INSTITUTO SUPERIOR TÉCNICO

Advanced Quantum Field Theory

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Contents

Preface

This is a text for an *Advanced Quantum Field Theory* course that I have been teaching for many years at Instituto Superior Técnico. This course was first written in Portuguese. Then, at a latter stage, I added some text in one-loop techniques in English. Then, I realized that this text could be more useful if it was all in English. As the process of revising the whole text shall take a long time, I decided to make available a *mixed language* text. At the moment, around 60% are in English, the rest in Portuguese. My goal is to change this gradually.

This last semester an effort was made to correct known misprints in the equations before the full translation gets done. However, I am certain that many more still remain. If you find errors or misprints, please send me an email.

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Chapter 1

Free Field Quantization

1.1 General formalism

1.1.1 Canonical quantization for particles

Before we study the canonical quantization of systems with an infinite number of degrees of freedom, as it is the case with fields, we will review briefly the quantization of systems with a finite number of degrees of freedom, like a system of particles.

Let us start with a system that consists of one particle with just one degree of freedom, like a particle moving in one space dimension. The classical equations of motion are obtained from the action,

$$S = \int_{t_1}^{t_2} dt L(q, \dot{q}) \ . \tag{1.1}$$

The condition for the minimization of the action, $\delta S = 0$, gives the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{1.2}$$

which are the equations of motion.

Before proceeding to the quantization, it is convenient to change to the Hamiltonian formulation. We start by defining the conjugate momentum p, to the coordinate q, by

$$p = \frac{\partial L}{\partial \dot{q}} \tag{1.3}$$

Then we introduce the Hamiltonian using the Legendre transform

$$H(p, q) = p\dot{q} - L(q, \dot{q}) \tag{1.4}$$

In terms of H the equations of motion are,

$$\{H, q\}_{\text{PB}} = \frac{\partial H}{\partial p} = \dot{q} \tag{1.5}$$

$$\{H, p\}_{\rm PB} = -\frac{\partial H}{\partial q} = \dot{p} \tag{1.6}$$

where the Poisson Bracket (PB) is defined by

$$\{f(p,q), g(p,q)\}_{\rm PB} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} \tag{1.7}$$

obviously satisfying

$$\{p, q\}_{\rm PB} = 1 \ . \tag{1.8}$$

The quantization is done by promoting p and q to hermitian operators that instead of Eq. (1.8) will satisfy the commutation relation ($\hbar = 1$),

$$[p, q] = -i \tag{1.9}$$

which is trivially satisfied in the coordinate representation where $p = -i\dfrac{\partial}{\partial q}$. The dynamics is the given by the Schrödinger equation

$$H(p,q)\,|\Psi_S(t)\rangle = i\frac{\partial}{\partial t}\,|\Psi_S(t)\rangle \tag{1.10}$$

If we know the state of the system in $t = 0, |\Psi_S(0)\rangle$, then Eq. (1.10) completely determines the state $|\Psi_s(t)\rangle$ and therefore the value of any physical observable. This description, where the states are time dependent and the operators, on the contrary, do not depend on time, is known as the Schrödinger representation. There exits and alternative description, where the time dependence goes to the operators and the states are time independent. This is called the Heisenberg representation. To define this representation, we formally integrate Eq. (1.10) to obtain

$$|\Psi_S(t)\rangle = e^{-iHt}\,|\Psi_S(0)\rangle = e^{-iHt}\,|\Psi_H\rangle \ . \tag{1.11}$$

The state in the Heisenberg representation, $|\Psi_H\rangle$, is defined as the state in the Schrödinger representation for $t = 0$. The unitary operator e^{-iHt} allows us to go from one representation to the other. If we define the operators in the Heisenberg representation as,

$$O_H(t) = e^{iHt} O_S e^{-iHt} \tag{1.12}$$

then the matrix elements are representation independent. In fact,

$$\begin{aligned}\langle \Psi_S(t)|O_S|\Psi_S(t)\rangle &= \left\langle \Psi_S(0)|e^{iHt}O_S e^{-iHt}|\Psi_S(0)\right\rangle &(1.13)\\ &= \langle \Psi_H|O_H(t)|\Psi_H\rangle \ . &(1.14)\end{aligned}$$

The time evolution of the operator $O_H(t)$ is then given by the equation

$$\frac{dO_H(t)}{dt} = i[H, O_H(t)] + \frac{\partial O_H}{\partial t} \tag{1.15}$$

which can easily be obtained from Eq. (1.12). The last term in Eq. (1.15) is only present if O_S explicitly depends on time.

In the non-relativistic theory the difference between the two representations is very small if we work with energy eigenfunctions. If $\psi_n(q,t) = e^{-i\omega_n t} u_n(q)$ is a Schrödinger wave function, then the Heisenberg wave function is simply $u_n(q)$. For the relativistic theory, the Heisenberg representation is more convenient, because it is easier to describe the time evolution of operators than that of states. Also, Lorentz covariance is more easily handled in the Heisenberg representation, because time and spatial coordinates are together in the field operators.

In the Heisenberg representation the fundamental commutation relation is now

$$[p(t), q(t)] = -i \tag{1.16}$$

The dynamics is now given by

$$\frac{dp(t)}{dt} = i[H, p(t)] \quad ; \quad \frac{dq(t)}{dt} = i[H, q(t)] \tag{1.17}$$

Notice that in this representation the fundamental equations are similar to the classical equations with the substitution,

$$\{,\}_{PB} \Longrightarrow i[,] \tag{1.18}$$

In the case of a system with n degrees of freedom Eqs. (1.16) and (1.17) are generalized to

$$[p_i(t), q_j(t)] = -i\delta_{ij} \tag{1.19}$$

$$[p_i(t), p_j(t)] = 0 \tag{1.20}$$

$$[q_i(t), q_j(t)] = 0 \tag{1.21}$$

and

$$\dot{p}_i(t) = i[H, p_i(t)] \ ; \ \dot{q}_i(t) = i[H, q_i(t)] \tag{1.22}$$

Because it is an important example let us look at the harmonic oscillator. The Hamiltonian is

$$H = \frac{1}{2}(p^2 + \omega_0^2 q^2) \tag{1.23}$$

The equations of motion are

$$\dot{p} = i[H, p] = -\omega_0^2 q \tag{1.24}$$

$$\dot{q} = i[H, q] = p \Longrightarrow \ddot{q} + \omega_0^2 q = 0 \ . \tag{1.25}$$

It is convenient to introduce the operators

$$a = \frac{1}{\sqrt{2\omega_0}}(\omega_0 q + ip) \ ; \ a^\dagger = \frac{1}{\sqrt{2\omega_0}}(\omega_0 q - ip) \tag{1.26}$$

The equations of motion for a and $a^\dagger$ are very simple:

$$\dot{a}(t) = -i\omega_0 a(t) \text{ e } \dot{a}^\dagger(t) = i\omega_0 a^\dagger(t) \ . \tag{1.27}$$

They have the solution

$$a(t) = a_0 e^{-i\omega_0 t} \ ; \ a^\dagger(t) = a_0^\dagger e^{i\omega_0 t} \tag{1.28}$$

and obey the commutation relations

$$[a, a^\dagger] = [a_0, a_0^\dagger] = 1 \tag{1.29}$$
$$[a, a] = [a_0, a_0] = 0 \tag{1.30}$$
$$[a^\dagger, a^\dagger] = [a_0^\dagger, a_0^\dagger] = 0 \tag{1.31}$$

In terms of $a, a^\dagger$ the Hamiltonian reads

$$\begin{aligned} H &= \frac{1}{2}\,\omega_0(a^\dagger a + a a^{\dagger)} = \frac{1}{2}\,\omega_0(a_0^\dagger a_0 + a_0 a_0^\dagger) && (1.32)\\ &= \omega_0 a_0^\dagger a_0 + \frac{1}{2}\,\omega_0 && (1.33)\end{aligned}$$

where we have used

$$[H, a_0] = -\omega_0 a_0, \quad [H, a_0^\dagger] = \omega_0 a_0^\dagger \tag{1.34}$$

We see that a_0 decreases the energy of a state by the quantity ω_0 while $a_0^\dagger$ increases the energy by the same amount. As the Hamiltonian is a sum of squares the eigenvalues must be positive. Then it should exist a ground state (state with the lowest energy), $|0\rangle$, defined by the condition

$$a_0 \,|0\rangle = 0 \tag{1.35}$$

The state $|n\rangle$ is obtained by the application of $\left(a_0^\dagger\right)^n$. If we define

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a_0^\dagger\right)^n |0\rangle \tag{1.36}$$

then

$$\langle m|n\rangle = \delta_{mn} \tag{1.37}$$

and

$$H\,|n\rangle = \left(n + \frac{1}{2}\right)\omega_0\,|n\rangle \tag{1.38}$$

We will see that, in the quantum field theory, the equivalent of a_0 and $a_0^\dagger$ are the creation and annihilation operators.

1.1.2 Canonical quantization for fields

Let us move now to field theory, that is, systems with an infinite number of degrees of freedom. To specify the state of the system, we must give for all space-time points one number (or more if we are not dealing with a scalar field). The equivalent of the coordinates $q_i(t)$ and velocities, $\dot{q}_i$, are here the fields $\varphi(\vec{x}, t)$ and their derivatives, $\partial^\mu\varphi(\vec{x}, t)$. The action is now

$$S = \int d^4x \mathcal{L}(\varphi, \partial^\mu\varphi) \tag{1.39}$$

where the Lagrangian density $\mathcal{L}$, is a functional of the fields φ and their derivatives $\partial^\mu\varphi$. Let us consider closed systems for which $\mathcal{L}$ does not depend explicitly on the coordinates x^μ (energy and linear momentum are therefore conserved). For simplicity let us consider systems described by n scalar fields $\varphi_r(x), r = 1, 2, \cdots n$. The stationarity of the action, $\delta S = 0$, implies the equations of motion, the so-called Euler-Lagrange equations,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} - \frac{\partial \mathcal{L}}{\partial \varphi_r} = 0 \qquad r = 1, \cdots n \tag{1.40}$$

For the case of real scalar fields with no interactions that we are considering, we can easily see that the Lagrangian density should be,

$$\mathcal{L} = \sum_{r=1}^{n} \left[\frac{1}{2} \partial^\mu \varphi_r \partial_\mu \varphi_r - \frac{1}{2} m^2 \varphi_r \varphi_r \right] \tag{1.41}$$

in order to obtain the Klein-Gordon equations as the equations of motion,

$$(\Box + m^2)\varphi_r = 0 \quad ; \quad r = 1, \cdots n \tag{1.42}$$

To define the canonical quantization rules we have to change to the Hamiltonian formalism, in particular we need to define the conjugate momentum $\pi(x)$ for the field $\varphi(x)$. To make an analogy with systems with n degrees of freedom, we divide the 3-dimensional space in cells with elementary volume ΔV_i. Then we introduce the coordinate $\varphi_i(t)$ as the average of $\varphi(\vec{x}, t)$ in the volume element ΔV_i, that is,

$$\varphi_i(t) \equiv \frac{1}{\Delta V_i} \int_{(\Delta V_i)} d^3x \varphi(\vec{x}, t) \tag{1.43}$$

and also

$$\dot{\varphi}_i(t) \equiv \frac{1}{\Delta V_i} \int_{(\Delta V_i)} d^3x \dot{\varphi}(\vec{x}, t) \ . \tag{1.44}$$

Then

$$L = \int d^3x \mathcal{L} \to \sum_i \Delta V_i \overline{\mathcal{L}}_i \ . \tag{1.45}$$

Therefore the canonical momentum is now

$$p_i(t) = \frac{\partial L}{\partial \dot{\varphi}_i(t)} = \Delta V_i \frac{\partial \overline{\mathcal{L}}_i}{\partial \dot{\varphi}_i(t)} \equiv \Delta V_i \pi_i(t) \tag{1.46}$$

and the Hamiltonian

$$H = \sum_i p_i \dot{\varphi}_i - L = \sum_i \Delta V_i (\pi_i \dot{\varphi}_i - \overline{\mathcal{L}}_i) \tag{1.47}$$

Going now into the limit of the continuum, we define the conjugate momentum,

$$\pi(\vec{x}, t) \equiv \frac{\partial \mathcal{L}(\varphi, \dot{\varphi})}{\partial \dot{\varphi}(\vec{x}, t)} \tag{1.48}$$

in such a way that its average value in ΔV_i is $\pi_i(t)$ defined in Eq. (1.46). Eq. (1.47) suggests the introduction of an Hamiltonian density such that

$$\begin{aligned} H &= \int d^3x \mathcal{H} && (1.49) \\ \mathcal{H} &= \pi\dot{\varphi} - \mathcal{L} \; . && (1.50) \end{aligned}$$

To define the rules of the canonical quantization we start with the coordinates $\varphi_i(t)$ and conjugate momenta $p_i(t)$. We have

$$\begin{aligned} &[p_i(t), \varphi_j(t)] = -i\delta_{ij} \\ &[\varphi_i(t), \varphi_j(t)] = 0 \\ &[p_i(t), p_j(t)] = 0 \end{aligned} \tag{1.51}$$

In terms of momentum $\pi_i(t)$ we have

$$[\pi_i(t), \varphi_j(t)] = -i\frac{\delta_{ij}}{\Delta V_i} \; . \tag{1.52}$$

Going into the continuum limit, $\Delta V_i \to 0$, we obtain

$$[\varphi(\vec{x}, t), \varphi(\vec{x}', t)] = 0 \tag{1.53}$$

$$[\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0 \tag{1.54}$$

$$[\pi(\vec{x}, t), \varphi(\vec{x}', t)] = -i\delta(\vec{x} - \vec{x}') \tag{1.55}$$

These relations are the basis of the canonical quantization. For the case of n scalar fields, the generalization is:

$$[\varphi_r(\vec{x}, t), \varphi_s(\vec{x}', t)] = 0 \tag{1.56}$$

$$[\pi_r(\vec{x}, t), \pi_s(\vec{x}', t)] = 0 \tag{1.57}$$

$$[\pi_r(\vec{x}, t), \varphi_s(\vec{x}', t)] = -i\delta_{rs}\delta(\vec{x} - \vec{x}') \tag{1.58}$$

where

$$\pi_r(\vec{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_r(\vec{x}, t)} \tag{1.59}$$

and the Hamiltonian is

$$H = \int d^3x \mathcal{H} \tag{1.60}$$

with

$$\mathcal{H} = \sum_{r=1}^{n} \pi_r \dot{\varphi}_r - \mathcal{L} \; . \tag{1.61}$$

1.1.3 Symmetries and conservation laws

The Lagrangian formalism gives us a powerful method to relate symmetries and conservation laws. At the classical level the fundamental result is the following theorem.

Noether's Theorem

To each continuous symmetry transformation that leaves $\mathcal{L}$ and the equations of motion invariant, corresponds one conservation law.

Proof:

Instead of making the proof for all cases, we will consider three very important particular cases:

i) Translations

Let us consider an infinitesimal translation

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu} \tag{1.62}$$

Then

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \varepsilon^{\mu}\frac{\partial\mathcal{L}}{\partial x^{\mu}} \tag{1.63}$$

and $\mathcal{L}'$ leads to the same equations of motion as $\mathcal{L}$, as they differ only by a 4-divergence. If $\mathcal{L}$ is invariant for translations, then Eq. (1.63) tells us that it can not depend explicitly on the coordinates x^{μ}. Therefore

$$\begin{aligned}
\delta\mathcal{L} &= \sum_r \left[\frac{\partial\mathcal{L}}{\partial\varphi_r}\delta\varphi_r + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\delta(\partial_\mu\varphi_r)\right] \\
&= \partial_\mu\left[\sum_r \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\varepsilon^{\nu}\partial_\nu\varphi_r\right]
\end{aligned} \tag{1.64}$$

where we have used the equations of motion, that is Eq. (1.40) and $\delta\varphi_r = \varepsilon^{\nu}\partial_\nu\varphi_r$. From Eqs. (1.63) and (1.64) and using the fact that ε^{μ} is arbitrary we get

$$\partial_\mu T^{\mu\nu} = 0 \tag{1.65}$$

where $T^{\mu\nu}$ is the energy-momentum tensor *defined by*

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \sum_r \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\partial^{\nu}\ \varphi_r \tag{1.66}$$

Using these relations we can define the conserved quantities

$$\begin{aligned}
P^{\mu} &\equiv \int d^3x T^{0\mu} \\
\frac{dP^{\mu}}{dt} &= 0
\end{aligned} \tag{1.67}$$

Noticing that $T^{00} = \mathcal{H}$, it is easy to realize that P^{μ} should be the 4-momentum vector. Therefore we conclude that invariance for translations leads to the conservation of energy and momentum.

ii) Lorentz transformations

Consider the infinitesimal Lorentz transformations

$$x'^\mu = x^\mu + \omega^\mu{}_\nu \, x^\nu \tag{1.68}$$

The transformation (1.68) for the coordinates will induce the following transformation for the fields,

$$\varphi'_r(x') = S_{rs}(\omega)\varphi_s(x) \tag{1.69}$$

For the case of scalar fields $S_{rs} = \delta_{rs}$ and for spinors we know that $S_{rs} = \delta_{rs} + \frac{1}{8}[\gamma_\mu, \gamma_\nu]_{rs}\omega^{\mu\nu}$. In general the variation of φ_r comes from two different effects. We have

$$\begin{aligned} \delta\varphi_r(x) &\equiv \varphi'_r(x) - \varphi_r(x) = S^{-1}_{rs}(\omega)\varphi_s(x') - \varphi_r(x) \\ &= -\frac{1}{2}\omega_{\alpha\beta}\left[(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\delta_{rs} + \Sigma^{\alpha\beta}_{rs}\right]\varphi_s \end{aligned} \tag{1.70}$$

where we have defined

$$S_{rs}(\omega) = \delta_{rs} + \frac{1}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta}_{rs} \ . \tag{1.71}$$

Then

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\delta\varphi_r\right] \tag{1.72}$$

which gives

$$\partial_\mu M^{\mu\alpha\beta} = 0 \tag{1.73}$$

with

$$M^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\Sigma^{\alpha\beta}_{rs}\varphi_s \ . \tag{1.74}$$

The conserved angular momentum is then

$$M^{\alpha\beta} = \int d^3x M^{0\alpha\beta} = \int d^3x \ \left[x^\alpha T^{0\beta} - x^\beta T^{0\alpha} + \sum_{r,s}\pi_r\Sigma^{\alpha\beta}_{rs}\varphi_s\right] \tag{1.75}$$

with

$$\frac{dM^{\alpha\beta}}{dt} = 0 \ . \tag{1.76}$$

iii) Internal symmetries

Let us consider that the Lagrangian is invariant for an infinitesimal internal symmetry transformation

$$\delta\varphi_r(x) = -i\varepsilon\lambda_{rs}\varphi_s(x) \tag{1.77}$$

Then we can easily show that (see Problem 1.2)

$$\partial_\mu J^\mu = 0 \tag{1.78}$$

$$J^\mu = -i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_r)}\lambda_{rs}\varphi_s \tag{1.79}$$

which leads to the conserved charge

$$Q(\lambda) = -i\int d^3x \pi_r \lambda_{rs}\varphi_s \quad ; \quad \frac{dQ}{dt} = 0 \tag{1.80}$$

These relations between symmetries and conservation laws were derived for the classical theory. Let us see now what happens when we quantize the theory. In the quantum theory the fields $\varphi_r(x)$ become operators acting on the Hilbert space of the states. The physical observables are related with the matrix elements of these operators. We have therefore to require Lorentz covariance for those matrix elements. This in turn requires that the operators have to fulfill certain conditions.

This means that the classical fields relation

$$\varphi'_r(x') = S_{rs}(a)\varphi_s(x) \tag{1.81}$$

should be in the quantum theory

$$\left\langle \Phi'_\alpha|\varphi_r(x')|\Phi'_\beta\right\rangle = S_{rs}(a)\left\langle \Phi_\alpha|\varphi_s(x)|\Phi_\beta\right\rangle \tag{1.82}$$

It should exist an unitary transformation $U(a,b)$ that should relate the two inertial frames

$$\left|\Phi'\right\rangle = U(a,b)\left|\Phi\right\rangle \tag{1.83}$$

where $a^\mu{}_\nu$ e b^μ are defined by

$$x'^\mu = a^\mu{}_\nu x^\nu + b^\mu \tag{1.84}$$

Using Eq. (1.83) in Eq. (1.82) we get that the field operators should transform as

$$U(a,b)\varphi_r(x)U^{-1}(a,b) = S^{(-1)}_{rs}(a)\varphi_s(ax+b) \tag{1.85}$$

Let us look at the consequences of this relation for translations and Lorentz transformations. We consider first the translations. Eq. (1.85) is then

$$U(b)\varphi_r(x)U^{-1}(b) = \varphi_r(x+b) \tag{1.86}$$

For infinitesimal translations we can write

$$U(\varepsilon) \equiv e^{i\varepsilon_\mu \mathcal{P}^\mu} \simeq 1 + i\varepsilon_\mu \mathcal{P}^\mu \tag{1.87}$$

where $\mathcal{P}^\mu$ is an hermitian operator. Then Eq. (1.86) gives

$$i[\mathcal{P}^\mu, \varphi_r(x)] = \partial^\mu \varphi_r(x) \tag{1.88}$$

The correspondence with classical mechanics and non relativistic quantum theory suggests that we identify $\mathcal{P}^\mu$ with the 4-momentum, that is, $\mathcal{P}^\mu \equiv P^\mu$ where P^μ has been defined in Eq. (1.67).

As we have an explicit expression for P^μ and we know the commutation relations of the quantum theory, the Eq. (1.88) becomes an additional requirement that the theory has to verify in order to be invariant under translations. We will see explicitly that this is indeed the case for the theories in which we are interested.

For Lorentz transformations $x'^\mu = a^\mu{}_\nu \, x^\nu$, we write for an infinitesimal transformation

$$a^\mu{}_\nu = g^\mu_\nu + \omega^\mu{}_\nu + O(\omega^2) \tag{1.89}$$

and therefore

$$U(\omega) \equiv 1 - \frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu} \tag{1.90}$$

We then obtain from Eq. (1.85) the requirement

$$i[\mathcal{M}^{\mu\nu}, \varphi_r(x)] = x^\mu \partial^\nu \varphi_r - x^\nu \partial^\mu \varphi_r + \Sigma^{\mu\nu}_{rs}\varphi_s(x) \tag{1.91}$$

Once more the classical correspondence lead us to identify $\mathcal{M}^{\mu\nu} = M^{\mu\nu}$ where the angular momentum $M^{\mu\nu}$ is defined in Eq. (1.75). For each theory we will have to verify Eq. (1.91) for the theory to be invariant under Lorentz transformations. We will see that this is true for the cases of interest.

1.2 Quantization of scalar fields

1.2.1 Real scalar field

The real scalar field described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi\varphi \tag{1.92}$$

to which corresponds the Klein-Gordon equation

$$(\Box + m^2)\varphi = 0 \tag{1.93}$$

is the simplest example, and in fact was already used to introduce the general formalism. As we have seen the conjugate momentum is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi} \tag{1.94}$$

and the commutation relations are

$$\begin{aligned}
&[\varphi(\vec{x},t), \varphi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0 \\
&[\pi(\vec{x},t), \varphi(\vec{x}',t)] = -i\delta^3(\vec{x}-\vec{x}')
\end{aligned} \tag{1.95}$$

The Hamiltonian is given by,

$$\begin{aligned}
H &= P^0 = \int d^3x \mathcal{H} \\
&= \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\varphi|^2 + \frac{1}{2}m^2\varphi^2\right]
\end{aligned} \tag{1.96}$$

and the linear momentum is

$$\vec{P} = -\int d^3x \pi \vec{\nabla}\varphi \tag{1.97}$$

Using Eqs. (1.96) and (1.97) it is easy to verify that

$$i[P^\mu, \varphi] = \partial^\mu \varphi \tag{1.98}$$

showing the invariance of the theory for the translations. In the same way we can verify the invariance under Lorentz transformations, Eq. (1.91), with $\Sigma^{\mu\nu}_{rs} = 0$ (spin zero).

In order to define the states of the theory it is convenient to have eigenstates of energy and momentum. To build these states we start by making a spectral Fourier decomposition of $\varphi(\vec{x}, t)$ in plane waves:

$$\varphi(\vec{x}, t) = \int \widetilde{dk} \,\left[a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right] \tag{1.99}$$

where

$$\widetilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2\omega_k} \;;\; \omega_k = +\sqrt{|\vec{k}|^2 + m^2} \tag{1.100}$$

is the Lorentz invariant integration measure. As in the quantum theory φ is an operator, also $a(k)$ e $a^\dagger(k)$ should be operators. As φ is real, then $a^\dagger(k)$ should be the hermitian conjugate to $a(k)$. In order to determine their commutation relations we start by solving Eq. (1.99) in order to $a(k)$ and $a^\dagger(k)$. Using the properties of the delta function, we get

$$\begin{aligned} a(k) &= i\int d^3x e^{ik\cdot x} \overleftrightarrow{\partial}_0 \varphi(x) \\ a^\dagger(k) &= -i\int d^3x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \varphi(x) \end{aligned} \tag{1.101}$$

where we have introduced the notation

$$a\overleftrightarrow{\partial}_0 b = a\frac{\partial b}{\partial t} - \frac{\partial a}{\partial t} b \tag{1.102}$$

The second member of Eq. (1.101) is time independent as can be checked explicitly (see Problem 1.3). This observation is important in order to be able to choose equal times in the commutation relations. We get

$$\begin{aligned} [a(k), a^\dagger(k')] &= \int d^3x \int d^3y \,\left[e^{ik\cdot x}\overleftrightarrow{\partial}_0\varphi(\vec{x}, t), e^{-ik'\cdot y}\overleftrightarrow{\partial}_0\varphi(\vec{y}, t)\right] \\ &= (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}') \end{aligned} \tag{1.103}$$

and

$$[a(k), a(k')] = [a^\dagger(k), a^\dagger(k')] = 0 \tag{1.104}$$

We then see that, except for a small difference in the normalization, $a(k)$ e $a^\dagger(k)$ should be interpreted as annihilation and creation operators of states with momentum k^μ. To show this, we observe that

$$H = \frac{1}{2}\int \widetilde{dk}\; \omega_k \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \tag{1.105}$$

$$\vec{P} = \frac{1}{2}\int \widetilde{dk}\; \vec{k} \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \tag{1.106}$$

Using these explicit forms we can then obtain

$$[P^\mu, a^\dagger(k)] = k^\mu a^\dagger(k) \tag{1.107}$$

$$[P^\mu, a(k)] = -k^\mu a(k) \tag{1.108}$$

showing that $a^\dagger(k)$ adds momentum k^μ and that $a(k)$ destroys momentum k^μ. That the quantization procedure has produced an infinity number of oscillators should come as no surprise. In fact $a(k), a^\dagger(k)$ correspond to the quantization of the normal modes of the classical Klein-Gordon field.

By analogy with the harmonic oscillator, we are now in position of finding the eigenstates of H. We start by defining the base state, that in quantum field theory is called the vacuum. We have

$$a(k)\left|0\right\rangle_k = 0 \quad ; \quad \forall_k \tag{1.109}$$

Then the vacuum, that we will denote by $|0\rangle$, will be formally given by

$$|0\rangle = \Pi_k \left|0\right\rangle_k \tag{1.110}$$

and we will assume that it is normalized, that is $\langle 0|0\rangle = 1$. If now we calculate the vacuum energy, we find immediately the first problem with infinities in Quantum Field Theory (QFT). In fact

$$\begin{aligned}
\langle 0|H|0\rangle &= \frac{1}{2}\int \widetilde{dk}\; \omega_k \left\langle 0\right| \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \left|0\right\rangle \\
&= \frac{1}{2}\int \widetilde{dk}\; \omega_k \left\langle 0\right| \left[a(k), a^\dagger(k)\right] \left|0\right\rangle \\
&= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\; \omega_k (2\pi)^3 2\omega_k \delta^3(0) \\
&= \frac{1}{2}\int d^3k\; \omega_k \delta^3(0) = \infty
\end{aligned} \tag{1.111}$$

This infinity can be understood as the the (infinite) sum of the zero point energy of all quantum oscillators. In the discrete case we would have, $\sum_k \frac{1}{2}\omega_k = \infty$. This infinity can be easily removed. We start by noticing that we only measure energies as differences with respect to the vacuum energy, and those will be finite. We will then define the energy of the vacuum as being zero. Technically this is done as follows. We define a new operator $P^\mu_{N.O.}$ as

$$\begin{aligned}
P^\mu_{N.O.} &\equiv \frac{1}{2}\int \widetilde{dk}\; k^\mu \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \\
&\quad -\frac{1}{2}\int \widetilde{dk}\; k^\mu \left\langle 0\right| \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] \left|0\right\rangle \\
&= \int \widetilde{dk}\; k^\mu a^\dagger(k)a(k)
\end{aligned} \tag{1.112}$$

Now $\left\langle 0|P^\mu_{N.O.}|0\right\rangle = 0$. The ordering of operators where the annihilation operators appear on the right of the creation operators is called *normal ordering* and the usual notation is

$$: \frac{1}{2}(a^\dagger(k)a(k) + a(k)a^\dagger(k)) :\equiv a^\dagger(k)a(k) \tag{1.113}$$

Therefore to remove the infinity of the energy and momentum corresponds to choose the normal ordering to our operators. We will adopt this convention in the following dropping the subscript "$N.O.$" to simplify the notation. This should not appear as an *ad hoc* procedure. In fact, in going from the classical theory where we have products of fields into the quantum theory where the fields are operators, we should have a prescription for the correct ordering of such products. We have just seen that this should be the normal ordering.

Once we have the vacuum we can build the states by applying the the creation operators $a^\dagger(k)$. As in the case of the harmonic oscillator, we can define the *number* operator,

$$N = \int \widetilde{dk}\; a^\dagger(k)a(k) \tag{1.114}$$

It is easy to see that N commutes with H and therefore the eigenstates of H are also eigenstates of N. The state with one particle of momentum k^μ is obtained as $a^\dagger(k)\left|0\right\rangle$. In fact we have

$$\begin{aligned} P^\mu a^\dagger(k)\left|0\right\rangle &= \int \widetilde{dk}' k'^\mu a^\dagger(k')a(k')a^\dagger(k)\left|0\right\rangle \\ &= \int d^3k' k'^\mu \delta^3(\vec{k}-\vec{k}')a^\dagger(k)\left|0\right\rangle \\ &= k^\mu a^\dagger(k)\left|0\right\rangle \end{aligned} \tag{1.115}$$

and

$$Na^\dagger(k)\left|0\right\rangle = a^\dagger(k)\left|0\right\rangle \tag{1.116}$$

In a similar way, the state $a^\dagger(k_1)...a^\dagger(k_n)\left|0\right\rangle$ would be a state with n particles. However, the sates that we have just defined have a problem. They are not normalizable and therefore they can not form a basis for the Hilbert space of the quantum field theory, the so-called Fock space. The origin of the problem is related to the use of plane waves and states with exact momentum. This can be solved forming states that are superpositions of plane waves

$$\left|1\right\rangle = \lambda \int \widetilde{dk} C(k) a^\dagger(k)\left|0\right\rangle \tag{1.117}$$

Then

$$\begin{aligned} \left\langle 1|1\right\rangle &= \lambda^2 \int \widetilde{dk}_1\widetilde{dk}_2 C^*(k_1)C(k_2)\left\langle 0|a(k_1)a^\dagger(k_2)|0\right\rangle \\ &= \lambda^2 \int \widetilde{dk}|C(k)|^2 = 1 \end{aligned} \tag{1.118}$$

and therefore

$$\lambda = \left(\int \widetilde{dk}\; |C(k)|^2\right)^{-1/2} \tag{1.119}$$

with the condition that $\int \widetilde{dk}\, |C(k)|^2 < \infty$. If k is only different from zero in a neighborhood of a given 4-momentum k^μ, then the state will have a well defined momentum (within some experimental error).

A basis for the Fock space can then be constructed from the n–particle normalized states

$$\begin{aligned} |n\rangle &= \left(n! \int \widetilde{dk}_1 \cdots \widetilde{dk}_n |C(k_1, \cdots k_n)|^2 \right)^{-1/2} \\ & \int \widetilde{dk}_1 \cdots \widetilde{dk}_n C(k_1, \cdots k_n) a^\dagger(k_1) \cdots a^\dagger(k_n)\, |0\rangle \end{aligned} \tag{1.120}$$

that satisfy

$$\langle n|n\rangle = 1 \tag{1.121}$$

$$N\,|n\rangle = n\,|n\rangle \tag{1.122}$$

Due to the commutation relations of the operators $a^\dagger(k)$ in Eq. (1.120), the functions $C(k_1 \cdots k_n)$ are symmetric, that is,

$$C(\cdots k_i, \cdots k_j, \cdots) = C(\cdots k_j \cdots k_i \cdots) \tag{1.123}$$

This shows that the quanta that appear in the canonical quantization of real scalar fields obey the Bose–Einstein statistics. This interpretation in terms of particles, with creation and annihilation operators, that results from the canonical quantization, is usually called *second quantization*, as opposed to the description in terms of wave functions (the *first quantization*.

1.2.2 Microscopic causality

Classically, the fields can be measured with an arbitrary precision. In a relativistic quantum theory we have several problems. The first, results from the fact that the fields are now operators. This means that the observables should be connected with the matrix elements of the operators and not with the operators. Besides this question, we can only speak of measuring φ in two space-time points x and y if $[\varphi(x), \varphi(y)]$ vanishes. Let us look at the conditions needed for this to occur.

$$\begin{aligned} [\varphi(x), \varphi(y)] &= \int \widetilde{dk}_1 \widetilde{dk}_2 \left\{ \left[a(k_1), a^\dagger(k_2)\right] e^{-ik_1\cdot x + ik_2 \cdot y} + \left[a^\dagger(k_1), a(k_2)\right] e^{ik_1 x - ik_2\cdot y} \right\} \\ &= \int \widetilde{dk}_1 \left(e^{-ik_1\cdot(x-y)} - e^{ik_1\cdot(x-y)} \right) \\ &\equiv i\Delta(x-y) \end{aligned} \tag{1.124}$$

The function $\Delta(x-y)$ is Lorentz invariant and satisfies the relations

$$(\Box_x + m^2)\Delta(x-y) = 0 \tag{1.125}$$

$$\Delta(x-y) = -\Delta(y-x) \tag{1.126}$$

$$\Delta(\vec{x} - \vec{y}, 0) = 0 \tag{1.127}$$

The last relation ensures that the equal time commutator of two fields vanishes. Lorentz invariance implies then,

$$\Delta(x - y) = 0 \quad ; \quad \forall\ (x - y)^2 < 0 \tag{1.128}$$

This means that for two points that can not be physically connected, that is for which $(x-y)^2 < 0$, the fields interpreted as physical observables, can then be independently measured. This result is known as *Microscopic Causality.* We note that

$$\partial^0 \Delta(x - y)|_{x^0 = y^0} = -\delta^3(\vec{x} - \vec{y}) \tag{1.129}$$

which ensures the canonical commutation relation, Eq. (1.95).

1.2.3 Vacuum fluctuations

It is well known from Quantum Mechanics that, in an harmonic oscillator, the coordinate is not well defined for the energy eigenstates, that is

$$\left\langle n|q^2|n\right\rangle \ > \ (\langle n|q|n\rangle)^2 = 0 \tag{1.130}$$

In Quantum Field Theory, we deal with an infinite set of oscillators, and therefore we will have the same behavior, that is,

$$\langle 0|\varphi(x)\varphi(y)|0\rangle \neq 0 \tag{1.131}$$

although

$$\langle 0|\varphi(x)|0\rangle = 0 \tag{1.132}$$

We can calculate Eq. (1.131). We have

$$\begin{aligned}
\langle 0|\varphi(x)\varphi(y)|0\rangle &= \int \widetilde{dk}_1 \widetilde{dk}_2 e^{-ik_1\cdot x} e^{ik_2\cdot y} \left\langle 0|a(k_1)a^\dagger(k_2)|0\right\rangle \\
&= \int \widetilde{dk}_1 e^{-ik\cdot(x-y)} \equiv \Delta_+(x-y)
\end{aligned} \tag{1.133}$$

The function $\Delta_+(x-y)$ corresponds to the positive frequency part of $\Delta(x-y)$. When $y \to x$ this expression diverges quadratically,

$$\left\langle 0|\varphi^2(x)|0\right\rangle = \Delta_+(0) = \int \widetilde{dk}_1 = \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \tag{1.134}$$

This divergence can not be eliminated in the way we did with the energy of the vacuum. In fact these vacuum fluctuations, as they are known, do have observable consequences like, for instance, the *Lamb shift.* We will be less worried with the result of Eq. (1.134), if we notice that for measuring the square of the operator φ at x we need frequencies arbitrarily large, that is, an infinite amount of energy. Physically only *averages* over a finite space-time region have meaning.

1.2.4 Charged scalar field

The description in terms of real fields does not allow the distinction between particles and anti-particles. It applies only the those cases were the particle and anti-particle are identical, like the π^0. For the more usual case where particles and anti-particles are distinct, it is necessary to have some charge (electric or other) that allows us to distinguish them. For this we need complex fields.

The theory for the scalar complex field can be easily obtained from two real scalar fields φ_1 and φ_2 with the same mass. If we denote the complex field φ by,

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \tag{1.135}$$

then

$$\mathcal{L} = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2) =: \partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi : \tag{1.136}$$

which leads to the equations of motion

$$(\Box + m^2)\varphi = 0 \ ; \ (\Box + m^2)\varphi^\dagger = 0 \tag{1.137}$$

The classical theory given in Eq. (1.136) has, at the classical level, a conserved current, $\partial_\mu J^\mu = 0$, with

$$J^\mu = i\varphi^\dagger \overleftrightarrow{\partial}^\mu \varphi \tag{1.138}$$

Therefore we expect, at the quantum level, the charge Q

$$Q = \int d^3x : i(\varphi^\dagger \dot{\varphi} - \dot{\varphi}^\dagger \varphi) : \tag{1.139}$$

to be conserved, that is, $[H, Q] = 0$. To show this we need to know the commutation relations for the field φ. The definition Eq. (1.135), and the commutation relations for φ_1 and φ_2 allow us to obtain the following relations for φ and $\varphi^\dagger$:

$$[\varphi(x), \varphi(y)] = [\varphi^\dagger(x), \varphi^\dagger(y)] = 0 \tag{1.140}$$

$$[\varphi(x), \varphi^\dagger(y)] = i\Delta(x - y) \tag{1.141}$$

For equal times we can get from Eq. (1.141)

$$[\pi(\vec{x}, t), \varphi(\vec{y}, t)] = [\pi^\dagger(\vec{x}, t), \varphi^\dagger(\vec{y}, t)] = -i\delta^3(\vec{x} - \vec{y}) \tag{1.142}$$

where

$$\pi = \dot{\varphi}^\dagger \quad ; \quad \pi^\dagger = \dot{\varphi} \tag{1.143}$$

The plane waves expansion is then

$$\varphi(x) = \int \widetilde{dk} \ \left[a_+(k) e^{-ik\cdot x} + a_-^\dagger(k) e^{ik\cdot x} \right]$$

$$\varphi^\dagger(x) = \int \widetilde{dk} \ \left[a_-(k) e^{-ik\cdot x} + a_+^\dagger(k) e^{ik\cdot x} \right] \tag{1.144}$$

where the definition of $a_\pm(k)$ is

$$a_\pm(k) = \frac{a_1(k) \pm i a_2(k)}{\sqrt{2}} \ ; \ a_\pm^\dagger = \frac{a_1^\dagger(k) \mp i a_2^\dagger(k)}{\sqrt{2}} \tag{1.145}$$

The algebra of the operators $a_\pm$ it is easily obtained from the algebra of the operators $a_{i'}s$. We get the following non-vanishing commutators:

$$[a_+(k), a_+^\dagger(k')] = [a_-(k), a_-^\dagger(k')] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}') \tag{1.146}$$

therefore allowing us to interpret a_+ and $a_+^\dagger$ as annihilation and creation operators of quanta of type $+$, and similarly for the quanta of type $-$. We can construct the number operators for those quanta:

$$N_\pm = \int \widetilde{dk}\ a_\pm^\dagger(k) a_\pm(k) \tag{1.147}$$

One can easily verify that

$$N_+ + N_- = N_1 + N_2 \tag{1.148}$$

where

$$N_i = \int \widetilde{dk}\ a_i^\dagger(k) a_i(k) \tag{1.149}$$

The energy-momentum operator can be written in terms of the $+$ and $-$ operators,

$$P^\mu = \int \widetilde{dk}\ k^\mu \left[a_+^\dagger(k) a_+(k) + a_-^\dagger(k) a_-(k)\right] \tag{1.150}$$

where we have already considered the normal ordering. Using the decomposition in Eq. (1.144), we obtain for the charge Q:

$$\begin{aligned} Q &= \int d^3x : i(\varphi^\dagger \dot{\varphi} - \dot{\varphi}^\dagger \varphi) : \\ &= \int \widetilde{dk}\ \left[a_+^\dagger(k) a_+(k) - a_-^\dagger(k) a_-(k)\right] \\ &= N_+ - N_- \end{aligned} \tag{1.151}$$

Using the commutation relation in Eq. (1.146) one can easily verify that

$$[H, Q] = 0 \tag{1.152}$$

showing that the charge Q is conserved. The Eq. (1.151) allows us to interpret the $\pm$ quanta as having charge ± 1. However, before introducing interactions, the theory is symmetric, and we can not distinguish between the two types of quanta. From the commutation relations (1.146) we obtain,

$$\begin{aligned} [P^\mu, a_+^\dagger(k)] &= k^\mu a_+^\dagger(k) \\ [Q, a_+^\dagger(k)] &= +a_+^\dagger(k) \end{aligned} \tag{1.153}$$

showing that $a_+^\dagger(k)$ creates a quanta with 4-momentum k^μ and charge $+1$. In a similar way we can show that $a_-^\dagger$ creates a quanta with charge -1 and that $a_\pm(k)$ annihilate quanta of charge ± 1, respectively.

