}) - a_{\text{out}}^\dagger(\vec{k}) &= +\frac{i}{\sqrt{Z}} \int d^4x \partial_0 \left(e^{-ik\cdot x} \overleftrightarrow{\partial_0}\, \phi(x)\right) \\
a_{\text{in}}(\vec{k}) - a_{\text{out}}(\vec{k}) &= -\frac{i}{\sqrt{Z}} \int d^4x \partial_0 \left(e^{+ik\cdot x} \overleftrightarrow{\partial_0}\, \phi(x)\right)
\end{aligned} \tag{6.60}$$

These combinations admit further simplifications which may be seen by working out the integrand and using the fact that $k^2 = m^2$,

$$\begin{aligned}
\int d^4x \partial_0 \left(e^{\pm ik\cdot x} \overleftrightarrow{\partial_0}\, \phi(x)\right) &= \int d^4x \left((k^0)^2 \phi(x) + \partial_0^2 \phi(x)\right) e^{\pm ik\cdot x} \\
&= \int d^4x e^{\pm ik\cdot x} (\Box + m^2)\phi(x)
\end{aligned} \tag{6.61}$$

Thus, we obtain the following formulas,

$$\begin{aligned}
a_{\text{in}}^\dagger(\vec{k}) - a_{\text{out}}^\dagger(\vec{k}) &= +\frac{i}{\sqrt{Z}} \int d^4x \; e^{-ik\cdot x} (\Box + m^2)\phi(x) \\
a_{\text{in}}(\vec{k}) - a_{\text{out}}(\vec{k}) &= -\frac{i}{\sqrt{Z}} \int d^4x \; e^{+ik\cdot x} (\Box + m^2)\phi(x)
\end{aligned} \tag{6.62}$$

Clearly, for a free field, satisfying $(\Box + m^2)\phi = 0$, there is no difference between the in- and out-operators, but as soon as interactions are turned on, they are no longer equal.

6.4.3 The reduction formulas

The next step is the evaluation of the operator difference (6.62) between the states $|\text{in } \alpha\rangle = |\text{in } p_1 \cdots p_{n-1}\rangle$ and $|\text{out } \beta\rangle = |\text{out } q_1 \cdots q_m\rangle$,

$$\langle \text{out } \beta| \left(a^\dagger_{\text{in}}(\vec{q}_m) - a^\dagger_{\text{out}}(\vec{q}_m) \right) |\text{in } \alpha\rangle = \frac{i}{\sqrt{Z}} \int d^4x \; e^{-iq_m \cdot x} (\Box + m^2) \langle \text{out } \beta|\phi(x)|\text{in } \alpha\rangle \tag{6.63}$$

Thus, we have

$$\begin{aligned} &\langle \text{out } p_1 \cdots p_n|\text{in } q_1 \cdots q_m\rangle \\ &\quad = \langle \text{out } p_1 \cdots p_n|a^\dagger_{\text{out}}(\vec{q}_m)|\text{in } q_1 \cdots q_{m-1}\rangle \\ &\quad\quad + \frac{i}{\sqrt{Z}} \int d^4x \; e^{-iq_m \cdot x} (\Box + m^2) \langle \text{out } p_1 \cdots p_n|\phi(x)|\text{in } q_1 \cdots q_{m-1}\rangle \end{aligned} \tag{6.64}$$

The first term on the rhs vanishes, unless it happens to be that $\langle \text{out } p_1 \cdots p_n|$ contain precisely the particle with momentum $\vec{q}_m$, in which case the amplitude contributes with strength unity. Therefore, this term corresponds to a disconnected contribution to the scattering process (see Fig.7). The disconnected contributions simply reduce to S-matrix elements with fewer particles and may be handled recursively.

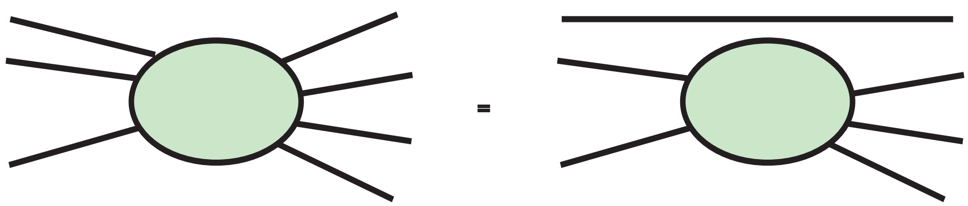


Figure 7: Contribution to the S-matrix with disconnected part.

Henceforth, we concentrate on the connected contributions, denoted with the corresponding suffix. The reduction formula for peeling off a single particle thus reduces to

$$\begin{aligned} &\langle \text{out } p_1 \cdots p_n|\text{in } q_1 \cdots q_m\rangle_{\text{conn}} \\ &\quad = \frac{i}{\sqrt{Z}} \int d^4x \; e^{-iq_m \cdot x} (\Box + m^2) \langle \text{out } p_1 \cdots p_n|\phi(x)|\text{in } q_1 \cdots q_{m-1}\rangle \end{aligned} \tag{6.65}$$

Clearly, the process may be repeated on all incoming particles, until all incoming particles have been removed,

$$\begin{aligned} &\langle \text{out } p_1 \cdots p_n|\text{in } q_1 \cdots q_m\rangle_{\text{conn}} \\ &\quad = \left(\frac{i}{\sqrt{Z}} \right)^m \prod_{j=1}^m \left(\int d^4y_j \; e^{-iq_j \cdot y_j} (\Box + m^2)_{y_j} \right) \langle \text{out } p_1 \cdots p_n|\phi(y_1) \cdots \phi(y_m)|\text{in } 0\rangle \end{aligned} \tag{6.66}$$

where the state $|\text{in } 0\rangle$ stands for the in-state with zero in-particles.

It remains to remove also the out-particles. However, now there is a new issue to be dealt with. The removal of the out particles must always occur to the left of the removal of the in-particles, since the two operator types will not have simple commutation relations. This ordering is achieved by ordering the later times to the left of the earlier times, since out-states correspond to $x^0 \to +\infty$ while in-states correspond to $x^0 \to -\infty$. Since out-states will have to be removed using annihilation operators, there is also an adjoint to be taken. Thus, the final formula is

$$\begin{aligned}
&\langle \text{out } p_1 \cdots p_n | \text{in } q_1 \cdots q_m \rangle_{\text{conn}} \\
&\quad = \frac{i^m(-i)^n}{\sqrt{Z}^{m+n}} \prod_{k=1}^{n} \left(\int d^4x_k \; e^{+ip_k \cdot x_k} (\Box + m^2)_{x_k} \right) \\
&\qquad \times \prod_{j=1}^{m} \left(\int d^4y_j \; e^{-iq_j \cdot y_j} (\Box + m^2)_{y_j} \right) \langle \text{out } 0 | T\phi(x_1) \cdots \phi(x_n)\phi(y_1) \cdots \phi(y_m) | \text{in } 0 \rangle
\end{aligned} \tag{6.67}$$

The key lesson to be learned from this formula is that the calculation of any scattering matrix element requires the calculation in QFT of only a single type of object, namely the time ordered correlation function, or *correlator*,

$$\langle 0 | T\phi(x_1) \cdots \phi(x_n) | 0 \rangle \tag{6.68}$$

Much of the remainder of the subject of quantum field theory will be concerned with the calculation of these quantities.

6.4.4 The Dirac Field

The derivation for the Dirac field is completely analogous, though the precise 1-particle wave functions are of course those associated with the Dirac equation instead of with the Klein-Gordon equation. Furthermore, one will have to be careful about taking the anti-commutative nature of the field ψ into account. For example, for in-fields we have

$$\psi_{\text{in}}(x) \;\; = \;\; \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda=\pm} \left(b_{\text{in}}(k,\lambda) u(k,\lambda) \; e^{-ik\cdot x} + d^\dagger_{\text{in}}(k,\lambda) v(k,\lambda) \; e^{ik\cdot x} \right) \tag{6.69}$$

which gives rise to the following expressions for the in- and out-operators,

$$\begin{aligned}
b_{\text{in}}(k,\lambda) &= \int d^3x \bar{u}(k,\lambda) \; e^{ik\cdot x} \gamma^0 \psi_{\text{in}}(x) \\
d^\dagger_{\text{in}}(k,\lambda) &= \int d^3x \bar{v}(k,\lambda) \; e^{-ik\cdot x} \gamma^0 \psi_{\text{in}}(x) \\
b^\dagger_{\text{in}}(k,\lambda) &= \int d^3x \bar{\psi}(x) \gamma^0 \; e^{-ik\cdot x} u(k,\lambda) \\
d_{\text{in}}(k,\lambda) &= \int d^3x \bar{\psi}(x) \gamma^0 \; e^{-ik\cdot x} v(k,\lambda)
\end{aligned} \tag{6.70}$$

The 1-particle state normalizations with the in-field ψ_{in} and with the fully interacting field ψ are as follows,

$$
\begin{aligned}
\langle 0|\psi_{\text{in}}(x)|\text{in } p\ \lambda\rangle &= u(p,\lambda)e^{-ip\cdot x} \\
\langle 0|\psi(x)|\text{in } p\ \lambda\rangle &= \sqrt{Z_2}u(p,\lambda)e^{-ip\cdot x}
\end{aligned}
\tag{6.71}
$$

You see that one can get a similar formula as for the bosonic scalar case, but now you have to be careful about where the spinor indices are contracted. Clearly, everything is again expressed in terms of time-ordered correlators for the fully interacting fermion fields,

$$
\langle 0|T\psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2)\ldots\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)\ldots\bar{\psi}_{\beta_p}(y_p)|0\rangle \tag{6.72}
$$

6.4.5 The Photon Field

Since the lontgitudinal photons are free, we are only interested in scattering transverse photons. $k_\mu\epsilon^\mu(k) = 0$

$$
\begin{aligned}
& \langle\beta(k_1\epsilon)\text{out}|\alpha\text{ in}\rangle = \beta\text{ out}|\alpha-(k\epsilon)\text{ in}\rangle \\
& +\langle\beta\text{ out}|\epsilon\cdot a^{\text{out}}(k)-\epsilon\cdot a^{\text{out}}(k)-\epsilon\cdot a^{\text{in}}(k)|\alpha\text{ in}\rangle \\
= \quad & -i\int_t d^3x e^{ikx}\overset{\leftrightarrow}{\partial}_o\langle\beta\text{ out}|\epsilon(k)\cdot A^{T\text{ out}}(x)-\epsilon(k)A^{R\text{ in}}(x)|\alpha\text{ in}\rangle \\
= \quad & -i\ Z_3^{1/2}\int_{t\to\infty} d^3x e^{ikx}\overset{\leftrightarrow}{\partial}_o\langle\beta\text{ out}|\epsilon(k)\cdot A^T(x)|\alpha\text{ in}\rangle \\
& +i\ Z_3^{-1/2}\int_{t\to-\infty} d^3x e^{ikx}\overset{\leftrightarrow}{\partial}_o\langle\beta\text{ out}|\epsilon(k)A^T(x)|\alpha\text{ in}\rangle \\
= \quad & -i\ Z_3^{-1/2}\int d^4x\left\{e^{ikx}\partial_o^2\langle\beta\text{ out}|\epsilon(k)\cdot A^T(x)|\alpha\text{ in}\rangle\right. \\
& \left.-(\partial_o^2e^{ikx})\langle\beta\text{ out}|\epsilon(k)\cdot A^T(x)|\alpha\text{ in}\rangle\right\} \\
= \quad & -i\ Z_3^{-1/2}\int d^4x\left\{e^{ikx}(\Box+\mu^2)_x\langle\beta\text{ out}|\epsilon(k)\cdot A^T(x)\alpha\text{ in}\rangle\right\}
\end{aligned}
\tag{6.73}
$$

Now:

$$
A_\mu^T(x) = A_\mu(x) + \frac{1}{m^2}\partial_\mu\partial\cdot A \tag{6.74}
$$

$$
\begin{aligned}
(\Box+\mu^2)_x A_\mu^T(x) &= (\Box+\mu^2)A_\mu + \frac{1}{m^2}\Box\partial_\mu(\partial\cdot A) + \frac{\mu^2}{m^2}\partial_\mu\partial\cdot A \\
&= (\Box+\mu^2)A_\mu - \partial_\mu(\partial\cdot A) + \lambda\partial_\mu(\partial\cdot A) \\
&= j_\mu\ , \qquad \text{our best friend!}
\end{aligned}
\tag{6.75}
$$

Hence, in the non-forward regime, we have the simple formula:

$$
\langle\beta(k,\epsilon)\text{ out}|\alpha\text{ in}\rangle == i\ Z_3^{1/2}\int d^4x e^{ikx}\langle\beta\text{ out}|\epsilon^\mu(k)j_\mu(x)|\alpha\text{ in}\rangle \tag{6.76}
$$

Now let's play this game over with two photons:

$$
\langle\beta(k_f,\epsilon_f)\text{out}|\alpha(k_i,\epsilon_i)\text{in}\rangle
$$

$$
\begin{aligned}
&= -Z_3^{-1}\int d^4x\int d^4y e^{i(k_f x - k_i y)}(\Box+\mu^2)_y \\
&\qquad \langle\beta \text{ out}|T\epsilon_f\cdot j(x)\left[\epsilon_i\cdot(A(y))+\frac{1}{m^2}\partial(\partial A)(y)\right]|\alpha \text{ in}\rangle
\end{aligned}
\tag{6.77}
$$

You see that this is *not just* the time ordered product of the currents, there will be some additional terms, arising however only at $x^o = y^o$. By locality, the contribution must now be entirely localized at $\vec{x} = \vec{y} = \vec{y}$ as well. One can show, after a lot of work that one exactly gets the T^* product. What else could it be?

$$
\begin{aligned}
& \langle\beta(k_f,\epsilon_f) \text{ out}|\alpha(k_i,\epsilon_i) \text{ in}\rangle \\
= \; & -Z_3^{-1}\int d^4x\int d^4y e^{i(k_f x-k_i y)} \\
& \langle\beta \text{ out}|T^*\epsilon_f\cdot j(x)\epsilon_i\cdot j(y)|\alpha \text{ in}\rangle
\end{aligned}
\tag{6.78}
$$

It is straightforward to generalize this formula.

6.5 Several Scalar Fields

Suppose we are interested in (not the most general case)

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\phi^a(M^2)_z{}^b\phi_b \qquad a,b=1,2,\ldots,N \tag{6.79}
$$

$$
\pi^a = \partial_o\phi^a \qquad [\pi^a(t,\vec{x}),\;\; \phi^b(t,\vec{x}')] = -i\delta^{ab}\delta^3(\vec{x}-\vec{x}') \tag{6.80}
$$

Since M^2 is symmetric and real, it may be diagonalized by an orthogonal transformation, which leaves the commutation relations unchanged

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial_\mu\phi^a - \frac{1}{2,{}_a^2\,\phi^a\phi^a} \tag{6.81}
$$

$$
= \text{completely decoupled scalar fields, so follow previous reasoning} \tag{6.82}
$$

One case is of special interest: Some of the m_a's are equal. Without loss of generality we can assume them all equal.

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}m^2\phi^a\phi^a \tag{6.83}
$$

This system is now invariant under "rotations" in the internal space labelled by the index "a"

$$
\delta\phi^a = \epsilon^a{}_b\phi^b \tag{6.84}
$$

$$
j^\mu = \partial^\mu\phi^a\epsilon^{ab}\phi^b \tag{6.85}
$$

$$
\tag{6.86}
$$

$$
\begin{aligned}
\delta x^\mu &= 0 \qquad \Rightarrow \delta\mathcal{L} = 0 & (6.87)\\
\partial_\mu j^\mu &= \Box\phi^a\epsilon^{ab}\phi^b & (6.88)\\
&= -m^2\phi^a\epsilon^{ab}\phi^b = 0 & (6.89)
\end{aligned}
$$

$$
Q = \int d^3x\epsilon^{ab} : \pi^a\phi^b := \int d^3x\epsilon^{ab}\pi^a\phi^b \tag{6.90}
$$

One has an *internal* symmetry $O(N)$.

Another case of interest is the complex scalar field for $N = 2$

$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi^1\partial^\mu\phi^q + \frac{1}{2}\partial_\mu\phi^2\partial^\mu\phi^2 - \frac{1}{2}m^2(\phi^1\phi^1 + \phi^2\phi^2) \tag{6.91}
$$

It is convenient to introduce $\phi = \frac{1}{sqrt2}(\phi^1 + i\phi^2)$

$$
\begin{aligned}
\mathcal{L} = \partial_\mu\phi^+\partial_\mu\phi &- m^2\phi^+\phi \Rightarrow \pi = \frac{\partial\mathcal{L}}{\partial(\partial_o\phi)} = \partial^o\phi^+ \quad \text{etc.}\\
[\phi(t,\vec{x}),\ \phi(t,\vec{x}')] &= [\phi^+(t,\vec{x}),\ phi^+(t,\vec{x}')] = 0\\
[\phi(t,\vec{x}),\ \pi(t,\vec{x}')] &= [\phi^+(t,\vec{x}),\ \pi^+(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{y})\\
[\phi(t,\vec{x}),\ \pi^+(t,\vec{x}')] &= [\phi^+(t,\vec{x}),\ \pi(t,\vec{x}')] = 0 & (6.92)
\end{aligned}
$$

$O(2)$ symmetry is equivalent to $\phi \to e^{i\theta}\phi \sim U(1)$.

6.5.1 The energy momentum tensor for the Dirac field

Recall that under Lorentz transformatiions, we had

$$
\begin{aligned}
\psi_L'(x') &= \left(1 + \frac{i}{2}\vec{\sigma}(\vec{\omega} - i\vec{\nu})\right)\psi_L(x) & (6.93)\\
\psi_L'(x') &= \left(1 + \frac{i}{2}\vec{\sigma}(\vec{\omega} + i\vec{\nu})\right)\psi_R(x) & (6.94)
\end{aligned}
$$

This defines the spin matrix for the spinor fields $S_{\mu\nu}$:

$$
\psi'(x') = \left(1 + \frac{i}{2}\epsilon^{\mu\nu}S_{\mu\nu}\right)\psi(x) \tag{6.95}
$$

Identifying the two we find

$$
S_{ij} = \epsilon_{ijk}\begin{bmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{bmatrix} \qquad S_{oi} = -i\begin{bmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{bmatrix} \tag{6.96}
$$

which can be united in

$$\boxed{S_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]} \qquad \text{“the spinor rep. of } O(3,1)\text{”} \tag{6.97}$$

e.g.

$$\mathcal{L} = \bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi - m\bar{\psi}\psi \tag{6.98}$$

The Noether current for Lorentz transformations is

$$j^\mu = \epsilon_{-kappa\lambda} x^\lambda T^{\mu\kappa} - \frac{1}{2}\bar{\psi}\gamma^\mu \epsilon_{\kappa\lambda} S^{\kappa\lambda}\psi \tag{6.99}$$

Can we write it in the improved form? Modify $T^{\mu\nu}$ by a term that is trivially conserved:

$$T^{\mu\nu} = \theta^{\mu\nu} - \partial_\sigma B^{\sigma\mu\nu} \qquad B^{\sigma\mu\nu} = -B^{\mu\sigma\nu} = -B^{\nu\mu\sigma} \tag{6.100}$$

$$\begin{aligned}
j^\mu &= \frac{1}{2}\epsilon_{\kappa\lambda}(x^\lambda\theta^{\mu\kappa} - x^\kappa\theta^{\mu\lambda}) && (6.101)\\
&\quad + \frac{1}{2}\epsilon_{\kappa\lambda}(-x^\lambda\partial_\sigma B^{\sigma\mu\kappa} + x^\kappa\partial_\sigma B^{\sigma\mu\lambda}) - \frac{1}{2}\epsilon_{\kappa\lambda}\bar{\psi}\gamma^\mu S^{\kappa\lambda}\psi && (6.102)\\
&= \frac{1}{2}\epsilon_{\kappa\lambda}(x^\lambda\theta^{\mu\kappa} - x^\kappa\theta^{\mu\lambda}) && (6.103)\\
&\quad + \frac{1}{2}\epsilon_{\kappa\lambda}\partial_\sigma(-x^\lambda B^{\sigma\mu\kappa} + x^\kappa B^{\sigma\mu\lambda}) && (6.104)\\
&\quad + \frac{1}{2}\epsilon_{\kappa\lambda}(-B^{\lambda\mu\kappa} + B^{\kappa\mu\lambda} - \bar{\psi}\gamma^\mu S^{\kappa\lambda}\psi) && (6.105)
\end{aligned}$$

Hence we should require $B^{\kappa\mu\lambda} = \frac{1}{2}\bar{\psi}\gamma^\mu S^{\kappa\lambda}\psi$.

$$\boxed{\theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\sigma(\bar{\psi}\gamma^\mu S^{\sigma\nu}\psi} \qquad \text{Belinfante tensor} \tag{6.106}$$

Note that

$$j^\mu = \frac{1}{2}\epsilon_{\kappa\lambda}(x^\lambda\theta^{\mu\kappa} - x^\kappa\theta^{\mu\lambda}) + \frac{1}{2}\epsilon_{\kappa\lambda}\partial_\sigma(-x^\lambda)B^{\sigma\nu\kappa} + x^\kappa B^{\sigma\mu\lambda} \tag{6.107}$$

The last term is manifestly conserved $\Rightarrow$

$$j^\mu = \frac{1}{2}\epsilon_{\kappa\lambda}(x^\lambda\theta^{\mu\kappa} - x^\kappa\theta^{\mu\lambda}) \tag{6.108}$$

is conserved, hence $\theta^{\mu\nu}$ is also symmetric since it is conserved.

6.6 The Photon Field

For the first time we encounter gauge invariance. One may proceed either in Coulomb gauge, where manifest Lorentz covariance is lost, or one may keep a Lorentz covariant formulation at the expense of introducing unphysical, negative norm states.

a) The Coulomb gauge treatment is given in Bjorken and Drell, and since we always want to insist on manifest Lorentz covariance, I will not present it here.

b) Thus, we now deal with the Lorentz covariant, indefinite formulation, and we assume that the photon is coupled to a conserved source j^μ:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 + \frac{1}{2}\mu^2 A_\mu A^\mu - j^\mu A_\mu \tag{6.109}$$

Here we have given the photon a mass μ, in order to distinguish it from the ghost particle. The equation of motion is

$$\Box A_\rho - \partial_\rho\partial_\sigma A^\sigma + \lambda\partial_\rho\partial_\sigma A^\sigma + \mu^2 A_\rho = j_\rho \tag{6.110}$$

Conservation of the current implies

$$(\Box + \mu^2)\partial\cdot A + (\lambda - 1)\Box\partial\cdot A = 0 \Rightarrow \Box\partial\cdot A + \frac{\mu^2}{\lambda}\partial A = 0 \tag{6.111}$$

Hence the field $\partial \cdot A$ is a **free** field of mass $m^2 = \mu^2/\lambda$, and does not couple to j^μ. It has negative norm, and we call it the *ghost*. Now we are going to use Lorentz covariance, causality and positive energy requirements to construct the most general expression for the commutator function:

$$\langle 0|[A_\nu(x), A_\rho(y)]|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \left[\pi(k^2)\eta_{\nu\rho} - \pi'(k^2)\frac{k_\nu k_\rho}{k^2}\right] \tag{6.112}$$

where we have used translation invariance and Lorentz invariance. Now we use the fact that only states with positive m^2 contribute to write

$$\begin{aligned}
\langle 0|[A_\nu(x), A_\rho(y)]|0\rangle &= \int_0^\infty dM^2 \int \frac{d^4k}{(2\pi)^4}\delta(k^2 - M^2) e^{ik(x-y)}[\quad] && (6.113)\\
&= \int_0^\infty dM^2 \int \frac{d^3k}{(2\pi)^3 2k^o}\left\{ e^{ik(x-y)}\left[\pi(k^2)\eta_{\nu\rho} - \pi'(k^2)\frac{k_\nu k_\rho}{k^2}\right]\right. && (6.114)\\
&= \left. + e^{-ik(x-y)}\left[\pi_-(k^2)\eta_{\nu\rho} - \pi'_-(k^2)\frac{k_\nu k_\rho}{k^2}\right]\right\} && (6.115)
\end{aligned}$$

where now $k^o = \sqrt{\vec{k}^2 + M^2}$. Now we insist on causality, and require that the equal time commutator vanishes (this is enough in view of Lorentz inv.)

$$\langle 0|[A_\nu(x),\, A_\rho(y)]|0\rangle|_{x^o=q^o} \;=\; \int_0^\infty dM^2 \int \frac{d^3k}{(2\pi)^3 2k^o} \left\{ e^{-i\vec{k}(\vec{x}-\vec{y})} \left[\pi(M^2)\eta_{\nu\rho} - \pi'(M^2)\frac{k_\nu k_\rho}{M^2} \right] \right. \tag{6.116}$$

$$\left. + \, e^{i\vec{k}(\vec{x}-\vec{y})} \left[\pi_-(M^2)\eta_{\nu\rho} - \pi'_-(M^2)\frac{k_\nu k_\rho}{M^2} \right] \right\} \tag{6.117}$$

By the independence of the different terms, it follows that

$$\pi_-(M^2) = -\pi(M^2) \ , \qquad \pi'_-(M^2) = -\pi'(M^2) \tag{6.118}$$

Hence we get

$$\langle 0| \quad [A_\nu(x), A_\rho(y)]|0\rangle \tag{6.119}$$

$$= \int_0^\infty dM^2 \int \frac{d^3k}{(2\pi)^3 2k^o} (e^{ik(x-y)} - e^{-ik(x-y)}) \left[\pi(M^2)\eta_{\nu\rho} - \pi'(M^2)\frac{k_\nu k_\rho}{M^2} \right] \tag{6.120}$$

However, this does not yet completely vanish in particular for the $\nu = 0, \rho = i$ components, and we must have

$$\int_0^\infty dM^2 \frac{1}{M^2}\pi'(M^2) = 0 \tag{6.121}$$

Furthermore, let us insist on the fact that the $\partial \cdot A$ field is free:

$$(\Box + m^2)_x \qquad \langle 0|[\partial \cdot A(x),\; A_\rho(y)]|0\rangle \tag{6.122}$$

$$= \; (\Box + m^2)_x i \int_0^\infty \int \frac{d^3k}{(2\pi)^3 2k^o} \left(e^{-ik(x-y)} + e^{ik(x-y)} \right) [\pi(M^2) - \pi'(M^2)] k_\rho \tag{6.123}$$

It is clear that this requires

$$(m^2 - M^2)[\pi(M^2) - \pi'(M^2)] = 0 \tag{6.124}$$

$$\boxed{\pi'(M^2) = \pi(M^2) - \alpha Z_3 \delta(M^2 - m^2)} \tag{6.125}$$

The relation on π' may now be rewritten as

$$\boxed{\int_0^\infty dM^2 \frac{1}{M^2}\pi(M^2) = \frac{\alpha Z_3}{m^2}} \tag{6.126}$$

Now from the commutator, we can compute the equal time commutator: Recall, $\pi^\rho = \partial^\rho A^o - \partial^o A^\rho - \lambda\eta^{\rho_o}\partial\cdot A$, so that

$$\begin{aligned}
\langle \quad 0 \quad & |[\pi^\nu(x), |A_\rho(y)]|0\rangle && (6.127)\\
= \quad & i \int_0^\infty dM^2 \int \frac{d^3k}{(2\pi)^3 2k^o}\left(e^{-ik(x-y)} + e^{ik(x-y)}\right) && (6.128)\\
& \Big[\pi(M^2)(k^\nu\eta_{\rho_o} - k^o\eta_{\rho_\nu} - \lambda\eta^{\nu_o}k_\rho) && (6.129)\\
& -\pi'(M^2)\left(\frac{k^\nu k_\rho k^o - k^o k_\rho k^\nu - \lambda\eta^{\nu_o}k_\rho M^2}{M^2}\right)\Big] && (6.130)
\end{aligned}$$

Now restrict to equal time:

$$\begin{aligned}
\langle \quad 0 \quad & |\,[\pi^\nu(x),\ A_\rho(y)]|0\rangle|_{x^o=y^o} && (6.131)\\
= \quad & i \int_0^\infty dM^2 \int \frac{d^3k}{(2\pi)^3 2k^o}(e^{i\vec{k}(\vec{x}-\vec{y})} + e^{-i\vec{k}(\vec{x}-\vec{y})}) && (6.132)\\
& \Big[\pi(M^2)(k^o\eta^{\nu_o}\eta_{\rho_o} - k^o\eta_{\rho_\nu} - \lambda\eta^{\nu_o}\eta_{\rho_o}k^o) && (6.133)\\
& +\pi'(M^2)\lambda\eta^{\nu_o}\eta_{\rho_o}k^o\Big] && (6.134)\\
= \quad & i\delta^3(\vec{x}-\vec{y})\int_0^\infty dM^2\left[\pi(M^2)\{\eta^{\nu_o}\eta_{\rho_o} - \eta^\nu_\rho\} - \alpha Z_3\lambda\delta(M^2-m^2)\eta^{\nu_o}\eta_{\rho_o}\right] && (6.135)
\end{aligned}$$

For this expression we may write

$$\langle 0|\,[\pi^\nu(x),\ A_\rho(y)]|0\rangle|_{x^o=y^o} = i\ \delta^3(\vec{x}-\vec{y})[-\eta^\nu_\rho + a\eta^\nu_o\eta_{o\rho}] \qquad (6.136)$$

In order to come as close as possible to canonical commutation relations

$$\int_0^\infty dM^2\pi(M^2) = 1 \qquad a = \int_0^\infty dM^2\pi(M^2) - \alpha\lambda Z_3 = 1 - \alpha\lambda Z_3 \qquad (6.137)$$

One may also show that $\pi(M^2) \geq 0$.

Now we are going to be interested directly in the current j^μ, which is gauge invariant. We may compute its commutator:

$$\begin{aligned}
\langle 0|[j_\rho(x), j_\nu(y)]|0\rangle \quad = \quad & \int_0^\infty dM^2\pi(M^2)(\mu^2-M^2)^2\int\frac{d^3k}{(2\pi)^3 2k_o} && (6.138)\\
& \left(e^{ik(x-y)} - e^{-ik(x-y)}\right)\left(\rho_{\rho\nu} - \frac{k_\rho k_\nu}{M^2}\right) && (6.139)
\end{aligned}$$

Conservation of j is explicit in these formulae. We may now evaluate their equal time commutators: $x^o = y^o$

$$\begin{aligned}
\langle 0|[j_o(x), j_o(y)]|0\rangle \quad &= \quad 0 \qquad \text{trivially} && (6.140)\\
\langle 0|[j_o(x), j_k(y)]|0\rangle \quad &= \quad \underbrace{\int_0^\infty dM^2\pi(M^2)(\mu^2-M^2)^2\frac{1}{M^2}}_{>0}\, i\ \partial^*_k\delta^3(\vec{x}-\vec{y}) && (6.141)\\
&= \quad \text{so-called Schwinger term} && (6.142)
\end{aligned}$$

As long as there is non-trivial interaction, this term does not vanish. In fact this may be seen from very general arguments:

$$\langle 0|[j_o(0,\vec{x}),\ j_k(0,\vec{y})]|0\rangle = C_k(\vec{x},\vec{y}) \tag{6.143}$$

Now differentiate in $\vec{y}$ and use current conservation:

$$\langle 0|[j_o(0,\vec{x}),\ -\partial_o j^o(0,\vec{y})]|0\rangle = \frac{\partial}{\partial y^k} C^k(\vec{x},\vec{y}) \tag{6.144}$$

Now the time derivative is gotten from a commutator with the Hamiltonian:

$$\partial_o j^o = i[H, j^o] \tag{6.145}$$

Hence

$$\langle 0|j_o(0,\vec{x}) H j_o(0,\vec{y}) + j_o(0,\vec{y}) H j_o(0,\vec{x})|0\rangle = i\frac{\partial}{\partial y^k} C^k(\vec{x},\vec{y}) \tag{6.146}$$

Now integrate over $\vec{x}$ and $\vec{y}$s, with some test function f:

$$F = \int d^3x f(\vec{x}) j^o(\vec{x}) \tag{6.147}$$

then we get

$$\begin{aligned}
\int d^3x \int d^3y f(x) f(y) i\frac{\partial}{\partial y^k} C^k(\vec{x},\vec{y}) &= 2\langle 0|FHF|0\rangle && (6.148)\\
&= 2\sum_n |\langle 0|F|n\rangle|^2 E_n && (6.149)\\
&\neq 0 && (6.150)
\end{aligned}$$

The next thing is that as soon as one has a Schwinger term, the ordinary time ordering symbol is no longer covariant: Let's do this in general, for any two operators:

$$[A(0,\vec{x}), B(0)] = C(0)\delta^3(\vec{x}) + D^i(0)\partial_i\delta^3(\vec{x}) \tag{6.151}$$

or also

$$[A(x), B(0)]\delta(t) = C(0)\delta^4(x) + S^i(0)\partial_i\delta^4(x) \tag{6.152}$$

This suggests to introduce a general equal time surface, given by its orthogonal vector n^μ, which is time-like:

$n^o > 0 \qquad n^2 = 1 \qquad P_{\alpha\beta} = \eta_{\alpha\beta} - n_\alpha n_\beta$

$$[\quad A(x) \quad , B(0)]\delta(x\cdot n) = C(n)\delta^4(x) + S^a(n)P_{\alpha\beta}\partial^\beta\delta^4(x) \tag{6.153}$$

Similarly the time-ordered product is

$$T(x;n) = \theta(x\cdot n)A(x)B(0) + \theta(-x\cdot n)B(0)A(x) \tag{6.154}$$

We are now going to give a constructive proof of the fact that T is *not* covariant whenever a Schwinger term exists, and we immediately construct the covariant T or $T*$

$$T^*(x) = T(x;n) + \tau(x;n) \tag{6.155}$$

Now we demand that T^* be invariant, = independent of n . $\rightarrow$ vary the independent directions of n^μ: $P_{\alpha\beta}\frac{\partial}{\partial\pi^\beta}$

$$0 = P^{\alpha\beta}\frac{\partial}{\partial\pi^\beta}T(x;n) + P^{\alpha\beta}\frac{\partial}{\partial\pi^\beta}\tau(x;n) \tag{6.156}$$

$$\begin{aligned}
P^{\alpha\beta}\frac{\partial}{\partial\pi^\beta}T(x;n) &= P^{\alpha\beta}x_\beta\delta(x\cdot n)A(x)B(0) - P^{\alpha\beta}x_\beta\delta(x\cdot n)B(0)A(x) && (6.157)\\
&= P^{\alpha\beta}x_\beta\delta x\cdot n)[A(x),B(0)] && (6.158)\\
&= P^{\alpha\beta}x_\beta\left(C(n)\delta^4(x) + D^\gamma P_{\gamma\delta}\partial^\delta\delta^4(x)\right) && (6.159)\\
&= -P^{\alpha\gamma}S_\gamma(n)\delta^4(x) \neq 0 && (6.160)
\end{aligned}$$

Now we had not defined the part along n^β of the Schwinger term in any case, so we may set

$$S_\beta(n)\delta^4(x) = \frac{\partial}{\partial n^\beta}\tau(x;n) + n_\beta f(x;n) \tag{6.161}$$

which determines $\tau(x;n)$, even though not uniquely. $\tau(x;n)$ is called a *covariantizing seagull*. Note that $\tau(x;n)$ only has support at $t=0$ and $x=0$ (or $x=y$)! So it does not alter the long distance properties of the commutator.

Asymptotic conditions:

Recall that $\partial\cdot A$ was a free field of mass m^2, hence this component of the field persists from in to out! We may define a transverse field:

$$A_\mu - \frac{1}{\Box}\partial_\mu(\partial\cdot A) \qquad \text{or} \qquad A_\mu + \frac{1}{m^2}\partial_\mu(\partial\cdot A) \tag{6.162}$$

and the asymptotic condition should only hold for this physical field:

$$A_\mu - \frac{1}{\Box}\partial_\mu(\partial\cdot A) \stackrel{t\to-\infty}{\rightarrow} Z_3^{1/2}\left(A_\mu^{\text{in}} - \frac{1}{\Box}(\partial\cdot A^{\text{in}})\right) \tag{6.163}$$

now of course $\partial A^{\rm in} = \partial A$ since the field is free:

$$\boxed{A_\mu \to Z_3^{1/2} A_\mu^{\rm in} + (1 - Z_3^{1/2}) \frac{1}{\Box} \partial_\mu (\partial \cdot A)^{\rm in}} \tag{6.164}$$

In particular

$$\langle 0 | A_\mu | 1 \rangle = Z_3^{1/2} \langle 0 | A_\mu^{{\rm in}T} | 1 \rangle + \left(-\frac{1}{m^2} \right) \langle 0 | \partial_\mu (\partial \cdot A^{\rm in}) | 1 \rangle \tag{6.165}$$

Hence a purely longitudinal photon is normalized to unity here, this corresponds to $\pi(M^2) = 0$.

$$\langle 0 | [A_\nu(x), \underbrace{A_\rho(y)}_{\text{longitudinal}}] | 0 \rangle = \alpha \int \frac{d^3k}{(2\pi)^3 2k_o} (e^{ik(x-y)} - e^{-ik(x-y)}) \frac{k_\nu k_\rho}{m^2} \tag{6.166}$$

From inserting a complete set of states, we get

$$\boxed{\alpha = 1} \tag{6.167}$$

Now look at the transverse states:

$$\langle 0 | [A_\nu(x), \underbrace{A_\rho(y)}_{\text{transverse}}] | 0 \rangle = \int_0^\infty dM^2 \pi(M^2) \int \frac{d^3k}{(2\pi)^3 2k^o} (e^{ik(x-y)} - e^{-ik(x-y)}) \left(\eta_{\nu\rho} - \frac{k_\nu k_\rho}{M^2} \right) \tag{6.168}$$

The contribution of the 1-particle *transverse* state is $\pi_1(M^2) = Z_3 \delta(M^2 - \mu^2)$. S in particular we have

$$\boxed{1 = Z_3 + \int_{\text{threshold}(2\mu^2)}^\infty dM^2 \pi(M^2)} \tag{6.169}$$

7 Functional Methods in Scalar Field Theory

The essential information of a quantum field theory is contained in its time ordered correlation functions. From these, the scattering matrix elements, the transition amplitudes and probabilities for any process may be computed.

In the preceding sections we have evaluated the correlation functions exactly for the free field theories containing scalar, spin 1/2 and spin 1 fields. The evaluation of the correlation functions in any interacting theory is a very difficult problem that can only rarely be carried out exactly. Frequently, one will have to resort to an approximate or perturbative treatment, for example based on an expansion in powers of the coupling constants. Even the perturbative series is quite complicated because the combinatorics as well as the space-time properties give rise to involved problems. Historically, perturbative calculations were first carried out in the operator formalism of QFT, by recasting each of the expansion coefficients in terms of free field theory correlation functions. For scalar field theory problems, these calculations were involved but manageable. For gauge theories, however, the operator methods very quickly become unmanageable. For non-Abelian Yang-Mills theory, operator methods are seldom useful.

The functional integral formulation (already derived for quantum mechanics in SA2) provides an alternative way of dealing with QFT. Historically, the functional integral formulation of QFT was regarded foremost merely as an efficient way of organizing perturbation theory. Its indispensible value was established permanently, however, with the advent of non-Abelian Yang-Mills theory. Its direct connection with statistical mechanics led Wilson and others to use the functional integral formulation as a non-perturbative formulation of QFT, and to a much deeper understanding of the properties of renormalization.

7.1 Functional integral formulation of quantum mechanics

In SA.2.6, the functional integral formulation of quantum mechanics was derived for a Hamiltonian $H(p,q)$ and the following formula was obtained,

$$\langle q'|e^{-itH}|q\rangle = \int \mathcal{D}p \int \mathcal{D}q \ \exp\left\{ i \int_0^t dt' \Big(p\dot{q} - H(p,q) \Big)(t') \right\} \tag{7.1}$$

with the boundary conditions $q(t'=0)=q$ and $q(t'=t)=q'$, and free boundary conditions on $p(t')$. Recall that the measure arose from a discretisation of the time interval $[0,t]$, for $t>0$, into N segments $[t_i, t_{i+1}]$, $t_0=0$, $t_N=t$, $i=0,N-1$, and was formally defined by

$$\mathcal{D}p\mathcal{D}q \equiv \lim_{N\to\infty} \prod_{i=1}^{N-1} \frac{dp(t_i)dq(t_i)}{2\pi} \tag{7.2}$$

Notice that the intermediate times are ordered, $t=t_0<t_1<\cdots<t_{N-1}<t_N=t$, so that the integral is really over *paths* rather than over functions.

7.2 The functional integral for scalar fields

We now generalize the above discussion to the case of one real scalar field ϕ, thus to an infinite number of degrees of freedom. We denote the momentum canonically conjugate to ϕ by π and the Hamiltonian by $H[\phi, \pi]$. The analogue of the *position basis of states* $|q\rangle$ and $|p\rangle$ is given by $|\phi(\vec{x})\rangle$ and by $|\pi(\vec{x})\rangle$. Remark that these states do not *depend* on $\vec{x}$; instead they are functionals of ϕ. The completeness relations are now

$$\begin{aligned} I &= \int \mathcal{D}\phi(\vec{x}) \quad |\phi(\vec{x})\rangle\langle\phi(\vec{x})| \\ I &= \int \mathcal{D}\pi(\vec{x}) \quad |\pi(\vec{x})\rangle\langle\pi(\vec{x})| \end{aligned} \tag{7.3}$$

Clearly, the derivation is formal and we shall postpone all issues of cutoff and renormalization until later. The derivation then follows identically the same path as we had for quantum mechanics and therefore, we have the following result,

$$\langle\phi'|e^{-itH}|\phi\rangle = \int \mathcal{D}\phi\mathcal{D}\pi \exp\{iI[\phi,\pi]\} \qquad \begin{cases} \phi(t,\vec{x}) = \phi'(\vec{x}) \\ \phi(0,\vec{x}) = \phi(\vec{x}) \end{cases} \tag{7.4}$$

where the action is defined by

$$I[\phi,\pi] \equiv \int_0^t dt' \left(\int d^3x \pi\dot{\phi} - H[\pi,\phi] \right)(t') \tag{7.5}$$

In the special case of the following simple but useful Hamiltonian,

$$H[\phi,\pi] = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi) \right) \tag{7.6}$$

one gets after performing the Gaussian integration over π

$$\langle\phi'|e^{-itH}|\phi\rangle = \int \mathcal{D}\phi \exp\left\{ i\int_0^t dt' \int d^3x \mathcal{L}(\phi,\partial\phi) \right\} \qquad \begin{cases} \phi(t,\vec{x}) = \phi'(\vec{x}) \\ \phi(0,\vec{x}) = \phi(\vec{x}) \end{cases} \tag{7.7}$$

Notice that, as an added bonus, this formulation is now manifestly Lorentz covariant, as it is given in terms of the Lorentz invariant Lagrangian

$$\mathcal{L}(\phi,\partial\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{7.8}$$

Preserving Lorentz invariance will prove invaluable later on.

7.3 Time ordered correlation functions

The above result may be applied directly to the calculation of the vacuum expectation value of the time ordered product of a string of fields ϕ,

$$\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \tag{7.9}$$

Notice that this object is a completely symmetric function of $x_1, \cdots x_n$. Hence, without loss of generality, the points x_i may be time-ordered, $x_1^o \geq x_2^o \geq x_3^o \cdots \geq x_n^o$ so that

$$\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \quad = \quad \langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \tag{7.10}$$

Next, one uses the evolution operator to bring out the time-dependence of the field operators $\phi(x_i)$ given by the following expression,

$$\phi(x_i^o, \vec{x}_i) = e^{ix_i^o H}\phi(0, \vec{x}_i)e^{-ix_i^o H} \tag{7.11}$$

Using this formula for the time-dependence of every inserted operator, we obtain,

$$\begin{aligned}
&\langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle \qquad\qquad (7.12)\\
&\quad = \langle 0|e^{ix_1^o H}\phi(0,\vec{x}_1)e^{-i(x_1^o - x_2^o)H}\phi(0,\vec{x}_2)e^{-i(x_2^o - x_3^o)H}\cdots e^{-i(x_{n-1}^o - x_n^o)H}\phi(0,\vec{x}_n)e^{-ix_n^o H}|0\rangle
\end{aligned}$$

Next, we insert the identity operator exactly n times and express the identity as a completeness relation in the position basis of states $|\phi(\vec{x})\rangle$ Since the insertions have to be carried out n times, we introduce n integration fields , which we denote by $\phi(x_i^o, \vec{x}_i)$, $i = 1, \cdots, n$. The operator insertions $\phi(0, \vec{x}_i)$ are diagonal in this basis and we obtain,

$$\begin{aligned}
= \quad & \int \mathcal{D}\phi(x_1^o,\vec{x}_1)\cdots\int D\phi(x_n^o,\vec{x}_n)\langle 0|e^{ix_1^o H}|\phi(x_1^o,\vec{x}_1)\rangle\phi(x_1^o,\vec{x}_1) \qquad (7.13)\\
&\times\langle\phi(x_1^o,\vec{x}_1)|e^{-i(x_1^o - x_2^o)H}|\phi(x_2^o,\vec{x}_2)\rangle\phi|x_2^o,\vec{x}_2\rangle\\
&\times\langle\phi(x_2^o,\vec{x}_2)|e^{-i(x_2^o - x_3^o)H}|\phi(x_3^o,\vec{x}_3)\rangle\cdots\langle\phi(x_n^o,\vec{x}_n)|e^{-ix_n^o H}|0\rangle
\end{aligned}$$

At this stage, we use the functional integral form for the matrix elements in the field basis of the evolution operator, as derived in (). It is immediately realized that the boundary conditions at each integration over $\phi(x_i^o, \vec{x}_i)$ simply requires integration over a path in time that is continuous. Furthermore, since the Hamiltonian acts on the ground state, assumed to have zero energy, the time integrations may be taken to $\pm\infty$ on both ends. Finally, there is a normalization factor $Z[0]$ that will guarantee that the ground state is properly normalized to $\langle 0|0\rangle = 1$;

$$\langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle = \frac{1}{Z[0]}\int \mathcal{D}\phi\ \phi(x_1)\cdots\phi(x_n)\exp\left\{i\int d^4x\mathcal{L}\right\} \tag{7.14}$$

It is clear that the right hand side is automatically a symmetric function of the space-time points $x_1, \cdots, x_n$. Therefore, the rhs in fact equals the time ordered expectation value

$$\begin{aligned}
\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle \quad &= \quad \frac{1}{Z[0]}\int \mathcal{D}\phi\ \phi(x_1)\cdots\phi(x_n)\exp\left\{i\int d^4x\mathcal{L}\right\}\\
Z[0] \quad &\equiv \quad \int \mathcal{D}\phi\ \exp\left\{i\int d^4x\mathcal{L}\right\} \qquad (7.15)
\end{aligned}$$

for any time ordering of the points x_i.

7.4 The generating functional of correlators

Now it is very easy to write down a generating functional for the time-ordered expectation values. Define

$$\begin{aligned} Z[J] &\equiv \int \mathcal{D}\phi \ \exp\left\{ i \int d^4x \phi(x) J(x) \right\} \ e^{iS[\phi]} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \Big(\prod_{j=1}^{n} \int d^4x_j J(x_j) \Big) \int \mathcal{D}\phi \ \phi(x_1) \cdots \phi(x_n) \ e^{iS[\phi]} \end{aligned} \tag{7.16}$$

so that

$$\begin{aligned} \frac{Z[J]}{Z[0]} &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \Big(\prod_{j=1}^{n} \int d^4x_j J(x_j) \Big) \langle 0 | T\phi(x_1) \cdots \phi(x_n) | 0 \rangle \\ &= \langle 0 | T \exp\left\{ i \int d^4x J(x) \phi(x) \right\} | 0 \rangle \end{aligned} \tag{7.17}$$

Equivalently, the n-point correlator may be expressed as an n-fold functional derivative with repsect to the source J,

$$\langle 0 | T\phi(x_1) \cdots \phi(x_n) | 0 \rangle = (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \left(\frac{Z[J]}{Z[0]} \right) \Big|_{J=0} \tag{7.18}$$

7.5 The generating Functional for free scalar field theory

It is instructive to see what these objects are in free field theory:

$$\mathcal{L}_o = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{7.19}$$

We may directly compute the free generating functional. Since the integral is Gaussian, all J-dependence may be obtained by shifting the integration as follows,

$$\begin{aligned} Z_o[J] &= \int \mathcal{D}\phi \ \exp\left\{ i \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J\phi \right) \right\} \qquad (7.20) \\ &= \int \mathcal{D}\phi \ \exp\left\{ i \int d^4x \left(-\frac{1}{2} \phi (\Box + m^2) \phi + J\phi \right) \right\} \\ &= \exp\left\{ +\frac{i}{2} J (\Box + m^2)^{-1} J \right\} \ \int \mathcal{D}\phi \ \exp\left\{ i \int d^4x \left(-\frac{1}{2} \tilde{\phi} (\Box + m^2) \tilde{\phi} \right) \right\} \end{aligned}$$

where we have used the notation $\tilde{\phi} = \phi - (\Box + m^2)^{-1} J$. By shifting the $\tilde{\phi}$ back to ϕ, it is shown that the integration factor in the last line is equal to $Z[0]$. Furthermore, the inverse of the operator $i(\Box + m^2)$ is nothing but the scalar Green function defined previously by the Feynman propagator $G(x-y; m^2) = G(x-y)$,

$$(\Box + m^2)_x G(x-y; m^2) = -i \delta^{(4)}(x-y) \tag{7.21}$$

so that the final answer may be put in the form

$$Z_o[J] = Z_o[0] \exp \left\{ -\frac{1}{2} \int d^4x \int d^4y J(x) G(x-y) J(y) \right\} \tag{7.22}$$

In particular, the two point function is given by

$$\langle 0|T\phi(x)\phi(x')|0\rangle = -\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(x')} \left(\frac{Z_o[J]}{Z_o[0]} \right) \Big|_{J=0} = G(x-x') \tag{7.23}$$

Similarly, the 4-point function is obtained by

$$\begin{aligned}
\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle &= \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \frac{\delta}{\delta J(x_3)} \frac{\delta}{\delta J(x_4)} \left(\frac{Z_o[J]}{Z_o[0]} \right) \Big|_{J=0} \\
&= \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_4)} \frac{1}{2} \left(-\frac{1}{2} \int d^4u \int d^4v J(u) G(u-v) J(v) \right)^2 \\
&= G(x_1-x_2)G(x_3-x_4) + G(x_1-x_3)G(x_2-x_4) \\
&\quad + G(x_1-x_4)G(x_2-x_3)
\end{aligned} \tag{7.24}$$

Recall that this may be represented graphically as follows:

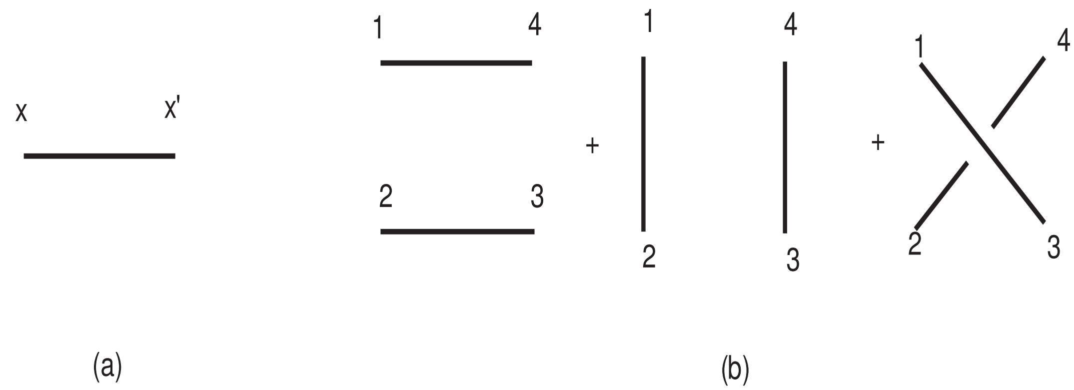


Figure 8: The scalar 2- and 4-point function in free field theory

More generally, the n-point function vanishes when n is odd (by $\phi \to -\phi$ symmetry), while $2n$-point functions are given by

$$\langle 0|T\phi(x_1)\cdots\phi(x_{2n})|0\rangle = \frac{1}{2^n\, n!} \sum_\sigma G(x_{\sigma(1)} - x_{\sigma(2)}) \cdots G(x_{\sigma(2n-1)} - x_{\sigma(2n)}) \tag{7.25}$$

The summation is effectively over all possible inequivalent pairings of the $2n$ points, in which each inequivalent term then appears with coefficient 1.

8 Perturbation Theory for Scalar Fields

For a general interacting theory, none of the Green functions can be computed exactly. Recall that in ordinary quantum mechanics, one can do this only in the cases where extra symmetries decouple the degrees of freedom, which is exceedingly rare. It is even more rare in quantum field theory and is known to take place in some 1+1-dimensional space-time field theories. Short of exact solutions, one needs to make approximations. In perturbation theory, one regards one or several parameters as small and such that the theory can be solved exactly (in practice this is always free field theory) when all these parameters are set to 0. Next, one expands in powers of these small parameters.

8.1 Scalar Field Theory with potential interactions

The prototypical scalar field theory we shall be interested in has an action of the form,

$$S[\phi] = \int d^4 z \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda V(\phi) \right) \tag{8.1}$$

where the potential $V(\phi)$ is an analytic function of ϕ admitting a powers series expansion in ϕ, and λ is a small parameter. It is convenient to isolate from $V(\phi)$ the terms that are linear and quadratic in ϕ, to shift ϕ so as to cancel the linear terms and to regroup the quadratic terms into a mass term, which may be treated exactly. Thus, we may assume that $V'(0) = V''(0) = 0$.

We consider the example of the quartic potential first, and set $V(\phi) = \phi^4/4!$. The correlation functions of interest are

$$\langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle \quad = \quad \frac{\int \mathcal{D}\phi \; \phi(x_1) \cdots \phi(x_n) \; e^{iS[\phi]}}{\int \mathcal{D}\phi \; e^{iS[\phi]}} \tag{8.2}$$

The simplest thing to compute is the denominator assuming that λ is small,

$$\begin{aligned} \int \mathcal{D}\phi \; e^{iS[\phi]} \quad &= \quad \int \mathcal{D}\phi \exp \left\{ i S_o[\phi] - \frac{i\lambda}{4!} \int d^4 z \phi^4(z) \right] \\ &= \quad \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{-i\lambda}{4!} \right)^p \int \mathcal{D}\phi \int d^4 z_1 \cdots \int d^4 z_p \phi^4(z_1) \cdots \phi^4(z_p) \; e^{iS_o[\phi]} \end{aligned} \tag{8.3}$$

This is now a problem of free ϕ-fields, governed by the free action $S_o[\phi]$. Therefore, *the expansion may be expressed in terms of free field vacuum expectation values*,

$$\frac{\int \mathcal{D}\phi \; e^{iS[\phi]}}{\int \mathcal{D}\phi \; e^{iS_o[\phi]}} = \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{-i\lambda}{4!} \right)^p \int d^4 z_1 \cdots d^4 z_p \langle 0 | T \phi^4(z_1) \cdots \phi^4(z_p) | 0 \rangle_o \tag{8.4}$$

Similarly, the numerator is expanded in powers of λ as well, and we have

$$\begin{aligned}
&\frac{\int \mathcal{D}\phi \ \phi(x_1)\cdots\phi(x_n)\ e^{iS[\phi]}}{\int \mathcal{D}\phi \ e^{iS_o[\phi]}} \\
&\qquad = \sum_{p=0}^{\infty} \frac{1}{p!}\left(\frac{-i\lambda}{4!}\right)^p \int d^4z_1 \cdots d^4z_p \langle 0|T\phi(x_1)\cdots\phi(x_n)\phi^4(z_1)\cdots\phi^4(z_p)|0\rangle_o
\end{aligned} \tag{8.5}$$

Again this has become a problem in free field theory. Therefore, the problem that we set out to solve is basically solved by this formula.