1.2.5 Time ordered product and the Feynman propagator

The operator $\varphi^\dagger$ creates a particle with charge +1 or annihilates a particle with charge −1. In both cases it adds a total charge +1. In a similar way φ annihilates one unit of charge. Let us construct a state of one particle (not normalized) with charge +1 by application of $\varphi^\dagger$ in the vacuum:

$$|\Psi_+(\vec{x},t)\rangle \equiv \varphi^\dagger(\vec{x},t)\,|0\rangle \tag{1.154}$$

The amplitude to propagate the state $|\Psi_+\rangle$ into the future to the point $(\vec{x}',t')$ with $t'>t$ is given by

$$\theta(t'-t)\left\langle\Psi_+(\vec{x}',t')|\Psi_+(\vec{x},t)\right\rangle = \theta(t'-t)\left\langle 0|\varphi(\vec{x}',t')\varphi^\dagger(\vec{x},t)|0\right\rangle \tag{1.155}$$

In $\varphi^\dagger(\vec{x},t)\,|0\rangle$ only the operator $a_+^\dagger(k)$ is active, while in $\langle 0|\,\varphi(\vec{x}',t')$ the same happens to $a_+(k)$. Therefore Eq. (1.155) is the matrix element that creates a quanta of charge +1 in $(\vec{x},t)$ and annihilates it in $(\vec{x}',t')$ with $t'>t$.

There exists another way of increasing the charge by +1 unit in $(\vec{x},t)$ and decreasing it by −1 in $(\vec{x}',t')$. This is achieved if we create a quanta of charge −1 in $\vec{x}'$ at time t' and let it propagate to $\vec{x}$ where it is absorbed at time $t>t'$. The amplitude is then,

$$\theta(t-t')\left\langle\Psi_-(\vec{x},t)|\Psi_-(\vec{x}',t')\right\rangle = \left\langle 0|\varphi^\dagger(\vec{x},t)\varphi(\vec{x}',t')|0\right\rangle\theta(t-t') \tag{1.156}$$

Since we can not distinguish the two paths we must sum of the two amplitudes in Eqs. (1.155) and (1.156). This is the so-called *Feynman propagator*. It can be written in a more compact way if we introduce the time ordered product. Given two operators $a(x)$ and $b(x')$ we define the time ordered product T by,

$$T a(x)b(x') = \theta(t-t')a(x)b(x') + \theta(t'-t)b(x')a(x) \tag{1.157}$$

In this prescription the older times are always to the right of the more recent times. It can be applied to an arbitrary number of operators. With this definition, the Feynman propagator reads,

$$\Delta_F(x'-x) = \left\langle 0|T\varphi(x')\varphi^\dagger(x)|0\right\rangle \tag{1.158}$$

Using the φ and $\varphi^\dagger$ decomposition we can calculate Δ_F (for free fields, of course)

$$\begin{aligned}
\Delta_F(x'-x) &= \int \widetilde{dk}\,\left[\theta(t'-t)e^{-ik\cdot(x'-x)} + \theta(t-t')e^{ik\cdot(x'-x)}\right] && (1.159)\\
&= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik\cdot(x'-x)} && (1.160)\\
&\equiv \int \frac{d^4k}{(2\pi)^4}\Delta_F(k)e^{-ik\cdot(x'-x)}
\end{aligned}$$

where

$$\Delta_F(k) \equiv \frac{i}{k^2-m^2+i\varepsilon} \tag{1.161}$$

$\Delta_F(k)$ is the propagator in momenta space (Fourier transform). The equivalence between Eq. (1.160) and Eq. (1.159) is done using integration in the complex plane of the

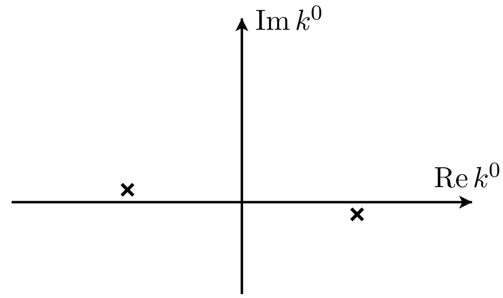


Figure 1.1: Integration in the complex k^0 plane.

time component k^0, with the help of the residue theorem. The contour is defined by the $i\varepsilon$ prescription, as indicated in Fig. (1.1). Applying the operator $(\Box'_x + m^2)$ to $\Delta_F(x'-x)$ in any of the forms of Eq. (1.159) one can show that

$$(\Box'_x + m^2)\Delta_F(x'-x) = -i\delta^4(x'-x) \tag{1.162}$$

that is, $\Delta_F(x'-x)$ is the Green's function for the Klein-Gordon equation with Feynman boundary conditions.

In the presence of interactions, Feynman propagator looses the simple form of Eq. (1.161). However, as we will see, the free propagator plays a key role in perturbation theory.

1.3 Second quantization of the Dirac field

Let us now apply the formalism of second quantization to the Dirac field. As we will see, something has to be changed, otherwise we would be led to a theory obeying Bose statistics, while we know that electrons have spin 1/2 and obey Fermi statistics.

1.3.1 Canonical formalism for the Dirac field

The Lagrangian density that leads to the Dirac equation is

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi \tag{1.163}$$

The conjugate momentum to ψ_α is

$$\pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha} = i\psi^\dagger_\alpha \tag{1.164}$$

while the conjugate momentum to $\psi^\dagger_\alpha$ vanishes. The Hamiltonian density is then

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi \tag{1.165}$$

The requirement of translational and Lorentz invariance for $\mathcal{L}$ leads to the tensors $T^{\mu\nu}$ and $M^{\mu\nu\lambda}$. Using the obvious generalizations of Eqs. (1.66) and (1.74) we get

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\mathcal{L} \tag{1.166}$$

and

$$M^{\mu\nu\lambda} = i\overline{\psi}\gamma^\mu(x^\nu\partial^\lambda - x^\lambda\partial^\nu + \sigma^{\nu\lambda})\psi - \left(x^\nu g^{\mu\lambda} - x^\mu g^{\nu\lambda}\right)\mathcal{L} \tag{1.167}$$

where

$$\sigma^{\nu\lambda} = \frac{1}{4}[\gamma^\nu, \gamma^\lambda] \tag{1.168}$$

The 4-momentum P^μ and the angular momentum tensor $M^{\nu\lambda}$ are then given by,

$$\begin{aligned} P^\mu &\equiv \int d^3x T^{0\mu} \\ M^{\mu\lambda} &\equiv \int d^3x M^{0\nu\lambda} \end{aligned} \tag{1.169}$$

or

$$\begin{aligned} H &\equiv \int d^3x \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi \\ \vec{P} &\equiv \int d^3x \psi^\dagger(-i\vec{\nabla})\psi \end{aligned} \tag{1.170}$$

If we define the angular momentum vector $\vec{J} \equiv (M^{23}, M^{31}, M^{12})$ we get

$$\vec{J} = \int d^3x \psi^\dagger \left(\vec{r}\times\frac{1}{i}\vec{\nabla} + \frac{1}{2}\vec{\Sigma}\right)\psi \tag{1.171}$$

which has the familiar aspect $\vec{J} = \vec{L} + \vec{S}$. We can also identify a conserved current, $\partial_\mu j^\mu = 0$, with $j^\mu = \overline{\psi}\gamma^\mu\psi$, which will give the conserved charge

$$Q = \int d^3x \psi^\dagger\psi \tag{1.172}$$

All that we have done so far is at the classical level. To apply the canonical formalism we have to enforce commutation relations and verify the Lorentz invariance of the theory. This will lead us into problems. To see what are the problems and how to solve them, we will introduce the plane wave expansions,

$$\psi(x) = \int \widetilde{dp}\,\sum_s \left[b(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v(p,s)e^{ip\cdot x}\right] \tag{1.173}$$

$$\psi^\dagger(x) = \int \widetilde{dp}\,\sum_s \left[b^\dagger(p,s)u^\dagger(p,s)e^{+ip\cdot x} + d(p,s)v^\dagger(p,s)e^{-ip\cdot x}\right] \tag{1.174}$$

where $u(p,s)$ and $v(p,s)$ are the spinors for positive and negative energy, respectively, introduced in the study of the Dirac equation and $b, b^\dagger, d$ and $d^\dagger$ are operators. To see what are the problems with the canonical quantization of fermions, let us calculate P^μ. We get

$$P^\mu = \int \widetilde{dk}\; k^\mu \sum_s \left[b^\dagger(k,s)b(k,s) - d(k,s)d^\dagger(k,s)\right] \tag{1.175}$$

where we have used the orthogonality and closure relations for the spinors $u(p,s)$ and $v(p,s)$. From Eq. (1.175) we realize that if we define the vacuum as $b(k,s)\left|0\right\rangle = d(k,s)\left|0\right\rangle = 0$ and if we quantize with commutators then particles b and particles d will contribute with opposite signs to the energy and the theory will not have a stable ground state. In fact, this was the problem already encountered in the study of the negative energy solutions of the Dirac equation, and this is the reason for the negative sign in Eq. (1.175). Dirac's hole theory required Fermi statistics for the electrons and we will see how spin and statistics are related.

To discover what are the relations that $b, b^\dagger, d$ and $d^\dagger$ should obey, we recall that at the quantum level it is always necessary to verify Lorentz invariance. This gives,

$$i[P_\mu, \psi(x)] = \partial_\mu \psi \ ; \ i[P_\mu, \overline{\psi}(x)] = \partial_\mu \overline{\psi} \tag{1.176}$$

Instead of imposing canonical quantization commutators and, as a consequence, verifying Eq. (1.176) we will do the other way around. We start with Eq. (1.176) and we will discover the appropriate relations for the operators. Using the expansions Eqs. (1.173) and (1.174) we can show that Eq. (1.176) leads to

$$[P_\mu, b(k,s)] = -k_\mu b(k,s) \ ; \ [P_\mu, b^\dagger(k,s)] = k_\mu b^\dagger(k,s) \tag{1.177}$$

$$[P_\mu, d(k,s)] = -k_\mu d(k,s) \ ; \ [P_\mu, d^\dagger(k,s)] = k_\mu d^\dagger(k,s) \tag{1.178}$$

Using Eq. (1.175) for P_μ we get

$$\sum_{s'} \left[\left(b^\dagger(p,s')b(p,s') - d(p,s')d^\dagger(p,s') \right), b(k,s) \right] = -(2\pi)^3 2k^0 \delta^3(\vec{k} - \vec{p}) b(k,s) \tag{1.179}$$

and three other similar relations. If we assume that

$$[d^\dagger(p,s')d(p,s'), b(k,s)] = 0 \tag{1.180}$$

the condition from Eq. (1.179) reads

$$\begin{aligned} \sum_{s'} \Big[b^\dagger(p,s')\{b(p,s'), b(k,s)\} - \{b^\dagger(p,s'), b(k,s)\} b(p,s') \Big] &= \\ = -(2\pi)^3 2k^0 \delta^3(\vec{p} - \vec{k}) b(k,s) & \end{aligned} \tag{1.181}$$

where the parenthesis $\{,\}$ denote anti-commutators. It is easy to see that Eq. (1.181) is verified if we impose the canonical commutation relations. We should have

$$\begin{aligned} \{b^\dagger(p,s), b(k,s)\} &= (2\pi)^3 2k^0 \delta^3(\vec{p} - \vec{k}) \delta_{ss'} \\ \{d^\dagger(p,s'), d(k,s)\} &= (2\pi)^3 2k^0 \delta^3(\vec{p} - \vec{k}) \delta_{ss'} \end{aligned} \tag{1.182}$$

and all the other anti-commutators vanish. Note that as b anti-commutes with d and $d^\dagger$, then it commutes with $d^\dagger d$ and therefore Eq. (1.180) is verified.

With the anti-commutator relations both contributions to P^μ in Eq. (1.175) are positive. As in boson case we have to subtract the zero point energy. This is done, as usual, by taking all quantities normal ordered. Therefore we have for P^μ,

$$P^\mu \ = \ \int \widetilde{dk} \ k^\mu \sum_s : \left(b^\dagger(k,s)b(k,s) - d(k,s)d^\dagger(k,s) \right) :$$

$$= \int \widetilde{dk}\ k^\mu \sum_s : \Big(b^\dagger(k,s)b(k,s) + d^\dagger(k,s)d(k,s)\Big) : \qquad (1.183)$$

and for the charge

$$\begin{aligned} Q &= \int d^3x : \psi^\dagger(x)\psi(x) : \\ &= \int \widetilde{dk}\ \sum_s \Big[b^\dagger(k,s)b(k,s) - d^\dagger(k,s)d(k,s)\Big] \end{aligned} \qquad (1.184)$$

which means that the quanta of b type have charge $+1$ while those of d type have charge -1. It is interesting to note that was the second quantization of the Dirac field that introduced the $-$ sign in Eq. (1.184), making the charge operator without a definite sign, while in Dirac theory was the probability density that was positive defined. The reverse is true for bosons. We can easily show that

$$\begin{array}{ll} [Q,b^\dagger(k,s)] = b^\dagger(k,s) & [Q,d(k,s)] = d(k,s) \\ [Q,b(k,s)] = -b(k,s) & [Q,d^\dagger(k,s)] = -d^\dagger(k,s) \end{array}$$

and then

$$[Q,\psi] = -\psi\ ;\ [Q,\overline{\psi}] = \overline{\psi} \qquad (1.185)$$

In `QED` the charge is given by eQ ($e<0$). Therefore we see that ψ creates positrons and annihilates electrons and the opposite happens with $\overline{\psi}$.

We can introduce the number operators

$$N^+(p,s) = b^\dagger(p,s)b(p,s) \quad ; \quad N^-(p,s) = d^\dagger(p,s)d(p,s) \qquad (1.186)$$

and we can rewrite

$$P^\mu = \int \widetilde{dk}\ k^\mu \sum_s (N^+(k,s) + N^-(k,s)) \qquad (1.187)$$

$$Q = \int \widetilde{dk}\ \sum_s (N^+(k,s) - N^-(k,s)) \qquad (1.188)$$

Using the anti-commutator relations in Eq. (1.182) it is now easy to verify that the theory is Lorentz invariant, that is (see Problem 1.4),

$$i[M^{\mu\nu},\psi] = (x^\mu\partial^\nu - x^\nu\partial^\mu)\psi + \Sigma^{\mu\nu}\psi\ . \qquad (1.189)$$

1.3.2 Microscopic causality

The anti-commutation relations in Eq. (1.182) can be used to find the anti-commutation relations at equal times for the fields. We get

$$\{\psi_\alpha(\vec{x},t), \psi^\dagger_\beta(\vec{y},t)\} = \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta} \qquad (1.190)$$

and

$$\{\psi_\alpha(\vec{x},t), \psi_\beta(\vec{y},t)\} = \{\psi^\dagger_\alpha(\vec{x},t), \psi^\dagger_\beta(\vec{y},t)\} = 0 \qquad (1.191)$$

These relations can be generalized to unequal times

$$\begin{aligned}
\{\psi_\alpha(x), \psi^\dagger_\beta(y)\} &= \int \widetilde{dp} \left[\left[(\not{p}+m)\gamma^0\right]_{\alpha\beta} e^{-ip\cdot(x-y)} - \left[(-\not{p}+m)\gamma^0\right]_{\alpha\beta} e^{ip\cdot(x-y)} \right] \\
&= \left[(i\not{\partial}_x + m)\gamma^0\right]_{\alpha\beta} i\Delta(x-y)
\end{aligned} \tag{1.192}$$

where the $\Delta(x-y)$ function was defined in Eq. (1.124) for the scalar field. The fact that γ^0 appears in Eq. (1.192) is due to the fact that in Eq. (1.192) we took $\psi^\dagger$ and not $\overline{\psi}$. In fact, if we multiply on the right by γ^0 we get

$$\{\psi_\alpha(x), \overline{\psi}_\beta(y)\} = (i\not{\partial}_x + m)_{\alpha\beta} i\Delta(x-y) \tag{1.193}$$

and

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\overline{\psi}_\alpha(x), \overline{\psi}_\beta(y)\} = 0 \tag{1.194}$$

We can easily verify the covariance of Eq. (1.193). We use

$$\begin{aligned}
&U(a,b)\psi(x)U^{-1}(a,b) = S^{-1}(a)\psi(ax+b) \\
&U(a,b)\overline{\psi}(x)U^{-1}(a,b) = \overline{\psi}(ax+b)S(a) \\
&S^{-1}\gamma^\mu S = a^\mu{}_\nu \gamma^\nu
\end{aligned} \tag{1.195}$$

to get

$$\begin{aligned}
&U(a,b)\{\psi_\alpha(x), \overline{\psi}_\beta(y)\}U^{-1}(a,b) = \\
&\qquad = S^{-1}_{\alpha\tau}(a)\{\psi_\tau(ax+b), \overline{\psi}_\lambda(ay+b)\}S_{\lambda\beta}(a) \\
&\qquad = S^{-1}_{\alpha\tau}(a)(i\not{\partial}_{ax}+m)_{\tau\lambda} i\Delta(ax-ay)S_{\lambda\beta}(a) \\
&\qquad = (i\not{\partial}+m)_{\alpha\beta} i\Delta(x-y)
\end{aligned} \tag{1.196}$$

where we have used the invariance of $\Delta(x-y)$ and the result $S^{-1} i\not{\partial}_{ax} S = i\not{\partial}_x$. For $(x-y)^2 < 0$ the anti-commutators vanish, because $\Delta(x-y)$ also vanishes. This result allows us to show that any two observables built as bilinear products of $\overline{\psi}$ e ψ commute for two spacetime points for which $(x-y)^2 < 0$. Therefore

$$\begin{aligned}
&\left[\overline{\psi}_\alpha(x)\psi_\beta(x), \overline{\psi}_\lambda(y)\psi_\tau(y)\right] = \\
&\qquad = \overline{\psi}_\alpha(x)\{\psi_\beta(x), \overline{\psi}_\lambda(y)\}\psi_\tau(y) - \{\overline{\psi}_\alpha(x), \overline{\psi}_\lambda(y)\}\psi_\beta(x)\psi_\tau(y) \\
&\qquad\quad + \overline{\psi}_\lambda(y)\overline{\psi}_\alpha(x)\{\psi_\beta(x), \psi_\tau(y)\} - \overline{\psi}_\lambda(y)\{\psi_\tau(y), \overline{\psi}_\alpha(x)\}\psi_\beta(x) \\
&\qquad = 0
\end{aligned} \tag{1.197}$$

for $(x-y)^2 < 0$. In this way the microscopic causality is satisfied for the physical observables, such as the charge density or the momentum density.

1.3.3 Feynman propagator

For the Dirac field, as in the case of the charged scalar field, there are two ways of increasing the charge by one unit in x' and decrease it by one unit in x (note that the electron has negative charge). These ways are

$$\theta(t'-t)\left\langle 0|\psi_\beta(x')\psi^\dagger_\alpha(x)|0\right\rangle \tag{1.198}$$

$$\theta(t-t')\left\langle 0|\psi^\dagger_\alpha(x)\psi_\beta(x')|0\right\rangle \tag{1.199}$$

In Eq. (1.198) an electron of positive energy is created at $\vec{x}$ in the instant t, propagates until $\vec{x}'$ where is annihilated at time $t' > t$. In Eq. (1.199) a positron of positive energy is created in x' and annihilated at x with $t > t'$. The Feynman propagator is obtained summing the two amplitudes. Due the exchange of ψ_β and $\overline{\psi}_\alpha$ there must be a minus sign between these two amplitudes. Multiplying by γ^0, in order to get $\overline{\psi}$ instead of $\psi^\dagger$, we get for the Feynman propagator,

$$\begin{aligned} S_F(x'-x)_{\alpha\beta} &= \theta(t'-t)\left\langle 0|\psi_\alpha(x')\overline{\psi}_\beta(x)|0\right\rangle \\ &\quad -\theta(t-t')\left\langle 0|\overline{\psi}_\beta(x)\psi_\alpha(x')|0\right\rangle \\ &\equiv \left\langle 0|T\psi_\alpha(x')\overline{\psi}_\beta(x)|0\right\rangle \end{aligned} \tag{1.200}$$

where we have defined the time ordered product for fermion fields,

$$T\eta(x)\chi(y) \equiv \theta(x^0-y^0)\eta(x)\chi(y) - \theta(y^0-x^0)\chi(y)\eta(x)\ . \tag{1.201}$$

Inserting in Eq. (1.200) the expansions for ψ and $\overline{\psi}$ we get,

$$\begin{aligned} S_F(x'-x)_{\alpha\beta} &= \int \widetilde{dk}\ \left[(\not{k}+m)_{\alpha\beta}\theta(t'-t)e^{-ik\cdot(x'-x)} + (-\not{k}+m)_{\alpha\beta}\theta(t-t')e^{ik\cdot(x'-x)}\right] \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m)_{\alpha\beta}}{k^2-m^2+i\varepsilon}e^{-ik\cdot(x'-x)} \\ &\equiv \int \frac{d^4k}{(2\pi)^4} S_F(k)_{\alpha\beta}e^{-ik\cdot(x'-x)} \end{aligned} \tag{1.202}$$

where $S_F(k)$ is the Feynman propagator in momenta space. We can also verify that Feynman's propagator is the Green function for the Dirac equation, that is (see Problem 1.7),

$$(i\not{\partial}-m)_{\lambda\alpha}\, S_F(x'-x)_{\alpha\beta} = i\delta_{\lambda\beta}\delta^4(x'-x) \tag{1.203}$$

1.4 Electromagnetic field quantization

1.4.1 Introduction

The free electromagnetic field is described by the classical Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.204}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.205}$$

The free field Maxwell equations are

$$\partial_\alpha F^{\alpha\beta} = 0 \tag{1.206}$$

that corresponds to the usual equations in 3-vector notation,

$$\vec{\nabla} \cdot \vec{E} = 0 \quad ; \quad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} \tag{1.207}$$

The other Maxwell equations are a consequence of Eq. (1.205) and can written as,

$$\partial_\alpha \widetilde{F}^{\alpha\beta} = 0 \quad ; \quad \widetilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \tag{1.208}$$

corresponding to

$$\vec{\nabla} \cdot \vec{B} = 0 \quad ; \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.209}$$

Classically, the quantities with physical significance are the fields $\vec{E}$ e $\vec{B}$, and the potentials A^μ are auxiliary quantities that are not unique due to the gauge invariance of the theory. In quantum theory the potentials A_μ are the ones playing the leading role as, for instance in the minimal prescription. We have therefore to formulate the quantum fields theory in terms of A^μ and not of $\vec{E}$ and $\vec{B}$.

When we try to apply the canonical quantization to the potentials A^μ we immediately run into difficulties. For instance, if we define the conjugate momentum as,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\dot{A}_{\mu)}} \tag{1.210}$$

we get

$$\begin{aligned} \pi^k &= \frac{\partial \mathcal{L}}{\partial(\dot{A}_k)} = -\dot{A}^k - \frac{\partial A^0}{\partial x^k} = E^k \\ \pi^0 &= \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \end{aligned} \tag{1.211}$$

Therefore the conjugate momentum to the coordinate A^0 vanishes and does not allow us to use directly the canonical formalism. The problem has its origin in the fact that the photon, that we want to describe, has only two degrees of freedom (positive or negative helicity) but we are using a field A^μ with four degrees of freedom. In fact, we have to impose constraints on A^μ in such a way that it describes the photon. This problem can be addressed in three different ways:

i) *Radiation Gauge*

Historically, this was the first method to be used. It is based in the fact that it is always possible to choose a gauge, called the *radiation gauge*, where

$$A^0 = 0 \quad ; \quad \vec{\nabla} \cdot \vec{A} = 0 \tag{1.212}$$

that is, the potential $\vec{A}$ is transverse. The conditions in Eq. (1.212) reduce the number of degrees of freedom to two, the transverse components of $\vec{A}$. It is then possible to apply the canonical formalism to these transverse components and quantize the electromagnetic field in this way. The problem with this method is that we loose explicit Lorentz covariance. It is then necessary to show that this is recovered in the final result. This method is followed in many text books, for instance in Bjorken and Drell [1].

ii) *Quantization of systems with constraints*

It can be shown that the electromagnetism is an example of an Hamilton generalized system, that is a system where there are constraints among the variables. The way to quantize these systems was developed by Dirac for systems of particles with n degrees of freedom. The generalization to quantum field theories is done using the formalism of path integrals. We will study this method in Chapter 6, as it will be shown, this is the only method that can be applied to non-abelian gauge theories, like the Standard Model.

iii) *Undefined metric formalism*

There is another method that works for the electromagnetism, called the formalism of the *undefined metric*, developed by Gupta and Bleuler [2, 3]. In this formalism, that we will study below, Lorentz covariance is kept, that is we will always work with the 4-vector A_μ, but the price to pay is the appearance of states with negative norm. We have then to define the Hilbert space of the physical states as a sub-space where the norm is positive. We see that in all cases, in order to maintain the explicit Lorentz covariance, we have to complicate the formalism. We will follow the book of Silvan Schweber [4].

1.4.2 Undefined metric formalism

To solve the difficulty of the vanishing of π^0, we will start by modifying the Maxwell Lagrangian introducing a new term,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2 \tag{1.213}$$

where ξ is a dimensionless parameter. The equations of motion are now,

$$\Box A^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\mu (\partial \cdot A) = 0 \tag{1.214}$$

and the conjugate momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} - \frac{1}{\xi} g^{\mu 0} (\partial \cdot A) \tag{1.215}$$

that is

$$\left\{ \begin{array}{ll} \pi^0 & = -\frac{1}{\xi}(\partial \cdot A) \\ \pi^k & = E^k \end{array} \right.$$

We remark that the Lagrangian of Eq. (1.212) and the equations of motion, Eq. (1.214), reduce to Maxwell theory in the gauge $\partial \cdot A = 0$. This why we say that the choice of Eq. (1.212) corresponds to a class of Lorenz gauges with parameter ξ. With this abuse of language (in fact we are not setting $\partial \cdot A = 0$, otherwise the problems would come back) the value of $\xi = 1$ is known as the *Feynman gauge* and $\xi = 0$ as the *Landau gauge*.

From Eq. (1.214) we get

$$\Box(\partial \cdot A) = 0 \tag{1.216}$$

implying that $(\partial \cdot A)$ is a massless scalar field. Although it would be possible to continue with a general ξ, from now on we will take the case of the so-called *Feynman gauge*, where $\xi = 1$. Then the equation of motion coincides with the Maxwell theory in the Lorenz gauge. As we do not have anymore $\pi^0 = 0$, we can impose the canonical commutation relations at equal times:

$$\begin{aligned}
&[\pi^\mu(\vec{x},t), A_\nu(\vec{y},t)] = -ig^\mu{}_\nu\, \delta^3(\vec{x}-\vec{y}) \\
&[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = [\pi_\mu(\vec{x},t), \pi_\nu(\vec{y},t)] = 0
\end{aligned} \tag{1.217}$$

Knowing that $[A_\mu(\vec{x},t), A_\mu(\vec{y},t)] = 0$ at equal times, we can conclude that the space derivatives of A_μ also commute at equal times. Then, noticing that

$$\pi^\mu = -\dot{A}^\mu + \text{ space derivatives} \tag{1.218}$$

we can write instead of Eq. (1.217)

$$\begin{aligned}
&[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = [\dot{A}_\mu(\vec{x},t), \dot{A}_\mu(\vec{y},t)] = 0 \\
&[\dot{A}_\mu(\vec{x},t), A_\nu(\vec{y},t)] = ig_{\mu\nu}\delta^3(\vec{x}-\vec{y})
\end{aligned} \tag{1.219}$$

If we compare these relations with the corresponding ones for the real scalar field, where the only one non-vanishing is,

$$[\dot{\varphi}(\vec{x},t), \varphi(\vec{y},t)] = -i\delta^3(\vec{x}-\vec{y}) \tag{1.220}$$

we see ($g_{\mu\nu} = \text{diag}(+,-,-,-)$ that the relations for space components are equal but they differ for the time component. This sign will be the source of the difficulties previously mentioned.

If, for the moment, we do not worry about this sign, we expand $A_\mu(x)$ in plane waves,

$$A^\mu(x) = \int \widetilde{dk} \sum_{\lambda=0}^{3} \left[a(k,\lambda)\varepsilon^\mu(k,\lambda)e^{-ik\cdot x} + a^\dagger(k,\lambda)\varepsilon^{\mu *}(k,\lambda)e^{ik\cdot x}\right] \tag{1.221}$$

where $\varepsilon^\mu(k,\lambda)$ are a set of four independent 4-vectors that we assume to real, without loss of generality. We will now make a choice for these 4-vectors. We choose $\varepsilon^\mu(1)$ and $\varepsilon^\mu(2)$ orthogonal to k^μ and n^μ, such that

$$\varepsilon^\mu(k,\lambda)\varepsilon_\mu(k,\lambda') = -\delta_{\lambda\lambda'} \text{ for } \lambda, \lambda' = 1, 2 \tag{1.222}$$

After, we choose $\varepsilon^\mu(k,3)$ in the plane (k^μ, n^μ) orthogonal to n^μ and normalized, that is

$$\varepsilon^\mu(k,3)n_\mu = 0 \quad ; \quad \varepsilon^\mu(k,3)\varepsilon_\mu(k,3) = -1 \tag{1.223}$$

Finally we choose $\varepsilon^\mu(k,0)=n^\mu$. The vectors $\varepsilon^\mu(k,1)$ and $\varepsilon^\mu(k,2)$ are called transverse polarizations, while $\varepsilon^\mu(k,3)$ and $\varepsilon^\mu(k,0)$ longitudinal and scalar polarizations, respectively. We can give an example. In the frame where $n^\mu=(1,0,0,0)$ and $\vec{k}$ is along the z axis we have

$$\begin{aligned}\varepsilon^\mu(k,0)&\equiv(1,0,0,0)\ ;\ \varepsilon^\mu(k,1)\equiv(0,1,0,0)\\ \varepsilon^\mu(k,2)&\equiv(0,0,1,0)\ ;\ \varepsilon^\mu(k,3)\equiv(0,0,0,1)\end{aligned}\tag{1.224}$$

In general we can show that

$$\begin{aligned}&\varepsilon(k,\lambda)\cdot\varepsilon^*(k,\lambda')=g^{\lambda\lambda'}\\ &\sum_\lambda g^{\lambda\lambda}\varepsilon^\mu(k,\lambda)\varepsilon^{*\nu}(k,\lambda)=g^{\mu\nu}\end{aligned}\tag{1.225}$$

Inserting the expansion (1.221) in (1.219) we get

$$[a(k,\lambda),a^\dagger(k',\lambda')]=-g^{\lambda\lambda'}2k^0(2\pi)^3\delta^3(\vec{k}-\vec{k}')\tag{1.226}$$

showing, once more, that the quanta associated with $\lambda=0$ has a commutation relation with the wrong sign. Before addressing this problem, we can verify that the generalization of Eq. (1.219) for arbitrary times is

$$[A_\mu(x),A_\nu(y)]=-ig_{\mu\nu}\Delta(x,y)\tag{1.227}$$

showing the covariance of the theory. The function $\Delta(x-y)$ is the same that was introduced before for scalar fields.

Therefore, up to this point, everything is as if we had 4 scalar fields. There is, however, the problem of the sign difference in one of the commutators. Let us now see what are the consequences of this sign. For that we introduce the vacuum state defined by

$$a(k,\lambda)\left|0\right\rangle=0\qquad\lambda=0,1,2,3\tag{1.228}$$

To see the problem with the sign we construct the one-particle state with scalar polarization, that is

$$\left|1\right\rangle=\int\widetilde{dk}\ f(k)a^\dagger(k,0)\left|0\right\rangle\tag{1.229}$$

and calculate its norm

$$\begin{aligned}\langle 1|1\rangle&=\int\widetilde{dk_1}\widetilde{dk_2}f^*(k_1)f(k_2)\left\langle 0|a(k_1,0)a^\dagger(k_2,0)|0\right\rangle\\ &=-\left\langle 0|0\right\rangle\int\widetilde{dk}\ |f(k)|^2\end{aligned}\tag{1.230}$$

where we have used Eq. (1.226) for $\lambda=0$. The state $|1\rangle$ has a negative norm. The same calculation for the other polarization would give well behaved positive norms. We therefore conclude that the Fock space of the theory has indefinite metric. What happens then to the probabilistic interpretation of quantum mechanics?

To solve this problem we note that we are not working anymore with the classical Maxwell theory because we modified the Lagrangian. What we would like to do is to impose the condition $\partial \cdot A = 0$, but that is impossible as an equation for operators, as that would bring us back to the initial problems with $\pi^0 = 0$. We can, however, require that condition on a weaker form, as a condition only to be verified by the physical states. More specifically, we require that the part of $\partial \cdot A$ that contains the annihilation operator (positive frequencies) annihilates the physical states,

$$\partial^\mu A_\mu^{(+)} \left|\psi\right\rangle = 0 \tag{1.231}$$

The states $|\psi\rangle$ can be written in the form

$$|\psi\rangle = |\psi_T\rangle \, |\phi\rangle \tag{1.232}$$

where $|\psi_T\rangle$ is obtained from the vacuum with creation operators with transverse polarization and $|\phi\rangle$ with scalar and longitudinal polarization. This decomposition depends, of course, on the choice of polarization vectors. To understand the consequences of Eq. (1.231) is enough to analyze the states $|\phi\rangle$ as $\partial^\mu A_\mu^{(+)}$ contains only scalar and longitudinal polarizations,

$$i\partial \cdot A^{(+)} = \int \widetilde{dk} \; e^{-ik\cdot x} \sum_{\lambda=0,3} a(k,\lambda) \; \varepsilon(k,\lambda) \cdot k \tag{1.233}$$

and therefore Eq. (1.231) becomes

$$\sum_{\lambda=0,3} k \cdot \varepsilon(k,\lambda) \; a(k,\lambda) \, |\phi\rangle = 0 \tag{1.234}$$

Condition (1.234) does not determine completely $|\phi\rangle$. In fact, there is much arbitrariness in the choice of the transverse polarization vectors, to which we can always add a term proportional to k^μ because $k \cdot k = 0$. This arbitrariness must reflect itself on the choice of $|\phi\rangle$. Condition (1.234) is equivalent to,

$$[a(k,0) - a(k,3)] \, |\phi\rangle = 0 \; . \tag{1.235}$$

We can construct $|\phi\rangle$ as a linear combination of states $|\phi_n\rangle$ with n scalar or longitudinal photons:

$$\begin{aligned} &|\phi\rangle = C_0 \, |\phi_0\rangle + C_1 \, |\phi_1\rangle + \cdots + C_n \, |\phi_n\rangle + \cdots \\ &|\phi_0\rangle \equiv |0\rangle \end{aligned} \tag{1.236}$$

The states $|\phi_n\rangle$ are eigenstates of the operator number for scalar or longitudinal photons,

$$N' \, |\phi_n\rangle = n \, |\phi_n\rangle \tag{1.237}$$

where

$$N' = \int \widetilde{dk} \; \left[a^\dagger(k,3) a(k,3) - a^\dagger(k,0) a(k,0) \right] \tag{1.238}$$

Then

$$n \langle \phi_n | \phi_n \rangle = \langle \phi_n | N' | \phi_n \rangle = 0 \tag{1.239}$$

where we have used Eq. (1.235). This means that

$$\langle \phi_n | \phi_n \rangle = \delta_{n0} \tag{1.240}$$

that is, for $n \neq 0$, the state $|\phi_n\rangle$ has zero norm. We have then for the general state $|\phi\rangle$,

$$\langle \phi | \phi \rangle = |C_0|^2 \geq 0 \tag{1.241}$$

and the coefficients $C_i, i = 1, \cdots n \cdots$ are arbitrary. We have to show that this arbitrariness does not affect the physical observables. The Hamiltonian is

$$\begin{aligned} H &= \int d^3x : \pi^\mu \dot{A}_\mu - \mathcal{L} : \\ &= \frac{1}{2} \int d^3x : \sum_{i=1}^{3} \left[\dot{A}_i^2 + (\vec{\nabla} A_i)^2 \right] - \dot{A}_0^2 - (\vec{\nabla} A_0)^2 : \\ &= \int \widetilde{dk}\ k^0 \left[\sum_{\lambda=1}^{3} a^\dagger(k,\lambda) a(k,\lambda) - a^\dagger(k,0) a(k,0) \right] \end{aligned} \tag{1.242}$$

It is easy to check that if $|\psi\rangle$ is a physical state we have

$$\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\left\langle \psi_T | \int \widetilde{dk}\ k^0 \sum_{\lambda=1}^{2} a^\dagger(k,\lambda) a(k,\lambda) | \psi_T \right\rangle}{\langle \psi_T | \psi_T \rangle} \tag{1.243}$$

and the arbitrariness on the physical states completely disappears when we take average values. Besides that, only the physical transverse polarizations contribute to the result. One can show (see Problem 1.10) that the arbitrariness in $|\phi\rangle$ is related with a gauge transformation within the class of Lorenz gauges.

It is important to note that although for the average values of the physical observables only the transverse polarizations contribute, the scalar and longitudinal polarizations are necessary for the consistency of the theory. In particular they show up when we consider complete sums over the intermediate states.

Invariance for translations is readily verified. For that we write,

$$P^\mu = \int \widetilde{dk}\ k^\mu \sum_{\lambda=0}^{3} (-g^{\lambda\lambda}) a^\dagger(k,\lambda) a(k,\lambda) \tag{1.244}$$

Then

$$\begin{aligned} i[P^\mu, A^\nu] &= \int \widetilde{dk}\ \widetilde{dk}' ik^\mu \sum_{\lambda,\lambda'} (-g^{\lambda\lambda}) \Big\{ \left[a^\dagger(k,\lambda) a(k,\lambda), a(k',\lambda') \right] \varepsilon^\nu(k',\lambda') e^{-ik\cdot x} \\ &\qquad + \left[a^\dagger(x,\lambda) a(k,\lambda), a^\dagger(k',\lambda') \right] \varepsilon^{*\nu}(k',\lambda') e^{ik'\cdot x} \Big\} \end{aligned}$$

$$\begin{aligned}
&= \int \widetilde{dk}\; ik^\mu \sum_\lambda \left[a(k,\lambda)\varepsilon^\nu(k,\lambda)e^{-ik\cdot x} - a^\dagger(k,\lambda)\varepsilon^\nu(k,\lambda)e^{ik\cdot x}\right] \\
&= \partial^\mu A^\nu
\end{aligned} \tag{1.245}$$

showing the invariance under translations. In a similar way, it can be shown the invariance for Lorentz transformations (see Problem 1.11). For that we have to show that

$$M^{jk} = \int d^3x : \left[x^j T^{0k} - x^k T^{0j} + E^j A^k - E^k A^j\right] : \tag{1.246}$$

$$M^{0i} = \int d^3x : \left[x^0 T^{0i} - x^i T^{00} - (\partial\cdot A)A^i - E^i A^0\right] : \tag{1.247}$$

where ($\xi = 1$)

$$\begin{aligned}
T^{0i} &= -(\partial\cdot A)\,\partial^i A^0 - E^k \partial^i A^k \\
T^{00} &= \sum_{i=1}^{3}\left[\dot{A}_i^2 + (\vec{\nabla} A_j)^2\right] - \dot{A}_0^2 - (\vec{\nabla} A_0)^2
\end{aligned} \tag{1.248}$$

Using these expressions one can show that the photon has helicity ± 1, corresponding therefore to spin one. For that we start by choosing the direction of $\vec{k}$ along the axis 3 (z axis) and take the polarization vector with the choice of Eq. (1.224). A one-photon physical state will then be (not normalized),

$$|k,\lambda\rangle = a^\dagger(k,\lambda)\,|0\rangle \qquad \lambda = 1,2 \tag{1.249}$$

Let us now calculate the angular momentum along the axis 3. This is given by

$$\begin{aligned}
M^{12}\,|k,\lambda\rangle &= M^{12} a^\dagger(k,\lambda)\,|0\rangle \\
&= [M^{12}, a^\dagger(k,\lambda)]\,|0\rangle
\end{aligned} \tag{1.250}$$

where we have used the fact that the vacuum state satisfies $M^{12}\,|0\rangle = 0$. The operator M^{12} has one part corresponding the orbital angular momenta and another corresponding to the spin. The contribution of the orbital angular momenta vanishes (angular momenta in the direction of motion) as one can see calculating the commutator. In fact the commutator with the orbital angular momenta is proportional to k^1 or k^2, which are zero by hypothesis. Let us then calculate the spin part. Using the notation,

$$A^\mu = A^{\mu(+)} + A^{\mu(-)} \tag{1.251}$$

where $A^{\mu(+)}(A^{\mu(-)})$ correspond to the positive (negative) frequencies, we get

$$: E^1 A^2 - E^2 A^1 := E^{1(+)} A^{2(+)} + E^{1(-)} A^{2(+)} + A^{2(-)} E^{1(+)} + E^{1(-)} A^{2(-)} - (1 \leftrightarrow 2) \tag{1.252}$$

Then

$$\left[: E^1 A^2 - E^2 A^1 :, a^\dagger(k,\lambda)\right] =$$

$$
\begin{aligned}
&= E^{1(+)}\left[A^2(+),a^\dagger(k,\lambda)\right]+\left[E^{1(+)},a^\dagger(k,\lambda)\right]A^{2(+)}\\
&\quad +E^1(-)\left[A^2(+),a^\dagger(k,\lambda)\right]+A^{2(-)}\left[E^{1(+)},a^\dagger(k,\lambda)\right]-(1\leftrightarrow 2)\\
&= E^1\left[A^{2(+)},a^\dagger(k,\lambda)\right]+A^2\left[E^{1(+)},a^\dagger(k,\lambda)\right]-(1\leftrightarrow 2)
\end{aligned}
\tag{1.253}
$$

Now (recall that $\lambda=1,2$)

$$
\begin{aligned}
[A^{2(+)},a^\dagger(k,\lambda)] &= \int \widetilde{dk}'\sum_{\lambda'}\varepsilon^2(k',\lambda')\;\left[a(k',\lambda'),a^\dagger(k,\lambda)\right]e^{-ik'\cdot x}\\
&= \varepsilon^2(k,\lambda)e^{-ik\cdot x}\\
[E^{1(+)},a^\dagger(k,\lambda)] &= \int \widetilde{dk}'\sum_{\lambda'}\left(ik'^0\varepsilon^0(k',\lambda')+ik'^1\varepsilon^0(k',\lambda')\right)\left[a(k',\lambda'),a^\dagger(k,\lambda)\right]\;e^{-ik'\cdot x}\\
&= ik^0\varepsilon^1(k,\lambda)e^{-ik\cdot x}
\end{aligned}
\tag{1.254}
$$

Therefore

$$
\begin{aligned}
&\int d^3x\left[:E^1A^2-E^2A^1:,a^\dagger(k,\lambda)\right]\\
&\quad= \int d^3xe^{-ik\cdot x}\left[E^1\varepsilon^2(k,\lambda)+A^2ik^0\varepsilon^1(k,\lambda)-E^2\varepsilon^1(k,\lambda)+A^1ik^0\varepsilon^2(k,\lambda)\right]\\
&\quad= \int d^3xe^{-ik\cdot x}\left[\varepsilon^1(k,\lambda)\overleftrightarrow{\partial}_0A^2(x)-\varepsilon^2(k,\lambda)\overleftrightarrow{\partial}_0A^1(x)\right]
\end{aligned}
\tag{1.255}
$$

where we have used the fact that $E^i=-\dot{A}^i,\quad i=1,2$, for our choice of frame and polarization vectors. On the other hand

$$
\begin{aligned}
a(k,\lambda) &= -i\int d^3xe^{ik\cdot x}\;\overleftrightarrow{\partial}_0\;\varepsilon^\mu(k,\lambda)A_\mu(x)\\
a^\dagger(k,\lambda) &= i\int d^3xe^{-ik\cdot x}\;\overleftrightarrow{\partial}_0\;\varepsilon^\mu(k,\lambda)A_\mu(x)
\end{aligned}
\tag{1.256}
$$

For our choice we get

$$
\begin{aligned}
a^\dagger(k,1) &= -i\int d^3xe^{-ik\cdot x}\;\overleftrightarrow{\partial}_0\;A^1(x)\\
a^\dagger(k,2) &= -i\int d^3xe^{-ik\cdot x}\;\overleftrightarrow{\partial}_0\;A^2(x)
\end{aligned}
\tag{1.257}
$$

and therefore

$$
[M^{12},a^\dagger(k,\lambda)]=i\varepsilon^1(k,\lambda)a^\dagger(k,2)-i\varepsilon^2(k,\lambda)a^\dagger(k,1) \tag{1.258}
$$

We find that the state $a^\dagger(k,\lambda)\left|0\right\rangle,\lambda=1,2$ is not an eigenstate of the operator M^{12}. However the linear combinations,

$$
a_R^\dagger(k) = \frac{1}{\sqrt{2}}\left[a^\dagger(k,1)+ia^\dagger(k,2)\right]
$$

$$a_L^\dagger(k) \;=\; \frac{1}{\sqrt{2}}\left[a^\dagger(k,1) - i a^\dagger(k,2)\right] \tag{1.259}$$

which correspond to right and left circular polarization, verify

$$[M^{12}, a_R^\dagger(k)] = a_R^\dagger(k) \; ; \; [M^{12}, a_L^\dagger(k)] = -a_L^\dagger(k) \tag{1.260}$$

showing that the photon has spin 1 with right or left circular polarization (negative or positive helicity).

1.4.3 Feynman propagator

The Feynman propagator is defined as the vacuum expectation value of the time ordered product of the fields, that is

$$\begin{aligned} G_{\mu\nu}(x,y) &\equiv \langle 0|T A_\mu(x) A_\nu(y)|0\rangle \\ &= \theta(x^0-y^0)\,\langle 0|A_\mu(x)A_\nu(y)|0\rangle + \theta(y^0-x^0)\,\langle 0|A_\nu(y)A_\mu(x)|0\rangle \end{aligned} \tag{1.261}$$

Inserting the expansions for $A_\mu(x)$ and $A_\nu(y)$ we get

$$\begin{aligned} G_{\mu\nu}(x-y) &= -g_{\mu\nu}\int \widetilde{dk}\;\left[e^{-ik\cdot(x-y)}\theta(x^0-y^0) + e^{ik\cdot(x-y)\theta(y^0-x^0)}\right] \\ &= -g_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\varepsilon}e^{-ik\cdot(x-y)} \\ &\equiv \int \frac{d^4k}{(2\pi)^4} G_{\mu\nu}(k) e^{-ik\cdot(x-y)} \end{aligned} \tag{1.262}$$

where $G_{\mu\nu}(k)$ is the Feynman propagator on the momentum space

$$G_{\mu\nu}(k) \equiv \frac{-ig_{\mu\nu}}{k^2+i\varepsilon} \tag{1.263}$$

It is easy to verify that $G_{\mu\nu}(x-y)$ is the Green's function of the equation of motion, that for $\xi = 1$ is the wave equation, that is

$$\Box_x G_{\mu\nu}(x-y) = i g_{\mu\nu}\delta^4(x-y) \tag{1.264}$$

These expressions for $G_{\mu\nu}(x-y)$ and $G_{\mu\nu}(k)$ correspond to the particular case of $\xi = 1$, the so-called *Feynman gauge*. For the general case where $\xi \neq 0$ the equation of motion reads

$$\left[\Box_x g^\mu_\rho - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial_\rho\right] A^\rho(x) = 0 \tag{1.265}$$

For this case the equal times commutation relations are more complicated (see Problem 1.12). Using those relations one can show that the Feynman propagator is still the Green's function of the equation of motion, that is

$$\left[\Box_x g^\mu_\rho - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial_\rho\right] \langle 0|T A^\rho(x) A^\nu(y)|0\rangle = i g^{\mu\nu}\delta^4(x-y) \tag{1.266}$$

Using this equation we can then obtain in an arbitrary ξ gauge (of the Lorenz type),

$$G_{\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2+i\varepsilon} + i(1-\xi)\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\ . \tag{1.267}$$

1.5 Discrete Symmetries

We know from the study of the Dirac equation the transformations like space inversion (Parity) and charge conjugation, are symmetries of the Dirac equation. More precisely, if $\psi(x)$ is a solution of the Dirac equation, then

$$\psi'(x) = \psi'(-\vec{x}, t) = \gamma_0 \psi(\vec{x}, t) \tag{1.268}$$

$$\psi^c(x) = C\overline{\psi}^T(x) \tag{1.269}$$

are also solutions (if we take the charge $-e$ for ψ^c). Similar operations could also be defined for scalar and vector fields.

With second quantization the fields are no longer functions, they become operators. We have therefore to find unitary operators $\mathcal{P}$ and $\mathcal{C}$ that describe those operations within this formalism. There is another discrete symmetry, time reversal, that in second quantization will be described by an anti-unitary operator $\mathcal{T}$. We will exemplify with the scalar field how to get these operators. We will leave the Dirac and Maxwell fields as exercises.

1.5.1 Parity

To define the meaning of the Parity operation we have to put the system in interaction with the measuring system, considered to be classical. This means that we will consider the system described by

$$\mathcal{L} \longrightarrow \mathcal{L} - j_\mu(x) A^\mu_{ext}(x) \tag{1.270}$$

where we have considered that the interaction is electromagnetic. $j_\mu(x)$ is the electromagnetic current that has the form,

$$\begin{aligned} j_\mu(x) &= ie : \varphi^* \overleftrightarrow{\partial}_\mu \varphi : \quad \text{scalar field} \\ j_\mu(x) &= e : \overline{\psi}\gamma_\mu\psi : \ \text{Dirac field} \end{aligned} \tag{1.271}$$

In a Parity transformation we invert the coordinates of the measuring system, therefore the classical fields are now

$$A^\mu_{ext} = (A^0_{ext}(-\vec{x}, t)), -\vec{A}_{ext}(-\vec{x}, t) = A^{ext}_\mu(-\vec{x}, t) \tag{1.272}$$

For the dynamics of the new system to be identical to that of the original system, which should be the case if Parity is conserved, it is necessary that the equations of motion remain the same. This is true if

$$\mathcal{P}\mathcal{L}(\vec{x}, t)\mathcal{P}^{-1} = \mathcal{L}(-\vec{x}, t) \tag{1.273}$$

$$\mathcal{P} j_\mu(\vec{x}, t)\mathcal{P}^{-1} = j^\mu(-\vec{x}, t) \tag{1.274}$$

Eqs. (1.273) and (1.274) are the conditions that a theory should obey in order to be invariant under Parity. Furthermore $\mathcal{P}$ should leave the commutation relations unchanged, so that the quantum dynamics is preserved. For each theory that conserves Parity should be possible to find an unitary operator $\mathcal{P}$ that satisfies these conditions.

Now we will find such an operator $\mathcal{P}$ for the scalar field. It is easy to verify that the condition

$$\mathcal{P}\varphi(\vec{x},t)\mathcal{P}^{-1} = \pm\varphi(-\vec{x},t) \tag{1.275}$$

satisfies all the requirements. The sign $\pm$ is the *intrinsic* parity of the particle described by the field φ, ($+$ for scalar and $-$ for pseudo-scalar). In terms of the expansion of the momentum, Eq. (1.275) requires

$$\mathcal{P}a(k)\mathcal{P}^{-1} = \pm a(-k) \quad ; \quad \mathcal{P}a^\dagger(k)\mathcal{P}^{-1} = \pm a^\dagger(-k) \tag{1.276}$$

where $-k$ means that we have changed $\vec{k}$ into $-\vec{k}$ (but k^0 remains intact, that is, $k^0 = +\sqrt{|\vec{k}|^2+m^2}$). It is easier to solve Eq. (1.276) in the momentum space. As $\mathcal{P}$ should be unitary, we write

$$\mathcal{P} = e^{iP} \tag{1.277}$$

Then

$$\begin{aligned} \mathcal{P}a(k)\mathcal{P}^{-1} &= a(k) + i[P,a(k)] + \cdots + \frac{i^n}{n!}[P,[\cdots,[P,a(k)]\cdots] + \cdots \\ &= -a(-k) \end{aligned} \tag{1.278}$$

where we have chosen the case of the pseudo-scalar field.

Eq. (1.278) suggests the form

$$[P,a(k)] = \frac{\lambda}{2}[a(k)+\varepsilon a(-k)] \tag{1.279}$$

where λ and $\varepsilon = \pm 1$ are to be determined. We get

$$[P,[P,a(k)]] = \frac{\lambda^2}{2}[a(k)+\varepsilon a(-k)] \tag{1.280}$$

and therefore

$$\begin{aligned} \mathcal{P}a(k)\mathcal{P}^{-1} &= a(k) + \frac{1}{2}\left[i\lambda + \frac{(i\lambda)^2}{2!} + \cdots + \frac{(i\lambda)^4}{n!} + \cdots\right](a(k)+\varepsilon a(-k)) \\ &= \frac{1}{2}[a(k)-\varepsilon a(-k)] + \frac{1}{2}e^{i\lambda}[a(k)+\varepsilon a(-k)] \\ &= -a(-k) \end{aligned} \tag{1.281}$$

We solve Eq. (1.281) if we choose $\lambda = \pi$ and $\varepsilon = +1$ ($\lambda = \pi$ and $\varepsilon = -1$ for the scalar case). It is easy to check that

$$P_{ps} = -\frac{\pi}{2}\int \widetilde{dk}\ \left[a^\dagger(k)a(k) + a^\dagger(k)a(-k)\right] = P_{ps}^\dagger \tag{1.282}$$

and it is solution of Eq. (1.279) for $\lambda = \pi$ and $\varepsilon = +1$. Therefore,

$$\mathcal{P}_{ps} = \exp\left\{-i\frac{\pi}{2}\int \widetilde{dk}\ \left[a^\dagger(k)a(k) + a^\dagger(k)a(-k)\right]\right\} \tag{1.283}$$

and for the scalar field

$$\mathcal{P}_s = \exp\left\{-i\frac{\pi}{2}\int \widetilde{dk}\ \left[a^\dagger(k)a(k) - a^\dagger(k)a(-k)\right]\right\} \tag{1.284}$$

For the case of the Dirac field, the condition equivalent to Eq. (1.275) is now

$$\mathcal{P}\psi(\vec{x},t)\mathcal{P}^{-1} = \gamma^0\psi(-\vec{x},t) \tag{1.285}$$

Repeating the same steps we get

$$\begin{aligned}\mathcal{P}_{Dirac} \quad &= \quad \exp\left\{-i\frac{\pi}{2}\int \widetilde{dp}\ \sum_s \left[b^\dagger(p,s)b(p,s) - b^\dagger(p,s)b(-p,s)\right.\right.\\ &\qquad \left.\left. + d^\dagger(p,s)d(p,s) + d^\dagger(p,s)d(-p,s)\right]\right\}\end{aligned} \tag{1.286}$$

The case of the Maxwell field is left as an exercise.

1.5.2 Charge conjugation

The conditions for charge conjugation invariance are now

$$\mathcal{C}\mathcal{L}(x)\mathcal{C}^{-1} = \mathcal{L} \quad ; \quad \mathcal{C}j_\mu\mathcal{C}^{-1} = -j_\mu \tag{1.287}$$

where j_μ is the electromagnetic current. Conditions (1.287) are verified for the charged scalar fields if

$$\mathcal{C}\varphi(x)\mathcal{C}^{-1} = \varphi^*(x) \quad ; \quad \mathcal{C}\varphi^*(x)\mathcal{C}^{-1} = \varphi(x) \tag{1.288}$$

and for the Dirac field if

$$\begin{aligned}\mathcal{C}\psi_\alpha(x)\mathcal{C}^{-1} &= C_{\alpha\beta}\ \overline{\psi}_\beta(x)\\ \mathcal{C}\overline{\psi}_\alpha(x)\mathcal{C}^{-1} &= -\psi_\beta(x)C^{-1}_{\beta\alpha}\end{aligned} \tag{1.289}$$

where C is the charge conjugation matrix.

Finally from the invariance of $j_\mu A^\mu$ we obtain the condition for the electromagnetic field,

$$\mathcal{C}A_\mu\mathcal{C}^{-1} = -A_\mu \tag{1.290}$$

By using a method similar to the one used in the case of the Parity we can get the operator $\mathcal{C}$ for the different theories. For instance, for the scalar field we get

$$\mathcal{C}_s = \exp\left\{i\frac{\pi}{2}\int \widetilde{dk}\ (a_+^\dagger - a_-^\dagger)(a_+ - a_-)\right\} \tag{1.291}$$

and for the Dirac field

$$\mathcal{C} = \mathcal{C}_1\mathcal{C}_2 \tag{1.292}$$

with

$$\mathcal{C}_1 \quad = \quad \exp\left\{-i\int \widetilde{dp}\ \sum_s \phi(p,s)\left[b^\dagger(p,s)b(p,s) - d^\dagger(p,s)d(p,s)\right]\right\}$$

$$\mathcal{C}_2 = \exp\left\{ i\frac{\pi}{2} \int \widetilde{dp} \sum_s \left[b^\dagger(p,s) - d^\dagger(p,s) \right] \left[b(p,s) - d(p,s) \right] \right\} \tag{1.293}$$

with

$$\begin{aligned} v(p,s) &= e^{i\phi(p,s)} \ u^c(p,s) \\ u(p,s) &= e^{i\phi(p,s)} \ v^c(p,s) \end{aligned} \tag{1.294}$$

where the phase $\phi(p,s)$ is arbitrary (see [5]).

1.5.3 Time reversal

Classically the meaning of the time reversal invariance it is clear. We change the sign of the time, the velocities change direction and the system goes from what was the final state to the initial state. This exchange between the initial and final state has as consequence, in quantum mechanics, that the corresponding operator must be anti-linear or anti-unitary. In fact $\langle f|i\rangle = \langle i|f\rangle^*$ and therefore if we want $\langle \mathcal{T}\varphi_f|\mathcal{T}\varphi_i\rangle = \langle \varphi_i|\varphi_f\rangle$ then $\mathcal{T}$ must include the complex conjugation operation. We can write

$$\mathcal{T} = \mathcal{U}K \tag{1.295}$$

where $\mathcal{U}$ is unitary and K is the instruction to tale the complex conjugate of all *c-numbers*. Then

$$\begin{aligned} \langle T\varphi_f|T\varphi_i\rangle &= \langle \mathcal{U}K\varphi_f|\mathcal{U}K\varphi_i\rangle \\ &= \langle \mathcal{U}\varphi_f|\mathcal{U}\varphi_i\rangle^* \\ &= \langle \varphi_f|\varphi_i\rangle^* = \langle \varphi_i|\varphi_f\rangle \end{aligned} \tag{1.296}$$

as we wanted. A theory will be invariant under time reversal if

$$\begin{aligned} &\mathcal{T}\mathcal{L}(\vec{x},t)\mathcal{T}^{-1} = \mathcal{L}(\vec{x},-t) \\ &\mathcal{T}j_\mu(\vec{x},t)\mathcal{T}^{-1} = j^\mu(\vec{x},-t) \end{aligned} \tag{1.297}$$

For the scalar field this condition will be verified if

$$\mathcal{T}\varphi(\vec{x},t)\mathcal{T}^{-1} = \pm\varphi(\vec{x},-t) \tag{1.298}$$

and for the electromagnetic field we must have.

$$\mathcal{T}A^\mu(\vec{x},t)\mathcal{T}^{-1} = A_\mu(\vec{x},-t) \tag{1.299}$$

making $j^\mu A_\mu$ invariant. For the case of the Dirac field the transformation is

$$\mathcal{T}\psi_\alpha(\vec{x},t)\mathcal{T}^{-1} = T_{\alpha\beta}\psi_\beta(\vec{x},-t) \tag{1.300}$$

In order that Eq. (1.297) is satisfied, the T matrix must satisfy

$$T\gamma_\mu T^{-1} = \gamma_\mu^T = \gamma^{\mu *} \tag{1.301}$$

with a solution, in the Dirac representation,

$$T = i\gamma^1\gamma^3 \tag{1.302}$$

Applying the same type of reasoning already used for $\mathcal{P}$ and $\mathcal{C}$ we can find $\mathcal{T}$, or equivalently, $\mathcal{U}$. For the Dirac field, noticing that

$$\begin{aligned} Tu(p,s) &= u^*(-p,-s)e^{i\alpha_+(p,s)} \\ Tv(p,s) &= v^*(-p,-s)e^{i\alpha_-(p,s)} \end{aligned} \tag{1.303}$$

we can write $\mathcal{U} = \mathcal{U}_1\mathcal{U}_2$ and obtain

$$\mathcal{U}_1 = \exp\left\{-i\int \widetilde{dp}\ \sum_s \left[\alpha_+ b^\dagger(p,s)b(p,s) - \alpha_- d^\dagger(p,s)d(p,s)\right]\right\} \tag{1.304}$$

and

$$\begin{aligned} \mathcal{U}_2 \;=\; \exp\Bigg\{-i\frac{\pi}{2}\int \widetilde{dp}\ \sum_s \Big[& b^\dagger(p,s)b(p,s) + b^\dagger(p,s)b(-p-s) \\ & - d^\dagger(p,s)d(p,s) - d^\dagger(p,s)d(-p,-s)\Big]\Bigg\} \end{aligned} \tag{1.305}$$

1.5.4 The $\mathcal{TCP}$ theorem

It is a fundamental theorem in Quantum Field Theory that the product $\mathcal{TCP}$ is an invariance of any theory that satisfies the following general conditions:

- The theory is local and covariant for Lorentz transformations.
- The theory is quantized using the usual relation between spin and statistics, that is, commutators for bosons and anti-commutators for fermions.

This theorem due to Lüdus, Zumino, Pauli e Schwinger has an important consequence that if one of the discrete symmetries is not preserved then another one must also be violated to preserve the invariance of the product. For a proof of the theorem see the books of Bjorken and Drell[1, 6] and Itzykson and Zuber[7].

Problems for Chapter 1

1.1 Verify, for the scalar field, the covariant relations for translations and Lorentz transformations,

$$\begin{aligned} i[P^\mu, \varphi] &= \partial^\mu \varphi \\ i[M^{\mu\nu}, \varphi] &= (x^\mu \partial^\nu - x^\nu \partial^\mu)\varphi \end{aligned} \tag{1.306}$$

1.2 Show that

$$\partial^0 \Delta(x-y)|_{x^0=y^0} = -\delta^3(\vec{x}-\vec{y}) \tag{1.307}$$

1.3 Show that

$$\begin{aligned} &\int \frac{d^4k}{(2\pi)^4} \; \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x-y)} = \\ &= \int \widetilde{dk} \; \left[\theta(x^0 - y^0) e^{-ik\cdot(x-y)} + \theta(y^0 - x^0) e^{ik\cdot(x-y)}\right] \end{aligned} \tag{1.308}$$

where $\widetilde{dk} \equiv \dfrac{d^3k}{(2\pi)^3 2k^0}$. **Hint:** Integrate in the complex plane of the variable dk^0 and use the prescription $i\varepsilon$ to define the contours.

1.4 Show that for the Dirac theory the requirements of Lorentz invariance are satisfied,

$$i[M^{\mu\nu}, \varphi] = (x^\mu \partial^\nu - x^\nu \partial^\mu)\varphi + \Sigma^{\mu\nu}\varphi \quad ; \quad \Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] \tag{1.309}$$

1.5 Show that

$$\{\psi_\alpha(\vec{x}, t), \psi^\dagger_\beta(\vec{y}, t)\} = \delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}) \tag{1.310}$$

1.6 Show that

$$\begin{aligned} S_F(x-y)_{\alpha\beta} \;\; &= \;\; \theta(x^0 - y^0) \left\langle 0|\psi_\alpha(x)\overline{\psi}_\beta(y)|0\right\rangle \\ &\quad -\theta(y^0 - x^0) \left\langle 0|\overline{\psi}_\beta(y)\psi_\alpha(x)|0\right\rangle \end{aligned} \tag{1.311}$$

corresponds to

$$S_F(x-y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\varepsilon}\, e^{-ip\cdot(x-y)} \tag{1.312}$$

Hint: Expand ψ_α e $\overline{\psi}_\beta$ in plane waves.

1.7 Show that

$$(i\not{\partial}_x - m)_{\alpha\beta} S_F(x-y)_{\beta\gamma} = i\delta_{\alpha\gamma}\delta^4(x-y) \tag{1.313}$$

1.8 Show that is is always possible to choose the electromagnetic potential A^μ such that

$$A^0 = 0\ , \vec{\nabla}\cdot\vec{A} = 0 \qquad\qquad \text{(Radiation gauge)} \tag{1.314}$$

1.9 Show that we have

$$[A_\mu(x), A_\nu(y)] = -ig_{\mu\nu}\Delta(x-y) \tag{1.315}$$

1.10 Consider the indefinite metric formalism for the electromagnetic field.

a) Consider the expectation value of A_μ in the state $|\phi\rangle$. Show that

$$\begin{aligned}\langle\phi|A_\mu|\phi\rangle &= C_0^* C_1 \int \widetilde{dk}\ e^{-ik\cdot x} \langle 0| \left[\varepsilon_\mu(k,3)a(k,3) + \varepsilon_\mu(k,0)a(k,0)\right] |\phi_1\rangle \\ &\quad + \text{h.c.}\end{aligned} \tag{1.316}$$

b) Choose the state $|\phi_1\rangle$ in the form

$$|\phi_1\rangle = \int \widetilde{dk}\ f(k) \left[a^\dagger(k,3) - a^\dagger(k,0)\right] |0\rangle \tag{1.317}$$

Show that

$$\langle\phi|A_\mu|\phi\rangle = \int \widetilde{dk}\ \left[\varepsilon_\mu(k,3) + \varepsilon_\mu(k,0)\right] \left(C_0^* C_1 e^{-ik\cdot x} f(k) + c.c.\right) \tag{1.318}$$

c) Choose $\varepsilon^\mu(k,\lambda)$ to be real. Show that

$$\varepsilon^\mu(k,3) + \varepsilon^\mu(k,0) = \frac{k^\mu}{(n\cdot k)} \tag{1.319}$$

d) Show that

$$\langle\phi|A_\mu|\phi\rangle = \partial_\mu \Lambda(x) \tag{1.320}$$

where

$$\Box\Lambda = 0 \tag{1.321}$$

Comment the result.

1.11 Show the covariance of the electromagnetism for the Lorentz transformations,

$$i[M^{\mu\nu}, A^{\lambda}] = (x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})A^{\lambda} + \Sigma^{\mu\nu,\lambda}{}_{\sigma}A^{\sigma} \tag{1.322}$$

where

$$\Sigma^{\mu\nu,\lambda\sigma} = g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\lambda\nu} \tag{1.323}$$

1.12 Show that for the general case of $\xi \neq 1$ we have

$$\begin{aligned}
&[A_{\mu}(\vec{x},t), A_{\nu}(\vec{y},t)] = 0 \\
&[\dot{A}_{\mu}(\vec{x},t), A_{\nu}(\vec{y},t)] = ig_{\mu\nu}\left[1 - (1-\xi)g_{\mu 0}\right]\delta^3(\vec{x}-\vec{y}) \\
&[\dot{A}_{i}(\vec{x},t), \dot{A}_{j}(\vec{y},t)] = [\dot{A}_{0}(\vec{x},t), \dot{A}_{0}(\vec{y},t)] = 0 \\
&[\dot{A}_{0}(\vec{x},t), \dot{A}_{i}(\vec{y},t)] = i(1-\xi)\partial_i\delta^3(\vec{x}-\vec{y})
\end{aligned} \tag{1.324}$$

1.13 Use the results of Problem 1.12 to show that, in the general gauge with $\xi \neq 1$ we have

$$\left[\Box_x g^{\mu}{}_{\rho} - \left(1 - \frac{1}{\xi}\right)\partial^{\mu}\partial_{\rho}\right]\langle 0|TA^{\rho}(x)A^{\nu}(y)|0\rangle = ig^{\mu\nu}\delta^4(x-y) \tag{1.325}$$

where

$$\left(\Box g^{\mu}\rho - \left(1 - \frac{1}{\xi}\right)\partial^{\mu}\partial_{\rho}\right)A^{\rho} = 0 \tag{1.326}$$

1.14 Find the operator $\mathcal{P}$ for the Dirac and Maxwell fields.

1.15 Find the operator $\mathcal{C}$ for the Dirac and Maxwell fields.

1.16 Show that

$$\mathcal{T}\psi_{\alpha}(\vec{x},t)T^{-1} = \mathcal{T}_{\alpha\beta}\psi_{\beta}(\vec{x},-t) \tag{1.327}$$

ensures that

$$\mathcal{T}\mathcal{L}(\vec{x},t)\mathcal{T}^{-1} = \mathcal{L}(\vec{x},-t) \tag{1.328}$$

if there is a matrix T such that $T\gamma_{\mu}T^{-1} = \gamma^{\mu *}$. Find T in the Dirac representation.

1.17 Find the operator $\mathcal{T}$ for the Dirac and Maxwell fields.

1.18 Consider the Lagrangian

$$\mathcal{L} = \overline{\psi}i\gamma^{\mu}D_{\mu}P_L\psi - m\overline{\psi}\psi \tag{1.329}$$

where

$$\begin{aligned}
D_{\mu} &= \partial_{\mu} + iA^a_{\mu}\frac{\tau^a}{2} \\
P_L &= \frac{1-\gamma_5}{2}
\end{aligned} \tag{1.330}$$

Show that the theory is neither invariant under $\mathcal{P}$ nor under $\mathcal{C}$ but it is invariant for the product $\mathcal{CP}$.

Chapter 2

Physical States. S Matrix. LSZ Reduction.

2.1 Physical states

In the previous chapter we saw, for the case of free fields, how to construct the space of states, the so-called *Fock space* of the theory. When we consider the real physical case, with interactions, we are no longer able to solve the problem exactly. For instance, the interaction between electrons and photons is given by a set of nonlinear coupled equations,

$$\begin{aligned}(i\not{\partial} - m)\psi &= e\not{A}\,\psi \\ \partial_\mu F^{\mu\nu} &= e\overline{\psi}\gamma^\nu\psi \end{aligned} \tag{2.1}$$

that do not have an exact solution. In practice we have to resort to approximation methods. In the following chapter we will learn how to develop a covariant perturbation theory. Here we are going just to study the general properties of the theory.

Let us start by the physical states. As we do not know how to solve the problem exactly, we can not prove the assumptions we are going to make about these states. However, these are *reasonable* assumptions, based essentially on Lorentz covariance. We choose our states to be eigenstates of energy and momentum, and of all the other observables that commute with P^μ. Besides that, we will also assume that

i) The eigenvalues of p^2 are non-negative and $p^0 > 0$.

ii) There exists one non-degenerate base state, with the minimum of energy, which is Lorentz invariant. This state is called the vacuum state $|0\rangle$ and by convention

$$p^\mu \,|0\rangle = 0 \tag{2.2}$$

iii) There exist one particle states $\left|p^{(i)}\right\rangle$, such that,

$$p^{(i)}_\mu p^{(i)\mu} = m_i^2 \tag{2.3}$$

for each stable particle with mass m_i.

iv) The vacuum and the one-particle states constitute the discrete spectrum of p^ν.

2.2 *In* states

As we are mainly interested in scattering problems, we should construct states that have a simple interpretation in the limit $t \to -\infty$. At that time, the particles that are going to participate in the scattering process have not interacted yet (we assume that the interactions are adiabatically switched off when $|t| \to \infty$ which is appropriate for scattering problems).

We look for operators that create one particle states with the physical mass. To be explicit, we start by an hermitian scalar field given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(x) \tag{2.4}$$

where $V(x)$ is an operator made of more than two interacting fields φ at point x. For instance, those interactions can be self-interactions of the type

$$V(x) = \frac{\lambda}{4!}\varphi^4(x) \tag{2.5}$$

The field φ satisfies the following equation of motion

$$(\Box + m^2)\varphi(x) = -\frac{\partial V}{\partial\varphi(x)} \equiv j(x) \tag{2.6}$$

and the equal time canonical commutation relations,

$$\begin{aligned}
&[\varphi(\vec{x},t)\varphi(\vec{y},t)] = [\pi(\vec{x},t)\pi(\vec{y},t)] = 0 \\
&[\pi(\vec{x},t),\varphi(\vec{y},t)] = -i\delta^3(\vec{x}-\vec{y})
\end{aligned} \tag{2.7}$$

where

$$\pi(x) = \dot{\varphi}(x) \tag{2.8}$$

if we assume that $V(x)$ has no derivatives. We designate by $\varphi_{in}(x)$ the operator that creates one-particle states. It will be a functional of the fields $\varphi(x)$. Its existence will be shown by explicit construction. We require that $\varphi_{in}(x)$ must satisfy the conditions:

i) $\varphi_{in}(x)$ and $\varphi(x)$ transform in the same way for translations and Lorentz transformations. For translations we have then

$$i\,[P^\mu, \varphi_{in}(x)] = \partial^\mu\varphi_{in}(x) \tag{2.9}$$

ii) The spacetime evolution of $\varphi_{in}(x)$ corresponds to that of a free particle of mass m, that is

$$(\Box + m^2)\varphi_{in}(x) = 0 \tag{2.10}$$

From these definitions it follows that $\varphi_{in}(x)$ creates one-particle states from the vacuum. In fact, let us consider a state $|n\rangle$, such that,

$$P^\mu\,|n\rangle = p_n^\mu\,|n\rangle \ . \tag{2.11}$$

Then

$$\begin{aligned}\partial^\mu \langle n|\varphi_{in}(x)|0\rangle &= i\langle n|\left[P^\mu,\varphi_{in}(x)\right]|0\rangle \\ &= ip_n^\mu \langle n|\varphi_{in}(x)|0\rangle \end{aligned} \tag{2.12}$$

and therefore

$$\Box \langle n|\varphi_{in}(x)|0\rangle = -p_n^2 \langle n|\varphi_{in}(x)|0\rangle \tag{2.13}$$

Then

$$(\Box + m^2)\langle n|\varphi_{in}(x)|0\rangle = (m^2 - p_n^2)\langle n|\varphi_{in}(x)|0\rangle = 0 \tag{2.14}$$

where we have used the fact that $\varphi_{in}(x)$ is a free field, Eq. (2.10). Therefore the states created from the vacuum by φ_{in} are those for which $p_n^2 = m^2$, that is, the one-particle states of mass m.

The Fourier decomposition of $\varphi_{in}(x)$ is then the same as for free fields, that is,

$$\varphi_{in}(x) = \int \widetilde{dk}\, \left[a_{in}(k)e^{-ik\cdot x} + a_{in}^\dagger(k)e^{ik\cdot x}\right] \tag{2.15}$$

where $a_{in}(k)$ and $a_{in}^\dagger(k)$ satisfy the usual algebra for creation and annihilation operators. In particular, by repeated use of $a_{in}^\dagger(k)$ we can create one state of n particles.

To express $\varphi_{in}(x)$ in terms of $\varphi(x)$ we start by introducing the retarded Green's function of the Klein-Gordon operator,

$$(\Box_x + m^2)\Delta_{ret}(x-y;m) = \delta^4(x-y) \tag{2.16}$$

where

$$\Delta_{ret}(x-y;m) = 0 \;\text{ if }\; x^0 < y^0 \tag{2.17}$$

We can then write

$$\sqrt{Z}\varphi_{in}(x) = \varphi(x) - \int d^4y \Delta_{ret}(x-y;m)j(y) \tag{2.18}$$

The field $\varphi_{in}(x)$, defined by Eq. (2.15), satisfies the two initial conditions. The constant $\sqrt{Z}$ was introduced to normalize φ_{in} in such a way that it has amplitude 1 to create one-particle states from the vacuum. The fact that $\Delta_{ret} = 0$ for $x_0 \to -\infty$, suggests that $\sqrt{Z}\varphi_{in}(x)$ is, in some way, the limit of $\varphi(x)$ when $x_0 \to -\infty$. In fact, as φ and φ_{in} are operators, the correct asymptotic condition must be set on the matrix elements of the operators. Let $|\alpha\rangle$ and $|\beta\rangle$ be two normalized states. We define the operators

$$\begin{aligned}\varphi^f(t) &= i\int d^3x f^*(x) \overleftrightarrow{\partial}_0 \varphi(x) \\ \varphi_{in}^f &= i\int d^3x f^*(x) \overleftrightarrow{\partial}_0 \varphi_{in}(x)\end{aligned} \tag{2.19}$$

where $f(x)$ is a normalized solution of the Klein-Gordon equation. By Green's theorem, φ_{in}^f does not depend on time (for plane waves $f = e^{-ik\cdot x}$ and $\varphi_{in}^f = a_{in}$). Then the asymptotic condition of Lehmann, Symanzik e Zimmermann (LSZ) [8], is

$$\lim_{t\to-\infty} \langle\alpha|\,\varphi^f(t)\,|\beta\rangle = \sqrt{Z}\,\langle\alpha|\,\varphi_{in}^f\,|\beta\rangle \tag{2.20}$$

2.3 Spectral representation for scalar fields

We saw that Z had a physical meaning as the square of the amplitude for the field $\varphi(x)$ to create one-particle states from the vacuum. Let us now find a formal expression for Z and show that $0 \leq Z \leq 1$.