It is useful to generalize this formula to any potential. It is also convenient to recast the result in terms of the generating functional, from which any correlator may be obtained. Thus, the quantity we are interested in is as follows,

$$Z[J] \equiv \int \mathcal{D}\phi \ \exp\left\{ iS[\phi] + i\int d^4x J(x)\phi(x)\right\} \tag{8.6}$$

Just as before, we expand the action into its free part and the interacting part,

$$S[\phi] = S_o[\phi] - \lambda \int d^4x V(\phi) \tag{8.7}$$

This allows us to expand also the generating functional in powers of λ,

$$\begin{aligned}
Z[J] &= \sum_{p=0}^{\infty} \frac{1}{p!} \int \mathcal{D}\phi \left(-i\lambda \int d^4x V(\phi)\right)^p \ \exp\left\{ iS_o[\phi] + i\int d^4x J(x)\phi(x)\right\} \qquad (8.8)\\
&= \sum_{p=0}^{\infty} \frac{1}{p!} \left\{-i\lambda \int d^4x V\left(-i\frac{\delta}{\delta J(x)}\right)\right\}^p \ \int \mathcal{D}\phi \ \exp\left\{ iS_o[\phi] + i\int d^4x J(x)\phi(x)\right\}
\end{aligned}$$

Hence we have a very neat shorthand for the expansion, as follows,

$$Z[J] = \exp\left\{-i\lambda \int d^4x V\left(-i\frac{\delta}{\delta J(x)}\right)\right\} Z_o[J] \tag{8.9}$$

and of course the generating functional for the free theory was known exactly in terms of the scalar Feynman propagator G,

$$Z_o[J] = \exp\left\{-\frac{1}{2}\int d^4x \int d^4y J(x) G(x-y) J(y)\right\} \tag{8.10}$$

Again, the problem of perturbation theory has been completely reformulated in terms of problems in free scalar field theory.

8.2 Feynman rules for ϕ^4-theory

Fermi and Feynman introduced a graphical expansion for perturbation theory that commonly goes under the name of Feynman rules. We begin by working this out for the case of $V(\phi) = \phi^4/4!$. Because of $\phi \to -\phi$ symmetry, correlators with an odd number of fields vanish identically, in the free as well as in the interacting theory. To order λ, the following correlators are required,

$$\begin{aligned}&\langle 0|T\phi(x_1)\cdots\phi(x_{2n})\frac{1}{4!}\phi^4(z)|0\rangle_o \\ &\qquad = \frac{1}{4!}\lim_{z_i\to z}\langle 0|T\phi(x_1)\cdots\phi(x_{2n})\phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4)|0\rangle_o \end{aligned} \tag{8.11}$$

The free field correlator of a string of canonical scalar fields is a quantity that we have learned how to evaluate previously. There are 3 different types of contributions.

1. All contractions of the $\phi(z_i)$ are carried out onto $\phi(z_j)$ (this is the only possibility if $n = 0$), and all contractions of $\phi(x_k)$ are carried out onto $\phi(x_l)$, which yields the graphical contribution of Fig 2 (a). There are 3 distinct double pairings amongst 4 points, thus the remaining combinatorial factor is 1/8.

2. Two $\phi(z_i)$ are contracted with one another, while all others are contracted with $\phi(x_j)$, which yields the graphical representation of Fig 2 (b). There are 6 possible pairs amongst 4 points, but the remaining 2 points not in the pair may be permuted freely yielding another factor of 2. Thus, the combinatorial factor is 1/2.

3. Finally, all $\phi(z_i)$ may be contracted with $\phi(x_j)$, yielding combinatorial factors of 1, as represented in Fig 2 (c).

In equations, we have

$$\begin{aligned}&\langle 0|T\phi(x_1)\cdots\phi(x_{2n})\frac{1}{4!}\phi^4(z)|0\rangle_o \\ &\quad = \frac{1}{8}G(0)^2\langle 0|T\phi(x_1)\cdots\phi(x_{2n})|0\rangle_o \\ &\quad +\frac{1}{2}G(0)\sum_{i\neq j=1}^{2n} G(z-x_i)G(z-x_j)\langle 0|T\phi(x_1)\cdots\hat{i}\hat{j}\cdots\phi(x_{2n})|0\rangle_o \\ &\quad + \sum_{i\neq j\neq k\neq l=1}^{2n} G(z-x_i)G(z-x_j)G(z-x_k)G(z-x_l) \\ &\qquad\qquad\qquad \times\langle 0|T\phi(x_1)\cdots\hat{i}\hat{j}\hat{k}\hat{l}\cdots\phi(x_{2n})|0\rangle_o\end{aligned} \tag{8.12}$$

Notice that the contribution from Fig 2 (a) is disconnected, namely the double bubble is disconnected from any of the external points x_i. But this bubble is common to the

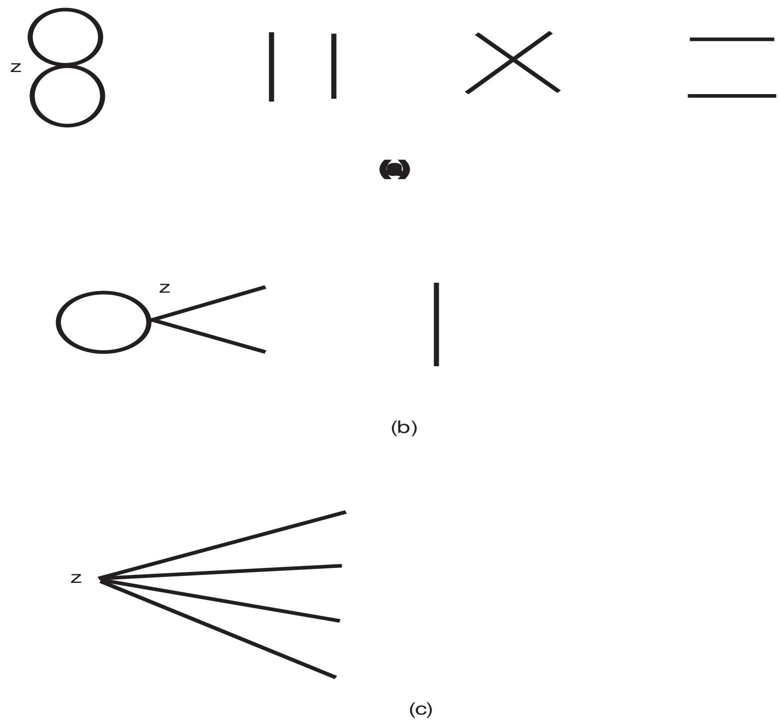


Figure 9: First order contributions to $\langle 0|T\phi(x_1)\cdots\phi(x_4)|0\rangle$

numerator and denominator in the functional integral and therefore cancels out. Thus, the only contributions arise from Fig 2 (b) and (c), or analytically,

$$
\begin{aligned}
&\langle 0|T\phi(x_1)\cdots\phi(x_{2n})|0\rangle \qquad\qquad (8.13)\\
&\quad = \langle 0|T\phi(x_1)\cdots\phi(x_{2n})|0\rangle_o \\
&\qquad -\frac{i\lambda}{2}\sum_{i\neq j=1}^{2n}\int d^4z G(z-z)G(z-x_i)G(z-x_j)\langle 0|T\phi(x_1)\cdots\hat{i}\hat{j}\cdots\phi(x_{2n})|0\rangle_o \\
&\qquad -i\lambda\sum_{i\neq j\neq k\neq l=1}^{2n}\int d^4z G(z-x_i)G(z-x_j)G(z-x_k)G(z-x_l) \\
&\qquad\qquad\qquad \times\langle 0|T\phi(x_1)\cdots\hat{i}\hat{j}\hat{k}\hat{l}\cdots\phi(x_{2n})|0\rangle_o
\end{aligned}
$$

Clearly, it becomes very cumbersome to write out these formulas completely, and a graphical expansion turns out to capture all the relevant information concisely.

8.3 The cancellation of vacuum graphs.

Let us define a *graph without vacuum parts* as a graph where all interaction vertices are connected to an external ϕ by at least one line. It is clear that

$$\left(\sum_{\text{all graphs}}\right) = \left(\sum_{\text{no vac parts}}\right) \times \left(\sum_{\text{bubble graphs}}\right) \tag{8.14}$$

Hence

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \sum_{\text{no vac parts}} \tag{8.15}$$

Thus in practice, one never has to compute the bubble graphs in computing the time ordered products. To illustrate this, take the 2-point function in $\lambda\phi^4$-theory. To order λ^2, we have

$$\begin{aligned}\langle 0|T\phi(x)\phi(y)|0\rangle &= G(x-y)+\frac{1}{2}(-i\lambda)\int d^4zG(0)G(x-z)G(y-z) \\ &+\frac{1}{4}(-i\lambda)^2\int d^2z_1\int d^2z_2G(x-z_1)G(0)G(z_1-z_2)G(0)G(z_2-y) \\ &+\frac{1}{4}(-i\lambda)^2\int d^2z_1\int d^2z_2G(x-z_1)G(z_1-z_2)^2G(0)G(z_1-y) \\ &+\frac{1}{6}(-i\lambda)^2\int d^2z_1\int d^2z_2G(x-z_1)G(z_1-z_2)^3G(z_2-y)+O(\lambda^3)\end{aligned} \tag{8.16}$$

The Feynman representation is given in Fig 3.

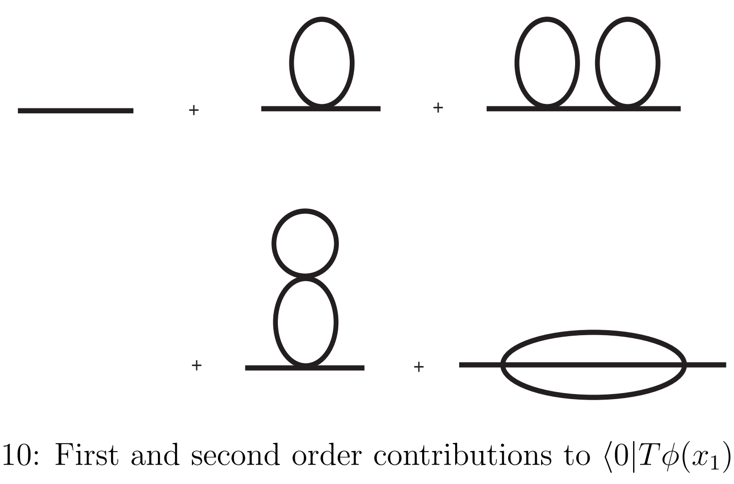


Figure 10: First and second order contributions to $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$

8.4 General Position Space Feynman Rules for scalar field theory

The starting point is the formula for the generating functional

$$\begin{aligned} Z[J] &= \exp\left\{-i\int d^4x V\left(-i\frac{\delta}{\delta J(x)}\right)\right\} Z_o[J] \\ Z_o[J] &= \exp\left\{-\frac{1}{2}\int d^4x\int d^4y J(x)G(x-y)J(y)\right\} Z_o[0] \end{aligned} \tag{8.17}$$

From the combinatorics of these formulae one deduces the following rules

1. Let $x_1, x_2 \cdots x_n$ be the external ϕ's in $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$;
2. Let $z_1 \cdots z_p$ be the internal interaction vertices (order p in V);
3. From every external $\phi(x_i)$ departs *one line*, which can connect to an external ϕ (except itself) or to an interaction vertex z_j;
4. From an interaction vertex of the type $\frac{1}{N!}\phi^N$, there depart N lines, which join an external ϕ, another interaction vertex or itself;
5. To every line is associated $G_F(x-y)$ between points x and y;
6. Every interaction vertex $\frac{1}{N!}\phi^N$ has a coupling constant factor of ig_N;
7. Every interaction vertex is integrated over $\int d^4z_i$;
8. There is an overall symmetry factor.

8.5 Momentum Space Representation

We begin by defining the Fourier transform of the n-point function,

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \prod_{j=1}^{n}\left(\int \frac{d^4p_j}{(2\pi)^4}\, e^{ip_j\cdot x_j}\right)\tilde{G}(p_1,\cdots,p_n) \tag{8.18}$$

as well as the inverse Fourier transform relation,

$$\tilde{G}(p_1,\cdots,p_n) = \prod_{j=1}^{n}\left(\int d^4x_j\, e^{-ip_j\cdot x_j}\right)\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle \tag{8.19}$$

Recall, however, that the scalar field theory action $S[\phi]$, as well as the vacuum $|0\rangle$ are translation invariant. Therefore, the time ordered expectation value $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$ only depends on the differences of the arguments. As a consequence, the Fourier transform

$\tilde{G}(p_1, \cdots, p_n)$ is non-vanishing only when overall momentum is conserved, $p_1 + \cdots + p_n = 0$ and is proportional to an overall δ-function,

$$\tilde{G}(p_1, \cdots, p_n) = (2\pi)^4 \delta^4(p_1 + \cdots + p_n) G(p_1, \cdots, p_n) \tag{8.20}$$

The Feynman rules in momentum space are easily derived, and are customarily formulated for the quantities $G(p_1, \cdots, p_n)$.

1. Draw all topologically distinct connected diagrams with n external legs of incoming momenta $p_1, \cdots p_n$. Denote by $k_1, \cdots k_\ell$ the momenta of internal lines.

2. To the j-th external line, assign the value of the propagator,

$$G_F(p_j) = \frac{i}{p_j^2 - m^2 + i\epsilon}, \qquad j = 1, \cdots, n \tag{8.21}$$

3. To every internal line, assign

$$\int \frac{d^4 k_\ell}{(2\pi)^4} \, \frac{i}{k_\ell^2 - m^2 + i\epsilon} \tag{8.22}$$

4. To each interaction vertex, assign

$$-i g_N (2\pi)^4 \delta^4(q) \tag{8.23}$$

   where q is the sum of incoming momenta to the vertex.

5. Integrate over the internal momenta, and multiply by the combinatorial factor.

8.6 Does perturbation theory converge ?

Having produced a perturbative expansion in the coupling constant, the question arises as to whether the corresponding power series is convergent or not. As a caricature of the perturbative expansion of the functional integral is to take an ordinary integral of the same type. Thus, we consider

$$Z(m, \lambda) \equiv \int_{-\infty}^{+\infty} d\phi \; \exp\Big\{ -m^2 \phi^2 - \lambda^2 \phi^4 \Big\} \tag{8.24}$$

There are now two possible expansions; the first in power of m, the second in powers of λ. The expansion in powers of m yields

$$\begin{aligned} Z(m, \lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} (-m^2)^n \int_{-\infty}^{+\infty} d\phi \; \phi^{2n} \; \exp\Big\{ -\lambda^2 \phi^4 \Big\} \\ &= \sum_{n=0}^{\infty} \frac{1}{2\sqrt{\lambda}} \frac{\Gamma(\frac{n}{2} + \frac{1}{4})}{\Gamma(n+1)} \left(-\frac{m^2}{\lambda} \right)^n \end{aligned} \tag{8.25}$$

while the expansion in powers of λ yields,

$$
\begin{aligned}
Z(m,\lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!}(-\lambda^2)^n \int_{-\infty}^{+\infty} d\phi\ \phi^{4n}\ \exp\Big\{-m^2\phi^2\Big\} \\
&= \sum_{n=0}^{\infty} \frac{1}{m}\frac{\Gamma(2n+\frac{1}{2})}{\Gamma(n+1)}\left(-\frac{\lambda^2}{m^4}\right)^n
\end{aligned}
\tag{8.26}
$$

Clearly, from analyzing the n-dependence of the ratios of Γ functions, the expansion in m is convergent (with ∞ radius of convergence) while the expansion in power of λ has zero radius of convergence. This comes as no surprise. If we had flipped the sign of m^2, the integral itself would still be fine and convergent. On the other hand, if we had flipped the sign of λ^2, the integral badly diverges, and therefore, it cannot have a finite radius of convergence around 0. The moral of the story is that perturbation theory around free field theory is basically always given by an asymptotic series which has zero radius of convergence. Nonetheless, and it is one of the most remarkable things about QFT, perturbation theory gives astoundingly accurate answers when it is applicable.

8.7 The loop expansion

We now derive the $\hbar$-dependence of the various orders of perturbation theory:

$$
\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = \int \mathcal{D}\phi\phi(x_1)\cdots\phi(x_n)e^{\frac{i}{\hbar}S[\phi]} \qquad \text{(connected)} \tag{8.27}
$$

A general diagram has

$$
\begin{cases}
\text{E} & \text{external legs} \\
\text{I} & \text{internal lines} \\
\text{V} & \text{vertices}
\end{cases}
\tag{8.28}
$$

The number of loops L is equal to the number of independent internal momenta, after all comservatioins have been imposed on vertices, minus one overall momentum conservation $L = I - (V-1)$. Now calculate to what order in $\hbar$ this graph contributes:

- each propagator contributes one power of $\hbar$;
- each vertex contributes one power of $1/\hbar$.

Hence a graph contributes

$$
\hbar^{E+I-V} = \hbar^{(E+L-1)} \tag{8.29}
$$

For fixed number of external lines, the $\hbar$ expansiion **is** a loop expansion, provided all the couplings in the classical actioin are independent of $\hbar$.

8.8 Truncated Diagrams

To simplify and systematize the enumeration of all diagrams that contribute to a given process, it is useful to identify special sub-categories of diagrams. We have already discussed the connected diagrams, from which all others may be recovered. We now introduce *truncated* and *1-particle irreducible diagrams* which allow to further cut down the number of key diagrams.

In the calculation of the S-matrix, the full Green function is not required; instead we only use it in the combination (scalar field)

$$\prod_{i=1}^{n}\left(\int d^4x_i e^{-ip_i\cdot x_i}(\Box+m^2)_{x_i}Z^{-1/2}\right)\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$$
$$= Z^{-n/2}\prod_{i=1}^{n}(-p_i^2+m^2)\tilde{G}^{(n)}(p_1,\cdots,p_n) \tag{8.30}$$

Here p_i are on-shell external momentum. As we have deduced from the spectral representation, the two-point function behaves like

$$G^{(2)}(p,-p)=\frac{Z}{p^2-m^2}+\text{reg.}\qquad \text{as}\quad p^2\to m^2 \tag{8.31}$$

Hence what really comes in the reduction formula is

$$\lim_{p_i^2\to m^2} Z^{n/2}\prod_{i=1}^{n}\left[G_c^{(2)}(p_i,-p_i)\right]^{-1}\; G^{(n)}(p_1,p_2\cdots p_n) \tag{8.32}$$

One defines the *truncated correlator* by

$$G^{(n)}_{\text{trun}}(p_1,\cdots,p_n)=\prod_{i=1}^{n}G_c^{(2)}(p_i,-p_i)^{-1}G^{(n)}(p_1,\cdots,p_n) \tag{8.33}$$

So that the S-matrix elements are given by

$$\langle q_1\cdots q_m\ \text{out}|p_1\cdots p_n\ \text{in}\rangle_c$$
$$=(Z^{1/2})^{m+n}(2\pi)^4\delta^4(q-p)G^{(n+m)}_{\text{trun}}(q_1,\cdots,q_m;p_1,\cdots,p_n) \tag{8.34}$$

where $\delta(q-p)$ stands for an overall momentum conservation δ-function. Truncation of the n-point correlator reduces the n-point function in that the entire set of diagrams that corresponds to the self-energy corrections of the external legs is to be omitted. Therefore, it is much simpler to enumerate the truncated diagrams than it is to enumerate all connected diagrams.

8.9 Connected graphs

A connected diagram is such that its graph is connected. The generating functional for connected graphs is the log of $Z[J]$;

$$Z[J] = Z[0]\ \exp\{iG_c[J]\} \tag{8.35}$$

and its expansion in powers of J yields the connected n-point functions,

$$G_c[J] = -i\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\cdots d^4x_n J(x_1)\cdots J(x_n)G_c^{(n)}(x_1,\cdots,x_n) \tag{8.36}$$

Often, connected vacuum expectation values are simply denoted by a subscript c as well, and thus we have

$$G_c^{(n)}(x_1,\cdots,x_n) = \langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle_c \tag{8.37}$$

The proof of the relation between $Z[J]$ and $G_c[J]$ is purely combinatorial. Denoting each separate connected contribution of degree i in J by X_i, $i=1,\cdots,\infty$, we have

$$\exp\sum_{i=1}^{\infty}X_i = \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{n_1+2n_2\cdots+pn_p=p}\frac{p!}{n_1!\cdots n_p!}X_1^{n_1}X_2^{n_2}\cdots X_p^{n_p} \tag{8.38}$$

which indeed reproduces the correct combinatorics for the full expansion order by order p in the power of J, appropriate for $Z[J]$.

8.10 One-Particle Irreducible Graphs and the Effectice Action

A one-particle irreducible diagram is a connected truncated graph which remains connected after *any one internal line is cut*. The Legendre transform of the generating functional for connected graphs $G_c[J]$ is the generating functional for 1PIR's. The Legendre transform is defined as follows. One begins by defining the vacuum expectation value $\varphi(x;J)$ of the field ϕ in the presence of the source J,

$$\varphi(x;J) \equiv \frac{\delta}{\delta J(x)}G_c(J) = \langle 0|\phi(x)|0\rangle_J \tag{8.39}$$

Now we assume that the relationship defining φ is invertible, and that J may be eliminated in terms of φ in a unique manner,

$$\varphi(x;J) = \varphi(x) \quad\Longleftrightarrow\quad J(x) = J(x;\varphi) \tag{8.40}$$

The Legendre transform $\Gamma[\varphi]$ of $G_c[J]$ is then defined by

$$\Gamma[\varphi] \equiv G_c[J] - \int d^4x J(x)\varphi(x)\Big|_{J=J(x;\varphi)} \tag{8.41}$$

where it is understood that J is eliminated in terms of φ. As always, it follows that

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = -J(x) \tag{8.42}$$

Now $J(x)$ should be thought of as a source which must be applied from the outside in order that the vacuum expectation value of ϕ be φ. When $J=0$, we recover the original theory, without source, so that

$$\left.\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}\right|_{\varphi_c} = 0 \qquad \text{at} \qquad J(x;\varphi_c) = 0 \tag{8.43}$$

Next we investigate which diagrams are generated by $\Gamma[\varphi]$: For simplicity, we assume for the moment that $\varphi_c = 0$ (the case of no spontaneous symmetry breaking)

$$\Gamma[\varphi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots \int d^4x_n \; \Gamma^{(n)}(x_1, \cdots x_n)\varphi(x_1)\cdots\varphi(x_n) \tag{8.44}$$

The effective field equation (8.43) then implies that $\Gamma^{(1)}(x) = 0$. In other words, the 1PIR 1-point function vanishes.

To obtain relations for more general $\Gamma^{(n)}$'s, we differentiate the relation defining φ with respect to φ. We use the functional chainrule to express the derivation with respect to φ in terms of the differentiation with respect to J,

$$\delta^4(x-y) = \frac{\delta}{\delta\varphi(y)}\left(\frac{\delta G_c[J]}{\delta J(x)}\right) = \int d^4z \; \frac{\delta J(z)}{\delta\varphi(y)} \; \frac{\delta^2 G_c[J]}{\delta J(z)\delta J(x)} \tag{8.45}$$

However, from (8.42), we have

$$-\frac{\delta J(z)}{\delta\varphi(y)} = \frac{\delta^2\Gamma[\varphi]}{\delta\varphi(z)\delta\varphi(y)} \tag{8.46}$$

Evaluating this relation at $\varphi = J = 0$, we find a relation between the connected and 1PIR 2-point functions,

$$-\delta^4(x-y) = \int d^4z \Gamma^{(2)}(x,z) G_c^{(2)}(z,y) \qquad \text{or} \qquad \Gamma^{(2)} = -\left\{G_c^{(2)}\right\}^{-1} \tag{8.47}$$

It is useful to illustrate this relation by working out schematically the corrections to the 2-point function. If we denote the 1PIR corrections to the self-energy by $\Sigma(p)$, then the full 2-point function is given by

$$\begin{aligned} G_c^{(2)}(p) &= \frac{1}{p^2-m^2} + \frac{1}{p^2-m^2}\Sigma(p)\frac{1}{p^2-m^2} + \frac{1}{p^2-m^2}\Sigma(p)\frac{1}{p^2-m^2}\Sigma(p)\frac{1}{p^2-m^2} + \cdots \\ &= \frac{1}{p^2-m^2-\Sigma(p)} = \Gamma^{(2)}(p)^{-1} \end{aligned} \tag{8.48}$$

which confirms our assertion that Γ generates 1PIR only. A graphical representation is given in Fig 5.

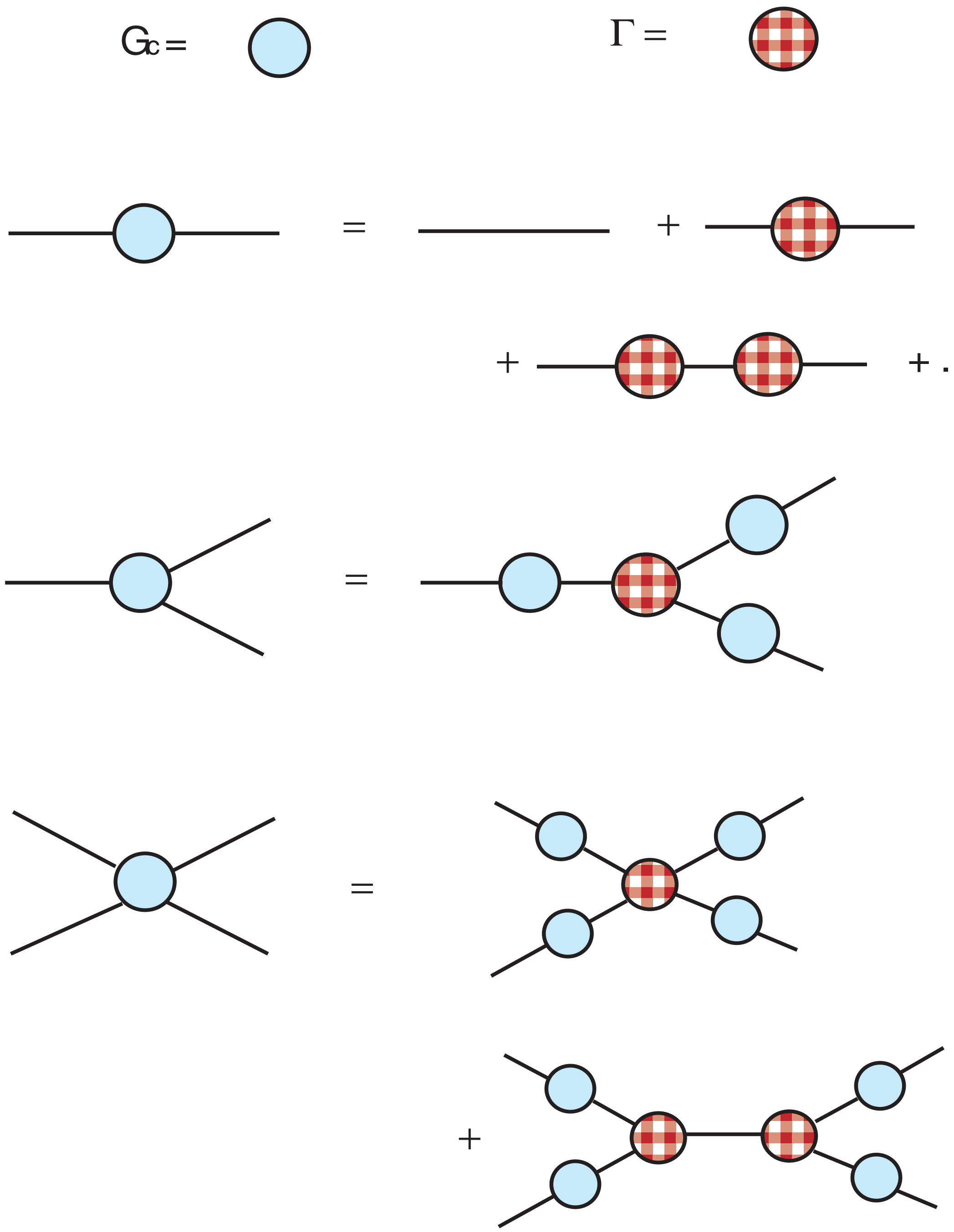


Figure 11: Relations between connected and 1PIR graphs in a theory with cubic and quartic couplings, and vanishing 1-point function.

By differentiating once more, and setting $\varphi = J = 0$, we obtain information on the 3-point function,

$$\int d^4u\Gamma^{(3)}(u, y, z)G_c^{(2)}(x, u) = \int\int d^4ud^4v\Gamma^{(2)}(u, y)\Gamma^{(2)}(v, z)G_c^3(x, u, v) \tag{8.49}$$

By multiplying by $G^{(2)}(y, \cdot)G^{(2)}(z, \cdot)$, we obtain

$$G^{(3)}(x, y, z) \quad = \quad \int d^4ud^4vd^4wG^{(2)}(x, u)G^{(2)}(y, v)G^{(2)}(z, w)\Gamma^{(3)}(u, v, w) \tag{8.50}$$

So, for the three-point function, the 1PIR part arises from truncating the graph. One may repeat the exercise for 4 legs. Graphical representations are given in Fig 11.

Clearly, the expansion is in terms of the *tree-level* graphs, $\Gamma^{(n)}$ used as *generalized vertices*. This is the *skeleton expansion* of any graph in terms of 1PIR's.

$\Gamma[\varphi]$ is the so-called effective action of the system. Why action? Here, it is useful to make a loop expansion

$$Z[J] = \int \mathcal{D}\phi \exp \frac{i}{\hbar}\left\{S[\phi] + \int J\phi\right\} = \exp\left\{\frac{i}{\hbar}G_c[J]\right\} \tag{8.51}$$

(note, here we have kept the J-independent term in G[J]). Now, keep only the leading term in $\hbar \to 0$, then the integral will be dominated by its saddle points. Let's assume that the saddle point is unique and denote it by ϕ_s, then we have

$$\frac{\delta S[\phi_s]}{\delta\phi(x)} + J(x) = 0 \tag{8.52}$$

But this is precisely the same equation as was satisfied by Γ itself. Thus, to leading order in $\hbar$, we have $\Gamma[\varphi] = S[\varphi]$. All Green functions can be reconstructed from it by the skeleton expansion so that $\Gamma[\varphi]$ is the essential quantity in a quantum field theory.

8.11 Schwinger Dyson Equations

We concentrate on a scalar field theory, with Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi) \tag{8.53}$$

Now we use the fact that the functional integral of a functional derivative vanishes,

$$\int \mathcal{D}\phi\frac{\delta}{\delta\phi(z)}\left\{\phi(x_1)\cdots\phi(x_n)e^{iS[\phi]}\right\} = 0 \tag{8.54}$$

Working out the actual derivatives under the integral, we obtain an hierarchy of equations, involving the correlators,

$$\sum_{i=1}^{n} \delta^4(x_i - z)\langle 0|T\phi(x_1)\cdots\widehat{\phi(x_i)}\cdots\phi(x_n)|0\rangle \tag{8.55}$$
$$-i(\Box + m^2)_z\langle 0|T\phi(x_1)\cdots\phi(x_n)\phi(z)|0\rangle - i\langle 0|T\phi(x_1)\cdots\phi(x_n)\frac{\partial V}{\partial\phi}(z)|0\rangle = 0$$

providing a recursive type of relationship that expresses the quantum equations of motion. These are the Schwinger-Dyson equations.

8.12 The Background Field Method for Scalars

Feynman diagrams and standard perturbation theory do not always provide the most efficient calculational methods. This is especially true for theories with complicated local invariances, such as gauge and reparametrization invariances. But the method also provides important short-cuts for scalar field theories.

The background field method is introduced first for a theory with a single scalar field ϕ. The classical action is denoted $S_0[\phi]$, and the full quantum action, including all renormalization counterterms is denoted by $S[\phi]$. The generating functional for connected correlators is given by

$$\exp\left\{\frac{i}{\hbar}G_c[J]\right\} \equiv \int D\phi\ \exp\left\{\frac{i}{\hbar}S[\phi] + \frac{i}{\hbar}\int d^4x J(x)\phi(x)\right\} \tag{8.56}$$

By the very construction of the renormalized action $S[\phi]$, $G_c[J]$ is a finite functional of J. The Legendre transform is defined by

$$\Gamma[\varphi] \equiv \int d^4x J(x)\varphi(x) - G_c[J] \tag{8.57}$$

where J and φ are related by

$$\varphi(x) = \frac{\delta G_c[J]}{\delta J(x)} \qquad\qquad J(x) = -\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} \tag{8.58}$$

Recall that $\Gamma[\varphi]$ is the generating functional for 1-particle irreducible correlators.

To obtain a recursive formula for Γ, one proceeds to eliminate J from (17.1) using the above relations,

$$\exp\frac{i}{\hbar}\left(\Gamma[\varphi] + \int J\varphi\right) = \int D\phi\ \exp\frac{i}{\hbar}\left(S[\phi] - \int dx\phi(x)\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}\right) \tag{8.59}$$

Moving the J-dependence from the left to the right side, and expressing J in terms of Γ in the process, we obtain,

$$\exp \frac{i}{\hbar}\Gamma[\bar{\phi}] = \int D\phi \ \exp \frac{i}{\hbar}\Big(S[\phi] - \int dx (\phi - \varphi)(x) \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} \Big) \tag{8.60}$$

The final formula is obtained by shifting the integration field $\phi \to \varphi + \sqrt{\hbar}\phi$,

$$\boxed{\exp \frac{i}{\hbar}\Gamma[\varphi] = \int D\phi \ \exp \frac{i}{\hbar}\Big(S[\varphi + \sqrt{\hbar}\phi] - \sqrt{\hbar} \int dx \phi(x) \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} \Big)} \tag{8.61}$$

At first sight, it may seem that little has been gained since both the left and right hand sides involve Γ. Actually, the power of this formula is revealed when it is expanded in powers of $\hbar$, since on the rhs, Γ enters to an order higher by $\sqrt{\hbar}$ than it enters on the lhs.

The $\hbar$-expansion is performed as follows,

$$\begin{aligned} S[\phi] &= S_0[\phi] + \hbar S_1[\phi] + \hbar^2 S_2[\phi] + \mathcal{O}(\hbar^3) \\ \Gamma[\varphi] &= \Gamma_0[\varphi] + \hbar \Gamma_1[\varphi] + \hbar^2 \Gamma_2[\varphi] + \mathcal{O}(\hbar^3) \end{aligned} \tag{8.62}$$

The recursive equation to lowest order clearly shows that $\Gamma_0[\varphi] = S_0[\varphi]$. The interpretation of the terms $S_i[\phi]$ for $i \geq 1$ is clearly in terms of counterterms to loop order i, while that of $\Gamma_i[\varphi]$ is that of i-loop corrections to the effective action. The 1-loop correction is readily evaluated for example,

$$\exp i\Gamma_1[\varphi] = e^{iS_1[\varphi]} \int D\phi \exp\Big\{ \frac{i}{2} \int d^4x \ \phi(x) \int d^4y \ \phi(y) \frac{\delta^2 S_0[\varphi]}{\delta\varphi(x)\delta\varphi(y)} \Big\} \tag{8.63}$$

For example, if S_0 describes a field ϕ with mass m and interaction potential $V(\phi)$, we have

$$S_0[\phi] = \int dx \Big(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi) \Big) \tag{8.64}$$

then we have

$$\begin{aligned} e^{i\Gamma_1[\varphi]} &= e^{iS_1[\varphi]} \int D\phi \ \exp\Big\{ \frac{i}{2} \int d^4x \phi \Big(\Box + m^2 + V''(\varphi) \Big) \phi \Big\} \\ \Gamma_1[\varphi] &= S_1[\varphi] + \frac{i}{2} \ln \mathrm{Det} \Big[\Box + m^2 + V''(\varphi) \Big] \end{aligned} \tag{8.65}$$

Upon expanding this formula in perturbation theory, the standard expressions may be recovered,

$$\Gamma_1[\varphi] = S_1[\varphi] + \Gamma_1[0] + \frac{i}{2} \ln \mathrm{Det} \Big[1 + V''(\varphi) \frac{1}{\Box + m^2} \Big] \tag{8.66}$$

which clearly reproduces the standard 1-loop expansion. Therefore, for scalar theories, the gain produced by the use of the background field method is usually only minimal.

9 Basics of Perturbative Renormalization

In this section, we begin by presenting some useful techniques for the calculation of Feynman diagrams. Next, we introduce some of the basic ideas and techniques of renormalization in perturbation theory. Finally, we study the analytic behavior of the 4-point diagrams to one loop order in $\lambda\phi^4$ as a function of external momenta.

9.1 Feynman parameters

One will often be confronted with loop integrals over internal momenta that involve a product of momentum space propagators. It would not be totally straightforward to do these integrals, if it were not for the use of Feynman parameters. The simplest type of integrals of this type may be taken to be

$$I_d(p) = \int \frac{d^dk}{(2\pi)^d} \, \frac{1}{(k+p)^2 - m^2 + i\epsilon} \times \frac{1}{k^2 - m^2 + i\epsilon} \tag{9.1}$$

We have considered this integral in any dimension of space-time d. Feynman's trick for dealing with the product of propagators is to represent the product in terms of an integral. The basic formula is as follows,

$$\frac{1}{ab} = \int_0^1 d\alpha \, \frac{1}{[a\alpha + b(1-\alpha)]^2} \tag{9.2}$$

Applying this formula to the integral $I_d(p)$, we obtain

$$I_d(p) = \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \, \frac{1}{[(k+\alpha p)^2 + \alpha(1-\alpha)p^2 - m^2 + i\epsilon]^2} \tag{9.3}$$

For $d \geq 4$, this integral actually diverges for any p due to the contributions at large k. We shall temporarily ignore this problem and consider $d < 4$. In a convergent integral, one may now safely shift $k \to k - \alpha p$, so that we end up with

$$I_d(p) = \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \, \frac{1}{[k^2 + \alpha(1-\alpha)p^2 - m^2 + i\epsilon]^2} \tag{9.4}$$

The great advantage of this form of the integral is that all momenta enter through k^2 and p^2, which are Lorentz scalars.

More generally, the product of several propagator denominators may be treated with multiple integrals. The easiest is to use the exponential representation first,

$$\frac{1}{a_1^{\mu_1} \cdots a_n^{\mu_n}} = \int_0^\infty d\lambda_1 \cdots \int_0^\infty d\lambda_n \, \frac{\lambda_1^{\mu_1 - 1} \cdots \lambda_n^{\mu_n - 1}}{\Gamma(\mu_1) \cdots \Gamma(\mu_n)} \, e^{-\lambda_1 a_1 - \cdots - \lambda_n a_n} \tag{9.5}$$

Putting $\lambda_i = \lambda \alpha_i$, we recover the Feynman parametrization,

$$\begin{aligned}\frac{1}{a_1^{\mu_1} a_2^{\mu_2} \cdots a_n^{\mu_n}} &= \frac{\Gamma(\mu_1 + \mu_2 + \cdots + \mu_n)}{\Gamma(\mu_1)\Gamma(\mu_2)\cdots\Gamma(\mu_n)} \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_n \delta(\alpha_1 + \cdots + \alpha_n - 1) \\ &\qquad \times \frac{\alpha_1^{\mu_1 - 1} \cdots \alpha_n^{\mu_n - 1}}{[\alpha_1 a_1 + \cdots + \alpha_n a_n]^{\mu_1 + \cdots + \mu_n}}\end{aligned} \tag{9.6}$$

It is of course possible to integrate out one of the α_i by using the δ-function, but the result is then less symmetrical in the a_i. For $n = 2$ and $\mu_1 = \mu_2 = 1$, we recover the earlier result this way.

9.2 Integrations over internal momenta

It remains to evaluate the integrations over internal momenta. First, we shall assume that Feynman parameters have been employed to put the integration into standard form,

$$I(M; d, n) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - M^2 + i\epsilon)^n} \tag{9.7}$$

We shall now evaluate this integral for all values of d and n where it converges and then analytically continue the result throughout the complex planes for both d and n. As $k \to \infty$, the integrals converge as long as $d < 2n$, and we assume this bound satisfied.

The first step is to analytically continue to Euclidean internal momenta, $k^o \to i k_E^o$m leaving M alone. This is possible, because the singularities in the complex k^o plane may be completely avoided during this process thanks to the $i\epsilon$ prescription of the propagator. As a result, we have $k^2 \to -k_E^2$ and $dk^o \to i dk_E^o$, so that

$$I(M; d, n) = i^{2n+1} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{(k_E^2 + M^2)^n} \tag{9.8}$$

where k_E^2 is the Euclidean momentum square. To carry out this integration, we use d-dimensional spherical coordinates and the formula for the volume of the sphere S^{d-1},

$$V(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{9.9}$$

This formula is most easily proven as follows. Consider the Gaussian inetgral in d dimensions, which is just a product of d Gaussian inetgrals in 1 dimension,

$$\int d^d \vec{k}\, e^{-\vec{k}^2} = \pi^{d/2} \tag{9.10}$$

Working out the same formula in spherical coordinates, we have

$$\int d^d \vec{k}\, e^{-\vec{k}^2} = V(S^{d-1}) \int_0^\infty dk\, k^{d-1}\, e^{-k^2} = \frac{1}{2} V(S^{d-1}) \Gamma(d/2) \tag{9.11}$$

which upon combination with the first yields the desired expression for $V(S^{d-1})$. It is easy to check that $V(S^1) = 2\pi$, $V(S^2) = 4\pi$ and $V(S^3) = 2\pi^2$ as expected. The above formula is, however, valid for any d. Introducing real positive k such that $k_E^2 = k^2$, we have

$$I(M; d, n) = \frac{2\ i^{2n+1}}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty dk \frac{k^{d-1}}{(k^2 + M^2)^n} \tag{9.12}$$

The remaining integral may be carried out as follows. First, we let $k^2 = M^2 t$, so that

$$\int_0^\infty dk \frac{k^{d-1}}{(k^2 + M^2)^n} = \frac{1}{2} \frac{1}{(M^2)^{n-d/2}} \int_0^\infty dt\ \frac{t^{d/2-1}}{(t+1)^n} \tag{9.13}$$

Next, we let $x = 1/(t+1)$, so that

$$\begin{aligned} \int_0^\infty dk \frac{k^{d-1}}{(k^2 + M^2)^n} &= \frac{1}{2} \frac{1}{(M^2)^{n-d/2}} \int_0^1 dx\ (1-x)^{d/2-1} x^{n-d/2-1} \\ &= \frac{1}{2} \frac{1}{(M^2)^{n-d/2}} \frac{\Gamma(d/2)\Gamma(n-d/2)}{\Gamma(n)} \end{aligned} \tag{9.14}$$

Therefore, we find

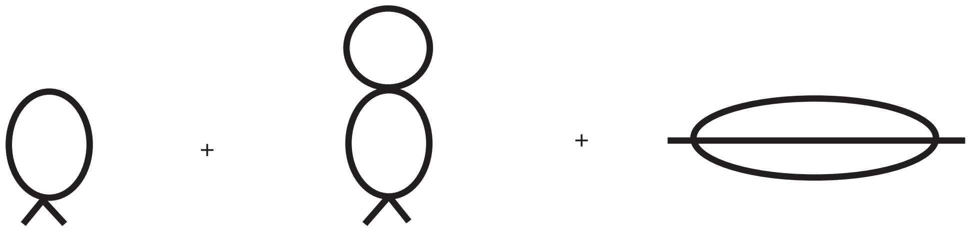
$$I(M; d, n) = \frac{i^{2n+1}\Gamma(n-d/2)}{(4\pi)^{d/2}\Gamma(n)}\ \frac{1}{(M^2)^{n-d/2}} \tag{9.15}$$

The calculation was carried out there where the integral was convergent. We see now however, that the analytic continuation of the expression that we have obtained is straightforward since the analytic continuation of the Γ-function is well-known. Thus, we view the result obtained above as THE answer !

9.3 The basic idea of renormalization

As soon as loops occur, the diagram is potentially divergent when the internal momentum integration fails to converge in the limit of large momenta. It would seem that the appearance of divergences renders the quantum field theory meaningless. Actually, the procedure of renormalization is required to organize and properly interpret the loop corrections that arise. We shall illustrate the salient features and ideas of renormalization within the context of $\lambda\phi^4$ theory, with Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{9.16}$$



(a) (b) (c)

Figure 12: 1PIR contributions to mass (a-b-c) and field (c) renormalization in $\lambda\phi^4$ theory up to order $\mathcal{O}(\lambda^2)$

9.3.1 Mass renormalization

We begin by considering the 2-point function to 1-loop order, given by the insertion of a single bubble graph, as in Fig 12 (a). Two loop contributions are also shown for later use.

The self-energy to this order is given by

$$\begin{aligned} \Gamma^{(2)}(p) &= p^2 - m^2 + \lambda G_o(m) + \mathcal{O}(\lambda^2) \\ G_o(m) &\equiv \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \end{aligned} \tag{9.17}$$

Clearly, the integral G_o for the bubble graph is UV divergent. The first thing we need to do is to *regularize* the integral. We shall later on give a long list of fancy regulators. For the time being, let's be simple minded and cutoff the Euclidean continuation of this integral by a cutoff $k_E^2 < \Lambda^2$. We then have

$$G_o(m, \Lambda) = \int_{k_E^2 < \Lambda^2} \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} = \frac{1}{(4\pi)^2} \left(\Lambda^2 - m^2 \ln \frac{\Lambda^2 + m^2}{m^2} \right) \tag{9.18}$$

The momentum independent G_o will have the effect of changing the mass of the partice, albeit by a divergent amount.

The first key insight offered by renormalization theory is that the mass parameter m that enters the Lagrangian is not equal to the mass of the physical particle created by the ϕ-field. The mass in the Lagrangian equals the mass of the physical ϕ particle only in the free field limit when $\lambda = 0$. If $\lambda \neq 0$, however, the relation between the mass in the Lagrangian and the physical mass will depend on the coupling.

The second key insight offered by renormalization theory is that the mass parameter appearing in the Lagrangian is not a physically observable quantity. Only the mass of the physical ϕ particle is observable – as a pole in the S-matrix. Therefore, the mass parameter in the Lagrangian, or *bare mass* m_o, should be left *unspecified* at first, to be determined later by the mass m of the physical particle.

The third insight offered by renormalization theory is that it is possible – in *so-called renormalizable quantum field theories* – to determine the bare mass m_o as a function of the physical mass m, the coupling λ and the cutoff Λ in such a way that m may be kept fixed as the cutoff is removed $\Lambda \to \infty$.

To illustrate the points made above on ϕ^4 theory to order λ, we are thus instructed to take as a starting point the Lagrangian with bare mass m_0,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_o^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{9.19}$$

and to recompute all of the above up to order λ. The slef-energy corrections are given by

$$\begin{aligned}
\Gamma^{(2)}(p) &= p^2 - m_o^2 + \lambda G_o(m_o) + \mathcal{O}(\lambda^2) \\
G_o(m_o, \Lambda) &= \frac{1}{(4\pi)^2}\left(\Lambda^2 - m_o^2 \ln \frac{\Lambda^2 + m_o^2}{m_o^2}\right)
\end{aligned} \tag{9.20}$$

In order for the mass of the ϕ particle to be m – to this order λ – it suffices to set

$$m^2 = m_o^2 - \frac{\lambda}{(4\pi)^2}\left(\Lambda^2 - m_o^2 \ln \frac{\Lambda^2 + m_o^2}{m_o^2}\right) + \mathcal{O}(\lambda^2) \tag{9.21}$$

Actually, this relation may be solved to this order by workin iteratively,

$$m_o^2 = m^2 + \frac{\lambda}{(4\pi)^2}\left(\Lambda^2 - m^2 \ln \frac{\Lambda^2 + m^2}{m^2}\right) + \mathcal{O}(\lambda^2) \tag{9.22}$$

The key lesson learned from this simple example is that as the cutoff Λ is sent to ∞, it is necessary to keep on adjusting m_o so that the physical mass m can remain constant. As $\Lambda \to \infty$, this actually requires $m_o \to \infty$ as well.

9.3.2 Coupling Constant Renormalization

The lowest 1PIR corrections to the 4-point function arise at order $\mathcal{O}(\lambda^2)$ and are depicted in Fig 13. The three diagrams are permutations in the external legs of the single basic diagram. In the study of any 4-point, it is very useful to introduce the Mandelstam variables, s, t, u, which are the Lorentz invariant combinations one can form out of the (all incoming) external momenta p_i, $i = 1, 2, 3, 4$. They are defined by

$$\begin{aligned}
s &\equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 \\
t &\equiv (p_1 + p_4)^2 = (p_2 + p_3)^2 \\
u &\equiv (p_1 + p_3)^2 = (p_2 + p_4)^2
\end{aligned} \tag{9.23}$$

The Mandelstam relations obey $s + t + u = p_1^2 + p_2^2 + p_3^2 + p_4^2$, and if the external particles are on-shell then the rhs equals $4m^2$.

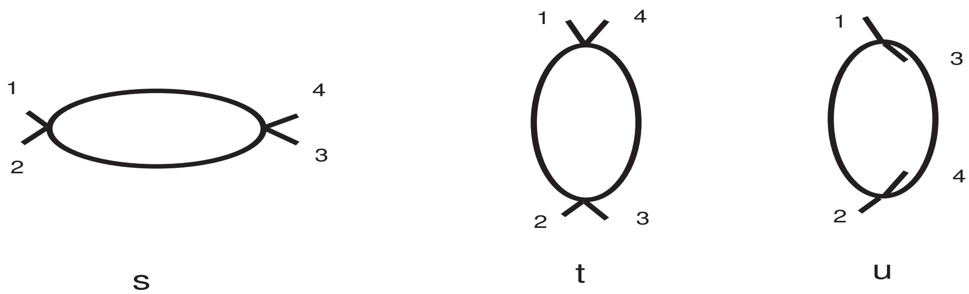


Figure 13: 1PIR $\mathcal{O}(\lambda^2)$ contributions to coupling constant renormalization in $\lambda\phi^4$ theory

The 4-point 1PIR contribution to this order is therefore obtained as follows,

$$\Gamma^{(4)}(p_i) = \lambda^2 \Big(I(s) + I(t) + I(u) \Big) \tag{9.24}$$

The remaining integrals is readily computed using Feynman rules and

$$I(s) = \frac{1}{2}(-i)^3 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(k + p_1 + p_2)^2 - m^2 + i\epsilon} \tag{9.25}$$

The integral $I(s)$ receives a logarithmically divergent contribution from $k \to \infty$. Remarkably, this divergence is the same for all external momenta, so that the difference $I(s) - I(s')$ is finite and the divergence is entirely contained in $I(0)$ for example.

The momentum independent divergent contribution may be regularized using an Euclidean momenta cutoff, and we find,

$$I(0) \;\; = \;\; \int_{k_E^2 < \Lambda^2} \frac{d^4k_E}{(2\pi)^4} \frac{1}{[k_E^2 + m^2]^2} = \frac{1}{32\pi^2} \left(\ln \frac{\Lambda^2 + m^2}{m^2} - \frac{\Lambda^2}{\Lambda^2 + m^2} \right) \tag{9.26}$$

This contribution is a constant and therefore of the same form as the original $\lambda\phi^4$ coupling in the Lagrangian. Thus the total 1PIR 4-point function is given as follows,

$$\Gamma^{(4)}(p_i) = -\lambda + 3\lambda^2 I(0) + \lambda^2 \Big\{ I(s) + I(t) + I(u) - 3I(0) \Big\} + \mathcal{O}(\lambda^3) \tag{9.27}$$

Now we repeat verbatim from mass renormalization, the instructions given by renormalization theory for the renormalization of the coupling constant.

The first key insight from renormalization theory is that the coupling parameter λ that enters the Lagrangian is not equal to the coupling of the physical ϕ particles. The coupling in the Lagrangian equals the coupling of the physical ϕ particle only in the free field limit when $\lambda = 0$. If $\lambda \neq 0$, however, the relation between the coupling in the Lagrangian and the coupling of the physical ϕ particles will depend on the coupling itself.

The second key insight from renormalization theory is that the coupling parameter appearing in the Lagrangian is not a physically observable quantity. Only the coupling of the physical ϕ particles is observable – as the transition probability in the S-matrix. Therefore, the coupling parameter in the Lagrangian, or *bare coupling* λ_o, should be left *unspecified* at first, to be determined later by the coupling λ of the physical ϕ particles.

The third insight from renormalization theory is that it is possible – in *so-called renormalizable quantum field theories* – to determine the bare coupling λ_o as a function of the physical coupling λ, the mass m and the cutoff Λ in such a way that λ may be kept fixed as the cutoff is removed $\Lambda \to \infty$.

To illustrate the points made above on ϕ^4 theory to order λ^2, we are thus instructed to take as a starting point the Lagrangian with bare mass m_0 and bare coupling λ_o,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_o^2\phi^2 - \frac{\lambda_o}{4!}\phi^4 \tag{9.28}$$

and to recompute all of the above up to order λ^2. The 4-point corrections are given by

$$\Gamma^{(4)}(p_i) \;\; = \;\; -\lambda_o + 3\lambda_o^2 I(0) + \mathcal{O}(\lambda_o^3) \tag{9.29}$$

In order for the coupling of the ϕ particles to be λ – to this order λ^2 – it suffices to set

$$-\lambda = -\lambda_o + 3\lambda_o^2 I(0) + \mathcal{O}(\lambda_o^3) \tag{9.30}$$

Actually, this relation may be solved to this order by working iteratively, (it is customary to omit terms that vanish in the limit $\Lambda \to \infty$, and we shall also do so here)

$$\lambda_o = \lambda + \frac{3\lambda^2}{32\pi^2}\ln\frac{\Lambda^2}{m^2} + \mathcal{O}(\lambda^3) \tag{9.31}$$

The key lesson learned from this simple example is that as the cutoff Λ is sent to ∞, it is necessary to keep on adjusting λ_o so that the physical coupling λ can remain constant. As $\Lambda \to \infty$, this actually requires $\lambda_o \to \infty$ as well. Notice that to higher orders, the renormalization of the mass and of the coupling will be intertwined and greater care will be required to extract the correct procedure.

9.3.3 Field renormalization

Considering the 2-point function still in $\lambda\phi^4$ theory, but now to order $\mathcal{O}(\lambda^2)$, we have the additional 1PIR diagrams of Fig 12 (b), (c). We encounter further mass renormalization effects, which may be dealt with by the methods explained above as well. We also encounter, through the diagram (c) in Fig 12, also a new type of divergence. This may be seen from the explicit expression of the diagram, (the $i\epsilon$ prescription is understood)

$$\Gamma_2^{(2)}(p) = \frac{\lambda^2}{6}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(k+p)^2-m^2}\,\frac{1}{(k+\ell)^2-m^2}\,\frac{1}{\ell^2-m^2} \tag{9.32}$$

This integral is quadratically divergent, but the quadratic divergence is p-independent. Thus, the differences $\Gamma_2^{(2)}(p)-\Gamma_2^{(2)}(p')$ are less divergent. By Lorentz invariance, we expect $\Gamma_2^{(2)}(p)$ to vanish to second order as $p\to 0$. The second derivative is easily computed,

$$\frac{\partial^2\Gamma_2^{(2)}(p)}{\partial p_\mu \partial p^\mu} = \frac{\lambda^2}{6}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4\ell}{(2\pi)^4}\,\frac{8m^2}{[(k+p)^2-m^2]^3}\,\frac{1}{(k+\ell)^2-m^2}\,\frac{1}{\ell^2-m^2} \tag{9.33}$$

This integral is indeed logarithmically divergent because the sub-integration over ℓ is logarithmically divergent. Recognizing the integration over ℓ as the integral $I(k^2)$ encountered at the time of coupling constant renormalization, we may write the above as

$$\frac{\partial^2\Gamma_2^{(2)}(p)}{\partial p_\mu \partial p^\mu} = i\frac{\lambda^2}{3}\int\frac{d^4k}{(2\pi)^4}\,\frac{8m^2}{[(k+p)^2-m^2]^3}\Big(I(0)+\{I(k^2)-I(0)\}\Big) \tag{9.34}$$

The integral over $\{I(k^2)-I(0)\}$ is convergent. The integral multiplying $I(0)$ is convergent and yields

$$\frac{\partial^2\Gamma_2^{(2)}(p)}{\partial p_\mu \partial p^\mu}\bigg|_{I(0)} = \frac{\lambda^2}{6\pi^2}I(0) = \frac{\lambda^2}{3(4\pi)^4}\ln\frac{\Lambda^2}{m^2} \tag{9.35}$$

Thus, we see that a divergent contribution arises which is proportional to p^2. At first sight, it would not appear that there is a coupling in the Lagrangian capable of absorbing this divergence. But in fact there is : the kinetic term is customarily normalized to 1/2 by choosing an appropriate normalization of the canonical field. The divergent correction found here indicates that quantum effects change this nromalization, and give rise to *field renormalization*, also sometimes improperly referred to as wave-function renormlization.

Therefore, the starting point ought to be the *bare Lagrangian* in which the field normalization, the mass and the coupling have all been left at their bare values. One often uses the following convention for the bare Lagrangian

$$\mathcal{L}_B = \frac{1}{2}Z\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}Zm_o^2\phi^2 - \frac{\lambda_o}{4!}Z^2\phi^4 \tag{9.36}$$

which, when re-expressed in terms of the *bare field*

$$\phi_o(x) \equiv \sqrt{Z}\phi(x) \tag{9.37}$$

takes the form of the classical Lagrangian, but now with bare mass and coupling,

$$\mathcal{L}_B = \frac{1}{2}\partial^\mu\phi_o\partial_\mu\phi_o - \frac{1}{2}m_o^2\phi_o^2 - \frac{\lambda_o}{4!}\phi_o^4 \tag{9.38}$$

The constant Z, to this order is given by

$$Z = 1 - \frac{\lambda^2}{3(4\pi)^4} \ln \frac{\Lambda^2}{m^2} + \mathcal{O}(\lambda^3) \tag{9.39}$$

Unlike mass and coupling renormalization in $\lambda\phi^4$ theory, field renormalization has no contributions to first order in the coupling.

9.4 Renormalization point and renormalization conditions

Now that we have understood how to extract finite answers from calculations involving divergent cutoffs, we need to make precise how to extract physical answers as well. This is achieved by imposing *renormalization conditions* on some of the correlators in the theory. To completely deteremine the theory, there are to be exactly as many independent renormalization conditions as there are bare parameters in the bare Lagrangian. For scalar $\lambda\phi^4$ theory, there are 3 parameters and thus there must be three independent renormalization conditions.

Mass and field renormalization will require two independent renormalization conditions of the 2-point function $\Gamma^{(2)}(p)$ while coupling constant renormalization will require one renormalization condition on the 4-point function $\Gamma^{(4)}(p_1, p_2, p_3, p_4)$. The resulting correlators will depend upon these renormalization condition and therefore they will have been renormalized, a fact that is shown by suffixing the letter R.

Clearly, there is freedom in choosing the momenta at which the renormalization conditions are to be imposed. This choice of momenta is referred to as the *renormalization point*. It may seem most natural to relate the renormalization conditions to the properties of the physical particles. For example, if m is to be the mass of the physical ϕ particle (the position of the pole in the S-matrix corresponding to the exchange of a s ingle particle), then the appropriate condition is an *on-shell renormalization condition*,

$$\Gamma_R^{(2)}(p)\Big|_{p^2=m^2} = 0 \tag{9.40}$$

But Lorentz invariance allows for a more general choice, such as for example this *off-shell renormalization condition*,

$$\Gamma_R^{(2)}(p)\Big|_{p^2=\mu^2} = \mu^2 - m^2 \tag{9.41}$$

The same choice of renormalization points arise when dealing with field renormalization.

Coupling constant renormalization, which involves the 4-point function, and thus 3 independent momenta, offers even more freedom of choice of renormalization points. Since

$s+t+u = p_1^2+p_2^2+p_3^2+p_4^2$, and on-shell renormalization consition with maximum symmetry amongst the momenta would be

$$\Gamma_R^{(4)}(p_1, p_2, p_3, p_4)\Big|_{s=t=u=4m^2/3} = -\lambda \tag{9.42}$$

but it is equally easy to take this condition off-shell to the point μ,

$$\Gamma_R^{(4)}(p_1, p_2, p_3, p_4)\Big|_{s=t=u=4\mu^2/3} = -\lambda \tag{9.43}$$

On-shell versus off-shell condition may be of use in turn in a variety of circumstances as we shall see later.

9.5 Renormalized Correlators

Let us impose the on-shell renormalization conditions

$$\begin{aligned}
\Gamma_R^{(2)}(p)\Big|_{p^2=m^2} &= 0 \\
\frac{\partial \Gamma_R^{(2)}(p)}{\partial p^2}\Big|_{p^2=m^2} &= 1 \\
\Gamma_R^{(4)}(p_1, p_2, p_3, p_4)\Big|_{s=t=u=4m^2/3} &= -\lambda
\end{aligned} \tag{9.44}$$

The full expressions of the renormalized correlators are then found to be given by

$$\begin{aligned}
\Gamma_R^{(2)}(p) &= p^2 - m^2 + \\
\Gamma_R^{(4)}(p_i) &= -\lambda + \lambda^2 \Big\{ I(s) + I(t) + I(u) - 3I\Big(\frac{4m^2}{3}\Big)\Big\}
\end{aligned} \tag{9.45}$$

It remains to evaluate more explicitly the function $I(s) - I(s_o)$.

The quantity $I(s) - I(s_o)$ may be evaluated using a single Feynman parameter, and may be recovered from setting $s_o = 0$. It is convenient to set $s = p^2$, and to omit the $i\epsilon$ symbols, so that

$$I(s) - I(0) = -\frac{i}{2}\int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4}\Big\{\frac{1}{[(k+\alpha p)^2 + \alpha(1-\alpha)s - m^2]^2} - \frac{1}{[k^2 - m^2]^2}\Big\} \tag{9.46}$$

This integral is convergent. We would like to shift the k-integration in the first integral. Since this integral is divergent, however, it is unclear that this procedure would be legitimate. Thus, we begin by proving a lemma.

Lemma 1 *One may shift under logarithmically divergent integrations;*

$$\int \frac{d^4k}{(2\pi)^4}\Big\{\frac{1}{[k^2-m^2+i\epsilon]^2}-\frac{1}{[(k+p)^2-m^2+i\epsilon]^2}\Big\}=0 \tag{9.47}$$

To prove this, notice that the equality holds at $p=0$. Then, differentiating with respect to p^μ for general p, we obtain

$$\int \frac{d^4k}{(2\pi)^4}\frac{4(k+p)_\mu}{[(k+p)^2-m^2+i\epsilon]^3} \tag{9.48}$$

But this integral is convergent and after shifting $k+p\to k$ is odd in $k\to -k$ and therefore vanishes automatically. This proves the lemma for all p.

Using lemma 1, carrying out the analytic continuation $k^o \to ik^o_E$, and carrying out the k_E integration, we find

$$\begin{aligned} I(s)-I(0) &= \frac{1}{2}\int_0^1 d\alpha \int \frac{d^4k_E}{(2\pi)^4}\Big\{\frac{1}{[k_E^2-\alpha(1-\alpha)s+m^2]^2}-\frac{1}{[k_E^2+m^2]^2}\Big\} \\ &= \frac{1}{32\pi^2}\int_0^1 d\alpha \ln\frac{m^2}{m^2-\alpha(1-\alpha)s} \end{aligned} \tag{9.49}$$

We shall now study this integral as a function of s. It is convenient to change integration variables to $2\alpha = 1+x$, so that

$$I(s)-I(0)=\frac{1}{32\pi^2}\ln\frac{4m^2}{s}-\frac{1}{32\pi^2}\int_0^1 dx\ln(x^2+\sigma^2) \tag{9.50}$$

where

$$\sigma^2 \equiv \frac{4m^2}{s}-1 \tag{9.51}$$

Using the relation

$$\int_0^1 dx\ln(x+i\sigma)=(1+i\sigma)\ln(1+i\sigma)-i\sigma\ln(i\sigma)-1 \tag{9.52}$$

we find

$$I(s)-I(0)=\frac{1}{32\pi^2}\; i\sigma\ln\frac{i\sigma-1}{i\sigma+1}=\frac{i}{32\pi^2}\sqrt{\frac{4m^2}{s}-1}\;\ln\frac{i\sqrt{\frac{4m^2}{s}-1}-1}{i\sqrt{\frac{4m^2}{s}-1}+1} \tag{9.53}$$

The integral is real as long as m^2 and s are real and $s<4m^2$. When $s>4m^2$, however, there is a branch cut, whose discontinuity is actuually most easily computed from the integral representation that we started from,

$$\begin{aligned} \mathrm{Im}\Big(I(s)-I(0)\Big) &= 2\pi i\frac{1}{32\pi^2}\int_0^1 d\alpha\theta(\alpha(1-\alpha)s-m^2) \\ &= \frac{i}{16\pi}\theta(s-4m^2)\sqrt{1-\frac{4m^2}{s}} \end{aligned} \tag{9.54}$$

9.6 Summary of regularization and renormalization procedures

Quantum field theories expanded in perturbation theory, will develop UV divergences. The first step in the process of defining the theory properly – without UV divergences – is to regularize it. Over the 80 years of dealing with quantum field theories, physicists have introduced and invented myriad regulators. Of course, it will be advantageous to regularize theories by preserving as many symmetries of the original theory as possible (e.g. Poincaré invariance, gauge invariance, conformal invariance). Here, we give a brief list. Regulators may be grouped according to whether they respect Poincaré invariance.

- **Poincaré non-invariant regulators**

  1. The lattice formulation of quantum field theory is an example of a short space-time distance cutoff. As we shall see later, the lattice is best formulated for *Euclidean QFT*, and provides a perturbative and non-perturbative regularization of the theory. There is a gauge invariant formulation.

  2. Cutoff on momentum space at high momenta. This regulator is perhaps the most intuitive one for qualitative studies in perturbation theory. It provides a perturbative or non-perturbative regularization. Gauge invariance is not maintained in general.

- **Poincaré invariant Regulators**

  1. Dimensional regularization in which the dimension d of space-time is analytically continued throughout the complex d plane. This regularization is only applicable to Feynman diagrams and is thus in essence perturbative. It is very convenient and preserves gauge invariance and reparametrization invariance.

     - Tensor algebra, $\delta_{\mu\nu}\delta^{\nu\mu} = \eta_{\mu\nu}\eta^{\nu\mu} = d$.
     - Spinor algebra

     There are some difficulties with fermions, essentially because spinor representations cannot naturally be continued to fractional dimensions. Therefore, the rules for spinors are a bit ad hoc. Here are some of the key rules,

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu} I \\
\gamma_\mu \gamma^\mu &= d \cdot I \\
\gamma_\mu \gamma_\nu \gamma^\mu &= (2-d)\gamma_\nu \\
\gamma_\mu \gamma_\nu \gamma_\kappa \gamma^\mu &= 2\gamma_\kappa \gamma_\nu + (d-2)\gamma_\nu \gamma_\kappa \\
\mathrm{tr}\ I &= f(d) \quad \text{as long as} \quad f(4) = 4
\end{aligned} \tag{9.55}$$

     There is no good definition of γ^5. Deeper significance for these problems will be derived when discussing the chiral anomaly.

2. Pauli-Villars regulators require introducing extra degrees of freedom with high masses (and the wrong causality kinetic term). This works well to one-loop order and preserves gauge invariance.

3. Heat-kernel and ζ-function regularizations are very powerful methods applicable mostly only to 1-loop, but which preserve gauge invariance as well as reparametrization invariance when formulated on general manifolds !

• **Renormalization**

Once the theory has been regularized, we have a UV finite, but regulator dependent theory. The question that renormalization theory addresses is *whether one can make sense* out of this theory, so as to get a predictive theory. The key to renormalization theory is that in renormalizable theories, all the divergences are of a special form, namely proportional to terms already present in the Lagrangian. Even when one has renormalized the theory to all orders in perturbatiion theory, the question still remains as to whether the full theory is renormalizable (non-perturbatively). Most of these theories are not resolved to this day, such as ϕ^4-theory, theories with broken symmetries with monopoles and so on. The next steps are those of renormalization.

1. Keeping the bare couplings and masses in the Lagrangian undetermined as a function of the physical parameters and the cutoff.

2. Imposition of certain **regulator independent** renormalization conditions at a renormalization point. There should be as many renormalization conditions as independent couplings in the Lagrangian.

3. Derive the bare coupling constants in the Lagrangian as a function of the cutoff and the physical couplings and masses.

4. Take the limit where the cutoff is removed.

10 Functional Integrals for Fermi Fields

Consider a theory of Dirac fermions whose Lagrangian of the Dirac type, i.e. it is bilinear in the Dirac fields and given in terms of a first order differential operator D by

$$\begin{aligned} \mathcal{L} &= \bar\psi D \psi \\ D &= i\partial\!\!\!/ - M \\ M(x) &= mI + \phi(x) I + \phi'(x)\gamma_5 + gA\!\!\!/(x) + g' A\!\!\!/'(x)\gamma_5 \end{aligned} \tag{10.1}$$

Here, M is a matrix-valued function that may include mass terms m, scalar fields ϕ and ϕ' and gauge fields amongst others. The operator D will be required to be self-adjoint.

Can one construct a functional integral representation for the time ordered Green functions, in analogy to the construction we had given for the scalar field ?

$$\langle 0|T\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar\psi^{\beta_1}(y_1)\cdots\bar\psi^{\beta_n}(y_n)|0\rangle \tag{10.2}$$

There is a fundamental difference with the bosonic case: the correlation function is anti-symmetric in the ψ fields. Any functional integral based on an integration over commuting fields will yields a result that is symmetric in the points x_i. The natural way out is via the use of anti-commuting or Grassmann variables, already encountered previously. A set of n anti-commuting variables may be defined as follows,

$$\theta_1, \theta_2 \cdots \theta_n \qquad \text{so that} \qquad \{\theta_i, \theta_j\} = 0 \tag{10.3}$$

In particular the most general analytic functiion of one θ is $F(\theta) = F_o + \theta F_1$. One may now define differentiation

$$\frac{\partial}{\partial\theta} F(\theta) \equiv F_1 \qquad\qquad \left\{ \frac{\partial}{\partial\theta}, \theta \right\} = 1 \tag{10.4}$$

Grassmann numbers i.e. functions of ordinary $x_1 \cdots x_p$ and θ_n, form a graded ring,

(1) Even grading : those which commute with all θ's. Example are the bosonic variables $x_1, \cdots, x_p$, the bilinears $\theta_i\theta_j$, the quadrilinears $\theta_i\theta_j\theta_k\theta_l$ etc.

(2) Odd grading : those which anti-commute with all θ's. Examples are the θ_i, the trilinears $\theta_i\theta_j\theta_k$, etc.

Leibnitz's rule of differentiation has to be adapted. Let F and G be two functions of grading f and g, then

$$\frac{\partial}{\partial\theta}(FG) = \frac{\partial F}{\partial\theta} G + (-)^f F \frac{\partial G}{\partial\theta} \tag{10.5}$$

so that $\frac{\partial}{\partial\theta}$ should be viewed as odd grading.

10.1 Integration over Grassmann variables

Integration over Grassmann variables is defined by the following rules:

(1) linearity

$$\int d\theta F(\theta) = \left(\int d\theta \ 1\right) F_o + \left(\int d\theta \ \theta\right) F_1 \tag{10.6}$$

(2) integral of a total derivative vanishes

Thus, only two integrals are required,

$$\int d\theta \ 1 = \int d\theta \frac{\partial}{\partial\theta}\theta = 0 \qquad\qquad \int d\theta \ \theta = 1 \tag{10.7}$$

We may now state things more generally, and introduce a space of n Grassmann variable θ_i, $i = 1, \cdots, n$,

$$\{\theta_i, \theta_j\} = \left\{\frac{\partial}{\partial\theta_i} \ , \ \frac{\partial}{\partial\theta_j}\right\} = 0 \qquad\qquad \left\{\frac{\partial}{\partial\theta_i} \ , \theta_j\right\} = \delta_{ij} \tag{10.8}$$

So one may view the set of θ_i's and $\frac{\partial}{\partial\theta_i}$'s as a set of Clifford generators.

Consider $\theta_1 \cdots \theta_n$ and their complex conjugates $\bar\theta_1, \cdots \bar\theta_n$ and calculate the following Gaussian integral,

$$\begin{aligned}\int d^n\theta \int d^n\bar\theta \ e^{\bar\theta_i M_{ij}\theta_j} &= \int d\theta_1 \cdots d\theta_n \int d\bar\theta_1 \cdots d\bar\theta_n \ e^{\bar\theta_i M_{ij}\theta_j} \\ &= \int d\theta_1 \cdots d\theta_n \int d\bar\theta_1 \cdots d\bar\theta_n \frac{1}{n!}\left(\bar\theta_i M_{ij}\theta_j\right)^n\end{aligned} \tag{10.9}$$

To calculate this, we use the anti-symmetry of the product of θ_i, and we obtain

$$\theta_{i_1} \cdots \theta_{i_n} = \epsilon_{i_1 \cdots i_n} \ \ \theta_1\theta_2 \cdots \theta_n \tag{10.10}$$

Therefore, the integrations may be carried out explicitly and we have

$$\int d^n\theta d^n\bar\theta \ e^{\bar\theta_i M_{ij}\theta_j} = \frac{1}{n!}\epsilon_{i_1\cdots i_n}\epsilon_{j_1\cdots j_n} M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_n j_n} = \det M \tag{10.11}$$

Compare this result with the analogous Gaussian integrals for commuting variables,

$$\int dz_1 \cdots dz_n \int d\bar z_1 \cdots d\bar z_n e^{\bar z_i M_{ij} z_j} = (\det M)^{-1} \tag{10.12}$$

Similarly, correlation functions may also be evaluated explicitly,

$$\begin{aligned}\int d^n\theta d^n\bar\theta \ \theta_k\bar\theta_\ell \ e^{\bar\theta_i M_{ij}\theta_j} &= (\det M)(M^{-1})_{k\ell} \\ \int d^n\theta d^n\bar\theta \ \theta_{k_1}\theta_{k_2} \ \bar\theta_{\ell_1}\bar\theta_{\ell_2} \ e^{\bar\theta_i M_{ij}\theta_j} &= (\det M)\left[(M^{-1})_{k_1\ell_1}(M^{-1})_{k_2\ell_2} - (M^{-1})_{k_1\ell_2}(M^{-1})_{k_2\ell_1}\right]\end{aligned} \tag{10.13}$$

So we have constructed time ordered products that are antisymmetric in these indistinguishable arguments.

10.2 Grassmann Functional Integrals

With the above tools in hand, functional integrals over Grassmann fields may be computed. The first object of interest is the 2-point function,

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \;\; \psi_\alpha(x)\bar{\psi}^\beta(y) \; e^{i\int d^4x\mathcal{L}} \tag{10.14}$$

The arguments used to calculate this object are identical for any Lagrangian that is of the Dirac type, so we shall treat the general case. To calculate (10.14), we diagonalize D, which is possible in terms of real eigenvalues since $D^\dagger = D$. We denote the eigenvalues by λ_n and the corresponding ortho-normalized eigenfunctions by ψ_n. The index n may have continuous and discrete parts and degenerate eigenvalues will be labelled by distinct n. The field may then be decomposed in terms of the basis of eigenfunctions ψ_n with Grassmann odd coefficients c_n as follows,

$$\begin{aligned}\psi(x) &= \sum_n c_n\psi_n(x)\\ \int d^4x\mathcal{L} &= \sum_n \bar{c}_n c_n\lambda_n\end{aligned} \tag{10.15}$$

Next, we change variables in the functional integral from the field $\psi(x)$ to the coefficients c_n. Since the eigenfunctions satisfy $(\psi_m, \psi_n) = \delta_{mn}$, the Jacobian of this change of variables is unity. Actually, *the expression in terms of the coefficients c_n may be viewed as the proper definition of the measure*,

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \equiv \int \prod_n dc_n d\bar{c}_n \tag{10.16}$$

Calculating the vacuum amplitude is now starightforward,

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \; e^{i\int d^4x\mathcal{L}} = \int \prod_n dc_n d\bar{c}_n \exp\left\{ i\sum_n \bar{c}_n c_n\lambda_n \right\} = \mathrm{Det}(iD) \tag{10.17}$$

The 2-point function may be computed by the same methods,

$$\begin{aligned}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\psi_\alpha(x)\bar{\psi}^\beta(y) \; e^{i\int d^4x\mathcal{L}} &= \sum_{m,n}\prod_p \int dc_p d\bar{c}_p \;\psi_{m\alpha}(x)\bar{\psi}^\beta_n(y)\; c_m\bar{c}_n \;\exp\left\{ i\sum_n \bar{c}_n c_n\lambda_n \right\}\\ &= (\mathrm{Det}iD)\sum_n \frac{\psi_{n\alpha}(x)\bar{\psi}^\beta_n(y)}{i\lambda_n}\end{aligned} \tag{10.18}$$

The object multiplying $\mathrm{Det}(iD)$ on the rhs is recognized to be the Dirac propagator, namely the inverse of the operator iD,

$$\begin{aligned}S_F(x,y;M) &\equiv \sum_n \frac{\psi_{n\alpha}(x)\bar{\psi}^\beta_n(y)}{i\lambda_n} \qquad (10.19)\\ \Big(DS_F(x,y;M)\Big)_\alpha{}^\beta &= -i\sum_n \psi_{n\alpha}(x)\bar{\psi}^\beta_n(y) = -i\delta_\alpha{}^\beta\delta^{(4)}(x-y)\end{aligned}$$

This is as should be expected. Since the field equations for ψ are $(i\partial\!\!\!/ - M)\psi = 0$, the 2-point function satisfies

$$(i\partial\!\!\!/ - M)_\alpha{}^\gamma \langle 0|T\psi_\gamma(x)\bar{\psi}^\beta(y)|0\rangle = -i\delta_\alpha{}^\beta \delta^{(4)}(x-y) \tag{10.20}$$

For the higher point-functions, the same techniques may be employed and we have

$$\begin{aligned} &\langle 0|T\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}^{\beta_1}(y_1)\cdots\bar{\psi}^{\beta_n}(y_n)|0\rangle \\ &= \frac{1}{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS[\psi]}} \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}^{\beta_1}(y_1)\cdots\bar{\psi}^{\beta_n}(y_n)e^{iS(\psi)} \end{aligned} \tag{10.21}$$

10.3 The generating functional for Grassmann fields

In particular, one may again derive a generating functional. The source term cannot however be taken to be an ordinary function, since the time ordered product is antisymmetric. One has to use a new set of Grassmann valued functions:

$$Z[\eta,\bar{\eta}] \equiv \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left\{iS[\psi] + \int d^4x(\bar{\psi}\eta + \bar{\eta}\psi)\right\} \tag{10.22}$$

Note that we used Lorentz invariant combinations. By expanding the generating functional in powers of the source terms, we have

$$\begin{aligned} Z[\eta,\bar{\eta}] &= \sum_{m,n=0}^{\infty} \frac{1}{m!}\frac{1}{n!} \int d^4x_1\cdots d^4x_m \int d^4y_1\cdots d^4y_n \bar{\eta}^{\alpha_1}(x_1)\cdots\bar{\eta}^{\alpha_n}(x_m)\eta_{\beta_1}(y_1)\cdots\eta_{\beta_n}(y_n) \\ &\quad \times \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_m}(x_m)\bar{\psi}^{\beta_1}(y_1)\cdots\bar{\psi}^{\beta_n}(y_n)e^{iS[\psi]} \end{aligned} \tag{10.23}$$

If the action is purely quadratic in ψ, then the number of ψ and $\bar{\psi}$'s must be equal

$$\begin{aligned} \frac{Z[\eta,\bar{\eta}]}{Z[0,0]} &= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int d^4x_1\cdots d^4x_n \int d^4y_1\cdots d^4y_n \bar{\eta}^{\alpha_1}(x_1)\cdots\bar{\eta}^{\alpha_n}(x_n)\eta_{\beta_1}(y_1)\cdots\eta_{\beta_n}(y_n) \\ &\quad \times\langle 0|T\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}^{\beta_1}(y_1)\cdots\bar{\psi}^{\beta_n}(y_n)|0\rangle \end{aligned} \tag{10.24}$$

For the free theory, the generating functional Z_o may be computed by completing the square in the Grassmann functional integral,

$$\begin{aligned} iS[\psi] + \int d^4x(\bar{\psi}\eta + \bar{\eta}\psi) &= \int d^4x \left(\bar{\psi}iD\psi + \bar{\psi}\eta + \bar{\eta}\psi\right) \\ &= \int d^4x \left\{(\overline{\psi - iD^{-1}\eta})iD(\psi - iD^{-1}\eta) + \bar{\eta}(iD)^{-1}\eta\right\} \end{aligned} \tag{10.25}$$

Hence the generating functional may be evaluated in terms of the Feynman propagator,

$$\frac{Z[\eta,\bar{\eta}]}{Z[0,0]} = \exp\left\{\int d^4x \int d^4y\, \bar{\eta}(x)S_F(x,y;M)\eta(y)\right\} \tag{10.26}$$

The time-ordered expectation value of the n-point functions are recovered analogously.

10.4 Lagrangian with multilinear terms

When the Lagrangian is not just quadratic in ψ one proceeds as follows. Say it also has quartic terms, e.g.

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{2}g^2(\bar{\psi}\psi)^2 \tag{10.27}$$

It is always possible to introduce a non-dynamical bosonic "auxiliary field" σ as well as a new Lagrangian $\mathcal{L}'$, such that elimination of the field σ using the field equations for σ yields the old Lagrangian. In the quartic case,

$$\begin{aligned} \mathcal{L}' &= \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\sigma - g\bar{\psi}\psi)^2 - \frac{1}{2}g^2(\bar{\psi}\psi)^2 \\ &= \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}\sigma^2 - g\sigma\bar{\psi}\psi \end{aligned} \tag{10.28}$$

When we compute a vacuum expectatiion value with the first Lagrangian

$$\begin{aligned} &\langle 0|T\psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}^{\beta_1}(y_1)\cdots\bar{\psi}^{\beta_n}(y_n)|0\rangle \qquad (10.29) \\ &= \frac{1}{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ e^{i\int\mathcal{L}}}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}^{\beta_1}(y_n)\cdots\bar{\psi}^{\beta_n}(y_n)e^{i\int\mathcal{L}} \end{aligned}$$

we may clearly replace it with

$$= \frac{1}{\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\sigma\ e^{i\int\mathcal{L}'}}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\sigma\ \psi_{\alpha_1}(x_1)\cdots\psi_{\alpha_n}(x_n)\bar{\psi}^{\beta_1}(y_1)\cdots\bar{\psi}^{\beta_n}(y_n)e^{i\int\mathcal{L}'} \tag{10.30}$$

The Lagrangian $\mathcal{L}'$ is now quadratic in ψ and can be treated with the preceding methods.

11 Quantum Electrodynamics : Basics

This is the theory of an Abelian gauge field, coupled to charged Dirac spinor fields. Here, we take the case of a single spinor field ψ, which may be thought of as the field decribing the electron and positron. We include the usual gauge fixing term. The action is as follows,

$$
\begin{aligned}
S[A,\psi] &= S_o[A,\psi] + S_I[A,\psi] \\
S_o[A,\psi] &= \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 + \bar\psi(i\partial\!\!\!/ - m)\psi \right) \\
S_I[A,\psi] &= -\int d^4x \; eA_\mu \bar\psi \gamma^\mu \psi
\end{aligned}
\tag{11.1}
$$

The electric charge e enters via the fine structure constant

$$
\alpha = \frac{e^2}{4\pi\hbar c} \sim \frac{1}{137.04}
\tag{11.2}
$$

which may be viewed as small. Thus, we expand directly in powers of the interaction S_I. The free propagators were computed previously,

$$
\begin{aligned}
S_\alpha{}^\beta(x-y) &\equiv \langle 0|T\psi_\alpha(x)\bar\psi^\beta(y)|0\rangle_o = \int \frac{d^4k}{(2\pi)^4} \left(\frac{i}{k\!\!\!/ - m}\right)_\alpha{}^\beta \; e^{-ik\cdot(x-y)} \\
G_{\mu\nu}(x-y) &\equiv \langle 0|TA_\mu(x)A_\nu(y)|0\rangle_o = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \left(\eta_{\mu\nu} + \xi \frac{k_\mu k_\nu}{k^2}\right) \; e^{-ik(x-y)}
\end{aligned}
\tag{11.3}
$$

It is convenient to use the abbreviation $\xi \equiv (1-\lambda)/\lambda$ when dealing with the gauge choice parameter λ.

11.1 Feynman rules in momentum space

The Feynman rules are completely analogous to (and much simpler than) the ones for the scalar field. The vertex is simply the multiplicative factor $-ie(\gamma^\mu)_\alpha{}^\beta$. Propagators and vertex are represented in Fig 4. Note that ψ and $\bar\psi$ are distinct fields (their change is opposite), so that the counting factors are modified. In addition, for closed fermion loops, there is a factor of -1 for each loop. In momentum space, the Feynman rules are as follows,

$$
\begin{aligned}
&\text{Incoming fermion} && \left(\frac{i}{p\!\!\!/ - m}\right)_\alpha{}^\beta \\
&\text{Outgoing fermion} && \left(\frac{i}{-p\!\!\!/ - m}\right)_\alpha{}^\beta \\
&\text{Photon} && \frac{-i}{p^2}\left(\eta_{\mu\nu} + \xi \frac{p_\mu p_\nu}{p^2}\right)
\end{aligned}
\tag{11.4}
$$

Henceforth, the $i\epsilon$ contributions to the propagators will be understood in all expressions.

The direction of the arrow indicates the flow of electric charge, taken from $\bar{\psi}$ to ψ.

$$\text{Vertex} \qquad -ie(\gamma_\mu)_\beta{}^\alpha (2\pi)^4 \delta^4(\text{total incoming momentum}) \tag{11.5}$$

Internal lines have an additional factor of momentum integration,

$$\begin{aligned} &\text{Fermion} && \frac{d^4p}{(2\pi)^4}\left(\frac{i}{\not{p}-m}\right)_\alpha{}^\beta \\ &\text{Photon} && \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2}\left(\eta_{\mu\nu}+\xi\frac{p_\mu p_\nu}{p^2}\right) \end{aligned} \tag{11.6}$$

A minus sign for each closed fermion loop.

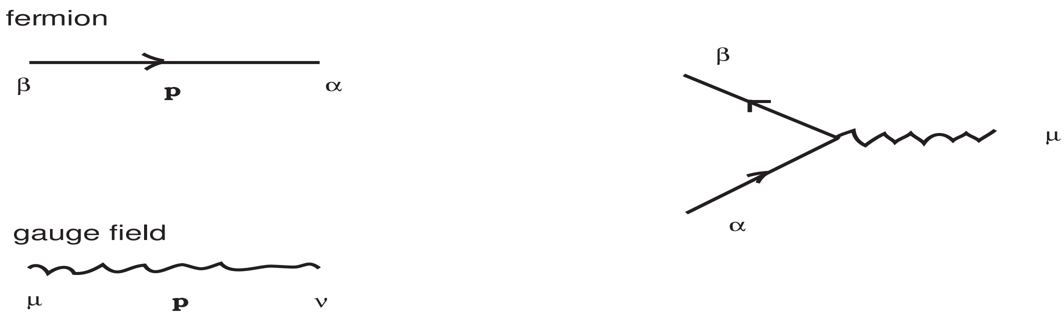


Figure 14: Feynman rules for QED

11.2 Verifying Feynman rules for one-loop Vacuum Polarization

$$\begin{aligned} \langle 0|TA_\mu(x)A_\nu(y)|0\rangle &= \left(\int \mathcal{D}A \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{i\int d^4x \mathcal{L}(A,\psi,\bar{\psi})} A_\mu(x)A_\nu(y)\right)_{\text{novac}} \\ &= \int \mathcal{D}A \int \mathcal{D}\bar{\psi}\mathcal{D}\psi A_\mu(x)A_\nu(y)\frac{1}{2}(-ie)^2\left(\int d^4z\bar{\psi}\gamma^\mu\psi A_\mu\right)^2 e^{i\int d^4x\mathcal{L}_o} \end{aligned} \tag{11.7}$$

The purely fermionic part that has to be computed here is

$$\begin{aligned} &\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; \bar{\psi}\gamma^\mu\psi(z_1)\; \bar{\psi}\gamma^\nu\psi(z_2)\; e^{i\int d^4x\mathcal{L}_o} \Big/ \int \mathcal{D}\bar{\psi}\mathcal{D}\bar{\psi}\; e^{i\int d^4x\mathcal{L}_o} \\ &\quad = \frac{\delta}{\delta\eta_\beta(z_1)}(\gamma^\mu)_\beta{}^\alpha \frac{\delta}{\delta\bar{\eta}_\alpha(z_1)}\frac{\delta}{\delta\eta_\gamma(z_2)}(\gamma^\nu)_\gamma{}^\delta\frac{\delta}{\delta\bar{\eta}^\delta(z_2)}\exp\left\{\int_x\int_y \bar{\eta}(x)S(x-y)\eta(y)\right\} \\ &\quad = \frac{\delta}{\delta\eta_\beta(z_1)}(\gamma^\mu)_\beta{}^\alpha \frac{\delta}{\delta\bar{\eta}_\alpha(z_1)}\frac{\delta}{\delta\eta_\gamma(z_2)}(\gamma^\nu)_\gamma{}^\delta \int_u S_\delta^\rho(z_2-u)\eta_\rho(u)\int_x\int_y \bar{\eta}(x)S(x-y)\eta(y) \\ &\quad = \frac{\delta}{\delta\eta_\beta(z_1)}(\gamma^\mu)_\beta{}^\alpha \frac{\delta}{\delta\bar{\eta}_\alpha(z_1)}(\gamma^\nu)_\gamma{}^\delta S_\delta{}^\gamma(0)\int_x\int_y \bar{\eta}(x)S(x-y)\eta(y) \\ &\quad\quad + \frac{\delta}{\delta\eta_\beta(z_1)}(\gamma^\mu)_\beta{}^\alpha \frac{\delta}{\delta\bar{\eta}_\alpha(z_1)}(\gamma^\nu)_\gamma{}^\delta \int_y S_\delta{}^\rho(z_2-y)\eta_\rho(y)\int_x \bar{\eta}^\sigma(x)S^\gamma_\sigma(x-z_2) \end{aligned} \tag{11.8}$$

The first term on the rhs of the last expression above must vanish by Lorentz invariance. The cancellation may also be seen directly from the expression for the propagator,

$$(\gamma^\nu)_\gamma{}^\delta S_\delta{}^\gamma(0) = \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}\{\gamma^\mu(\not k + m)\}}{k^2 - m^2 + i\epsilon} = 0 \tag{11.9}$$

The remainder of the calculation proceeds as follows,

$$\begin{aligned}
&= -\frac{\delta}{\delta\eta_\beta(z_1)} (\gamma^\mu)_\beta{}^\alpha (\gamma^\nu)_\gamma{}^\delta \int_y S_\delta{}^\rho(z_2 - y)\eta_\rho(y) \;\; S_\alpha{}^\gamma(z_1 - z_2) \\
&= -(\gamma^\mu)_\beta{}^\alpha (\gamma^\nu)_\gamma{}^\delta \; S_\delta{}^\beta(z_2 - z_1) \; S_\alpha{}^\gamma(z_1 - z_2) \\
&= -\mathrm{tr}\Big(\gamma^\mu S(z_1 - z_2)\gamma^\nu S(z_2 - z_1)\Big)
\end{aligned} \tag{11.10}$$

Hence

$$\begin{aligned}
\langle 0|TA_\mu(x)A_\nu(y_|0\rangle &= \frac{1}{2}e^2 \int d^2z_1 \int d^2z_2 \; \langle 0|TA_\mu(x)A_\nu(y)A_\kappa(z_1)A_\lambda(z_2)|0\rangle_o \\
&\qquad \times \mathrm{tr}(\gamma^\kappa S(z_1 - z_2)\gamma^\lambda S(z_2 - z_1))
\end{aligned} \tag{11.11}$$

Of the three possible contributions to the tree level 4-point function for the gauge fields, one is disconnected and corresponds to a vacuum graph which does not contribute to the correlator. The remaining two connected contributions are equal and yield

$$\begin{aligned}
\langle 0|TA_\mu(x)A_\nu(y)|0\rangle &= G_{\mu\nu}(x-y) + e^2 \int d^4z_1 \int d^4z_2 G_{\mu\kappa}(x - z_1)G_{\nu\lambda}(y - z_2) \\
&\qquad \times \mathrm{tr}(\gamma^\kappa S(z_1 - z_2)\gamma^\lambda S(z_2 - z_1))
\end{aligned} \tag{11.12}$$

11.3 General fermion loop diagrams

The first class of diagrams, to one loop order, consist of a single fermion loop with an arbitrary number of external A_μ field insertions. In 1PIR graphs, the external photon propagators are truncated away, a fact that is indicated by using *short photon propagator lines*. The graphs up to 5 external insertions are depicted in Fig 15.

These diagrams represent the perturbative expansion in powers of eA_μ of the fermion effective action, defined as follows,

$$\Gamma[A_\mu] = -i \ln \mathrm{Det}\Big(i\not\partial - e\not A - m\Big) \tag{11.13}$$

11.4 Pauli-Villars Regulators

By power counting, all diagrams with $n \leq 4$ are expected to be UV divergent and therefore need to be regularized. It is convenient to use a regulator that preserves Poincaré as well

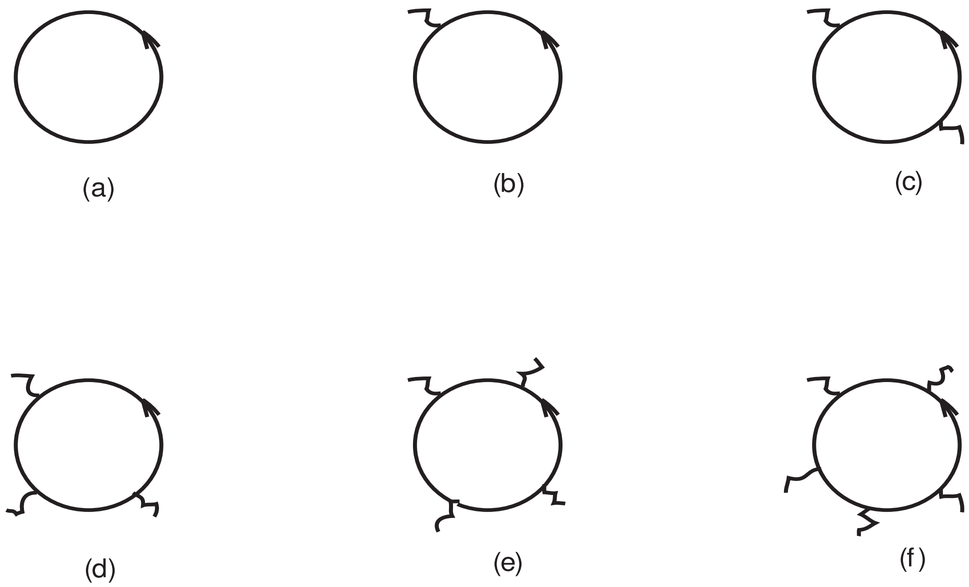


Figure 15: One-Fermion-Loop graphs

as gauge invariance. Pauli-Villars regulators introduces extra fermions fields with identical gauge couplings, but very large masses, on the order of a regulator scale Λ. Since the Pauli-Villars regulator masses are very large there is no modification to the physics at distance scales large compared to the regulator scale Λ. We denoted the Pauli-Villars regulators by ψ_s and their masses by M_s; the Lagrangian is then

$$\mathcal{L}_{\rm PV} = \mathcal{L} + \sum_{s=1}^{S} C_s \bar\psi_s (i\partial\!\!\!/ - eA\!\!\!/ - M_s)\psi_s \tag{11.14}$$

The number S of regulators, and the constants C_s and M_s are to be determined by the requirement that the regularized correlators should be finite. These data are not unique; a change in choice leads to a finite change in the definition of the regularized correlators.

11.5 Furry's Theorem

Furry's Theorem states that the 1-fermion loop diagrams for QED with an odd number of current insertions vanishes identically in view of charge conjugation symmetry. Recall that the charge conjugate ψ^c of the field ψ is defined by

$$\psi^c = \eta_c C \bar\psi^t = \eta_c C(\gamma^o)^T \psi^* \tag{11.15}$$

where the charge conjugation matrix is defined to satisfy

$$C\gamma_\mu^t C^{-1} = -\gamma_\mu \qquad\qquad C^\dagger C = I \tag{11.16}$$

As a result, the charge conjugate of the electric current reverses its sign, as might be expected. The fermion propagator in momentum space transforms as follows,

$$C \left(\frac{1}{\not k - m} \right)^t C^{-1} \quad = \quad \frac{1}{-\not k - m} \tag{11.17}$$

and this implies that its transformation in position space is given by Hence

$$C \ S(x,y)^t C^{-1} = S(y,x) \tag{11.18}$$

For a given ordering of momenta and μ labels, two graphs with opposite charge flow (opposite orientation of the arrow) contribute and must be added up. They are given by

$$\begin{aligned} \Gamma^{(n)} \quad &= \quad \mathrm{tr}\Big(\gamma^{\mu_1} S(x_1,x_2) \gamma^{\mu_2} S(x_2,x_3) \gamma^{\mu_3} \cdots \gamma^{\mu_n} S(x_n,x_1) \Big) \\ &\quad + \mathrm{tr}\Big(\gamma^{\mu_1} S(x_1,x_n) \gamma^{\mu_n} S(x_n,x_{n-1}) \gamma^{\mu_{n-1}} \cdots \gamma^{\mu_2} S(x_2,x_1) \Big) \end{aligned} \tag{11.19}$$

Taking the transpose inside the trace leaves this expression invariant, and re-expressing the transposes with the help of the charge conjugation matrix C yields

$$\begin{aligned} \Gamma^{(n)} \quad &= \quad \mathrm{tr}\Big(S(x_n,x_1)^t (\gamma^{\mu_n})^t \cdots (\gamma^{\mu_3})^t S(x_2,x_3)^t (\gamma^{\mu_2})^t S(x_1,x_2)^t (\gamma^{\mu_1})^t \Big) \qquad (11.20) \\ &\quad + \mathrm{tr}\Big(S(x_2,x_1)^t (\gamma^{\mu_2})^t \cdots (\gamma^{\mu_{n-1}})^t S(x_n,x_{n-1})^t (\gamma^{\mu_n})^t S(x_1,x_n)^t (\gamma^{\mu_1})^t \Big) \\ &= \quad (-)^n \mathrm{tr}\Big(S(x_1,x_n) \gamma^{\mu_n} \cdots \gamma^{\mu_3} S(x_3,x_2) \gamma^{\mu_2} S(x_2,x_1) \gamma^{\mu_1} \Big) \qquad (11.21) \\ &\quad + (-)^n \mathrm{tr}\Big(S(x_1,x_2) \gamma^{\mu_2} \cdots \gamma^{\mu_{n-1}} S(x_{n-1},x_n) \gamma^{\mu_n} S(x_n,x_1) \gamma^{\mu_1} \Big) \\ &= \quad (-)^n \Gamma^{(n)} \qquad (11.22) \end{aligned}$$

This completes the proof of Furry's theorem. Therefore, diagrams (b) and (d) in Fig 15 vanish identically, leaving only those with even n.

11.6 Ward-Takahashi identities for the electro-magnetic current

Whenever there exists a current $j^\mu(x)$ which is satisfies the conservation equation $\partial_\mu j^\mu = 0$, certain special relations, generally called *Ward identities*, will hold between certain correlation functions of local operators. In the case at hand, the relevant conserved current is the electro-magnetic current $j^\mu(x) \equiv \bar\psi(x)\gamma^\mu\psi(x)$. Our derivation of the Ward identities will, however, be more general than this case.

The conservation of the current j^μ implies the existence of a time-independent charge Q. Local observables may be organized in eigenspaces of definite Q-charge, and it suffices to examine correlators of operators of definite charges

$$[Q, \mathcal{O}_i(y)] = q_i \mathcal{O}_i(y) \qquad\qquad Q = \int d^3\vec{x}\; j^0(t, \vec{x}) \tag{11.23}$$

Since the commutator $[j^0(t, \vec{x}), \mathcal{O}_i(t, \vec{y})]$ can receive contributions only from $\vec{x} = \vec{y}$ in view of the locality of the operators, the above commutator with the charge requires the following local commutator to hold,

$$[j^0(x), \mathcal{O}_i(y)]\; \delta(x^0 - y^0) = q_i \delta^{(4)}(x - y) \mathcal{O}_i(y) \tag{11.24}$$

In specific theories, this relation will in fact result from the explicit form of the current j^μ and the local operators $\mathcal{O}_i(y)$ in terms of canonical fields and the use of the canonical commutation relations of the canonical fields. For example in QED, the relation may be worked out for the commutator $[j^0(x), \psi_\alpha(y)]\delta(x^0 - y^0)$.

Next, consider a correlator of a product of local operators and the conserved current $j^\mu(x)$. The interesting quantity to consider is the divergence of this correlator,

$$\partial_\mu^{(x)} \langle \emptyset | T j^\mu(x) \mathcal{O}_1(y_1) \cdots \mathcal{O}_n(y_n) | \emptyset \rangle \tag{11.25}$$

Were it not for the time-ordering instruction, this divergence would be identically 0 as long as the current is conserved. From the time ordering Heaviside functions on $x^0 - y_i^0$, however, one picks up non-vanishing terms. This may be seen first from the time-ordering on two fields,

$$\partial_\mu^{(x)} T j^\mu(x) \mathcal{O}_i(y_i) = \delta(x^0 - y_i^0)[j^0(x), \mathcal{O}_i(y)] = q_i \delta(x - y) \mathcal{O}_i(y_i) \tag{11.26}$$

and readily generalizes to n fields. Inserting this result in the correlator of interest yields,

$$\partial_\mu^{(x)} \langle \emptyset | T j^\mu(x) \mathcal{O}_1(y_1) \cdots \mathcal{O}_n(y_n) | \emptyset \rangle = \sum_{i=1}^n q_i\; \delta(x - y_i) \langle \emptyset | T \mathcal{O}_1(y_1) \cdots \mathcal{O}_n(y_n) | \emptyset \rangle \tag{11.27}$$

This is a very important equation, as it states that the *longitudinal part of the current correlator is expressed in terms of correlators without the insertion of the current.*

We give two examples in QED. The first applies to the correlator of electro-magnetic currents only. Since the electric charge of the current itself vanishes, one has

$$\partial_\mu^{(x)} \langle \emptyset | T j^\mu(x) j^{\nu_1}(y_1) \cdots j^{\nu_n}(y_n) | \emptyset \rangle = 0 \tag{11.28}$$

In other words, the current correlators are transverse in each external leg. The second example involves one current and two fermions,

$$\begin{aligned} \partial_\mu^{(x)} \langle \emptyset | T j^\mu(x) \psi_\alpha(y) \bar{\psi}^\beta(z) | \emptyset \rangle &= +\delta(x - y) \langle \emptyset | T \psi_\alpha(y) \bar{\psi}^\beta(z) | \emptyset \rangle \\ &\quad -\delta(x - z) \langle \emptyset | T \psi_\alpha(y) \bar{\psi}^\beta(z) | \emptyset \rangle \end{aligned} \tag{11.29}$$

Feynman diagrammatic representations of both equations are given in Fig.

Here and elsewhere, the superscript (x) will indicate that the derivatve is applied in the variable x.

11.7 Regularization and gauge invariance of vacuum polarization

The vacuum polarization graph corresponds to $n = 2$ and contributes to $\Gamma^{(2)}$. The unregularized Feynman diagram for vacuum polarization is given by

$$\Gamma_o^{\mu\nu}(k,m) \equiv (-)(-i)(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{tr} \left(\gamma^\mu \frac{i}{\not p - m} \gamma^\nu \frac{i}{\not p + \not k - m} \right) \tag{11.30}$$

The overall $-$ sign is for a closed fermion loop; the factor of $(-i)$ from thedefinition of Γ; the factors of $(-ie)$ from two vertex insertions. This integral is quadratically divergent. We introduce $s = 1, \cdots, S$ Pauli-Villars regulators with masses $M_s \sim \Lambda$ and define the *regularized vacuum polarization tensor* by

$$\Gamma^{\mu\nu}(k,m;\Lambda) \equiv ie^2 \int \frac{d^4p}{(2\pi)^4} \sum_{s=0}^{S} C_s \mathrm{tr} \left(\gamma^\mu \frac{1}{\not p - M_s} \gamma^\nu \frac{1}{\not p + \not k - M_s} \right) \tag{11.31}$$

where $M_0 = m$ and $C_0 = 1$. In practice, it suffices to take $S = 3$, but we shall leave this number arbitrary at this stage.

Gauge invariance requires conservation of the electro-magnetic current and by the Ward identities derived in the preceding subsection, transversality of the 2-point function, $k_\mu \Gamma^{\mu\nu}(k,m;\Lambda) = 0$. This property is readily checked using the following simple, but very useful general formula,

$$\begin{aligned}
\frac{1}{\not p - M_s} \not k \frac{1}{\not p + \not k - M_s} &= \frac{1}{\not p - M_s} \Big\{ (\not p + \not k - M_s) - (\not p - M_s) \Big\} \frac{1}{\not p + \not k - M_s} \\
&= \frac{1}{\not p - M_s} - \frac{1}{\not p + \not k - M_s}
\end{aligned} \tag{11.32}$$

As a result,

$$\begin{aligned}
k_\nu \Gamma^{\mu\nu}(k,m;\Lambda) &= ie^2 \int \frac{d^4p}{(2\pi)^4} \sum_{s=0}^{S} C_s \mathrm{tr}\, \gamma^\mu \left(\frac{1}{\not p - M_s} \not k \frac{1}{\not p + \not k - M_s} \right) \\
&= ie^2 \int \frac{d^4p}{(2\pi)^4} \sum_{s=0}^{S} C_s \mathrm{tr}\, \gamma^\mu \left(\frac{1}{\not p - M_s} - \frac{1}{\not p + \not k - M_s} \right)
\end{aligned} \tag{11.33}$$

Since the integrals have been rendered convergent by the Pauli-Villars regulators, one may shift the momentum integration by k in the second term so that the difference of the two integrals vanishes, confirming gauge invariance.

Gauge invariance thus amounts to transversality of the 2-point function. This property in turn leads to a dramatic simplification of the expression for vacuum polarization. Since

$\Gamma^{\mu\nu}$ is a Lorentz tensor which depends only on the vector k^μ, its general form must be

$$\Gamma^{\mu\nu}(k,m;\Lambda) = A\eta^{\mu\nu} + Bk^\mu k^\nu \tag{11.34}$$

Transversality requires $A + k^2 B = 0$, so that vacuum polarization is proportional to a single scalar function,

$$\Gamma^{\mu\nu}(k,m;\Lambda) = (k^2\eta^{\mu\nu} - k^\mu k^\nu)\bar{\Gamma}(k^2,m;\Lambda) \tag{11.35}$$

The fact that two powers of momentum are extracted by gauge invariance implies, just on dimensional grounds, that the remaining vacuum polarization function $\bar{\omega}$ will be only logarithmically divergent. Thus, we encounter for the first time, a general paradigm that gauge invariance improves UV convergence.

11.8 Calculation of vacuum polarization

The first step is to rationalize the fermion propagators.

$$\Gamma^{\mu\nu}(k,m;\Lambda) = ie^2 \int \frac{d^4p}{(2\pi)^4} \sum_{s=0}^{S} C_s \frac{\mathrm{tr}\ \gamma^\mu(\not p + M_s)\gamma^\nu(\not k + \not p + M_s)}{(p^2 - M_s^2)((k+p)^2 - M_s^2)} \tag{11.36}$$

The second step is to introduce a Feynman parameter,

$$\Gamma^{\mu\nu}(k,m;\Lambda) = ie^2 \int_0^1 d\alpha \int \frac{d^4p}{(2\pi)^4} \sum_{s=0}^{S} C_s \frac{\mathrm{tr}\ \gamma^\mu(\not p + M_s)\gamma^\nu(\not k + \not p + M_s)}{(p^2 - M_s^2 + 2\alpha k\cdot p + \alpha k^2)^2} \tag{11.37}$$

The third step is to combine the denominator as follows,

$$p^2 + 2\alpha k\cdot p + \alpha k^2 = (p+\alpha k)^2 + \alpha(1-\alpha)k^2 \tag{11.38}$$

The fourth step is to shift $p + \alpha k \to p$, which is permitted because the combined integrand leads to a convergent integral,

$$\Gamma^{\mu\nu}(k,m;\Lambda) = ie^2 \int_0^1 d\alpha \int \frac{d^4p}{(2\pi)^4} \sum_{s=0}^{S} C_s \frac{\mathrm{tr}\ \gamma^\mu(\not p - \alpha\not k + M_s)\gamma^\nu((1-\alpha)\not k + \not p + M_s)}{(p^2 - M_s^2 + \alpha(1-\alpha)k^2)^2} \tag{11.39}$$

The fifth step is to work out the trace, discard terms odd in p since those cancel by $p \to -p$ symmetry of the remaining integrand, and replace

$$p^\mu p^\nu \to \frac{1}{4}p^2\eta^{\mu\nu} \tag{11.40}$$

by using Lorentz symmetry of the integrand,

$$\mathrm{tr}\ \gamma^\mu \not p \gamma^\nu \not p \ \to \ -2p^2\eta^{\mu\nu} \tag{11.41}$$

Combining these result,

$$\Gamma^{\mu\nu}(k,m;\Lambda) = -ie^2 \int_0^1 d\alpha \int \frac{d^4p}{(2\pi)^4} \sum_{s=0}^{S} C_s \frac{2p^2\eta^{\mu\nu} - \alpha(1-\alpha)[4k^2\eta^{\mu\nu} - 8k^\mu k^\nu] - 4M_s^2\eta^{\mu\nu}}{(p^2+\alpha(1-\alpha)k^2 - M_s^2)^2} \tag{11.42}$$

This result is not manifestly transverse and we shall now show that the longitudinal piece vanishes by intgeration by part. To do so, we split this up as follows,

$$\begin{aligned}\Gamma^{\mu\nu}(k,m;\Lambda) \;&=\; (\eta^{\mu\nu}k^2 - k^\mu k^\nu)\bar{\Gamma}(k^2,m;\Lambda) \\ &\quad -ie^2\eta^{\mu\nu}\int_0^1 d\alpha \int \frac{d^4p}{(2\pi)^4} \sum_{s=0}^{S} C_s \frac{2p^2 + 4k^2\alpha(1-\alpha) - 4M_s^2}{(p^2+\alpha(1-\alpha)k^2 - M_s^2)^2}\end{aligned} \tag{11.43}$$

The integrand of the second term is a total derivative,

$$\frac{2p^2 + 4k^2\alpha(1-\alpha) - 4M_s^2}{(p^2+\alpha(1-\alpha)k^2 - M_s^2)^2} = \frac{\partial}{\partial p^\mu}\Big(\frac{p^\mu}{p^2+\alpha(1-\alpha)k^2 - M_s^2}\Big) \tag{11.44}$$

and therefore the p-integral vanishes since the sum over all Pauli-Villars regulator contributions is convergent. The remaining scalar integral is given as follows,

$$\bar{\Gamma}(k^2,m;\Lambda) = 8ie^2 \int_0^1 d\alpha \int \frac{d^4p}{(2\pi)^4} \sum_{s=0}^{S} C_s \frac{\alpha(1-\alpha)}{(p^2+\alpha(1-\alpha)k^2 - M_s^2)^2} \tag{11.45}$$

It suffices to take $S = 1,\ M_1 = \Lambda$. The p-integral encountered are scalar and may be carried out easily with the help of the formulas derived earlier,

$$\bar{\Gamma}(k^2,m,\Lambda) = \frac{8e^2}{(4\pi)^2}\int_0^1 d\alpha\ \alpha(1-\alpha)\ln\frac{m^2 - \alpha(1-\alpha)k^2}{\Lambda^2} \tag{11.46}$$

Next, we shall derive a physical interpretation for this result and carry out the required renormalizations.

11.9 Resummation and screening effect

Repeated insertion of the 1-loop vacuum polarization effect into the photon propagator is easily carried out. The tree level photon propagator is given by,

$$G^o_{\mu\nu}(k) \;=\; \frac{-i}{k^2}\left(\eta_{\mu\nu} + \frac{1-\lambda}{\lambda}\frac{k_\mu k_\nu}{k^2}\right) \tag{11.47}$$

Since the vacuum polarization effect is transverse, it is clear that the longitudinal part of the photon propagator is unmodified to all orders. A single insertion therefore produces

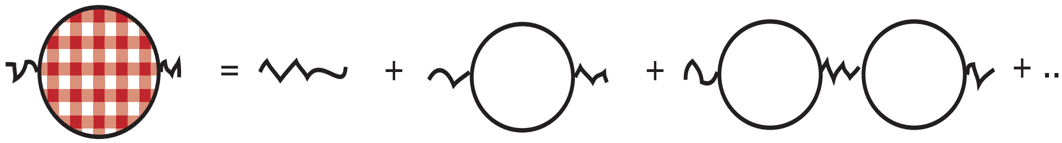


Figure 16: Resummation of 1-loop vacuum polarization

a purely transverse effect,

$$
\begin{aligned}
G^1_{\mu\nu}(k) &= \left(\frac{-i}{k^2}\right)^2 (k^2\eta_{\mu\nu} - k_\mu k_\nu) i\bar{\Gamma}(k^2, m; \Lambda) \\
&= \frac{-i}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\bar{\Gamma}(k^2, m; \Lambda)
\end{aligned}
\tag{11.48}
$$

Hence the full propagator becomes

$$
G_{\mu\nu}(k) = \frac{-i}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\bar{\Gamma}(k^2, m; \Lambda)}
\tag{11.49}
$$

The effect of this correction is to modify Coulomb's law. In the limit where $k \to 0$, for example, we have

$$
\begin{aligned}
\frac{e^2}{\vec{k}^2} &\to \frac{e^2_{\text{eff}}}{\vec{k}^2} \equiv \frac{e^2}{\vec{k}^2} \times \frac{1}{1-\bar{\Gamma}(0, m; \Lambda)} \\
1-\bar{\Gamma}(0, m; \Lambda) &= 1 + \frac{e^2}{12\pi^2}\ln\frac{\Lambda^2}{m^2} + \mathcal{O}(e^4)
\end{aligned}
\tag{11.50}
$$

Starting out with the charge e in the Lagrangian, we see that the *effective charge* observed in the Coulomb law is *reduced*, at distances long compared to $1/\Lambda$. Quantitatively, the effective charge is given by

$$
e^2_{\text{eff}} = \frac{e^2}{1+\frac{e^2}{12\pi^2}\ln\frac{\Lambda^2}{m^2}}
\tag{11.51}
$$

11.10 Physical interpretation of the effective charge

Vacuum polarization is produced by the screening effects of *virtual electron-positron* pairs. We now explain how this can come about. As is shown schematically by the vacuum polarization Feynman diagram itself, the presence of the photon gives rise to an electromagnetic field which creates a virtual pair. The QED vacuum is therefore not quite an empty space. Instead, in the QED vacuum, electron positron pairs are continuously beeing created and annihilated. Each pair acts like a small short-lived electric dipole. The presence of these electric dipoles makes the QED vacuum into a di-electric medium, analogous to the di-electric media created by matter, such as water, air etc.