We start by calculating the expectation value in the vacuum of the commutator of two fields,

$$i\Delta'(x,y) \equiv \langle 0| \left[\varphi(x), \varphi(y)\right] |0\rangle \tag{2.21}$$

As we do not know how to solve the equations for the interacting fields φ, we can not solve exactly the problem of finding the Δ', in contrast with the free field case. We can, however, determine its form using general arguments of Lorentz invariance and the assumed spectra for the physical states. We introduce a complete set of states between the two operators in Eq. (2.21) and we use the invariance under translations in order to obtain,

$$\begin{aligned} \langle n|\varphi(y)|m\rangle &= \langle n|\, e^{iP\cdot y}\varphi(0)e^{-iP\cdot y}\, |m\rangle \\ &= e^{i(p_n-p_m)\cdot y}\, \langle n|\varphi(0)|m\rangle \end{aligned} \tag{2.22}$$

Therefore we get

$$\begin{aligned} \Delta'(x,y) &= -i\sum_n \langle 0|\varphi(0)|n\rangle \, \langle n|\varphi(0)|0\rangle \, (e^{-ip_n\cdot(x-y)} - e^{ip_n\cdot(x-y)}) \\ &\equiv \Delta'(x-y) \end{aligned} \tag{2.23}$$

that is, like in the free field case, Δ' is only a function of the difference $x-y$. Introducing now

$$1 = \int d^4q\ \delta^4(q-p_n) \tag{2.24}$$

we get

$$\begin{aligned} \Delta'(x-y) &= -i\int \frac{d^4q}{(2\pi)^3}\left[(2\pi)^3\sum_n \delta^4(p_n-q)|\,\langle 0|\varphi(0)|n\rangle\,|^2\right](e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}) \\ &= -i\int \frac{d^4q}{(2\pi)^3}\rho(q)(e^{-iq\cdot(x-y)} - e^{iq\cdot(x-y)}) \end{aligned} \tag{2.25}$$

where we have defined the density $\rho(q)$ (spectral amplitude),

$$\rho(q) = (2\pi)^3\sum_n \delta^4(p_n-q)|\,\langle 0|\varphi(0)|n\rangle\,|^2 \tag{2.26}$$

This spectral amplitude measures the contribution to Δ' of the states with 4-momentum q^μ. $\rho(q)$ is Lorentz invariant (as can be shown using the invariance of $\varphi(x)$ and the properties of the vacuum and of the states $|n\rangle$) and vanishes when q is not in future light cone, due the assumed properties of the physical states. Then we can write

$$\rho(q) = \overline{\rho}(q^2)\theta(q^0) \tag{2.27}$$

and we get

$$
\begin{aligned}
\Delta'(x-y) &= -i\int \frac{d^4q}{(2\pi)^3}\,\overline{\rho}(q^2)\theta(q^0)(e^{-iq\cdot(x-y)}-e^{iq\cdot(x-y)}) \\
&= -i\int \frac{d^4q}{(2\pi)^3}\int d\sigma^2\delta(q^2-\sigma^2)\overline{\rho}(\sigma^2)\theta(q^0)\left[e^{-iq\cdot(x-y)}-e^{iq\cdot(x-y)}\right] \\
&= \int_0^\infty d\sigma^2\overline{\rho}(\sigma^2)\Delta(x-y;\sigma) \qquad (2.28)
\end{aligned}
$$

where

$$
\Delta(x-y;\sigma) = -i\int \frac{d^4q}{(2\pi)^3}\delta(q^2-\sigma^2)\theta(q^0)(e^{-iq\cdot(x-y)}-e^{iq\cdot(x-y)}) \qquad (2.29)
$$

is the invariant function defined for the commutator of free fields with mass σ.

The Eq. (2.28) is known as the spectral decomposition of the commutator of two fields. This expression will allow us to show that $0 \le Z < 1$. To show that, we separate the states of one-particle from the sum in Eq. (2.26). Let $|p\rangle$ be a one-particle state with momentum p. Then

$$
\begin{aligned}
\langle 0|\varphi(x)|p\rangle &= \sqrt{Z}\,\langle 0|\varphi_{in}(x)|p\rangle + \int d^4y\Delta_{ret}(x-y;m)\,\langle 0|j(y)|p\rangle \\
&= \sqrt{Z}\,\langle 0|\varphi_{in}(x)|p\rangle \qquad (2.30)
\end{aligned}
$$

where we have used

$$
\begin{aligned}
\langle 0|j(y)|p\rangle &= \left\langle 0|(\Box+m^2)\varphi(y)|p\right\rangle = \\
&= (\Box+m^2)e^{-ip\cdot y}\,\langle 0|\varphi(0)|p\rangle \\
&= (m^2-p^2)e^{-ip\cdot y}\,\langle 0|\varphi(0)|p\rangle = 0 \qquad (2.31)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\langle 0|\varphi_{in}(x)|p\rangle &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}e^{-ik\cdot x}\,\langle 0|a_{in}(k)|p\rangle \\
&= e^{-ip\cdot x} \qquad (2.32)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\rho(q) &= (2\pi)^3\int \widetilde{dp}\;\delta^4(p-q)Z + \text{ contributions from more than one particle} \\
&= Z\delta(q^2-m^2)\theta(q^0)+\cdots \qquad (2.33)
\end{aligned}
$$

Therefore

$$
\Delta'(x-y) = Z\Delta(x-y;m) + \int_{m_1^2}^\infty d\sigma^2\overline{\rho}(\sigma^2)\Delta(x-y;\sigma) \qquad (2.34)
$$

where m_1 is the mass of the lightest state of two or more particles. Finally noticing that

$$
\frac{\partial}{\partial x^0}\Delta'(x-y)|_{x^0=y^0} = \frac{\partial}{\partial x^0}\Delta(x-y;\sigma)|_{x^0=y^0} = -\delta^3(\vec{x}-\vec{y}) \qquad (2.35)
$$

we get the relation

$$1 = Z + \int_{m_1^2}^{\infty} d\sigma^2 \overline{\rho}(\sigma^2) \tag{2.36}$$

which means

$$0 \leq Z < 1 \tag{2.37}$$

where this last step results from the assumed positivity of $\overline{\rho}(\sigma^2)$.

2.4 *Out* states

In the same way as we reduced the dynamics of $t \to -\infty$ to the free fields φ_{in}, it is also possible to define in the limit $t \to +\infty$ the corresponding free fields, $\varphi_{out}(x)$. These free fields will be the final state of a scattering problem. The formalism is copied from the case of φ_{in}, and therefore we will present the results without going into the details of the derivations. $\varphi_{out}(x)$ obey the following relations:

$$\begin{aligned} i\,[P^\mu, \varphi_{out}] &= \partial^\mu \varphi_{out} \\ (\Box + m^2)\varphi_{out} &= 0 \end{aligned} \tag{2.38}$$

and has the expansion

$$\varphi_{out}(x) = \int \widetilde{dk}\ \left[a_{out}(k) e^{-ik\cdot x} + a^\dagger_{out}(k) e^{ik\cdot x} \right] \tag{2.39}$$

The asymptotic condition is now

$$\lim_{t\to\infty} \langle\alpha|\, \varphi^f(t)\, |\beta\rangle = \sqrt{Z}\, \langle\alpha|\, \varphi^f_{out}\, |\beta\rangle \tag{2.40}$$

and

$$\sqrt{Z}\varphi_{out}(x) = \varphi(x) - \int d^4y \Delta_{\rm adv}(x-y;m) j(y) \tag{2.41}$$

where the Green's functions $\Delta_{\rm adv}$ satisfy

$$\begin{aligned} &(\Box_x + m^2)\Delta_{\rm adv}(x-y;m) = \delta^4(x-y) \\ &\Delta_{\rm adv}(x-y;m) = 0 \quad ; \quad x^0 > y^0\ . \end{aligned} \tag{2.42}$$

For one-particle states we get

$$\begin{aligned} \langle 0|\varphi(x)|p\rangle &= \sqrt{Z}\,\langle 0|\varphi_{out}(x)|p\rangle \\ &= \sqrt{Z}\,\langle 0|\varphi_{in}(x)|p\rangle \\ &= \sqrt{Z} e^{-ip\cdot x} \end{aligned} \tag{2.43}$$

2.5 S matrix

We have now all the formalism needed to study the transition amplitudes from one initial state to a given final state, the so-called S matrix elements. Let us start by an initial state with n non-interacting particles (we suppose that initially they are well separated),

$$|p_1 \cdots p_n \,; in\rangle \equiv |\alpha \,; in\rangle \tag{2.44}$$

where $p_1 \cdots p_n$ are the 4-momenta of the n particles. Other quantum numbers are assumed but not explicitly written. The final state will be, in general, a state with m particles

$$\left|p'_1 \cdots p'_m \,; out\right\rangle \equiv |\beta \,; out\rangle \tag{2.45}$$

The S matrix element $S_{\beta\alpha}$ is defined by the amplitude

$$S_{\beta\alpha} \equiv \langle\beta \,; out|\alpha \,; in\rangle \tag{2.46}$$

The S matrix is an operator that induces an isomorphism between the *in* and *out* states, that by assumption are a complete set of states,

$$\begin{aligned}
&\langle\beta \,; out| = \langle\beta \,; in|\, S \\
&\langle\beta \,; in| = \langle\beta \,; out|\, S^{-1} \\
&\langle\beta \,; out|\alpha \,; in\rangle = \langle\beta \,; in|S|\alpha \,; in\rangle = \langle\beta \,; out|S|\alpha; out\rangle
\end{aligned} \tag{2.47}$$

From the assumed properties for the states we can show the following results for the S matrix.

i) $\langle 0|S|0\rangle = \langle 0|0\rangle = 1$ (stability and unicity of the vacuum)

ii) The stability of the one-particle states gives

$$\langle p \,; in|S|p \,; in\rangle = \langle p \,; out|p \,; in\rangle = \langle p \,; in|p \,; out\rangle = 1 \tag{2.48}$$

because $|p \,; in\rangle = |p \,; out\rangle$.

iii) $\varphi_{in}(x) = S\varphi_{out}(x)S^{-1}$

iv) The S matrix is unitary. To show this we have

$$\delta_{\beta\alpha} = \langle\beta \,; out|\alpha \,; out\rangle = \left\langle\beta \,; in|SS^{\dagger}|\alpha \,; in\right\rangle \tag{2.49}$$

and therefore

$$SS^{\dagger} = 1 \tag{2.50}$$

v) The S matrix is Lorentz invariant. In fact we have

$$\begin{aligned} \varphi_{in}(ax+b) &= U(a,b)\varphi_{in}(x)U^{-1}(a,b) = US\varphi_{out}(x)S^{-1}U^{-1} \\ &= USU^{-1}\varphi_{out}(ax+b)US^{-1}U^{-1} \ . \end{aligned} \tag{2.51}$$

But

$$\varphi_{in}(ax+b) = S\varphi_{out}(ax+b)S^{-1} \ , \tag{2.52}$$

and therefore we get finally

$$S = U(a,b)SU^{-1}(a,b) \ . \tag{2.53}$$

2.6 Reduction formula for scalar fields

The S matrix elements are the quantities that are directly connected to the experiment. In fact, $|S_{\beta\alpha}|^2$ represents the transition probability from the initial state $|\alpha \, ; in\rangle$ to the final $|\beta \, ; out\rangle$. We are going in this section to use the previous formalism to express these matrix elements in terms of the so-called Green functions for the interacting fields. In this way the problem of the calculation of these probabilities is transferred to the problem of calculating these Green functions. These, of course, can not be evaluated exactly, but we will learn in the next chapter how to develop a covariant perturbation theory for that purpose.

Let us then proceed to the derivation of the relation between the S matrix elements and the the Green functions of the theory. This technique is known as the *LSZ reduction* from the names of Lehmann, Symanzik e Zimmermann [8] that have introduced it. By definition

$$\langle p_1 \cdots ; out | q_1 \cdots ; in \rangle = \left\langle p_1, \cdots ; out | a^\dagger_{in}(q_1) | q_2, \cdots ; in \right\rangle \tag{2.54}$$

Using

$$a^\dagger_{in}(q_1) = -i \int_t d^3x e^{-iq_1\cdot x} \overleftrightarrow{\partial}_0 \varphi_{in}(x) \tag{2.55}$$

where the integral is time-independent, and therefore can be calculated for an arbitrary time t. If we take $t \to -\infty$ and use the asymptotic condition for the *in* fields, Eq. (2.20), we get

$$\langle p_1 \cdots ; out | q_1 \cdots ; in \rangle = - \lim_{t\to -\infty} iZ^{-1/2} \int_t d^3x e^{-iq_1\cdot x} \overleftrightarrow{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in \rangle \tag{2.56}$$

In a similar way one can show that

$$\begin{aligned} &\left\langle p_1 \cdots ; out | a^\dagger_{out}(q_1) | q_2 \cdots ; in \right\rangle = \\ &\qquad = - \lim_{t\to \infty} iZ^{-1/2} \int_t d^3x e^{-iq_1\cdot x} \overleftrightarrow{\partial}_0 \langle p_1 \cdots ; out | \varphi(x) | q_2 \cdots ; in \rangle \ . \end{aligned} \tag{2.57}$$

Then, using the result,

$$\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right)\int d^3x f(\vec{x},t) = \lim_{t_f\to\infty, t_i\to-\infty}\int_{t_i}^{t_f} dt\frac{\partial}{\partial t}\int d^3x f(\vec{x},t)$$
$$= \int d^4x \partial_0 f(\vec{x},t) \qquad (2.58)$$

and subtracting Eq. (2.57) from Eq. (2.56) we get

$$\langle p_1\cdots;out|q_1\cdots;in\rangle = \left\langle p_1\cdots;out|a^\dagger_{out}(q_1)|q_2\cdots;in\right\rangle$$
$$+iZ^{-1/2}\int d^4x\ \partial_0\left[e^{-iq_1\cdot x}\overleftrightarrow{\partial}_0\langle p_1\cdots;out|\varphi(x)|q_2\cdots;in\rangle\right] \qquad (2.59)$$

The first term on the right-hand side of Eq. (2.59) corresponds to a sum of disconnected terms, in which at least one of the particles is not affected by the interaction (it will vanish if none of the initial momenta coincides with one of the final momenta). Its form is

$$\left\langle p_1\cdots;out|a^\dagger_{out}(q_1)|q_2\cdots;in\right\rangle =$$
$$= \sum_{k=1}^{n}(2\pi)^3 2p^0_k\,\delta^3(\vec{p}_k-\vec{q}_1)\,\langle p_1,\cdots,\widehat{p}_k,\cdots;out|q_2,\cdots;in\rangle \qquad (2.60)$$

where $\widehat{p}_k$ means that this momentum was taken out from that state. For the second term we write,

$$\int d^4x\ \partial_0\left[e^{-iq_i x}\overleftrightarrow{\partial}_0\langle p_1\cdots;out|\varphi(x)|q_2\cdots;in\rangle\right]$$
$$= \int d^4x\left[-\partial_0^2 e^{-iq_1\cdot x}\langle\cdots\rangle + e^{-iq_1\cdot x}\partial_0^2\langle\cdots\rangle\right]$$
$$= \int d^4x\left[(-\Delta^2+m^2)e^{-iq_1\cdot x}\langle\cdots\rangle + e^{-iq_1\cdot x}\partial_0^2\langle\cdots\rangle\right]$$
$$= \int d^4x e^{-iq_1\cdot x}(\Box+m^2)\langle p_1\cdots;out|\varphi(x)|q_2\cdots;in\rangle \qquad (2.61)$$

where we have used $(\Box+m^2)e^{-iq_1\cdot x}=0$, and have performed an integration by parts (whose justification would imply the substitution of plane waves by wave packets).

Therefore, after this first step in the reduction we get,

$$\langle p_1,\cdots p_n;out|q_1\cdots q_\ell;in\rangle =$$
$$= \sum_{k=1}^{n}2p^0_k(2\pi)^3\delta^3(\vec{p}_k-\vec{q}_1)\,\langle p_1,\cdots\widehat{p}_k;\cdots p_n;out|q_2\cdots q_2\cdots q_\ell;in\rangle$$
$$+iZ^{-1/2}\int d^4x e^{-iq_1 x}(\Box+m^2)\langle p_1\cdots p_n;out|\varphi(x)|q_2\cdots q_\ell;in\rangle \qquad (2.62)$$

We will proceed with the process until all the momenta in the initial and final state are exchanged by the field operators. To be specific, let us now remove one momentum in the final state. From now on we will no longer consider the disconnected terms, because in practice we are only interested in the cases where *all* the particles interact[1]. We have then

$$
\begin{aligned}
&\langle p_1 \cdots p_n; out|\varphi(x_1)|q_2 \cdots q_\ell; in\rangle = \langle p_2 \cdots p_n; out|a_{out}(p_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle \\
&= \lim_{y_1^0 \to \infty} iZ^{-1/2} \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle \\
&= \langle p_2 \cdots p_n; out|\varphi(x_1)a_{in}(p_1)|q_2 \cdots q_\ell; in\rangle \\
&\quad + \lim_{y_1^0 \to \infty} iZ^{-1/2} \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle \\
&\quad - \lim_{y_1^0 \to -\infty} iZ^{-1/2} \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|\varphi(x_1)\varphi(y_1)|q_2 \cdots q_\ell; in\rangle \\
&= \langle p_2 \cdots p_n; out|\varphi(x_1)a_{in}(p_1)|q_2 \cdots q_\ell; in\rangle \\
&\quad + iZ^{-1/2} \left(\lim_{y_1^0 \to \infty} - \lim_{y_1^0 \to -\infty} \right) \int d^3y_1\, e^{ip_1\cdot y_1} \overleftrightarrow{\partial}_{y_1^0} \langle p_2 \cdots p_n; out|T\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle
\end{aligned} \tag{2.63}
$$

where we have used the properties of the time-ordered product, Eq. (1.157). Applying the same procedure that lead to Eq. (2.61) we obtain,

$$
\begin{aligned}
&\langle p_1 \cdots p_n, ; out|\varphi(x_1)|q_2 \cdots q_\ell; in\rangle = \text{disconnected terms} \\
&\quad + iZ^{-1/2} \int d^4y_1 e^{ip_1\cdot y_1} (\Box_{y_1} + m^2) \langle p_2 \cdots p_n; out|T\varphi(y_1)\varphi(x_1)|q_2 \cdots q_\ell; in\rangle
\end{aligned} \tag{2.64}
$$

It is not very difficult to generalize this method to obtain the final reduction formula for scalar fields,

$$
\begin{aligned}
&\langle p_1 \cdots p_n; out|q_1 \cdots q_\ell; in\rangle = \text{disconnected terms} \\
&\quad + \left(\frac{i}{\sqrt{Z}}\right)^{n+\ell} \int d^4y_1 \cdots d^4y_n d^4x_1 \cdots d^4x_\ell e^{\left[i\sum_1^n p_k\cdot y_k - i\sum_1^\ell q_r\cdot x_r\right]} \\
&\quad (\Box_{y_1} + m^2) \cdots (\Box_{x_\ell} + m^2) \langle 0|T\varphi(y_1) \cdots \varphi(y_n)\varphi(x_1) \cdots \varphi(x_\ell)|0\rangle
\end{aligned} \tag{2.65}
$$

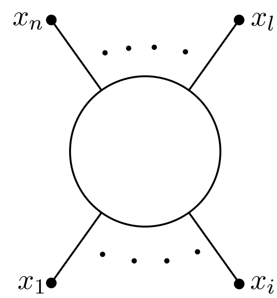
This last equation is the fundamental equation in quantum field theory. It allows us to relate the transition amplitudes to the Green functions of the theory. The quantity

$$
\langle 0|T\varphi(x_1) \cdots \varphi(x_n)|0\rangle \equiv G(x_1 \cdots x_n) \tag{2.66}
$$

[1]Once we know the cases where all the particles interact, we can always calculate situations where some of the particles do not participate in the scattering.

is known as the complete green function for $r = m + \ell$ particles and we will introduce the following diagrammatic representation for it,

$$G(x_1 \cdots x_n) = \qquad (2.67)$$



The factors $(\Box + m^2)$ in Eq. (2.65) force the external particles to be on-shell. In fact, in momentum space $(\Box + m^2) \to (-p^2 + m^2)$. Therefore, Eq. (2.65) will vanish unless the propagators of the external legs are on-shell, as in that case they will have a pole, $\frac{1}{p^2-m^2}$. Eq. (2.65) will then give the residue of that pole. We conclude that for the transition amplitudes only the truncated Green functions will contribute, that is the ones with the external legs removed. In the next chapter we will learn how to evaluate these Green functions in perturbation theory.

2.7 Reduction formula for fermions

2.7.1 States *in* and *out*

The definition of the *in* and *out* follows exactly the same steps as in the case of the scalar fields. We will therefore, for simplicity, just state the results with the details.

The states $\psi_{in}(x)$ satisfy the conditions,

$$\begin{aligned}
&(i\partial\!\!\!/ - m)\psi_{in}(x) = 0 \\
&[P_\mu, \psi_{in}(x)] = -i\partial_\mu \psi_{in}(x) \ .
\end{aligned} \qquad (2.68)$$

The states $\psi_{in}(x)$ will create one-particle states and they are related with the fields $\psi(x)$ by,

$$\sqrt{Z_2}\psi_{in}(x) = \psi(x) - \int d^4y S_{\rm ret}(x-y,m) j(y) \qquad (2.69)$$

where $\psi(x)$ satisfies the Dirac equation,

$$(i\partial\!\!\!/ - m)\psi(x) = j(x) \qquad (2.70)$$

and $S_{\rm ret}$ is the retarded Green function for the Dirac equation,

$$\begin{aligned}
&(i\partial\!\!\!/_x - m) S_{\rm ret}(x-y,m) = \delta^4(x-y) \\
&S_{ret}(x-y) = 0 \ ; \ x^0 < y^0
\end{aligned} \qquad (2.71)$$

The fields $\psi_{in}(x)$, as free fields, have the Fourier expansion,

$$\psi_{in}(x) = \int \widetilde{dp} \sum_s \left[b_{in}(p,s) u(p,s) e^{-ip\cdot x} + d^\dagger_{in}(p,s) v(p,s) e^{ip\cdot x} \right] \tag{2.72}$$

where the operators b_{in}, d_{in} satisfy exactly the same algebra as in the free field case. The asymptotic condition is now,

$$\lim_{t\to -\infty} \langle\alpha|\, \psi^f(t)\, |\beta\rangle = \sqrt{Z_2}\, \langle\alpha|\, \psi^f_{in}\, |\beta\rangle \tag{2.73}$$

where $\psi^f(t)$ and ψ^f_{in} have a meaning similar to Eq. (2.19).

For the ψ_{out} fields we get essentially the same expressions with ψ_{in} substituted by ψ_{out}. The main difference is in the asymptotic condition that now reads,

$$\lim_{t\to \infty} \langle\alpha|\, \psi^f(t)\, |\beta\rangle = \sqrt{Z_2}\, \langle\alpha|\, \psi^f_{out}\, |\beta\rangle \tag{2.74}$$

implying the following relation between the fields ψ_{out} and ψ,

$$\sqrt{Z_2}\psi_{out} = \psi(x) - \int d^4y S_{adv}(x-y;m) j(y) \tag{2.75}$$

where

$$\begin{aligned} &(i\partial\!\!\!/_x - m) S_{adv}(x-y;m) = \delta^4(x-y) \\ &S_{adv}(x-y;m) = 0 \qquad x^0 > y^0\ . \end{aligned} \tag{2.76}$$

2.7.2 Spectral representation fermions

Let us consider the vacuum expectation value of the anti-commutator of two Dirac fields,

$$\begin{aligned} S'_{\alpha\beta}(x,y) &\equiv i\, \langle 0|\, \{\psi_\alpha(x), \overline{\psi}_\beta(y)\}\, |0\rangle \\ &= i \sum_n \Big[\langle 0|\, \psi_\alpha(0)\, |n\rangle\, \langle n|\, \overline{\psi}_\beta(0)\, |0\rangle\, e^{-ip_n(x-y)} \\ &\qquad + \langle 0|\, \overline{\psi}_\beta(0)\, |n\rangle\, \langle n|\, \psi_\alpha(0)\, |0\rangle\, e^{ip_n\cdot(x-y)} \Big] \\ &\equiv S'_{\alpha\beta}(x-y) \end{aligned} \tag{2.77}$$

where we have introduced a complete set of eigen-states of the 4-momentum. As before we introduce the spectral amplitude $\rho_{\alpha\beta}(q)$,

$$\rho_{\alpha\beta}(q) \equiv (2\pi)^3 \sum_n \delta^4(p_n - q)\, \langle 0|\, \psi_\alpha(0)\, |n\rangle\, \langle n|\, \overline{\psi}_\beta(0)\, |0\rangle \tag{2.78}$$

We will now find the most general form for $\rho_{\alpha\beta}(q)$ using Lorentz invariance arguments. $\rho_{\alpha\beta}(q)$ is a 4×4 matrix in Dirac space, and can therefore be written as

$$\rho_{\alpha\beta}(q) = \overline{\rho}(q)\delta_{\alpha\beta} + \rho_\mu(q)\gamma^\mu_{\alpha\beta} + \rho_{\mu\nu}(q)\sigma^{\mu\nu}_{\alpha\beta} + \tilde{\rho}(q)\gamma^5_{\alpha\beta} + \tilde{\rho}_\mu(q)(\gamma^\mu\gamma^5)_{\alpha\beta} \tag{2.79}$$

Lorentz invariance arguments restrict the form of the coefficients $\overline{\rho}(q)$, $\rho_\mu(q)$, $\rho_{\mu\nu}(q)$, $\tilde{\rho}(q)$ and $\tilde{\rho}_\mu(q)$. Under Lorentz transformations the fields transform as

$$
\begin{aligned}
&U(a)\psi_\alpha(0)U^{-1}(a) = S^{-1}_{\alpha\lambda}(a)\overline{\psi}_\lambda(0)\\
&U(a)\overline{\psi}_\alpha(0)U^{-1}(a) = \overline{\psi}_\lambda(0)S_{\lambda\alpha}(a)\\
&S^{-1}\gamma^\mu S = a^\mu{}_\nu\gamma^\nu
\end{aligned}
\tag{2.80}
$$

Then we can show that the matrix (in Dirac space), $\rho_{\alpha\beta}$ must obey the relation,

$$
\rho(q) = S^{-1}(a)\rho(qa^{-1})S(a) \tag{2.81}
$$

where we have used a matrix notation. This relation gives the properties of the different coefficients on Eq. (2.79). For instance,

$$
\rho^\mu(q) = a^\mu{}_\nu\rho^\nu(qa^{-1}) \tag{2.82}
$$

which means that ρ^μ transform as a 4−vector.

Using the fact that $\rho_{\alpha\beta}$ is a function of q and vanishes outside the future light cone, we can finally write

$$
\rho_{\alpha\beta}(q) = \rho_1(q^2)\not{q}_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta} + \tilde{\rho}_1(q^2)(\not{q}\gamma^5)_{\alpha\beta} + \tilde{\rho}_2(q^2)\gamma^5_{\alpha\beta} \tag{2.83}
$$

that is, $\rho_{\alpha\beta}(q)$ is determined up to four scalar functions of q^2. Requiring invariance under parity transformations we get, instead of Eq. (2.81),

$$
\rho_{\alpha\beta}(\vec{q}, q_0) = \gamma^0_{\alpha\lambda}\rho_{\lambda\delta}(-\vec{q}, q^0)\gamma^0_{\delta\beta} \tag{2.84}
$$

and inserting in Eq. (2.83) we obtain,

$$
\widetilde{\rho}_1 = \widetilde{\rho}_2 = 0 \tag{2.85}
$$

Therefore for the Dirac theory, that preserves parity, we get,

$$
\rho_{\alpha\beta}(q) = \rho_1(q^2)\not{q}_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta} \tag{2.86}
$$

Repeating the steps of the scalar case we write,

$$
\begin{aligned}
S'_{\alpha\beta}(x-y) \;\; = \;\; & \int_0^\infty d\sigma^2 \left\{\rho_1(\sigma^2)S_{\alpha\beta}(x-y;\sigma)+\right.\\
& \left. + \left[\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2)\right]\delta_{\alpha\beta}\Delta(x-y;\sigma)\right\}
\end{aligned}
\tag{2.87}
$$

where Δ and $S_{\alpha\beta}$ are the functions defined for free fields. We can then show that

i) ρ_1 e ρ_2 are real

ii) $\rho_1(\sigma^2) \geq 0$

iii) $\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2) \geq 0$

Using the previous relations we can extract of the one-particle states from Eq. (2.87). We get,

$$\begin{aligned} S'_{\alpha\beta}(x-y) &= Z_2 S_{\alpha\beta}(x-y;m) \\ &\quad + \int_{m_1^2}^{\infty} d\sigma^2 \left\{ \rho_1(\sigma^2) S_{\alpha\beta}(x-y;\sigma) \right. \\ &\qquad \left. + \left[\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2)\right] \delta_{\alpha\beta}\Delta(x-y;\sigma) \right\} \end{aligned} \tag{2.88}$$

where m_1 is the threshold for the production of two or more particles. Evaluating Eq. (2.88) at equal times we can obtain

$$1 = Z_2 + \int_{m_1^2} d\sigma^2 \rho_1(\sigma^2) \tag{2.89}$$

that is

$$0 \le Z_2 < 1 \tag{2.90}$$

2.7.3 Reduction formula fermions

To get the reduction formula for fermions we will proceed as in the scalar case. The only difficulty has to do with the spinor indices. The creation and annihilation operators can be expressed in terms of the fields ψ_{in} by the relations,

$$\begin{aligned} b_{in}(p,s) &= \int d^3x \overline{u}(p,s) e^{ip\cdot x} \gamma^0 \psi_{in}(x) \\ d^\dagger_{in}(p,s) &= \int d^3x \overline{v}(p,s) e^{-ip\cdot x} \gamma^0 \psi_{in}(x) \\ b^\dagger_{in}(p,s) &= \int d^3x \overline{\psi}_{in}(x) \gamma^0 e^{-ip\cdot x} u(p,s) \\ d_{in}(p,s) &= \int d^3x \overline{\psi}_{in}(x) \gamma^0 e^{ip\cdot x} v(p,s) \end{aligned} \tag{2.91}$$

with the integrals being time dependent. In fact, to be more rigorous we should substitute the plane wave solutions by wave packets, but as in the scalar case, to simplify matter we will not do it. To establish the reduction formula we start by extracting one electron from the initial state,

$$\begin{aligned} \langle \beta; out | (ps)\alpha; in \rangle &= \left\langle \beta; out | b^\dagger_{in}(p,s) | \alpha, in \right\rangle \\ &= \langle \beta - (p,s); out | \alpha; in \rangle + \left\langle \beta; out | b^\dagger_{in}(p,s) - b^\dagger_{out}(p,s) | \alpha; in \right\rangle \\ &= \text{disconnected terms} \\ &\quad + \int d^3x \left\langle \beta; out | \overline{\psi}_{in}(x) - \overline{\psi}_{out}(x) | \alpha; in \right\rangle \gamma^0 e^{-ip\cdot x} u(p,s) \end{aligned}$$

$$
\begin{aligned}
&= \text{disconnected terms} \\
&\quad - \left(\lim_{t\to+\infty} - \lim_{t\to-\infty} \right) \frac{1}{\sqrt{Z_2}} \int d^3x \left\langle \beta; out | \overline{\psi}(x) | \alpha; in \right\rangle \gamma^0 e^{-ip\cdot x} u(p,s) \\
&= \text{disconnected terms} \\
&\quad - Z_2^{-1/2} \int d^4x \left[\left\langle \beta; out | \partial_0 \overline{\psi}(x) | \alpha; in \right\rangle \gamma^0 e^{-ip\cdot x} u(p,s) \right. \\
&\qquad\qquad \left. + \left\langle \beta; out | \overline{\psi}(x) | \alpha; in \right\rangle \gamma^0 \partial_0 (e^{-ip\cdot x} u(p,s)) \right]
\end{aligned}
\tag{2.92}
$$

Using now

$$
(i\gamma^0 \partial_0 + i\gamma^i \partial_i - m)(e^{-ip\cdot x} u(p,s)) = 0 \tag{2.93}
$$

we get, after an integration by parts,

$$
\begin{aligned}
\left\langle \beta; out | b^\dagger_{in}(\rho, s) | \alpha; in \right\rangle &= \text{disconnected terms} \\
&\quad - i Z_2^{-1/2} \int d^4x \left\langle \beta; out | \overline{\psi}(x) | \alpha; in \right\rangle (-i \overleftarrow{\not\partial}_x - m) e^{-ip\cdot x} u(p,s)
\end{aligned}
\tag{2.94}
$$

In a similar way the reduction of an anti-particle from the initial state gives,

$$
\begin{aligned}
\left\langle \beta; out | d^\dagger_{in}(p, s) | \alpha; in \right\rangle &= \text{disconnected terms} \\
&\quad + i Z_2^{-1/2} \int d^4x e^{-ip\cdot x} \overline{v}(p,s) (i \not\partial_x - m) \left\langle \beta; out | \psi(x) | \alpha; in \right\rangle
\end{aligned}
\tag{2.95}
$$

while the reduction of a particle or of an anti-particle from the final state give, respectively,

$$
\begin{aligned}
\left\langle \beta; out | b_{out}(p, s) | \alpha; in \right\rangle &= \text{disconnected terms} \\
&\quad - i Z_2^{-1/2} \int d^4x e^{ip\cdot x} \overline{u}(p,s) (i \not\partial_x - m) \left\langle \beta; out | \psi(x) | \alpha; in \right\rangle
\end{aligned}
\tag{2.96}
$$

and

$$
\begin{aligned}
\left\langle \beta; out | d_{out}(p, s) | \alpha; in \right\rangle &= \text{disconnected terms} \\
&\quad + i Z_2^{-1/2} \int d^4x \left\langle \beta; out | \overline{\psi}(x) | \alpha; in \right\rangle (-i \overleftarrow{\not\partial}_x - m) v(p,s) e^{ip\cdot x}
\end{aligned}
\tag{2.97}
$$

Notice the formal relation between one electron in the initial state and a positron in the final state. To go from one to the other one just has to do,

$$
u(p,s) e^{-ip\cdot x} \to -v(p,s) e^{ip\cdot x} \tag{2.98}
$$

The following steps in the reduction are similar, one only has to pay attention to signs because of the anti-commutation relations for fermions. To write the final expression we denote the momenta in the state $\langle in|$ by p_i or $\overline{p_i}$, respectively for particles or anti-particles, and those in the state $\langle out|$ by $p'_i, \overline{p}'_i$. We also make the following conventions (needed to define the global sign),

$$|(p_1,s_1),\cdots,(\overline{p}_1,\overline{s}_1);\cdots;in\rangle = b^\dagger_{in}(p_1,s_1)\cdots d^\dagger_{in}(\overline{p}_1,\overline{s}_1)\cdots|0\rangle \tag{2.99}$$

and

$$\left\langle out;(p'_1,s'_1)\cdots,(\overline{p}'_1,\overline{s}'_1)\cdots\right| = \langle 0|\cdots d_{out}(\overline{p}'_1,\overline{s}'_1),\cdots b_{out}(p'_1,s'_1) \tag{2.100}$$

Then, if $n(n')$ denotes the total number of particles (anti-particles), we get

$$\begin{aligned}
&\left\langle out;(p'_1,s'_1)\cdots,(\overline{p}'_1,\overline{s}'_1)\cdots|(p_1,s_1),\cdots(\overline{p}_1,\overline{s}_1),\cdots;in\right\rangle = \text{disconnected terms}\\
&+(-iZ_2^{-1/2})^n(iZ_2^{-1/2})^{n'}\int d^4x_1\cdots d^4y_1\cdots d^4x'_1\cdots d^4y'_1\cdots\\
&\qquad e^{-i\sum(p_i\cdot x_i)-i\sum(\overline{p}_i\cdot y_i)}e^{+i\sum(p'_i\cdot x'_i)+i\sum(\overline{p}'_i\cdot y'_i)}\\
&\qquad \overline{u}(p'_1,s'_1)(i\partial\!\!\!/_{x'_1}-m)\cdots\overline{v}(\overline{p}_1,\overline{s}_1)(i\partial\!\!\!/_{y_1}-m)\\
&\qquad \langle 0|\,T(\cdots\overline{\psi}(y'_1)\cdots\psi(x'_1)\overline{\psi}(x_1)\cdots\psi(y_1)\cdots|0\rangle\\
&\qquad (-i\overleftarrow{\partial\!\!\!/}_{x_1}-m)u(p_1,s_1)\cdots(-i\overleftarrow{\partial\!\!\!/}_{y'_1}-m)v(\overline{p}'_1,\overline{s}'_1)
\end{aligned} \tag{2.101}$$

Eq. (2.101) is the fundamental expression that allows ti relate the elements of the S matrix with the Green functions of the theory. The operators within the time-ordered product can be reordered, modulo some minus sign. The sign and ordering shown correspond to the conventions in Eqs. (2.99) and (2.100). In terms of diagrams, we represent the Green function,

$$\langle 0|\,T\left[\overline{\psi}(y'_{m'})\cdots\overline{\psi}(y'_1)\psi(x'_{\ell'})\cdots\psi(x'_1)\overline{\psi}(x_1)\cdots\overline{\psi}(x_\ell)\psi(y_1)\cdots\psi(y_m)\right]|0\rangle \tag{2.102}$$

by the diagram[2] of Fig. (2.1).

The operators $(i\partial\!\!\!/-m)$ e $(-i\overleftarrow{\partial\!\!\!/}-m)$ force the particles to be on-shell and remove the propagators from the external lines (truncated Green functions). In the next chapter we will learn how to determine these functions in perturbation theory.

2.8 Reduction formula for photons

The LSZ formalism for photons, has some difficulties connected with the problems in quantizing the electromagnetic field. When one adopts a formalism (radiation gauge) where the only components of the field A^μ are transverse (as in Ref.[6]), the problems arise in showing the Lorentz and gauge invariance of the S matrix. In the formalism of the undefined metric, that we adopted in section 1.4.2, the difficulties are connected with the states of negative norm, besides the gauge invariance.

[2]With lepton number conservation, the number of particles minus anti-particles is conserved, that is

$$\ell - m = \ell' - m'$$

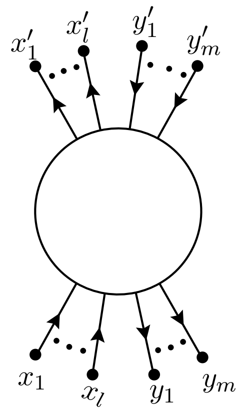


Figure 2.1: Green function for fermions.