It is a well-known effect of a di-electric medium that an electric charge is screened by the presence of the electric dipoles in the medium. Qualitatively, in the presence of a + charge, the dipoles will align their negative ends towards the + charge and the net effect of this is that the effective charge decreases with distance. This is schematically depicted in Fig 17.

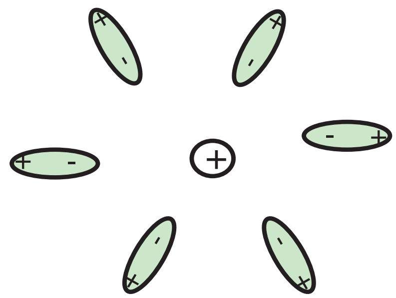


Figure 17: Polarization of a charge by virtual electron-positron pairs

The electric charge observed in the Coulomb law is the effective charge at long distances, much larger than $1/m_e$. If the charge at long distances is to be e, then the charge placed at the center must be larger. We shall denote that charge by e_0. The relation now becomes

$$e^2 = \frac{e_o^2}{1 + \frac{e_o^2}{12\pi^2} \ln \frac{\Lambda^2}{m^2}} \qquad \text{or} \qquad e_o^2 = e^2 + \frac{e^4}{12\pi^2} \ln \frac{\Lambda^2}{m^2} + O(e^6) \tag{11.52}$$

The k-dependent part of $\bar{\Gamma}(k, m, \Lambda)$ also affects the Coulomb interaction in a computable way. The value of the charge will now depend on the momentum. For example in the approximation of momenta large compared to m, we shall have

$$\begin{aligned} e_{\text{eff}}^2(k) = \frac{e_o^2}{1 - \bar{\Gamma}(k^2, m, \Lambda)} &= \frac{e_o^2}{1 - \bar{\Gamma}(0, m, \Lambda) + -\Gamma_R(k^2, m, \Lambda)} \\ &= \frac{e^2}{1 - \bar{\Gamma}_R(k^2, m; \Lambda)} \end{aligned} \tag{11.53}$$

with

$$\bar{\Gamma}_R(k^2, m, \Lambda) = \frac{e^2}{2\pi^2} \int_0^1 d\alpha \; \alpha(1-\alpha) \ln \frac{m^2 - \alpha(1-\alpha)k^2}{m^2} \tag{11.54}$$

In the regime $-k^2 \gg m^2$, referred to as the *deep inelastic scattering region*, we have

$$\bar{\Gamma}_R(k^2, m, \Lambda) = \frac{e^2}{12\pi^2} \ln \frac{-k^2}{m^2} + \mathcal{O}(e^4) \tag{11.55}$$

and the effective charge as a function of k^2 is given by

$$e_{\text{eff}}^2(k) = \frac{e^2}{1 - \frac{e^2}{12\pi^2} \ln \frac{-k^2}{m^2}} \tag{11.56}$$

This formula confirms yet once more that *as one probes smaller and smaller distances (large $-k^2$), the effective charge increases*; this is again consistent with our physical picture for vacuum polarization.

11.11 Renormalization conditions and counterterms

Our key conclusion is that vacuum polarization modifies the large distance Coulomb law behavior of the photon propagator. But *the Coulomb law provides the standard definition of the electric charge*. Thus, we shall reverse the order in which we impose known data.

1. We normalize the Coulomb law at long distances to be governed by the standard electric charge e;

2. Therefore, the bare parameters in the Lagrangian will not be the ones observed at long distance;

3. Imposing the condition that the Coulomb law at long distances is standard amounts to a *renormalization condition*, imposed at the zero momentum *renormalization point*. The modified Lagrangian used as a starting point is the *bare Lagrangian*. Its derivation from the physical Lagrangian is given by *counterterms*.

In the bare Lagrangian $\mathcal{L}_o$, the charge is replaced by the bare charge e_o, the fermion mass is replaced by the bare mass m_o and the fields A_μ and ψ are subject to field renormalization factors $\sqrt{Z_3}$ and $\sqrt{Z_2}$ respectively. It is customary to introduce a renormalization constant Z_1 for the vertex as well,

$$\begin{aligned}
\mathcal{L}_o &= -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 + Z_2\bar{\psi}(i\partial\!\!\!/ - m_o)\psi - eZ_1\bar{\psi}A\!\!\!/\psi \qquad (11.57)\\
\mathcal{L}_{\rm ct} &= -\frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu} + (Z_2-1)\bar{\psi}i\partial\!\!\!/\psi - (Z_2 m_o - m)\bar{\psi}\psi - e(Z_1-1)\bar{\psi}A\!\!\!/\psi
\end{aligned}$$

From vacuum polarization the only term we have encountered so far is Z_3.

The renormalization condition suitable to recovering the physical Coulomb law is that at long distance (i.e. zero momentum k), the full truncated 2-point function

$$\begin{aligned}
\Gamma_R^{\mu\nu}(k,m) &= (\eta^{\mu\nu}k^2 - k^\mu k^\nu)\bar{\Gamma}_R(k^2,m)\\
\bar{\Gamma}_R(0,m) &= 1 \qquad (11.58)
\end{aligned}$$

We know the contributions to $\bar{\Gamma}_R(k,m)$. The bare coefficient is $Z_3 = 1 + O(e^2)$; the one-loop correction of vacuum polarization $\bar{\Gamma}(k,m,\Lambda)$

$$\Gamma(k,m) = Z_3 - \bar{\Gamma}(k^2,m,\Lambda) + O(e^4) \qquad (11.59)$$

The renormalization condition implies the requirement

$$Z_3 = 1 + \bar{\Gamma}(0, m, \Lambda) = 1 - \frac{e^2}{12\pi^2} \ln \frac{\Lambda^2}{m^2} + O(e^4) \tag{11.60}$$

Notice that the Ward-Takahashi identities relating the electron propagator and the vertex function imply the relation $Z_1 = Z_2$. In other words, the combination $eA_\mu = e_0 A_{0\mu}$ is not renormalized.

11.12 The Gell-Mann – Low renormalization group

We shall deal in detail with the renormalization group later on, but historically, the subject started with the following key observation by Gell-Mann and Low. One really is not forced to renormalize the photon 2-point function at *zero* photon momentum. In fact, when one imagines a theory with vanishing fermion mass $m = 0$, it is *impossible to renormzalize at zero momentum*. To solve this problem, Gell-Mann and Low propose to choose an arbitrary scale, called the *renormalizatiion scale* μ such that

$$\bar{\Gamma}_R(k^2, m; \mu)\Big|_{k^2=-\mu^2} = 1 \tag{11.61}$$

Using the explicit form of Z_3:

$$Z_3 = 1 + \frac{e^2}{2\pi^2} \int_0^1 d\alpha \; \alpha(1-\alpha) \ln \frac{m^2 + \alpha(1-\alpha)\mu^2}{\Lambda^2} + \mathcal{O}(e^4) \tag{11.62}$$

Now if the bare charge is e_o, we have

$$e(\mu) = Z_3^{1/2}(\mu, \Lambda, m) e_o(\Lambda) \tag{11.63}$$

The charge $e(\mu)$, for given bare charge e_o, and given cutoff, now depends upon μ. The limit $m \to 0$ is now regular.

The dependence of $e(\mu)$ on μ defines the Gell-Mann–Low β-function,

$$\beta(e) \equiv \frac{\partial e(\mu)}{\partial \ln \mu}\Big|_{e_o, \Lambda \text{ fixed}} \tag{11.64}$$

The β-function may be evaluated to one loop order with the help of the results on Z_3 that we have just derived and one finds,

$$\beta(e) = e(\Lambda) \frac{\partial Z_3^{1/2}}{\partial \ln \mu}\Big|_{e_o \;\Lambda} = e(\mu) \frac{1}{2} \frac{\partial \ln Z_3}{\partial \ln \mu}\Big|_{e_o, \Lambda} = \frac{e^3}{12\pi^2} + \mathcal{O}(e^5) \tag{11.65}$$

12 QED: Radiative Corrections

In the previous section, one-loop contributions due to dynamical fermions only have been calculated. In these problems, the gauge potential may effectively be treated as an external field, since the photon was not quantized there. In this section, the gauge field is also quantized. Two important special cases are studied in detail : the fermion propagator and the vertex function, from which the one-loop anomalous magnetic moment is also deduced. A discussion of infrared singularities is also given.

12.1 A first look at the Electron Self-Energy

By Lorentz and parity invariance of QED, the electron self-energy must be of the following general form,

$$\Sigma(p,m) = A(p^2)\not{p} + mB(p^2)I \tag{12.1}$$

The coefficients A, B may, of course, also depend on the mass and on any regulators, which will not be exhibited. Note that terms of the type $C(p^2)\not{p}\gamma^5 + D(p^2)m\gamma^5$ are allowed by Lorentz invariance, but violate parity. The presence of the extra factor of m in the B-term results from approximate chiral symmetry of the theory with small m.

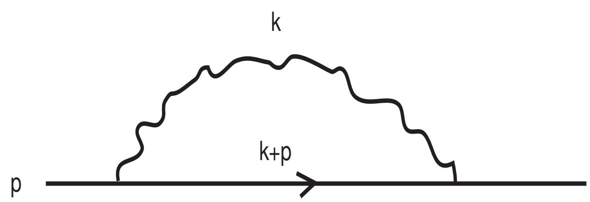


Figure 18: 1-loop 1PIR contribution to the electron self-energy

The 1-loop contribution to the electron self-energy is shown in Fig 18, and takes the form,

$$-i(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \left(\eta_{\mu\nu} + \xi \frac{k_\mu k_\nu}{k^2} \right) \gamma^\mu \frac{i}{\not{p} + \not{k} - m} \gamma^\nu \tag{12.2}$$

Contrarily to vacuum polarization, the electron self-energy is dependent upon the gauge parameter ξ. For simplicity, choose Feynman gauge $\xi = 0$ first. The graph has a linear superficial UV divergence. By Lorentz symmetry, however, this divergence is reduced to logarithmic. A convenient UV regulator is by replacing the photon propagator à la Pauli-Villars, (another choice could be dimensional regularization),

$$\frac{1}{k^2} \to \frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2} \tag{12.3}$$

where Λ is a large regulator mass scale. To evaluate $\Sigma^{(1)}$, we start by rationalizing the fermion propagator, and using the γ-matrix identities $\gamma_\mu \gamma^\mu = 4I$, $\gamma_\mu \gamma^\alpha \gamma^\mu = -2\gamma^\alpha$,

$$\Sigma^{(1)}(p, m, \Lambda) = ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{-2\not{p} - 2\not{k} + 4m}{[(p+k)^2 - m^2]} \left(\frac{1}{k^2} - \frac{1}{k^2 - \Lambda^2} \right) \tag{12.4}$$

Introducing one Feynman parameter α, shifting $k \to k - \alpha p$, recognizing the cancellation of a $\not{k}$ term in the numerator, Wick rotating to Euclidean momenta k_E, and decomposing according to its Lorentz properties, we find

$$\begin{aligned} A(p^2) &= +2e^2 \int_0^1 d\alpha \int \frac{d^4k_E}{(2\pi)^4} \left(\frac{1-\alpha}{[k_E^2 + M^2]^2} - \frac{1-\alpha}{[k_E^2 + M^2 + (1-\alpha)\Lambda^2]^2} \right) \\ B(p^2) &= -4e^2 \int_0^1 d\alpha \int \frac{d^4k_E}{(2\pi)^4} \left(\frac{1}{[k_E^2 + M^2]^2} - \frac{1}{[k_E^2 + M^2 + (1-\alpha)\Lambda^2]^2} \right) \end{aligned} \tag{12.5}$$

where $M^2 \equiv -\alpha(1-\alpha)p^2 + \alpha m^2$. The k_E-integrations are finite and may be evaluated using,

$$\int \frac{d^4k_E}{(2\pi)^4} \left(\frac{1}{[k_E^2 + M_0^2]^2} - \frac{1}{[k_E^2 + M_1^2]^2} \right) = \frac{1}{(4\pi)^2} \ln \frac{M_1^2}{M_0^2} \tag{12.6}$$

As a result, we have (making use of the fact that $m^2, p^2 \ll \Lambda^2$),

$$\begin{aligned} A(p^2) &= +\frac{2e^2}{(4\pi)^2} \int_0^1 d\alpha (1-\alpha) \ln \left(\frac{(1-\alpha)\Lambda^2}{-\alpha(1-\alpha)p^2 + \alpha m^2} \right) \\ B(p^2) &= -\frac{4e^2}{(4\pi)^2} \int_0^1 d\alpha \ln \left(\frac{(1-\alpha)\Lambda^2}{-\alpha(1-\alpha)p^2 + \alpha m^2} \right) \end{aligned} \tag{12.7}$$

The UV divergences are easily extracted, and we have

$$A(p^2)\Big|_{\rm div} = -\frac{e^2}{16\pi^2} \ln \Lambda^2 \qquad\qquad B(p^2)\Big|_{\rm div} = +\frac{e^2}{4\pi^2} \ln \Lambda^2 \tag{12.8}$$

Both are independent of the momentum p and therefore require counterterms precisely of the same form as already present in the Lagrangian, namely renormalization of Z_2 and m.

It remains to specify suitable renormalization point and renormalization conditions. Proceeding to impose renormalization conditions when the electrons are on-shell,

$$\Sigma_R^{(1)}(p, m)\Big|_{\not{p}=m} = 0 \qquad\qquad \frac{\partial \Sigma_R^{(1)}(p, m)}{\partial \not{p}}\Bigg|_{\not{p}=m} = 1 \tag{12.9}$$

we encounter what at first might seem like a surprising difficulty, Indeed, the derivative of $\Sigma_R^{(1)}(p)$ with respect to p produces (amongst other things) the p-derivative of the function

$A(p^2)$, which is given by

$$\frac{\partial A(p^2)}{\partial p^2} = +\frac{2e^2}{(4\pi)^2}\int_0^1 d\alpha \frac{(1-\alpha)^2}{-(1-\alpha)p^2+m^2} \tag{12.10}$$

The integral is singular when we let $p^2 \to m^2$, as the integrand behaves like $1/\alpha$.

This divergence has nothing to do with UV behavior. Its origin may be traced to the analytic structure of $A(p^2)$ as a function of p^2. The argument of the logarithm becomes negative when $(1-\alpha)p^2 > m^2$ for at least some $\alpha \in [0, 1]$. This will happen as soon as $p^2 > m^2$. It implies that as soon as the CM energy exceeds m, it is possible to create a real electron and a real photon. The reason the branch cut for this process starts right at $p^2 = m^2$ is that the photon is massless. The massless nature of the photon allows its creation at any positive energy, no matter how small. Thus, the divergence found here is associated with a low energy or infrared (IR) problem. Its resolution is *not* through renormalization, because its significance is physical. The solution to this IR problem will have to await our study of the vertex function. It is possible to choose a renormalization point where the electrons are off-shell. Actually, this is not so natural and it is inconvenient to do so while maintaining Lorentz invariance. Instead, we shall introduce a small photon mass to regularize the IR problems and defer to later a proper understanding of the zero photon mass limit.

12.2 Photon mass and removing infrared singularities

Does gauge invariance prevent having a massive photon? One might think yes. Actually Abelian gauge symmetry, such as in QED, allows for a non-zero photon mass. Here is why. Assume the mass is being introduced as a simple term in the Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 + \frac{\mu^2}{2}A_k A^k - A_\mu j^\mu \tag{12.11}$$

The classical field equations read

$$\partial_\mu F^{\mu\nu} + \lambda \partial^\nu (\partial \cdot A) + \mu^2 A^\nu - j^\nu = 0 \tag{12.12}$$

Combining current conservation, $\partial_\nu j^\nu = 0$ with the field equations, we obtain a field equation purely for the longitudinal component of the gauge field,

$$\lambda \Box (\partial \cdot A) + \mu^2 (\partial \cdot A) = 0 \tag{12.13}$$

Remarkably, the longitudinal part of the photon field is still a free field, as it had been in the massless case, and thus it decouples from the true dynamics of the theory. By contrast, in non-Abelian gauge theories the longitudinal part will not be a free field and therefore will not decouple. Masses for non-Abelian gauge fields cannot be given in this simple manner.

The associated massive photon propagator is considerably more complicated than the massless one,

$$\frac{-i}{k^2 - \mu^2}\left(\eta_{\mu\nu} + \xi \frac{k_\mu k_\nu}{k^2 - \mu^2/\lambda}\right) \tag{12.14}$$

Here, we continue to use the abbreviation $\xi = (1-\lambda)/\lambda$. Clearly, for the massive theory, it is very advantageous to choose Feynman gauge $\lambda = 1$ and thus $\xi = 0$.

12.3 The Electron Self-Energy for a massive photon

The Lorentz decomposition of the electron self-energy in (12.1) is unmodified by the introduction of a photon mass. The functions A, B become,

$$\begin{aligned} A(p^2) &= +\frac{2e^2}{(4\pi)^2}\int_0^1 d\alpha (1-\alpha) \ln\left(\frac{(1-\alpha)\Lambda^2}{-\alpha(1-\alpha)p^2 + \alpha m^2 + (1-\alpha)\mu^2}\right) \\ B(p^2) &= -\frac{4e^2}{(4\pi)^2}\int_0^1 d\alpha \ln\left(\frac{(1-\alpha)\Lambda^2}{-\alpha(1-\alpha)p^2 + \alpha m^2 + (1-\alpha)\mu^2}\right) \end{aligned} \tag{12.15}$$

It is readily verified that $\partial A(p^2)/\partial p^2$ now has a finite limit as $p^2 \to m^2$. The branch cut is now supported at $p^2 > m^2/(1-\alpha) + \mu^2/\alpha$, (with $\alpha \in [0,1]$), which starts at $p^2 = (m+\mu)^2$. This is as expected : physical particles can be produced as soon as the rest energy $m + \mu$ is available.

12.4 The Vertex Function : General structure

The 1-loop 1PIR contribution to the vertex function is shown in Fig 19. Its unregularized Feynman integral expression is given by

$$\Gamma_\mu^{(1)}(p', p) = -ie^3 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(\eta_{\rho\sigma} + \xi\frac{k_\rho k_\sigma}{k^2}\right)\gamma^\rho \frac{1}{\not p' - \not k - m}\gamma_\mu \frac{1}{\not p - \not k - m}\gamma^\sigma \tag{12.16}$$

The k-integral has a logarithmic UV divergence, which will be regularized by a Pauli Villars regulator on the photon propagator, just as we had used already for the electron-self-energy. The divergence itself is clearly proportional to the bare interaction vertex γ^μ, consistent with 1-loop renormalizability.

Assuming that this regularization has been carried out, it is instructive to verify explicitly the Ward identity between the vertex function and the electron self-energy $\Sigma^{(1)}$.

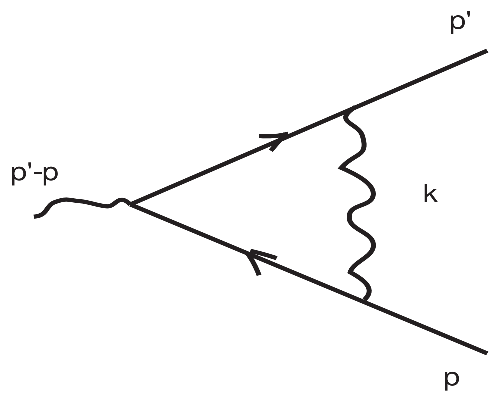


Figure 19: 1-loop 1PIR contribution to the vertex function

The Ward identity takes the form,

$$(p'-p)^\mu \; \Gamma^{(1)}_\mu(p',p) = +e\left(\Sigma^{(1)}(p') - \Sigma^{(1)}(p)\right) \tag{12.17}$$

It is proven to one-loop using the simple fact that $(p'-p)^\mu\gamma_\mu = (\not p' + \not k - m) - (\not p + \not k - m)$. Substituting this relation into the integrand of the vertex function one finds,

$$(p'-p)^\mu \frac{1}{\not p' + \not k - m}\gamma_\mu \frac{1}{\not p + \not k - m} = -e\left(\frac{1}{\not p + \not k - m} - \frac{1}{\not p' + \not k - m}\right) \tag{12.18}$$

Actually, the result holds to all orders in e. To see how this works, let $S(p)$ be the full fermion propagator, whose inverse is related to the regularized electron self-energy,

$$S(p)^{-1} = -i(\not p - m) - i\Sigma(p) \tag{12.19}$$

and $\Gamma_\mu(p',p)$ be the full vertex function, then we have

$$(p'-p)^\mu \Gamma_\mu(p',p) = -ie[S^{-1}(p) - S^{-1}(p')] \tag{12.20}$$

The Ward identities imply that the renormalizations of $\Gamma^{(1)}_\mu(p',p)$ and $\Sigma^{(1)}(p)$ are related to one another. Actually, the mass renormalization does not involve the momenta, so only Z_2 should affect $\Gamma^{(1)}_\mu(p',p)$.

We can make this connection explicit to one-loop order, where we have already established the structure of the divergences and required renormalization counterterms,

$$\begin{aligned} \Gamma^{(1)}_\mu(p',p) &= -e\gamma_\mu Z_1 + \Gamma^{(1)}_{R\mu}(p',p) \\ \Sigma^{(1)}(p) &= Z_2(\not p - m) + (Z_2 m_o - m) + \Sigma^{(1)}_R(p) \end{aligned} \tag{12.21}$$

Recall the standard on-shell renormalization conditions on the electron self-energy,

$$\Sigma^{(1)}_R(p)\Big|_{\not p = m} = 0 \qquad\qquad \frac{\partial \Sigma^{(1)}_R}{\partial \not p}(p)\Big|_{\not p = m} = 1 \tag{12.22}$$

For the vertex, the standard renormalization condition is also on-shell for the electrons, and imposes the vanishing of $\Gamma_\mu^{R(1)}$ at vanishing photon momentum,

$$\bar{u}(p')\Gamma^{(1)}_{R\mu}(p',p)u(p)\Big|_{p'=p;p^2=m^2} = 0 \tag{12.23}$$

Next, letting $p' \to p$ in the Ward identity (12.17) provides the following relation,

$$\Gamma_\mu^{(1)}(p,p) = -e\frac{\partial \Sigma^{(1)}(p)}{\partial p^\mu} \tag{12.24}$$

In terms of the renormalized quantities, we have

$$-e\gamma_\mu Z_1 + \Gamma^{(1)}_{R\mu}(p,p) = -e\gamma_\mu Z_2 - e\frac{\partial \Sigma_R^{(1)}(p)}{\partial p^\mu} \tag{12.25}$$

Evaluating this relation on-shell for the electrons, at zero momentum for the photon and using the renormalization conditions, it is immediate that the contributions of $\Gamma^{(1)}_{R\mu}$ and $\Sigma_R^{(1)}$ cancel, leaving the famous Ward identity

$$Z_1 = Z_2 \tag{12.26}$$

a result which is valid to all orders in e.

12.5 Calculating the off-shell Vertex

We begin by calculating the vertex function for general off-shell momenta p, p'. For the sake of generality, and because the result will be needed later as well, we shall include a non-zero photon mass μ in the calculation from the start,

$$\Gamma_\mu^{(1)}(p',p) = -ie^3\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\mu^2}\left(\eta_{\rho\sigma} + \xi\frac{k_\rho k_\sigma}{k^2-\frac{\mu^2}{\lambda}}\right)\gamma^\rho\frac{1}{\not p' - \not k - m}\gamma_\mu\frac{1}{\not p - \not k - m}\gamma^\sigma \tag{12.27}$$

It will always be assumed that the vertex integral has been UV regularized by using a Pauli Villars regulator Λ on the photon, just as was used for the electron self-energy. This fact will be indicated by the subscript Λ on the k-integral. To begin, we work in Feynman gauge with $\xi = 0$; we shall later also discuss the contributions from $\xi \neq 0$. Rationalizing the fermion propagators gives,

$$\Gamma_\mu^{(1)}(p',p) = -ie^3\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{\gamma^\sigma(\not p' - \not k + m)\gamma_\mu(\not p - \not k + m)\gamma_\sigma}{(k^2-\mu^2)((p'-k)^2-m^2)((p-k)^2-m^2)} \tag{12.28}$$

Introducing two Feynman parameters α, β for the 3 factors in the denominator, and using the following formula,

$$\frac{1}{ABC} = 2\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\frac{1}{[(1-\alpha-\beta)A + \alpha B + \beta C]^3} \tag{12.29}$$

we obtain

$$\Gamma^{(1)}_\mu(p',p) = -2ie^3 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{\gamma^\sigma(\not p' - \not k + m)\gamma_\mu(\not p - \not k + m)\gamma_\sigma}{((k-\alpha p' - \beta p)^2 - M^2(\mu))^3} \tag{12.30}$$

where we have defined the function

$$M^2(\mu) = (\alpha+\beta)m^2 + (1-\alpha-\beta)\mu^2 - \alpha p'^2 - \beta p^2 + (\alpha p' + \beta p)^2 \tag{12.31}$$

Upon shifting $k \to k + \alpha p' + \beta p$, we find

$$\Gamma^{(1)}_\mu(p',p) = -2ie^3 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{\mathcal{N}_\mu}{(k^2 - M^2(\mu))^3} \tag{12.32}$$

where the numerator is abbreviated by

$$\mathcal{N}_\mu = \gamma^\sigma\Big((1-\alpha)\not p' - \beta\not p - \not k + m\Big)\gamma_\mu\Big((1-\beta)\not p - \alpha\not p' - \not k + m\Big)\gamma_\sigma \tag{12.33}$$

The numerator may be simplified by using the following identities,

$$\begin{aligned}
\gamma^\sigma\gamma_\mu\gamma_\sigma &= -2\gamma_\mu \\
\gamma^\sigma\gamma_\mu\gamma_\nu\gamma_\sigma &= +4\eta_{\mu\nu} I \\
\gamma^\sigma\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma &= -2\gamma_\sigma\gamma_\nu\gamma_\mu
\end{aligned} \tag{12.34}$$

After regrouping and using the result of angular averaging $k_\mu k_\nu \to 1/4k^2\eta_{\mu\nu}$, one finds the following numerator

$$\begin{aligned}
\mathcal{N}_\mu &= -k^2\gamma_\mu + N_\mu \qquad\qquad (12.35)\\
N_\mu &= -2m^2\gamma_\mu + 4m(1-2\beta)p_\mu + 4m(1-2\alpha)p'_\mu \\
&\quad +2\beta(1-\beta)\ \not p\gamma_\mu\not p + 2\alpha(1-\alpha)\ \not p'\gamma_\mu\not p' \\
&\quad -2(1-\alpha)(1-\beta)\ \not p\gamma_\mu\not p' - 2\alpha\beta\ \not p'\gamma_\mu\not p
\end{aligned}$$

The k-integrals are readily evaluated,

$$\begin{aligned}
\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-M^2(\mu))^3} &= -\frac{i}{32\pi^2 M^2(\mu)} \\
\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{k^2}{(k^2-M^2(\mu))^3} &= \frac{i}{16\pi^2}\ln\frac{M^2(\Lambda)}{M^2(\mu)}
\end{aligned} \tag{12.36}$$

In the last integral, one may use $m, p, p' \ll \Lambda$ to carry out the following simplification, $M^2(\Lambda) \sim (1-\alpha-\beta)\Lambda^2$. In summary, the full off-shell vertex function is given by

$$\Gamma^{(1)}_\mu(p',p) = -\frac{e^3}{(4\pi)^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left(2\gamma_\mu \ln\frac{M^2(\Lambda)}{M^2(\mu)} + \frac{N_\mu}{M^2(\mu)}\right) \tag{12.37}$$

The integration over the Feynman parameters is fairly involved. Fortunately, it will not be needed here in full.

It remains to make a suitable choice of renormalization conditions. It is natural to take the fermions to be on-shell and the photon to have zero momentum, which implies $p' = p$,

$$\bar{u}(p)\left(-eZ_1\gamma_\mu + \Gamma^{(1)}_\mu(p,p)\right)u(p)\Big|_{p^2=m^2} = \bar{u}(p)(-e\gamma_\mu)u(p)\Big|_{p^2=m^2} + \mathcal{O}(e^5) \qquad (12.38)$$

Substituting the data of the renormalization point in the function M^2, we find,

$$M^2(\mu) = (1-\alpha-\beta)\mu^2 + (\alpha+\beta)^2 m^2 \qquad (12.39)$$

It is clear that the integration over α, β would fail to converge for zero photon mass $\mu = 0$, where an IR divergence occurs. As is the case of the electron self-energy, the physical origin of this IR divergence resides in the diverging probability for producing ultra-low energy photons. With $\mu \neq 0$, however, the integrals are convergent and the regulator dependent part of Z_1 may be deduced, (it arises only from the ln term),

$$Z_1 = 1 - \frac{e^2}{16\pi^2}\ln\Lambda^2 + \mathcal{O}(e^4) \qquad (12.40)$$

The finite renormalization parts may also be computed by carrying out the α, β-integrals, and the dependence on the gauge parameter may be incorporated as well, (Itzykson and Zuber)

$$Z_1 = 1 - \frac{e^2}{(4\pi)^2}\left(\frac{1}{\lambda}\ln\frac{\Lambda^2}{m^2} + 3\ln\frac{\mu^2}{m^2} - \frac{1}{\lambda}\ln\frac{\mu^2}{\lambda m^2} + \frac{9}{4}\right) + \mathcal{O}(e^4) \qquad (12.41)$$

12.6 The Electron Form Factor and Structure Functions

The renormalized vertex function resulting from the preceding subsection is quite a complicated object. In particular, the numerator N_μ generates 7 different spinor/tensor structures. Depending on the ultimate purpose of the calculations, not all of this information may actually be required. Such is the case for the calculation of the anomalous magnetic moment of the electron, on which we shall now focus attention.

A simpler object than the full vertex function is obtained when both electrons are restricted to being specific on-shell electron states. Equivalently, one is then evaluating not the full correlator $\langle\emptyset|j^\mu(x)\psi_\alpha(y)\bar{\psi}^\beta(z)|\emptyset\rangle$ but rather the *electron form factor*,

$$\langle u(p')|j^\mu(q)|u(p)\rangle = \bar{u}(p')\Gamma_\mu(p',p)u(p) \qquad (12.42)$$

Here, $|u(p)\rangle$ stands for an (on-shell) electron state with wave function $u(p)$, and we have $q = p' - p$ and $p^2 = p'^2 = m^2$. The photon is *not on-shell*, i.e. q^2 is left general. The reason this object is called a form factor is because it parametrizes the distributions of charge and current "inside" the electron. Even though the electron is assumed to be a "point-like particle", renormalization effects spread out its charge and gives the electron structure.

Despite these restrictions, the form factor actually still contains a wealth of information, such as the anomalous magnetic moment, and yet its form is much simpler than that of the full vertex. This may be seen by listing all possible Lorentz invariant structures that may occur. The Lorentz structure is determined by the Dirac matrix insertion between $\bar{u}(p')$ and $u(p)$, as well as by the explicit dependence on the external momenta. The matrix elements $\bar{u}(p')\gamma^5 u(p)$ and $\bar{u}(p')\gamma^5 S_{\mu nu} u(p)$ violate parity symmetry and are not allowed in QED. This leaves the matrix elements $\bar{u}(p')u(p)$, $\bar{u}(p')\gamma_\mu u(p)$ and $\bar{u}(p')S_{\mu\nu}u(p)$ only. The dependence on momenta also significantly simplifies, since $p^2 = p'^2 = m^2$ and $2p \cdot p' = 2m^2 - q^2$. Thus, the general Lorentz structure of the form factor is

$$\bar{u}(p')\Gamma_\mu(p', p)u(p) = \bar{u}(p')\left(\gamma_\mu f_1(q^2) + S_{\mu\nu}q^\nu f_2(q^2) + (p_\mu + p'_\mu)f_3(q^2)\right)u(p) \tag{12.43}$$

Actually, the *Gordon identity* provides a relation between all three coefficients,

$$(p + p')_\mu \bar{u}(p')u(p) = 2m\bar{u}(p')\gamma_\mu u(p) - 2iq^\nu \bar{u}(p')S_{\mu\nu}u(p) \tag{12.44}$$

the identity may be proven using

$$\begin{aligned} \gamma_\mu \not{p} &= p_\mu - 2iS_{\mu\nu}p^\nu & \qquad S_{\mu\nu} &= \frac{i}{4}[\gamma_\mu, \gamma_\nu] \\ \not{p}'\gamma_\mu &= p'_\mu + 2iS_{\mu\nu}p'^\nu & \qquad [\gamma_\mu, \gamma_\nu] &= -4iS_{\mu\nu} \end{aligned} \tag{12.45}$$

It is convenient to eliminate the term in $\bar{u}(p')u(p)$ in favor of the other two. It is customary to define the form factor in terms of the two *structure functions* $F_1(q^2)$ and $F_2(q^2)$,

$$\bar{u}(p')\Gamma_\mu(p', p)u(p) = \bar{u}(p')\gamma_\mu u(p)F_1(q^2) + \frac{iq^\nu}{m}\bar{u}(p')S_{\mu\nu}u(p)F_2(q^2) \tag{12.46}$$

In the following subsection, it will be shown that the anomalous magnetic moment can be computed solely from the value of $F_2(0)$.

12.7 Calculation of the structure functions

The ξ-dependent part of the form factor gives a contribution $F_1^{(\xi)}(q^2)$ to the structure function $F_1(q^2)$ and does not contribute to $F_2(q^2)$. Indeed, carrying out the contraction of $k_\rho k_\sigma$, we have the following simplification,

$$\bar{u}(p')\not{k}\frac{1}{\not{p}' - \not{k} - m}\gamma_\mu \frac{1}{\not{p} - \not{k} - m}\not{k}u(p) = \bar{u}(p')\gamma_\mu u(p) \tag{12.47}$$

This is established using the Dirac equations $(\not{p} - m)u(p) = 0$ and $\bar{u}(p')(\not{p}' - m) = 0$, with the help of which we have $\not{k}u(p) = -(\not{p} - \not{k} - m)u(p)$ and $\bar{u}(p')\not{k} = -\bar{u}(p')(\not{p}' - \not{k} - m)$. As a result, we have,

$$F_1^{(\xi)}(q^2) = -ie^3\xi \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-\mu^2)(k^2-\mu^2/\lambda)} = \frac{e^3\xi}{(4\pi)^2} \ln \frac{\Lambda^2}{\mu^2} \tag{12.48}$$

The remainder of the form factor may be evaluated for $\xi = 0$.

To calculate the remaining form factor, one evaluates the full off-shell vertex on shell and sorts the various Lorentz invariants according to the decomposition (**??**). The function $M^2(\mu)$ simplifies considerably by putting the electrons on-shell and we have

$$M^2(\mu) = (1-\alpha-\beta)\mu^2 + (\alpha+\beta)^2 m^2 - \alpha\beta q^2 \tag{12.49}$$

This requires evaluation of the matrix element $\bar u(p')N_\mu u(p)$, which follows from the following simple results,

$$\begin{aligned}
\bar u(p')\, \not p \gamma_\mu \not p\, u(p) &= \bar u(p') \Big(m^2\gamma_\mu - mq_\mu - 2imq^\nu S_{\mu\nu}\Big) u(p) \\
\bar u(p')\, \not p' \gamma_\mu \not p'\, u(p) &= \bar u(p') \Big(m^2\gamma_\mu + mq_\mu - 2imq^\nu S_{\mu\nu}\Big) u(p) \\
\bar u(p')\, \not p' \gamma_\mu \not p\, u(p) &= \bar u(p') \Big(m^2\gamma_\mu \Big) u(p) \\
\bar u(p')\, \not p \gamma_\mu \not p'\, u(p) &= \bar u(p') \Big(m^2\gamma_\mu + q^2\gamma_\mu - 4miq^\nu S_{\mu\nu}\Big) u(p)
\end{aligned} \tag{12.50}$$

Therefore, the matrix elements of N_μ are found to be,

$$\begin{aligned}
\bar u(p')N_\mu u(p) = \bar u(p')\Big(& 2mq_\mu \{\alpha(1-\alpha) - \beta(1-\beta)\} - 2(1-\alpha)(1-\beta)q^2\gamma_\mu \\
& + m^2\gamma_\mu \Big\{4 - 4(\alpha+\beta) - 2(\alpha+\beta)^2\Big\} \\
& - 4i(\alpha+\beta)(1-\alpha-\beta)mq^\nu S_{\mu\nu}\Big) u(p)
\end{aligned} \tag{12.51}$$

Since the function $M^2(\mu)$ is a symmetric function of α and β and since the integration measure is symmetric as well, the numerator is effectively symmetrized, which cancels the term in mq_μ. The remainder is readily decomposed onto the structure functions F_1 and F_2, and we have

$$\begin{aligned}
F_1(q^2) &= -\frac{e^3}{8\pi^2} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \Bigg(\ln \frac{M^2(\Lambda)}{M^2(\mu)} \\
&\qquad + \frac{m^2\{2-2\alpha-2\beta-(\alpha+\beta)^2\} - q^2(1-\alpha)(1-\beta)}{(1-\alpha-\beta)\mu^2 + (\alpha+\beta)^2 m^2 - \alpha\beta q^2} \Bigg) \\
F_2(q^2) &= \frac{e^3}{4\pi^2} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \frac{m^2(\alpha+\beta)(1-\alpha-\beta)}{(1-\alpha-\beta)\mu^2 + m^2(\alpha+\beta)^2 - \alpha\beta q^2}
\end{aligned} \tag{12.52}$$

The function F_1 is singular as $\mu \to 0$ signaling an IR divergence. If the renormalization conditions are taken on-shell for the electrons and at $q^2 = 0$ for the photons (which is actually off-shell since $\mu \neq 0$), then the renormalized structure function is given by

$$
\begin{aligned}
F_1(q^2) &= -\frac{e^3}{8\pi^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \Bigg(\ln \frac{(1-\alpha-\beta)\mu^2 + (\alpha+\beta)^2 m^2}{(1-\alpha-\beta)\mu^2 + (\alpha+\beta)^2 m^2 - \alpha\beta q^2} \\
&\qquad + \frac{m^2\{2-2\alpha-2\beta-(\alpha+\beta)^2\} - q^2(1-\alpha)(1-\beta)}{(1-\alpha-\beta)\mu^2 + (\alpha+\beta)^2 m^2 - \alpha\beta q^2} \\
&\qquad - \frac{m^2\{2-2\alpha-2\beta-(\alpha+\beta)^2\}}{(1-\alpha-\beta)\mu^2 + (\alpha+\beta)^2 m^2} \Bigg)
\end{aligned}
\tag{12.53}
$$

The interpretation of the infrared divergence will be given later. The physical interpretation of F_1 is to describe the charge distribution of the electron.

In $F_2(q^2)$, the limit $\mu^2 \to 0$ is finite and may be taken. This quantity is properly renormalized all by itself. Important is the $q^2 \to 0$ limit, corresponding to low energy photons and homogeneous static fields,

$$
F_2(0) = \frac{e^3}{4\pi^2}\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left(\frac{1}{\alpha+\beta} - 1 \right) = \frac{e^3}{8\pi^2} \tag{12.54}
$$

The physical interpretation of F_2 is to describe the distribution of magnetic dipole moments of the electron. Next, this result is applied to the calculation of the anomalous magnetic moment of the electron, viewed customarily in response to a constant magnetic field, i.e. at $q = 0$.

12.8 The One-Loop Anomalous Magnetic Moment

The form of the renormalized vertex function, evaluated on-shell for the electrons, is in the small q approximation,

$$
\bar u(p')\Gamma_\mu(p',p)u(p) = \bar u(p') \left(e\gamma_\mu + \frac{e^3}{8\pi^2}\frac{iq^\nu}{m} S_{\mu\nu} \right) u(p) \tag{12.55}
$$

To compute the anomalous magnetic moment, we need to incorporate the effect of the above correction in the Dirac equation and extract the proper magnetic moment term. First, multiply by the gauge field A_μ. Translating to position space, we use

$$
iq^\nu A^\mu S_{\mu\nu} = \frac{i}{2}(q^\nu A^\mu - q^\mu A^\nu) S_{\mu\nu} \to \frac{1}{2} F^{\mu\nu} S_{\mu\nu} \tag{12.56}
$$

The modified Dirac equation is therefore,

$$
\left(i\partial\!\!\!/ - eA\!\!\!/ - \frac{e^3}{8\pi^2}\frac{1}{2m} F^{\mu\nu} S_{\mu\nu} - m \right) \psi = 0 \tag{12.57}
$$

To extract the magnetic moment, we use a standard spinor formula,

$$(i\partial\!\!\!/ - eA\!\!\!/)^2 = (i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - eF^{\mu\nu}S_{\mu\nu} \tag{12.58}$$

which may be derived by decomposing the product of two γ-matrices on the lhs into a sum of their commutator and anti-commutator. Multiplying the effective Dirac equation to the left by $(i\partial\!\!\!/ - eA\!\!\!/ + m)$ and using the fact that in the absence of A we have $ip\!\!\!/ = m$, we have

$$\begin{aligned}
&(i\partial\!\!\!/ - eA\!\!\!/ + m)\left(i\partial\!\!\!/ - eA\!\!\!/ - \frac{e^3}{8\pi^2}\frac{1}{2m}F^{\mu\nu}S_{\mu\nu} - m\right)\psi \\
&= \left((i\partial_\mu - eA_\mu)^2 - m^2 - \left(e + \frac{e^3}{8\pi^2}\right)F^{\mu\nu}S_{\mu\nu}\right)\psi = 0
\end{aligned} \tag{12.59}$$

Thus, the electron magnetic moment receives a finite correction:

$$\begin{aligned}
\mu &= \frac{e}{m}\left(1 + \frac{\alpha}{2\pi}\right) + O(\alpha^2) \qquad\qquad \alpha = \frac{e^2}{4\pi} \\
\frac{\alpha}{2\pi} &= 0.0011614\ \cdots \\
expt.: &\quad 0.0011597\ : \qquad \text{other effects; QCD, weak } \cdots
\end{aligned}$$

13 Functional Integrals For Gauge Fields

The functional methods appropriate for gauge fields present a number of complications, especially for non-Abelian gauge theories. In this chapter, we begin with a rapid and somewhat sketchy discussion of the Abelian case and then move onto a systematic functional integral quantization of the Abelian and nonAbelian cases.

13.1 Functional integral for Abelian gauge fields

The starting point will be taken to be the Hamiltonian, supplemented by Gauss' law. It is easiest to describe the system in temporal gauge $A_0 = 0$, where we have,

$$H = \int d^3x \left(\frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 - \vec{A} \cdot \vec{J} \right) \tag{13.1}$$

where $\vec{J}$ is a conserved external current and we the following expressions for electric and magnetic fields,

$$\vec{E} = \frac{\partial \vec{A}}{\partial t} \qquad\qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{13.2}$$

Furthermore, configuration space is described by all $A_i(\vec{x})$, hence a "position basis" of Fock space consists of all states of the form

$$\mathcal{F} \equiv \Big\{ |A_i(\vec{x})\rangle \quad \text{for all} \quad A_i(\vec{x}) \Big\} \tag{13.3}$$

The Hilbert space of physical states (recall the difference between the Fock space and the physical Hilbert space, as discussed for free spin 1 fields) is obtained by enforcing on the states the constraint of Gauss' law

$$G(A) \equiv \vec{\nabla} \cdot \vec{E} - J^o \tag{13.4}$$

Thus, we have

$$\mathcal{H}_{\rm phys} \equiv \Big\{ |\psi\rangle \in \mathcal{F} \quad \text{such that} \quad G(A)|\psi\rangle = 0 \Big\} \tag{13.5}$$

Of course, we are only interested in the physical states. In particular, we are interested only in time-evolution on the subspace of physical states. Therefore, we introduce the projection operator onto $\mathcal{H}_{\rm phys}$ as follows,

$$\mathcal{P} \equiv \sum_{|\psi\rangle \in \mathcal{H}_{\rm phys}} |\phi\rangle\langle\psi| \tag{13.6}$$

and the projected evolution operator may be decomposed as follows,

$$e^{-itH}\Big|_{\mathcal{H}_{\rm phys}} = \lim_{N\to\infty} \Big(\mathcal{P}\ e^{-itH/N} \Big)^N \tag{13.7}$$

We always assume that the full Hamiltonian, using the dynammics of j is used, so that it is time independent. Otherwise T-order!

The projection operator onto physical states may also be constructed as a functionla intgeral, though admittedly, this construction is a bit formal,

$$\mathcal{P} = \frac{1}{\int \mathcal{D}\Lambda} \int \mathcal{D}\Lambda \exp\left(i \int d^3x \Lambda(\vec{x})(\vec{\nabla}\cdot\vec{E} - J^o)(\vec{x})\right) \tag{13.8}$$

Clearly $\mathcal{P}$ is the identity on $\mathcal{H}_{\rm phys}$, and zero anywhere else! Summing over only physical states is equivalent to summing over all states $|A_i(\vec{x})\rangle$ and using the projector $\mathcal{P}$ to project the sum onto physical states only.

$$\begin{aligned} e^{-itH}\Big|_{\mathcal{H}_{\rm phys}} &= \frac{1}{\int \mathcal{D}\Lambda} \int \mathcal{D}A_i \int \mathcal{D}E_i \int \mathcal{D}\Lambda \qquad (13.9) \\ & \exp\left\{ i \int_0^t dt \int d^3x \left(\vec{E}\cdot\partial_0\vec{A} - \frac{1}{2}\vec{E}^2 - \frac{1}{2}\vec{B}^2 + \Lambda(\vec{\nabla}\cdot\vec{E} - J^o) + \vec{A}\cdot\vec{J}\right)\right\} \end{aligned}$$

where it is understood that the usual boundary conditions on the functional integral variables are implemented at time 0 and t. Now rebaptize Λ as A_o, and express the result in terms of a generating functional for the source J^μ,

$$Z[J] \equiv \frac{1}{\int \mathcal{D}\Lambda} \int \mathcal{D}A_\mu \ \exp\left\{ i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu\right)\right\} \tag{13.10}$$

However, this is really only a formal expression. In particular, the volume $\int \mathcal{D}\Lambda$ of the space of all gauge transformation is wildly infinite and must be dealt with before one can truly make sense of this expression.

Recall that the expressions that came in always assumed the form $Z[J^\mu]/Z[0]$ so that an overall factor of the volume of the gauge transformations cancels out. Thus in the expression for $Z[J^\mu]$ itself, we may *ab initio* fix a gauge, so that the volume of the gauge orbit is eliminated. We shall give a full discussion of this phenomenon when dealing with the non-Abelian case in the next subsection; here we take an informal approach. One of the most convenient choices is given by a linear gauge condition,

$$\int \mathcal{D}A_\mu \prod_x \delta(\partial^\mu A_\mu - c) \exp\left\{ i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right)\right\} \tag{13.11}$$

Since no particular slice is preferred, it is natural to integrate with respect to c, say with a Gaussian damping factor,

$$\int \mathcal{D}c \ e^{-i\lambda/2 \int d^4x c^2(x)} \prod_x \delta(\partial^\mu A_\mu(x) - c(x)) = e^{-i\lambda/2 \int d^4x (\partial^\mu A_\mu)^2} \tag{13.12}$$

and we recover our good old friend:

$$Z[J] = \int \mathcal{D}A_\mu \ \exp\left\{ i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\lambda(\partial\cdot A)^2 - J^\mu A_\mu\right)\right\} \tag{13.13}$$

The J-dependence of $Z[J^\mu]$ is independent of λ, so we can formally take $\lambda \to 0$ and regain our original gauge invariant theory. It is straightforward to evaluate the free field result,

$$\begin{aligned}
\langle 0|TA_\mu(x)A_\nu(y)|0\rangle &= i\int \frac{d^4k}{(2\pi)^4}\left(\eta_{\mu\nu}+\frac{1-\lambda}{\lambda}\frac{k_\mu k_\nu}{k^2+i\epsilon}\right)\frac{e^{-ik(x-y)}}{k^2+i\epsilon} \\
&= G_{\mu\nu}(x-y) \qquad (13.14)
\end{aligned}$$

So that

$$\frac{Z[j]}{Z[0]} = \exp\left\{\frac{1}{2}\int d^4x\int d^4y J^\mu(x)G_{\mu\nu}(x-y)J^\nu(y)\right\} \qquad (13.15)$$

which, provided J^μ is conserved, is independent of λ. In particular when any gauge independent quantity is evaluated, the dependence on λ will disappear. When $\lambda \to \infty$ one recovers Landau gauge, which is manifestly transverse; when $\lambda \to 1$, one recovers another very convenient Feynman gauge.

13.2 Functional Integral for Non-Abelian Gauge Fields

We consider a Yang-Mills theory based on a compact gauge group G with structure constants f^{abc}, $a,b,c = 1,\cdots,\mathrm{dim}G$. As discussed previously, if G is a direct product of n simple and n' $U(1)$ factors, then there will be $n+n'$ independent coupling constants describing $n+n'$ different gauge theories. Here, we shall deal with only one such component. Thus, we assume that G is simple or that it is just an Abelian theory with $G = U(1)$. The basic field in the theory is the Yang-Mills field A^a_μ. The starting point is the classical action,

$$\begin{aligned}
S[A;J] &= -\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{a\mu\nu} - \int d^4x J^{a\mu}A^a_\mu \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \qquad (13.16)
\end{aligned}$$

where summation over repeated gauge indices a is assumed.

For this action to be gauge invariant, the current must be covariantly conserved $D_\mu J^\mu = 0$, which may render the J^μ dependent on the field A_μ itself. This complicates the discussion, as compared to the Abelian case. Actually, we shall make one of two possible assumptions.

1. The current J^μ arises from the coupling of other fields (such as fermions or scalars) in a complete theory that is fully gauge invariant.

2. The current J^μ is an external source, but will be used to produce correlation functions of gauge invariant operators, such as $\mathrm{tr}F_{\mu\nu}F^{\mu\nu}$.

In both cases, the full theory will really be gauge invariant.

13.3 Canonical Quantization

Canonical quantization is best carried out in temporal gauge with $A_0 = 0$. To choose this gauge, the equation $\partial_0 U = igUA_0$ must be solved, which may always be done in terms of the time-ordered exponential

$$U(t, \vec{x}) = T \exp\Big\{ ig \int_{t_0}^{t} dt' A_0(t', \vec{x}) \Big\} U(t_0, \vec{x}) \tag{13.17}$$

In flat space-time with no boundary conditions in the time direction, this solution always exists. If time is to be period (as in the case of problems in statistical mechanics at finite temperature) then one cannot completely set $A_0 = 0$.

It is customary to introduce Yang-Mills electric and magnetic fields by

$$\begin{aligned} B_i^a &\equiv \frac{1}{2}\epsilon_{ijk} F_{jk}^a \\ E_i^a &\equiv F_{0i}^a = \partial_0 A_i^a \end{aligned} \tag{13.18}$$

and E_i^a is the canonical momentum conjugate to A_i^a. The Hamiltonian and Gauss' law are given by

$$\begin{aligned} H[E, A; J] &= \int d^3x \Big\{ \frac{1}{2} E_i^a E_i^a + \frac{1}{2} B_i^a B_i^a - J_i^a A_i^a \Big\} \\ G^a(E, A) &= \partial_i E_i^a + g f^{abc} A_i^b E_i^c \end{aligned} \tag{13.19}$$

The canonical commutation relations are

$$[E_i^a(x), A_j^b(y)]\delta(x^0 - y^0) = -i\delta^{ab}\delta_{ij}\delta^{(4)}(x - y) \tag{13.20}$$

The operator G^a commutes with the Hamiltonian (for a covariantly conserved J^μ). The action of G^a on the fields is by gauge transformation, as may be seen by taking the relevant commutator,

$$[G^a(x), A_i^b(y)]\delta(x^0 - y^0) = \partial_i \delta^{(4)}(x - y)\delta^{ab} + f^{abc} A_i^c \delta^{(4)}(x - y) \tag{13.21}$$

or better even in terms of the the integrated form of G^a,

$$\begin{aligned} G_\omega &\equiv \int d^3x G^a(x) \omega_a(x) \\ [G_\omega, A_i^b(y)] &= (D_i \omega)^b(y) \end{aligned} \tag{13.22}$$

Gauge transformations form a Lie algebra as usual,

$$[G^a(x), G^b(y)]\delta(x^0 - y^0) = i f^{abc} G^c(x) \delta^{(4)}(x - y) \tag{13.23}$$

Thus, the operator G^a generates infinitesimal gauge transformations.

As in the case of the Abelian gauge field, Gauss' law cannot be imposed as an operator equation because it would contradict the canonical commutation relations between A_i and E_i. Thus, it is a constraint that should be imposed as an initial condition. In quantum field theory, it must be imposed on the physical states. The Fock and physical Hilbert spaces are defined in turn by

$$\begin{aligned} \mathcal{F} &\equiv \Big\{ |A_i^a(\vec{x})\rangle \Big\} \\ \mathcal{H}_{\rm phys} &\equiv \Big\{ |\psi\rangle \in \mathcal{F} \quad \text{such that} \quad G^a(E,A)|\psi\rangle = 0 \Big\} \end{aligned} \tag{13.24}$$

It is interesting to note the physical significance of Gauss' law.

13.4 Functional Integral Quantization

This proceeds very similarly to the case of the quantization of the Abelian gauge field. The starting points are the Hamiltonian in $A_0 = 0$ gauge and the constraint of Gauss' law which commuted with the Hamiltonian. We construct the evolution operator projected onto the physical Hilbert space by repeated (and redundant) insertions of the projection operator. Recall the definition and functional integral representation of this projector,

$$\begin{aligned} \mathcal{P} &\equiv \sum_{|\psi\rangle \in \mathcal{H}_{\rm phys}} |\psi\rangle\langle\psi| \\ &= \frac{1}{\int \mathcal{D}\Lambda^a} \int \mathcal{D}\Lambda^a(\vec{x}) \exp\Big\{ i \int d^3x \Lambda^a(\vec{x}) \Big(D_i E_i^a - J_0^a \Big) \Big\} \end{aligned} \tag{13.25}$$

As in the case of the Abelian gauge fields, we construct the evolution operator projected onto the physical Hilbert space by repeated insertions of $\mathcal{P}$,

$$\begin{aligned} e^{-itH}\Big|_{\rm phys} &= \lim_{N\to\infty} \Big(\mathcal{P}\ e^{-i\frac{t}{N}H} \Big)^N \qquad (13.26) \\ &= \frac{1}{\int \mathcal{D}\Lambda^a} \int \mathcal{D}\Lambda^a \int \mathcal{D}A_i^a \int \mathcal{D}E_i^a \\ &\qquad \times \exp\Big\{ i \int_0^t dt' \int d^3x \Big(E_i^a \partial_0 A_i^a - \frac{1}{2} E_i^a E_i^a - \frac{1}{2} B_i^a B_i^a \\ &\qquad\qquad + A_i^a J_i^a + \Lambda^a (D_i E_i^a - J_0^a) \Big) \Big\} \end{aligned}$$

together with the customary boundary conditions on the field A_i^a and free boundary conditions on the canonically conjugate momentum E_i^a.

Next, we recall that Gauss' law resulted from the variation of the non-dynamical field A_0, which had been set to 0 in this gauge. But here, Λ^a plays again the role of a non-dynamical Lagrange multiplier and we rebaptize it as follows,

$$\Lambda^a(x) \quad \to \quad A_0^a(x) \tag{13.27}$$

The integration over the non-dynamical canonical momentum E_i^a may be carried out in an algebraic way by completing the square of the Gaussian,

$$\begin{aligned}
E_i^a \partial_0 A_i^a - \frac{1}{2} E_i^a E_i^a - \frac{1}{2} B_i^a B_i^a + A_i^a J_i^a - D_i A_0^a E_i^a - A_0^a J_0^a \\
= -\frac{1}{2}(E_i^a - \partial_0 A_i^a + D_i A_0)^2 + \frac{1}{2}(\partial_0 A_i^a - D_i A_0)^2 - \frac{1}{2} B_i^a B_i^a - A_\mu^a J^{a\mu}
\end{aligned} \tag{13.28}$$

The intgeration measure over $\mathcal{D}E_i^a$ is invariant under (functional) shifts in E_i^a, and we use this property to realize that the remaining Gaussian integral over E_i^a is independent of A_μ^a and J_μ^a, and therefor produces a constant. The remainder is easily recognized as the Lagrangian density with source that was used as a starting point.

As a concrete formula, we record the matrix element of the evolution operator taken between (non-physical) states in the Fock space,

$$\begin{aligned}
\langle A_2(\vec{x})|e^{-itH}|A_1(\vec{x})\rangle \;\; = \;\; & c \int \mathcal{D}A_\mu^a \exp\Big\{ i \int_0^t dt' \int d^3x \Big(-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - A_\mu^a J^{a\mu} \Big) \Big\} \\
& \text{boundary conditions} \left\{ \begin{array}{l} A(0,\vec{x}) = A_1(\vec{x}) \\ A(t,\vec{x}) = A_2(\vec{x}) \end{array} \right.
\end{aligned} \tag{13.29}$$

Time ordered correlation functions are deduced as usual and may be summarized by the following generating functional,

$$e^{iG[J]} \equiv \int \mathcal{D}A_\mu^a \exp\Big\{ -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - J^{a\mu} A_\mu^a \Big\} \tag{13.30}$$

Of course, this integral is formal, since gauge invariance leads to the integration of equivalent copies of A_μ^a. Therefore, we need to gauge fix, a process outlined in the next subsection.

13.5 The redundancy of integrating over gauge copies

We wish to make sense out of the functional intgeral of any gauge invariant combination of operators which we shall collectively denote by $\mathcal{O}$. For example, the following operators are gauge invariant,

$$\begin{aligned}
\mathcal{O}_+(x) &\equiv F_{\mu\nu}^a F^{a\mu\nu}(x) \\
\mathcal{O}_-(x) &\equiv F_{\mu\nu}^a \tilde{F}^{a\mu\nu}(x)
\end{aligned} \tag{13.31}$$

An infinite family of operators $\mathcal{O}$ may be formed out of a time ordered product of a string of $\mathcal{O}_+$ and $\mathcal{O}_-$ opertors,

$$\mathcal{O} = T\mathcal{O}_+(x_1) \cdots \mathcal{O}_+(x_m) \mathcal{O}_-(y_1) \cdots \mathcal{O}_+(y_n) \tag{13.32}$$

We have the general formula

$$\langle 0|\mathcal{O}|0\rangle = \frac{\int \mathcal{D}A_\mu^a \; \mathcal{O} \; e^{iS[A]}}{\int \mathcal{D}A_\mu^a \; e^{iS[A]}} \tag{13.33}$$

The key characteristic of this formula is that $\mathcal{O}$ is a gauge invariant operator.

The measure $\mathcal{D}A_\mu^a$ is also gauge invariant. The way to see this is to notice that a measure results from a metric on the space of all gauge fields. Once this metric has been found and properly defined, the measure will forllow. The metric may be taken to be given by the following norm,

$$|| \ \delta A_\mu^a \ ||^2 \equiv \int d^4x \ \delta A_\mu^a \ \delta A^{a\mu} \tag{13.34}$$

The variation of a gauge field transforms homogeneously and thus the metric is invariant. Therefore, we can construct an invariant measure.

Thus, we realize now that all pieces of the functional integral for $\langle|\mathcal{O}|0\rangle$ are invariant under local gauge transformations. But this renders the numerator and the denominator infinite, since in each the integration extends over all gauge copies of each gauge field. However, since the integrations are over gauge invariant integrands, the full integrations will factor out a volume factor of the space of all gauge transformations times the integral over the quotient of all gauge fields by all gauge transformations. Concretely,

$$\begin{aligned} \mathcal{A} &\equiv \{ \text{ all } A_\mu^a(x) \ \} \\ \mathcal{G} &\equiv \{ \text{ all } \Lambda^a(x) \ \} \end{aligned} \tag{13.35}$$

Thus we have symbolically,

$$\int_{\mathcal{A}} \mathcal{D}A_\mu^a \ \mathcal{O} \ e^{iS[A]} = \mathrm{Vol}(\mathcal{G}) \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A_\mu^a \ \mathcal{O} \ e^{iS[A]} \tag{13.36}$$

and inparticular, the volume of the space of all gauge transformations cancels out in the formula for $\langle 0|\mathcal{O}|0\rangle$,

$$\langle 0|\mathcal{O}|0\rangle = \frac{\int_{\mathcal{A}} \mathcal{D}A_\mu^a \ \mathcal{O} \ e^{iS[A]}}{\int_{\mathcal{A}} \mathcal{D}A_\mu^a \ e^{iS[A]}} = \frac{\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A_\mu^a \ \mathcal{O} \ e^{iS[A]}}{\int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A_\mu^a \ e^{iS[A]}} \tag{13.37}$$

The second formula does not have the infinite redundancy integrating over all gauge transformations of any gauge field.

In the case of the Abelian gauge theories, the problem was manifested through the fact that the longitudinal part of the gauge field does not couple to the $F_{\mu\nu}F^{\mu\nu}$ term in the action, a fact that renders the kinetic operator non-invertible. A gauge fixing term $\lambda(\partial_\mu A^\mu)^2$ was added; its effect on the physical dynamics of the theory is none since the current is conserved. For the non-Abelian theory, things are not as simple. First of all there are now interaction terms in the $F_{\mu\nu}^a F^{a\mu\nu}$ term, but also, the current is only covariantly conserved. The methods used for the Abelian case do not generalize to the non-Abelian case and we have to invoke more powerful methods.

13.6 The Faddeev-Popov gauge fixing procedure

Geometrically, the problem may be understood by considering the space of all gauge fields $\mathcal{A}$ and studying the action on this space of the gauge transformations $\mathcal{G}$, as depicted in Fig 20. The meaning of the figures is as follows

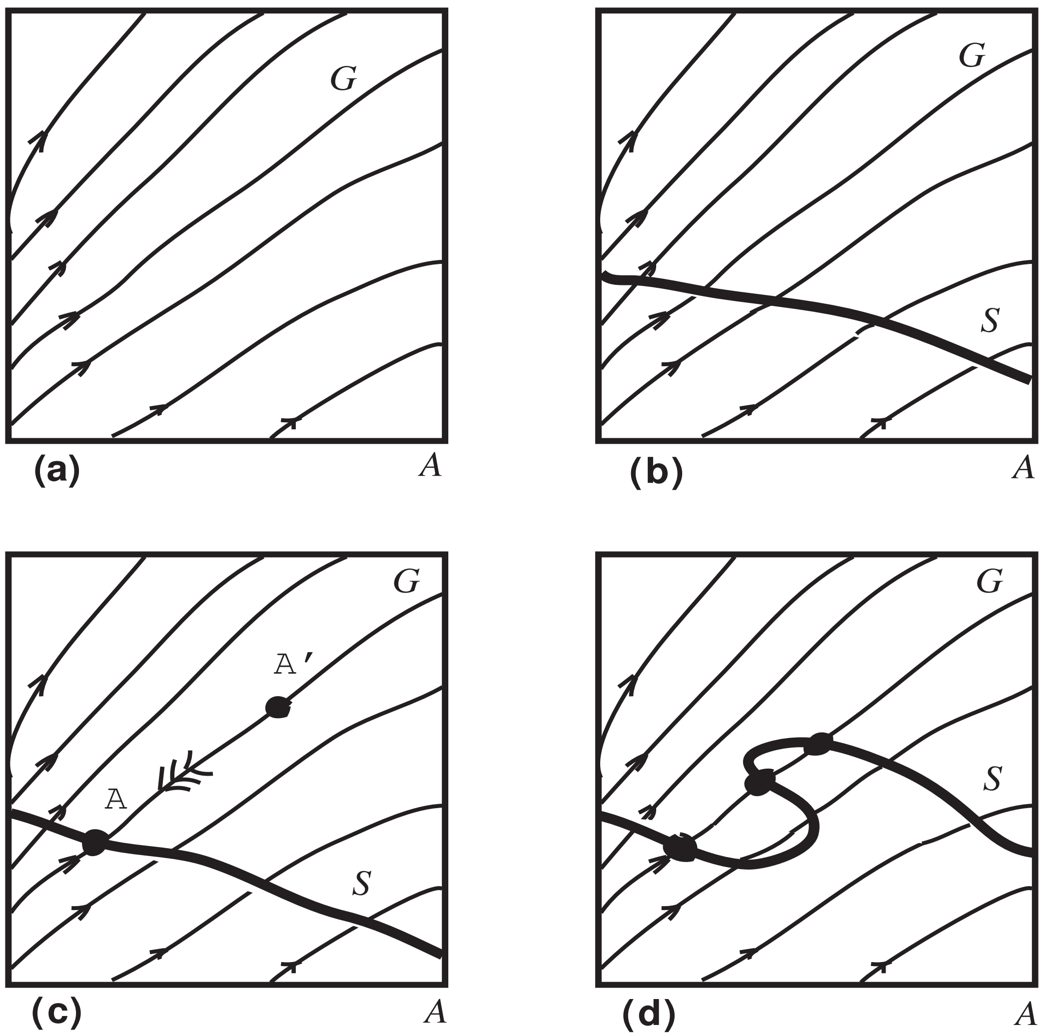


Figure 20: Each square represents the space $\mathcal{A}$ of all gauge field; each slanted thin line represents an orbit of $\mathcal{G}$; each fat line represents a gauge slice $\mathcal{S}$. Every gauge field A' may be brought back to the slice $\mathcal{S}$ by a gauge transformation, shown in (c). A gauge slice may intersect a given orbit several times, shown in (d) – this is the Gribov ambiguity.

- (a) The space $\mathcal{A}$ of all gauge fields is schematically represented as a square and the action of the group of gauge transformations is indicated by the thin slanted lines.

(Mathematically, $\mathcal{A}$ is the space of all connections on fiber bundles with structure group G over $\mathbf{R}^4$.)

- (b) There is no canonical way of representing the coset $\mathcal{A}/\mathcal{G}$. Therefore, one must choose a *gauge slice*, represented by the fat line $\mathcal{S}$. To avoid degeneracies, the gauge slice should be taken to be *transverse* to the orbits of the gauge group $\mathcal{G}$. (Mathematically, the gauge slice is a local section of the fiber bundle.)

- (c) Transversality guarantees – at least locally – that every point A' in gauge field space $\mathcal{A}$ is the gauge transformation of a point A on the gauge slice $\mathcal{S}$.

- (d) It is a tricky problem whether – globally – every gauge orbit intersects the gauge slice just once or more than once. In physics, this problem is usually referred to as the *Gribov ambiguity*. (Mathematically, it can be proven that interesting slices produce multiple intersections.) Fortunately, in perturbation theory, the fields remain small and the Gribov copies will be *infinitely far away* in field space and may thus be ignored. Non-perturbatively, one would have to deal with this issue; fortunately, the best method is lattice gauge theory, where one never is forced to gauge fix.

The choice of gauge slice is quite arbitrary. To remain within the context of QFT's governed by local Poincaré invariant dynamics, we shall most frequently choose a gauge compatible with these assumptions. Specifically, we choose the gauge slice $\mathcal{S}$ to be given by a *local equation* $\mathcal{F}^a(A)(x) = 0$, where $\mathcal{F}(A)$ involves A and derivatives of A in a local manner. Possible choices are

$$\mathcal{F}^a(A) = \partial_\mu A^{a\mu} \qquad\qquad \mathcal{F}^a = A_0^a \tag{13.38}$$

But many other choices could be used as well.

To reduce the functional integral to an integral along the gauge slice, we need to factorize the measure $\mathcal{D}A_\mu^a$ into a measure along the gauge slice $\mathcal{S}$ times a measure along the orbits of the gauge transformations $\mathcal{G}$. There is a nice gauge-invariant measure $\mathcal{D}U$ on $\mathcal{G}$ resulting from the following gauge invariant metric on $\mathcal{G}$,

$$|| \ \delta U \ ||^2 \equiv - \int d^4x \mathrm{tr}\Big(U^{-1}\delta U \ U^{-1}\delta U \Big) \tag{13.39}$$

If the orbits $\mathcal{G}$ were everywhere orthogonal to $\mathcal{S}$, then the factorization would be easy to carry out. In general it is not possible to make such a choice for $\mathcal{S}$ and a non-trivial Jacobian factor will emerge. To compute it, we make use of the Faddeev-Popov trick.

Denote a gauge transformation U of a gauge field A_μ by

$$A_\mu^U \equiv U A_\mu U^{-1} + \frac{i}{g} \partial_\mu U U^{-1} \tag{13.40}$$

Notice that the action of gauge transformations on A_μ satisfies the group composition law of gauge transformations,

$$\begin{aligned} (A_\mu^V)^U &= U\Big(V A_\mu V^{-1} + \frac{i}{g}\partial_\mu V V^{-1}\Big)U^{-1} + \frac{i}{g}\partial_\mu U U^{-1} \\ &= (UV) A_\mu (UV)^{-1} + \frac{i}{g}\partial_\mu (UV)(UV)^{-1} \\ &= A_\mu^{UV} \end{aligned} \tag{13.41}$$

which will be useful later on.

Let $\mathcal{F}(A)$ denote the loval gauge slice function, the gauge slice being determined by $\mathcal{F}^a(A) = 0$. The quantity $\mathcal{F}^a(A^U)$ will vanish when A^U belongs to $\mathcal{S}$, and will be non-zero otherwise. We define the quantity $\mathcal{J}[A]$ by

$$1 = \mathcal{J}[A] \int \mathcal{D}U \ \prod_x \delta\Big(\mathcal{F}^a(A^U)(x)\Big) \tag{13.42}$$

The functional δ-function looks a bit menacing, but should be understood as a Fourier integral,

$$\prod_x \delta(\mathcal{F}^a(A^U)(x)\Big) \equiv \int \mathcal{D}\Lambda^a \ \exp\Big\{i \int d^4x \Lambda^a(x) \mathcal{F}^a(A^U)(x)\Big\} \tag{13.43}$$

Using the composition law $(A_\mu^U)^V = A_\mu^{UV}$ and the gauge invariance of the measure $\mathcal{D}U$, it follows that $\mathcal{J}$ is gauge invariant. This is shown as follows,

$$\begin{aligned} \mathcal{J}[A^V]^{-1} &= \int \mathcal{D}U \ \prod_x \delta\Big(\mathcal{F}^a((A^V)^U)(x)\Big) \\ &= \int \mathcal{D}U \ \prod_x \delta\Big(\mathcal{F}^a(A^{UV})(x)\Big) \\ &= \int \mathcal{D}(U'V^{-1}) \ \prod_x \delta\Big(\mathcal{F}^a(A^{U'})(x)\Big) \\ &= \mathcal{J}[A]^{-1} \end{aligned} \tag{13.44}$$

The deinition of the Jacobian $\mathcal{J}$ readily allows for a factorization of the measure of $\mathcal{A}$ into a measure over the splice $\mathcal{S}$ times a measure over the gauge orbits $\mathcal{G}$. This is done as follows,

$$\begin{aligned}
\int_{\mathcal{A}} \mathcal{D}A_\mu \; \mathcal{O} \; e^{iS[A]} &= \int_{\mathcal{A}} \mathcal{D}A_\mu \int_{\mathcal{G}} \mathcal{D}U \; \prod_x \Big(\mathcal{F}^a(A^U)(x) \Big) \mathcal{J}[A] \; \mathcal{O} \; e^{iS[A]} \\
&= \int_{\mathcal{G}} \mathcal{D}U \int_{\mathcal{A}} \mathcal{D}A_\mu \prod_x \Big(\mathcal{F}^a(A^U)(x) \Big) \mathcal{J}[A] \; \mathcal{O} \; e^{iS[A]} \\
&= \Big(\int_{\mathcal{G}} \mathcal{D}U \Big) \int_{\mathcal{A}} \mathcal{D}A_\mu \prod_x \Big(\mathcal{F}^a(A)(x) \Big) \mathcal{J}[A] \; \mathcal{O} \; e^{iS[A]} \qquad (13.45)
\end{aligned}$$

In passing from the first to the second line, we have simply inverted the orders of integration over $\mathcal{A}$ and $\mathcal{G}$; from the second to the third, we have used the gauge invariance of the measure $\mathcal{D}A_\mu$, of $\mathcal{O}$ and $S[A]$; from the third to the fourth, we have used the fact that the functional δ-function restricts the integration over all of $\mathcal{A}$ to one just over the slice $\mathcal{S}$. Notice that the integration over all gauge transformation function $\int \mathcal{D}U$ is independent of A and is an irrelevant constant.

The gauge fixed formula is independent of the gauge function $\mathcal{F}$.

Therefore, we end up with the following final formula for the gauge-fixed functional intgeral for Abelian or non-Abelian gauge fields,

$$\langle 0|\mathcal{O}|0\rangle = \frac{\int_{\mathcal{A}} \mathcal{D}A_\mu \prod_x \Big(\mathcal{F}^a(A)(x) \Big) \mathcal{J}[A] \; \mathcal{O} \; e^{iS[A]}}{\int_{\mathcal{A}} \mathcal{D}A_\mu \prod_x \Big(\mathcal{F}^a(A)(x) \Big) \mathcal{J}[A] \; e^{iS[A]}} \qquad (13.46)$$

It remains to compute $\mathcal{J}$ and to deal with the functional δ-function.

13.7 Calculation of the Faddeev-Popov determinant

At first sight, the Jacobian $\mathcal{J}$ would appear to be given by a very complicated functional integral over U. The key observation that makes a simple evaluation of $\mathcal{J}$ possible is that $\mathcal{F}^a(A)$ only vanishes (i.e. its functional δ-function makes a non-vanishing contribution) on the gauge slice. To compute the Jacobian, it suffices to linearlize the action of the gauge transformations around the gauge slice. Therefore, we have

$$\begin{aligned}
U &= I + i\omega^a T^a + \mathcal{O}(\omega^2) \\
A^U_\mu &= A_\mu - D_\mu \omega + \mathcal{O}(\omega^2) \\
D_\mu \omega^a &= \partial_\mu \omega^a - g f^{abc} A^b_\mu \omega^c \qquad (13.47)
\end{aligned}$$

and the gauge function also linearizes,

$$\begin{aligned}
\mathcal{F}^a(A^U) &= \mathcal{F}^a(A_\mu - D_\mu\omega)(x) + \mathcal{O}(\omega^2) \\
&= \mathcal{F}^a(A)(x) - \int d^4y \left(\frac{\delta \mathcal{F}^a(A)(x)}{\delta A^b_\mu(y)} \right) D_\mu \omega^b(y) + \mathcal{O}(\omega^2) \\
&= \mathcal{F}^a(A)(x) + \int d^4y \mathcal{M}^{ab}(x,y)\omega^b(y) + \mathcal{O}(\omega^2) \qquad (13.48)
\end{aligned}$$

where the operator $\mathcal{M}$ is defined by

$$\mathcal{M}^{ab}(x,y) \equiv D_\mu^y \left(\frac{\delta \mathcal{F}^a(A)(x)}{\delta A_\mu^b(y)} \right) \tag{13.49}$$

Now the calculation of $\mathcal{J}$ becomes feasible.

$$\begin{aligned} \mathcal{J}[A]^{-1} &= \int_{\mathcal{G}} \mathcal{D}U \prod_x \delta\Big(\mathcal{F}^a(A^U)(x)\Big) \\ &= \int \mathcal{D}\omega^a \prod_x \delta\left(\mathcal{F}^a(A)(x) + \int d^4y \mathcal{M}^{ab}(x,y)\omega^b(y)\right) \\ &= (\mathrm{Det}\mathcal{M})^{-1} \end{aligned} \tag{13.50}$$

Inverting this relation, we have

$$\mathcal{J}[A] = \mathrm{Det}\mathcal{M} = \mathrm{Det}\left(D_\mu^y \frac{\delta \mathcal{F}(A)(x)}{\delta A_\mu(y)} \right) \tag{13.51}$$

Given a gauge slice function $\mathcal{F}$, this is now a quite explicit formula. In QFT, however, we were always dealing with local actions, local operators and these properties are obscured in the above determinant formula.

13.8 Faddeev-Popov ghosts

We encountered a determinant expression in the numerator once before when delaing with intgerations over fermions. This suggests that the Faddeev-Popov determinant may also be cast in the form of a functional intgeration over two complex conjugate Grassmann odd fields, which are usually referred to as the Faddeev-Popov ghost fields and are denoted by $c^a(x)$ and $\bar{c}^a(x)$. The expression for the determinant is then given by a functional integral over the ghost fields,

$$\begin{aligned} \mathrm{Det}\mathcal{M} &= \int \mathcal{D}c^a \mathcal{D}\bar{c}^a \exp\Big\{ iS_{\text{ghost}}[A;c,\bar{c}] \Big\} \\ S_{\text{ghost}}[A;c,\bar{c}] &\equiv \int d^4x \int d^4y \; \bar{c}^a(x) \mathcal{M}^{ab}(x,y) c^b(y) \end{aligned} \tag{13.52}$$

If the function $\mathcal{F}$ is a local function of A_μ, then the support of $\mathcal{M}^{ab}(x,y)$ is at $x=y$ and the above double integral collapses to a single and thus local intgeral involving the ghost fields.

13.9 Axial and covariant gauges

First of all, there is a class of *axial gauges* where no ghosts are ever needed ! The gauge function must be linear in A_μ^a and involve no derivatives on A_μ. They involve a constant vector n^μ, and are therefore never Lorentz covariant, $\mathcal{F}^a(A)(x) = A^a_\mu(x) n^\mu$,

$$\begin{aligned}
\frac{\delta F^a(A)(x)}{\delta A^b_\mu(y)} &= \delta^{ab}\delta^{(4)}(x-y)n^\mu \\
D^y_\mu \frac{\delta F(A)(x)}{\delta A_\mu(y)} &= \partial^y_\mu \delta^{(4)}(x-y)n^\mu = \mathcal{M}(x,y)
\end{aligned} \tag{13.53}$$

Therefore, in axial gauges, the operator $\mathcal{M}$ is independent of A and thus a constant which may be ignored. Thus, no ghost are required as the Faddeev-Popov formula would just inform us that they decouple. Actually, since axial gauges are not Lorentz covariant, they are much less useful in practice than it might appear.

It turns out that Lorentz covariant gauges are much more useful. The simplest such gauges are specified by a function $f^a(x)$ as follows,

$$\mathcal{F}^a(A)(x) \equiv \partial_\mu A^{a\mu}(x) - f^a(x) \tag{13.54}$$

This choice gives rise to the following functional integral,

$$Z_\mathcal{O} = \int_\mathcal{A} \mathcal{D}A_\mu \prod_x \delta(\partial_\mu A^{a\mu} - f^a) \ \mathrm{Det}\mathcal{M} \ \mathcal{O} \ e^{iS[A]} \tag{13.55}$$

Since this functional integral is independent of the choice of f^a, we may actually integrate over all f^a with any weight factor we wish. A convenient choice is to take the weight factor to be Gaussian in f^a,

$$Z_\mathcal{O} = \frac{\int \mathcal{D}f^a \ \exp\Big\{-\frac{i}{2}\int d^4x f^a f^a\Big\} \int_\mathcal{A} \mathcal{D}A_\mu \prod_x \delta(\partial_\mu A^{a\mu} - f^a) \ \mathrm{Det}\mathcal{M} \ \mathcal{O} \ e^{iS[A]}}{\int \mathcal{D}f^a \ \exp\Big\{-\frac{i}{2}\int d^4x f^a f^a\Big\}} \tag{13.56}$$

The denominator is just again a A_μ-independent constant which may be omitted. The f^a-intgeral in the numerator may be carried out since the functional δ-function instructs us to set $f^a = \partial_\mu A^{a\mu}$, so that

$$Z_\mathcal{O} = \int_\mathcal{A} \mathcal{D}A_\mu \int Dc^a \int \mathcal{D}\bar{c}^a \mathcal{O} \ \exp\Big\{iS[A] + iS_{\mathrm{gf}}[A] + iS_{\mathrm{ghost}}[A;c,\bar{c}]\Big\} \tag{13.57}$$

where the gauge fixing terms in the action are given by

$$S_{\mathrm{gf}}[A] = \int d^4x \Big\{-\frac{1}{2\xi}(\partial_\mu A^{a\mu})^2\Big\} \tag{13.58}$$

It remains to obtain a more explicit form for the ghost action. To this end, we first compute $\mathcal{M}$.

$$\begin{aligned}
\mathcal{M}^{ab}(x,y) &= D^y_\mu \Big(\frac{\delta \partial^\nu_x A_\nu(x)}{\delta A_\mu(y)}\Big)^{ab} \\
&= (D^y_\mu)^{ab} \partial^\mu_x \delta^{(4)}(x-y)
\end{aligned} \tag{13.59}$$

Inserting this operator into the expression for the ghost action, we obtain,

$$
\begin{aligned}
S_{\text{ghost}}[A; c, \bar{c}] &\equiv \int d^4x \int d^4y \; \bar{c}^a(x) \mathcal{M}^{ab}(x, y) c^b(y) \\
&= \int d^4x \int d^4y \bar{c}(x) D^y_\mu \partial^\mu_x \delta(x - y) c(y) \\
&= \int d^4x \partial_\mu \bar{c}^a(x) (D_\mu c)^a
\end{aligned}
\tag{13.60}
$$

Thus, the total action for these gauges is given as follows,

$$
S_{\text{tot}}[A; c, \bar{c}] = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2 + \partial_\mu \bar{c}^a (D^\mu c)^a \tag{13.61}
$$

and we have the following general formula for vacuum expectation values of gauge invariant operators,

$$
\langle 0 | \mathcal{O} | 0 \rangle = \frac{\int \mathcal{D}A_\mu \int \mathcal{D}c \mathcal{D}\bar{c} \; \mathcal{O} \; e^{iS_{\text{tot}}[A; c, \bar{c}]}}{\int \mathcal{D}A_\mu \int \mathcal{D}c \mathcal{D}\bar{c} \; e^{iS_{\text{tot}}[A; c, \bar{c}]}} \tag{13.62}
$$

In the section on perturbation theory, we shall show how to use this formula in practice.

14 Anomalies in QED

Initial exposure to renormalization may give the impression that the procedure is an ad hoc remedy for a putative ill of QFT, that should not be required in some better formulation of the theory. This is not true. The presence of anomalies and the existence of the renormalization group will prove that the renormalization procedure has deep, physically relevant meaning. Questions that are often posed in this context are

1. Are UV divergences a consequence of the perturbative expansion ?
2. If so, is it possible to re-sum them to finite results in the full theory?
3. Are there any physical consequences of renormalization ?

To answer these questions, we are taking a detour to anomalies.

14.1 Massless 4d QED, Gell-Mann–Low Renormalization

Recall the form of the 2-point vacuum polarization tensor to one-loop order

$$\begin{aligned}\Pi_{\mu\nu}(k,m,\Lambda) &= (k^2\eta_{\mu\nu}-k_\mu k_\nu)\Pi(k,m,\Lambda) \\ \Pi(k,m,\Lambda) &= 1-\frac{e^2}{2\pi^2}\int_0^1 d\alpha\alpha(1-\alpha)\ln\frac{m^2-\alpha(1-\alpha)k^2}{\Lambda^2}+O(e^4)\end{aligned} \tag{14.1}$$

When the electron mass $m \neq 0$, it is natural (though not necessary) to renormalize at 0 photon momentum, and to require $\Pi_R(0,m)=1$, so that

$$\Pi_R(k,m) = 1-\Pi(0,m,\Lambda)+\Pi(k,m,\Lambda) \tag{14.2}$$

However, when $m=0$, $\Pi(0,m,\Lambda)=\infty$, and this renormalization is not possible. This led Gell-Mann–Low to renormalize "off-shell", at a mass-scale μ that has no direct relation with the electron mass or photon mass,

$$\Pi_R(k,m,\mu)\Big|_{k^2=-\mu^2} = 1 \tag{14.3}$$

So that

$$\Pi_R(k,m,\mu) = 1-\frac{e^2}{2\pi^2}\int_0^1 d\alpha(1-\alpha)\alpha\ln\frac{m^2-\alpha(1-\alpha)k^2}{m^2+\alpha(1-\alpha)\mu^2}+O(e^4) \tag{14.4}$$

Now, we may safely let $m \to 0$,

$$\Pi_R(k,0,\mu) = 1-\frac{e^2}{12\pi^2}\ln\left(\frac{-k^2}{\mu^2}\right)+O(e^4) \tag{14.5}$$

The cost of renormalizing the massless theory is the introduction of a new scale μ, which was not originally present in the theory.

14.2 Scale Invariance of Massless QED

But now think for a moment about what this means in terms of scaling symmetry. When $m = 0$, classical electrodynamics is scale invariant,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\xi}{2}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\not{\partial} - e\not{A})\psi \tag{14.6}$$

A scale transformation $x^\mu \to x'^\mu = \lambda x^\mu$, transforms a field $\phi_\Delta(x)$ of dimension Δ as follows

$$\phi_\Delta(x) \to \phi'_\Delta(x') = \lambda^{-\Delta}\phi_\Delta(x) \qquad \begin{cases} A_\mu : & \Delta = 1 \\ \psi \;\; : & \Delta = \frac{3}{2} \end{cases} \tag{14.7}$$

The classical theory will be invariant since the effects of scale transformations on the various terms in the Lagrangian are as follows,

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu}(x) &\;\to\; \lambda^{-4}F_{\mu\nu}F^{\mu\nu}(x) \\ (\partial_\mu A^\mu)^2 &\;\to\; \lambda^{-4}(\partial_\mu A^\mu)^2 \\ \bar{\psi}(i\not{\partial} - e\not{A})\psi &\;\to\; \lambda^{-4}\bar{\psi}(i\not{\partial} - e\not{A})\psi \end{aligned} \tag{14.8}$$

where we have assumed that e and ξ do not transform under scale transformations. Thus $\mathcal{L} \to \lambda^{-4}\mathcal{L}$, and the action $S = \int d^4x\mathcal{L}$ is invariant. Actually, you can show that $\mathcal{L}$ is invariant under all *conformal transformations* of x^μ; infinitesimally $\delta x^\mu = \delta\lambda x^\mu$ for a scale transformation; $\delta x^\mu = x^2\delta c^\mu - x^\mu(\delta c \cdot x)$ for a special conformal transformation. Thus, *classical massless QED is scale invariant*; nothing in the theory sets a scale.

A fermion mass term, however, spoils scale invariance since $\bar{\psi}\psi \to \lambda^{-3}\bar{\psi}\psi$, assuming that m is being kept fixed. Similarly, a photon mass would also spoil scale invariance.

Now, think afresh of the one-loop renormalization problem: when $m = 0$, one-loop renormalization *forced us to introduce a scale* μ in order to carry out renormalization. One *cannot* quantize QED without introducing a scale, and thereby breaking scale invariance.

What we have just identified is a field theory, massless QED, whose classical scale symmetry has been broken by the process of renormalization, i.e. by quantum effects.

Whenever a symmetry of the classical Lagrangian is destroyed by quantum effects caused by UV divergences, this classical symmetry is said to be *anomalous*. In this case, it is the scaling anomaly, also referred to as the conformal anomaly.

14.3 Chiral Symmetry

In case the above result is insufficiently convincing at this time, we give an example of another symmetry that has an anomaly, seen in a calculation that is *finite*. Take again massless QED,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi \tag{14.9}$$

Setting $m = 0$ has enlarged the symmetry to include conformal invariance (classically). It also produces another new symmetry: *chiral symmetry*. To see this, decompose ψ into left and right spinors, in a chiral basis where γ^5 is diagonal,

$$\psi = \psi_L \oplus \psi_R \qquad \gamma^5\psi_L = +\psi_L, \quad \gamma^5\psi_R = -\psi_R \tag{14.10}$$

The Lagrangian then becomes,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}_L(i\partial\!\!\!/ - eA\!\!\!/)\psi_L + \bar{\psi}_R(i\partial\!\!\!/ - eA\!\!\!/)\psi_R \tag{14.11}$$

Clearly, ψ_L and ψ_R may be rotated by two independent phases,

$$U(1)_L \times U(1)_R \qquad \begin{cases} \psi_L \to e^{i\theta_L}\psi_L \\ \psi_R \to e^{i\theta_R}\psi_R \end{cases} \tag{14.12}$$

and the Lagrangian is invariant. Now, we re-arrange these transformations as follows

$$U(1)_V : \begin{cases} \psi_L \to e^{i\theta_+}\psi_L \\ \psi_R \to e^{i\theta_+}\psi_R \end{cases} \qquad 2\theta_+ = \theta_L + \theta_R \tag{14.13}$$

$$U(1)_A : \begin{cases} \psi_L \to e^{i\theta_-}\psi_L \\ \psi_R \to e^{-\theta_-}\psi_R \end{cases} \qquad 2\theta_- = \theta_L - \theta_R \tag{14.14}$$

or, in 4-component spinor notation

$$\begin{cases} U(1)_V : & \psi \to e^{i\theta_+}\psi \\ U(1)_A : & \psi \to e^{i\gamma_5\theta_-}\psi \end{cases} \tag{14.15}$$

As symmetries, they have associated conserved currents.

- the vector current $\qquad j^\mu \equiv \bar{\psi}\gamma^\mu\psi$
- the axial vector current $\qquad j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma_5\psi$

Conservation of these currents may be checked directly by using the Dirac field equations, in the following form,

$$\begin{aligned} i\gamma^\mu\partial_\mu\psi &= e\gamma^\mu A_\mu\psi \\ -i\partial_\mu\bar{\psi}\gamma^\mu &= e\bar{\psi}\gamma^\mu A_\mu \end{aligned} \tag{14.16}$$

One then has,

$$\partial_\mu j_5^\mu = \partial_\mu\bar{\psi}\gamma^\mu\gamma_5\psi - \bar{\psi}\gamma_5\gamma^\mu\partial_\mu\psi = ie\bar{\psi}\gamma^\mu A_\mu\gamma_5\psi + ie\bar{\psi}\gamma_5\gamma^\mu A_\mu\psi = 0 \tag{14.17}$$

Any symmetry that transforms left and right spinors differently is referred to as chiral.

14.4 The axial anomaly in 2-dimensional QED

It is very instructive to consider the fate of chiral symmetry in the much simpler 2-dimensional version of QED, given by the same Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi \tag{14.18}$$

Contrarily to 4-dim QED, this theory is not scale invariant, even for vanishing fermion mass, since the dimensions of the various fields and couplings are given as follows, $[e] = 1$, $[A] = 0$, $[\psi] = 1/2$. But it is chiral invariant, just as 4-dim QED was, and we have two classically conserved currents,

$$j^\mu \equiv \bar{\psi}\gamma^\mu\psi \qquad j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma_5\psi \tag{14.19}$$

A considerable simplification arises from the fact that the Dirac matrices are much simpler:

$$\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^5 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14.20}$$

As a result of this simplicity, the axial and vector currents are *related* to one another. This fact may be established by computing the products $\gamma^\mu\gamma_5$ explicitly,

$$\begin{cases} \gamma^0\gamma_5 = \sigma^1\sigma^3 = -i\sigma^2 = -\gamma^1 = +\gamma_1 \\ \gamma^1\gamma_5 = i\sigma^2\sigma^3 = -\sigma^1 = -\gamma^0 = -\gamma_0 \end{cases} \tag{14.21}$$

Introducing the rank 2 antisymmetric tensor $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$, normalized to $\epsilon^{01} = 1$, the above relations may be recast in the following form,

$$\gamma^\mu\gamma_5 = \epsilon^{\mu\nu}\gamma_\nu \quad \Longrightarrow \quad j_5^\mu = \epsilon^{\mu\nu} j_\nu \tag{14.22}$$

Consider the calculation of the vacuum polarization to one-loop, but this time in 2-dim QED. The vacuum polarization diagram is just the correlator of the two vector currents in the free theory, $\Pi^{\mu\nu}(k) = e\langle\emptyset|T j^\mu(k) j^\nu(-k)|\emptyset\rangle$, and is given by,

$$\Pi^{\mu\nu}(k) = -e \int \frac{d^2p}{(2\pi)^2}\, \mathrm{tr}\left(\gamma^\mu \frac{i}{k\!\!\!/ + p\!\!\!/}\gamma^\nu \frac{i}{p\!\!\!/}\right) \tag{14.23}$$

A priori, the integral is logarithmically divergent, and must be regularized. We choose to regularize in a manner that preserves gauge invariance and thus conservation of the current j^μ. As a result, we have $\Pi^{\mu\nu}(k) = (k^2\eta^{\mu\nu} - k^\mu k^\nu)\Pi(k^2)$. Gauge invariant regularization may be achieved using a single Pauli-Villars regulator with mass M, for example. Just as in 4-dim QED, current conservation lowers the degree of divergence and in 2-dim, vacuum polarization is actually finite.

$$\Pi(k^2) = 4e\int_0^1 d\alpha \int \frac{d^2p}{(2\pi)^2}\left[\frac{\alpha(1-\alpha)}{(p^2+\alpha(1-\alpha)k^2)^2} - \frac{\alpha(1-\alpha)}{(p^2+\alpha(1-\alpha)k^2-M^2)^2}\right] \quad (14.24)$$

Carrying out the p-integration,

$$\Pi(k^2) = \frac{ie}{\pi}\int_0^1 d\alpha \left[\frac{\alpha(1-\alpha)}{-\alpha(1-\alpha)k^2} - \frac{\alpha(1-\alpha)}{-\alpha(1-\alpha)k^2+M^2}\right] \quad (14.25)$$

Clearly, as $M \to \infty$, the limit is finite and the contribution from the Pauli-Villars regulator actually cancels. One finds the finite result,

$$\Pi(k^2) = -\frac{ie}{\pi k^2} \quad (14.26)$$

Notice that, even though the final result for vacuum polarization is finite, the Feynman integral was superficially logarithmically divergent and requires regularization. It is during this regularization process that we ensure gauge invariance by insisting on the conservation of the vector current j^μ. Later, we shall see that one might have chosen *NOT* to conserve the vector current, yielding a different result.

Next, we study the correlator of the axial vector current, $\Pi_5^{\mu\nu}(k) = e\langle\emptyset|j_5^\mu(k)j^\nu(-k)|\emptyset\rangle$. Generally, $\Pi_5^{\mu\nu}$ and $\Pi^{\mu\nu}$ would be completely independent quantities. In 2-dim, however, the axial current is related to the vector current by (14.22), and we have

$$\Pi_5^{\mu\nu}(k) = \epsilon^{\mu\rho}\Pi_\rho{}^\nu(k) = (\epsilon^{\mu\nu}k^2 - \epsilon^{\mu\rho}k_\rho k^\nu)\Pi(k^2) \quad (14.27)$$

Conservation of the vector current requires that $k_\nu\Pi_5^{\mu\nu}(k) = 0$, which is manifest in view of the transversality of $\Pi^{\mu\nu}(k)$. The divergence of the axial vector current may also be calculated. The contribution from the correlator with a single vector current insertion is

$$k_\mu\Pi_5^{\mu\nu}(k) = k_\mu\epsilon^{\mu\nu}k^2\Pi(k^2) = -\frac{ie}{\pi}k_\mu\epsilon^{\mu\nu} \quad (14.28)$$

which is manifestly non-vanishing. As a result, the *axial vector is not conserved* and chiral symmetry in 2-dim QED is not a symmetry at the quantum level : *chiral symmetry suffers an anomaly*. Translating this into an operator statement:

$$\boxed{\partial_\mu j_5^\mu = -\frac{e}{2\pi}F_{\mu\nu}\epsilon^{\mu\nu}} \quad (14.29)$$

Remarkably, this result is exact (Adler-Bardeen theorem). Higher point correlators are convergent and classical conservation valid.

14.5 The axial anomaly and regularization

The above calculation of the axial anomaly was based on our explicit knowledge of the vector current two-point function (or vacuum polarization) to one-loop order. Actually, the anomaly equation (14.29) is and *exact quantum field theory operator equation, which in particular is valid to all orders in perturbation theory.* To establish this stronger result, there must be a more direct way to proceed than from the vacuum polarization result.

First, consider the issue of conservation of the axial vector current when only the fermion is quantized and the gauge field is treated as a classical background. The correlator $\langle \emptyset | j_5^\mu | \emptyset \rangle_A$ is then given by the sum of all 1-loop diagrams shown in figure xx. Consider first the graphs with $n \geq 2$ external A-fields. Their contribution to $\langle \emptyset | j_5^\mu | \emptyset \rangle_A$ is given by

$$\Pi_5^{\mu;\nu_1\cdots\nu_n}(k;k_1,\cdots,k_n) = -e^n \int \frac{d^2p}{(2\pi)^2} \text{tr}\left(\gamma^\mu\gamma_5 \frac{1}{\not p + \not \ell_1}\gamma^{\nu_1}\cdots\frac{1}{\not p + \not \ell_n}\gamma^{\nu_n}\frac{1}{\not p + \not \ell_{n+1}}\right) \quad (14.30)$$

where we use the notation $l_{i+1} - l_i = k_i$, As we assumed $n \geq 2$, this integral is UV convergent, and may be computed without regularization. In particular, the conservation of the axial vector current may be studied by using the following simple identity, (and the fact that $l_{n+1} - l_1 = -k$ by overall momentum conservation $k + k_1 + \cdots + k_n = 0$),

$$\frac{1}{\not p + \not \ell_{n+1}} \not k \gamma_5 \frac{1}{\not p + \not \ell_1}\gamma^{\nu_1} = -\gamma_5 \frac{1}{\not p + \not \ell_1} + \gamma_5 \frac{1}{\not p + \not \ell_{n+1}} \quad (14.31)$$

Hence the divergence of the axial vector current correlator becomes,

$$\begin{aligned} k_\mu \Pi_5^{\mu;\nu_1\cdots\nu_n}(k;k_1,\cdots,k_n) &= e^n \int \frac{d^2p}{(2\pi)^2} \text{tr}\left(\gamma_5 \frac{1}{\not p + \not \ell_1}\gamma^{\nu_1}\cdots\frac{1}{\not p + \not \ell_n}\gamma^{\nu_n}\right) \\ &\quad -e^n \int \frac{d^2p}{(2\pi)^2} \text{tr}\left(\gamma_5 \frac{1}{\not p + \not \ell_2}\gamma^{\nu_2}\cdots\frac{1}{\not p + \not \ell_n}\gamma^{\nu_n}\frac{1}{\not p + \not \ell_{n+1}}\gamma^{\nu_1}\right) \end{aligned} \quad (14.32)$$

When $n \geq 3$, each term separately is convergent. Using boson symmetry in the external A-field (i.e. the symmetry under the permutations on i of (k_i, ν_i)), combined with shift symmetry in the integration variable p, all contributions are found to cancel one another. When $n = 2$, each term separately is logarithmically divergent, and shift symmetry still applies. Thus, we have shown that $\partial_\mu j_5^\mu$ receives contributions only from terms first order in A, which are precisely the ones that we have computed already in the preceding subsection.

To see how the contribution that is first order in A arises, we need to deal with a contribution that is superficially logarithmically divergent, and requires regularization. One possible regularization scheme is by Pauli-Villars, regulators. The logarithmically divergent integral requires just a single regulator with mass M, and we have

$$\Pi_5^{\mu\nu}(k,M) = -e \int \frac{d^2p}{(2\pi)^2} \text{tr}\left(\gamma^\mu\gamma_5 \left\{\frac{1}{\not p + \not k}\gamma^\nu \frac{1}{\not p} - \frac{1}{\not p + \not k - M}\gamma^\nu\frac{1}{\not p - M}\right\}\right) \quad (14.33)$$

The conservation of the vector current, i.e. gauge invariance, is readily verified, using the identity

$$\frac{1}{\not p+\not k}\not k\frac{1}{\not p} - \frac{1}{\not p+\not k-M}\not k\frac{1}{\not p-M} = \frac{1}{\not p} - \frac{1}{\not p+\not k} - \frac{1}{\not p+\not k-M} + \frac{1}{\not p-M} \tag{14.34}$$

and the fact that one may freely shift p in a logarithmically divergent integral. The divergence of the axial vector current is investigated using a similar identity,

$$\begin{aligned}
&\mathrm{tr}\not k\gamma_5\left\{\frac{1}{\not p+\not k}\gamma^\nu\frac{1}{\not p} - \frac{1}{\not p+\not k-M}\gamma^\nu\frac{1}{\not p-M}\right\} \\
&\qquad = \mathrm{tr}\gamma_5\gamma^\nu\left(\frac{1}{\not p+\not k} - \frac{1}{\not p} - \frac{1}{\not p+\not k-M} + \frac{1}{\not p-M}\right) \\
&\qquad\quad +2M\mathrm{tr}\left(\gamma_5\frac{1}{\not p+\not k-M}\gamma^\nu\frac{1}{\not p-M}\right)
\end{aligned} \tag{14.35}$$

The first line of the integrand on the rhs may be shifted in the integral and cancels. The second part remains, and we have

$$k_\mu\Pi_5^{\mu\nu}(k,M) = -2eM\int\frac{d^2p}{(2\pi)^2}\mathrm{tr}\left(\gamma_5\frac{1}{\not p+\not k-M}\gamma^\nu\frac{1}{\not p-M}\right) \tag{14.36}$$

Of course, this integral is superficially UV divergent, because we have used only a single Pauli-Villars regulator. To obtain a manifestly finite result, at least two PV regulators should be used; it is understood that this has been done. To evaluate (14.36), introduce a single Feynman parameter, α and shift the internal momentum p accordingly,

$$k_\mu\Pi_5^{\mu\nu}(k,M) = -2eM\int_0^1 d\alpha\int\frac{d^2p}{(2\pi)^2}\frac{\mathrm{tr}\gamma_5(\not p+(1-\alpha)\not k+M)\gamma^\nu(\not p-\alpha\not k+M)}{(p^2-M^2+\alpha(1-\alpha)k^2)^2} \tag{14.37}$$

The trace $\mathrm{tr}\gamma_5\gamma^{\mu_1}\cdot\gamma^{\mu_n}$ vanishes whenever n is odd, while for $n=2$, we have $\mathrm{tr}\gamma_5\gamma^\mu\gamma^\nu = 2\epsilon^{\mu\nu}$. In the numerator of (14.37), the contributions quadratic in p, those quadratic in k and those quadratic in M cancel because they involve traces with odd n; those linear in p cancel because of $p\to -p$ symmetry of the denominator. This leaves only terms linear in M and linear in k. Finally, using $\mathrm{tr}\gamma_5\not k\gamma^\nu = 2\epsilon^{\mu\nu}k_\mu$ and the familiar integral

$$\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2-M^2+\alpha(1-\alpha))^2} = \frac{i}{4\pi}\;\frac{1}{M^2-\alpha(1-\alpha)k^2} \tag{14.38}$$

we have

$$k_\mu\Pi_5^{\mu\nu}(k,M) = -\frac{ie}{\pi}\epsilon^{\mu\nu}k_\mu\int_0^1 d\alpha\frac{M^2}{M^2-\alpha(1-\alpha)k^2} \tag{14.39}$$

As M is viewed as a Pauli-Villars regulator, we consider the limit $M \to \infty$, which is smooth, and we find,

$$\lim_{M\to\infty} k_\mu \Pi_5^{\mu\nu}(k, M) = -\frac{ie}{\pi}\epsilon^{\mu\nu} k_\mu \tag{14.40}$$

which is in precise agreement with (14.28). Clearly, the anomaly contribution arises here solely from the Pauli-Villars regulator.

14.6 The axial anomaly and massive fermions

If a fermion has a mass, then chiral symmetry is violated by the mass term. This may be seen directly from the Dirac equation for a massive fermion in QED,

$$\begin{aligned} i\gamma^\mu \partial_\mu \psi &= m\psi + e\gamma^\mu A_\mu \psi \\ -i\partial_\mu \bar\psi \gamma^\mu &= m\bar\psi + e\bar\psi A_\mu \gamma^\mu \end{aligned} \tag{14.41}$$

Chiral transformations may still be defined on the fermion fields and the axial vector current may still be defined by the same expression. For massive fermions, however, chiral transformations are not symmetries any more and the axial vector current fails to be conserved. Its divergence is easy to compute classically from the Dirac equation, $\partial_\mu j_5^\mu = 2im\bar\psi\gamma_5\psi$. Hence the Pauli-Villars regulators violate the conservation of j_5^μ, and this effect survives as $M \to \infty$. In fact, one readily establishes the axial anomaly equation for a fermion of mass m,

$$\partial_\mu j_5^\mu = 2im\bar\psi\gamma_5\psi - \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} \tag{14.42}$$

In fact, it is *impossible* to find a regulator of UV divergences which is gauge invariant and preserves chiral symmetry. Dimensional regularization makes $2 \to$ d, but now we have a problem defining $\gamma^5 = \gamma^0\gamma^1$.

14.7 Solution to the massless Schwinger Model

Two-dimensional QED is a theory of electrically charged fermions, interacting via a static Coulomb potential, with no dynamical photons. The potential is just the Green function $G(x,y)$ for the 1-dim Laplace operator, satisfying $\partial_x^2 G(x,y) = \delta(x,y)$, and given by

$$G(x,y) = \frac{1}{2}|x-y| \tag{14.43}$$

The force due to this potential is constant in space. It would take an infinite amount of energy to separate two charges by an infinite distance, which means that the charges will be *confined*. Interestingly, this simple model of QED in 2-dim is a model for confinement !

Remarkably, 2-dim QED was completely solved by Schwinger in 1958. To achieve this solution, one proceeds from the equations for the conservation of the vector current, the anomaly of the axial vector current, and Maxwell's equations,

$$\begin{aligned}
\partial_\mu j^\mu &= 0 \\
\partial_\mu j_5^\mu = \epsilon^{\mu\nu}\partial_\mu j_\nu &= -\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} \\
\partial_\mu F^{\mu\nu} &= e j^\nu
\end{aligned} \tag{14.44}$$

Of course, it is crucial to the general solution that these equations hold exactly. Notice that this is a system of linear equations, a fact that is at the root for the existence of an exact solution. We recast the axial anomaly equation in the following form,

$$\partial_\mu j_\nu - \partial_\nu j_\mu = -\frac{e}{\pi}F_{\mu\nu} \tag{14.45}$$

Finally, taking the divergence in ∂_μ, using $\partial_\mu j^\mu = 0$ and Maxwell's equations, we obtain,

$$\Box j^\nu = -\frac{e}{2\pi}\partial_\mu F^{\mu\nu} = -\frac{e^2}{\pi}j^\nu \tag{14.46}$$

which is the field equation for a free massive boson, with mass $m^2 = e^2/\pi$. Actually, this boson is a single real scalar ϕ, which may be obtained by solving explicitly the conservation equation $\partial_\mu j^\mu = 0$ in terms of ϕ by $j^\mu = \epsilon^{\mu\nu}\partial_\nu\phi$. Notice that the gauge field may also be expressed in terms of this scalar, (or the current), by solving the chiral anomaly equation,

$$A_\mu = -\frac{\pi}{e}j_\mu + \partial_\mu\theta = -\frac{\pi}{e}\epsilon_{\mu\nu}\partial^\nu\phi + \partial_\mu\theta \tag{14.47}$$

where θ is any gauge transformation. This is just the London equation of superconductivity, expressing the fact that the photon field is massive now. This last equation is at the origin of the boson-fermion correspondence in 2-dim.

Remarkably, the physics of the (massless) Schwinger model is precisely what we would expect from a theory describing confined fermions. The salient features are as follows,

- The asymptotic spectrum contains neither free quarks, nor free gauge particles;
- The spectrum contains only fermion bound states, which are gauge neutral;
- The spectrum has a mass gap, even though the fermions were massless.

It is also interesting to include a mass to the fermions, but this model is much more complicated, and no longer exactly solvable. Its bosonized counterpart is the massive Sine-Gordon theory.

14.8 The axial anomaly in 4 dimensions

In d=4, the 2-point function $\langle \emptyset | T j_5^\mu j^\nu | \emptyset \rangle$ vanishes. However, there are now triangle diagrams, which do not vanish. The same problems arise with regularization and insisting on gauge invariance, one derives the anomaly equation for the axial vector current,

$$\boxed{\partial_\mu j_5^\mu = \frac{e^2}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + 2im\bar{\psi}\gamma_5\psi} \qquad \tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda} F^{\kappa\lambda} \tag{14.48}$$

Its calculation is analogous to that in two dimensions.

15 Perturbative Non-Abelian Gauge Theories

In this section, the Feynman rules will be derived for a gauge theory based on a simple group G, governed by a single gauge coupling constant g. The pure gauge action, and the gauge fixing terms and Faddeev-Popov ghost contribution may be expressed in terms of the fundamental fields as follows,

$$\begin{aligned}
\mathcal{L}_{\rm gauge} &= -\frac{1}{2}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)\partial^\mu A^{\nu a} + \frac{1}{2}g(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) f^{abc} A^{\mu b} A^{\nu c} \\
&\qquad -\frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} \\
\mathcal{L}_{\rm ghost} &= -\frac{\lambda}{2}\partial_\mu A^{\mu a}\partial_\nu A^{\nu a} + \partial_\mu \bar{c}^a \partial^\mu c^a - g f^{abc}\partial_\mu \bar{c}^a A^{\mu b} c^c
\end{aligned} \tag{15.1}$$

Fermions ψ and (complex) bosons ϕ, minimally coupled to the gauge field in representations T_ψ and T_ϕ respectively, may also be included. The corresponding actions are

$$\begin{aligned}
\mathcal{L}_\psi &= \bar{\psi}^i \left(i\partial\!\!\!/ \delta_i{}^j - gA\!\!\!/^a (T_\psi^a)_i{}^j - m_\psi \delta_i{}^j \right) \psi_j \qquad (15.2) \\
\mathcal{L}_\phi &= \partial_\mu \phi^{*i}\partial^\mu \phi_i + igA_\mu^a \left(\partial^\mu \phi^{*i} (T_\phi^a)_i{}^j \phi_j - \phi^{*i} (T_\phi^a)_i{}^j \partial^\mu \phi_j \right) \\
&\quad - g^2 A_\mu^a A^{\mu b} \phi^{*i} (T_\phi^a)_i{}^j (T_\phi^b)_j{}^k \phi_k - V(\phi)
\end{aligned}$$

Here, f^{abc} are the structure constants of the Lie algebra of G and T_ϕ^a and T_ψ^a are the representation matrices corresponding to the representations T_ϕ and T_ψ of G under which ϕ and ψ transform. They obey the structure relations,

$$\begin{aligned}
[T_\phi^a, T_\phi^b] &= i f^{abc} T_\phi^c \\
[T_\psi^a, T_\psi^b] &= i f^{abc} T_\psi^c
\end{aligned} \tag{15.3}$$

Finally, $V(\phi)$ is a potential which we shall assume to be bounded from below. In this section, we shall specialize to developing perturbation theory in a so-called *unbroken phase*, characterized by the fact that the minimum of the potential we choose to expand about is symmetric under the group G. (The *spontaneously broken, or Higgs phase* will be discussed in later sections.) By a suitable shift in ϕ_i by constants, this minimum may be chosen to be at $\phi_i = 0$, so that

$$\frac{\partial V}{\partial \phi_i}(0) = \frac{\partial V}{\partial \phi^{*i}}(0) = 0 \tag{15.4}$$

Furthermore, by adding an unobservable constant shift to the Lagrangian, the potential at the minimum may be chosen to vanish, $V(0) = 0$. Invariance of the minimum under G finally requires that the double derivative of the potential at the minimum be proportional to the identity,

$$\frac{\partial^2 V}{\partial\phi^{*i}\partial\phi_j}(0) = m_\phi^2\,\delta_i{}^j \tag{15.5}$$

The parameter m_ϕ is the mass of the scalar in this unbroken phase.