Here we are going to ignore these difficulties[3] and assume that we can define the *in* fields by the relation,

$$\sqrt{Z_3} A^{\mu}_{in}(x) = A^{\mu}(x) - \int d^4y D^{\mu\nu}_{\rm ret}(x-y) j_{\nu}(y) \tag{2.103}$$

and in the same way for the *out* fields,

$$\sqrt{Z_3} A^{\mu}_{out}(x) = A^{\mu}(x) - \int d^4y D^{\mu\nu}_{\rm adv}(x-y) j_{\nu}(y) \tag{2.104}$$

where

$$\begin{aligned} &\Box A^{\mu}_{in} = \Box A^{\mu}_{out} = 0 \\ &\Box A^{\mu} = j^{\mu} \\ &\Box D^{\mu\nu}_{\rm adv,\ ret} = \delta^{\mu\nu}\delta^{\mu}(x-y) \end{aligned} \tag{2.105}$$

The fields *in* and *out* are free fields, and therefore they have a Fourier expansion in plane waves and creation and annihilation operators of the form

$$A^{\mu}_{in}(x) = \int \widetilde{dk} \sum_{\lambda=0}^{3} \left[a_{in}(k,\lambda)\varepsilon^{\mu}(k,\lambda)e^{-ik\cdot x} + a^{\dagger}_{in}(k,\lambda)\varepsilon^{\mu *}(k,\lambda)e^{ik\cdot x} \right] \tag{2.106}$$

and therefore

$$\begin{aligned} a_{in}(k,\lambda) &= -i \int d^3x e^{ik\cdot x} \overleftrightarrow{\partial}_0 \varepsilon^{\mu}(k,\lambda) A^{in}_{\mu}(x) \\ a^{\dagger}_{in}(k,\lambda) &= i \int d^3x e^{-ik\cdot x} \overleftrightarrow{\partial}_0 \varepsilon^{\mu *}(k,\lambda) A^{in}_{\mu}(x) \end{aligned} \tag{2.107}$$

[3]We will see in chapter 6 a more satisfactory procedure to quantize all gauge theories, including Maxwell theory of the electromagnetic field. We will see that the resulting perturbation theory coincides with the one we get here. This is our justification to be less precise here.

where, as usual, $a_{in}(k, \lambda)$ and $a^\dagger_{in}(k, \lambda)$ are time independent. In Eq. (2.106) all the polarizations appear, but as the elements of the S matrix are between physical states, we are sure that the longitudinal and scalar polarizations do not contribute. In this formalism what is difficult to show is the spectral decomposition. We are not going to enter those details, just state that we can show that Z_3 is gauge independent and satisfies $0 \leq Z_3 < 1$. The reduction formula is easily obtained. We get

$$\begin{aligned}
\langle\beta; out|(k\lambda)\alpha; in\rangle &= \langle\beta - (k,\lambda); out|\alpha; in\rangle + \left\langle\beta; out|a^\dagger_{in}(k,\lambda) - a^\dagger_{out}(k,\lambda)|\alpha; in\right\rangle \\
&= \text{disconnected terms} \\
&\quad +i\int d^3x e^{-ik\cdot x}\overleftrightarrow{\partial}_0 \varepsilon^*_\mu(k,\lambda)\, \langle\beta; out|A^\mu_{in}(x) - A^\mu_{out}(x)|\alpha; in\rangle \\
&= \text{disconnected terms} \\
&\quad -i(\lim_{t\to+\infty} - \lim_{t\to-\infty})Z_3^{-1/2}\int d^3x e^{-ik\cdot x}\overleftrightarrow{\partial}_0\, \langle\beta; out|A^\mu(x)|\alpha; in\rangle\, \varepsilon^*_\mu(k,\lambda) \\
&= \text{disconnected terms} \\
&\quad -iZ_3^{-1/2}\int d^4x e^{-ik\cdot x}\overleftrightarrow{\partial}_0\, \langle\beta; out|A^\mu(x)|\alpha; in\rangle\, \varepsilon^*_\mu(k,\lambda) \\
&= \text{disconnected terms} \\
&\quad -iZ_3^{-1/2}\int d^4x e^{-ik\cdot x}\vec{\Box}_x\, \langle\beta; out|A^\mu(x)|\alpha; in\rangle\, \varepsilon^*_\mu(k,\lambda)
\end{aligned} \tag{2.108}$$

The final formula for photons is then

$$\begin{aligned}
\left\langle k'_1\cdots k'_n; out|k_1\cdots k_\ell; in\right\rangle &= \text{disconnected terms} \\
&+\left(\frac{-i}{\sqrt{Z_3}}\right)^{n+\ell}\int d^4y_1\cdots d^4y_n d^4x_1\cdots d^4x_\ell\; e^{\left[i\sum^n k'_i\cdot y_i - i\sum^\ell k_i\cdot x_i\right]} \\
&\qquad \varepsilon^{\mu_1}(k_1,\lambda_1)\cdots\varepsilon^{\mu_\ell}(k_\ell,\lambda_\ell)\varepsilon^{*\mu'_1}(k'_1,\lambda'_1)\cdots\varepsilon^{*\mu'_n}(k'_n,\lambda'_n) \\
&\qquad \Box_{y_1}\cdots\Box_{x_\ell}\,\langle 0|\, T(A_{\mu'_1}(y_1)\cdots A_{\mu'_n}(y_n)A_{\mu_1}(x_1)\cdots A_{\mu_\ell}(x_\ell)\,|0\rangle
\end{aligned} \tag{2.109}$$

and corresponds to the diagram of Fig. (2.2).

2.9 Cross sections

The reduction formulas, Eqs.(2.65), (2.101) and (2.109), are the fundamental results of this chapter. They relate the transition amplitudes from the initial to the final state with the Green functions of the theory. In the next chapter we will show how to evaluate these Green functions setting up the so-called covariant perturbation theory. Before we close this chapter, let us indicate how these transition amplitudes

$$S_{fi} \equiv \langle f; out|i; in\rangle \tag{2.110}$$

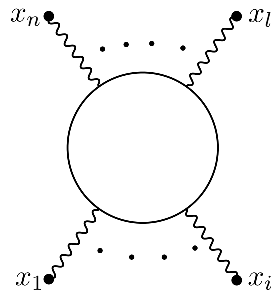


Figure 2.2: Green function for photons.

are related with the quantities that are experimentally accessible, the cross sections. Then the path between experiment (cross sections) and theory (Green functions) will be established.

As we have seen in the reduction formulas there is always a trivial contribution to the S matrix, that corresponds to the so-called *disconnected terms*, when the system goes from the initial to the final state without interaction. The subtraction of this trivial contribution leads us to introduce the T matrix with the relation,

$$S_{fi} = 1_{fi} - i(2\pi)^4\delta^4(P_f - P_i)T_{fi} \tag{2.111}$$

where we have factorized explicitly the delta function expressing the 4-momentum conservation. If we neglect the trivial contribution, the transition probability from the initial to the final state will be given by

$$W_{f\leftarrow i} = \left|(2\pi)^4\delta^4(P_f - P_i)T_{fi}\right|^2 \tag{2.112}$$

To proceed we have to deal with the meaning of a square of a delta function. This appears because we are using plane waves. To solve this problem we can normalize in a box of volume V and consider that the interaction has a duration of T. Then

$$(2\pi)^4\delta^4(P_f - P_i) = \lim_{\substack{V\to\infty \\ T\to\infty}} \int_V d^3x \int_{-T/2}^{T/2} dx^0 e^{i(P_f-P_i)\cdot x} \ . \tag{2.113}$$

However

$$F \equiv \int_V d^3x \int_{-T/2}^{T/2} dx^0 e^{i(P_f-P_i)\cdot x} = V\delta_{\vec{P}_f\vec{P}_j}\ \frac{2}{|E_f - E_i|}\sin\left|\frac{T}{2}(E_f - E_i)\right| \tag{2.114}$$

and the square of the last expression can be done, giving,

$$|F|^2 = V^2\delta_{\vec{P}_f,\vec{P}_i}\ \frac{4}{|E_f - E_i|^2}\sin^2\left|\frac{T}{2}(E_f - E_i)\right| \ . \tag{2.115}$$

If we want the transition rate by unit of volume (and unit of time) we divide by VT. Then

$$\Gamma_{fi} = \lim_{\substack{V\to\infty \\ T\to\infty}} V\delta_{\vec{P}_f,\vec{P}_i}\ 2\ \frac{\sin^2 \frac{T}{2}(E_f - E_i)}{\frac{T}{2}(E_f - E_i)^2}|T_{fi}|^2 \tag{2.116}$$

Using now the results

$$\lim_{V\to\infty} V\delta_{\vec{P}_f\vec{P}_j} = (2\pi)^3\delta^3(\vec{P}_f - \vec{P}_i)$$

$$\lim_{T\to\infty} 2\,\frac{\sin^2\frac{T}{2}(E_f - E_i)}{\frac{T}{2}(E_f - E_i)^2} = (2\pi)\,\delta(E_f - E_i) \tag{2.117}$$

we get for the transition rate by unit volume and unit time,

$$\Gamma_{fi} \equiv (2\pi)^4\delta^4(P_f - P_i)|T_{fi}|^2 \tag{2.118}$$

To get the cross section we have to further divide by the incident flux, and normalize the particle densities to one particle per unit volume. Finally, we sum (integrate) over all final states in a certain energy-momentum range. We get,

$$d\sigma = \frac{1}{\rho_1\rho_2}\frac{1}{|\vec{v}_{12}|}\Gamma_{fi}\prod_{j=3}^{n}\frac{d^3p_j}{2p_j^0(2\pi)^3} \tag{2.119}$$

where

$$\rho_1 = 2E_1 \quad ; \quad \rho_2 = 2E_2 \tag{2.120}$$

An equivalent way of writing this equation is

$$d\sigma = \frac{1}{4\left[(p_i\cdot p_2)^2 - m_1^2m_2^2\right]^{1/2}}(2\pi)^4\delta^4(P_f - P_i)|T_{fi}|^2\prod_{j=3}^{n}dp_j \tag{2.121}$$

that exhibits well the Lorentz invariance of each part that enters the cross section[4]. The incident flux and phase space factors are purely kinematics. The physics, with its interactions, is in the matrix element T_{fi}.

We note that with our conventions, fermion and boson fields have the same normalization, that is, the one-particle states obey

$$\langle p|p'\rangle = 2p^0(2\pi)^3\delta^3(\vec{p} - \vec{p}') \tag{2.122}$$

differing in this way from some older books like Ref.[1].

[4]It is assumed that, in the case of two beams they are in the same line. Then the cross section, being a *transverse area*, is invariant for Lorentz transformations along that direction.

Problems for Chapter 2

2.1 Show that the spectral representation for fermions, $\rho_{\alpha\beta}(q)$, satisfies,

a) $\rho(q) = S^{-1}(a)\rho(qa^{-1})S(a)$

b) $\rho_{\alpha\beta}(\vec{q}, q^0) = \gamma^0_{\alpha\lambda}\rho_{\lambda\delta}(-\vec{q}, q^0)\gamma^0_{\delta\beta}$

2.2 Use the results of the previous problem to show that, in a theory that preserves Parity, like QED, we have

$$\rho_{\alpha\beta}(q) = \rho_1(q^2)\not{q}_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta} \tag{2.123}$$

2.3 Show that the functions ρ_1 and ρ_2 defined in problem 2.2 satisfy the following properties:

i) ρ_1 and ρ_2 are real

ii) $\rho_1(\sigma^2) \geq 0$

iii) $\sigma\rho_1(\sigma^2) - \rho_2(\sigma^2) \geq 0$

2.4 Show that for the Dirac field we have

$$1 = Z_2 + \int_{m_1^2}^{\infty} d\sigma^2 \rho_1(\sigma^2) \tag{2.124}$$

2.5 Show that

$$\langle 0| \left[\varphi_{in}(x), \varphi_{out}(y)\right] |0\rangle = i\Delta(x-y; m) \tag{2.125}$$

Chapter 3

Covariant Perturbation Theory

3.1 U matrix

In this chapter we are going to develop a method to evaluate the Green functions of a given theory. From what we have seen in the two previous chapters, we realize that we only know how to calculate for free fields, like the *in* an *out* fields. However, the Green functions we are interested in, are given in terms of the physical interacting fields, and we do not know how to operate with these. We are going to see how to express the physical fields as perturbative series in terms of free *in* fields. In this way we will be able to evaluate the Green functions in perturbation theory.

We start by defining the U matrix. To simplify matters, we will be considering, for the moment, only scalar fields. In the end we will return to the other cases. The physical interacting fields $\varphi(\vec{x},t)$ and their conjugate momenta $\pi(\vec{x},t)$, satisfy the same equal time commutation relations than the *in* fields, $\varphi_{in}(\vec{x},t)$ and their $\pi_{in}(\vec{x},t)$. Also, both φ and φ_{in} form a complete set of operators, in the sense that any state, free or interacting, can be obtained by application of φ_{in} or φ in the vacuum. This implies that there should be an unitary transformation $U(t)$ that relates φ with φ_{in}, that is,

$$
\begin{aligned}
\varphi(\vec{x},t) &= U^{-1}(t)\varphi_{in}(\vec{x},t)U(t) \\
\pi(\vec{x},t) &= U^{-1}(t)\pi_{in}(\vec{x},t)U(t)
\end{aligned}
\tag{3.1}
$$

The dynamics of U can be obtained from the equations of motion for $\varphi(x)$ and $\varphi_{in}(x)$. These are,

$$
\begin{aligned}
\frac{\partial \varphi_{in}}{\partial t}(x) &= i[H_{in}(\varphi_{in},\pi_{in}),\varphi_{in}] \\
\frac{\partial \pi_{in}}{\partial t}(x) &= i[H_{in}(\varphi_{in},\pi_{in}),\pi_{in}]
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t}(x) &= i[H(\varphi,\pi),\varphi] \\
\frac{\partial \pi}{\partial t}(x) &= i[H(\varphi,\pi),\pi]
\end{aligned}
\tag{3.3}
$$

Then from Eqs. (3.2) and (3.1) we get,

$$\begin{aligned}\dot{\varphi}_{in}(x) &= \frac{\partial}{\partial t}\left[U(t)\varphi(x)U^{-1}(t)\right] \\ &= \left[\dot{U}(t)U^{-1}(t), \varphi_{in}\right] + i\left[H(\varphi_{in}, \pi_{in}), \varphi_{in}(x)\right] \\ &= \dot{\varphi}_{in}(x) + \left[\dot{U}U^{-1} + iH_I(\varphi_{in}, \pi_{in}), \varphi_{in}\right] \end{aligned} \tag{3.4}$$

where

$$H_I(\varphi_{in}, \pi_{in}) = H(\varphi_{in}, \pi_{in}) - H_{in}(\varphi_{in}, \pi_{in}) \equiv H_I(t) \tag{3.5}$$

and in a similar way

$$\dot{\pi}_{in}(x) = \dot{\pi}_{in} + \left[\dot{U}U^{-1} + iH_I(\varphi_{in}, \pi_{in}), \pi_{in}\right] \tag{3.6}$$

From Eqs. (3.4) and (3.6) we obtain,

$$i\dot{U}U^{-1} = H_I(t) + E_0(t) \tag{3.7}$$

where $E_0(t)$ commutes with φ_{in} and π_{in} and is therefore a time dependent c-number, not an operator. Defining

$$H_I'(t) = H_I(t) + E_0(t) \tag{3.8}$$

we get a differential equation for $U(t)$, that reads,

$$i\frac{\partial U(t)}{\partial t} = H_I'(t)U(t) \tag{3.9}$$

The solution of this equation in terms of the *in* fields, is the basis of the covariant perturbation theory.

To integrate Eq. (3.9) we need an initial condition. For that we introduce the operator

$$U(t, t') \equiv U(t)U^{-1}(t') \tag{3.10}$$

where $t \geq t'$, and that obviously satisfies

$$U(t, t) = 1 \tag{3.11}$$

It is easy to see that $U(t, t')$ also satisfies Eq. (3.9), that is,

$$i\frac{\partial U(t, t')}{\partial t} = H_I'(t)U(t, t') \tag{3.12}$$

and has the initial condition, Eq. (3.11). To proceed we start by transforming Eq. (3.12) in an equivalent integral equation, that is,

$$U(t, t') = 1 - i\int_{t'}^{t} dt_1 H_I'(t_1)U(t_1, t') \tag{3.13}$$

Notice that we have not solved the problem because $U(t,t')$ appears on both sides of the equation. However, we can iterate the equation to get the expansion,

$$\begin{aligned} U(t,t') &= 1 - i\int_{t'}^{t} dt_1 H'_I(t_1) + (-i)^2 \int_{t'}^{t} dt_1 H'_I(t_1) \int_{t'}^{t_1} dt_2 H'_I(t_2) \\ &\quad + \cdots + (-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n H'_I(t_1) \cdots H'_I(t_n) \\ &\quad + \cdots \end{aligned} \tag{3.14}$$

Of course this expansion can only be useful if H_I contains a small parameter and, because of that, we can truncate the expansion at certain order in that parameter. Coming back to Eq. (3.14), as $t_1 \geq t_2 \geq, \cdots t_n$, the product is time-ordered and we can therefore write

$$U(t,t') = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 \cdots \int_{t'}^{t_{n-1}} dt_n T(H'_I(t_1) \cdots H'_I(t_n)) \tag{3.15}$$

Using the symmetry t_1, t_2 we can write,

$$\begin{aligned} \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 T(H'_I(t_1) H'_I(t_2)) &= \int_{t'}^{t} dt_2 \int_{t'}^{t_2} dt_1 T(H'_I(t_1) H'_I(t_2)) \\ &= \frac{1}{2} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 T(H'_I(t_1) H'_2(t_2)) \end{aligned} \tag{3.16}$$

In general, for n integrations, instead of $\frac{1}{2}$ we will have $\frac{1}{n!}$, and we get,

$$\begin{aligned} U(t,t') &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} dt_1 \cdots \int_{t'}^{t} dt_n T(H'_I(t_1) \cdots H'_I(t_n)) \\ &\equiv T\left(\exp[-i \int_{t'}^{t} dt H'_I(t)] \right) \\ &= T\left(\exp[-i \int_{t'}^{t} d^4x \mathcal{H}_I(\varphi_{in})] \right) \end{aligned} \tag{3.17}$$

where the time-ordered product is to be interpreted expanding the exponential.

The operators U satisfy the following multiplication rule

$$U(t,t') = U(t,t'')U(t'',t') \tag{3.18}$$

which can seen using the definition, Eq. (3.10), or from the explicit expression, Eq. (3.17). From Eq. (3.18), we can obtain,

$$U(t,t') = U^{-1}(t',t) \tag{3.19}$$

3.2 Perturbative expansion of Green functions

As we saw in the previous chapter, the LSZ technique reduces the evaluation of the elements of the S matrix to a basic ingredient, the so-called Green functions of the theory. These are expectation values of time-ordered products of the Heisenberg fields, $\varphi(x)$,

$$G(x_1, \cdots, x_n) \equiv \langle 0| \, T\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n) \, |0\rangle \tag{3.20}$$

The basic idea for the evaluation of the Green functions consists in expressing the fields $\varphi(x)$ in terms of the fields $\varphi_{in}(x)$, using the operator U. We get

$$\begin{aligned} G(x_1, \cdots, x_n) &= \langle 0| \, T(U^{-1}(t_1)\varphi_{in}(x_1)U(t_1,t_2)\varphi_{in}(x_2)U(t_2,t_3)\cdots \\ &\quad \cdots U(t_{n-1},t_n)\varphi_{in}(x_n)U(t_n)) \, |0\rangle \\ &= \langle 0| \, T(U^{-1}(t)U(t,t_1)\varphi_{in}(x_1)U(t_1,t_2)\cdots \\ &\quad \cdots U(t_{n-1},t_n)\varphi_{in}(x_n)U(t_n,-t)U(-t)) \, |0\rangle \end{aligned} \tag{3.21}$$

where t is a time that we will let go to ∞. When $t \to \infty$, t is later than all the t_i and $-t$ is earlier than all the times t_i. Therefore we can take $U^{-1}(t)$ e $U(-t)$ out of the time-ordered product. Using the multiplicative property of the operator U we can then write,

$$G(x_1, \cdots, x_n) = \langle 0| \, U^{-1}(t) T\left(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\exp[-i\int_{-t}^{t} H'_I(t')dt']\right) U(-t) \, |0\rangle \tag{3.22}$$

where the time-ordered product T is meant to be applied after expanding the exponential. If it were not for the presence of the operators $U^{-1}(t)$ and $U(-t)$, we would have been successful in expressing the Green function $G(x_1 \cdots x_n)$ completely in terms of the *in* fields. Now we are going to show that the vacuum is an eigenstate of the operator $U(t)$. For that we consider an arbitrary state $|\alpha p; \; in\rangle$ that contains one particle of momentum p, all the other quantum numbers being denoted collectively by α. To simplify, we continue considering the case of the scalar field. We have then,

$$\begin{aligned} \langle \alpha p; in|U(-t)|0\rangle &= \langle \alpha; in|a_{in}(p)U(-t)|0\rangle \\ &= -i\int d^3x f_p^*(\vec{x},-t')\left(\frac{\overrightarrow{\partial}}{\partial t'} - \frac{\overleftarrow{\partial}}{\partial t'}\right)\langle \alpha; in| \, \varphi_{in}(\vec{x},-t')U(-t) \, |0\rangle \end{aligned} \tag{3.23}$$

where $f_p(\vec{x},t) = e^{-ip\cdot x}$. We use now Eq. (3.1) to express $\varphi_{in}(\vec{x},-t)$ in terms of $\varphi(\vec{x},-t)$. We get,

$$\begin{aligned} &\langle \alpha p; in|U(-t)|0\rangle = \\ &\quad = -i\int d^3x f_p^*(\vec{x},-t')\overleftrightarrow{\partial}'_0 \langle \alpha; in|U(-t')\varphi(\vec{x},-t')U^{-1}(-t')U(-t)|0\rangle \\ &\quad = -i\int d^3x f_p^*(\vec{x},-t')\Big[-\overleftarrow{\partial}_{0'} \langle \alpha; in| \, U(-t')\varphi(\vec{x},-t')U^{-1}(-t')U(-t) \, |0\rangle \end{aligned}$$

$$+ \langle \alpha; in| \, U(-t')\dot{\varphi}(\vec{x}, -t')U^{-1}(-t')U(-t) \, |0\rangle \Big]$$

$$+i \int' d^3x f_p^*(\vec{x}, -t') \, \langle \alpha; in| \, \dot{U}(-t')\varphi(\vec{x}, -t')U^{-1}(-t')U(-t) \, |0\rangle$$

$$+i \int d^3x f_p^*(\vec{x}, -t') \, \langle \alpha; in| \, U(-t')\varphi(\vec{x}, -t')\dot{U}^{-1}(-t')U(-t) \, |0\rangle \qquad (3.24)$$

We take now the $t = t' \to \infty$ limit. Then

$$\langle \alpha p; in| \, U(-t) \, |0\rangle = \sqrt{Z} \, \langle \alpha; in| \, U(-t)a_{in}(p) \, |0\rangle$$
$$+ \langle \alpha; in| \, \dot{U}(-t)\varphi(\vec{x}, -t) + U(-t)\varphi(\vec{x}, -t)\dot{U}^{-1}(-t)U(-t) \, |0\rangle \qquad (3.25)$$

Now the first term in Eq. (3.25) vanishes because $a_{in}(p) \, |0\rangle = 0$. The second term also vanishes because we have (we omit the arguments to simplify the notation),

$$\begin{aligned} \dot{U}\varphi + U\varphi\dot{U}^{-1}U &= \dot{U}U^{-1}\varphi_{in}U + \varphi_{in}U\dot{U}^{-1}U \\ &= \dot{U}U^{-1}\varphi_{in}U - \varphi_{in}\dot{U}U^{-1}U \\ &= [\dot{U}U^{-1}, \varphi_{in}]U = -i[H_I, \varphi_{in}]U = 0 \qquad (3.26) \end{aligned}$$

where we have used Eq. (3.7) and assumed that the interactions have no derivative[1]. We conclude then that,

$$\lim_{t\to\infty} \langle \alpha p; in|U(-t)|0\rangle = 0 \qquad (3.27)$$

for all states *in* that contain at least one particle. This means that,

$$\lim_{t\to\infty} U(-t) \, |0\rangle = \lambda_- \, |0\rangle \qquad (3.28)$$

In a similar way we could show that,

$$\lim_{t\to\infty} U(t) \, |0\rangle = \lambda_+ \, |0\rangle \qquad (3.29)$$

Returning now to the expression for the Green function, we can write,

$$G(x_1, \cdots x_n) = \lambda_- \lambda_+^* \, \langle 0| \, T \left(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \exp \left[-i \int_{-t}^{t} H_I'(t')dt' \right] \right) |0\rangle \qquad (3.30)$$

The dependence in the operator U disappeared from the expectation value. To proceed, let us evaluate the constants $\lambda_\pm$, or more to the point, the combination $\lambda_-\lambda_+^*$ that appears in Eq. (3.30). We get (in the limit $\to \infty$),

$$\lambda_-\lambda_+^* \;\; = \;\; \langle 0| \, U(-t) \, |0\rangle \, \langle 0| \, U^{-1}(t) \, |0\rangle$$

[1]The study of theories with derivatives was not trivial before the quantization via path integral was introduced. As we will be viewing this method for gauge theories, we can avoid here the complications of the derivatives. The quantization via path integral is the only method that is available for non-abelian gauge theories as we will be discussing in chapter 6.

$$
\begin{aligned}
&= \langle 0| U(-t)U^{-1}(t) |0\rangle = \langle 0| U(-t,t) |0\rangle \\
&= \langle 0| T\left(\exp\left[+i\int_{-t}^{t} dt' H_I'(t')\right]\right) |0\rangle \\
&= \langle 0| T\left(\exp\left[-i\int_{-t}^{t} dt' H_I'(t')\right]\right) |0\rangle^{-1}
\end{aligned} \tag{3.31}
$$

Using this result we can write the Green function of Eq. (3.30) in the form,

$$
G(x_1,\cdots,x_n) = \frac{\langle 0| T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\exp[-i\int_{-t}^{t} dt' H_I'(t')]) |0\rangle}{\langle 0| T(\exp[-i\int_{-t}^{t} dt' H_I'(t')) |0\rangle} \tag{3.32}
$$

when $t \to \infty$. Before we write the final expression, we can now introduce the number $E_0(t)$. For that we recall that,

$$
H_I' = H_I + E_0 \tag{3.33}
$$

and noticing that E_0 is not an operator, we get a factor $\exp[-i\int_{-t}^{t} dt' E_0(t')]$ both in the numerator and denominator, canceling out in the final result. The final result can then be obtained from Eq. (3.32), just substituting H_I' by H_I. We get,

$$
\begin{aligned}
G(x_1\cdots,x_n) &= \frac{\langle 0| T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\exp[-i\int_{-t}^{t} dt' H_I(t')]) |0\rangle}{\langle 0| T(\exp[-i\int_{-t}^{t} dt' H_I(t')) |0\rangle} \\
&= \frac{\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{\infty} d^4y_1\cdots d^4y_m \langle 0| T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)\mathcal{H}_I(y_1)\cdots\mathcal{H}_I(y_m) |0\rangle}{\sum_{m=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty} d^4y_1\cdots d^4y_m \langle 0| T(\mathcal{H}_I(y_1)\cdots\mathcal{H}_I(y_m)) |0\rangle}
\end{aligned} \tag{3.34}
$$

This equation is the fundamental result. The Green functions have been expressed in terms of the *in* fields whose algebra we know. It is therefore possible to reduce Eq. (3.34) to known quantities. In this reduction plays an important role the Wick's theorem, to which we now turn.

3.3 Wick's theorem

To evaluate the amplitudes that appear in Eq. (3.34) we have to move the annihilation operators to the right until they act on the vacuum. The final result from these manipulations can be stated in the form of a theorem, known as Wick's theorem, which reads,

$$
\begin{aligned}
&T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n)) = \\
&= \,:\varphi_{in}(x_1)\cdots\varphi_{in}(x_n): + [\langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0\rangle :\varphi_{in}(x_3)\cdots\varphi_{in}(x_n): + \text{perm.} \\
&+ \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)) |0\rangle \langle 0| T(\varphi_{in}(x_3)\varphi_{in}(x_4)) |0\rangle :\varphi_{in}(x_5)\cdots\varphi_{in}(x_n): + \text{perm.} \\
&+\cdots
\end{aligned}
$$

$$+\begin{cases} \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2))|0\rangle \cdots \langle 0| T(\varphi_{in}(x_{n-1})\varphi_{in}(x_n))|0\rangle + \text{perm.} \\ \hfill \text{n even} \\ \\ \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)|0\rangle \cdots \langle 0| T(\varphi_{in}(x_{n-2})\varphi_{in}(x_{n-1}))|0\rangle\, \varphi_{in}(x_n) + \text{perm.} \\ \hfill \text{n odd} \end{cases} \tag{3.35}$$

Proof:

The proof of the theorem is done by induction. For $n = 1$ it is certainly true (and trivial). Also for $n = 2$ we can shown that

$$T(\varphi_{in}(x_1)\varphi_{in}(x_2)) =: \varphi_{in}(x_1)\varphi_{in}(x_2) : + \text{c-number} \tag{3.36}$$

where the *c-number* comes from the commutations that are needed to move the annihilation operators to the right. To find this constant, we do not have to do any calculation, just to notice that

$$\langle 0| : \cdots : |0\rangle = 0 \tag{3.37}$$

Then

$$T(\varphi_{in}(x_1)\varphi_{in}(x_2)) =: \varphi_{in}(x_1)\varphi_{in}(x_2) : + \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2))|0\rangle \tag{3.38}$$

which is in agreement with Eq. (3.35).

Continuing with the induction, let us assume that Eq. (3.35) is valid for a given n. We have to show that it remains valid for $n+1$. Let us consider then $T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_{n+1}))$ and let us assume that t_{n+1} is the earliest time. Then

$$\begin{aligned} T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_{n+1})) &= \\ &= T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n))\varphi_{in}(x_{n+1}) \\ &= :\varphi_{in}(x_1)\cdots\varphi_{in}(x_n):\varphi_{in}(x_{n+1}) \\ &\quad + \sum_{\text{perm}} \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2))|0\rangle :\varphi_{in}(x_3)\cdots\varphi_{in}(x_n):\varphi_{in}(x_{n+1}) \\ &\quad + \cdots \end{aligned} \tag{3.39}$$

To write Eq. (3.39) in the form of Eq. (3.35) it is necessary to find the rule showing how to introduce $\varphi_{in}(x_{n+1})$ inside the normal product. For that, we introduce the notation,

$$\varphi_{in}(x) = \varphi_{in}^{(+)}(x) + \varphi_{in}^{(-)}(x) \tag{3.40}$$

where $\varphi_{in}^{(+)}(x)$ contains the annihilation operator and $\varphi_{in}^{(-)}(x)$ the creation operator. Then we can write,

$$:\varphi_{in}(x_1)\cdots\varphi_{in}(x_n) := \sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\prod_{j\in B}\varphi_{in}^{(+)}(x_j) \tag{3.41}$$

where the sum runs over all the sets A, B that constitute partitions of the n indices. Then

$$
\begin{aligned}
&: \varphi_{in}(x_1)\cdots\varphi_{in}(x_n) : \varphi_{in}(x_{n+1}) = \\
&\quad = \sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\prod_{j\in B}\varphi_{in}^{(+)}(x_j)[\varphi_{in}^{(+)}(x_{n+1})+\varphi_{in}^{(-)}(x_{n+1})] \\
&\quad = \sum_{A,B}\prod_{i\epsilon A}\varphi_{in}^{(-)}(x_i)\prod_{j\in B}\varphi_{in}^{(+)}(x_j)\varphi_{in}^{(+)}(x_{n+1}) \\
&\qquad + \sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\varphi_{in}^{(-)}(x_{n+1})\prod_{j\in B}\varphi_{in}^{(+)}(x_j) \\
&\qquad + \sum_{A,B}\prod_{i\in A}\varphi_{in}^{(-)}(x_i)\sum_{k\in B}\prod_{j\in B j\neq k}\varphi_{in}^{(+)}(x_j)\,\langle 0|\,\varphi_{in}^{(+)}(x_k)\varphi_{in}^{(-)}(x_{n+1})\,|0\rangle
\end{aligned}
\tag{3.42}
$$

we can now write,

$$
\begin{aligned}
\langle 0|\,\varphi_{in}^{(+)}(x_k)\varphi_{in}^{(-)}(x_{n+1})\,|0\rangle &= \langle 0|\,\varphi_{in}(x_k)\varphi_{in}(x_{n+1})\,|0\rangle \\
&= \langle 0|\,T(\varphi_{in}(x_k)\varphi_{in}(x_{n+1}))\,|0\rangle
\end{aligned}
\tag{3.43}
$$

where we have used the fact that t_{n+1} is the earliest time. We can then write Eq. (3.42) in the form,

$$
\begin{aligned}
&: \varphi_{in}(x_1)\cdots\varphi_{in}(x_n) : \varphi_{in}(x_{n+1}) =: \varphi_{in}(x_1)\cdots\varphi_{in}(x_{n+1}) : \\
&\quad + \sum_k : \varphi_{in}(x_1)\cdots\varphi_{in}(x_{k-1})\varphi_{in}(x_{k+1})\cdots\varphi_{in}(x_n) : \langle 0|\,T(\varphi_{in}(x_k)\varphi_{in}(x_{n+1}))\,|0\rangle
\end{aligned}
\tag{3.44}
$$

With this result, Eq. (3.39) takes the general form of Eq. (3.35) for the $n+1$ case, ending the proof of the theorem. To fully understand the theorem, it is important to do in detail the case $n=4$, to see how things work. The importance of the Wick's theorem for the applications comes from the following two corollaries.

Corollary 1 : If n is odd, then $\langle 0|\,T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n))\,|0\rangle = 0$, as results trivially from Eqs. (3.35) and (3.37) and from,

$$
\langle 0|\,\varphi_{in}(x)\,|0\rangle = 0 \tag{3.45}
$$

Corollary 2: If n is even

$$
\begin{aligned}
&\langle 0|\,T(\varphi_{in}(x_1)\cdots\varphi_{in}(x_n))\,|0\rangle = \\
&\quad = \sum_{\text{perm}}\delta_p\,\langle 0|\,T(\varphi_{in}(x_1)\varphi_{in}(x_2))\,|0\rangle\cdots\langle 0|\,T(\varphi_{in}(x_{n-1})\varphi_{in}(x_n))\,|0\rangle
\end{aligned}
\tag{3.46}
$$

where δ_p is the sign of the permutation that is necessary to introduce in case of fermion fields. This result, that in practice is the most important one, also results from Eqs. (3.35), (3.37) and (3.45).

Therefore the vacuum expectation value of the time-ordered product of n operators that appear in the general formula, Eq. (3.34), are obtained considering all the vacuum expectation values of the fields taken two by two (*contractions*) in all possible ways. Now these *contractions* are nothing else than the Feynman propagators for free fields. For instance,

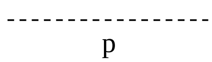
$$\begin{aligned}
&\langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4))|0\rangle \\
&\quad = \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2))|0\rangle \langle 0| T(\varphi_{in}(x_3)\varphi_{in}(x_4)|0\rangle \\
&\quad\quad + \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_3))|0\rangle \langle 0| T(\varphi_{in}(x_2)\varphi_{in}(x_4))|0\rangle \\
&\quad\quad + \langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_4))|0\rangle \langle 0| T(\varphi_{in}(x_2)\varphi_{in}(x_3))|0\rangle \\
&\quad = \Delta_F(x_1-x_2)\Delta_F(x_3-x_4)+\Delta_F(x_1-x_3)\Delta_F(x_2-x_4) \\
&\quad\quad +\Delta_F(x_1-x_4)\Delta_F(x_2-x_3)
\end{aligned} \tag{3.47}$$

where

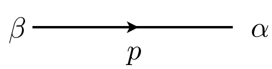
$$\Delta_F(x-y)=\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik(x-y)} \tag{3.48}$$

is the Feynman propagator for the free field theory in the case of scalar fields.

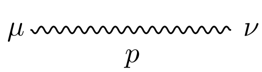
It is convenient to use a graphical (diagrammatic) representation for these propagators. We have in configuration space,



$$\Delta_F(x-y)=\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik\cdot(x-y)} \tag{3.49}$$



$$S_F(x-y)_{\alpha\beta}=\int \frac{d^4p}{(2\pi)^4}\frac{i(\not p+m)_{\alpha\beta}}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)} \tag{3.50}$$



$$D_F^{\mu\nu}(x-y)=\int \frac{d^4k}{(2\pi)^4}\frac{-ig^{\mu\nu}}{k^2+i\epsilon}e^{-ik\cdot(x-y)} \tag{3.51}$$

respectively for scalar, spinor and photon (in the Feynman gauge) fields.

As the interaction Hamiltonian is normal ordered, there will be no contractions between the fields that appear in $\mathcal{H}_I$. The fields in $\mathcal{H}_I$ can only contract with fields outside. In this way the contractions will connect the points corresponding to $\mathcal{H}_I$, the so-called vertices, to either external points or points in another $\mathcal{H}_I$, corresponding to another vertex. To illustrate this point let us consider the $\lambda\varphi^4$ theory where,

$$\mathcal{H}_I(x)=\frac{1}{4!}\lambda : \varphi_{in}^4(x) : \tag{3.52}$$

Then a contribution of order λ^2 to $G(x_1,x_2,x_3,x_4)$ comes from the term,

$$\frac{\lambda^2}{(4!)^2}\langle 0| T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \varphi_{in}^4(y_1) :: \varphi_{in}^4(y_2) : |0\rangle \tag{3.53}$$

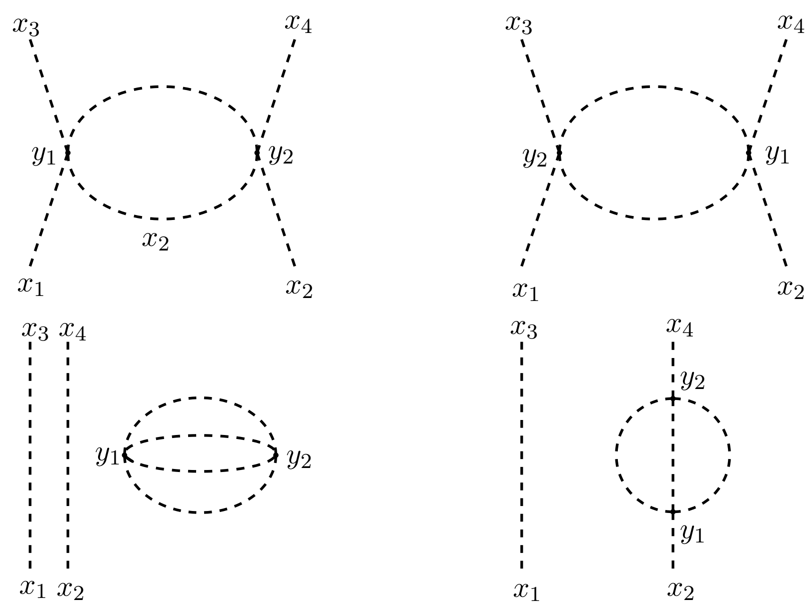


Figure 3.1: Some of the diagrams resulting from Eq (3.53).

and leads to the diagrams in Fig. (3.1). In these diagrams, the interaction is represented by four lines coming from one point, y_1 or y_2. These lines are contractions between one field from one $\mathcal{H}_I$ with other field that might belong either to another $\mathcal{H}_I$, or be one of the external fields in $G(x_1 \cdots x_4)$. To obtain the Feynman rules we are left with a combinatorial problem. We are not going to find them here, as they are much easier to express in momentum space, as we will see in the following.

In Fig. (3.1) the diagrams a), b) and d) are called *connected* while the diagram c) is called *disconnected*. One diagram is *disconnected* when there is a part of the diagram that is not connected in any way to an external line. We will see in the following that these diagrams do not contribute to the Green functions. Diagram d) is *connected* but is also called *reducible* because it can be obtained by multiplication of simpler Green functions. As we will see only the *irreducible* diagrams are important.

3.4 Vacuum–Vacuum amplitudes

We have seen in the previous section examples of the numerator of Eq. (3.34). Let us now look at the denominator, the so-called vacuum-vacuum amplitudes. Continuing with the example of $\lambda\varphi^4$, some of the diagrams contributing for these amplitudes are shown in Fig. (3.2). The diagrams associated with the numerator of Eq. (3.34) can be separated into connected and disconnected parts. For all diagrams that have as connected part a contribution of order s in the interaction $\mathcal{H}_I$, the numerator of $G(x_1 \cdots x_n)$ takes the form,

$$\sum_{p=0}^{\infty} \frac{(-i)^p}{p!} \int d^4y_1 \cdots d^4y_p \left\langle 0 \right| T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \mathcal{H}_I(y_1) \cdots \mathcal{H}_I(y_s)) \left| 0 \right\rangle_c$$

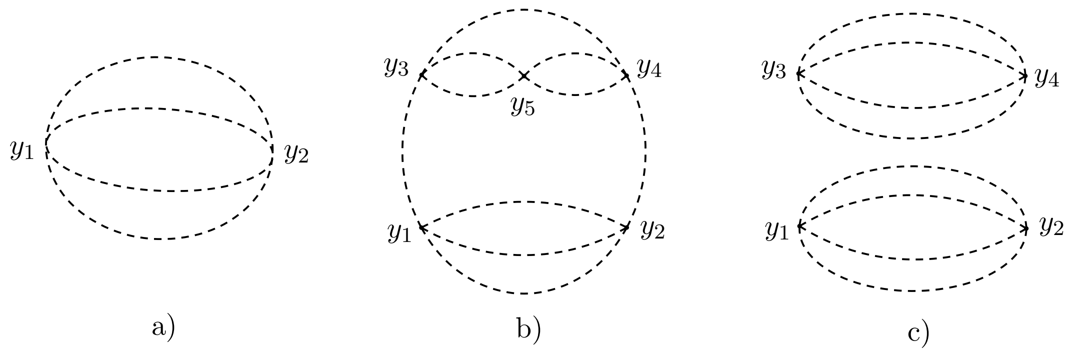


Figure 3.2: Some vacuum–vacuum amplitudes in Eq. (3.54).

$$\times \frac{p!}{s!(p-s)!} \langle 0| T(\mathcal{H}_I(y_{s+1}) \cdots \mathcal{H}_I(y_p)) |0\rangle \tag{3.54}$$

where the subscript c indicates that only the connected parts are included. The combinatorial factor

$$\begin{pmatrix} p \\ s \end{pmatrix} = \frac{p!}{s!(p-s)!} \tag{3.55}$$

is the number of ways in which we can extract s terms $\mathcal{H}_I$ from a set of p terms. We write then Eq. (3.54) in the form,

$$\begin{aligned} &\frac{(-i)^s}{s!} \int d^4y_1 \cdots d^4y_s \langle 0| T(\varphi_{in}(x_1) \cdots \varphi_{in}(x_n) \mathcal{H}_I(y_1) \cdots \mathcal{H}(y_s)) |0\rangle_c \\ &\qquad \times \sum_{r=0}^{\infty} \frac{(-i)^r}{r!} \int d^4z_1 \cdots d^4z_r \langle 0| T(\mathcal{H}_I(z_1) \cdots \mathcal{H}_I(z_r)) |0\rangle \end{aligned} \tag{3.56}$$

This equation has the form of a connected diagram of order s times an infinite series of vacuum-vacuum amplitudes, that cancels exactly against the denominator. This is true for all orders, and therefore we can write,

$$\begin{aligned} G(x_1, \cdots x_n) &= \frac{\sum_i G_i(x_1 \cdots x_n)}{\sum_k D_k} = \frac{(\sum_i G_i^c(x_1, \cdots x_n))(\sum_k D_k)}{\sum_k D_k} \\ &= \sum_i G_i^c(x_1 \cdots x_n) \end{aligned} \tag{3.57}$$

where G_i^c are the connected diagrams and D_k the disconnected ones. This result means that we can simply ignore completely the disconnected diagrams and consider only the connected ones when evaluating the Green functions. These are simply the sum of all connected diagrams, simplifying enormously the structure of Eq. (3.34).

3.5 Feynman rules for $\lambda\varphi^4$

To understand how the Feynman rules appear, let us consider the case of a real scalar field with an interaction of the form,

$$\mathcal{H}_I = \frac{\lambda}{4!} : \varphi_{in}^4 := -\mathcal{L}_I \tag{3.58}$$

To be more precise we consider two particles in the initial and final state. Then the S matrix element is,

$$\begin{aligned} S_{fi} &= \left\langle p_1' p_2'; out | p_1 p_2; in \right\rangle \\ &= (i)^4 \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + ip_i' x_3 + ip_2' \cdot x_4} \\ &\qquad (\Box_{x_1} + m^2) \cdots (\Box_{x_4} + m^2) \left\langle 0 \right| T(\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)) \left| 0 \right\rangle \end{aligned} \tag{3.59}$$

For the Green function we use the expressions in Eqs. (3.34) and (3.57) and we obtain,

$$\begin{aligned} &G(x_1, x_2, x_3, x_4) = \sum_{p=0}^{\infty} \frac{(-i\lambda)^p}{p!} \int d^4z_1 \cdots d^4z_p \\ &\left\langle 0 \right| T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \frac{\varphi_{in}^4(z_1)}{4!} : \cdots : \frac{\varphi_{in}^4(z_p)}{4!} :) \left| 0 \right\rangle_c \end{aligned} \tag{3.60}$$

As the case $p = 0$ is trivial (there is no interaction) we begin by the $p = 1$ case.

- $p = 1$

Then the Green function is,

$$\begin{aligned} G(x_1, x_2, x_3, x_4) &= (-i\lambda) \int d^4z \left\langle 0 \right| T\left(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4) : \frac{\varphi_{in}^4(z)}{4!} : \right) \left| 0 \right\rangle \\ &= (-i\lambda)\frac{4!}{4!} \int d^4z \Delta_F(x_1 - z)\Delta_F(x_2 - z)\Delta_F(x_3 - z)\Delta_F(x_4 - z) \end{aligned} \tag{3.61}$$

to which corresponds, in the configuration space, the diagram of Fig. (3.3). To proceed, we introduce the Fourier transform of the propagators, that is,

$$\Delta_F(x_1 - z) = \int \frac{d^4q_1}{(2\pi)^4} e^{-iq_1\cdot(x_1 - z)} \Delta_F(q_1) \tag{3.62}$$

where

$$\Delta_F(q_1) = \frac{i}{q_1^2 - m^2} \tag{3.63}$$

then

$$G(x_1, \cdots x_4) = (-i\lambda) \int d^4z \frac{d^4q_1}{(2\pi)^4} \cdots \frac{d^4q_4}{(2\pi)^4}\ e^{-iq_1\cdot x_1 - iq_2 x_2 - iq_3 x_3 - iq_4 x_4 + i(q_1+q_2+q_3+q_4)\cdot z}$$

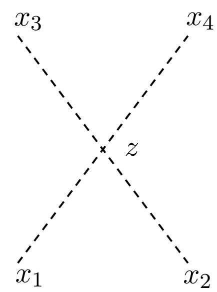


Figure 3.3: Vertex in the $\lambda\phi^4$ theory.

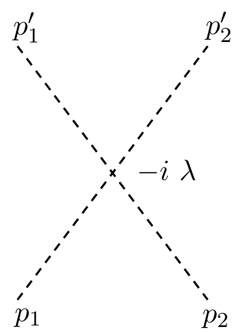


Figure 3.4: Vertex in momentum space.

$$
\begin{aligned}
&\Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4)\\
= \ (-i\lambda)\int &\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_4}{(2\pi)^4}\ e^{-iq_1\cdot x_1-iq_2\cdot x_2-iq_3\cdot x_3-iq_4\cdot x_4}\\
&(2\pi)^4\delta^4(q_1+q_2+q_3+q_4)\Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4)
\end{aligned}
\tag{3.64}
$$

If we now introduce the T matrix transition amplitude, defined by

$$S_{fi} = \delta_{fi} - i(2\pi)^4\delta(P_f - P_i)\ T_{fi} \tag{3.65}$$

we obtain

$$-iT_{fi} = (-i\lambda) \tag{3.66}$$

for this amplitude we draw the Feynman diagram of Fig. (3.4), and we associate to the vertex the number $(-i\lambda)$.

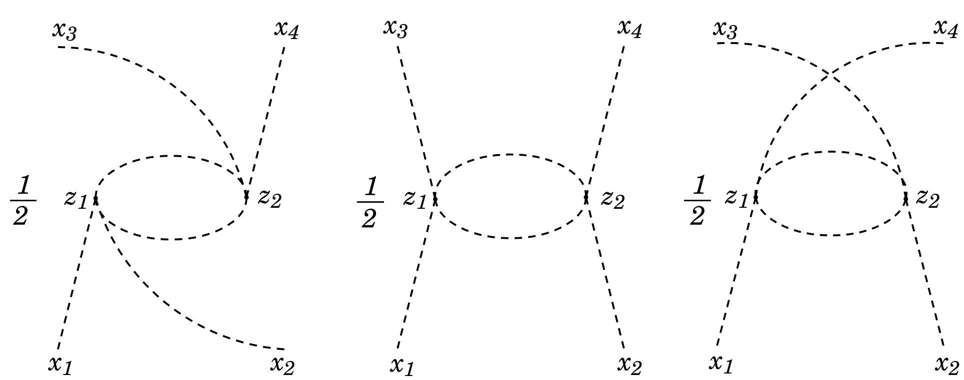
- $p = 2$

Let us consider now a more complicated case, the evaluation of $G(x_1\cdots x_4)$ in second order in the coupling λ. After this exercise we will be in position to be able to state the

Feynman rules in momentum space with all generality. From Eq. (3.60) we get in second order in λ,

$$
\begin{aligned}
&G(x_1,\cdots x_4)=\\
&=\frac{(-i\lambda)^2}{2!}\int d^4z_1d^4z_2\,\langle 0|\,T\bigg(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4):\frac{\varphi_{in}^4(z_1)}{4!}::\frac{\varphi_{in}^4(z_2)}{4!}:\bigg)|0\rangle_c\\
&=\frac{(-i\lambda)^2}{2!}\int d^4z_1d^4z_2\left(\frac{4\times 3}{4!}\right)\times\left(\frac{4\times 3}{4!}\right)\times 2\\
&\quad\Big\{\Delta_F(x_1-z_1)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_2-x_3)\Delta_F(z_2-x_4)\\
&\quad+\Delta_F(x_1-z_1)\Delta_F(x_2-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-x_3)\Delta_F(z_2-x_4)\\
&\quad+\Delta_F(x_1-z_1)\Delta_F(x_2-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-x_4)\Delta_F(z_2-x_3)\\
&\quad+\Delta_F(x_1-z_2)\Delta_F(x_2-z_2)\Delta_F(z_2-z_1)\Delta_F(z_2-z_1)\Delta_F(z_1-x_3)\Delta_F(z_1-x_4)\\
&\quad+\Delta_F(x_1-z_2)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_2-z_1)\Delta_F(z_1-x_3)\Delta_F(z_2-x_4)\\
&\quad+\Delta_F(x_1-z_2)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)\Delta_F(z_1-x_4)\Delta_F(z_1-x_3)\Big\}\\
&=\frac{(-i\lambda)^2}{2!}\int d^4z_1d^4z_2
\end{aligned}
$$

$\Big\{$



$$+\,(z_1\leftrightarrow z_2)\Big\}\tag{3.67}$$

Let us now go into momentum space, by introducing the Fourier transform of the propagators. We start by diagram a),

$$
\begin{aligned}
&G^{(a)}(x_1,x_2,x_3,x_4)=\\
&\quad=\frac{(-i\lambda)^2}{2!}\frac{1}{2}\int d^4z_1d^4z_2\Delta_F(x_1-z_1)\Delta_F(x_2-z_1)\Delta_F(z_1-z_2)\Delta_F(z_1-z_2)
\end{aligned}
$$

$$
\begin{aligned}
& \qquad \Delta_F(z_2 - x_3)\Delta_F(z_2 - x_4) \\
= & \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int d^4z_1 d^4z_2 \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{d^4q_3}{(2\pi)^4}\frac{d^4q_4}{(2\pi)^4}\frac{d^4q_5}{(2\pi)^4}\frac{d^4q_6}{(2\pi)^4} \\
& \qquad e^{i[(q_1\cdot x_1+q_2\cdot x_2-q_3\cdot x_3-q_4\cdot x_4)+z_1\cdot(q_5-q_1-q_2+q_6)+z_2\cdot(q_3+q_4-q_5-q_6)]} \\
& \qquad \Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4)\Delta_F(q_5)\Delta_F(q_6) \\
= & \frac{(-i\lambda)^2}{2!}\frac{1}{2}(2\pi)^4\int \frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_5}{(2\pi)^4}\delta^4(q_1+q_2-q_3-q_4)\ e^{i[q_1\cdot x_1+q_2\cdot x_2-q_2\cdot x_3-q_4\cdot x_4]} \\
& \qquad \Delta_F(q_1)\Delta_F(q_2)\Delta_F(q_3)\Delta_F(q_4)\Delta_F(q_5)\Delta_F(q_1+q_2-q_5)
\end{aligned}
\tag{3.68}
$$

Now we insert the last equation into the reduction formula. We get

$$
\begin{aligned}
S_{fi}^{(a)} &= (i)^4\int d^4x_1\cdots d^4x_4 e^{-i[p_1\cdot x_1+p_2\cdot x_2-p'_1\cdot x_3-p'_2\cdot x_4]} \\
& \qquad (\Box_{x_1}+m^2)\cdots(\Box_{x_4}+m^2)G^{(a)}(x_1,\cdots,x_4)
\end{aligned}
\tag{3.69}
$$

The only dependence of $G^{(a)}$ in the coordinates, $x_i (i = 1, \cdots 4)$, is in the exponential, therefore,

$$
(\Box_{x_i}+m^2)\to(-q_i^2+m^2) \tag{3.70}
$$

and using

$$
(-q_i^2+m^2)\Delta_F(q_i) = -i \tag{3.71}
$$

we get

$$
\begin{aligned}
S_{fi}^{(a)} &= \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int d^4x_1\cdots d^4x_4\int\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_5}{(2\pi)^4}(2\pi)^4\delta^4(q_1+q_2-q_3-q_4) \\
& \qquad e^{-i[x_1\cdot(p_1-q_1)+x_2\cdot(p_2-q_2)-x_3\cdot(p'_1-q_3)-x_4\cdot(p'_2-q_4)]}\Delta_F(q_5)\Delta_F(q_1+q_2-q_5) \\
&= \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int\frac{d^4q_1}{(2\pi)^4}\cdots\frac{d^4q_5}{(2\pi)^4}(2\pi)^4\delta^4(q_1+q_2-q_3-q_4)(2\pi)^4\delta^4(p_1-q_1) \\
& \qquad (2\pi)^4\delta^4(p_2-q_2)(2\pi)^4\delta^4(p'_1-q_3)(2\pi)^4\delta^4(p'_2-q_4)\Delta_F(q_5)\Delta_F(q_1+q_2-q_5) \\
&= \frac{(-i\lambda)^2}{2!}\frac{1}{2}\int\frac{d^4q_5}{(2\pi)^4}(2\pi)^4\delta^4(p_1+p_2-p'_1-p'_2)\Delta_F(q_5)\Delta_F(p_1+p_2-q_5)
\end{aligned}
\tag{3.72}
$$

This expression can be written in the form

$$
S_{fi}^{(a)} = (2\pi)^4\delta^4(p_1+p_2-p'_1-p'_2)\frac{(-i\lambda)^2}{2!}\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)\Delta_F(p_1+p_2-q) \tag{3.73}
$$

If we denote by a') the diagram a) with the interchange $z_1 \leftrightarrow z_2$ and redo the calculation we get exactly the same result as in Eq. (3.73). Therefore,

$$
S_{fi}^{(a+a')} = (2\pi)^4\delta^4(p_1+p_2-p'_1-p'_2)(-i\lambda)^2\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Delta_F(q)\Delta_F(p_1+p_2-q) \tag{3.74}
$$

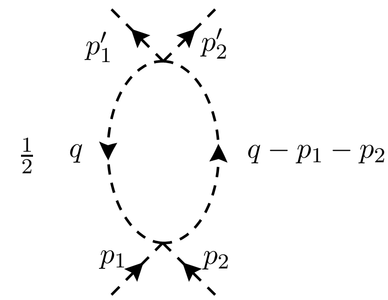


Figure 3.5:

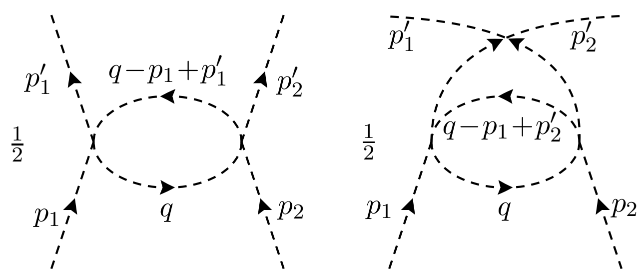


Figure 3.6:

or in terms of the T_{fi} matrix,

$$-iT_{fi}^{(a+a')} = (-i\lambda)^2 \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q)\Delta_F(p_1 + p_2 - q) \tag{3.75}$$

To encode this result we draw the Feynman diagram of Fig. (3.5), that has the same topology as a) and a') but in momentum space. We find that in order to evaluate the $-iT$ matrix, we associate to each vertex a factor $(-i\lambda)$, to each internal line a propagator Δ_F and for each loop the integral $\int \frac{d^4q}{(2\pi)^4}$. Besides that we have 4-momentum conservation at each vertex. Finally there is a symmetry factor (see below) which takes the value $\frac{1}{2}$ for this diagram.

If we repeat the calculations for diagrams $b) + b'$) and $c) + c'$) it is easy to see that we get,

$$-iT_{fi}^{(b+b')} = (-i\lambda)^2 \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q)\Delta_F(q - p_1 + p_1') \tag{3.76}$$

and

$$-iT_{fi}^{(c+c')} = (-i\lambda)^2 \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Delta_F(q)\Delta_F(q - p_1 + p_2') \tag{3.77}$$

to which correspond the diagrams of Fig. (3.6).

After this exercise we are in position to state the Feynman rules with all generality for the $\lambda\varphi^4$ theory. These are rules for the $-iT$ matrix, that is, after we factorize $(2\pi)^4\delta^4(\cdots)$. These are (for a process with n external legs):

1. Draw all topologically distinct diagrams with n external legs.
2. At each vertex multiply by the factor $(-i\lambda)$.
3. To each internal line associate a propagator $\Delta_F(q)$.
4. For each loop include an integral $\int \frac{d^4q}{(2\pi)^4}$. The direction of this momentum is irrelevant, but we have to respect 4-momentum conservation ate each vertex.
5. Multiply by the symmetry factor of the diagram. This is defined by,

$$S = \frac{\text{\# of distinct ways of connecting the vertices to the external legs}}{\text{Permutations of each vertex} \times \text{Permutations of equal vertices}} \tag{3.78}$$

6. Add the contributions of all the topologically distinct diagrams. The result is the $-iT$ matrix amplitude that enters the formula for the cross section.

3.6 Feynman rules for QED

We now turn to the case of `QED`. Like $\lambda\phi^4$, it is a theory without derivatives and therefore,

$$\mathcal{L}_I = -\mathcal{H}_I = -e\; Q\overline{\psi}_{in}\gamma^\mu\psi_{in}A_\mu^{in} \tag{3.79}$$

where e is the absolute value of the electron charge, or the proton charge. For the electron the sign enters explicitly in $Q = -1$. This way of writing in Eq. (3.79), allows for obvious generalizations for particles with other charges, like for instance the quarks. For `QED` we have then,

$$\mathcal{L}_I^{QED} = e\; \overline{\psi}_{in}\gamma^\mu\psi_{in}A_\mu^{in} \tag{3.80}$$

Due to the electric charge conservation, the Green functions that we have to deal with have an equal number of ψ and $\overline{\psi}$ fields. In general we have,

$$\begin{aligned} &G(x_1\cdots x_n x_{n+1}\cdots x_{2n}; y_1\cdots y_p) = \\ &\qquad = \langle 0|\, T(\psi(x_1)\cdots\psi(x_n)\overline{\psi}(x_{n+1})\cdots\overline{\psi}(x_{2n})A^{\mu_1}(y_1)\cdots A^{\mu_p}(y_p))\,|0\rangle \end{aligned} \tag{3.81}$$

where, for simplicity, we omit the spinorial indices in the fermion fields. This equation is written in terms of the physical fields. Following a similar procedure to the scalar field case, we can obtain an expression for G in terms of the *in* fields. This will be,

$$\begin{aligned} G(x_1\cdots x_{2n}; y_1\cdots y_p) \;&=\; \frac{\langle 0|\, T\psi_{in}(x_1)\cdots\overline{\psi}_{in}(x_{2n})A_{in}^{\mu_1}(y_1)\cdots A_{in}^{\mu_p}(y_p)\; e^{[i\int d^4z\mathcal{L}_I(z)]}\,|0\rangle}{\langle 0|\, T\exp[i\int d^4z\mathcal{L}_I(z)]\,|0\rangle} \\ &=\; \langle 0|\, T\psi_{in}(x_1)\cdots\overline{\psi}_{in}(x_{2n})A_{in}^{\mu_1}(y_1)\cdots A_{in}^{\mu_p}(y_p)\; e^{[i\int d^4t\mathcal{L}_I(z)]}\,|0\rangle_c \end{aligned} \tag{3.82}$$

where the fields in $\mathcal{L}_I$ are normal ordered, and $\langle 0|\cdots|0\rangle_c$ means that we only consider the connected diagrams. To get the Feynman rule we will evaluate a few simple processes.

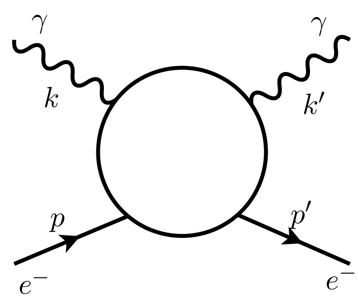


Figure 3.7: Kinematics for the Compton scattering.

3.6.1 Compton scattering

Compton scattering corresponds to the following process,

$$e^- + \gamma \to e^- + \gamma \tag{3.83}$$

and we choose the kinematics in Fig. (3.7). The S matrix element to evaluate is therefore,

$$S_{fi} = \left\langle (p', s'), k'; out | (p, s), k; in \right\rangle \tag{3.84}$$

Using the LSZ reduction formula Eq. (2.101), and Eq. (2.109) we can write,

$$S_{fi} = \int d^4x d^4x' \int d^4y d^4y' e^{-i[p\cdot x + k\cdot y - p'\cdot x' - k'\cdot y']} \varepsilon^{\mu}(k) \varepsilon^{*\mu'}(k')$$

$$\overline{u}(p', s')_{\alpha'} (i\overrightarrow{\not\partial}_{x'} - m)_{\alpha'\beta'} \Box_y \Box_{y'} \langle 0 | T(\psi_{\beta'}(x') \overline{\psi}_{\beta}(x) A_{\mu}(y) A_{\mu'}(y')) | 0 \rangle (-i\overleftarrow{\not\partial}_x - m)_{\beta\alpha} u_{\alpha}(p, s) \tag{3.85}$$

Our task is therefore to evaluate the Green function

$$G(x', x, y, y') \equiv \langle 0 | T(\psi_{\beta'}(x') \overline{\psi}_{\beta}(x) A_{\mu}(y) A_{\mu'}(y')) | 0 \rangle \tag{3.86}$$

If we use Eq. (3.82) and the fact that the interaction has an odd number of fields, we find that the lowest contribution is quadratic in the interaction[2]. We get

$$\begin{aligned}
G(x, x', y, y') = & \\
= \; & \frac{(ie)^2}{2!} \int d^4z_1 d^4z_2 \langle 0 | T(\psi^{in}_{\beta'}(x') \overline{\psi}^{in}_{\beta}(x) A^{in}_{\mu}(y) A^{in}_{\mu'}(y') \\
& \qquad : \overline{\psi}_{in}(z_1) \gamma^{\sigma} \psi_{in}(z_1) A^{in}_{\sigma}(z_1) :: \overline{\psi}^{in}(z_2) \gamma^{\rho} \psi^{in}(z_2) A^{in}_{\rho}(z_2) :) | 0 \rangle \\
= \; & \frac{(ie)^2}{2!} (\gamma^{\sigma})_{\gamma\delta} \, (\gamma^{\rho})_{\gamma'\delta'} \int d^4z_1 d^4z_2 \langle 0 | T(\psi^{in}_{\beta'}(x') \overline{\psi}^{in}_{\beta}(x) A^{in}_{\mu}(y) A^{in}_{\mu'}(y')
\end{aligned}$$

[2]By Wick's theorem the expectation value of an odd number of fields vanishes.

$$:\overline{\psi}^{in}_{\gamma}(z_1)\psi^{in}_{\delta}(z_1)A^{in}_{\sigma}(z_1)::\overline{\psi}^{in}_{\gamma'}(z_2)\psi^{in}_{\delta'}(z_2)A^{in}_{\rho}(z_2))\,|0\rangle \tag{3.87}$$

Now we use Wick's theorem to write $\langle 0|\,T(\cdots)\,|0\rangle$ in terms of the propagators. We get,

$$\begin{aligned}
&\langle 0|\,T(\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\beta}(x)A^{in}_{\mu}(y)A^{in}_{\mu'}(y'):\overline{\psi}^{in}_{\gamma}(z_1)\psi^{in}_{\delta}(z_1)A^{in}_{\sigma}(z_1)::\overline{\psi}^{in}_{\gamma'}(z_2)\psi^{in}_{\delta'}(z_2)A^{in}_{\rho}(z_2)):)\,|0\rangle\\
=\quad&\langle 0|\,T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma}(z_1)\,|0\rangle\,\langle 0|\,T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\beta}(x)\,|0\rangle\,\langle 0|\,T\psi^{in}_{\delta}(z_1)\overline{\psi}_{\gamma'}(z_2)\,|0\rangle\\
&\quad\langle 0|\,T(A^{in}_{\mu}(y)A^{in}_{\sigma}(z_1))\,|0\rangle\,\langle 0|\,TA^{in}_{\mu'}(y')A^{in}_{\rho}(z_2)\,|0\rangle\\
&+\langle 0|\,T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma}(z_1)\,|0\rangle\,\langle 0|\,T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\beta}(x)\,|0\rangle\,\langle 0|\,T\psi^{in}_{\delta}(z_1)\overline{\psi}_{\gamma'}(z_2)|0\\
&\quad\langle 0|\,TA^{in}_{\mu}(y)A^{in}_{\rho}(z_2)\,|0\rangle\,\langle 0|\,TA^{in}_{\mu'}(y')A^{in}_{\sigma}(z_1)\,|0\rangle\\
&+\langle 0|\,T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma'}(z_2)\,|0\rangle\,\langle 0|\,T\psi^{in}_{\delta}(z_1)\overline{\psi}^{in}_{\beta}(x)\,|0\rangle\,\langle 0|\,T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\gamma}(z_1)\,|0\rangle\\
&\quad\langle 0|\,TA^{in}_{\mu}(y)A^{in}_{\sigma}(z_1)\,|0\rangle\,\langle 0|\,TA^{in}_{\mu'}(y')A^{in}_{\rho}(z_2)\,|0\rangle\\
&+\langle 0|\,T\psi^{in}_{\beta'}(x')\overline{\psi}^{in}_{\gamma'}(z_2)\,|0\rangle\,\langle 0|\,T\psi^{in}_{\delta}(z_1)\overline{\psi}^{in}_{\beta}(x)\,|0\rangle\,\langle 0|\,T\psi^{in}_{\delta'}(z_2)\overline{\psi}^{in}_{\gamma}(z_1)\,|0\rangle\\
&\quad\langle 0|\,TA^{in}_{\mu}(y)A^{in}_{\rho}(z_2)\,|0\rangle\,\langle 0|\,TA^{in}_{\mu'}(y')A^{in}_{\sigma}(z_1)\,|0\rangle\\
=\quad&S_{F\beta'\gamma}(x'-z_1)S_{F\delta'\beta}(z_2-x)S_{F\delta\gamma'}(z_1-z_2)D_{F\mu\sigma}(y-z_1)D_{F\mu'\rho}(y'-z_2)\\
&+S_{F\beta'\gamma}(x'-z_1)S_{F\delta'\beta}(z_2-x)S_{F\delta\gamma'}(z_1-z_2)D_{F\mu\rho}(y-z_2)D_{F\mu'\sigma}(y'-z_1)\\
&+S_{F\beta'\gamma'}(x'-z_2)S_{F\delta\beta}(z_1-x)S_{F\delta'\gamma}(z_2-z_1)D_{F\mu\sigma}(y-z_1)D_{F\mu'\rho}(y'-z_2)\\
&+S_{F\beta'\gamma'}(x'-z_2)S_{F\delta\beta}(z_1-x)S_{F\delta'\gamma}(z_2-z_1)D_{F\mu\rho}(y-z_2)D_{F\mu'\sigma}(y'-z_1)
\end{aligned} \tag{3.88}$$

To better understand Eq. (3.88) it is useful to draw the corresponding diagrams in configuration space. We show them in Fig. (3.8). From this figure it is clear that $b)\equiv c)$ and $a)\equiv d)$ because z_1 and z_2 are irrelevant labels. From this we get a factor of 2 that is going to cancel the $\frac{1}{2!}$ in Eq. (3.87)[3]. We have then only two distinct diagrams that we take as c) and d). Then, including already the factor of 2, we get for diagrama c)

$$\begin{aligned}
G^{(c)}(x,x',y,y') \;=\;& (ie)^2(\gamma^{\sigma})_{\gamma\delta}(\gamma^{\rho})_{\gamma'\delta'}\int d^4z_1d^4z_2\; S_{F\beta'\gamma'}(x'-z_2)S_{F\delta\beta}(z_1-x)\\
&S_{F\delta'\gamma}(z_2-z_1)D_{F\mu\sigma}(y-z_1)D_{F\mu'\rho}(y'-z_2)
\end{aligned} \tag{3.89}$$

To proceed we could, like in the case of $\lambda\varphi^4$, introduce the Fourier transform of the propagators. However, it is easier to get rid of the external legs using the results,

$$\begin{aligned}
&(i\partial\!\!\!/_x-m)_{\alpha\lambda}S_{F\lambda\beta}(x-y)=i\delta_{\alpha\beta}\delta^4(x-y)\\
&S_{F\alpha\lambda}(x-y)(-i\overleftarrow{\partial\!\!\!/}_y-m)_{\lambda\beta}=i\delta_{\alpha\beta}\delta^4(x-y)
\end{aligned} \tag{3.90}$$

[3]In fact this result is general, for n vertices we have $n!$ that cancels against the factor $\frac{1}{n!}$ from the expansion of the exponential.

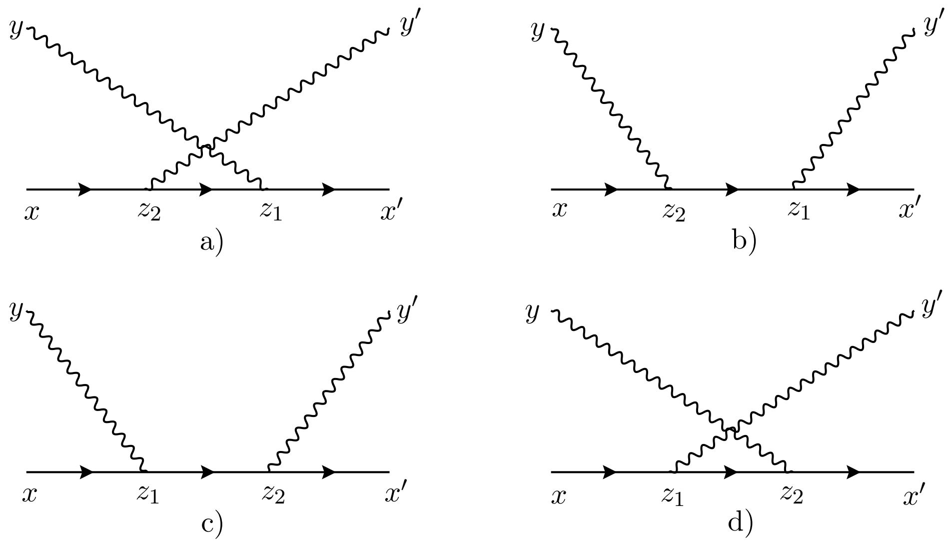


Figure 3.8: Diagrams for Compton scattering in configuration space.

and

$$\Box_x D_{F\mu\nu}(x-y) = i g_{\mu\nu}\delta^4(x-y) \tag{3.91}$$

We get therefore,

$$\begin{aligned}
S_{fi}^{(c)} &= (ie)^2 \int d^4x d^4x' d^4y d^4y' e^{-i(p\cdot x + k\cdot y - p'\cdot x' - k'\cdot y')} \varepsilon^{\mu}(k) g_{\mu\sigma} \varepsilon^{*\mu'}(k') g_{\mu'\rho} \\
&\qquad (\gamma^{\sigma})_{\gamma\delta}(\gamma^{\rho})_{\gamma'\delta'}\ \overline{u}(p',s')_{\alpha'}\delta_{\alpha'\gamma'} u_{\alpha}(p,s)\ \delta_{\delta\alpha} \\
&\qquad \int d^4z_1 d^4z_2 \delta^4(x'-z_2)\delta^4(x-z_1)\delta^4(y-z_1)\delta^4(y'-z_2)\ S_{F\delta'\gamma}(z_2-z_1) \\
&= (ie)^2 \int d^4z_1 d^4z_2 e^{-i(p\cdot z_1 + k\cdot z_1 - p'\cdot z_2 - k'\cdot z_2)} \varepsilon^{\mu}(k)\varepsilon^{*\mu'}(k') \\
&\qquad \overline{u}(p',s')_{\alpha'}(\gamma_{\mu'})_{\alpha'\delta'} S_{F\delta'\gamma}(z_2-z_1)(\gamma_{\mu})_{\gamma\alpha} u_{\alpha}(p,s)
\end{aligned} \tag{3.92}$$

Finally we use

$$\begin{aligned}
S_F(z_2-z_1) &= \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q}+m)}{q^2-m^2+i\epsilon} e^{-iq\cdot(z_2-z_1)} \\
&\equiv \int \frac{d^4q}{(2\pi)^4} S_F(q) e^{-iq\cdot(z_2-z_1)}
\end{aligned} \tag{3.93}$$

to get

$$\begin{aligned}
S_{fi}^{(c)} &= \int \frac{d^4q}{(2\pi)^4} d^4z_1 d^4z_2 e^{-iz_1\cdot(p+k-q) + iz_2\cdot(p'+k'-q)} \\
&\qquad \varepsilon^{\mu}(k)\varepsilon^{\mu'*}(k')\overline{u}(p',s')(ie\gamma_{\mu'})S_F(q)(ie\gamma_{\mu})u(p,s)
\end{aligned}$$

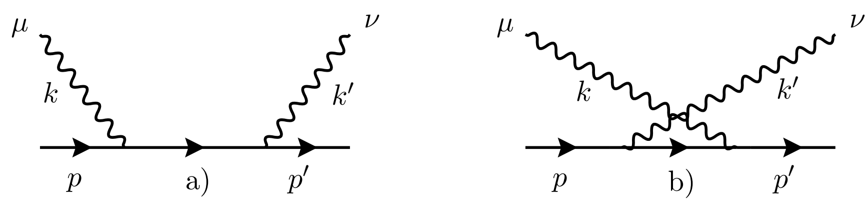


Figure 3.9: Diagrams for Compton scattering.

$$
\begin{aligned}
&= (2\pi)^4\delta^{(4)}(p+k-p'-k)\cdot \\
&\qquad \varepsilon^{\mu}(k)\varepsilon^{\mu'*}(k')\overline{u}(p',s')(ie\gamma_{\mu'})S_F(p+k)(ie\gamma_{\mu})u(p,s) \qquad (3.94)
\end{aligned}
$$

Therefore, the T matrix transition amplitude is,

$$
-iT_{fi}^{(c)} = \varepsilon^{\mu}(k)\varepsilon^{\mu'*}(k')\overline{u}(p',s')(ie\gamma_{\mu'})S_F(p+k)(ie\gamma_{\mu})u(p,s) \qquad (3.95)
$$

corresponding to the diagram on the left panel of Fig. (3.9). In Eq. (3.95) we factor out the quantity $(ie\gamma_\mu)$, because it will be clear that this quantity will be the Feynman rule for the vertex. The arrows in these diagrams correspond to the flow of electric charge. Notice that to an electron in the initial state we associate a spinor $u(p,s)$ and for an electron in the final state we associate the spinor $\overline{u}(p',s')$. Since the electron line as to be a c-number, we start writing the line in the reverse order of that of the arrows.

In a similar way for diagram d) we will get the diagram represented in Fig. (3.9), that corresponds to the following expression,

$$
-iT_{fi}^{(d)} = \varepsilon^{\mu}(k)\varepsilon^{*\mu'}(k')\overline{u}(p',s')(ie\gamma_{\mu})S_F(p-k')(ie\gamma_{\mu'})u(p,s) \qquad (3.96)
$$

Looking at Eqs. (3.95) and (3.96) we are almost in a position to state the Feynman rules for `QED`. Before that we will look at a case where we have positrons.

3.6.2 Electron–positron elastic scattering (Bhabha scattering)

We will consider electron-positron elastic scattering, the so-called *Bhabha scattering*,

$$
e^-(p) + e^+(q) \to e^-(p') + e^+(q') \qquad (3.97)
$$

This example will teach us two things. First, how positrons (that is the anti-particles) enter in the amplitudes. Secondly we will learn that, sometimes, due the anti-commutation rules of the fermions, we will get relative minus signs between different diagrams. We have,

$$
S_{fi} = \big\langle (p',s'),(q',\overline{s'}); out | (p,s),(q,\overline{s}); in \big\rangle \qquad (3.98)
$$

corresponding to the kinematics in Fig. (3.10). Notice that the arrows are in the direction of flow of charge of the electron, but the momenta do correspond to the real momenta of the particles or antiparticles in that frame: p entering and p' exiting for the electron,

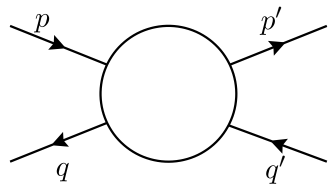


Figure 3.10: Diagram with positrons

and q entering and q' exiting for the positron. In the following we will not show the spin dependence in order to simplify the notation. Then using Eq. (3.98) we write,

$$\begin{aligned} S_{fi} &= \int d^4x d^4y d^4x' d^4y' e^{-i[p\cdot x+q\cdot y-p'\cdot x'-q'\cdot y']} \\ &\quad \overline{u}(p')_\alpha (i\overrightarrow{\not\partial}_{x'}-m)_{\alpha\beta}\, \overline{v}_\gamma(q)(i\overrightarrow{\not\partial}_y-m)_{\gamma\delta} \\ &\quad \langle 0|\, T\overline{\psi}_{\delta'}(y')\psi_\beta(x')\overline{\psi}_{\beta'}(x)\psi_\delta(y)\,|0\rangle \\ &\quad (-i\overleftarrow{\not\partial}_x-m)_{\beta'\alpha'}u_{\alpha'}(p)(-i\overleftarrow{\not\partial}_{y'}-m)_{\delta'\gamma'}v_{\gamma'}(q') \end{aligned} \tag{3.99}$$

We have, therefore, to evaluate the Green function

$$G(y',x',x,y) \equiv \langle 0|\, T\overline{\psi}_{\delta'}(y')\psi_\beta(x')\overline{\psi}_{\beta'}(x)\psi_\delta(y)\,|0\rangle \tag{3.100}$$

The lowest order contribution is of second order in the coupling[4]. We have (to simplify we omit the label *in*),

$$\begin{aligned} G(y',x',x,y) &= \frac{(ie)^2}{2}(\gamma^\mu)_{\epsilon\epsilon'}(\gamma^\nu)_{\varphi\varphi'}\int d^4z_1 d^4z_2 \\ &\quad \langle 0|\, T\overline{\psi}_{\delta'}(y')\psi_\beta(x')\overline{\psi}_{\beta'}(x)\psi_\delta(y) :\overline{\psi}_\epsilon(z_1)\psi_{\epsilon'}(z_1)A_\mu(z_1)::\overline{\psi}_\varphi(z_2)\psi_{\varphi'}(z_2)A_\nu(z_2):|0\rangle \\ &= \frac{(ie)^2}{2}(\gamma^\mu)_{\varepsilon\varepsilon'}(\gamma^\nu)_{\varphi\varphi'}\int d^4z_1 d^4z_2 \\ &\quad \Big[-S_{F\beta\epsilon}(x'-z_1)S_{F\epsilon'\beta'}(z_1-x)S_{F\delta\varphi}(y-z_2)S_{F\varphi'\delta'}(z_2-y')D_{F\mu\nu}(z_1-z_2) \\ &\quad +S_{F\delta\epsilon}(y-z_1)S_{F\epsilon'\beta'}(z_1-x)S_{F\beta\varphi}(x'-z_2)S_{F\varphi'\delta'}(z_2-y')D_{F\mu\nu}(z_1-z_2) \\ &\quad +(z_1\leftrightarrow z_2)\Big] \end{aligned} \tag{3.101}$$

Once more the exchange $(z_1 \leftrightarrow z_2)$ compensates for the factor $\frac{1}{2!}$ and we have two diagrams with a relative minus sign, as it is shown in Fig. (3.11). Let us look at the contribution of diagram a),

$$S_{fi}^{(a)} = -\int d^4x d^4y d^4x' d^4y' d^4z_1 d^4z_2 (ie)^2 (\gamma^\mu)_{\varepsilon\varepsilon'}(\gamma^\nu)_{\varphi\varphi'} e^{-i[p\cdot x+q\cdot y-p'\cdot x'-q'\cdot y']}$$

[4]There is, of course, a contribution without interaction, but that corresponds to disconnected terms in which we are not interested.