15.1 Feynman rules for Non-Abelian gauge theories

The corresponding Feynman rules are as follows. Propagators are given by

$$\begin{aligned}
\text{gauge} \qquad & = \frac{-i}{k^2+i\epsilon}\delta_{ab}\left(\eta_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2+i\epsilon}\right) \qquad\qquad \xi = 1-\frac{1}{\lambda}\\
\text{ghost} \qquad & = \frac{i}{k^2+i\epsilon}\delta_{ab}\\
\text{fermion} \qquad & = \left(\frac{i}{\not{k}-m_\psi+i\epsilon}\right)_{\alpha\beta}\delta_{ij}\\
\text{scalar} \qquad & = \frac{i}{k^2-m_\phi^2+i\epsilon}\delta_{ij}
\end{aligned} \tag{15.6}$$

The vertices are

$$\begin{aligned}
\text{gauge cubic} \qquad & = g f^{abc}(2\pi)^4\delta(p+q+r)\Big[\eta_{\mu\nu}(p-q)_\rho + \eta_{\nu\rho}(q-r)_\mu + \eta_{\rho\mu}(r-p)_\nu\Big]\\
\text{gauge quartic} \qquad & = -ig^2(2\pi)^4\delta(p+q+r+s)\Big[f^{eab}f^{ecd}(\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\sigma}\eta_{\nu\rho})\\
& \qquad\qquad +f^{eac}f^{edb}(\eta_{\mu\sigma}\eta_{\rho\nu}-\eta_{\mu\nu}\eta_{\rho\sigma})\\
& \qquad\qquad +f^{ead}f^{ebc}(\eta_{\mu\nu}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\sigma\nu})\Big]\\
\text{ghost} \qquad & = -g(2\pi)^4\delta(p-q+k)f^{abc}p_\mu\\
\text{fermion} \qquad & = -ig(2\pi)^4\delta(p-q-k)(\gamma_\mu)_{\alpha\beta}(T^a_\psi)_{ij}\\
\text{scalar cubic} \qquad & = g(2\pi)^4\delta(p-q-k)(T^a_\phi)_{ij}(p_\mu+q_\mu)\\
\text{scalar quartic} \qquad & = -ig^2(2\pi)^4\delta(p+q+r+s)\eta_{\mu\nu}\{T^a_\phi, T^b_\phi\}_{ij}
\end{aligned} \tag{15.7}$$

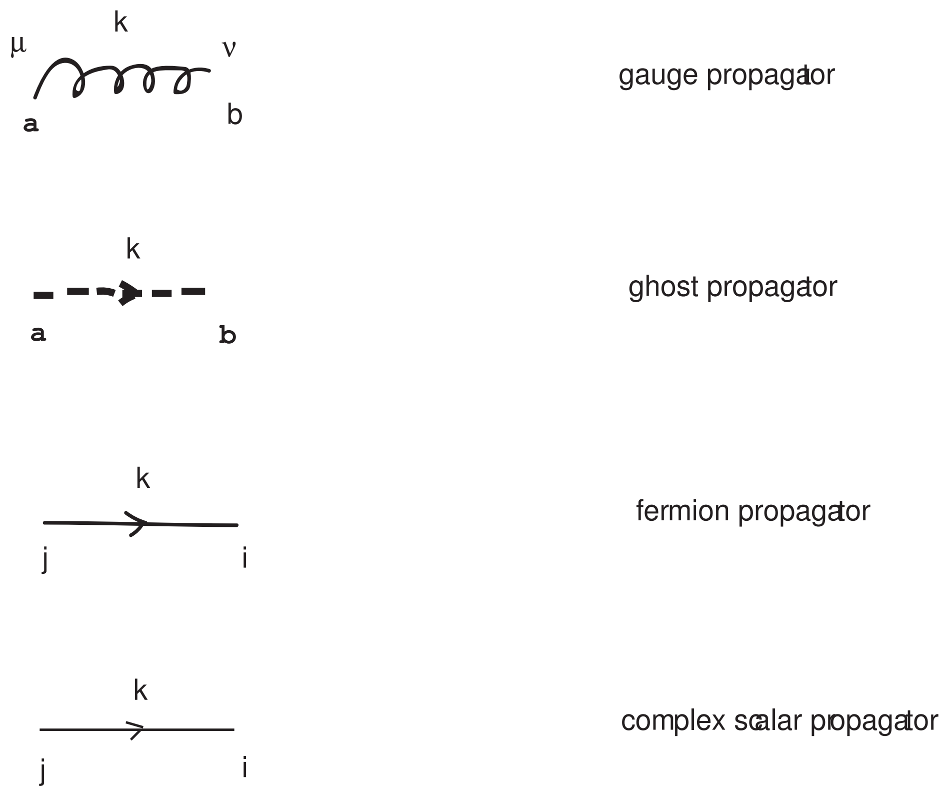


Figure 21: Feynman rules : propagators

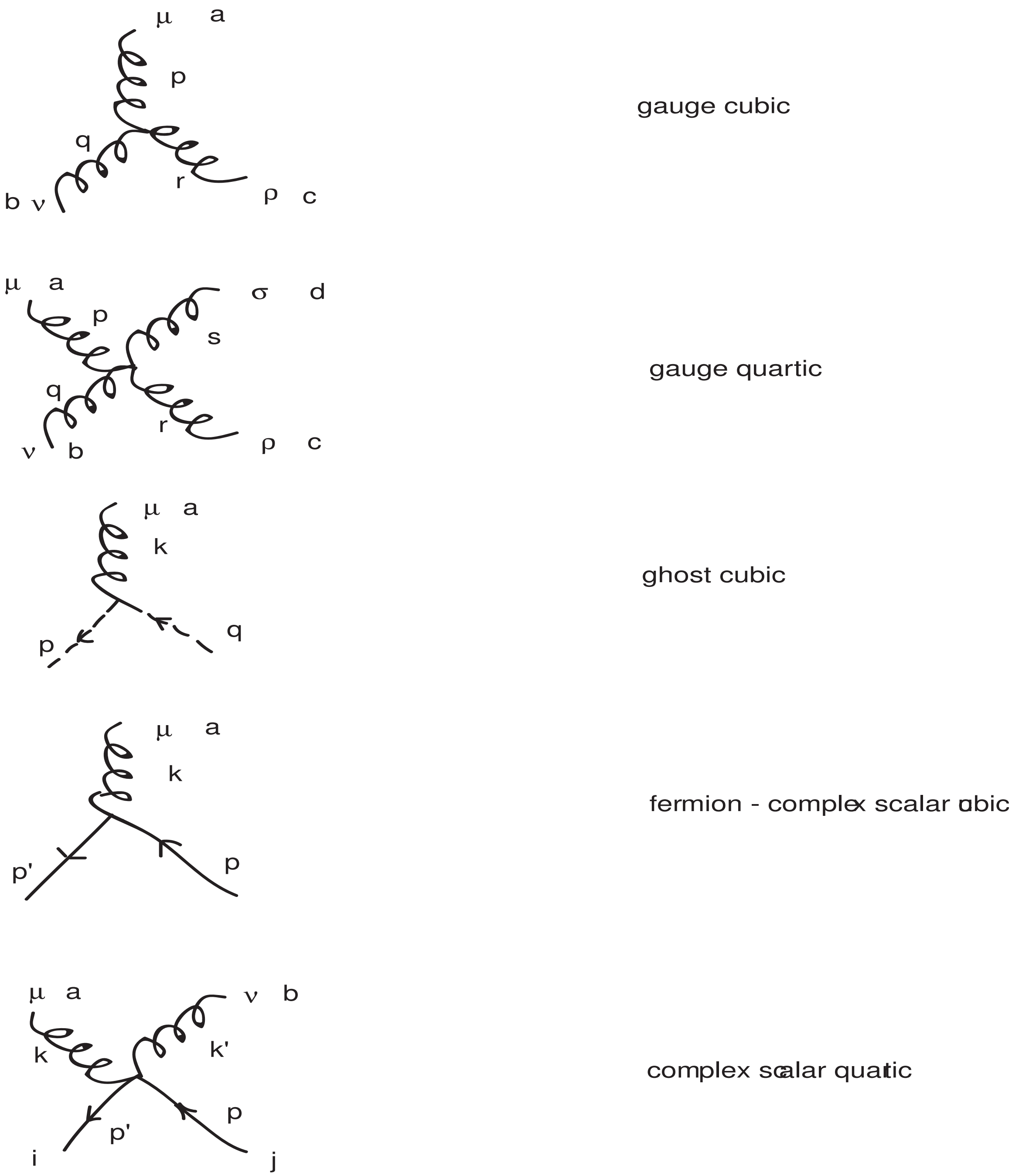


Figure 22: Feynman rules : vertices

15.2 Renormalization, gauge invariance and BRST symmetry

The point of departure for the quantization and renormalization of non-Abelian gauge theories is a gauge invariant Lagrangian, such as in the pure Yang-Mills case,

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \tag{15.8}$$

To renormalize this theory, one might be tempted to guess that only gauge-invariant counterterms will be required. The existence of gauge invariant regulators (such as dimensional regularization) would seem to further justify this guess. Perturbative quantization, however, forces us add a gauge fixing term, which in turn requires a Faddeev-Popov ghost,

$$\mathcal{L}_{\text{ghost}} = -\frac{\lambda}{2} \partial_\mu A^{\mu a} \partial_\nu A^{\nu a} + \partial_\mu \bar{c}^a D^\mu c^a \tag{15.9}$$

Neither of these contributions is gauge invariant. As soon as some terms in the Lagrangian fail to be invariant, it is no longer viable to use gauge invariance to restrict the form of the counterterms.

Luckily, the gauge and ghost Lagrangians actually exhibit a more subtle symmetry, referred to as BRST symmetry (after Becchi, Rouet, Stora; Tuytin 1975). This symmetry will effectively play the role of gauge invariance in the gauge fixed theory. To begin, the ghost fields are decomposed into real and imaginary parts,

$$c^a = \frac{1}{\sqrt{2}}(\rho^a + i\sigma^a) \qquad\qquad \bar{c}^a = \frac{1}{\sqrt{2}}(\rho^a - i\sigma^a) \tag{15.10}$$

Both ρ^a and σ^a are real odd Grassmann-valued Lorentz scalars. In terms of these real fields, the Lagrangian becomes

$$\mathcal{L}_{\text{ghost}} = -\frac{\lambda}{2} \partial_\mu A^{\mu a} \partial_\nu A^{\nu a} + i \partial_\mu \rho^a D^\mu \sigma^a \tag{15.11}$$

We now introduce a real odd Grassmann-valued *constant* ω. The product $\omega\sigma^a$ is then a real even Grassmann-valued Lorentz scalar function, and it is this combination that will be used to make infinitesimal gauge transformations on the non-ghost fields in the theory,

$$\begin{aligned} \delta A^a_\mu &= -\left(D_\mu(\omega\sigma)\right)^a \\ \delta\psi &= ig\left(\omega\sigma^a T^a_\psi\right)\psi \\ \delta\phi &= ig\left(\omega\sigma^a T^a_\phi\right)\phi \end{aligned} \tag{15.12}$$

One refers to these as *field-dependent gauge transformations*. Clearly, the gauge part of the Lagrangian $\mathcal{L}_{\text{gauge}}$ as well as the fermion and boson parts $\mathcal{L}_\psi$ and $\mathcal{L}_\phi$ are invariant.

Remarkably, local transformation laws for the ghost fields may be found so that also the ghost part of the Lagrangian $\mathcal{L}_{\rm ghost}$ is invariant,

$$\begin{aligned}
\delta\rho^a &= -i\lambda\,\omega\,\partial^\mu A^a_\mu \\
\delta\sigma^a &= -\frac{1}{2}g\omega f^{abc}\sigma^b\sigma^c
\end{aligned} \tag{15.13}$$

To establish invariance of $\mathcal{L}_{\rm ghost}$, we first establish invariance of $D_\mu\sigma$, under the transformation laws given above,

$$\begin{aligned}
\delta\left(D_\mu\sigma\right)^a &= \partial_\mu\delta\sigma^a - gf^{abc}(\delta A^b_\mu\sigma^c + A^b_\mu\delta\sigma^c) \\
&= -g\omega f^{abc}\partial_\mu\sigma^b\sigma^c + \frac{1}{2}gf^{abc}A^b_\mu g\omega f^{cde}\sigma^d\sigma^e \\
&\quad +gf^{abc}\left(\partial_\mu\omega\sigma^b - gf^{bde}A^d_\mu\omega\sigma^e\right)\sigma^c
\end{aligned} \tag{15.14}$$

The derivative terms in $\partial_\mu\sigma^b$ manifestly cancel, and the remaining terms may be regrouped as follows,

$$\delta\left(D_\mu\sigma\right)^a = \frac{1}{2}g^2A^b_\mu\omega\sigma^d\sigma^e\left\{f^{abc}f^{cde} + f^{acd}f^{cbe} - f^{ace}f^{cbd}\right\} = 0 \tag{15.15}$$

This expression vanishes by the Jacobi identity satisfied by the structure constants. Using the invariance of $D_\mu\sigma$, is now straightforward to complete the proof of invariance of $\mathcal{L}_{\rm ghost}$, and we have

$$\begin{aligned}
\delta\mathcal{L}_{\rm ghost} &= -\lambda\partial^\mu\delta A^a_\mu(\partial_\nu A^{\nu a}) + i\partial^\mu\delta\rho^a(D_\mu\sigma)^a \\
&= \lambda\partial^\mu(D_\mu\omega\sigma)^a(\partial_\nu A^{\nu a}) + \lambda\partial^\mu(\omega\partial_\nu A^{\nu a})(D_\mu\sigma)^a \\
&= \partial^\mu\Big(\lambda(\partial_\nu A^{\nu a})(D_\mu\sigma)^a\Big)
\end{aligned} \tag{15.16}$$

Since the Lagrangian transforms with a total derivative, the BRST transformation is actually a symmetry.

15.3 Slavnov-Taylor identities

Assuming that BRST symmetry is preserved under renormalization, will imply that the F^2 terms in the Lagrangian continue to appear in a gauge invariant combination, and that the only ghost contributions of the Lagrangian are of the form given in $\mathcal{L}_{\rm ghost}$. Thus, the form of the bare Lagrangian will be

$$\begin{aligned}
\mathcal{L}_b &= -\frac{1}{2}(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu})\partial^\mu A^{\nu a}_0 - \frac{\lambda_0}{2}\partial_\mu A^{\mu a}_0\partial_\nu A^{\nu a}_0 \\
&\quad +\frac{1}{2}g_0(\partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu})f^{abc}A^{\mu b}_0A^{\nu c}_0 - \frac{1}{4}g_0^2f^{abc}f^{ade}A^b_{0\mu}A^c_{0\nu}A^{\mu d}_0A^{\nu e}_0 \\
&\quad +\partial_\mu\bar c^a_0\partial^\mu c^a_0 - g_0f^{abc}\partial_\mu\bar c^a_0A^{\mu b}_0c^c_0 \\
&\quad +\bar\psi_0\left(i\partial\!\!\!/ - g_0A\!\!\!/^a_0T^a_\psi - m_0\right)\psi_0
\end{aligned} \tag{15.17}$$

Here, we have included the contribution of a fermion field, but omitted scalars. Multiplicative renormalization of all the fields yields the following relation between the bare fields and the renormalized fields,

$$A^a_{0\mu} = Z_3^{\frac{1}{2}} A^a_\mu \qquad c^a_0 = \tilde{Z}_3^{\frac{1}{2}} c^a \qquad \psi_0 = Z_2^{\frac{1}{2}} \psi \tag{15.18}$$

The vertices are also given independent Z-factors, so that the starting point for the bare Lagrangian, expressed in renormalized fields, is given by

$$\begin{aligned}
\mathcal{L}_b = & -\frac{1}{2} Z_3 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \partial^\mu A^{\nu a} - \frac{1}{2} \lambda Z_\lambda \partial_\mu A^{\mu a} \partial_\nu A^{\nu a} \\
& + \frac{1}{2} g \hat{Z}_1 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) f^{abc} A^{\mu b} A^{\nu c} - \frac{1}{4} g^2 Z_4 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} \\
& + \tilde{Z}_3 \partial_\mu \bar{c}^a \partial^\mu c^a - g \tilde{Z}_1 f^{abc} \partial_\mu \bar{c}^a A^{\mu b} c^c \\
& + Z_2 \bar{\psi}\, i\partial\!\!\!/ \psi - g Z_1 \bar{\psi}\, A\!\!\!/^a T^a_\psi \psi - m Z_m \bar{\psi} \psi
\end{aligned} \tag{15.19}$$

The identification between the two formulations of the bare Lagrangian give the following relations,

$$\begin{aligned}
g_0 &= g \hat{Z}_1\, Z_3^{-3/2} & g_0^2 &= g^2 Z_4 Z_3^{-2} \\
g_0 &= g \tilde{Z}_1\, \tilde{Z}_3^{-1} Z_3^{-1/2} & g_0 &= g Z_1 Z_2^{-1} Z_3^{-1}
\end{aligned} \tag{15.20}$$

Elimination of g and g_0 gives the Slavnov-Taylor identities,

$$\hat{Z}_1^2 = Z_3 Z_4 \qquad \tilde{Z}_1 \tilde{Z}_3 = \hat{Z}_1 Z_3^{-1} \qquad \hat{Z}_1 Z_3^{-1} = Z_1 Z_2^{-1} \tag{15.21}$$

A gauge/BRST-invariant regularization/renormalization will preserve these identities.

16 One-Loop renormalization of Yang-Mills theory

In the figures 23, 24, 26 and 27 below, a complete list is given of all one-loop contributions to Yang-Mills theory, with fermions, but without scalars, that can contribute to renormalization at one-loop order. Contributions to the 1-point function are given by figure 23. They must vanish by translation and Lorentz invariance, as may indeed be verified explicitly by making use of the Feynman rules.

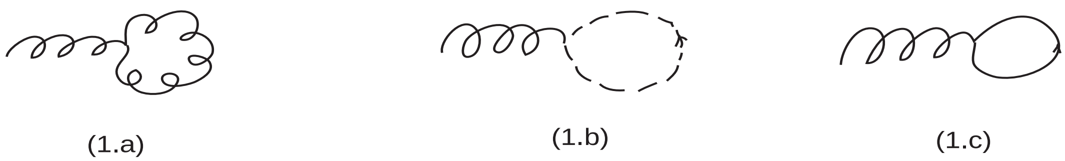


Figure 23: One-loop Yang-Mills : the 1-point functions

16.1 Calculation of Two-Point Functions

Contributions to the 2-point function are given by figure 24. Only diagram (2.b) in figure 24 manifestly vanishes in dimensional regularization. The fermion self-energy graph (2.e) may be obtained from the same graph in QED by tagging on the group theory factors.

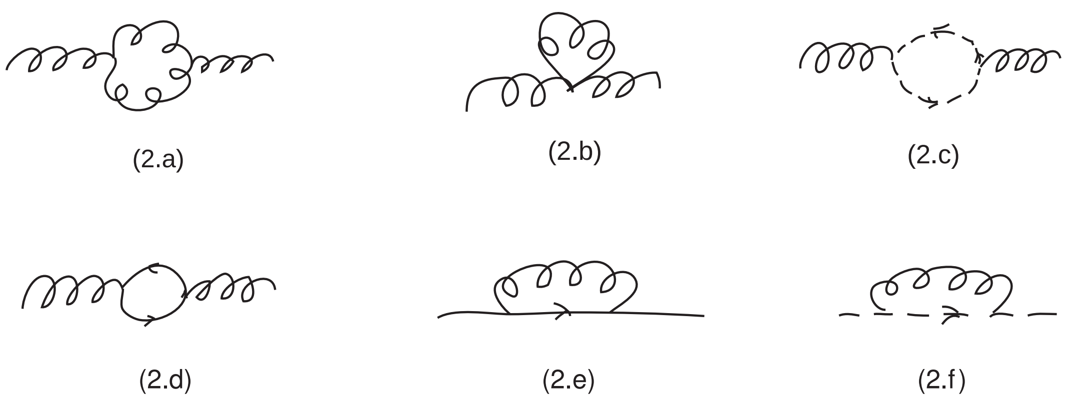


Figure 24: One-loop Yang-Mills : the 2-point functions

The contributions (2.a), (2.b), (2.c) and (2.d) are vacuum polarization effects for the Yang-Mills particle, and will yield the renormalization factor Z_3. Their contribution is expected to be transverse in view of gauge invariance. Graphs (2.e) and (2.f) will yield the renormalization constants Z_2 and $\tilde{Z}_3$ respectively.

16.1.1 Vacuum polarization by gluons

The novel diagrams are redrawn in figure 25 and given detailed labeling conventions.

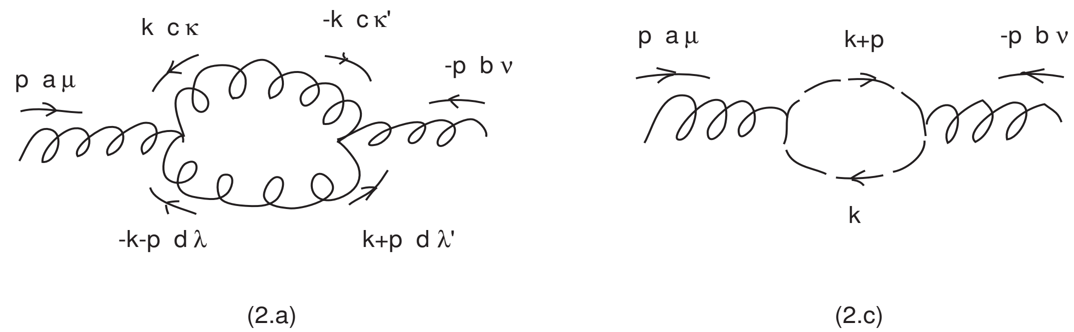


Figure 25: Detailed labeling of vacuum polarization by gluons and ghosts graphs

The gluon contribution is given by graph (2.a), as follows,

$$\begin{aligned}
\Gamma^{ab}_{(a)\mu\nu}(p) &= \frac{1}{2} g^2 \mu^{2\epsilon} f^{adc} f^{bcd} \int \frac{d^dk}{(2\pi)^d} \frac{-i}{k^2} \frac{-i}{(k+p)^2} \\
&\quad \times \left[\eta^{\kappa\kappa'} - \xi \frac{k^\kappa k^{\kappa'}}{k^2} \right] \left[\eta^{\lambda\lambda'} - \xi \frac{(k+p)^\lambda (k+p)^{\lambda'}}{(k+p)^2} \right] \\
&\quad \times [\eta_{\mu\lambda}(2p+k)_\kappa + \eta_{\kappa\lambda}(-2k-p)_\mu + \eta_{\kappa\mu}(k-p)_\lambda] \\
&\quad \times [\eta_{\nu\lambda'}(2p+k)_{\kappa'} + \eta_{\kappa'\lambda'}(-2k-p)_\nu + \eta_{\kappa'\nu}(k-p)_{\lambda'}]
\end{aligned} \tag{16.1}$$

It is useful to arrange this calculation according to the powers of ξ,

$$\Gamma^{ab}_{(a)\mu\nu}(p) = \frac{1}{2} g^2 \mu^{2\epsilon} f^{acd} f^{bcd} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2(k+p)^2} \left[M_{\mu\nu} - \xi \frac{N_{\mu\nu}}{k^2} - \xi \frac{P_{\mu\nu}}{(k+p)^2} + \xi^2 \frac{Q_{\mu\nu}}{k^2(k+p)^2} \right]$$

with the following expressions,

$$\begin{aligned}
M_{\mu\nu} &= (d-6)p_\mu p_\nu + (4d-6)p_{\{\mu}k_{\nu\}} + (4d-6)k_\mu k_\nu + \{2k^2 + 2kp + 5p^2\}\eta_{\mu\nu} \\
N_{\mu\nu} &= k^2 p_\mu p_\nu + (-6k^2 - 6kp)p_{\{\mu}k_{\nu\}} + (-k^2 - 6kp + p^2)k_\mu k_\nu + (k^2 + 2kp)^2 \eta_{\mu\nu} \\
P_{\mu\nu} &= (2k^2 - p^2)p_\mu p_\nu - 2pk p_{\{\mu}k_{\nu\}} + (2p^2 - k^2)k_\mu k_\nu + (k^2 - p^2)^2 \eta_{\mu\nu} \\
Q_{\mu\nu} &= (p^2 \eta_{\mu\sigma} - p_\mu p_\sigma)(p^2 \eta_{\nu\rho} - p_\nu p_\rho) k^\sigma k^\rho
\end{aligned} \tag{16.2}$$

We see that the integral involving $Q_{\mu\nu}$ is UV convergent and will not participate in renormalization. Notice that it is also manifestly transverse.

We shall compute in detail only the contribution involving $M_{\mu\nu}$. First, the term in $M_{\mu\nu}$ which is proportional to $\eta_{\mu\nu}$ may be re-expressed as follows, $2k^2 + 2kp + 5p^2 = k^2 + (k+p)^2 + 4p^2$; the terms in k^2 and $(k+p)^2$ cancel in dimensional regularization. Second, we introduce a single Feynman parameter, so that

$$\mathcal{I}_{\mu\nu} = \int \frac{d^dk}{(2\pi)^d} \frac{M_{\mu\nu}(k,p)}{k^2(k+p)^2} = \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{M_{\mu\nu}(k-\alpha p, p)}{(k^2 + \alpha(1-\alpha)p^2)^2} \tag{16.3}$$

Terms odd in $M_{\mu\nu}(k-\alpha p,p)$ which are odd in k do not contribute to the integral, and we are left with

$$M_{\mu\nu}(k-\alpha p,p)\sim(d-6-\alpha(1-\alpha)(4d-6))p_\mu p_\nu+4\eta_{\mu\nu}p^2+(4d-6)k_\mu k_\nu \tag{16.4}$$

The integration is now Lorentz invariant and we may replace $k_\mu k_\nu$ by $\eta_{\mu\nu}k^2/d$. The remaining k-integrations may be carried out by first analytically continuing to Euclidean time $k^0\to ik_E^0$ and then using the Euclidean integral formulas,

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+M^2)^n}=\frac{\Gamma(n-d/2)}{(4\pi)^{d/2}\Gamma(n)}\frac{1}{(M^2)^{n-d/2}} \tag{16.5}$$

As a result, we have

$$\begin{aligned}\mathcal{I}_{\mu\nu} &= i\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\int_0^1 d\alpha\Big(-\alpha(1-\alpha)p^2\Big)^{d/2-2} \\ &\quad\times\bigg((d-6)p_\mu p_\nu-\alpha(1-\alpha)(4d-6)p_\mu p_\nu+4p^2\eta_{\mu\nu}+\frac{4d-6}{2-d}\alpha(1-\alpha)p^2\eta_{\mu\nu}\bigg)\end{aligned} \tag{16.6}$$

The α-integration may be carried out using the Euler B-function, and we obtain,

$$\begin{aligned}\mathcal{I}_{\mu\nu} &= i\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}(-p^2)^{d/2-2}\bigg\{\Big((d-6)p_\mu p_\nu+4p^2\eta_{\mu\nu}\Big)\frac{\Gamma(d/2-1)^2}{\Gamma(d-2)} \\ &\qquad+\bigg(-(4d-6)p_\mu p_\nu+\frac{4d-6}{2-d}p^2\eta_{\mu\nu}\bigg)\frac{\Gamma(d/2)^2}{\Gamma(d)}\bigg\}\end{aligned} \tag{16.7}$$

Picking up only the pole part in ε where $d=4-2\varepsilon$, and using $\Gamma(2-d/2)=\Gamma(\varepsilon)\sim 1/\varepsilon$, we get

$$\mathcal{I}_{\mu\nu}=\frac{i}{(4\pi)^2\,\varepsilon}(-p^2)^{-\varepsilon}\left\{\frac{19}{6}p^2\eta_{\mu\nu}-\frac{22}{6}p_\mu p_\nu\right\} \tag{16.8}$$

The entire diagram is evaluated using analogous calculations, and we find,

$$\Gamma^{ab}_{(a)\mu\nu}(p) = \frac{ig^2(-p^2/\mu^2)^{-\varepsilon}}{32\pi^2\,\varepsilon}f^{acd}f^{bcd}\left\{\frac{19}{6}p^2\eta_{\mu\nu}-\frac{22}{6}p_\mu p_\nu+\xi(p^2\eta_{\mu\nu}-p_\mu p_\nu)\right\} \tag{16.9}$$

Notice that the gluon exchange graph by itself fails to be transverse, for any choice of gauge ξ.

16.1.2 Vacuum polarization by ghosts

The ghost contribution is given as follows,

$$\begin{aligned}\Gamma^{ab}_{(c)\mu\nu}(p) &= -g^2\mu^{2\epsilon}f^{adc}f^{bcd}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2}\frac{i}{(k+p)^2}k_\mu(k+p)_\nu \\ &= -g^2\mu^{2\epsilon}f^{acd}f^{bcd}\int_0^1 d\alpha\int\frac{d^dk}{(2\pi)^d}\frac{\eta_{\mu\nu}k^2/d-\alpha(1-\alpha)p_\mu p_\nu}{(k^2+\alpha(1-\alpha)p^2)^2}\end{aligned} \tag{16.10}$$

Retaining only the pole parts in ε, we find,

$$\Gamma^{ab}_{(c)\mu\nu}(p) = \frac{ig^2(-p^2/\mu^2)^{-\varepsilon}}{32\pi^2\varepsilon} f^{acd} f^{bcd} \left\{ \frac{1}{6} p^2 \eta_{\mu\nu} + \frac{2}{6} p_\mu p_\nu \right\} \tag{16.11}$$

Putting together the gluon and ghost contributions to vacuum polarization, we obtain,

$$\Gamma^{ab}_{(a,b,c)\mu\nu}(p) = \frac{ig^2(-p^2/\mu^2)^{-\varepsilon}}{32\pi^2\varepsilon} f^{acd} f^{bcd} \Big(p^2 \eta_{\mu\nu} - p_\mu p_\nu \Big) \left\{ \frac{13}{3} - (1-\xi) \right\} \tag{16.12}$$

16.1.3 Vacuum polarization by fermions

The vacuum polarization contribution due to fermions is given by graph (2.d) of figure 24, and is proportional to the value in QED. We express this as follows,

$$\begin{aligned} \Gamma^{ab}_{(d)\mu\nu}(p) &= g^2 \mathrm{tr}(iT^a_\psi)(iT^b_\psi)\Gamma_{\mu\nu}(p) \\ \Gamma_{\mu\nu}(p) &= -\int \frac{d^dk}{(2\pi)^d} \frac{i}{\not k - m} \frac{i}{\not k + \not p - m} \end{aligned} \tag{16.13}$$

The pole part of this contribution is given by

$$\Gamma^{ab}_{(d)\mu\nu}(p) = \frac{-ig^2(-p^2/\mu^2)^{-\varepsilon}}{32\pi^2\varepsilon} \frac{8}{3} \,\mathrm{tr}(T^a_\psi T^b_\psi) \Big(p^2 \eta_{\mu\nu} - p_\mu p_\nu \Big) \tag{16.14}$$

16.2 Indices and Casimir Operators

Some group theory is helpful in organizing the final presentation of the renormalization effects, to one-loop order and beyond. First, the generators of a simple Lie algebra $\mathcal{G}$ in a unitary representation r will be denoted by T^a_r. They satisfy the Lie algebra structure relations, $[T^a_r, T^b_r] = if^{abc}T^c_r$, and will be normalized as follows. We begin by denoting the dimension of r by $d(r)$. For any simple Lie algebra, $\mathcal{G}$, and any representation r, the trace $\mathrm{tr}T^a_r T^b_r$ is proportional to δ^{ab}, and the sum $T^a_r T^a_r$ is proportional to the identity matrix I_r in the representation r. The corresponding coefficient are referred to as the *quadratic index* $T(R)$, and the quadratic Casimir $C(r)$,

$$\begin{aligned} \mathrm{tr}\, I_r &= d(r) \\ \mathrm{tr}(T^a_r T^b_r) &= T(r)\delta^{ab} \\ T^a_r T^a_r &= C(r) I_r \end{aligned} \tag{16.15}$$

It suffices to normalize either $T(r)$ or $C(r)$ is a single representation of $\mathcal{G}$ to fix the normalizations of the structure constants f^{abc} and $T(r)$ and $C(r)$ for all r. For Lie algebras $SU(N)$, $SO(N)$ and $Sp(2N)$, it is customary to use the *defining representations* $r = D$ of dimensions N, N, and $2N$ respectively, and we require $T(D) = 1/2$. For exceptional groups, one uses one of the *fundamental representations* to fix the normalization.

The index and the Casimir are related to one another, as may be derived by evaluating $\mathrm{tr}(T_r^a T_r^a)$ respectively from the equations defining $T(r)$ and $C(r)$, and we obtain,

$$T(r)d(\mathcal{G}) = C(r)d(r) \tag{16.16}$$

where $d(\mathcal{G}) = \dim(\mathcal{G})$ denotes the dimension of the Lie algebra, which is, of course, the dimension of the adjoint representation. Under direct sums and tensor products, the dimension, index and Casimir behave as follows,

$$\begin{aligned} d(r_1 \oplus r_2) &= d(r_1) + d(r_2) & \qquad d(r_1 \otimes r_2) &= d(r_1)d(r_2) \\ T(r_1 \oplus r_2) &= T(r_1) + T(r_2) & \qquad T(r_1 \otimes r_2) &= d(r_1)T(r_2) + T(r_1)d(r_2) \end{aligned} \tag{16.17}$$

from which we deduce that

$$C(r_1 \oplus r_2) = \frac{d(r_1)C(r_1) + d(r_2)C(r_2)}{d(r_1) + d(r_2)} \qquad\qquad C(r_1 \otimes r_2) = C(r_1) + C(r_2) \tag{16.18}$$

Using these rules, we may evaluate some of these quantities for $\mathcal{G} = SU(N)$, and the fundamental representation D, and the adjoint representation $\mathcal{G}$,

$$\begin{aligned} d(D) &= N & \qquad T(D) &= \frac{1}{2} & \qquad C(D) &= \frac{N^2-1}{2N} \\ d(\mathcal{G}) &= N^2 - 1 & \qquad T(\mathcal{G}) &= N & \qquad C(\mathcal{G}) &= N \end{aligned} \tag{16.19}$$

The representation matrices of the adjoint representation are just the structure constants $(T_{\mathcal{G}}^a)_b{}^c = (i\, f^a)_b{}^c$. Here, we have been careful to include a factor of i in order to make $T_{\mathcal{G}}$ hermitian, as required by our earlier conventions. As a result, we have

$$\mathrm{tr}(T_{\mathcal{G}}^a T_{\mathcal{G}}^b) = (i\, f^a)_c{}^d (i\, f^b)_d{}^c = f^{acd} f^{bcd} = T(\mathcal{G})\delta^{ab} \tag{16.20}$$

We recognize the combination that entered the one-loop vacuum polarization corrections by gluons and ghosts.

16.3 Summary of two-point functions

We collect the pole contributions to the renormalization factors for 2-point functions,

$$\begin{aligned} Z_2 &= 1 + \frac{g^2}{32\pi^2\varepsilon}\{-(2-2\xi)C(\psi)\} \\ Z_3 &= 1 + \frac{g^2}{32\pi^2\varepsilon}\left\{\left(\frac{13}{3} - (1-\xi)\right)C(\mathcal{G}) - \frac{8}{3}T(\psi)\right\} \\ \tilde{Z}_3 &= 1 + \frac{g^2}{32\pi^2\varepsilon}\left\{\left(1 + \frac{1}{2}\xi\right)C(\mathcal{G})\right\} \end{aligned} \tag{16.21}$$

Even with the help of the Slavnov-Taylor identities, we do not have sufficient information, so far, to compute the relation between g and g_0.

16.4 Three-Point Functions

The following renormalization factors are determined by these graphs,

$$\begin{aligned}
Z_1 &= 1 + \frac{g^2}{32\pi^2\varepsilon}\left\{\left(-2+\frac{1}{2}\xi\right)C(\mathcal{G}) - (2-2\xi)C(\psi)\right\} \\
\hat{Z}_1 &= 1 + \frac{g^2}{32\pi^2\varepsilon}\left\{\left(\frac{4}{3}+\frac{3}{2}\xi\right)C(\mathcal{G}) - \frac{8}{3}T(\psi)\right\} \\
\tilde{Z}_1 &= 1 + \frac{g^2}{32\pi^2\varepsilon}\left\{-(1-\xi)C(\mathcal{G})\right\}
\end{aligned} \tag{16.22}$$

Some explanation of the group theoretic factors is in order, for graphs (3.e), (3.f), (3.g) and (3.h). In graphs (3.e) and (3.g), the group theoretic factor is

$$\begin{aligned}
f^{abc}T_r^bT_r^c &= \frac{1}{2}f^{abc}[T_r^b, T_r^c] \\
&= \frac{i}{2}f^{abc}f^{bcd}T_r^d \\
&= \frac{i}{2}C(\mathcal{G})T_r^a
\end{aligned} \tag{16.23}$$

where $r = \mathcal{G}$ for (3.e) and $r = \psi$ for (3.g). In graphs (3.f) and (3.h), the group theoretic factor is

$$\begin{aligned}
\sum_b T_r^bT_r^aT_r^b &= \sum_b T_r^bT_r^bT_r^a + \sum_b T_r^b[T_r^a, T_r^b] \\
&= C(r)T_r^a + i\sum_{b,c} f^{abc}T_r^bT_r^c \\
&= C(r)T_r^a - \frac{1}{2}T(r)T_r^a
\end{aligned} \tag{16.24}$$

where $r = \mathcal{G}$ for (3.f) and $r = \psi$ for (3.h).

16.5 Four-Point Functions

The graphs contributing to the 4-point functions are given in figure 27. The graphs (4.a-e) contribute to the Yang-Mills 4-point function and are expected to received contributions that need to be renormalized. These are summarized by the renormalization factor Z_4, which is

$$Z_4 = 1 + \frac{g^2}{32\pi^2\varepsilon}\left\{\left(-\frac{2}{3}+2\xi\right)C(\mathcal{G}) - \frac{8}{3}T(\psi)\right\} \tag{16.25}$$

On the other hand, the graphs (4.f-i), as well as (4.j-k) are superficially log divergent, yet would require counterterms not already represented in the Lagrangian. These contributions are actually UV convergent, essentially because on at least one ghost vertex,

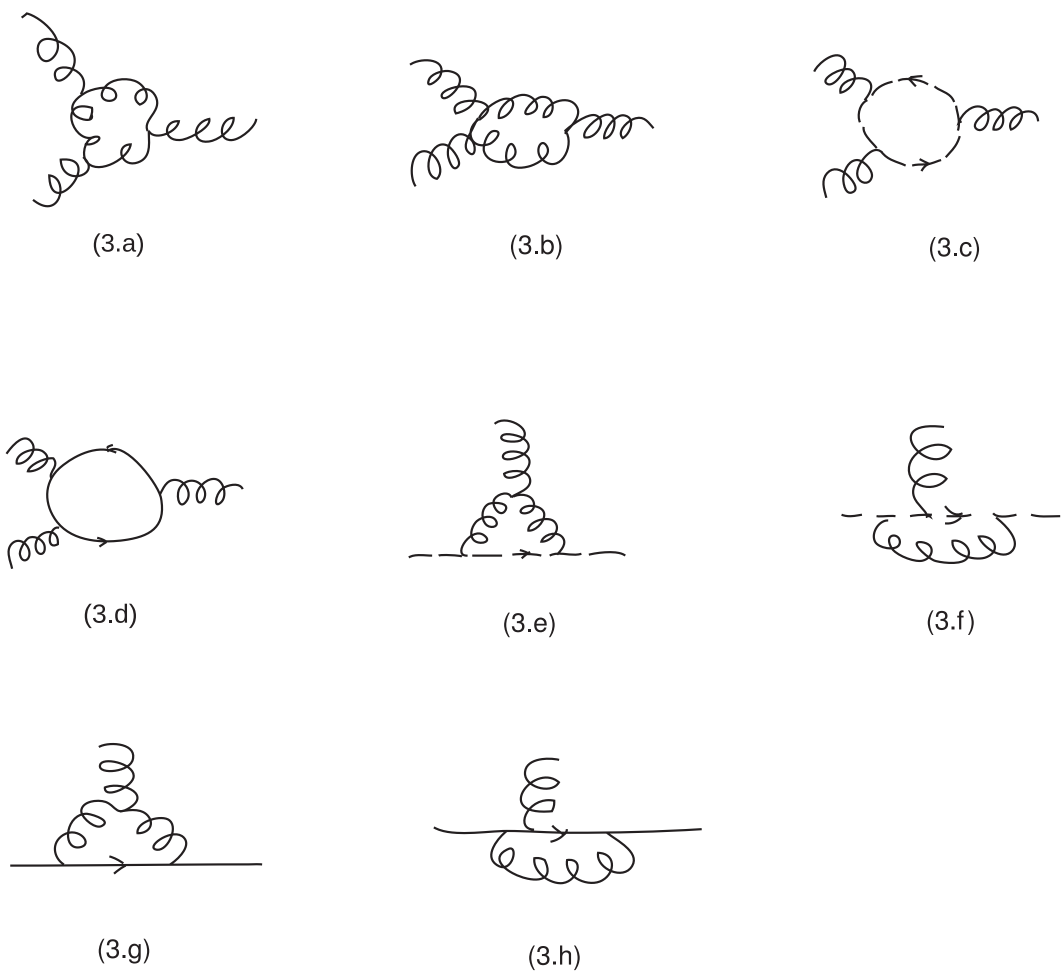


Figure 26: One-loop Yang-Mills : the 3-point functions

the vertex momentum factor is acting on the external leg and does not contribute to the loop momentum counting. Finally, the graphs (4.l-o) are UV convergent because their superficial degree of divergence is -2. Thus, Z_4, listed above, is in fact the only required renormalization for 4-point functions.

16.6 One-Loop Renormalization

The above results for the renormalization factors verify the Slavnov-Taylor identities, as may be seen by explicit calculation. There are 3 identities to be checked, which we shall express in terms of 3 combinations predicted to be 1 by the Slavnov-Taylor identities,

$$\begin{aligned} \hat{Z}_1^2\, Z_3^{-1}\, Z_4^{-1} &= 1 + \mathcal{O}(g^4) \\ \tilde{Z}_1\, \tilde{Z}_3^{-1}\, \hat{Z}_1^{-1}\, Z_3 &= 1 + \mathcal{O}(g^4) \\ Z_1\, Z_2^{-1}\, \hat{Z}_1^{-1}\, Z_3 &= 1 + \mathcal{O}(g^4) \end{aligned} \tag{16.26}$$

The quantity Z_λ is not renormalized at all.

Next, we derive the relation between the bare coupling g_0 and the renormalized coupling g. The starting point is the relation

$$g_0 = g\, \mu^\varepsilon\, \hat{Z}_1\, Z_3^{-3/2} \tag{16.27}$$

Using the expressions found above to one-loop order, we get

$$\begin{aligned} g_0 &= g\, \mu^\varepsilon \left(1 + \frac{g^2\, b_1}{32\pi^2\, \varepsilon} + \mathcal{O}(g^4) \right) \\ b_1 &= -\frac{11}{3} C(\mathcal{G}) + \frac{4}{3} T(\psi) \end{aligned} \tag{16.28}$$

Inverting and squaring both sides, up to this order in g gives an equivalent relation,

$$\frac{1}{g_0^2} = \frac{\mu^{-2\varepsilon}}{g^2} - \frac{\mu^{-2\varepsilon}}{16\pi^2\, \varepsilon} b_1 + \mathcal{O}(g^2) \tag{16.29}$$

The proper interpretation of this relation is as follows. If the bare coupling g_0 and the regulator ε are kept fixed, then the relation gives the dependence of the renormalized coupling g on the renormalization scale μ. Therefore, it is appropriate to denote the renormalized coupling by $g(\mu)$ instead of by g. The renormalized coupling should have a finite limit as $\varepsilon \to 0$, keeping μ fixed. This requires a special choice for the dependence of the bare coupling g_0 on the regulator ε,

$$\frac{1}{g_0^2} = \Lambda^{-2\varepsilon} \left(-\frac{b_1}{16\pi^2 \varepsilon} + c \right) \tag{16.30}$$

Here, the pole in ε is required by the requirement that $g(\mu)$ should have a finite limit as $\varepsilon \to 0$, but the constant c is arbitrary. A scale Λ is required on dimensional grounds.

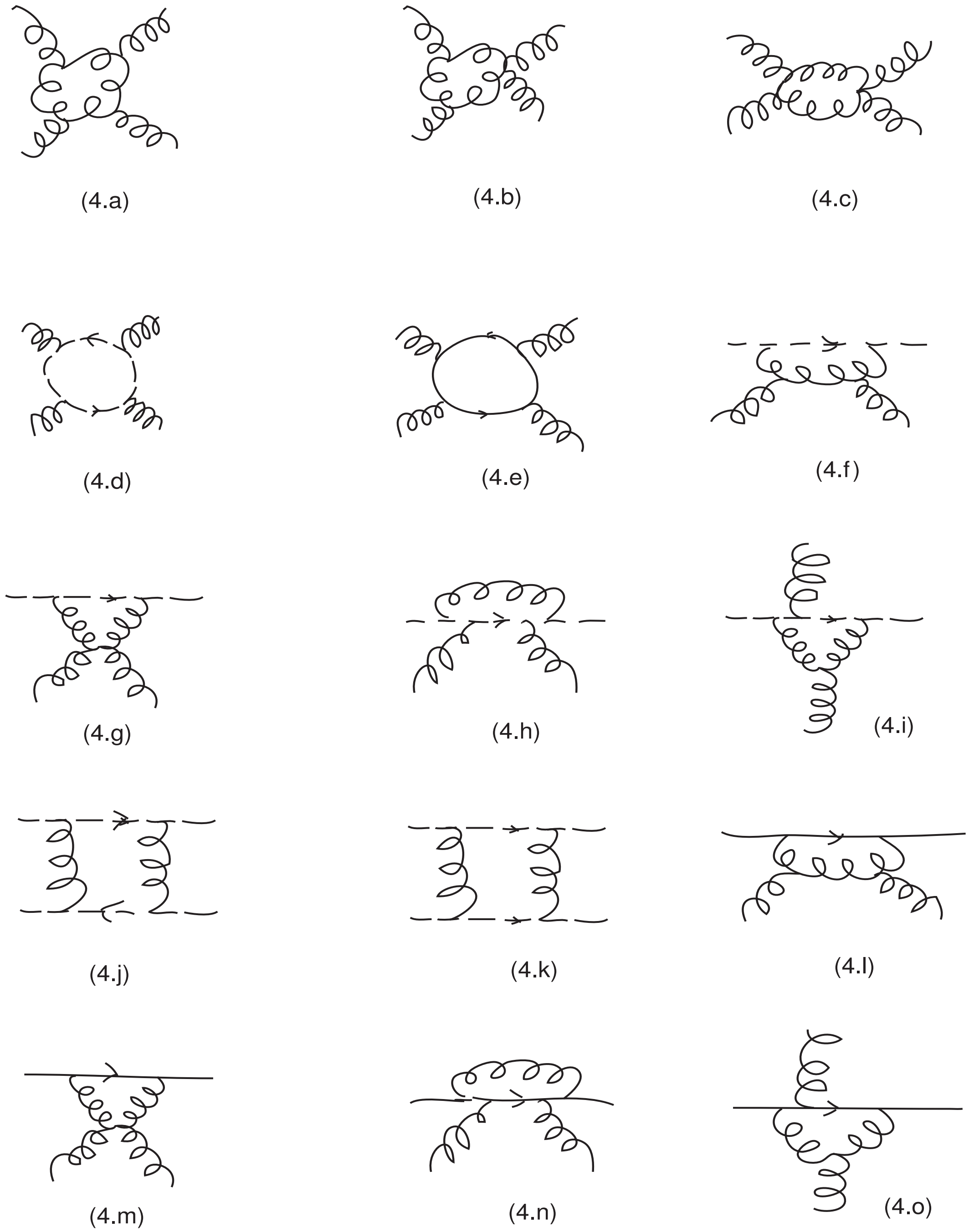


Figure 27: One-loop Yang-Mills : the 4-point functions

Notice that the choice of g_0 is independent of μ and does not involve $g(\mu)$. Using this choice and multiplying through by $\mu^{2\varepsilon}$ gives

$$\frac{1}{g(\mu)^2} = \frac{b_1}{16\pi^2\,\varepsilon}\left\{1 - \left(\frac{\mu}{\Lambda}\right)^{2\varepsilon}\right\} + c\left(\frac{\mu}{\Lambda}\right)^{2\varepsilon} \tag{16.31}$$

This expression for $g(\mu)$ now has a finite limit as $\varepsilon \to 0$, but it involves the unknown constant c. Taking the difference for two different values of μ eliminates this unknown and we get

$$\frac{1}{g(\mu)^2} - \frac{1}{g(\mu')^2} = \lim_{\varepsilon\to 0}\left(\frac{b_1}{16\pi^2\,\varepsilon}\left\{\left(\frac{\mu}{\Lambda}\right)^{2\varepsilon} - \left(\frac{\mu'}{\Lambda}\right)^{2\varepsilon}\right\}\right) \tag{16.32}$$

The dependence on Λ drops out, and we find

$$\boxed{\frac{1}{g(\mu)^2} - \frac{1}{g(\mu')^2} = \frac{b_1}{8\pi^2}\ln\left(\frac{\mu}{\mu'}\right) + \mathcal{O}(g^2) \qquad\qquad b_1 = -\frac{11}{3}C(\mathcal{G}) + \frac{4}{3}T(\psi)} \tag{16.33}$$

As a check, the coupling constant renormalization of QED may be recovered from this relation by setting the gauge group to be $\mathcal{G} = U(1)$, and $g = e$, so that $C(\mathcal{G}) = 0$ and $T(\psi) = \sum_j q_j^2$, for Dirac fermions j with charge $Q_j g$. We indeed recover

$$\frac{1}{e(\mu)^2} - \frac{1}{e(\mu')^2} = \frac{1}{12\pi^2}\sum_j q_j^2\,\ln\left(\frac{\mu}{\mu'}\right)^2 + \mathcal{O}(e^2) \tag{16.34}$$

This agrees precisely with the formulas derived in the section on QED.

16.7 Asymptotic Freedom

For a genuinely non-Abelian Yang-Mills theory, with simple gauge algebra $\mathcal{G}$, the Casimir $C(\mathcal{G})$ is positive, so that, unlike in QED, the coefficient b_1 may become negative. When $b_1 < 0$, the above equation implies that

$$\mu\, >\, \mu' \qquad \Rightarrow \qquad g(\mu)\, <\, g(\mu') \tag{16.35}$$

This means that as the energy scale at which the coupling g is probed is increased, the coupling decreases. This effect is the opposite of *electric screening* which took place in QED. In particular, it implies that as the energy is taken to be asymptotically high, the coupling will vanish, hence the name *asymptotic freedom*. Notice that the perturbative calculations in an asymptotically free theory should become more reliable as the energy is increased.

Conversely, as the energy is lowered, the coupling increases. Of course, at some point, the coupling will then fail to be in the perturbative regime and higher order corrections, as well as truly non-perturbative effects will have to be included. Nonetheless, the trend is indicative that the theory may develop a genuinely large coupling as the energy scale is lowered. For the strong interactions, such increase in coupling is certainly a necessary condition to have *quark confinement*. The phenomenon of growing coupling with lower energy scales is often referred to as *infrared slavery*.

To get a quantitative insight into the theories which are asymptotically free, we shall examine more closely the case of Yang-Mills theory for the gauge group $\mathcal{G} = SU(N)$ with N_f Dirac fermions in the defining representation of $SU(N)$, which is of dimension N. Group theory informs us that

$$C(\mathcal{G}) = N \qquad\qquad T(\psi) = \frac{1}{2} N_f \tag{16.36}$$

so that

$$b_1 = -\frac{11}{3} N + \frac{2}{3} N_f \tag{16.37}$$

Asymptotic freedom is realized as long as $b_1 < 0$, or

$$11\, N - 2\, N_f > 0 \tag{16.38}$$

in other words, when there are *not too many fermions*. Specifically for the color gauge group of QCD, we have $N = 3$, and the number of quark flavors is $N_f = 6$, namely, the u, d, c, s, t, b. We see that $11\, N - 2\, N_f = 21 > 0$ and QCD with 6 flavors of quarks is indeed asymptotically free.

17 Use of the background field method

Feynman diagrams and standard perturbation theory do not always provide the most efficient calculational methods. This is especially true for theories with complicated local invariances, such as gauge and reparametrization invariances. In this chapter, the background field method will be introduced and discussed. It will be utilized to provide a complete calculation of the renormalization effects of the Yang-Mills coupling.

17.1 Background field method for scalar fields

The background field method is introduced first for a theory with a single scalar field ϕ. The calssical action is denoted $S_0[\phi]$, and the full quantum action, including all renormalization counterterms is denoted by $S[\phi]$. The generating functional for connected correlators is given by

$$\exp\left\{-\frac{i}{\hbar}W[J]\right\} \equiv \int D\phi \ \exp\left\{\frac{i}{\hbar}S[\phi] - \frac{i}{\hbar}\int dx J(x)\phi(x)\right\} \tag{17.1}$$

By the very construction of the renormalized action $S[\phi]$, $W[J]$ is a finite functional of J. The Legendre transform is defined by

$$\Gamma[\bar\phi] \equiv \int J\bar\phi - W[J] \tag{17.2}$$

where J and $\bar\phi$ are related by

$$\bar\phi(x) = \frac{\delta W[J]}{\delta J(x)} \qquad\qquad J(x) = \frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi(x)} \tag{17.3}$$

Recall that $\Gamma[\bar\phi]$ is the generating functional for 1-particle irreducible correlators.

To obtain a recursive formula for Γ, one proceeds to eliminate J from (17.1) using the above relations,

$$\exp\frac{i}{\hbar}\Big(\Gamma[\bar\phi] - \int J\bar\phi\Big) = \int D\phi \ \exp\frac{i}{\hbar}\Big(S[\phi] - \int dx\phi(x)\frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi(x)}\Big) \tag{17.4}$$

Moving the J-dependence from the left to the right side, and expressing J in terms of Γ in the process, we obtain,

$$\exp\frac{i}{\hbar}\Gamma[\bar\phi] = \int D\phi \ \exp\frac{i}{\hbar}\Big(S[\phi] - \int dx(\phi-\bar\phi)(x)\frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi(x)}\Big) \tag{17.5}$$

The final formula is obtained by shifting the integration field $\phi \to \bar\phi + \sqrt{\hbar}\varphi$,

$$\boxed{\exp\frac{i}{\hbar}\Gamma[\bar\phi] = \int D\varphi \ \exp\frac{i}{\hbar}\Big(S[\bar\phi+\sqrt{\hbar}\varphi] - \sqrt{\hbar}\int dx\varphi(x)\frac{\delta\Gamma[\bar\phi]}{\delta\bar\phi(x)}\Big)} \tag{17.6}$$

At first sight, it may seem that little has been gained since both the left and right hand sides involve Γ. Actually, the power of this formula is revealed when it is expanded in powers of $\hbar$, since on the rhs, Γ enters to an order higher by $\sqrt{\hbar}$ than it enters on the lhs.

The $\hbar$-expansion is performed as follows,

$$\begin{aligned} S[\phi] &= S_0[\phi] + \hbar S_1[\phi] + \hbar^2 S_2[\phi] + \mathcal{O}(\hbar^3) \\ \Gamma[\bar\phi] &= \Gamma_0[\bar\phi] + \hbar \Gamma_1[\bar\phi] + \hbar^2 \Gamma_2[\bar\phi] + \mathcal{O}(\hbar^3) \end{aligned} \tag{17.7}$$

The recursive equation to lowest order clearly shows that $\Gamma_0[\bar\phi] = S_0[\bar\phi]$. The interpretation of the terms $S_i[\phi]$ for $i \geq 1$ is clearly in terms of counterterms to loop order i, while that of $\Gamma_i[\varphi]$ is that of i-loop corrections to the effective action. The 1-loop correction is readily evaluated for example,

$$\exp i\Gamma_1[\bar\phi] = e^{iS_1[\bar\phi]} \int D\varphi \exp\Big\{ \frac{i}{2} \int dx\ \varphi(x) \int dy\ \varphi(y) \frac{\delta^2 S_0[\bar\phi]}{\delta\bar\phi(x)\delta\bar\phi(y)} \Big\} \tag{17.8}$$

For example, if S_0 describes a field ϕ with mass m and interaction potential $V(\phi)$, we have

$$S_0[\phi] = \int dx \Big(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi) \Big) \tag{17.9}$$

then we have

$$\begin{aligned} \exp i\Gamma_1[\bar\phi] &= e^{iS_1[\bar\phi]} \int D\varphi\ \exp\Big\{ \frac{i}{2} \int dx \varphi \Big(\Box + m^2 + V''(\bar\phi) \Big) \varphi \Big\} \\ \Gamma_1[\bar\phi] &= \frac{i}{2} \ln \mathrm{Det} \Big[\Box + m^2 + V''(\bar\phi) \Big] + S_1[\bar\phi] \end{aligned} \tag{17.10}$$

Upon expanding this formula in perturbation theory, the standard expressions may be recovered,

$$\Gamma_1[\bar\phi] = \Gamma_1[0] + \frac{i}{2} \ln \mathrm{Det} \Big[1 + V''(\bar\phi)(\Box + m^2)^{-1} \Big] + S_1[\bar\phi] \tag{17.11}$$

which clearly reproduces the standard 1-loop expansion. Therefore, for scalar theories, the gain produced by the use of the background field method is usually only minimal.