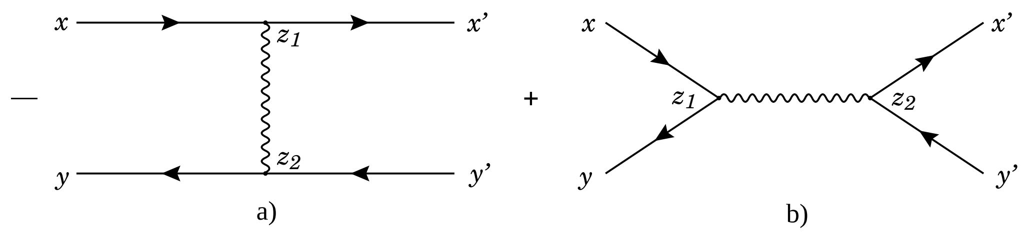


Figure 3.11:

$$
\begin{aligned}
& \overline{u}(p')_\alpha(i\overrightarrow{\partial\!\!\!/}'_x - m)_{\alpha\beta}\overline{v}_\gamma(q)(i\overrightarrow{\partial\!\!\!/}_y - m)_{\gamma\delta} \\
& S_{F\beta\varepsilon}(x'-z_1)S_{F\varepsilon'\beta'}(z_1-x)S_{F\delta\varphi}(y-z_2)S_{F\varphi'\delta'}(z_2-y') \\
& (-i\overleftarrow{\partial\!\!\!/}_x - m)_{\beta'\alpha'}u_{\alpha'}(p)(-i\overleftarrow{\partial\!\!\!/}'_y - m)_{\delta'\gamma'}v_{\gamma'}(q')D_{F\mu\nu}(z_1-z_2) \\
= & -\int d^4z_1 d^4z_2 e^{-i[p\cdot z_1 + q\cdot z_2 - p'\cdot z_1 - q'\cdot z_2]} \\
& \overline{u}(p')(ie\gamma^\mu)u(p)\overline{v}(q)(ie\gamma^\nu)v(q')D_{F\mu\nu}(z_1-z_2) \qquad (3.102)
\end{aligned}
$$

Using now the Fourier transform of the photon propagator,

$$
\begin{aligned}
D_{F\mu\nu}(z_1-z_2) &= \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}}{k^2+i\varepsilon}e^{-ik\cdot(z_1-z_2)} \\
&\equiv \int \frac{d^4k}{(2\pi)^u}D_{F\mu\nu}(k)e^{-ik\cdot(z_1-z_2)} \qquad (3.103)
\end{aligned}
$$

we get

$$
\begin{aligned}
S^{(a)}_{fi} &= -\overline{u}(p')(ie\gamma^\mu)u(\rho)\overline{v}(q)(ie\gamma^\nu)v(q') \\
& \qquad \int d^4z_1 d^4z_2 \frac{d^4k}{(2\pi)^4}D_{F\mu\nu}(k)e^{-iz_1\cdot(p-p'+k)}e^{-iz_2\cdot(q-q'-k)} \\
&= -(2\pi)^4\delta^4(p+q-p'-q')\overline{u}(p')(ie\gamma^\nu)u(p)\overline{v}(q)(ie\gamma^\mu)v(q')D_{F\mu\nu}(p'-p) \\
& \qquad (3.104)
\end{aligned}
$$

and therefore the T matrix element is,

$$
-iT^{(a)}_{fi} = -\overline{u}(p')(ie\gamma^\mu)u(p)D_{F\mu\nu}(p'-p)\overline{v}(q)(ie\gamma^\nu)v(q') \qquad (3.105)
$$

to which corresponds the Feynman diagram of Fig. (3.12).

In a similar way we would get

$$
-iT^{(b)}_{fi} = \overline{v}(q)(ie\gamma^\mu)u(p)D_{F\mu\nu}(p+q)\overline{u}(p')(ie\gamma^\nu)v(q') \qquad (3.106)
$$

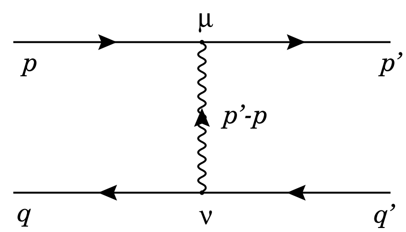


Figure 3.12:

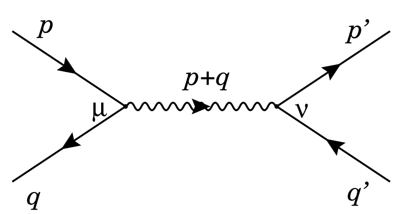


Figure 3.13:

that corresponds to the diagram of Fig. (3.13). Which of the diagrams has the minus sign is irrelevant, because this is the lowest order diagram. It depends on the conventions determining how to build the *in* state that lead to Eq. (3.98). Only the relative sign is important. However, higher order terms have to respect the same conventions.

3.6.3 Fermion Loops

Before we summarize the Feynman rules for `QED` let us look at what happens with fermion loops. One such example is the second order correction to the photon propagator shown in Fig. (3.14). First of all, the loop orientation it is only relevant if it leads to topological

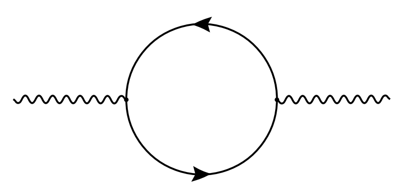


Figure 3.14: Vacuum polarization

different diagrams. Therefore the diagrams of Fig. (3.15) are topologically equivalent and only one should be considered. However the diagrams in Fig. (3.16) are topologically distinct and both should be considered.

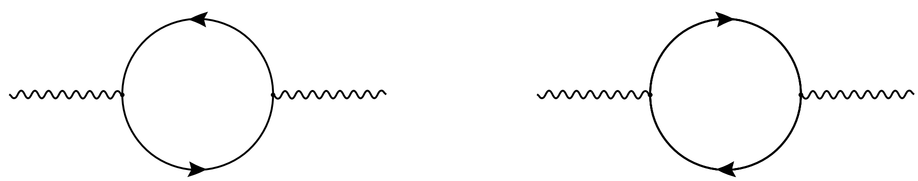


Figure 3.15: Topologically equivalent diagrams

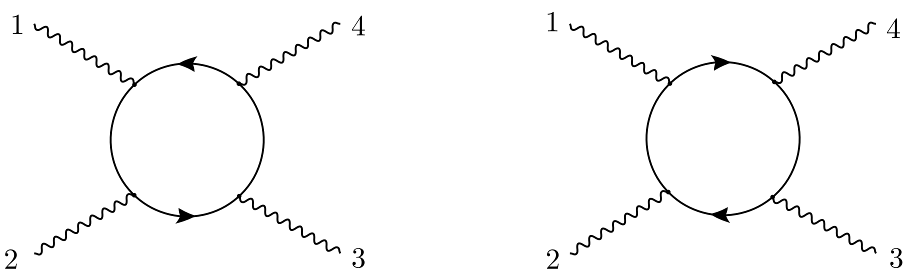


Figure 3.16: Topologically distinct diagrams

The second aspect that is relevant is a possible sign coming from the anti-commutation of the fermion fields, that should affect some diagrams, and in particular the fermion loop. To understand this sign we should note that by definition of loop, the internal lines are not connected to external fermion lines, they should originate only in the interaction. Therefore they should come from terms of the form

$$\langle 0|\, T \cdots : \overline{\psi}(z_1) \not{A}(z_1) \psi(z_1) : \cdots : \overline{\psi}(z_n) \not{A}(z_n) \psi(z_n) : \cdots |0\rangle \ . \tag{3.107}$$

Now it is clear that in order to make the appropriate contractions of the fermion fields to bring them to the form of the Feynman propagator, $\langle 0|\, T\psi(z_1)\overline{\psi}(z_2)\, |0\rangle$, it is necessary to make an odd number of permutations of the fermion fields, and therefore we get a $(-)$ sign for the loops. This sign is physically relevant because there is a lowest order diagram where the photons do not interact, corersponding to the free propagator. So the minus sign is defined in relation to this lowest order diagram and therefore it is not arbitrary (see the difference with respect to the discussion of the Bahbha scattering).

3.6.4 Feynman rules for QED

We are now in position to state the Feynman rules for `QED`

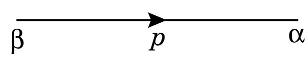
1. For a given process, draw all topologically distinct diagrams.

2. For each electron entering a diagram a factor $u(p,s)$. If it leaves the diagram a factor $\overline{u}(p,s)$.

3. For each positron leaving the diagram (final state) a factor $v(p,s)$. It it enters the diagram (initial state) then we have a factor $\overline{v}(p,s)$.

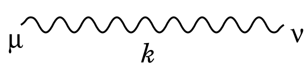
4. For each photon in the initial state we have the vector $\varepsilon^{\mu}(k)$ and in the final state $\varepsilon^{*\mu}(k)$.

5. For each internal fermionic line the propagator



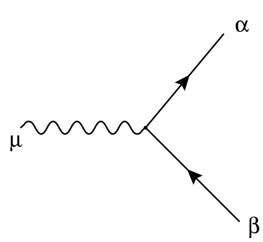
$$S_{F_{\alpha\beta}}(p) = i\frac{(\not p + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon} \tag{3.108}$$

6. For each virtual photon the propagator (Feynman gauge)



$$D_{F\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2} \tag{3.109}$$

7. For each vertex the factor



$$ie\gamma^{\mu} \tag{3.110}$$

8. For each internal momentum, not fixed by conservation of momenta, as in the case of loops, a factor

$$\int \frac{d^4q}{(2\pi)^4} \tag{3.111}$$

9. For each loop of fermions a -1 sign.

10. A factor of -1 between diagrams that differ by exchange of fermionic lines. In doubt, revert to Wick's theorem.

Comments

- In QED there are no symmetry factors, that is, they are always equal to 1.

- In our discussion we did not consider the Z factors that come in the reduction formulas, like in Eq. (2.65). This is true in lowest order in perturbation theory. They can be calculated also in perturbation theory. Their definition is (for instance for the electron),

$$\lim_{\not p \to m} S'_F(p) = Z_2 S_F(p) \tag{3.112}$$

where $S'_F(p)$ is the propagator of the theory with interactions. Then we can obtain, in perturbation theory,

$$Z_2 = 1 + O(\alpha) + \cdots \tag{3.113}$$

In higher orders it is necessary to correct the external lines with these $\sqrt{Z}$ factors.

3.7 General formalism for getting the Feynman rules

After showing how to obtain the Feynman rules for $\lambda\phi^4$ and QED, we are going to present here, without proof, a general method to obtain the Feynman rules of any theory, including the case when the interactions have derivatives, that we have excluded up to now, and that is very important for the Standard Model. This method can only be fully justified with the methods of Chapter 5. For simplicity we will consider only scalar fields.

The starting point is the action taken as a functional of the fields,

$$\Gamma_0[\varphi] \equiv \int d^4x \mathcal{L}[\varphi] \cdot \tag{3.114}$$

In fact, $\Gamma_0[\varphi]$ is the generating functional of the one particle irreducible Green functions in lowest order, as we will see in Chapter 5. The rules are as follows:

Propagators

1. Start by evaluating $\Gamma_0^{(2)}(x_i, x_j) \equiv \dfrac{\delta^2 \Gamma_0[\varphi]}{\delta\varphi(x_i)\delta\varphi(x_j)}$

2. Then evaluate the Fourier Transform (FT) to get $\Gamma_0^{(2)}(p_i, p_j)$ defined by the relation

$$(2\pi)^4 \delta^4(p_i + p_j) \Gamma_0^{(2)}(p_i, p_j) \equiv \int d^4x_i d^4x_j e^{-i(p_i \cdot x_i + p_j \cdot x_j)} \Gamma_0^{(2)}(x_i, x_j) \tag{3.115}$$

   where all the momenta are *incoming*.

3. The Feynman propagator is then

$$G^{(0)}_{Fij} = i[\Gamma_0^{(2)}(p_i, p_j)]^{-1} \ . \tag{3.116}$$

   Do not forget that $p_i = -p_j$.

Vertices

1. Evaluate $\Gamma_0^{(n)}(x_1 \cdots x_n) = \dfrac{\delta^n \Gamma_0[\varphi]}{\delta\varphi(x_1) \cdots \delta\varphi(x_n)}$

2. Then take the Fourier Transform to obtain

$$(2\pi)^4\delta^4(p_1+p_2+\cdots+p_n)\Gamma_0^{(n)}(p_1\cdots p_n)$$
$$\equiv \int d^4x_1\cdots d^4x_n e^{-i(p_1\cdot x_1+\cdots p_n\cdot x_n)}\Gamma_0^{(n)}(x_1\cdots x_n) \qquad (3.117)$$

3. The vertex in momenta space is then given by the rule

$$i\Gamma_0^{(n)}(p_1,\cdots p_n) \qquad (3.118)$$

Comments

- For fermionic fields it is necessary to take care with the order of the derivation. The convention that we take is

$$\frac{\delta^2}{\delta\psi_\alpha(x)\delta\overline{\psi}_\beta(y)}\left(\overline{\psi}(z)\Gamma\psi(z)\right) \equiv \Gamma_{\beta\alpha}\delta^4(z-x)\delta^4(z-y) \qquad (3.119)$$

  $\psi_\alpha(x)$ e $\psi_\beta(x)$ are here taken as classical anti-commuting fields (Grassmann variables, see Chapter 5).

- The functional derivatives are defined by

$$\frac{\delta\varphi_i(x)}{\delta\varphi_k(y)} \equiv \delta_{ik}\delta^4(x-y) \qquad (3.120)$$

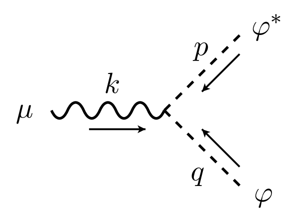


Figure 3.17: Cubic vertex in scalar QED.

3.7.1 Example: scalar electrodynamics

The Lagrangian is

$$\mathcal{L} = (\partial_\mu - ieQA_\mu)\varphi^*(\partial^\mu + ieQA^\mu)\varphi - m\varphi^*\varphi + \mathcal{L}_{\text{Maxwell}} - \frac{\lambda}{4}(\varphi^*\varphi)^2 \tag{3.121}$$

Therefore

$$\mathcal{L}_{int} = -ieQ\varphi^* \overleftrightarrow{\partial}_\mu \varphi A^\mu + e^2Q^2\varphi^*\varphi A_\mu A^\mu \tag{3.122}$$

The propagators are the usual ones, let us consider only the vertices. There are two vertices. The cubic vertex is

$$\Gamma^{(3)}_\mu(x_1, x_2, x_3) = -ieQ \int d^4z\delta^4(z - x_1)(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\delta^4(z - x_2)\delta^4(z - x_3) \tag{3.123}$$

therefore

$$\begin{aligned}
(2\pi)^4\delta^4(p + k + q)\Gamma^{(3)}_\mu(p, q, k) &\equiv -ieQ \int d^4zd^4x_1d^4x_2d^4x_3e^{-i(x_1\cdot p+x_2\cdot q+x_3\cdot k)} \\
&\qquad \delta^4(z - x_1)\ (\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu)\delta^4(z - x_2)\delta^4(z - x_3) \\
&= -ieQ \int d^4zd^4x_2e^{-i[(p+k)\cdot z+q\cdot x_2]}\partial_\mu\delta^4(z - x_2) \\
&\quad +ieQ \int d^4zd^4x_1e^{-i[p\cdot x_1+(q+k)\cdot z]}\partial_\mu\delta^4(z - x_1) \\
&= -ieQ(ip_\mu - iq_\mu)(2\pi)^4\delta^4(p + q + k)
\end{aligned} \tag{3.124}$$

Therefore we obtain for this vertex

$$i\Gamma_\mu(p, q, k) = ieQ\ (p_\mu - q_\mu) = -ieQ\ (q_\mu - p_\mu) \tag{3.125}$$

The other vertex is
We obtain,

$$\Gamma^{(4)}_{\mu\nu}(x_1, x_2, x_3, x_4) = 2e^2Q^2\delta^4(x_1 - x_2)\delta^4(x_1 - x_3)\delta^4(x_1 - x_4)g_{\mu\nu} \tag{3.126}$$

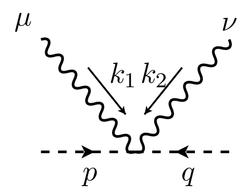


Figure 3.18: Quartic vertex in scalar QED (seagull).

and

$$\Gamma^{(4)}_{\mu\nu}(p, q, k_1, k_2) = 2(eQ)^2 g_{\mu\nu} \tag{3.127}$$

and we finally get for the Feynman rule.

$$i2e^2Q^2 \; g_{\mu\nu} \tag{3.128}$$

Comment

- From the above results we can enunciate a simple rule for interactions that have derivatives of fields.

  Consider that we have one field in the Lagrangian that has a derivative, say $\partial_\mu \phi$. Then the rule is

  $$\partial_\mu \phi \to -i \; (\textit{incoming momentum})_\mu \tag{3.129}$$

  In the end do not forget to multiply the result by i.

- As an example consider the following term in the Lagrangian for scalar electrodynamics

  $$\mathcal{L} = ieQ\partial_\mu \varphi^* \varphi A^\mu + \cdots \tag{3.130}$$

  If p is the incoming momentum of the line associated with the field φ^*, see Fig. 3.17, we have

  $$\text{Vertex} = i \times (ieQ) \times (-ip_\mu) = i\, e\, Q\, p_\mu \tag{3.131}$$

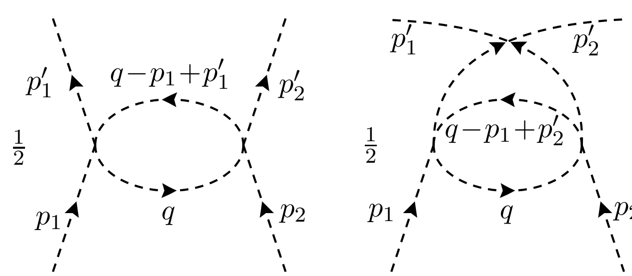
  in agreement with Eq. (3.125).

Problems for Chapter 3

3.1 Show explicitly that Wick's theorem is valid for the case of 4 fields, that is

$$T(\varphi_{in}(x_1)\varphi_{in}(x_2)\varphi_{in}(x_3)\varphi_{in}(x_4)) =: \varphi_{in}(x_1)\cdots\varphi_{in}(x_4) : + \cdots \tag{3.132}$$

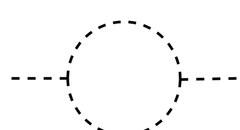
3.2 For the case of the $\lambda\varphi^4$ theory verify the Feynman rules for the diagrams



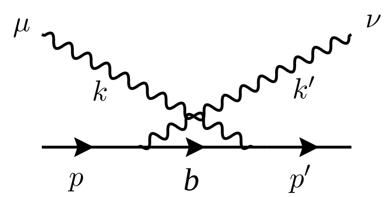
3.3 Consider a theory with the following interaction Lagrangian

$$\mathcal{L}_I = -\frac{\lambda}{3!}\varphi_{in}^3 \tag{3.133}$$

- a) Find the Feynman rules for this theory.
- b) Find the symmetry factor for the diagram



3.4 Verify that for Compton scattering the diagram



gives the result of Eq. (3.96).

3.5 Verify Eq. (3.106).

3.6 Show that in QED the symmetry factors are always 1.

3.7 Explicitly calculate the T matrix element for the process $e^+e^- \to \gamma\gamma$ and verify that is in agreement with the general rules.

3.8 Show that the amplitudes for $e^+e^- \to \gamma\gamma$ and $e^-\gamma \to e^-\gamma$ are related. How can one obtain one from the other?

Chapter 4

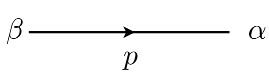
Radiative Corrections

4.1 QED Renormalization at one-loop

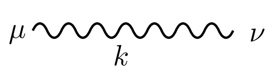
We will consider the theory described by the Lagrangian

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial \cdot A)^2 + \overline{\psi}(i\not{\partial} + e\not{A} - m)\psi \ . \tag{4.1}$$

The free propagators are

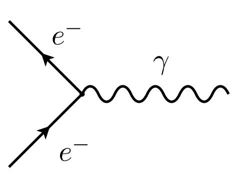


$$\left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{\beta\alpha} \equiv S^0_{F\beta\alpha}(p) \tag{4.2}$$



$$\begin{aligned} & -i\left[\frac{g_{\mu\nu}}{k^2 + i\varepsilon} + (1-\xi)\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}\right] \\ = \ & -i\left\{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2+i\varepsilon} + \xi\frac{k_\mu k_\nu}{k^4}\right\} \\ \equiv \ & G^0_{F\mu\nu}(k) \end{aligned} \tag{4.3}$$

and the vertex



$$+ie(\gamma_\mu)_{\beta\alpha} \qquad e = |e| > 0 \tag{4.4}$$

We will now consider the one-loop corrections to the propagators and to the vertex. We will work in the Feynman gauge ($\xi = 1$).

4.1.1 Vacuum Polarization

In first order the contribution to the photon propagator is given by the diagram of Fig. 4.1 that we write in the form

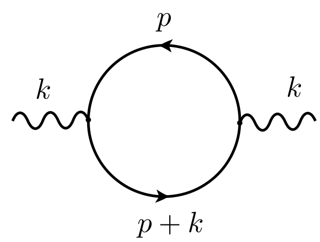


Figure 4.1: Vacuum polarization.

$$G^{(1)}_{\mu\nu}(k) \equiv G^0_{\mu\mu'}(k)\ i\,\Pi^{\mu'\nu'}(k) G^0_{\nu'\nu}(k) \tag{4.5}$$

where

$$\begin{aligned} i\,\Pi_{\mu\nu}(k) &= -(+ie)^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\mu(\not p + m)\gamma_\nu(\not p + \not k + m)]}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \\ &= -4e^2 \int \frac{d^4p}{(2\pi)^4} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \end{aligned} \tag{4.6}$$

Simple power counting indicates that this integral is quadratically divergent for large values of the internal loop momenta. In fact the divergence is milder, only logarithmic. The integral being divergent we have first to regularize it and then to define a renormalization procedure to cancel the infinities. For this purpose we will use the method of dimensional regularization. For a value of d small enough the integral converges. If we define $\epsilon = 4 - d$, in the end we will have a divergent result in the limit $\epsilon \to 0$. We get therefore[1]

$$\begin{aligned} i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)]}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \\ &= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \end{aligned} \tag{4.7}$$

where

$$N_{\mu\nu}(p,k) = 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2) \tag{4.8}$$

To evaluate this integral we first use the Feynman parameterization to rewrite the denominator as a single term. For that we use (see Appendix)

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \tag{4.9}$$

[1] Where μ is a parameter with dimensions of a mass that is introduced to ensure the correct dimensions of the coupling constant in dimension d, that is, $[e] = \frac{4-d}{2} = \frac{\epsilon}{2}$. We take then $e \to e\mu^{\frac{\epsilon}{2}}$. For more details see the Appendix.

to get

$$\begin{aligned}
i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\,\frac{N_{\mu\nu}(p,k)}{[x(p+k)^2 - xm^2 + (1-x)(p^2-m^2) + i\varepsilon]^2} \\
&= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\,\frac{N_{\mu\nu}(p,k)}{[p^2 + 2k\cdot px + xk^2 - m^2 + i\varepsilon]^2} \\
&= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\,\frac{N_{\mu\nu}(p,k)}{[(p+kx)^2 + k^2x(1-x) - m^2 + i\varepsilon]^2}
\end{aligned} \tag{4.10}$$

For dimension d sufficiently small this integral converges and we can change variables

$$p \to p - kx \tag{4.11}$$

We then get

$$i\,\Pi_{\mu\nu}(k,\epsilon) = -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\,\frac{N_{\mu\nu}(p-kx,k)}{[p^2 - C + i\epsilon]^2} \tag{4.12}$$

where

$$C = m^2 - k^2x(1-x) \tag{4.13}$$

$N_{\mu\nu}$ is a polynomial of second degree in the loop momenta as can be seen from Eq. (4.8). However as the denominator in Eq. (4.12) only depends on p^2 is it easy to show that

$$\begin{aligned}
&\int \frac{d^dp}{(2\pi)^d}\,\frac{p^\mu}{[p^2-C+i\epsilon]^2} = 0 \\
&\int \frac{d^dp}{(2\pi)^d}\,\frac{p^\mu p^\nu}{[p^2-C+i\epsilon]^2} = \frac{1}{d}\,g^{\mu\nu}\int \frac{d^dp}{(2\pi)^d}\,\frac{p^2}{[p^2-C+i\epsilon]^2}
\end{aligned} \tag{4.14}$$

This means that we only have to calculate integrals of the form

$$\begin{aligned}
I_{r,m} &= \int \frac{d^dp}{(2\pi)^d}\,\frac{(p^2)^r}{[p^2-C+i\epsilon]^m} \\
&= \int \frac{d^{d-1}p}{(2\pi)^d}\int dp^0 \frac{(p^2)^r}{[p^2-C+i\epsilon]^m}
\end{aligned} \tag{4.15}$$

To make this integration we will use integration in the plane of the complex variable p^0 as described in Fig. 4.2. The deformation of the contour corresponds to the so called Wick rotation,

$$p^0 \to ip_E^0 \qquad ; \qquad \int_{-\infty}^{+\infty} \to i\int_{-\infty}^{+\infty} dp_E^0 \tag{4.16}$$

and $p^2 = (p^0)^2 - |\vec{p}|^2 = -(p_E^0)^2 - |\vec{p}|^2 \equiv -p_E^2$, where $p_E = (p_E^0, \vec{p})$ is an euclidean vector, that is

$$p_E^2 = (p_E^0)^2 + |\vec{p}|^2 \tag{4.17}$$

We can then write (see the Appendix for more details),

$$I_{r,m} = i(-1)^{r-m}\int \frac{d^dp_E}{(2\pi)^d}\,\frac{p_E^{2r}}{\left[p_E^2 + C\right]^m} \tag{4.18}$$

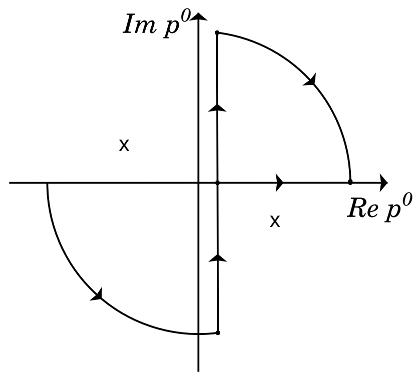


Figure 4.2: Wick rotation.

where we do not need the $i\epsilon$ anymore because the denominator is positive definite[2] $(C > 0)$. To proceed with the evaluation of $I_{r,m}$ we write,

$$\int d^d p_E = \int d\overline{p}\ \overline{p}^{\ d-1}\, d\Omega_{d-1} \tag{4.19}$$

where $\overline{p} = \sqrt{(p_E^0)^2 + |\vec{p}|^2}$ is the length of of vector p_E in the euclidean space with d dimensions and $d\Omega_{d-1}$ is the solid angle that generalizes spherical coordinates. We can show (see Appendix) that

$$\int d\Omega_{d-1} = 2\,\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{4.20}$$

The $\overline{p}$ integral is done using the result,

$$\int_0^\infty dx\ \frac{x^p}{(x^n + a^n)^q} = \pi(-1)^{q-1}\, a^{p+1-nq}\ \frac{\Gamma(\frac{p+1}{n})}{n\sin(\pi\,\frac{p+1}{n})\,\Gamma(\frac{p+1}{2} - q + 1)} \tag{4.21}$$

and we finally get

$$I_{r,m} = iC^{r-m+\frac{d}{2}}\ \frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}}\ \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})}\ \frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)} \tag{4.22}$$

Note that the integral representation of $I_{r,m}$, Eq. (4.15) is only valid for $d < 2(m-r)$ to ensure the convergence of the integral when $\overline{p} \to \infty$. However the final form of Eq. (4.22) can be analytically continued for all the values of d except for those where the function $\Gamma(m-r-d/2)$ has poles, which are (see section C.6),

$$m - r - \frac{d}{2} \neq 0, -1, -2, \ldots \tag{4.23}$$

For the application to dimensional regularization it is convenient to write Eq. (4.22) after making the substitution $d = 4 - \epsilon$. We get

$$I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^2}\ \left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}}\ C^{2+r-m}\ \frac{\Gamma(2+r-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\ \frac{\Gamma(m-r-2+\frac{\epsilon}{2})}{\Gamma(m)} \tag{4.24}$$

[2]The case when $C < 0$ is obtained by analytical continuation of the final result.

that has poles for $m - r - 2 \leq 0$ (see section C.6).

We now go back to calculate $\Pi_{\mu\nu}$. First we notice that after the change of variables of Eq. (4.11) we get, neglecting terms that vanish due to Eq. (4.14),

$$N_{\mu\nu}(p-kx,k) = 2p_\mu p_\nu + 2x^2 k_\mu k_\nu - 2x k_\mu k_\nu - g_{\mu\nu}\left(p^2 + x^2k^2 - xk^2 - m^2\right) \tag{4.25}$$

and therefore

$$\begin{aligned}\mathcal{N}_{\mu\nu} &\equiv \mu^\epsilon \int \frac{d^dp}{(2\pi)^d}\, \frac{N_{\mu\nu}(p-kx,k)}{[p^2 - C + i\epsilon]^2}\\ &= \left(\frac{2}{d}-1\right) g_{\mu\nu}\mu^\epsilon I_{1,2} + \Big[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu}m^2\Big]\mu^\epsilon I_{0,2} \end{aligned}\tag{4.26}$$

Using now Eq. (4.24) we can write

$$\begin{aligned}\mu^\epsilon I_{0,2} &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)}\\ &= \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\end{aligned}\tag{4.27}$$

where we have used the expansion of the Γ function, Eq. (C.47),

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \tag{4.28}$$

γ being the Euler constant and we have defined, Eq. (C.50),

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi \tag{4.29}$$

In a similar way

$$\begin{aligned}\mu^\epsilon I_{1,2} &= -\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} C\, \frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\, \frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(2)}\\ &= \frac{i}{16\pi^2}C\left(1 + 2\Delta_\epsilon - 2\ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\end{aligned}\tag{4.30}$$

Due to the existence of a pole in $1/\epsilon$ in the previous equations we have to expand all quantities up to $\mathcal{O}(\epsilon)$. This means for instance, that

$$\frac{2}{d} - 1 = \frac{2}{4-\epsilon} - 1 = -\frac{1}{2} + \frac{1}{8}\epsilon + \mathcal{O}(\epsilon^2) \tag{4.31}$$

Substituting back into Eq. (4.26), and using Eq. (4.13), we obtain

$$\begin{aligned}\mathcal{N}_{\mu\nu} &= g_{\mu\nu}\left[-\frac{1}{2} + \frac{1}{8}\epsilon + \mathcal{O}(\epsilon^2)\right]\left[\frac{i}{16\pi^2}C\left(1 + 2\Delta_\epsilon - 2\ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]\\ &\quad + \Big[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu}m^2\Big]\left[\frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]\end{aligned}$$

$$
\begin{aligned}
= & -\frac{i}{16\pi^2}k_\mu k_\nu \left[\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) 2x(1-x)\right] \\
& + \frac{i}{16\pi^2} g_{\mu\nu} k^2 \left[\Delta_\epsilon \Big(x(1-x) + x(1-x)\Big) + \ln\frac{C}{\mu^2}\Big(- x(1-x) - x(1-x)\Big) \right. \\
& \qquad \left. + x(1-x)\left(\frac{1}{2} - \frac{1}{2}\right)\right] \\
& + \frac{i}{16\pi^2} g_{\mu\nu} m^2 \left[\Delta_\epsilon(-1+1) + \ln\frac{C}{\mu^2}(1-1) + (-\frac{1}{2} + \frac{1}{2})\right]
\end{aligned}
\tag{4.32}
$$

and finally

$$
\mathcal{N}_{\mu\nu} = \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right) 2x(1-x) \tag{4.33}
$$

Now using Eq. (4.7) we get

$$
\begin{aligned}
\Pi_{\mu\nu}(k) &= -4e^2 \frac{1}{16\pi^2}\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right) \int_0^1 dx\ 2x(1-x)\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) \\
&= -\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\Pi(k^2, \epsilon)
\end{aligned}
\tag{4.34}
$$

where

$$
\Pi(k^2,\epsilon) \equiv \frac{2\alpha}{\pi}\int_0^1 dx\ x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2 - x(1-x)k^2}{\mu^2}\right] \tag{4.35}
$$

This expression clearly diverges as $\epsilon \to 0$. Before we show how to renormalize it let us discuss the meaning of $\Pi_{\mu\nu}(k)$. The full photon propagator is given by the series represented in Fig. 4.3, where

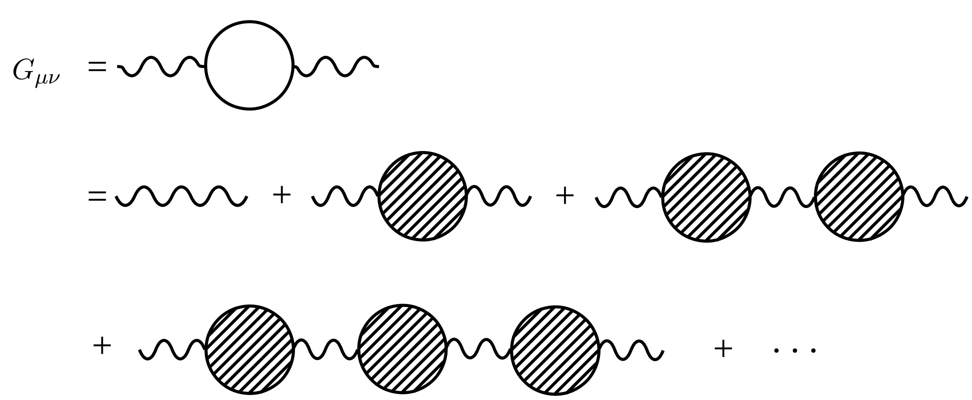
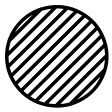


Figure 4.3: Full photon propagator.



$$\equiv i\,\Pi_{\mu\nu}(k) = \text{ sum of all one-particle irreducible } (\textit{proper}) \text{ diagrams to } \textit{all} \text{ orders} \tag{4.36}$$

In lowest order we have the contribution represented in Fig. 4.4, which is what we have

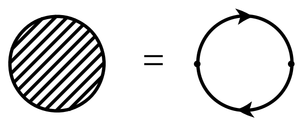


Figure 4.4: Lowest order contribution.

just calculated. To continue it is convenient to rewrite the free propagator of the photon (in an arbitrary gauge ξ) in the following form

$$\begin{aligned} iG^0_{\mu\nu} &= \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} = P^T_{\mu\nu}\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} \\ &\equiv iG^{0T}_{\mu\nu} + iG^{0L}_{\mu\nu} \end{aligned} \tag{4.37}$$

where we have introduced the transversal projector $P^T_{\mu\nu}$ defined by

$$P^T_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{4.38}$$

obviously satisfying the relations,

$$\begin{cases} k^\mu P^T_{\mu\nu} = 0 \\ P^{T\,\nu}_{\mu} P^T_{\nu\rho} = P^T_{\mu\rho} \end{cases} \tag{4.39}$$

The full photon propagator can also in general be written separating its transversal an longitudinal parts

$$G_{\mu\nu} = G^T_{\mu\nu} + G^L_{\mu\nu} \tag{4.40}$$

where $G^T_{\mu\nu}$ satisfies

$$G^T_{\mu\nu} = P^T_{\mu\nu} G_{\mu\nu} \tag{4.41}$$

Eq. (4.34) means that, to first order, the vacuum polarization tensor is transversal, that is

$$i\,\Pi_{\mu\nu}(k) = -ik^2 P^T_{\mu\nu}\,\Pi(k) \tag{4.42}$$

This result is in fact valid to all orders of perturbation theory, a result that can be shown using the Ward-Takahashi identities. This means that the longitudinal part of the photon propagator is not renormalized,

$$G^L_{\mu\nu} = G^{0L}_{\mu\nu} \tag{4.43}$$

For the transversal part we obtain from Fig. 4.3,

$$\begin{aligned} iG^T_{\mu\nu} &= P^T_{\mu\nu}\frac{1}{k^2} + P^T_{\mu\mu'}\frac{1}{k^2}(-i)k^2 P^{T\mu'\nu'}\Pi(k^2)(-i)P^T_{\nu'\nu}\frac{1}{k^2} \\ &\quad + P^T_{\mu\rho}\frac{1}{k^2}(-i)k^2 P^{T\rho\lambda}\ \Pi(k^2)(-i)P^T_{\lambda\tau}\frac{1}{k^2}(-i)k^2 P^{T\tau\sigma}\ \Pi(k^2)(-i)P^T_{\sigma\nu}\frac{1}{k^2} + \cdots \\ &= P^T_{\mu\nu}\frac{1}{k^2}\left[1 - \Pi(k^2) + \Pi^2(k^2) + \cdots\right] \end{aligned} \tag{4.44}$$

which gives, after summing the geometric series,

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\frac{1}{k^2\left[1+\Pi(k^2)\right]} \tag{4.45}$$

All that we have done up to this point is formal because the function $\Pi(k)$ diverges. The most satisfying way to solve this problem is the following. The initial lagrangian from which we started has been obtained from the classical theory and nothing tell us that it should be exactly the same in quantum theory. In fact, as we have just seen, the normalization of the wave functions is changed when we calculate *one-loop* corrections, and the same happens to the physical parameters of the theory, the charge and the mass. Therefore we can think that the correct lagrangian is obtained by adding corrections to the classical lagrangian, order by order in perturbation theory, so that we keep the definitions of charge and mass as well as the normalization of the wave functions. The terms that we add to the lagrangian are called *counterterms*[3]. The total lagrangian is then,

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, ...) + \Delta\mathcal{L} \tag{4.46}$$

Counterterms are defined from the normalization conditions that we impose on the fields and other parameters of the theory. In QED we have at our disposal the normalization of the electron and photon fields and of the two physical parameters, the electric charge and the electron mass. The normalization conditions are, to a large extent, arbitrary. It is however convenient to keep the expressions as close as possible to the free field case, that is, without radiative corrections. We define therefore the normalization of the photon field as,

$$\lim_{k\to 0} k^2 iG^{RT}_{\mu\nu} = 1\cdot P^T_{\mu\nu} \tag{4.47}$$

where $G^{RT}_{\mu\nu}$ is the renormalized propagator (the transversal part) obtained from the lagrangian $\mathcal{L}_{\text{total}}$. The justification for this definition comes from the following argument.

[3]This interpretation in terms of quantum corrections makes sense. In fact we can show that an expansion in powers of the coupling constant can be interpreted as an expansion in $\hbar^L$, where L is the number of the loops in the expansion term.