17.2 Background field method for Yang-Mills theories

For gauge theories, the background field method presents important calculational advantages, which we illustrate by working out the coupling constant renormalization effects in Yang-Mills theory. The starting point is the classical Yang-Mills action,

$$S_0[A] = -\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{17.12}$$

We begin by expanding the quantum field A_μ around a non-dynamical background $\bar{A}_\mu$,

$$\begin{aligned} A_\mu &= \bar{A}_\mu + g\sqrt{\hbar}a_\mu \\ F_{\mu\nu} &= \bar{F}_{\mu\nu} + g\sqrt{\hbar}(\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu) + ig^2\hbar[a_\mu, a_\nu] \end{aligned} \tag{17.13}$$

where the background field covariant derivative is defined by

$$\bar{D}_\mu a_\nu = \partial_\mu a_\nu + i[\bar{A}_\mu, a_\nu] \tag{17.14}$$

The expansion of the classical action is given by

$$\begin{aligned} \frac{1}{\hbar}S_0[A] - \frac{1}{\hbar}S_0[\bar{A}] &= -\int \mathrm{tr}\Big[2\frac{1}{\sqrt{\hbar}g}\bar{F}_{\mu\nu}\bar{D}^\mu a^\nu\Big] \qquad (17.15) \\ &+ \int \mathrm{tr}\Big[-(\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)\bar{D}^\mu a^\nu - i\bar{F}^{\mu\nu}[a_\mu, a_\nu]\Big] \\ &+ \int \mathrm{tr}\Big[-2i\sqrt{\hbar}g\bar{D}_\mu a_\nu[a^\mu, a^\nu] + \frac{1}{2}\hbar g^2[a^\mu, a^\nu][a_\mu, a_\nu]\Big] \end{aligned}$$

The action of gauge transformations on A_μ is unique, but its action on $\bar{A}_\mu$ and a_μ is not. The most useful way of decomposing the gauge invariance of A_μ is to let the quantum field a_μ transform homogeneously,

$$\begin{aligned} \bar{A}_\mu &\rightarrow \bar{A}'_\mu = U\bar{A}_\mu U^{-1} + i\partial_\mu U\ U \\ a_\mu &\rightarrow a'_\mu = Ua_\mu U^{-1} \end{aligned} \tag{17.16}$$

Notice that under this action of the gauge transformation each term in the expansions above transform homogeneously. As a resul, the structure of the counterterms is uniquely fixed by this background field gauge invariance. To one-loop order, we have

$$S_1[\bar{A}] = c\int \mathrm{tr}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu} \tag{17.17}$$

where c is to be determined by the renormalization conditions

Background gauge fixing and ghosts

In analogy with the quantization of Yang-Mills theory in trivial background, here also the kinetic part for the a_μ field fails to be invertible and therefore gauge fixing is required. It is convenient to work in background field Lorentz covariant gauges, such as the one defined by the following gauge function

$$\mathcal{F}(A) = \bar{D}_\mu a^\mu - f \qquad\qquad \frac{\delta\mathcal{F}(A)(x)}{\delta a^\mu(y)} = \bar{D}_\mu\delta(x-y) \tag{17.18}$$

where all terms are valued in the adjoint representation of the gauge algebra. The external function f is to be integrated over. The Faddeev-Popov operator is defined by

$$\mathcal{M}(x,y) = D_\mu\bar{D}^\mu\delta(x-y) \tag{17.19}$$

Notice that the first covrainat derivative is with respect to the full gauge field A_μ, while the second one is only with respect to the background gauge field $\bar{A}_\mu$. The corresponding Faddeev-Popov ghost Lagrangian is thus given by

$$
\begin{aligned}
S_{gh}[A,c,\bar{c}] &= 2\int \mathrm{tr}\Big(D_\mu \bar{c}\ \bar{D}^\mu c\Big) \\
&= 2\int \mathrm{tr}\Big(\bar{D}_\mu \bar{c}\ \bar{D}^\mu c + ig\sqrt{\hbar}\ [a_\mu, \bar{c}]\ \bar{D}^\mu c\Big) \qquad (17.20)
\end{aligned}
$$

while the gauge fixing Lagrangian that results from integrating out f gives rise to

$$
S_{gf}[A] = \frac{1}{\xi}\int \mathrm{tr}\Big(\bar{D}_\mu a^\mu\Big)^2 \qquad (17.21)
$$

The kinetic term for a_μ may now be combined with the gauge fixing term, using the following identities,

$$
\begin{aligned}
\int \mathrm{tr}\Big(\bar{D}_\nu a_\mu \bar{D}^\mu a^\nu\Big) &= -\int \mathrm{tr}\Big(a_\mu \bar{D}_\nu \bar{D}^\mu a^\nu\Big) \\
&= \int \mathrm{tr}\Big(a_\mu[\bar{D}^\mu, \bar{D}^\nu]a_\nu\Big) - \int \mathrm{tr}\Big(a_\mu \bar{D}^\mu \bar{D}^\nu a_\nu\Big) \\
&= -i\int \mathrm{tr}\Big(\bar{F}^{\mu\nu}[a_\mu, a_\nu]\Big) + \int \mathrm{tr}(\bar{D}_\mu a^\mu)^2 \qquad (17.22)
\end{aligned}
$$

Finally, it is convenient to choose Feynman gauge, where $\xi = 1$, so that

$$
\begin{aligned}
\frac{1}{\hbar}S_0[A] - \frac{1}{\hbar}S_0[\bar{A}] &= -\int \mathrm{tr}\Big[2\frac{1}{\sqrt{\hbar}g}\bar{F}_{\mu\nu}\bar{D}^\mu a^\nu\Big] \qquad (17.23) \\
&\quad + \int \mathrm{tr}\Big[-\bar{D}_\mu a_\nu \bar{D}^\mu a^\nu - 2i\bar{F}^{\mu\nu}[a_\mu, a_\nu]\Big] \\
&\quad + \int \mathrm{tr}\Big[-2i\sqrt{\hbar}g\bar{D}_\mu a_\nu[a^\mu, a^\nu] + \frac{1}{2}\hbar g^2[a^\mu, a^\nu][a_\mu, a_\nu]\Big]
\end{aligned}
$$

Notice that the kinetic term for the a_μ field may be viewed as the kinetic term for 4 scalar bosons, indexed by the Lorentz label ν.

Coupling scalars and spinors

Let T^a_s, T^a_f, and $T^a_{\mathcal{G}}$ denote representation matrices in the representations of the gauge algebra for the scalars (s), spinors (f) and gauge fields (i.e. the adjoint representation $\mathcal{G}$). We denote the associated background covariant derivatives, acting on Lorentz scalars, by

$$
\bar{D}_{R\mu} = \partial_\mu + i\bar{A}^a_\mu T^a_R \qquad\qquad R = s,\ f,\ \mathcal{G} \qquad (17.24)
$$

The Lagrangians for scalars and fermions coupled to the gauge field A are then given by

$$
\begin{aligned}
S_s[\phi, A] &= \int\Big(\frac{1}{2}(\bar{D}^\mu_s\phi)^t\bar{D}_{s\mu}\phi + ig\sqrt{\hbar}a^a_\mu\phi^t T^a_s\bar{D}^\mu_s\phi - \frac{1}{2}g^2\hbar a^{a\mu}a^b_\mu\phi^t T^a_s T^b_s\phi\Big) \\
S_f[\psi, \bar{\psi}, A] &= \int\Big(\bar{\psi}(i\gamma^\mu\bar{D}_{f\mu} - m)\psi - g\sqrt{\hbar}a^a_\mu\bar{\psi}\gamma^\mu T^a_f\psi\Big) \qquad (17.25)
\end{aligned}
$$

One-Loop contributions

The 1-loop contributions are all given by Gaussian integrals, and thus functional determinants,

$$\begin{aligned}
i\Gamma_1^s &= -\frac{1}{2}\ln \mathrm{Det}(-\bar{D}_s^\mu \bar{D}_{s\mu}) \\
i\Gamma_1^f &= +\ln \mathrm{Det}(i\gamma^\mu \bar{D}_{f\mu}) \\
i\Gamma_1^{gauge} &= -\frac{1}{2}\ln \mathrm{Det}(-\bar{D}_{\mathcal{G}}^\mu \bar{D}_{\mathcal{G}\mu}\delta_\nu{}^\kappa + 2i\bar{F}[\cdot,\cdot]) \\
i\Gamma_1^{ghost} &= +\ln \mathrm{Det}(-\bar{D}_{\mathcal{G}}^\mu \bar{D}_{\mathcal{G}\mu})
\end{aligned} \tag{17.26}$$

The fermion determinant may be put into a form more akin of the gauge contribution by squaring the background Dirac operator,

$$(i\gamma^\mu \bar{D}_{f\mu})^2 = -\bar{D}_{f\mu}\bar{D}_f^\mu - \frac{i}{4}[\gamma^\mu, \gamma^\nu]\bar{F}_{\mu\nu}^a T_f^a \tag{17.27}$$

and thus

$$i\Gamma_1^f = \frac{1}{2}\ln \mathrm{Det}\Big(-\bar{D}_{f\mu}\bar{D}_f^\mu - \frac{i}{4}[\gamma^\mu, \gamma^\nu]\bar{F}_{\mu\nu}^a T_f^a)\Big) \tag{17.28}$$

The interpretation of the various contributions to the effective action are then as follows. The square of the background covariant derivative on scalar represents the coupling of a charged scalar particle to the Yang-Milsl field. The effect on the coupling is diamagnetic. The term coupling $\bar{F}$ to spin is paramagnetic.

In the renormalization of the coupling, we shall only be interested in the contributions proportional to $S_0[A]$. Therefore, the contributions of the diamagnetic and paramagnetic pieces do not interfere, and we may expand the corrections as follows.

$$\begin{aligned}
i\Gamma_1^f &= 2\ln \mathrm{Det}(-\bar{D}_f^\mu \bar{D}_{f\mu}) \\
&\quad -\frac{1}{64}\int dx \int dx' \bar{F}_{\mu\nu}^a(x)\bar{F}_{\mu'\nu'}^{a'}(x')\langle\bar{\psi}[\gamma^\mu, \gamma^\nu]T_f^a\psi(x)\bar{\psi}[\gamma^{\mu'}, \gamma^{\nu'}]T_f^{a'}\psi(x')\rangle \\
i\Gamma_1^{gauge} &= -2\ln \mathrm{Det}(-\bar{D}_{\mathcal{G}}^\mu \bar{D}_{\mathcal{G}\mu}) \\
&\quad -\frac{1}{2}\int dx \int dx' \bar{F}_{\mu\nu}^a(x)\bar{F}_{\mu'\nu'}^{a'}(x')\langle f^{abc}a_\mu^b a_\nu^c(x) f^{a'b'c'}a_\mu^{b'}a_\nu^{c'}(x')\rangle
\end{aligned} \tag{17.29}$$

The coefficients in front of the double integrals arise as follows. Γ_1^f has a $-$ sign because it contains a closed fermion loop; it has a factor $i^2/2$ from second order perturbation theory; a factor of $(-i/4)^2$ from the $\bar{F}$-spin coupling in the action and an overall factor of $1/2$ because we took the square. Γ_1^{gauge} has a factor $i^2/2$ from second order perturbation theory; a factor of $(2i)^2$ from the $\bar{F}$-spin coupling in the action and a factor of $(i/2)^2$ from the normalization of the structure constants f^{abc}.

The correlators are easily evaluated. We have

$$\begin{aligned}&\langle\bar\psi[\gamma^\mu,\gamma^\nu]T_f^a\psi(x)\bar\psi[\gamma^{\mu'},\gamma^{\nu'}]T_f^{a'}\psi(x')\rangle \\ &\quad = -t_2(f)\delta^{aa'}\mathrm{tr}[\gamma^\mu,\gamma^\nu][\gamma^{\mu'},\gamma^{\nu'}]\int\frac{d^dk}{(2\pi)^d}\ e^{-ik(x-x')}\int\frac{d^dp}{(2\pi)^d}\frac{i}{p^2}\cdot\frac{i}{(p+k)^2}\end{aligned} \tag{17.30}$$

Using the γ-trace formula,

$$\mathrm{tr}[\gamma^\mu,\gamma^\nu][\gamma^{\mu'},\gamma^{\nu'}] = -16(\eta^{\mu\mu'}\eta^{\nu\nu'}-\eta^{\mu\nu'}\eta^{\nu\mu'}) \tag{17.31}$$

evaluating the Feynman integral in dimensional regularization, and retaining only the divergent part, we have

$$\langle\bar\psi[\gamma^\mu,\gamma^\nu]T_f^a\psi(x)\bar\psi[\gamma^{\mu'},\gamma^{\nu'}]T_f^{a'}\psi(x')\rangle = \frac{2i}{\pi^2\epsilon}t_2(f)\delta^{aa'}(\eta^{\mu\mu'}\eta^{\nu\nu'}-\eta^{\mu\nu'}\eta^{\nu\mu'})\delta(x-x') \tag{17.32}$$

The correlator of the a_μ-fields is similarly evaluated,

$$\langle f^{abc}a_\mu^b a_\nu^c(x)f^{a'b'c'}a_\mu^{b'}a_\nu^{c'}(x')\rangle = -\frac{i}{8\pi^2\epsilon}t_2(\mathcal{G})\delta^{aa'}(\eta^{\mu\mu'}\eta^{\nu\nu'}-\eta^{\mu\nu'}\eta^{\nu\mu'})\delta(x-x') \tag{17.33}$$

Furthermore, the group theory dependence of the scalar determinants may be factored out as follows. For any representation r of $\mathcal{G}$, we have

$$it_2(r)\Gamma_d = -\frac{1}{2}\ln\mathrm{Det}(-\bar D_r^\mu\bar D_{r\mu}) \tag{17.34}$$

Combining the above result, we have

$$\begin{aligned}\Gamma_1^s &= t_2(s)\Gamma_d \\ \Gamma_1^{ghost} &= -2t_2(\mathcal{G})\Gamma_d \\ \Gamma_1^f &= -4t_2(f)\Gamma_d - \frac{1}{16\pi^2\epsilon}t_2(f)\int\bar F_{\mu\nu}^a\bar F^{a\mu\nu} \\ \Gamma_1^{gauge} &= +4t_2(\mathcal{G})\Gamma_d + \frac{1}{8\pi^2\epsilon}t_2(\mathcal{G})\int\bar F_{\mu\nu}^a\bar F^{a\mu\nu}\end{aligned} \tag{17.35}$$

The normalization of the fermion vacuum polarization graph was computed long ago, and we have

$$\begin{aligned}\Gamma_1^f &= \frac{1}{4g^2}\delta Z_3\int\bar F_{\mu\nu}^a\bar F^{a\mu\nu} \\ \delta Z_3 &= -\frac{g^2}{6\pi^2\epsilon}t_2(f)\end{aligned} \tag{17.36}$$

Therefore,

$$\Gamma_d = -\frac{i}{192\pi^2\epsilon}\int\bar F_{\mu\nu}^a\bar F^{a\mu\nu} \tag{17.37}$$

From this we calculate,

$$
\begin{aligned}
\frac{1}{g^2} - \frac{1}{g_0^2} &= \frac{1}{48\pi^2\epsilon}\Big(+2t_2(\mathcal{G}) - 4t_2(f) + t_2(s)\Big) && \text{(diamagnetic)} \\
&\quad + \frac{1}{48\pi^2\epsilon}\Big(-24t_2(\mathcal{G}) + 12t_2(f)\Big) && \text{(paramagnetic)} \\
&= \frac{1}{48\pi^2\epsilon}\Big(-22t_2(\mathcal{G}) + 8t_2(f) + t_2(s)\Big) && (17.38)
\end{aligned}
$$

18 Non-perturbative Renormalization

Renormalization, as introduced in IV, was defined only in perturbation theory, order by order in the coupling. The Wilsonian approach starts from a non-perturbative definition of quantum field theory. Wilson used the lattice in position space. One may alternatively use a cutoff on momentum space. In either cases, it is best to define the theory with Euclidean signature; e.g. momenta

$$\begin{aligned} k^2 &\leq \Lambda^2 \qquad \text{Minkowski: unbounded} \\ k_E^2 &\leq \Lambda^2 \qquad \text{Euclidean: bounded} \end{aligned}$$

The partition function and correlators may be defined in the regularized (cutoff) theory by cutting off the fields

$$\begin{aligned} [\mathcal{D}\phi]_\Lambda &\equiv \prod_{|k|\leq\Lambda} d\phi(k) \\ Z[J] &\equiv \int [\mathcal{D}\phi]_\Lambda e^{-S[\phi]+\int d^4x \phi(x) J(x)} \end{aligned} \tag{18.1}$$

and the action includes bare parameters

$$S[\phi] = \int d^4x \left[\frac{1}{2} Z \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} Z m_o^2 \phi^2 + \frac{1}{4!} \lambda_o Z^2 \phi^4\right] \tag{18.2}$$

The theory is convergent in the UV since the momentum ranges have been cut off at high $|k|$. It is normally convergent in the IR for massive theories. If not, the system could be put in a finite box of volume V.

18.1 Integrating out High Momentum Shell

We are interested in non-perturbative dynamics, so we shall *not* necessarily assume $\lambda \ll 1$. We may, however, use the fact that Λ is large compared to any physical length scale. To this end, we divide up the integration region of all momenta (in the presence of the cutoff Λ) as follows,

$$\{|k| \leq \Lambda\} \quad = \quad \{|k| \leq b\Lambda\} \quad \cup \quad \{b\Lambda < |k| \leq \Lambda\} \tag{18.3}$$

where $0 < b < 1$ and independent of Λ. One may take $b = 1/2$ for example. The component $\{b\Lambda < |k| < \Lambda\}$ is a *high momentum shell* in that all its momenta are of order Λ and thus large compared to any physical energy momentum scales.

The fields may be organized according to the same partitioning,

$$\phi(x) = \varphi(x) + \hat{\phi}(x) \qquad \begin{cases} \varphi(k) \neq 0 & \text{support on} \quad |k| \leq b\Lambda \\ \hat{\phi}(k) \neq 0 & \text{support on} \quad b\Lambda \leq |k| \leq \Lambda \end{cases} \tag{18.4}$$

The functional integration measure factorizes,

$$[\mathcal{D}\phi]_\Lambda = [\mathcal{D}\varphi]_{b\Lambda} \times [\mathcal{D}\hat{\phi}] \tag{18.5}$$

The action has no cross terms between φ and $\hat{\phi}$ at the quadratic level because the momentum space integration separates on the two components of the partition,

$$\begin{aligned}\int_0^{|k|\leq\Lambda} d^4k\ \phi(-k)\frac{1}{2}Z(k^2+m^2)\phi(k) &= \int_0^{|k|\leq b\Lambda} d^4k\ \varphi(-k)\frac{1}{2}Z(k^2+m^2)\varphi(k) \\ &+ \int_{|k|>b\Lambda}^{|k|\leq\Lambda} d^4k\ \hat{\phi}(-k)\frac{1}{2}Z(k^2+m^2)\hat{\phi}(k)\end{aligned} \tag{18.6}$$

Hence the action splits:

$$S[\phi] = S[\varphi] + \hat{S}[\varphi;\hat{\phi}] \tag{18.7}$$

where $S[\varphi]$ is the original bare action, but now evaluated on the restricted field φ and $\hat{S}[\varphi;\hat{\phi}]$ is given as follows,

$$\hat{S}[\varphi;\hat{\phi}] = \int d^4x \left[\frac{1}{2}Z(\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} + m^2\hat{\phi}^2) + \frac{1}{4!}\lambda_o Z^2(\hat{\phi}^4 + 4\hat{\phi}^3\varphi + 6\hat{\phi}^2\varphi^2 + 4\hat{\phi}\varphi^3)\right] \tag{18.8}$$

This action is reminiscent of the one used in the background field method. The difference is that in the background field method, the quantum field was allowed to run over all possible field values, while here the range is restricted because the corresponding momenta are restricted to the high momentum shell.

The requirement that the cutoff scale is much larger than the physical scales of the problem amounts to requiring that the source field $J(x)$ used to probe the correlators should be considered only within this same restricted shell,

$$J(k) \neq 0 \quad \text{only when} \quad 0 < |k| < b\Lambda \tag{18.9}$$

and $J(k) = 0$ when $b\Lambda < |k| < \Lambda$.

We are now ready to analyze the full functional integral, defined by

$$e^{-W[J]} \equiv \int [\mathcal{D}\phi]_\Lambda e^{-S[\phi]} \tag{18.10}$$

Applying the partitioning of the momenta, of the fields and of the source to this integral, we get

$$e^{-W[J]} = \int [\mathcal{D}\varphi]_{b\Lambda} \exp\left\{-S[\varphi] - \Delta S[\varphi] + \int d^4x \varphi J\right\} \tag{18.11}$$

where the effective action $\Delta S[\varphi]$ is defined by

$$e^{-\Delta S[\varphi]} \equiv \int [\mathcal{D}\hat{\phi}] \exp\left\{-\hat{S}[\varphi;\hat{\phi}]\right\} \tag{18.12}$$

Notice that the effective action is calculated independently from the source J and is thus *universal*. In the present case, we also have the following form vertices. Feynman rules

(for the computation of last integral)

$$\begin{aligned} &: \quad \text{external } \tilde{\phi} \text{ field} \\ &: \quad \text{internal } \hat{\phi} \text{ field} \end{aligned}$$

The $m^2\hat{\phi}^2$ term is small compared to the kinetic terms, and may be treated perturbatively.

$$k \qquad p = \frac{1}{2}k^2(2\pi)^4\delta(k+p)\theta(k)$$

$$\theta(k) = \begin{cases} l & b\Lambda \leq |k| \leq \Lambda \\ 0 & \text{otherwise} \end{cases}$$

Investigate possible connections

$$\int_{b\Lambda}^{\Lambda} d^4k \frac{1}{k^2} = 2\pi^2 \int_{b\Lambda}^{\Lambda} k^3 dk \frac{1}{k^2} = \pi^2\Lambda^2(1-b^2) \tag{18.13}$$

Thus, we have a schematic expression for $\Delta S[\tilde{p}]$:

$$\begin{aligned}\Delta S[\tilde{\phi}] &= \int d^4x \left[\frac{1}{2}\delta Z \partial_\mu \tilde{\phi}\gamma^\mu \tilde{\phi} + \frac{1}{2}(\delta Z m_o^2 + Z\delta m_o^2)\tilde{\phi}^2 + \frac{1}{4!}(2Z\tilde{Z}\lambda_o + Z^2\delta\lambda_o)\tilde{\phi}^4 \right. \\ &+ \left. \text{higher dim operators } \Lambda^{\text{negative}}\right]\end{aligned}$$

e.g. $\epsilon\tilde{\phi}^6\Lambda^{-2} + \cdots$

Now rescale the theory by the factor b so as to return to the original cutoff scale:

$$k' = k/b \qquad \begin{aligned} x' &= xb \\ \partial' &= \partial b^{-1}\end{aligned} \qquad \phi' = b^{-1}\tilde{\phi}$$

$$\begin{aligned}S[\tilde{\phi}] &+ \Delta S[\tilde{\phi}] = \int d^4x \left[\frac{1}{2}\tilde{Z}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{1}{2}\tilde{Z}\tilde{m}_o^2\tilde{\phi}^2 \right. \\ &+ \left. \frac{1}{4!}\tilde{Z}^2\tilde{\lambda}_o\tilde{\phi}^4 + \text{higher dim}\right]\end{aligned}$$

$$\begin{cases}\tilde{Z} = Z + \delta Z \\ \tilde{m}_o^2 = m_o^2 + \delta m_o^2 \\ \tilde{\lambda}_o = \lambda_o + \delta\lambda_o\end{cases}$$

Rescale field renormalization: After scaling

$$\begin{aligned}S[\tilde{\phi}] &+ \Delta S[\tilde{\phi}] = \int d^4x' \left[\frac{1}{2}Z'\partial'_\mu\phi'\partial'_\mu\phi' + \frac{1}{2}m_o'^2\phi'^2 Z' \right. \\ &+ \left. \frac{1}{4!}\lambda'_o Z'^2\phi'^4 + \cdots\right] + \cdots + \epsilon'\phi^6\end{aligned}$$

$$\begin{cases}Z' = Z \\ m_o'^2 = \tilde{m}_o^2 b^{-2} \\ \lambda'_o = \tilde{\lambda}_o b^o \\ \epsilon' = \tilde{\epsilon} b^2\end{cases}$$

It would appear that we have integrated out the momentum shell

$$b\Lambda \leq |\vec{k}| \leq \Lambda \tag{18.14}$$

at the cost of introducing an ∞ number of (new) terms in the Lagrangian. However, one should think of repeating this process in order to come closer to the long distance scales.

$$\begin{array}{rll} \text{“relevant” operators (like } \phi^2) & b^{-2} \ : & \text{grow as } b \to 0 \\ \text{“irrelevant” operators (like } \phi^6) & b^2 \ : & \to 0 \text{ as } b \to 0 \\ \text{“marginal” operators (like } \partial_\mu\phi\partial^\mu\phi, \phi^4) & b^o & \left(\text{often } \log\frac{1}{b}\right)\end{array}$$

Irrelevant operators play negligible roles as the renormalization procedure is repeated.

18.2 Lattice QFT

19 The Renormalization Group

Quantum field theory is Poincaré invariant, but it is not, in general, scale invariant. Even when the classical Lagrangian is scale invariant, quantum mechanically, scale transformations may have an anomaly so that the full quantum theory fails to be scale invariant. Therefore, in general, quantum field theory changes with scale and it is of fundamental importance to know how.

The renormalization group equations (sometimes also referred to as the Callan-Symanzik equations) encode the scale dependence of a QFT. There are two key approaches.

1. The Gell-Mann–Low, Stückelberg and Petermann, Callan-Symanzik approach. The bare theory is kept fixed while the renormalization scale is varied. These renormalization group equations express the change of the renormalized correlation functions and coupling constants as the renormalization scale is varied.

2. The Wilsonian approach. The renormalized theory is kept fixed while the cutoff scale is varied. These renormalization group equations express the change of the correlations functions and effective action of the bare theory as the cutoff is varied. In practice, the Wilsonian approach is set up so that high energy/momenta are systematically integrated out near the cutoff scale and their effects on the remaining theory are summarized by including corrections in the remaining Lagrangian.

In this chapter, we treat 1, reserving 2 for the next.

19.1 Bare and renormalized correlators

Consider ϕ^4-theory, with bare Lagrangian

$$\mathcal{L}_o = \frac{1}{2}\partial_\mu \phi_o \partial^\mu \phi_o - \frac{1}{2} m_o^2 \phi_o^2 - \frac{\lambda_o}{4!}\phi_o^4 \tag{19.1}$$

Renormalization of the correlators of the bare canonical field ϕ_o proceeds multiplicatively $\phi(x) = Z_\phi^{-1/2}\ \phi_o(x)$. As a result, we have the following relations between the correlators of the bare and of the renormalized fields,

$$\begin{aligned} G_R^{(n)}(p_i; m, \lambda, \mu) &= Z_\phi^{-n/2}\ G_o^{(n)}(p_i; m_o, \lambda_o, \Lambda) \\ \Gamma_R^{(n)}(p_i; m, \lambda, \mu) &= Z_\phi^{+n/2}\ \Gamma_o^{(n)}(p_i; m_o, \lambda_o, \Lambda) \end{aligned} \tag{19.2}$$

The definition of $G_o^{(n)}$ (and thus of $\Gamma_o^{(n)}$ is straightforward. Starting from the bare Lagrangian, in the presence of a regulator Λ, the correlator $G_o^{(n)}(p_i; m_o, \lambda_o, \Lambda)$ is the Fourrier transform of $\langle \emptyset | T\phi_o(x_1)\cdots\phi_o(x_n)|\emptyset\rangle$.

To properly define $G_R^{(n)}$, we need to impose a certain number of renormalization conditions – actually 3 in this case. If we want those to be valid for massive or massless ϕ^4, we shouuld renormalize off-shell, at a renormalization scale μ. A standard choice is as follows. Mass and field renormalizations are carried out on the 2-point functions,

$$\Gamma_R^{(2)}(p;m,\lambda;\mu)\Big|_{p^2=-\mu^2} = -\mu^2 - m^2 \qquad\qquad \frac{\partial \Gamma_R^{(2)}}{\partial p^2}\Big|_{p^2=-\mu^2} = 1 \tag{19.3}$$

while coupling constant renormalization is governed by the 4-point function,

$$\Gamma_R^{(4)}(p_i;m,\lambda;\mu)\Big|_{\mathcal{R}} = -\lambda \qquad\qquad \mathcal{R} = \begin{cases} p_i^2 = -\mu^2 \\ s = t = u = -4\mu^2/3 \end{cases}$$

Imposing these conditions provides the following implicit relations between the bare and the renormalized parameters,

$$\begin{cases} m_o(m,\lambda,\Lambda,\mu) \\ \lambda_o(m,\lambda,\Lambda,\mu) \end{cases} \qquad \text{or} \qquad \begin{cases} m(m_o,\lambda,\Lambda,\mu) \\ \lambda(m_o,\lambda_o,\Lambda,\mu) \end{cases} \tag{19.4}$$

The field renormalization Z_ϕ may be expressed either as a function of m, λ, Λ, μ or as a function of $m_o, \lambda_o, \Lambda, \mu$.

19.2 Derivation of the renormalization group equations

To derive an equation that describes the variation in renormalization point μ, we use the following ingredients;

1. For fixed bare parameters, m_o, λ_o, Λ a change in renormalization scale will produce a change in the renormalized parameters m, and λ. Thus, for fixed bare parameters, m and λ are implicit functions of μ.

2. For fixed bare parameters m_o, λ_o, Λ, the bare correlators $G_o^{(n)}(p_i; m_o, \lambda_o, \Lambda)$ and $\Gamma_o^{(n)}(p_i; m_o, \lambda_o, \Lambda)$ are independent of μ, since their very definition never appealed to μ in the first place. This fact may be expressed in terms of simple differential equations for the bare correlators,

$$\begin{aligned} \mu \frac{d}{d\mu} G_o^{(n)}(p_i; m_o, \lambda_o, \Lambda) &= 0 \\ \mu \frac{d}{d\mu} \Gamma_o^{(n)}(p_i; m_o, \lambda_o, \Lambda) &= 0 \end{aligned} \tag{19.5}$$

   Here, the notation of *total differential* $d/d\mu$ is used to make it clear that the variation in μ applies to the explicit μ-dependence as well as to the implicit μ dependence such as of the renormalized mass m and coupling λ.

The momenta p_i will always be taken to be fixed when varying the renormalization scale.

3. These equations may be recast in terms of equations for $G_R^{(n)}$ and $\Gamma_R^{(n)}$, by using the relation (19.2) between bare and renormalized correlators,

$$\begin{aligned}
\mu\frac{d}{d\mu}\left[Z_\phi^{+n/2}G_R^{(n)}(p_i;m,\lambda,\mu)\right]\Big|_{m_o\lambda_o,\Lambda} &= 0 \\
\mu\frac{d}{d\mu}\left[Z_\phi^{-n/2}\Gamma_R^{(n)}(p_i;m,\lambda,\mu)\right]\Big|_{m_o\lambda_o,\Lambda} &= 0
\end{aligned} \tag{19.6}$$

The vertical bar with the subscript m_o, λ_o, Λ has been added to remind the reader that the total variation in μ is taken while keeping the bare parameters fixed.

The implicit dependence of λ, m, and Z_ϕ on μ gives rise to the following functions

$$\begin{aligned}
\beta(\lambda) &\equiv \mu\,\frac{\partial\lambda}{\partial\mu}\Big|_{m_o,\lambda_o,\Lambda} \\
\gamma_m(\lambda) &\equiv \mu\,\frac{\partial m}{\partial\mu}\frac{1}{m}\Big|_{m_o,\lambda_o,\Lambda} \\
\gamma_\phi(\lambda) &\equiv -\frac{1}{2}\mu\frac{\partial\ln Z_\phi}{\partial\mu}\Big|_{m_o,\lambda_o,\Lambda}
\end{aligned} \tag{19.7}$$

Notice that these functions do not depend upon the precise correlator under consideration. Instead they are universal functions that only depend on the theory under consideration. They are referred to as *the beta-function* β, the *mass anomalous dimension* γ_m, and the *anomalous dimension of the ϕ-field* γ_ϕ.

Finally, the renormalization group equations (or RG equations for short) are obtained by expressing (19.6) in terms of the β-function and anomalous dimensions,

$$\begin{aligned}
\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+\gamma_m m\frac{\partial}{\partial m}-n\gamma_\phi\right]G_R^{(n)}(p_i;m,\lambda,\mu) &= 0 \\
\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}+\gamma_m m\frac{\partial}{\partial m}+n\gamma_\phi\right]\Gamma_R^{(n)}(p_i;m,\lambda,\mu) &= 0
\end{aligned} \tag{19.8}$$

From their very construction β, γ_m and γ_ϕ must be finite as $\Lambda \to \infty$, because G_R never depended on Λ in the first place. Since these quantities have no dimension, their dependence must be as follows,

$$\beta(\lambda,\mu/m) \qquad \gamma_m(\lambda,\mu/m) \qquad \gamma_\phi(\lambda,\mu/m) \tag{19.9}$$

Often though, mass-independent renormalization may be carried out and the dependence is only on λ.

19.3 Massless QED reconsidered

Each independent field has a field renormalization factor,

$$\begin{aligned} \psi_o(x) &= \sqrt{Z_2}\,\psi(x) \\ A_{\mu o}(x) &= \sqrt{Z_3}\,A_\mu(x) \end{aligned} \tag{19.10}$$

Consider vacuum polarization first. Using gauge invariance, the vacuum 2-point functions may be expressed in terms of a scalar only,

$$\begin{aligned} \Gamma^{(2)}_{o\ \mu\nu}(p;e_o,\Lambda) &= (\eta_{\mu\nu}p^2 - p_\mu p_\nu)\Gamma^{(2)}_o(p^2;e_o,\Lambda) \\ \Gamma^{(2)}_{R\ \mu\nu}(p;e,\mu) &= (\eta_{\mu\nu}p^2 - p_\mu p_\nu)\Gamma^{(2)}_R(p^2;e,\mu) \end{aligned} \tag{19.11}$$

The bare 2-point function was calculated long ago and found to be given by

$$\Gamma^{(2)}_o(p^2;e_o,\Lambda) = 1 + \frac{e_o^2}{2\pi^2}\int_0^1 d\alpha\ \alpha(1-\alpha)\ln\frac{-\Lambda^2}{\alpha(1-\alpha)p^2} + \mathcal{O}(e_o^4) \tag{19.12}$$

The 2-point function must be renormalized off-shell since the fermions are massless, and it is convenient to take to renormalizaton conditions as follows,

$$\Gamma^{(2)}_R(-\mu^2;e,\mu) = 1 \tag{19.13}$$

The renormalized 2-point function is then determined to one-loop order and given for all p^2 by the familiar expression,

$$\Gamma^{(2)}_R(p^2;e,\mu) = 1 - \frac{e^2}{12\pi^2}\ln\frac{-p^2}{\mu^2} + O(e^4) \tag{19.14}$$

Making use of the fact that the difference between e^2 and e_o^2 in the above expressions amounts to a higher order correction, which has been omitted anyway, we may readily compute Z_3,

$$\begin{aligned} Z_3 &= 1 + \frac{e_o^2}{2\pi^2}\int_0^1 d\alpha\ \alpha(1-\alpha)\ln\frac{\alpha(1-\alpha)\mu^2}{\Lambda^2} + \mathcal{O}(e_o^4) \\ &= 1 - \frac{e^2}{12\pi^2}\ln\frac{\Lambda^2}{\mu^2} + e^2 C + \mathcal{O}(e^4) \end{aligned} \tag{19.15}$$

where C is a constant, given by

$$C = \frac{1}{2\pi^2}\int_0^1 d\alpha\ \alpha(1-\alpha)\ln\alpha(1-\alpha) \tag{19.16}$$

The relation between the bare and renormalized charges is given by

$$e = e_o Z_3^{1/2} = e_o\left(1 - \frac{e_o^2}{24\pi^2}\ln\frac{\Lambda^2}{\mu^2} + \frac{1}{2}e_o^2 C\right) + O(e_o^5) \tag{19.17}$$

The β-function may be calculated as follows. By definition,

$$\beta(e) = \mu \frac{\partial e}{\partial \mu}\bigg|_{e_o,\Lambda} \tag{19.18}$$

Using the above relation for the renormalized charge in terms of the bare charge, we have

$$\mu \frac{\partial e}{\partial \mu}\bigg|_{e_o,\Lambda} = \frac{e_o^3}{12\pi^2} + \mathcal{O}(e_o^5) \tag{19.19}$$

The term involving the constant C does not contribute, since it does not involve μ. The β-function should be expressed in terms of the renormalized coupling, since it enters into the RG equations for the renormalized correlators. This is easily up to this order, since the difference between e and e_o will be of order e_o^5. Collecting also the result for the anomalous dimensions of the field A_μ and the field ψ (the latter in Feynman gauge), we have

$$\begin{aligned} \beta(e) = \; &= +\frac{e^3}{12\pi^2} + \mathcal{O}(e^5) \\ \gamma_A(e) = \; &= -\frac{e^2}{12\pi^2} + \mathcal{O}(e^4) \\ \gamma_\psi(e) \; &= -\frac{e^2}{16\pi^2} + \mathcal{O}(e^4) \end{aligned} \tag{19.20}$$

19.4 The Running Coupling "Constant"

Consider now a more general theory with a single coupling, denoted g. We shall assume that the theory is massless or that mass-independent renormalization has been carried out. In this case, the β-function only depends upon the coupling g, and the definition of the β-function then becomes a simple first order differential equation for the renormalized coupling g as a function of the renormalization scale μ,

$$\mu \frac{\partial g}{\partial \mu} = \beta(g) \tag{19.21}$$

Thus, knowledge of the β-function gives us the dependence of the renormalized coupling on the scale. Under the flow $\mu \to \bar{\mu}(t)$, the coupling will depend on t via $g \to \bar{g}(t)$ with $\bar{g}(0) = g$. The equation is solved as follows,

$$dt = \frac{d\bar{g}}{\beta(\bar{g})} \qquad \Rightarrow \qquad \ln \frac{\bar{\mu}(t)}{\mu} = \int_g^{\bar{g}(t)} \frac{dx}{\beta(x)} \tag{19.22}$$

(1) For example in QED, we have $\beta(e) = e^3/12\pi^2 + \mathcal{O}(e^5)$; the solution to the β-function equation yields

$$\frac{1}{\bar{e}(t)^2} - \frac{1}{e^2} \;=\; -\frac{1}{6\pi^2} \ln \frac{\bar{\mu}(t)}{\mu} \tag{19.23}$$

This equation may also be cast in the form,

$$\bar{e}(t)^2 = \frac{e^2}{1 - \frac{e^2}{6\pi^2} \ln \frac{\bar{\mu}(t)}{\mu}} \tag{19.24}$$

As indicated previously, $\bar{e}(t)$ increases as t increases. If this RG improved one-loop formula were to be taken seriously for all energy scales, then the running coupling would diverge at the *Landau pole* value, which is given by

$$\frac{\bar{\mu}_L}{\mu} = \exp\left\{\frac{6\pi^2}{e^2}\right\} \sim \exp\left\{\frac{3\pi}{2\alpha}\right\} \tag{19.25}$$

The actual value of the Landau Pole mass scale is therefore enormous. If $\mu \sim m_e$, then $\bar{\mu}_L \sim 10^{277} GeV$, which is 258 orders of magnitude larger than the Planck mass.

19.5 Scaling properties of correlators

We investigate the implications for correlators. (1) In QED, consider the vacuum polarization,

$$\Gamma_o^{(2)}(p; e_o, \Lambda) = Z_3\left(\bar{e}(t), \Lambda/\bar{\mu}(t)\right)^{-1} \Gamma_R^{(2)}(p; \bar{e}(t); \bar{\mu}(t)) \tag{19.26}$$

This relation holds for any value of t, since the left side is manifestly independent of t. Therefore, it may also be recast in terms of the renormalized correlator only, at different values of t,

$$\Gamma_R^{(2)}(p; \bar{e}(t); \bar{\mu}(t)) = \frac{Z_3\left(\bar{e}(t), \frac{\Lambda}{\bar{\mu}(t)}\right)}{Z_3\left(e, \frac{\Lambda}{\mu}\right)} \Gamma_R^{(2)}(p; e; \mu) \tag{19.27}$$

The ratio

$$Z_3\left(\bar{e}(t), \frac{\Lambda}{\mu(t)}\right) Z_3\left(e, \frac{\Lambda}{\mu}\right)^{-1} \equiv Z(e; t) \tag{19.28}$$

cannot possibly depend upon Λ, because neither $\Gamma_R^{(2)}$ depends upon Λ.

$$\Gamma_R^{(2)}(p; \bar{e}(t); \bar{\mu}(t)) = Z(e; t)\Gamma_R^{(2)}(p; e; \mu) \tag{19.29}$$

Now trace ordinary scaling dimensions: ($\Gamma^{(2)}$ dimensionless)

$$\Gamma_R^{(2)}(p \exp(t); e, \mu e^t) = \Gamma_R^{(2)}(p; e; \mu) \tag{19.30}$$

This equation is valid for any e, in particular $\bar{e}(t)$, so we also have

$$\Gamma_R^{(2)}(p \exp(t); \bar{e}(t), \mu \exp(t)) = \Gamma_R^{(2)}(p; \bar{e}(t); \mu) \tag{19.31}$$

Using the relation (19.29), we find

$$\Gamma_R^{(2)}(p\exp(t); e; \mu) = Z(e;t)^{-1}\Gamma_R^{(2)}(p; \bar{e}(t); \mu) \tag{19.32}$$

Thus, as momentum is increased $t > 0$, the effect is to replace the coupling by the running coupling $\bar{e}(t)$. The generalization to n-point functions is straightforward

$$\Gamma_R^{(n)}(e^t p_i; e; \mu) = Z(e;t)^{n/2}) e^{(4-n)t}\Gamma_R^{(n)}(p_i; \bar{e}(t); \mu) \tag{19.33}$$

(2) Consider $\lambda\phi^4$ theory. To keep things simple, we assume that the renormalized theory is massless (or that we have performed mass-independent renormalization).

$$\begin{aligned} &\Gamma_R^{(2)}(0;\lambda;\mu)\Big|_{p^2=\mu^2} = -\mu^2 \qquad\qquad \left.\frac{\partial\Gamma_R^{(2)}}{\partial p^2}\right|_{p^2=-\mu^2} = 1 \\ &\Gamma_R^{(4)}(p_i;\lambda,\mu)\Big|_{\substack{p_i^2=-\mu^2 \\ s=t=\mu=-\frac{4}{3}\mu^2}} = -\lambda \end{aligned} \tag{19.34}$$

We have the usual relation between the renormalized and bare correlators,

$$\Gamma_R^{(2)}(p,\lambda,\mu) = Z_\phi\ \Gamma_o^{(2)}(p;\lambda_o,m_o,\Lambda) \tag{19.35}$$

The three equations fix the three unknowns Z_ϕ, λ_o, m_o as a function of λ, μ.

- $\Gamma_o^{(2)}(p,\lambda_o,m_o,\Lambda)$ is computed directly from the base Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_o\partial^\mu\phi_o - \frac{1}{2}m_o^2\phi_o^2 - \frac{\lambda_o}{4!}\phi_o^4 \tag{19.36}$$

  It is independent of the renormalization conditions and μ.

- Z_ϕ is determined from the renormalization conditions μ, Λ

$$Z_\phi(\lambda, \mu/\Lambda) \tag{19.37}$$

- The RG equation just states that $\Gamma_o^{(2)}$ is independent of μ.

$$\left[\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - n\gamma(\lambda)\right]\Gamma_R^{(n)}(p_i;\lambda,\mu) = 0 \tag{19.38}$$

$$\beta(\lambda) \equiv \left.\frac{\partial\lambda}{\partial\ln\mu}\right|_{\lambda_o,\Lambda} \qquad \gamma(\lambda) \equiv \frac{\partial}{\partial\ln\mu}\ln\sqrt{Z_\phi}\Big|_{\lambda_o\Lambda} \tag{19.39}$$

Let's integrate:

$$\begin{aligned}
\ln \frac{\bar{\mu}}{\mu} &= \int_{\lambda}^{\bar{\lambda}} \frac{dx}{\beta(x)} \\
\frac{1}{2} \ln Z_\phi &= \int^{\bar{\mu}} \frac{dx}{x} \gamma(\lambda(x))
\end{aligned}$$

$$\boxed{\Gamma_R^{(n)}(e^t p_i; \lambda, \mu) = \exp\left\{(4-n)t - n\int_0^t dt' \gamma(\lambda(t'))\right\} \Gamma_R^{(n)}(p_i; \bar{\lambda}(t); \mu)} \tag{19.40}$$

19.6 Asymptotic Behaviors for a Theory with One Coupling

- $\gamma(\lambda)$ indicates the scaling anomalous dimension of the field.
- $\beta(\lambda)$ provides information on the *asymptotic behavior* of the coupling as a function of scale.

$$\bar{\mu} = e^t \mu \ : \quad \begin{cases} t \to +\infty & \text{UV behavior} \\ t \to -\infty & \text{IR behavior} \end{cases} \tag{19.41}$$

Three behaviors to be distinguished for UV

(i) $\bar{\lambda}(t) \to \infty \quad$ as $\quad t \to +\infty$
strongly coupled

(ii) $\bar{\lambda}(t) \to \infty \neq 0 \quad$ as $\quad t \to +\infty$
UV fixed point

(iii) $\bar{\lambda}(t) \to 0 \quad$ as $\quad t \to +\infty$
Asymptotic freedom
(the coupling at large energy vanishes asymptotically, and the theory becomes free)

The asymptotic behavior may be obtained by linearizing around the fixed point.

(iii) $\bar{\lambda}(t) \to 0$

$$\begin{aligned}
\beta(\lambda) &= \text{given by perturbation theory} \\
&= \begin{cases} \sim \lambda^3 \text{ for } \lambda^2\phi^4 \ - \text{theory} \\ \sim \lambda^3 \text{ for } e = \lambda \text{ in QED} \end{cases} \\
\beta(\lambda) &= b_1 \lambda^3 + \text{higher order} \\
\frac{d\bar{\lambda}(t)}{dt} &= b_1 \bar{\lambda}^3 \qquad \frac{d\bar{\lambda}(t)}{\bar{\lambda}^3} = b_1 dt \\
& \quad - \frac{1}{\bar{\lambda}(t)^2} = 2 b_1 t
\end{aligned}$$

Since $\bar{\lambda}(t)^2 > 0$: $\qquad \boxed{b_1 < 0} \quad$ is required.

(ii) $\bar{\lambda}(t) \to \lambda_* \neq 0$

$$
\begin{aligned}
&\beta\lambda = \beta(\lambda_*) + (\lambda - \lambda_*)\beta' + \frac{1}{2}(\lambda - \lambda_*)^2\beta''(\lambda_*) + \cdots \\
&\frac{d\bar{\lambda}(t)}{dt} = \beta(\lambda_*) + (\bar{\lambda} - \lambda_*)\beta'(\lambda_*) + \cdots \\
&\Rightarrow \quad \boxed{\beta(\lambda_*) = 0} \quad \lambda_* \text{ is a fixed point in UV} \\
&\frac{d\bar{\lambda}}{\bar{\lambda} - \lambda_*} = \beta'(\lambda_*)dt + \cdots \\
&\ln(\bar{\lambda} - \lambda_*) = \beta'(\lambda_*)t \qquad \bar{\lambda}(t) = \lambda_* + e^{\beta'(\lambda_*)t} \\
&\text{Since } t \to +\infty,\ \lambda(t) \to \lambda_*\ , \quad \boxed{\beta'(\lambda_*) < 0}
\end{aligned}
$$

Three behaviors for IR

(i) $\bar{\lambda}(t) \to \infty \quad \text{as} \quad t \to -\infty$
strongly coupled

(ii) $\bar{\lambda}(t) \to \lambda_* \neq 0 \quad \text{as} \quad t \to -\infty$
IR fixed point

(iii) $\bar{\lambda}(t) \to 0 \quad \text{as} \quad t \to -\infty$
Gaussian fixed point (or IR free)(the coupling vanishes at small energies)

(iii) $\bar{\lambda}(t) \to 0$ again given by point theory

$$
-\frac{1}{\bar{\lambda}(t)^2} = 2b_1 t \qquad \boxed{bZ_1 > 0} \tag{19.42}
$$

(ii) $\bar{\lambda}(t) \to \lambda_* \neq 0$
$\bar{\lambda}(t) = \lambda_* + e^{\beta'(\lambda_*)t} \qquad \boxed{\beta'(\lambda_*) > 0}$
$\longrightarrow = t \to +\infty$
$\longrightarrow = t \to -\infty$

Renormalization scheme dependence

A change in renormalization conditions produces a change in the definition of the coupling constants. How do such changes affect the nature of the asymptotics?

Assume a theory with one coupling λ, and let's suppose the coupling λ' arises in a different scheme.

$$
\begin{aligned}
(\lambda')^2 &= G'^2(\lambda^2) = \lambda^2 + O(\lambda^4) \\
\lambda' &= G(\lambda) = \lambda + O(\lambda^3)
\end{aligned}
$$

e.g. in $\lambda^2\phi^4$ theory, a change in renormalization conditions may be of the form

$$
\begin{aligned}
\Gamma_R^{(4)}(p_i;\lambda;\mu\Big|_{\substack{p_i^2=-\mu^2 \\ s=t=u=-\frac{4}{3}\mu^2}} &= \lambda^2 \\
\Gamma_R^{(4)}(p_i;\lambda';\mu)\Big|_{\substack{p_i^2=-\mu^2 \\ s=t=-\mu^2, u=-2\mu^2}} &= \lambda'^2
\end{aligned}
$$

We assume that $G(\lambda)$ is analytic and (at least locally) monotonic: $\partial G/\partial\lambda \neq 0$

$$
\beta(\lambda) \equiv \left.\frac{\partial\lambda}{\partial\ln\mu}\right|_{\lambda_o,\Lambda} \qquad \beta'(\lambda') \equiv \left.\frac{\partial\lambda'}{\partial\ln\mu}\right|_{\lambda_o,\Lambda} \tag{19.43}
$$

The β-functions are simply related:

$$
\beta'(\lambda') = \left.\frac{\partial\lambda}{\partial\ln\mu}\right|_{\lambda_o,\Lambda} \quad \frac{\partial G(\lambda'}{\partial\lambda} = \frac{\partial G(\lambda)}{\partial\lambda}\beta(\lambda) \tag{19.44}
$$

We have the following results:

1. If $\beta(\lambda)=0$ then $\beta'(\lambda')=0$– the fixed points are *universal*
2. $\left.\frac{\partial\beta(\lambda)}{\partial\lambda}\right|_{\lambda_\star}$ is *universal*
3. $\gamma'(\lambda'_\star) = \gamma(\lambda_\star)$
   the anomalous dimension at the fixed point is also univiersal
4. If the perturbative expansion starts as

$$
\beta(\lambda) = -b_1\lambda^3 - b_2\lambda^5 + \cdots \tag{19.45}
$$

   then the coefficients b_1 and b_2 are *universal*.

1. is obvious since $\partial G/\partial\lambda \neq 0$.

2.

$$
\begin{aligned}
\frac{\partial}{\partial\lambda'}\beta'(\lambda'_\star) &= \frac{\partial\lambda}{\partial\lambda'}\cdot\frac{\partial}{\partial\lambda}\left(\frac{\partial G}{\partial\lambda}\beta(\lambda)\right)\Bigg|_{\lambda=\lambda_\star} \\
&= \frac{\partial\lambda}{\partial\lambda'}\cdot\frac{\partial\lambda'}{\partial\lambda}\cdot\frac{\partial\beta(\lambda)}{\partial\lambda} + \frac{\partial\lambda}{\partial\lambda'}\frac{\partial^2 G}{\partial\lambda^2}\beta(\lambda)\Bigg|_{\lambda=\lambda_\star} \\
&= \frac{\partial\beta(\lambda_\star)}{\partial\lambda}
\end{aligned}
$$

3. is analogous.

4. is very important, so we shall also prove it.

$$\beta'(\lambda') = \beta(\lambda)\frac{\partial G}{\partial \lambda} \tag{19.46}$$

assume

$$\begin{aligned}
G(\lambda) &= \lambda + g\lambda^3 + \cdots \\
\beta'(\lambda') &= (-b_1\lambda^3 - b_2\lambda^5 + \cdots)(1 + 3g\lambda^2 + \cdots) \\
&= -b_1\lambda^3 - (b_2 + 3gb_1)\lambda^5 + \cdots
\end{aligned}$$

now express result in terms of λ':

$$\begin{aligned}
\lambda &= \lambda' - g\lambda'^3 + \cdots \\
&= -b_1(\lambda' 0 g\lambda'^3 + \cdots)^3 - (b_2 + 3gb_1)\lambda'^5 + \cdots \\
&= -b_1\lambda'^3 + 3gb_1\lambda'^5 + \cdots \\
&= \text{same as } \beta(\lambda)
\end{aligned}$$

As a result of 4, one may imagine a renormalization scheme in which all higher order terms have been compensated by a suitable redefinition of λ, and

$$\beta(\lambda) = -b_1\lambda^3 - b_2\lambda^5 \tag{19.47}$$

is exact (at least perturbatively). The form graphs above are now reproduced by the 4 signs of b_1, b_2.

Conformal Invariant QFT

The two fixed points (Gaussian and asymptotic freedom) correspond to massless free field theory. At a fixed point $\lambda_\star \neq 0$, we have a fully interacting theory (since $\lambda \neq 0$), but it is scale invariant. Let's examine the corresponding behavior of the vertex functions.

$$\Gamma_R^{(n)}(e^t p_i; \lambda; \mu) = \exp\left\{(4-n)t - n\int_0^t dt'\gamma(\bar{\gamma}(t'))\right\}\Gamma_R^{(n)}(p_i, \bar{\lambda}(t); \mu) \tag{19.48}$$

As $\bar{\lambda}(t) \to \lambda^\star$ with $t \to \pm\infty$, then the asymptotic formula simplifies:

$$\boxed{\Gamma_R^{(n)}(e^t p_i; \lambda_\star; \mu) = \exp\{(4-n)t - n\gamma(\lambda_\star)t\}\Gamma_R^{(n)}(p_i, \lambda_\star; \mu)} \tag{19.49}$$

The interpretation is as follows:

1. μ is now irrelevant

2. $\lambda_\star$ is unchanged (scale invariance)

3. the dimension of the field ϕ is no longer 1, but there is an anomalous dimension $\gamma(\lambda_\star)$. The total dimension is $1 + \gamma(\lambda_\star)$.

Remark: scale invariant theories at non-trivial fixed pooints are fairly rare, with some famous examples occurring in supersymmetric field theory ($\mathcal{N} = 4$).

20 Global Internal Symmetries

In Nature, different quantum states often have similar physical properties and may be grouped together into multiplets of a certain group. This procedure is familiar, of course, from space rotations or Poincaré transformation symmetries. Internal symmetries were first introduced by Heisenberg, who noticed that – from the point of view of the strong interactions – there is little distinction between a proton and a neutron. For one, their masses are surprisingly close. Therefore, the proton and the neutron may be assembled into a doublet of isospin $SU(2)_I$,

$$\begin{pmatrix} p \\ n \end{pmatrix} \qquad \begin{array}{l} m_p = 938.2 \text{ MeV} \\ m_n = 939.5 \text{ MeV} \end{array} \tag{20.1}$$

The strong interactions are insensitive to $SU(2)_I$ isospin rotations of the proton - neutron doublet. The group $SU(2)_I$ will be a *global symmetry of the strong interactions*.

Of course, from the point of view of electromagnetism, the proton and neutron are very different, since the proton is electrically charged while the neutron is neutral, and therefore $SU(2)_I$ isospin is not a symmetry of the electromagnetic interactions. Global symmetries may be useful even if they are only approximate. In the case of $SU(2)_I$, broken by mass and electromagnetism.

More modern examples are found by considering quarks instead of hadrons.

$$u^c \quad d^c \quad s^c \quad c^c \quad b^c \quad t^c \qquad\qquad c = \text{color} = \{r \; w \; b\} \tag{20.2}$$

The strong interactions are sensitive only to the color of the quarks and not to their flavor. Extending the notion of isospin to six quarks, the symmetry group would naturally be $SU(6)$. Given that the masses of the quarks are approximately as follows,

$$\begin{array}{lll} m_u = 5 \text{ MeV} & m_d = 7 \text{ MeV} & m_s = 100 \text{ MeV} \\ m_c = 1.5 \text{ GeV} & m_b = 4.9 \text{ GeV} & m_t = 170 \text{ GeV} \end{array} \tag{20.3}$$

it is clear that $SU(6)$ is very badly broken and little or no trace of it is left in physics. The u, d and s quarks may still be considered alike from the point of view of the strong interactions as their masses are small compared to the typical strong interaction scale of the proton for example, and the associated $SU(3)$ symmetry is useful.

20.1 A more general discussion of symmetry

In quantum field theory, elementary particles are described by local fields, whose dynamics is governed by a Lagrangian. Thus, we shall consider field theories that are invariant under global symmetries.

In classical mechanics, a symmetry is a transformation of the fields that maps any solution of the associated Euler-Lagrange equations into a solution of the same equations. The symmetry will map a given general solution to a different solution. It is however possible that a specific solution is transformed into itself and is invariant under the transformation. Two distinct concepts of symmetry emerge from the above discussion; the symmetry of an equation versus the symmetry of a solution to the equation. Taking for example a simple equation with more than one solution, $x^3 - x = 0$, we have

- symmetry of equations : $x \to -x$
- symmetry of a particular solution : $0 \to 0$
- transformation of general solutions : $x = +1 \leftrightarrow x = -1$

The concept of symmetry for an object, such as a solution, dates back to ancient Greek times. The concept of symmetry for an equation, however, was first considered by Galois in the early 19-th century.

In quantum mechanics, for systems with a finite number of degree of freedom, a symmetry is realized via a unitary transformation U in Hilbert space. Internal symmetries, which commute with the Hamiltonian, obey

$$U^\dagger H U = H \tag{20.4}$$

Under an internal symmetry, the eigenstates of the Hamiltonian transform under a unitary (finite-dimensional) representation,

$$\begin{aligned} H|E,\alpha\rangle &= E|E,\alpha\rangle \\ H\;U|E,\alpha\rangle &= U\;H|E,\alpha\rangle = E\;U|E,\alpha\rangle \\ \Rightarrow \quad U|E,\alpha\rangle &= \sum_\beta C_{\alpha\beta}|E,\beta\rangle \qquad C^\dagger C = I \end{aligned} \tag{20.5}$$

Thus internal symmetries organize the quantum states at given energy into representations of the symmetry group. This realization is often referred to as *the symmetry is realized in the Wigner mode*.

In *quantum field theory*, there are essentially three possibilities. The starting point is a classical Lagrangian, assumed to have a symmetry in the classical sense. Upon quantization, it may be that this symmetry survives quantization, or alternatively that it suffers an anomaly and therefore ceases to be a symmetry of the system at the quantum level. $U(1)$ gauge invariance in QED is an example of the first case, while $U(1)_A$ symmetry in QED is an example of the second case. Recall that anomalies manifest themselves when regularization and renormalization effects necessarily destroy the symmetry.

When the symmetry survives quantization, there are two further possibilities, which are distinguished by the properties of the vacuum.

1. The symmetry leaves the vacuum invariant. This case is very much like the Wigner mode in quantum mechanics : the states in Hilbert space transform under unitary transformations. $U(1)$ symmetry in QED is again an example of this mode; Poincaré invariance is another, since all states are unitary mutliplets of ISO(1,3).

2. The symmetry does not leave the vacuum invariant. This case has no counterpart in quantum mechanics (except when supersymmetry is considered). The symmetry is said to be *spontaneously broken* or *in the Goldstone mode*.

20.2 Symmetries in systems with two scalar fields

Start with the case of two real scalar fields (or one complex).

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2\phi_2^2 - \frac{\lambda_1^2}{4!}\phi_1^4 - \frac{\lambda^2}{12}\phi_1^2\phi_2^2 - \frac{\lambda_2^2}{4!}\phi_2^4 \qquad (20.6)$$

Generically, the theory has only discrete symmetries :

$$R_1 \begin{cases} \phi_1 & \rightarrow & -\phi_1 \\ \phi_2 & \rightarrow & +\phi_2 \end{cases} \qquad\qquad R_2 \begin{cases} \phi_1 & \rightarrow & +\phi_1 \\ \phi_2 & \rightarrow & -\phi_2 \end{cases} \qquad (20.7)$$

The symmetry group is $Z_2 \times Z_2 \quad Z_2 = \{1, -1\}$.

For special arrangements of masses and couplings, however, the symmetry is actually larger than $Z_2 \times Z_2$. Specifically, if we require $m_1^2 = m_2^2$ and $\lambda_1^2 = \lambda_2^2$, we have a further Z_2 which acts by interchanging ϕ_1 and ϕ_2.

Furthermore, with $m_1^2 = m_2^2$ and $\lambda_1 = \lambda = \lambda_2$, the Lagrangian has a continuous $SO(2) \sim U(1)$ symmetry as well,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}(x) \rightarrow \begin{pmatrix} \cos\alpha & -\sin\alpha \\ +\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}(x) \qquad\qquad \alpha \in [0, 2\pi] \qquad (20.8)$$

One way of seeing this is because $\mathcal{L}$ may be written in terms of a potential which only depends on the $\phi_1^2 + \phi_2^2$. More generally, any Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V(\phi_1^2 + \phi_2^2) \qquad (20.9)$$

is invariant under $SO(2)$. Alternatively, it is often convenient to regroup the two real fields $\phi_{1,2}$ in a single complex field ϕ, in terms of which the $U(1)$ transformation law and the

Lagrangian may be expressed as below,

$$\begin{aligned}
\phi(x) &\equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)(x) \qquad\qquad \phi(x) \to e^{i\alpha}\phi(x) \\
\mathcal{L} &= \partial_\mu \bar\phi \partial^\mu \phi - V(2\bar\phi\phi)
\end{aligned} \tag{20.10}$$

Henceforth, the following $SO(2)$-invariant potential will be considered,

$$V(\phi_1^2 + \phi_2^2) = \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) + \frac{\lambda^2}{4!}(\phi_1^2 + \phi_2^2)^2 \tag{20.11}$$

Throughout, the assumption will be made that $\lambda^2 > 0$. Then, the potential is bounded from below for all values of m^2, including $m^2 < 0$. (For $\lambda^2 < 0$, the potential is unbounded from below, which is physically unacceptable.) Next, the semi-classical vacuum configuration will be determined and its symmetry properties will be examined in light of the preceding discussion.

(1) The symmetric or unbroken phase $m^2 > 0$

The form of the potential for $m^2 > 0$ is represented in Fig 1 (a). The potential has a unique minimum at $\phi_1 = \phi_2 = 0$. This configuration is the semi-classical approximation to the vacuum of the theory, and is invariant under $SO(2)$ rotations. Thus, the system has unbroken symmetry. To study the spectrum, the field is decomposed into creation and annihilation operators. Recall the Fock space construction. The starting point is the vacuum, corresponding here to the classical configuration $\phi_i = 0$. The field is then decomposed as follows,

$$\phi_i(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left(c_i(\vec k)e^{-ik\cdot x} + c_i^\dagger(\vec k)e^{ik\cdot x}\right) \qquad\qquad i = 1,2 \tag{20.12}$$

The ground state is defined by $c_i(\vec k)|0\rangle = 0$; these equations are invariant under $SO(2)$ and so is $|0\rangle$. The excited states (to lowest order in the interaction) are constructed by applying creation operators to the vacuum,

$$c_{i_1}^\dagger(\vec k_1)\cdots c_{i_n}^\dagger(\vec k_n)|0\rangle \tag{20.13}$$

Since $|0\rangle$ is invariant under $SO(2)$ it is manifest that the states transform under unitary representations of $SO(2)$. This is the Wigner mode.

(2) The spontaneously broken phase $m^2 < 0$

The form of the potential for $m^2 < 0$ is represented in Fig. 1 (b). The minimum of the potential is governed by the equations,

$$\frac{\partial V}{\partial \phi_i} = m^2\phi_i + \frac{\lambda^2}{6}(\phi_1^2 + \phi_2^2)\phi_i = 0 \qquad\qquad i = 1,2 \tag{20.14}$$

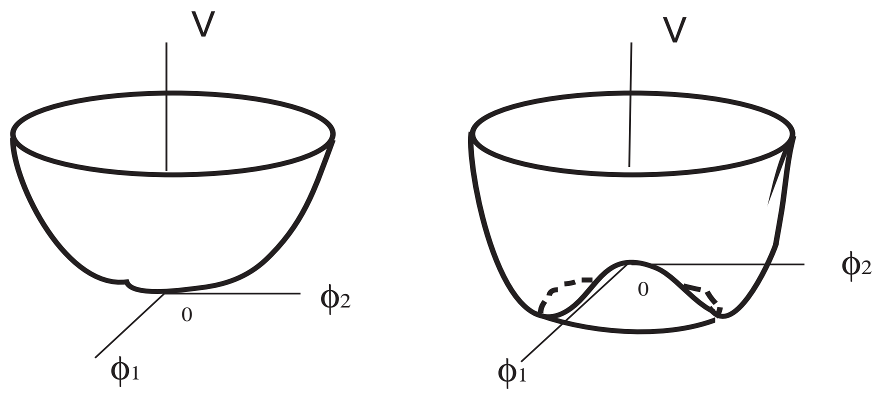


Figure 28: Quartic Potentials for (a) $m^2 > 0$ and (b) $m^2 < 0$

The first solution $\phi_{1,2} = 0$ corresponds to a local maximum, while the true minimum of the potential is given by

$$\phi_1^2 + \phi_2^2 = v^2 \qquad\qquad v^2 = -6\frac{m^2}{\lambda^2} > 0 \tag{20.15}$$

Clearly, the minimum of the potential exhibits a continuous degeneracy. The entire set of all minimum potential configurations is invariant under $SO(2)$, but no single choice of ϕ_i in this set is invariant.

Let us make the assumption that one can pick a single value of ϕ_i and build a Fock space and thus a Hilbert space on this configuration. (The validity of this assumption will be discussed later.) For example, the following choice may be made,

$$\phi_1 = v, \qquad \phi_2 = 0 \qquad\qquad v^2 = -6m^2/\lambda^2 \tag{20.16}$$

This configuration does *not* preserve $SO(2)$ symmetry. The system is in the *Goldstone* mode. The particle excitation spectrum built from this ground state by the Fock construction does *not* transform according to unitary representation of $SO(2)$.

The study of the semi-classical spectrum reveals a characteristic feature that will be generic when the vacuum breaks a continuous symmetry. The spontaneously broken symmetry produces a massless boson, often referred to as a *Nambu-Goldstone boson*. To see this, at the semi-classical level, one proceeds to shifting the fields,

$$\varphi_1 = \phi_1 + v \qquad\qquad \varphi_2 = \phi_2 \tag{20.17}$$

and rewriting the Lagrangian in terms of the φ_i fields, we find,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - \frac{\lambda^2}{24}\Big(4v^2\varphi_1^2 + 4v\varphi_1(\varphi_1^2+\varphi_2^2) + (\varphi_1^2+\varphi_2^2)^2\Big) \tag{20.18}$$

Manifestly, the field φ_2 is *massless*. A look at Fig 1 (b) readily reveals the origin of this phenomenon. As the semiclassical vacuum configuration has $\phi_1 = v$ but $\phi_2 = 0$, the fluctuations around this point purely in the direction of $\phi_2 = \varphi_2$ encounter no potential barrier at all and proceed along the flat direction of the potential, remaining at its minimum. This is the Nambu-Goldstone boson.

The treatment so far has been semi-classical. In the full quantum field theory, one is concerned with the full ground state of the Hamiltonian. In terms of the energy expectation value of any given state $|\psi>$,

$$E(\psi) \equiv \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \tag{20.19}$$

the ground state is defined by the requirement $E(\psi_0) \leq E(\psi)$ for all $\psi \in \mathcal{H}$. The following possibilities arise,

1. $U|\psi_0\rangle = |\psi_0\rangle$ Wigner mode
2. $U|\psi_0\rangle \neq |\psi_0\rangle$ Goldstone mode

The analysis of the full quantum problem is more involved. We first treat an exactly solvable example to illustrate the general ideas and to arrive at the important Coleman-Mermin-Wagner Theorem.

20.3 Spontaneous symmetry breaking in massless free field theory

Consider massless free field theory of a single scalar field $\phi(x)$ in flat d dimensional space-time $\mathbf{R}^d$, with Lagrangian,

$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi \tag{20.20}$$

The Lagrangian is invariant under shifts of ϕ by a real constant c,

$$\phi(x) \to \phi'(x) = \phi(x) + c \tag{20.21}$$

The corresponding Noether current and charge are

$$j^\mu = \partial^\mu\phi \qquad\qquad P = \int d^{d-1}x\, j^0 = \int d^{d-1}x\, \partial^0\phi \tag{20.22}$$

Since the theory is massless, it is also invariant under scale and conformal transformations.

As usual, we expect that the Hilbert space will arise from the Fock space construction on a unique Poincaré invariant vacuum, say $|\emptyset\rangle$. Is this vacuum invariant under the translations of ϕ ? Consider the 1-point function $\langle \emptyset | \phi(x) | \emptyset \rangle$. By Poincaré invariance, this is an x-independent real constant φ. If $|\emptyset\rangle$ were invariant under a translation $\phi \to \phi + c$, then φ should also be invariant under this transformation. The transformation is by $\varphi \to \varphi + c$, which leaves no finite φ invariant.

20.3.1 Adding a small explicit symmetry breaking term

The nature of the vacuum is always an IR problem. It is natural to try and modify our theory in the IR by a term in the Lagrangian that breaks the symmetry $\phi \to \phi + c$ explicitly. In this case, we can actually add a small mass term that will preserve the free field nature of the problem and hence allow for an exact solution. The new Lagrangian will be (for Euclidean metric),

$$\mathcal{L}_{m,\varphi} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 (\phi - \varphi)^2 \tag{20.23}$$

where φ is a real constant and $m^2 > 0$ is to be taken to 0 in order to recover $\mathcal{L}_0$. For given φ, the minimum of the potential of $\mathcal{L}_{m,\varphi}$ is at φ for all $m^2 > 0$. Thus, the 1- and 2-point functions are given by

$$\begin{aligned} \langle \emptyset | \phi(x) | \emptyset \rangle &= \varphi \\ \langle \emptyset | \phi(x)\phi(y) | \emptyset \rangle &= \varphi^2 + G_d(r) \qquad\qquad r^2 = (x-y)^2 \end{aligned} \tag{20.24}$$

and the Green function is given by

$$\begin{aligned} G_d(r) &= \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{k^2 + m^2} \, e^{ik\cdot(x-y)} \\ &= \int_0^\infty dt \int \frac{d^d k}{(2\pi)^d} \, e^{ik\cdot(x-y)} e^{-t(k^2+m^2)} \end{aligned} \tag{20.25}$$

Carrying out the Gaussian integration in k, we find,

$$G_d(r) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty dt \; t^{-d/2} e^{-tm^2 - r^2/4t} = \frac{1}{(2\pi)^{d/2}} \left(\frac{m}{r}\right)^{d/2-1} K_{d/2-1}(mr) \tag{20.26}$$

Actually, $G_d(r)$ always decays exponentially when $r \to \infty$ for fixed $m > 0$. We are most interested, however, in the regime which is closest to the massless regime : this is when $0 < mr \ll 1$ instead. The leading behavior in this regime for the cases is given by

$$\begin{aligned} G_1(r) &= \frac{1}{2m} e^{-mr} \\ G_2(r) &\sim -\frac{1}{2\pi} \ln(mr/2) \\ G_d(r) &\sim \frac{\Gamma(d/2)}{2(d-2)\pi^{d/2} \; r^{d-2}} \qquad\qquad d \geq 3 \end{aligned} \tag{20.27}$$

We recognize the harmonic oscillator case with $d = 1$ and the Yukawa potential for $d = 3$. The cases $d \geq 3$ actually admit finite limits when $m \to 0$ for fixed r, and the resulting limits fall of to 0 as $r \to \infty$. This behavior demonstrates that as $d \geq 3$, the system gets locked into the vacuum expectation value $\langle \emptyset | \phi(x) | \emptyset \rangle = \varphi$ and thus breaks the translation symmetry in the field ϕ at all m and also in the limit $m \to 0$.

The cases $d = 1$ and $d = 2$ do not obey this pattern, as no finite limits exist. The case $d = 1$ is ordinary quantum mechanics. The ground state for $m = 0$ is indeed a state uniformly spread out in ϕ, which will produce divergent $\langle \emptyset | \phi | \emptyset \rangle$. The case $d = 2$ is also important. The failure of being able to take the limit $m \to 0$ is at the origin of the Coleman-Mermin-Wagner Theorem that no continuous symmetry can be spontaneously broken in 2 space-time dimensions.

20.3.2 Canonical analysis

To determine the vacuum of the theory, we construct the Hamiltonian H_0 in terms of the field ϕ and its canonical momentum π. Upon quantization, UV divergences may be renormalized by normal ordering. Decomposing the field in creation and annihilation operators,

$$
\begin{aligned}
\phi(x) &= \int_{\vec{k}} \left(a^\dagger(\vec{k}) e^{-ik\cdot x} + a(\vec{k}) e^{+ik\cdot x} \right) \\
H_0 &= \int_{\vec{k}} |\vec{k}| a^\dagger(\vec{k}) a(\vec{k}) \\
P &= \frac{1}{2i} \left(a(0) - a^\dagger(0) \right)
\end{aligned}
\tag{20.28}
$$

Here, a general notation is used for the summation over the momenta $\vec{k}$. In d-dimensional flat space, the summation corresponds to an integral over the continuous variable $\vec{k}$,

$$
\int_{\vec{k}} = \int \frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}\ 2|\vec{k}|}
\tag{20.29}
$$

but we shall also consider quantization on a torus, where the sum will be discrete.

The Hamiltonian is positive and lowest energy is achieved when a state is annihilated by the operators $a(\vec{k})$. But, we observe a variant of the usual (for example for massive fields) situation, since the operator $a(0)$ corresponds to zero energy. Thus, minimum, i.e. zero energy, requires only that all the operators $a(\vec{k})$ with $\vec{k} \neq 0$ annihilate the minimum energy state. Its behavior under $a(0)$ is undetermined by the minimum energy condition.

The translation charge P only depends upon $a(0) - a^\dagger(0)$ and commutes with H_0. Therefore, the zero energy states are distinguished by their P-charge and may be labeled by this charge, p. The conjugate variable is $Q = (a(0) + a^\dagger(0))/2$, and satisfies $[Q, P] = iV$, where V is the volume of space.

20.4 Why choose a vacuum ?

This question is justified because in ordinary quantum mechanics such degeneracies get lifted. For example, in the Hamiltonian

$$H = \frac{1}{2m}\vec{p}^2 + \frac{1}{2}m\lambda^2(x^2 - v^2)^2 \tag{20.30}$$

a state localized around $x = -v$ has a non-zero probability to tunnel to a state localized around $x = +v$. In fact the transition probability is given by the imaginary time path integral, familiar from the WKB approximation,

$$\langle x = +v | x = -v \rangle = \int \mathcal{D}x(t)\ e^{-S[x]/\hbar} \tag{20.31}$$

with the boundary conditions $x(-\infty) = -v; x(+\infty) = +v$. The Euclidean action is given by,

$$S[x] = \int_{-\infty}^{\infty} dt \left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\lambda^2(x^2 - v^2)^2 \right) \tag{20.32}$$

Semi-classically, $\hbar$ is small and the integral will be dominated by extrema of S (satisfying the initial and final boundary conditions). Solutions are given by the first integral:

$$\frac{1}{2}m\dot{x}^2 - \frac{m\lambda^2}{2}(x^2 - v^2)^2 = 0 \tag{20.33}$$

whose solutions are

$$x_o(t) = \pm v\ \tanh\Big(\lambda v(t - t_o)\Big) \tag{20.34}$$

The corresponding action is finite and given by $S[x_o] = 4/3m\lambda v^3$; the transition amplitude in this approximation is non-zero and given by

$$\langle x = +v | x = -v \rangle = e^{-S[x_o]/\hbar} = e^{\frac{4m\lambda v^3}{3\hbar}} \tag{20.35}$$

These objects are called *instantons*. The true ground state is a superposition of the states localized at v and $-v$. It would thus be absurd to choose one configuration above the other.

What happens in QFT? Can one still tunnel from one configuration to the other? Let's investigate this with the help of the semi-classical approximation to the imaginary time path integral.

$$S[\phi] = \int d^dx \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda^2}{4}(\phi^2 - v^2)^2 \right\} \tag{20.36}$$

Assume there exists a solution $\phi_o(x)$, which is a minimum of the classical action. Consider its scaled version

$$\phi_o^\sigma(x) \equiv \phi_o(\sigma x) \tag{20.37}$$

$$S[\phi^\sigma] = \int d^d x \left\{ \frac{1}{2}\sigma^{2-d}\partial_\mu\phi_o\partial^\mu\phi_o + \frac{\lambda^2}{4}\sigma^{-d}(\phi_o^2 - v^2)^2 \right\} \tag{20.38}$$

Now $\sigma = 1$ better be a stationary point.

$$\left.\frac{\partial S[\phi^\sigma]}{\partial\sigma}\right|_{\sigma=1} = \int d^d x \left\{ \partial_\mu\phi_o\partial^\mu\phi_o(2-d) - d\frac{\lambda^2}{4}(\phi_o^2 - v^2)^2 \right\} \tag{20.39}$$

But for $d \geq 2$, both terms are negative, and cancellation would require

$$\partial_\mu\phi_o\partial^\mu\phi_o = 0 \quad \text{and} \quad \phi_o^2 = v^2 \tag{20.40}$$

Only for $d = 1$ can there be non-trivial solutions. This result is often referred to as *Derrick's theorem*. Thus, for a scalar potential theory with $d \geq 2$, it is impossible to tunnel from one localized state to the other via instantons.

Even when vacuum degeneracy occurs for space-time dimension larger than 1, it may still be possible to have tunneling between the degenerate vacua. Famous examples of this phenomenon are precisely when *instantons* with finite action exist in space-time dimension larger than 1,

- Non-linear σ-models in $d = 2$;
- Higgs model with $U(1)$ gauge symmetry in $d = 2$;
- QCD in $d = 4$ with θ-vacua.

Finally, the Mermin-Wagner-Coleman Theorem states that in $d = 2$, a continuous (bosonic) symmetry can never be spontaneously broken. The reason for this resides in the fact that the massless Goldstone bosons that would arise through this breaking cannot define a properly interacting field theory, since their infrared behavior is too singular.

20.5 Generalized spontaneous symmetry breaking (scalars)

Consider a more general field theory of N real scalar fields ϕ_a, $a = l, \cdots, N$. For simplicity, assume that the Lagrangian has a standard kinetic term as well as a potential V.

$$\mathcal{L} = \sum_a \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - V(\phi_a) \tag{20.41}$$

It will be further assumed that $V(\phi_a)$ is bounded from below, and has its minima at finite values of ϕ_a. (Runaway potentials such as e^ϕ generate special problems.)

- It is assumed that $\mathcal{L}$ is invariant under a symmetry group G acting linearly on ϕ; for this reason, these models are often referred to as *linear σ-models*;
- Invariance of the kinetic term requires that G be a subgroup of $SO(N)$;
- The potential $V(\phi_a)$ must be invariant under the action of G;
- The ground state of $V(\phi_a)$ is assumed to be degenerate; any given minimum ϕ_0 is invariant under a subgroup H of G. (H may be the identity.) The symmetry group G of the Lagrangian is then spontaneously broken to the symmetry group H of the vacuum.

Goldstone's Theorem

The Goldstone bosons associated with the spontaneous symmetry breaking from $G \to H$ are described by the coset space G/H, and their number is $\dim G/H = \dim G - \dim H$.

To prove the theorem, we use the fact that G acts transitively on G/H; i.e. without fixed points. Every point in G/H may be obtained by applying some $g \in G$ to any given reference point in G/H. Actually, this is true for the action of G on G (a special case with $H = 1$). Indeed, take a reference point $g_o$in G then to reach a point $g \in G$, it suffices to multiply to the left by $g\ g_o^{-1}$. Thus, it suffices to analyze the problem around any point on G/H, and then carry the result to all of G/H.

Take the reference point in G/H to be ϕ^o which satisfies

$$\left.\frac{\partial V}{\partial \phi_a}\right|_{\phi_a=\phi_a^o} = 0 \tag{20.42}$$

Now the set of all solutions transform under H. Decompose the Lie algebra $\mathcal{G}$ of G into its subalgebra $\mathcal{H}$ and the orthogonal complement under the Cartan-Killing form, $\mathcal{M}$:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M} \qquad \left\{ \begin{array}{ll} \rho(T^i)\phi^o = 0 & \qquad T^i \in \mathcal{H} \\ \rho(T^j)\phi^o \neq 0 & \qquad T^j \in \mathcal{M} \end{array} \right. \tag{20.43}$$

The Goldstone boson fields now correspond to the directions of the potential from ϕ^o where ϕ^o is charged, but the potential is still at its minimum value. These are precisely the directions in G/H represented here by its tangent space $\mathcal{M}$.

Example 1 : $\phi_a, \quad a = 1, \cdots, N$

$$\mathcal{L} = \frac{1}{2}\sum_a \partial_\mu \phi_a \partial^\mu \phi_a - V(\phi_a\phi_a) \tag{20.44}$$

invariant under $SO(N)$. Assume the minimum of v occurs at $\phi_a\phi_a = v^2 > 0$. The symmetry of the Lagrangian is $G = SO(N)$, but the symmetry of any vacuum configuration is only $H = SO(N-1)$ so that $G/H = S^{N-1}$ and there are $N-1$ Goldstone Bosons.

Example 2 : M real $n \times n$ matrix valued field,

$$\mathcal{L} = \frac{1}{2} \text{ tr } \partial_\mu M^T \partial^\mu M - \frac{1}{2} m^2 \text{tr} M^T M - \frac{\lambda^2}{4!} \text{ tr}(M^T M)^2 \tag{20.45}$$

This Lagrangian is invariant under the following $G = O(n)_L \times O(n)_R$ transformations,

$$M \to g_L M g_R^T \qquad\qquad g_L \in O(n)_L, \ g_R \in O(n)_R \tag{20.46}$$

Any stationnary point M of the potential obeys the equation $(6m^2 I_n + \lambda^2 M^T M) M^T = 0$. When $m^2 > 0$, the solution $M = 0$ is unique and the full G symmetry leaves the vacuum invariant. The interesting case is when $m^2 < 0$. The eigenvalues of $M^T M$ must either vanish or equal v^2, where $6m^2 = -\lambda^2 v^2$. The general solution to this equation may be organized according to the number n_* of vanishing eigenvalues, $M^T M = v^2 I_{n_*}$. For each n_*, the configuration defines a stationnary point of the potential,but only one case $n_* = n$ is an absolute minimum of the potential, which corresponds to the vacuum for the theory. This may be verified by substituting the solutions for $M^T M$ into the potential and evaluating the potential. The general solution for M to the equation $M^T M = v^2 I_n$ is given by $M = vg$ with $g \in SO(n)$, which under an $O(n)_L \times O(n)_R$ transformation is equivalent to $M = vI_n$. The residual symmetry of $M = vI_n$ is calculated as follows,

$$g_L \ (vI_n) \ g_R^T = (vI_n) \qquad \Rightarrow \qquad g_L = g_R \in O(n)_D \tag{20.47}$$

Hence the dynamics of this model shows the following pattern,

$$O(n)_L \times O(n)_R \to O(n)_D + \frac{1}{2} n(n-1) \quad \text{massless G.B.} \tag{20.48}$$

20.6 The non-linear sigma model

Often one is interested in describing *only* the dynammics of the Goldstone bosons at low energy/momentum. All other excitations are then massive and decouple. In this limit, the remaining Lagrangian consists of the kinetic term on the ϕ_a, subject to the constraint that the potential on these fields equals its minimum value,

$$\mathcal{L} = \sum_a \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a \qquad\qquad V(\phi_a) = V(\phi_a^o) \tag{20.49}$$

The Lagrangian is still invariant under G, and so is the condition for the minimum. But G is spontaneously broken to H, and the model only describes the Goldstone bosons.

The true dynamical fields are no longer the fields ϕ_a, since they are related to one another by the constraint $V(\phi_a) = V(\phi_a^o)$. Introducing true dynamical fields may be done by solving the constraint.

It is best to illustrate the procedure with an example. Consider a vector of N real scalar fields $\phi_a,\ a = 1, \cdots, N$ with a quartic potential,

$$V(\phi_a) = \frac{1}{2}m^2(\phi_a\phi_a) + \frac{\lambda}{4!}(\phi_a\phi_a)^2 \qquad v^2 = -\frac{6m^2}{\lambda} > 0 \tag{20.50}$$

Let the vacuum expectation value of ϕ_a be $\phi_a^o = v\delta_{a,N}$. The constraint $V(\phi_a) = V(\phi_a^o)$ is given by $\phi_a\phi_a = v^2$. Eliminating the only non-Goldstone field ϕ_N using $\phi_N = \sqrt{v^2 - \phi_i\phi_i}$, $i = 1, \cdots, N-1$. The resulting Lagrangian is now a *nonlinear sigma model*, with Lagrangian

$$\mathcal{L}_{GB} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} + \frac{1}{2}\frac{(\vec{\phi}\cdot\partial_\mu\vec{\phi})^2}{v^2 - \vec{\phi}^2} \tag{20.51}$$

A number of remarks:

1. $G/H = S^{N-1}$; indeed there are $N-1$ independent fields $\vec{\phi}$.
2. $H = SO(N-1)$ is realized *linearly* on $\vec{\phi}$ by rotations.
3. The Lagrangian is now *non-polynomial* in $\vec{\phi}$.
4. This implies that it is *non-renormalizable* (in $d = 4$). This is no surprise since we discarded higher energy parts of the theory.
5. Before eliminating ϕ_N, the Lagrangian $\mathcal{L}$ was invariant under $SO(N)$. What happened with this symmetry? Has it completely disappeared or is its presence just more difficult to detect ?

We shall now address the last point. Since $SO(N-1)$ leaves ϕ_N invariant, the only concern is with the transformations in $SO(N)$ modulo those in $SO(N-1)$. Using $SO(N-1)$ symmetry, these are equivalent to rotations g between ϕ_1 and ϕ_N, which leave ϕ_i, $i = 2, 3, \cdots, N-1$ invariant,

$$g\begin{pmatrix}\phi_1\\ \phi_N\end{pmatrix} \equiv \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_N\end{pmatrix} \tag{20.52}$$

Since ϕ_N has been eliminated from the Lagrangian $\mathcal{L}_{GB}$, only the action of g on ϕ_1 is of interest in examining the symmetries of $\mathcal{L}_{GB}$,

$$g(\phi_1) = \phi_1\cos\alpha + \sqrt{v^2 - \vec{\phi}^2}\ \ \sin\alpha \tag{20.53}$$

Notice that from this relation the one for ϕ_N follows:

$$g(\phi_N) = \sqrt{v^2 - \vec{\phi}^2 + \phi_1^2 - (\phi_1\cos\alpha + \phi_N\sin\alpha)^2}\ = \phi_N\cos\alpha - \phi_1\sin\alpha \tag{20.54}$$

As a result, the Lagrangian is still invariant $g(\mathcal{L}_{GB}) = \mathcal{L}_{GB}$. But, the novelty is that the symmetry is realized in a *nonlinear* fashion; hence the name.

20.7 The general G/H nonlinear sigma model

The wonderful thing about Goldstone boson dynammics is that, given the symmetries G and H, the corresponding Lagrangian $\mathcal{L}_{GB}$ is unique up to an overall constant. The basic assumption producing this uniqueness is that $\mathcal{L}_{GB}$ has a total of two derivatives. Thus, within this approximation, the dynamics of Goldstone bosons is independent of the precise microscopic theory and mechanism that produced the symmetry breaking and the Goldstone bosons. In other words, given G/H, the Goldstone boson dynamics is *universal*.

The construction is as follows. The group G is realized in terms of matrices for which we still use the notation g. The (independent) Goldstone fields in the theory are taken to be $\varphi_i,\ i = 1, \cdots,\ \dim \mathcal{M} = \dim G - \dim H$. The Lie algebra $\mathcal{G}$ admits the following orthogonal decomposition, $\mathcal{G} = \mathcal{H} + \mathcal{M}$. Instead of working with the independent fields φ_i, it is much more convenient to parametrize the Goldstone fields by the group elements $g \in G$, $g(\varphi_a)$. Of course, g contains too many fields, which will have to be projected out. To do this, one proceeds geometrically.

The combinations $g^{-1}\partial_\mu g$ take values in $\mathcal{G}$. In fact, they are *left-invariant 1-forms on* G, since under left-multiplication of the field g by a constant element $\gamma \in G$ we have,

$$\omega_\mu(g) \equiv g^{-1}\partial_\mu g \qquad \omega_\mu(\gamma g) = g^{-1}\gamma^{-1}\partial_\mu \gamma g = \omega_\mu(g) \tag{20.55}$$

On the other hand, *under right-multiplication by a function* $h \in H$, $\omega_\mu(g)$ *transforms as a connection or gauge field*,

$$\omega_\mu(gh) = h^{-1}g^{-1}\partial_\mu(gh) = h^{-1}\omega_\mu(g)h + h^{-1}\partial_\mu h \tag{20.56}$$

Therefore, the ortogonal projection of $\omega_\mu(g)$ onto $\mathcal{M}$ transforms homogeneously under H,

$$\omega_\mu^{\mathcal{M}}(g) \equiv g^{-1}\partial_\mu g\Big|_{\mathcal{M}} \tag{20.57}$$

and the following universal sigma model Lagrangian is

$$\mathcal{L}_{GB} = \frac{f^2}{2}\mathrm{tr}\left(\omega_\mu^{\mathcal{M}}(g)\omega^{\mathcal{M}\mu}(g)\right) \tag{20.58}$$

is invariant under left multiplication of the field g by constant $\gamma \in G$ and under right multiplication of g by any function in h. Therefore, $\mathcal{L}_{GB}$ precisely describes the dynamics of Goldstone bosons in the right coset G/H. The parameter f is a constant with the dimensions of mass in $d = 4$ but dimension 0 in the important case of $d = 2$.

20.8 Algebra of charges and currents

A local classical Lagrangian is invariant under a continuous symmetry $\delta_a \phi_i$ provided the Lagrangian changes by a total divergence of a local quantity X^μ, i.e. $\delta_a \mathcal{L} = \partial_\mu X^\mu_a$. There exists an associated *conserved current* constructed as follows,

$$j^\mu_a = \sum_i \frac{\partial \mathcal{L}}{\partial_\mu \phi_i} \, \delta_a \phi_i - X^\mu_a \qquad\qquad \partial_\mu j^\mu_a = 0 \tag{20.59}$$

and the associated Noether charge is time-independent,

$$Q_a = \int d^3 x j^o_a \qquad\qquad \dot{Q}_a = 0 \tag{20.60}$$

The commutator with the charge reproduces the field transformation,

$$\delta_a \phi_i(x) = [Q_a, \ \phi_i(x)] \tag{20.61}$$

When the symmetry has no anomalies, all these results generalize to the full QFT. The symmetry charges form a representation of a Lie algebra $\mathcal{G}$ acting on the field operators and on the Hilbert space. We have the following structure relations where f_{abc} are the structure constants of $\mathcal{G}$,

$$\begin{aligned}
[Q_a, Q_b] &= i \, f_{abc} Q_c \\
[Q_a, \ j^\mu_b(x)] &= i \, f_{abc} j^\mu_c(x) \\
[j^o_a(t, \vec{x}), \ \phi_i(t, \vec{y})] &= \delta^{(3)}(\vec{x} - \vec{y}) (T_a)_i{}^j \phi_j(t, \vec{y})
\end{aligned} \tag{20.62}$$

One may ask what the algebra of the currents is. The form of the time components of the currents was conjectured by Gell-Mann; that of a time and a space component of the current was worked out by Schwinger,

$$\begin{aligned}
[j^o_a(t, \vec{x}), \ j^o_b(t, \vec{y})] &= i \, f_{abc} \delta^{(3)}(x, \vec{y}) j^o_c(t, \vec{x}) \\
[j^o_a(t, \vec{x}), \ j^i_b(t, \vec{y})] &= i \, f_{abc} \delta^{(3)}(\vec{x} - \vec{y}) j^i_c(t, \vec{x}) + S^{ij}_{ab} \partial^j \delta^{(3)}(\vec{x} - \vec{y})
\end{aligned} \tag{20.63}$$

S^{ij}_{ab} is a set of constants, referred to as the *Schwinger* terms.

Example 1 : Free massless fermions

$$\mathcal{L} = \sum_{i=1}^N \bar{\psi}^i i \partial\!\!\!/ \psi_i = \sum_{i=1}^N (\bar{\psi}^i_L i \partial\!\!\!/ \psi_{iL} + \bar{\psi}^i_R i \partial\!\!\!/ \psi_{iR}) \tag{20.64}$$

is invariant under

1. $SU(N)_L$: $\quad \psi_{iL} \to \psi'_{iL} = (U_L)_i{}^j \psi_{jL} \qquad U_L^\dagger U_L = I$
   $\quad \psi_{iR} \to \psi_{iR}$

2. $SU(N)_R$: $\quad \psi_{iL} \to \psi_{iL}$

$$\psi_{iR} \to \psi_{iR} = (U_R)_i{}^j \psi_{jR} \qquad U_R^\dagger U_R = I$$

3. $U(1)_L$: $\quad \psi_{iL} \to \psi'_{iL} = e^{i\theta_L}\psi_{iL};$

4. $U(1)_R$: $\quad \psi_{iR} \to \psi'_{iR} = e^{i\theta_R}\psi_{iR};$

The corresponding currents are

$$\begin{aligned}
SU(N)_L &\qquad j_L^{\mu a} \equiv \bar{\psi}_L^i (T^a)_i{}^j \gamma^\mu \psi_{Lj} \\
SU(N)_R &\qquad j_R^{\mu a} \equiv \bar{\psi}_R^i (T^a)_i{}^j \gamma^\mu \psi_{Rj} \\
U(1)_L &\qquad j_L^{\mu} \equiv \bar{\psi}_L^i \gamma^\mu \psi_{Li} \\
U(1)_R &\qquad j_R^{\mu} \equiv \bar{\psi}_R^i \gamma^\mu \psi_{Ri}
\end{aligned} \tag{20.65}$$

Example 2 : Free massive fermions

A non-zero mass term will always spoil the full chiral symmetries. For example, if equal mass $m \neq 0$ is given to all components of ψ,

$$\mathcal{L}_m = \mathcal{L} - \sum_i m\bar{\psi}^i\psi_i \tag{20.66}$$

then $\mathcal{L}_m$ is invariant only under $SU(N)_V \times U(1)_V$, defined by $U_L = U_R$.

Example 3 : Real scalars

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - V(\phi_i\phi_i) \tag{20.67}$$

the Lagrangian is invariant under $SO(N)$ rotations :

$$\delta_a\phi_i(x) = T^a_{ij}\phi_j(x) \qquad T^a_{ij} = -T^a_{ji} \qquad j^\mu_a = \partial_\mu\phi_i T^a_{ij}\phi_j(x) \tag{20.68}$$

20.9 Proof of Goldstone's Theorem

We shall denote the fields collectively as $\phi_i(x)$, and the symmetry acts by

$$\delta_a\phi_i(x) = [Q_a, \phi_i(x)] \tag{20.69}$$

The transformation properties of correlation functions is now readily investigated.

$$\begin{aligned}
\delta_a\langle 0|T\phi_{i_1}(x_1) \quad \cdots \quad & \phi_{i_n}(x_n)|0\rangle \\
= \quad & \langle 0|T\delta_a\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)|0\rangle + \cdots + \langle 0|T\phi_{i_1}(x_1)\cdots\delta_a\phi_{i_n}(x_n)|0\rangle \\
= \quad & \langle 0|T[Q_a,\phi_{i_1}(x_1)]\cdots\phi_{i_n}(x_n)|0\rangle + \cdots + \langle 0|T\phi_{i_1}(x_1)\cdots[Q_a,\phi_{i_n}(x_n)]|0\rangle \\
= \quad & \langle 0|TQ_a\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)|0\rangle - \langle 0|T\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)Q_a|0\rangle
\end{aligned} \tag{20.70}$$

1. If $Q_a|0\rangle = 0$ the symmetry is unbroken, we have

$$\delta_a \langle 0|T\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)|0\rangle = 0 \tag{20.71}$$

2. If $Q_a|0\rangle \neq 0$, there will exist at least one correlator that is not invariant (otherwise one can show that $Q_a = 0$).

So now, let $Q_a|0\rangle \neq 0$, and

$$\delta_a \langle 0|T\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)|0\rangle \neq 0 \tag{20.72}$$

Consider the associated correlator with the current j_a^μ inserted, and take its divergence,

$$\partial_\mu^z \langle 0|Tj_a^\mu(z)\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)|0\rangle \tag{20.73}$$

Since the current is conserved, the only non-zero contributions come from the T product. This may be seen by working out the effect on a single $\phi_{i_k}(x_k)$ as follows,

$$\begin{aligned}
\partial_\mu^z\Big(T\ j_a^\mu(z)\phi_{i_k}(x_k)\Big) &= \partial_\mu^z\Big(\theta(z^o - x_k^o)j_a^\mu(z)\phi_{i_k}(x_k) - \theta(x_k^o - z^o)\phi_i(x_k)j_a^\mu(z)\Big) \\
&= \delta(z^o - x_k^o)[j_a^o(z),\ \phi_{i_k}(x_k)] = \delta^{(4)}(z - x_k)(T_a)_{i_k}^j\phi_j(t_i\vec{x}_k) \\
&= \delta^{(4)}(z - x_k)[Q_a, \phi_{i_k}(x_k)]
\end{aligned} \tag{20.74}$$

Thus

$$\begin{aligned}
&\partial_\mu^z \langle 0|Tj_a^\mu(z)\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)|0\rangle \\
&= \sum_{k=1}^n \delta^{(4)}(z - x_k)\langle 0|T\phi_{i_1}(x_1)\cdots[Q_a, \phi_{i_k}(x_k)]\cdots\phi_{i_n}(x_n)|0\rangle
\end{aligned} \tag{20.75}$$

Fourier transform in z

$$\begin{aligned}
&iq_\mu \langle 0|Tj_a^\mu(q)\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)|0\rangle \\
&= \sum_{k=1}^n e^{-iq\cdot x_k}\langle 0|T\phi_{i_1}(x_1)\cdots[Q_a, \phi_{i_k}(x_k)]\cdots\phi_{i_n}(x_n)|0\rangle
\end{aligned} \tag{20.76}$$

The right-hand side admits a finite, and non-vanishing, limit at $q \to 0$:

$$\lim_{q\to 0} iq_\mu \langle 0|Tj_a^\mu(q)\phi_{i_1}\cdots\phi_{i_n}|0\rangle = \delta_a\langle 0|T\phi_{i_1}\cdots\phi_{i_n}|0\rangle \neq 0 \tag{20.77}$$

How can the left-hand side be $\neq 0$? Only if the correlators

$$\langle 0|Tj_a^\mu(q)\phi_{i_1}\cdots\phi_{i_n}|0\rangle \tag{20.78}$$

has a pole in q as $q \to 0$:

$$\langle 0|Tj_a^\mu(q)\phi_{i_1}\cdots\phi_{i_n}|0\rangle = -i\frac{q^\mu}{q^2}\delta_a\langle 0|T\phi_{i_1}(x_1)\cdots\phi_{i_n}(x_n)|0\rangle + \mathcal{O}(q^0) \tag{20.79}$$

This necessarily signals the presence of a massless bosonic particle with the same internal quantum numbers as the current j_a^μ whose couplings necessarily involve a first derivative.

20.10 The linear sigma model with fermions

This model, invented by Gell-Mann and Levy (1960) provides a simple and beautiful illustration of the spontaneous symmetry breaking mechanisms, both for bosons and for fermions. The model served as a prototype for the later construction of the Weinberg-Salam model (1967).

$$\psi \equiv \begin{pmatrix} p \\ n \end{pmatrix} \qquad\qquad (\pi^+, \pi^o, \pi^-, \sigma) \tag{20.80}$$

Under *strong isospin*, $SU(2)_I$, the (p, n), (π^+, π^o, π^-) and (σ) multiplets transform respectively as a doublet, a triplet and a singlet. The Lagrangian is the sum of the bosonic $\mathcal{L}_\sigma$ and fermionic $\mathcal{L}_\psi$ parts,

$$\begin{aligned}
\mathcal{L}_\sigma &\equiv \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{\lambda^2}{4!}\Big((\sigma^2+\vec{\pi}^2) - f_\pi^2\Big)^2 \\
\mathcal{L}_\psi &= \bar{\psi}\Big(i\,\partial\!\!\!/ + g(\sigma + i\pi^a\tau^a\gamma_5)\Big)\psi
\end{aligned} \tag{20.81}$$

It is convenient to recast this in a more compact and transparent way, by defining the 2×2 matrix-valued fields Σ,

$$\Sigma \equiv \sigma I_2 + i\pi^a\tau^a = \begin{pmatrix} \sigma + i\pi^3 & i\pi^1 + \pi^2 \\ i\pi^1 - \pi^2 & \sigma - i\pi^3 \end{pmatrix} \tag{20.82}$$

Clearly, we have $\Sigma^\dagger\Sigma = (\sigma^2 + \vec{\pi}^2)I_2$, $\det\Sigma = \sigma^2 + \vec{\pi}^2$, and

$$\begin{aligned}
\mathcal{L}_\sigma &= \frac{1}{4}\,\mathrm{tr}\partial_\mu\Sigma^\dagger\partial^\mu\Sigma - \frac{\lambda^2}{48}\,\mathrm{tr}\Big(\Sigma^\dagger\Sigma - f_\pi^2 I_2\Big)^2 \\
\mathcal{L}_\psi &= \bar{\psi}_L i\partial\!\!\!/\psi_L + i\bar{\psi}_R\partial\!\!\!/\psi_R + g\bar{\psi}_R\Sigma^\dagger\psi_L + g\bar{\psi}_L\Sigma\psi_R
\end{aligned} \tag{20.83}$$

Both Lagrangians are manifestly invariant under chiral symmetry:

$$\begin{aligned}
\Sigma &\to g_L\Sigma g_R^+ \qquad\qquad g_L \in SU(2)_L \quad g_R \in SU(2)_R \\
\psi_L &\to g_L\,\psi_L \\
\psi_R &\to g_R\,\psi_R
\end{aligned} \tag{20.84}$$

The diagonal $U(1)_V$ of fermion number (or baryon number for a model of nucleons) acts on both ψ_L and ψ_R but does not act on Σ. The associated currents may be arranged in *vector* and *axial vector* currents:

$$\begin{cases} j_a^\mu = \frac{1}{2}(j_{La}^\mu + j_{Ra}^\mu) = \epsilon_{abc}\pi_b\partial^\mu\pi_c + \bar{\psi}\gamma^\mu\frac{\tau_a}{2}\psi \\ j_{5a}^\mu = \frac{1}{2}(j_{La}^\mu - j_{Ra}^\mu) = \sigma\partial^\mu\pi_a - \pi_a\partial^\mu\sigma + \bar{\psi}\gamma^\mu\gamma_5\frac{\tau_a}{2}\psi \end{cases} \tag{20.85}$$

Notice that σ is a singlet under $SU(2)_V$, so j_a^μ cannot involve σ. Both currents are classically conserved $\partial_\mu j^{\mu a} = \partial_\mu j_5^{\mu a} = 0$, although, depending upon the theory, there may be anomalies in the axial currents.

Unbroken phase: $\langle 0|\sigma|0\rangle = \langle 0|\pi^a|0\rangle = 0 \qquad (-f_\pi^2 \geq 0)$

Then π^a and σ have the same mass^2: $-\frac{\lambda}{12}f_\pi^2$ and ψ is massless.

Broken phase: $\langle 0|\sigma|0\rangle = f_\pi \quad \langle 0|\pi^a|0\rangle = 0.$

- σ-field is massive
- π^a are Goldstone bosons of $SU(2)_L \times SU(2)_R \to SU(2)_V$
- ψ are massive: $M_\psi = g f_\pi$

Finally, it is useful to exhibit the effects of small *explicit* symmetry breaking as well. For example, one may introduce a term that conserves $SU(2)_V$ but explicitly breaks the axial symmetry,

$$\mathcal{L}_b = -c\sigma \qquad\qquad c \quad \text{constant} \tag{20.86}$$

which is *not* invariant under $SU(2)_L \times SU(2)_R$. Thus, the vector current is still conserved, but not the axial vector current is not. Its violation may be evaluated classically,

$$\partial_\mu j^{\mu a} = 0 \qquad \partial_\mu j^\mu_{5a} = c\pi_a \tag{20.87}$$

Before this linear σ-model Lagrangian had been proposed, theories were developed using the dynamics only of symmetry currents. The above results in the linear model were then advocated as basic independent assumptions. These hypotheses went under the names of the *conserved vector current hypothesis* (CVC) and the *partially conserved axial vector current* (PCAC) respectively, and were proposed as such by Feynman and Gell-Mann. Computing the mass of the π^a field proceeds as follows. One starts with the axial vector current,

$$j^\mu_{5a} = \sigma\partial^\mu\pi_a - \pi_a\partial^\mu\sigma + \bar\psi\gamma^\mu\gamma_5\frac{\tau_a}{2}\psi \tag{20.88}$$

Its matrix element between the vacuum and the one-π state is obatined using the approximation that $\langle 0|\sigma\partial^\mu\pi_a|\pi^b(p)\rangle = \langle 0|\sigma|0\rangle\langle 0|\partial^\mu\pi_a|\pi^b(p)\rangle$, so that

$$\langle 0|j^\mu_{5a}(x)|\pi^b(p)\rangle = \delta_{ab} i\ p^\mu f_\pi e^{-ipx} \tag{20.89}$$

Taking the divergence of this equation,

$$\langle 0|\partial^\mu j^\mu_{5a}(x)|\pi_b(p)\rangle = \delta_{ab}\ p^2 f_\pi e^{-ipx} = \langle 0|c\pi_a(x)|\pi_b(p)\rangle = c\ \delta_{ab}\ e^{-ipx} \tag{20.90}$$

As a result, we find

$$m_\pi^2 f_\pi = c \tag{20.91}$$

This expression gives an explicit relation between the mass of the π pseudo-Goldstone bosons and the amount c of explicit symmetry breaking. Clearly, the use of *approximate symmetries* is still very helpful in solving certain quantum field theory problems.

20.11 Topological Defects–Solitons-Vortices

Spontaneous symmetry breaking (either of continuous or discrete symmetries) often implies the existence of topological defects. Let is examine broken ϕ^4-theory in $d = 2$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda^2}{2}(\phi^2 - v^2)^2 \tag{20.92}$$

Look for solutions that are time-independent and interpolate between $\phi \to -v$ and $\phi \to +v$. (cf. instanton problem in $d = 1$.) Again, there is an integral of motion,

$$\phi'' - 2\lambda^2\phi(\phi^2 - v^2) = 0 \qquad \frac{1}{2}\phi'^2 - \frac{\lambda^2}{2}(\phi^2 - v^2)^2 = \text{const.} \tag{20.93}$$

But $\phi \to \pm v$ solution,

$$\frac{d\phi(x)}{\phi^2 - v^2} = \pm\lambda dx \qquad \phi_o(x) \equiv \pm v \tanh v\lambda(x - x_o) \tag{20.94}$$

The energy of the solution is finite:

$$E = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}(\partial_x\phi)^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2\right) = \int_{-\infty}^{\infty} dx \frac{\lambda}{2}(\phi^2 - v^2)^2 \sim \frac{v^3}{\lambda} \tag{20.95}$$

Since the field equations are Poincaré invariant, it is manifest how to construct solutions in motion, moving at velocity u,

$$\phi_v(x,t) = \phi_o\left(\gamma(x - ut)\right) \qquad \gamma = \frac{1}{\sqrt{1 - u^2}} \tag{20.96}$$

This behavior is of course analogous to that of elementary particles. For weak coupling, these solutions correspond to particles with very large masses.

More generally, in dimension $d = 4$, topological defects are classified by the vanishing of a scalar field along some sub-space, as follows,

$$\begin{array}{lll} \phi(\vec{x}) = 0 & \text{at a point:} & \sim \text{ particle} \\ \phi(\vec{x}) = 0 & \text{along line:} & \sim \text{ vortex} \\ \phi(\vec{x}) = 0 & \text{along plane:} & \sim \text{ domain wall} \end{array}$$

21 The Higgs Mechanism

In an Abelian gauge theory, massive gauge particles and fields may be achieved my simply adding a mass term $m^2 A_\mu A^\mu$ to the Lagrangian. This addition is not gauge invariant. Yet, the longitudinal part of the gauge particle decouples from any conserved external current and the on-shell theory is effectively gauge invariant.

In a non-Abelian gauge theory, the addition of a mass term $m^2 A^a_\mu A^{\mu a}$ has a more dramatic effect and the resulting gauge theory fails to be unitary. Therefore, the problem of producing massive gauge fields and particles in a manner consistent with unitarity is an important and incompletely settled problem. The Higgs mechanism is one such way to render gauge particles massive. It requires the addition of elementary scalar fields to the theory. It was invented by Brout, Englert (1960) and Higgs (1960). As we shall discuss later, it is also possible to produce massive gauge particles with scalar fields that are not elementary, but rather arise as composites out of a more basic interaction. This mechanism has a very long history, starting with Anderson (1958), Jackiw and Johnson (1973), Cornwall and Norton (1973), and Weinberg (1976). Susskind revived the mechanism in a simpler and more general setting (1978) and the mechanism is usually referred to as "Technicolor" now.

21.1 The Abelian Higgs mechanism

The simplest model originates in the theory of superconductivity. The Landau-Ginzburg model uses a $U(1)$ gauge field and a complex scalar. The gauge field may be thought of as the photon field while the scalar arises as the order parameter associated with the condensation of Cooper pairs (which are electrically charged). A simple relativistic Lagrangian that encompasses these properties is as follows,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |i\partial_\mu \phi - eA_\mu \phi|^2 - \frac{\lambda^2}{4}(|\phi|^2 - v^2)^2 \tag{21.1}$$

We assume that $\lambda^2, v^2 > 0$. When $e = 0$, the model is easily analyzed, since it separates into a free Maxwell theory and a complex scalar theory which exhibits spontaneous symmetry breaking. The particle contents is thus

1. Two massless photon states;

2. One massless Goldstone scalar;

3. One massive scalar.

The scalar fields do mutually interact.

When $e \neq 0$, the gauge and scalar fields interact, and we want to determine the masses of the various particles. To isolate the Goldstone field, we assume that the scalar takes on a vacuum expectation value in a definite direction of the minimum of the potential, $\langle \emptyset | \phi | \emptyset \rangle = v$, and we introduce associated real fields $\phi_{1,2}$ as follows.

$$\phi = v + \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \langle \emptyset | \phi_{1,2} | \emptyset \rangle = 0 \tag{21.2}$$

In terms of these fields, the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial^\mu \phi_1 - eA_\mu \phi_2)^2 + \frac{1}{2}(\partial^\mu \phi_2 + ev\sqrt{2}A_\mu + eA_\mu \phi_1)^2 \\ & \qquad -\frac{\lambda^2}{4}(\sqrt{2}v\phi_1 + \frac{1}{2}\phi_1^2 + \frac{1}{2}\phi_2^2)^2 \end{aligned} \tag{21.3}$$

The spectrum of masses is governed by the quadratic approximation to the full action, which is readily identified,

$$\mathcal{L}_{\text{quad}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial^\mu \phi_1 \partial_\mu \phi_1 + \frac{1}{2}(\partial^\mu \phi_2 + ev\sqrt{2}A_\mu)^2 - \frac{\lambda^2 v^2}{2}\phi_1^2 \tag{21.4}$$

Effectively, the field ϕ_2 becomes unphysical, as any ϕ_2 bat be changed into any other ϕ_2' by a gauge transformation. One particular gauge choice is $\phi_2 = 0$, in which case, we have

$$\mathcal{L}_{\text{quad}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial^\mu \phi_1 \partial_\mu \phi_1 + e^2 v^2 A_\mu A^\mu - \frac{\lambda^2 v^2}{2}\phi_1^2 \tag{21.5}$$

The gauge field has mass $\sqrt{2}ev$, the scalar ϕ_1 remains a massive scalar and the Goldstone field ϕ_2 has disappeared completely. Actually, to make the gauge field A_μ massive, a third degree of polarization was required and this is provided precisely by the field ϕ_2. One says that *the Goldstone boson has been eaten by the gauge field.*

To avoid the impression that the above conclusions might be t and artifact of the quadratic approximation, we shall treat the problem now more generally. To this end, we use the following decomposition

$$\phi(x) = \rho(x)e^{i\theta(x)} \qquad \langle \emptyset | \rho | \emptyset \rangle = v \tag{21.6}$$

The Lagrangian in terms of these fields is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial^\mu \rho \partial_\mu \rho + (\partial_\mu \theta - eA_\mu)^2 - \frac{\lambda^2}{4}(\rho^2 - v^2)^2 \tag{21.7}$$

It is clear now that, to all orders in perturbation theory, the phase $\theta(x)$ is a gauge artifact and may be rotated away freely.