Consider the Coulomb scattering to all orders of perturbation theory. We have then the situation described in Fig. 4.5. Using the Ward-Takahashi identities one can show that

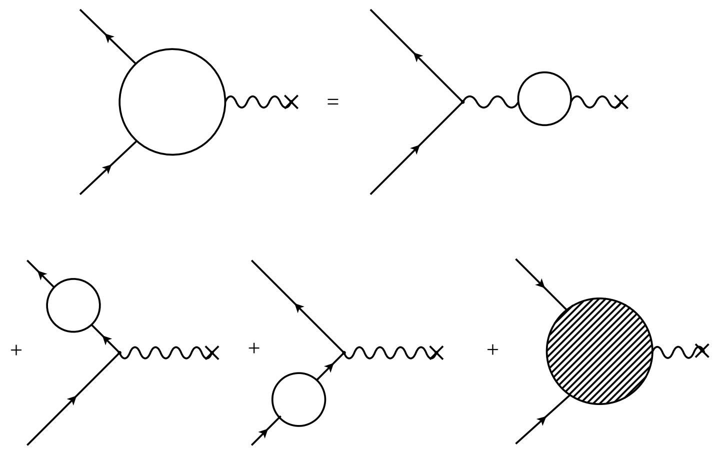


Figure 4.5: Corrections to Coulomb scattering.

the last three diagrams cancel in the limit $q = p' - p \to 0$. Then the normalization condition, Eq. (4.47), means that we have the situation described in Fig. 4.6, that is, the experimental value of the electric charge is determined in the limit $q \to 0$ of the Coulomb scattering.

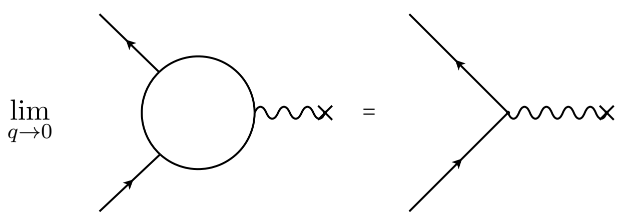


Figure 4.6: Definition of the electric charge.

The counterterm lagrangian has to have the same form as the classical lagrangian to respect the symmetries of the theory. For the photon field it is traditional to write

$$\Delta\mathcal{L} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\delta Z_3\, F_{\mu\nu}F^{\mu\nu} \tag{4.48}$$

corresponding to the following Feynman rule

$$\mu \;\overset{k}{\sim\!\sim\!\sim\!\times\!\sim\!\sim\!\sim}{}^{k}\; \nu \qquad -i\,\delta Z_3 k^2\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{4.49}$$

We have then

$$i\Pi_{\mu\nu} \quad = \quad i\Pi^{loop}_{\mu\nu} - i\,\delta Z_3 k^2\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

$$= -i\,\left(\Pi(k,\epsilon) + \delta Z_3\right) k^2\, P^T_{\mu\nu} \tag{4.50}$$

Therefore we should make the substitution

$$\Pi(k,\epsilon) \to \Pi(k,\epsilon) + \delta Z_3 \tag{4.51}$$

in the photon propagator. We obtain,

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\,\frac{1}{k^2}\,\frac{1}{1+\Pi(k,\epsilon)+\delta Z_3} \tag{4.52}$$

The normalization condition, Eq. (4.47), implies

$$\Pi(0,\epsilon) + \delta Z_3 = 0 \tag{4.53}$$

from which one determines the constant δZ_3. We get

$$\begin{aligned} \delta Z_3 &= -\Pi(0,\epsilon) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right] \\ &= -\frac{\alpha}{3\pi}\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right] \end{aligned} \tag{4.54}$$

The renormalized photon propagator can then be written as [4]

$$iG_{\mu\nu}(k) = \frac{P^T_{\mu\nu}}{k^2[1+\Pi(k,\epsilon)-\Pi(0,\epsilon)]} + i\,G^L_{\mu\nu} \tag{4.55}$$

The *finite* radiative corrections are given through the function

$$\begin{aligned} \Pi^R(k^2) &\equiv \Pi(k^2,\epsilon) - \Pi(0,\epsilon) \\ &= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 - x(1-x)k^2}{m^2}\right] \\ &= -\frac{\alpha}{3\pi}\left\{\frac{1}{3} + 2\left(1+\frac{2m^2}{k^2}\right)\left[\left(\frac{4m^2}{k^2}-1\right)^{1/2}\cot^{-1}\left(\frac{4m^2}{k^2}-1\right)^{1/2} - 1\right]\right\} \end{aligned} \tag{4.56}$$

where the last equation is valid for $k^2 < 4m^2$. For values $k^2 > 4m^2$ the result for $\Pi^R(k^2)$ can be obtained from Eq. (4.56) by analytical continuation. Using ($k^2 > 4m^2$)

$$\cot^{-1} iz = i\left(-\tanh^{-1} z + \frac{i\pi}{2}\right) \tag{4.57}$$

and

$$\left(\frac{4m^2}{k^2}-1\right)^{1/2} \to i\sqrt{1-\frac{4m^2}{k^2}} \tag{4.58}$$

[4]Notice that the photon mass is not renormalized, that is the pole of the photon propagator remains at $k^2 = 0$.

we get

$$
\begin{aligned}
\Pi^R(k^2) &= -\frac{\alpha}{3\pi}\left\{\frac{1}{3}+2\left(1+\frac{2m^2}{k^2}\right)\left[-1+\sqrt{1-\frac{4m^2}{k^2}}\tanh^{-1}\left(1-\frac{4m^2}{k^2}\right)^{1/2}\right.\right. &(4.59)\\
&\left.\left.\qquad -i\frac{\pi}{2}\sqrt{1-\frac{4m^2}{k^2}}\right]\right\} &(4.60)
\end{aligned}
$$

The imaginary part of Π^R is given by

$$
Im\ \Pi^R(k^2)=\frac{\alpha}{3}\left(1+\frac{2m^2}{k^2}\right)\sqrt{1-\frac{4m^2}{k^2}}\theta\left(1-\frac{4m^2}{k^2}\right) \tag{4.61}
$$

and it is related to the pair production that can occur [5] for $k^2>4m^2$.

4.1.2 Self-energy of the electron

The electron full propagator is given by the diagrammatic series of Fig. 4.7, which can be written as,

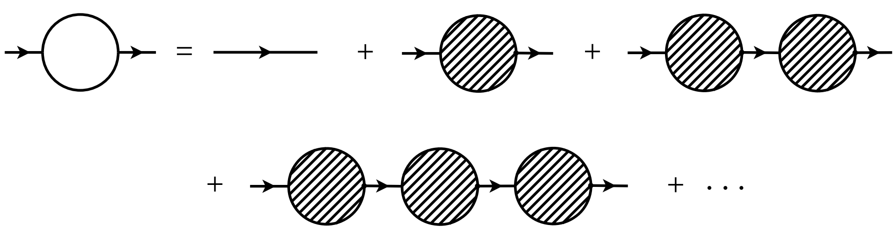
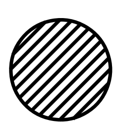


Figure 4.7: Full electron propagator

$$
\begin{aligned}
S(p) &= S^0(p)+S^0(p)\Big(-i\,\Sigma(p)\Big)S^0(p)+\cdots\\
&= S^0(p)\Big[1-i\,\Sigma(p)S(p)\Big] &(4.62)
\end{aligned}
$$

where we have identified



$$
\equiv -i\,\Sigma(p) \tag{4.63}
$$

Multiplying on the left with $S_0^{-1}(p)$ and on the right with $S^{-1}(p)$ we get

$$
S_0^{-1}(p)=S^{-1}(p)-i\,\Sigma(p) \tag{4.64}
$$

which we can rewrite as

[5] For $k^2>4m^2$ there is the possibility of producing one pair e^+e^-. Therefore on top of a virtual process (vacuum polarization) there is a real process (pair production).

$$S^{-1}(p) = S_0^{-1}(p) + i\,\Sigma(p) \tag{4.65}$$

Using the expression for the free field propagator,

$$S_0(p) = \frac{i}{\not p - m} \Longrightarrow S_0^{-1}(p) = -i(\not p - m) \tag{4.66}$$

we can then write

$$\begin{aligned} S^{-1}(p) &= S_0^{-1}(p) + i\Sigma(p) \\ &= -i\left[\not p - (m + \Sigma(p))\right] \end{aligned} \tag{4.67}$$

We conclude that it is enough to calculate $\Sigma(p)$ to all orders of perturbation theory to obtain the full electron propagator. The name *self-energy* given to $\Sigma(p)$ comes from the fact that, as can be seen in Eq. (4.67), it comes as an additional (momentum dependent) contribution to the mass.

In lowest order there is only the diagram of Fig. 4.8 contributing to $\Sigma(p)$ and therefore we get,

$$-i\Sigma(p) = (+ie)^2 \int \frac{d^4k}{(2\pi)^4} (-i) \frac{g_{\mu\nu}}{k^2 - \lambda^2 + i\varepsilon} \gamma^\mu \frac{i}{\not p + \not k - m + i\varepsilon} \gamma^\nu \tag{4.68}$$

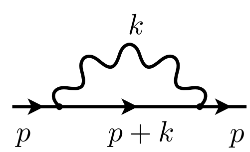


Figure 4.8: Self-energy of the electron

where we have chosen the Feynman gauge ($\xi = 1$) for the photon propagator and we have introduced a small mass for the photon λ, in order to control the infrared divergences (IR) that will appear when $k^2 \to 0$ (see below). Using dimensional regularization and the results of the Dirac algebra in dimension d,

$$\begin{aligned} \gamma_\mu(\not p + \not k)\gamma^\mu &= -(\not p + \not k)\gamma_\mu\gamma^\mu + 2(\not p + \not k) = -(d-2)(\not p + \not k) \\ m\gamma_\mu\gamma^\mu &= m\,d \end{aligned} \tag{4.69}$$

we get

$$\begin{aligned} -i\,\Sigma(p) &= -\mu^\epsilon e^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - \lambda^2 + i\varepsilon} \gamma_\mu \frac{\not p + \not k + m}{(p+k)^2 - m^2 + i\varepsilon} \gamma^\mu \\ &= -\mu^\epsilon e^2 \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)(\not p + \not k) + m\,d}{[k^2 - \lambda^2 + i\varepsilon]\,[(p+k)^2 - m^2 + i\varepsilon]} \\ &= -\mu^\epsilon e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)(\not p + \not k) + m\,d}{[(k^2 - \lambda^2)\,(1-x) + x(p+k)^2 - xm^2 + i\varepsilon]^2} \end{aligned}$$

$$
\begin{aligned}
&= -\mu^{\epsilon} e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{-(d-2)(\not{p}+\not{k}) + m\,d}{\left[(k+px)^2 + p^2 x(1-x) - \lambda^2(1-x) - xm^2 + i\varepsilon\right]^2} \\
&= -\mu^{\epsilon} e^2 \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{-(d-2)\left[\not{p}(1-x) + \not{k}\right] + m\,d}{\left[k^2 + p^2 x(1-x) - \lambda^2(1-x) - xm^2 + i\varepsilon\right]^2} \\
&= -\mu^{\epsilon} e^2 \int_0^1 dx \left[-(d-2)\not{p}(1-x) + m\,d \right] I_{0,2} \qquad (4.70)
\end{aligned}
$$

where[6]

$$
I_{0,2} = \frac{i}{16\pi^2} \left[\Delta_\epsilon - \ln\left[-p^2 x(1-x) + m^2 x + \lambda^2(1-x)\right]\right] \qquad (4.71)
$$

The contribution from the *loop* in Fig. 4.8 to the electron self-energy $\Sigma(p)$ can then be written in the form,

$$
\Sigma(p)^{loop} = A(p^2) + B(p^2)\,\not{p} \qquad (4.72)
$$

with

$$
\begin{aligned}
A &= e^2 \mu^{\epsilon}(4-\epsilon) m \frac{1}{16\pi^2} \int_0^1 dx \,\left[\Delta_\epsilon - \ln\left[-p^2 x(1-x) + m^2 x + \lambda^2(1-x)\right]\right] \\
B &= -e^2 \mu^{\epsilon}(2-\epsilon) \frac{1}{16\pi^2} \int_0^1 dx\,(1-x) \Big[\Delta_\epsilon \\
&\qquad\qquad - \ln\left[-p^2 x(1-x) + m^2 x + \lambda^2(1-x)\right]\Big] \qquad (4.73)
\end{aligned}
$$

Using now the expansions

$$
\begin{aligned}
\mu^{\epsilon}(4-\epsilon) &= 4\left[1 + \epsilon\left(\ln\mu - \frac{1}{4}\right) + \mathcal{O}(\epsilon^2)\right] \\
\mu^{\epsilon}(4-\epsilon)\Delta_\epsilon &= 4\left[\Delta_\epsilon + 2\left(\ln\mu - \frac{1}{4}\right) + \mathcal{O}(\epsilon)\right] \\
\mu^{\epsilon}(2-\epsilon) &= 2\left[1 + \epsilon\left(\ln\mu - \frac{1}{2}\right) + \mathcal{O}(\epsilon^2)\right] \\
\mu^{\epsilon}(2-\epsilon)\Delta_\epsilon &= 2\left[\Delta_\epsilon + 2\left(\ln\mu - \frac{1}{2}\right) + \mathcal{O}(\epsilon)\right] \qquad (4.74)
\end{aligned}
$$

we can finally write,

$$
A(p^2) = \frac{4\,e^2 m}{16\pi^2} \int_0^1 dx \left[\Delta_\epsilon - \frac{1}{2} - \ln\left[\frac{-p^2 x(1-x) + m^2 x + \lambda^2(1-x)}{\mu^2}\right]\right] \qquad (4.75)
$$

and

$$
B(p^2) = -\frac{2\,e^2}{16\pi^2} \int_0^1 dx\,(1-x) \left[\Delta_\epsilon - 1 - \ln\left[\frac{-p^2 x(1-x) + m^2 x + \lambda^2(1-x)}{\mu^2}\right]\right] \qquad (4.76)
$$

[6]The linear term in k vanishes.

To continue with the renormalization program we have to introduce the counterterm lagrangian and define the normalization conditions. We have

$$\Delta\mathcal{L} = i\,(Z_2-1)\,\overline{\psi}\gamma^\mu\partial_\mu\psi - (Z_2-1)\,m\,\overline{\psi}\psi + Z_2\delta m\,\overline{\psi}\psi + (Z_1-1)e\,\overline{\psi}\gamma^\mu\psi A_\mu \qquad (4.77)$$

and therefore we get for the self-energy

$$-i\,\Sigma(p) = -i\,\Sigma^{loop}(p) + i\,(\not{p}-m)\,\delta Z_2 + i\,\delta m \qquad (4.78)$$

Contrary to the case of the photon we see that we have two constants to determine. In the *on-shell* renormalization scheme that is normally used in QED the two constants are obtained by requiring that the pole of the propagator corresponds to the physical mass (hence the name of *on-shell* renormalization), and that the residue of the pole of the renormalized electron propagator has the same value as the free field propagator. This implies,

$$\begin{aligned}
\Sigma(\not{p}=m) = 0 \quad &\rightarrow \quad \delta m = \Sigma^{loop}(\not{p}=m) \\
\left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m} = 0 \quad &\rightarrow \quad \delta Z_2 = \left.\frac{\partial\Sigma^{loop}}{\partial\not{p}}\right|_{\not{p}=m}
\end{aligned} \qquad (4.79)$$

We then get for δm,

$$\begin{aligned}
\delta m &= A(m^2) + m\,B(m^2) \\
&= \frac{2\,me^2}{16\pi^2}\int_0^1 dx\,\left\{\left[2\Delta_\epsilon - 1 - 2\ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\right. \\
&\qquad\qquad \left. -(1-x)\left[\Delta_\epsilon - 1 - \ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\right\} \\
&= \frac{2\,m\,e^2}{16\pi^2}\left[\frac{3}{2}\Delta_\epsilon - \frac{1}{2} - \int_0^1 dx\,(1+x)\ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right] \\
&= \frac{3\alpha m}{4\pi}\left[\Delta_\epsilon - \frac{1}{3} - \frac{2}{3}\int_0^1 dx\,(1+x)\ln\left(\frac{m^2x^2}{\mu^2}\right)\right]
\end{aligned} \qquad (4.80)$$

where in the last step in Eq. (4.80) we have taken the limit $\lambda \to 0$ because the integral does not diverge in that limit[7]. In a similar way we get for δZ_2,

$$\delta Z_2 = \left.\frac{\partial\Sigma^{loop}}{\partial\not{p}}\right|_{\not{p}=m} = \left.\frac{\partial A}{\partial\not{p}}\right|_{\not{p}=m} + B + m\left.\frac{\partial B}{\partial\not{p}}\right|_{\not{p}=m} \qquad (4.81)$$

where

$$\begin{aligned}
\left.\frac{\partial A}{\partial\not{p}}\right|_{\not{p}=m} &= \frac{4\,e^2\,m^2}{16\pi^2}\int_0^1 dx\,\frac{2(1-x)x}{-m^2x(1-x)+m^2x+\lambda^2(1-x)} \\
&= \frac{2\,\alpha\,m^2}{\pi}\int_0^1 dx\,\frac{(1-x)x}{m^2x^2+\lambda^2(1-x)}
\end{aligned}$$

[7] δm is not IR divergent.

$$
\begin{aligned}
B &= -\frac{\alpha}{2\pi}\int_0^1 dx\,(1-x)\left[\Delta_\epsilon - 1 - \ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right] \\
m\left.\frac{\partial B}{\partial \not{p}}\right|_{\not{p}=m} &= -\frac{\alpha}{2\pi}\,m^2\int_0^1 dx\;\frac{2x(1-x)^2}{m^2x^2+\lambda^2(1-x)} \qquad (4.82)
\end{aligned}
$$

Substituting Eq. (4.82) in Eq. (4.81) we get,

$$
\begin{aligned}
\delta Z_2 &= -\frac{\alpha}{2\pi}\left[\frac{1}{2}\Delta_\epsilon - \frac{1}{2} - \int_0^1 dx\,(1-x)\ln\left(\frac{m^2x^2}{\mu^2}\right) - 2\int_0^1 dx\;\frac{(1+x)(1-x)xm^2}{m^2x^2+\lambda^2(1-x)}\right] \\
&= \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln\frac{m^2}{\mu^2} - 2\ln\frac{\lambda^2}{m^2}\right] \qquad (4.83)
\end{aligned}
$$

where we have taken the $\lambda \to 0$ limit in all cases that was possible. It is clear that the final result in Eq. (4.83) diverges in that limit, therefore implying that Z_2 is IR divergent. This is not a problem for the theory because δZ_2 is not a physical parameter. We will see in section 4.4.2 that the IR diverges cancel for real processes. If we had taken a general gauge ($\xi \neq 1$) we would find out that δm would not be changed but that Z_2 would show a gauge dependence. Again, in physical processes this should cancel in the end.

4.1.3 The Vertex

The diagram contributing to the QED vertex at one-loop is the one shown in Fig. 4.9. In

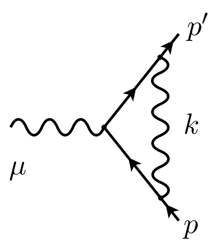


Figure 4.9: The QED vertex.

the Feynman gauge ($\xi = 1$) this gives a contribution,

$$
\begin{aligned}
ie\,\mu^{\epsilon/2}\Lambda_\mu^{loop}(p',p) &= (ie\,\mu^{\epsilon/2})^3\int\frac{d^dk}{(2\pi)^d}(-i)\frac{g_{\rho\sigma}}{k^2-\lambda^2+i\varepsilon} \\
&\qquad \gamma^\sigma\frac{i[(\not{p}'+\not{k})+m]}{(p'+k)^2-m^2+i\varepsilon}\gamma_\mu\frac{i[(\not{p}+\not{k})+m]}{(p+k)^2-m^2+i\varepsilon}\gamma^\rho \qquad (4.84)
\end{aligned}
$$

where Λ_μ is related to the full vertex Γ_μ through the relation

$$
\begin{aligned}
i\Gamma_\mu &= ie\,(\gamma_\mu + \Lambda_\mu^{loop} + \gamma_\mu\delta Z_1) \\
&= ie\,\left(\gamma_\mu + \Lambda_\mu^R\right) \qquad (4.85)
\end{aligned}
$$

The integral that defines $\Lambda_\mu^{loop}(p',p)$ is divergent. As before we expect to solve this problem by regularizing the integral, introducing counterterms and normalization conditions. The counterterm has the same form as the vertex and is already included in Eq. (4.85). The normalization constant is determined by requiring that in the limit $q = p' - p \to 0$ the vertex reproduces the tree level vertex because this is what is consistent with the definition of the electric charge in the $q \to 0$ limit of the Coulomb scattering. Also this should be defined for on-shell electrons. We have therefore that the normalization condition gives,

$$\overline{u}(p)\left(\Lambda_\mu^{loop} + \gamma_\mu \delta Z_1\right) u(p)\Big|_{\not p = m} = 0 \tag{4.86}$$

If we are interested only in calculating δZ_1 and in showing that the divergences can be removed with the normalization condition then the problem is simpler. It can be done in two ways.

1st method

We use the fact that δZ_1 is to be calculated on-shell and for $p = p'$. Then

$$i\Lambda_\mu^{loop}(p,p) = e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\rho \frac{1}{\not p + \not k - m + i\varepsilon}\gamma_\mu \frac{1}{\not p + \not k - m + i\varepsilon}\gamma^\rho \tag{4.87}$$

However we have

$$\frac{1}{\not p + \not k - m + i\varepsilon}\gamma_\mu \frac{1}{\not p + \not k - m + i\varepsilon} = -\frac{\partial}{\partial p^\mu}\,\frac{1}{\not p + \not k - m + i\varepsilon} \tag{4.88}$$

and therefore

$$\begin{aligned} i\Lambda_\mu^{loop}(p,p) &= -e^2\mu^\epsilon \frac{\partial}{\partial p^\mu}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\rho \frac{\not p + \not k + m}{(p+k)^2 - m^2 + i\varepsilon}\gamma^\rho && (4.89)\\ &= -i\frac{\partial}{\partial p^\mu}\,\Sigma^{loop}(p) && (4.90)\end{aligned}$$

We conclude then, that $\Lambda_\mu^{loop}(p,p)$ is related to the self-energy of the electron[8],

$$\Lambda_\mu^{loop}(p,p) = -\frac{\partial}{\partial p^\mu}\,\Sigma^{loop} \tag{4.91}$$

On-shell we have

$$\Lambda_\mu^{loop}(p,p)\Big|_{\not p = m} = -\left.\frac{\partial \Sigma^{loop}}{\partial p^\mu}\right|_{\not p = m} = -\delta Z_2 \gamma_\mu \tag{4.92}$$

and the normalization condition, Eq. (4.86), gives

$$\delta Z_1 = \delta Z_2 \tag{4.93}$$

As we have already calculated δZ_2 in Eq. (4.83), then δZ_1 is determined.

[8]This result is one of the forms of the Ward-Takahashi identity.

2nd method

In this second method we do not rely in the Ward identity but just calculate the integrals for the vertex in Eq. (4.84). For the moment we do not put $p' = p$ but we will assume that the vertex form factors are to be evaluated for on-shell spinors. Then we have

$$\begin{aligned}
i\,\overline{u}(p')\Lambda_\mu^{loop}u(p) &= e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{\overline{u}(p)\gamma_\rho\,[\not p' + \not k + m)]\,\gamma_\mu\,[\not p + \not k + m)]\,\gamma^\rho u(p)}{D_0D_1D_2} \\
&= e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\,\frac{\mathcal{N}_\mu}{D_0D_1D_2}
\end{aligned} \tag{4.94}$$

where

$$\begin{aligned}
\mathcal{N}_\mu &= \overline{u}(p)\Big[(-2+d)k^2\gamma_\mu + 4p\cdot p'\gamma_\mu + 4(p+p')\cdot k\,\gamma_\mu + 4m\,k_\mu \\
&\qquad - 4\not k\,(p+p')_\mu + 2(2-d)\not k k_\mu\Big]\,u(p)
\end{aligned} \tag{4.95}$$

$$D_0 = k^2 - \lambda^2 + i\epsilon \tag{4.96}$$

$$D_1 = (k+p')^2 - m^2 + i\epsilon \tag{4.97}$$

$$D_2 = (k+p)^2 - m^2 + i\epsilon \tag{4.98}$$

Now using the results of section C.7.3 with

$$r_1^\mu = p'^\mu \quad ; \quad r_2^\mu = p^\mu \tag{4.99}$$

$$P^\mu = x_1p'^\mu + x_2p^\mu \tag{4.100}$$

$$C = (x_1+x_2)^2m^2 - x_1x_2\,q^2 + (1-x_1-x_2)\lambda^2 \tag{4.101}$$

where

$$q = p' - p \tag{4.102}$$

we get,

$$\begin{aligned}
i\,\overline{u}(p')\Lambda_\mu^{loop}u(p) &= i\,\frac{\alpha}{4\pi}\Gamma(3)\int_0^1 dx_1\int_0^{1-x_1} dx_2\,\frac{1}{2C} \\
&\Bigg\{\overline{u}(p')\gamma_\mu u(p)\Bigg[-(-2+d)(x_1^2m^2 + x_2^2m^2 + 2x_1x_2p'\cdot p) - 4p'\cdot p \\
&+4(p+p')\cdot(x_1p' + x_2p) + \frac{(2-d)^2}{2}C\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\Bigg] \\
&+\overline{u}(p)u(p)m\,\Big[4(x_1p' + x_2p)_\mu - 4(p'+p)_\mu(x_1+x_2) \\
&- 2(2-d)(x_1+x_2)(x_1p' + x_2p)_\mu\Big]\Bigg\}
\end{aligned} \tag{4.103}$$

$$= \quad i\,\overline{u}(p)\left[G(q^2)\,\gamma_\mu + H(q^2)\,(p+p')\right]u(p) \tag{4.104}$$

where we have defined[9],

$$\begin{aligned}
G(q^2) \equiv\ & \frac{\alpha}{4\pi}\left[\Delta_\epsilon - 2 - 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2\, \ln \frac{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2}{\mu^2}\right. \\
& + \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left(\frac{-2(x_1+x_2)^2 m^2 - x_1 x_2 q^2 - 4m^2 + 2q^2}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2} \right. \\
& \left.\left. + \frac{2(x_1+x_2)(4m^2 - q^2)}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2} \right)\right]
\end{aligned} \tag{4.105}$$

$$H(q^2) \equiv \frac{\alpha}{4\pi}\left[\int_0^1 dx_1 \int_0^{1-x_1} dx_2\, \frac{-2m\,(x_1+x_2) + 2m\,(x_1+x_2)^2}{(x_1+x_2)^2 m^2 - x_1 x_2 q^2 + (1-x_1-x_2)\lambda^2}\right] \tag{4.106}$$

Now, using the definition of Eq. (4.85), we get for the renormalized vertex,

$$\overline{u}(p')\Lambda^R_\mu(p',p)u(p) = \overline{u}(p')\left[\left(G(q^2) + \delta Z_1\right)\gamma_\mu + H(q^2)\,(p+p')_\mu\right]u(p) \tag{4.107}$$

As δZ_1 is calculated in the limit of $q = p' - p \to 0$ it is convenient to use the Gordon identity to get rid of the $(p'+p)^\mu$ term. We have

$$\overline{u}(p')\left(p'+p\right)_\mu u(p) = \overline{u}(p')\left[2m\gamma_\mu - i\sigma_{\mu\nu}\, q^\nu\right]u(p) \tag{4.108}$$

and therefore,

$$\begin{aligned}
\overline{u}(p')\Lambda^R_\mu(p',p)u(p) \quad &= \quad \overline{u}(p')\left[\left(G(q^2) + 2mH(q^2) + \delta Z_1\right)\gamma_\mu - i\,H(q^2)\,\sigma_{\mu\nu}\,q^\nu\right]u(p) \\
&= \quad \overline{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2)\right]u(p)
\end{aligned} \tag{4.109}$$

where we have introduced the usual notation for the vertex form factors,

$$F_1(q^2) \quad \equiv \quad G(q^2) + 2mH(q^2) + \delta Z_1 \tag{4.110}$$

$$F_2(q^2) \quad \equiv \quad -2mH(q^2) \tag{4.111}$$

The normalization condition of Eq. (4.86) implies $F_1(0) = 0$, that is,

$$\delta Z_1 = -G(0) - 2m\,H(0) \tag{4.112}$$

We have therefore to calculate $G(0)$ and $H(0)$. In this limit the integrals of Eqs. (4.105) and (4.106) are much simpler. We get (we change variables $x_1 + x_2 \to y$),

$$G(0) \quad = \quad \frac{\alpha}{4\pi}\left[\Delta_\epsilon - 2 - 2\int_0^1 dx_1 \int_{x_1}^1 dy\, \ln\frac{y^2 m^2 + (1-y)\lambda^2}{\mu^2}\right.$$

[9] To obtain Eq. (4.106) one has to show that the symmetry of the integrals in $x_1 \leftrightarrow x_2$ implies that the coefficient of p is equal to the coefficient of p'.

$$+\int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2y^2m^2 - 4m^2 + 8ym^2}{y^2m^2 + (1-y)\lambda^2}\Bigg] \tag{4.113}$$

$$H(0) = \frac{\alpha}{4\pi} \int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2m\,y + 2m\,y^2}{y^2m^2 + (1-y)\lambda^2} \tag{4.114}$$

Now using

$$\int_0^1 dx_1 \int_{x_1}^1 dy\, \ln \frac{y^2m^2 + (1-y)\lambda^2}{\mu^2} = \frac{1}{2}\left(\ln \frac{m^2}{\mu^2} - 1\right) \tag{4.115}$$

$$\int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2y^2m^2 - 4m^2 + 8ym^2}{y^2m^2 + (1-y)\lambda^2} = 7 + 2\ln\frac{\lambda^2}{m^2} \tag{4.116}$$

$$\int_0^1 dx_1 \int_{x_1}^1 dy\, \frac{-2m\,y + 2m\,y^2}{y^2m^2 + (1-y)\lambda^2} = -\frac{1}{m} \tag{4.117}$$

(where we took the limit $\lambda \to 0$ if possible) we get,

$$G(0) = \frac{\alpha}{4\pi}\left[\Delta_\epsilon + 6 - \ln\frac{m^2}{\mu^2} + 2\ln\frac{\lambda^2}{m^2}\right] \tag{4.118}$$

$$H(0) = -\frac{\alpha}{4\pi}\frac{1}{m} \tag{4.119}$$

Substituting the previous expressions in Eq. (4.112) we get finally,

$$\delta Z_1 = \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln\frac{m^2}{\mu^2} - 2\ln\frac{\lambda^2}{m^2}\right] \tag{4.120}$$

in agreement with Eq. (4.83) and Eq. (4.93). The general form of the form factors $F_i(q^2)$, for $q^2 \neq 0$, is quite complicated. We give here only the result for $q^2 < 0$ (in section C.10.3 we will give a general formula for numerical evaluation of these functions),

$$\begin{aligned} F_1(q^2) &= \frac{\alpha}{4\pi}\left\{\left(2\ln\frac{\lambda^2}{m^2} + 4\right)(\theta\coth\theta - 1) - \theta\,\tanh\frac{\theta}{2} - 8\coth\theta \int_0^{\theta/2} \beta\tanh\beta d\beta\right\} \\ F_2(q^2) &= \frac{\alpha}{2\pi}\frac{\theta}{\sinh\theta} \end{aligned} \tag{4.121}$$

where

$$\sinh^2\frac{\theta}{2} = -\frac{q^2}{4m^2}. \tag{4.122}$$

In the limit of zero transferred momenta ($q = p' - p = 0$) we get

$$\begin{cases} F_1(0) = 0 \\ F_2(0) = \dfrac{\alpha}{2\pi} \end{cases} \tag{4.123}$$

a result that we will use in section 4.4.1 while discussing the anomalous magnetic moment of the electron.

4.2 Ward-Takahashi identities in QED

In the study of the `QED` vertex, in one of the methods, we used the Ward identity

$$\Lambda_\mu(p,p) = -\frac{\partial}{\partial p^\mu}\Sigma(p) \tag{4.124}$$

We are going to derive here the general form for these identities. The following discussion is formal in the sense that the various Green functions are divergent. We have to prove that we can find a regularization scheme that preserves the identities. This happens when one uses a regularization that preserves the gauge invariance of the theory. Examples are dimensional regularization and the Pauli-Villars regularization.

Ward identities are a consequence of the gauge invariance of the theory, as will be fully discussed in chapters 5 and 6. Here we are only going to use the fact that there is a conserved current,

$$\begin{aligned} &j_\mu = e\overline{\psi}\gamma_\mu\psi \\ &\partial_\mu j^\mu = 0 \end{aligned} \tag{4.125}$$

We are interested in calculating the quantity

$$\partial_x^\mu \langle 0|\, T j_\mu(x)\psi(x_1)\overline{\psi}(y_1)\cdots\psi(x_n)\overline{\psi}(y_n)A_{\nu_1}(z_1)\cdots A_{\nu_p}(z_p)\,|0\rangle \tag{4.126}$$

This quantity does not vanish, despite the fact that $\partial^\mu j_\mu = 0$. This happens because in the time ordered product we have θ functions that depend on the coordinate x^0. For instance, for the field $\psi(x_i)$ we should have a contribution of the form,

$$\begin{aligned} &\partial_x^0\big[\theta(x^0-x_i^0)j_0(x)\psi(x_i)+\theta(x_i^0-x^0)\psi(x_i)j_0(x)\big] \\ &\quad=\ \delta(x^0-x_i^0)j_0(x)\psi(x_i)-\delta(x^0-x_i^0)\psi(x_i)j_0(x) \\ &\quad=\ \big[j_0(x),\psi(x_i)\big]\delta(x^0-x_i^0) \end{aligned} \tag{4.127}$$

In this way we get ($\widehat{\ }$ means that we omit that term from the sum),

$$\begin{aligned} &\partial_x^\mu \langle 0|\, T j_\mu(x)\psi(x_1)\cdots\overline{\psi}(y_n)A_{\nu_1}(z_1)\cdots A_{\nu_p}(z_p)\,|0\rangle \\ &=\sum_{i=1}^{n}\langle 0|\,T\left\{[j_0(x),\psi(x_i)]\,\delta(x^0-x_i^0)\overline{\psi}(y_i)\right. \\ &\quad\left.+\psi(x_i)\left[j_0(x),\overline{\psi}(y_i)\right]\delta(x^0-y_i^0)\right\}\psi(x_1)\overline{\psi}(y_1)\cdots\widehat{\psi(x_i)\overline{\psi}}(y_i)\cdots A_{\nu_p}(z_p)\,|0\rangle \\ &\quad+\sum_{j=1}^{p}\langle 0|\,T\psi(x_1)\cdot\overline{\psi}(y_n)A_{\nu_p}(z_1)\cdots[j_0(x),A_{\nu_j}(z_j)]\delta(x^0-z_j^0)\cdots A_{\nu_p}(z_p)\,|0\rangle \end{aligned} \tag{4.128}$$

Using now the equal time commutation relations,

$$\begin{aligned} &\big[j_0(x),\psi(x')\big]\delta(x^0-x'^0) = -e\psi(x)\delta^4(x-x') \\ &\big[j_0(x),\overline{\psi}(x')\big]\delta(x^0-x'^0) = e\overline{\psi}(x)\delta^4(x-x') \\ &\big[j_0(x),A_\mu(x')\big]\delta(x^0-x'^0) = 0 \end{aligned} \tag{4.129}$$

that express that $\psi, \overline{\psi}$ and A_μ create *quanta* with charge $Q = \int d^3x j^0(x)$ equal to $-e, +e$ and zero, respectively, we get,

$$\begin{aligned} &\partial_x^\mu \langle 0| T j_\mu(x) \psi(x_1) \cdots \overline{\psi}(y_n) A_{\nu_1}(z_1) \cdots A_{\nu_p}(z_p) |0\rangle \\ &= e \langle 0| T \psi(y_1) \cdots A_{\nu_p}(z_p) |0\rangle \sum_{i=1}^{n} \left[\delta^4(x-y_i) - \delta^4(x-x_i)\right] \end{aligned} \tag{4.130}$$

Taking different values for n and p we get different relations among the Green functions of the theory. We will consider in the following, two important cases.

4.2.1 Transversality of the photon propagator $n = 0$, $p = 1$

The Green function $\langle 0| T j_\mu(x) A_\nu(y) |0\rangle$ corresponds to the Feynman diagram of Fig. 4.10, and it is related with the full photon propagator shown in Fig. 4.11, by the diagrammatic

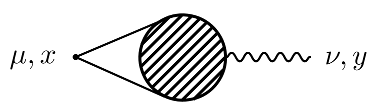


Figure 4.10: Green function $\langle 0| T j_\mu(x) A_\nu(y) |0\rangle$.

relation shown in Fig. 4.12 known as the Dyson-Schwinger equation for `QED`. It can be written as

$$G_{\mu\nu}(x-y) = G^0_{\mu\nu}(x-y) - i \int d^4x' G^0_{\mu\rho}(x-x') \langle 0| T j^\rho(x') A_\nu(y) |0\rangle \tag{4.131}$$

We apply now the derivative ∂_x^μ to get,

$$\partial_x^\mu G_{\mu\nu}(x-y) = \partial_x^\mu G^0_{\mu\nu}(x-y) - i \int d^4x' \partial_x^\mu G^0_{\mu\rho}(x-x') \langle 0| T j^\rho(x') A_\mu(y) |0\rangle \tag{4.132}$$

The free photon propagator is given by,

$$G^0_{\mu\rho}(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-i(x-x')\cdot p} G^0_{\mu\rho}(p) \tag{4.133}$$

where

$$G^0_{\mu\nu}(p) = -i \left[\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{p^2} + \xi \frac{p_\mu p_\nu}{p^4} \right] . \tag{4.134}$$

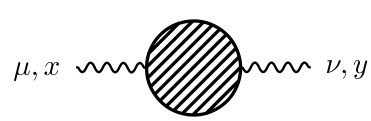


Figure 4.11: Photon propagator

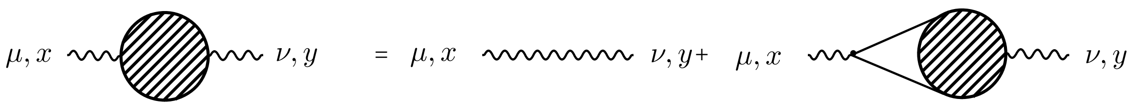


Figure 4.12: Dyson-Schwinger equation.

Therefore

$$\begin{aligned}
\partial_x^\mu G^0_{\mu\rho}(x-x') &= \int \frac{d^4p}{(2\pi)^4} e^{-i(x-x')\cdot p}(-ip^\mu) G^0_{\mu\rho}(p) \\
&= \int \frac{d^4p}{(2\pi)^4} e^{-i(x-x')\cdot p}(-ip_\rho) F(p^2) \\
&= -\partial_\rho^{x'} \int \frac{d^4p}{(2\pi)^4} e^{-i(x-x')\cdot p} F(p^2) \\
&= -\partial_\rho^{x'} \widetilde{F}(x-x') \qquad (4.135)
\end{aligned}$$

and we get

$$\begin{aligned}
\partial_x^\mu G_{\mu\nu}(x-y) &= \partial_x^\mu G^0_{\mu\nu}(x-y) + i\int d^4x' \partial^\rho_{x'} \widetilde{F}(x-x') \langle 0| T j_\rho(x') A_\nu(y) |0\rangle \\
&= \partial_x^\mu G^0_{\mu\nu}(x-y) - i\int d^4x' \widetilde{F}(x-x') \partial_\rho^{x'} \langle 0| T j^\rho(x') A_\nu(y) |0\rangle \\
&= \partial_x^\mu G^0_{\mu\nu}(x-y) \qquad (4.136)
\end{aligned}$$

where we have made an integration by parts and used the Ward-Takashashi identity for $n=0$, $p=1$. We have then

$$\partial_x^\mu G_{\mu\nu}(x-y) = \partial_x^\mu G^0_{\mu\nu}(x-y) \qquad (4.137)$$

which in momenta space implies

$$p^\mu G_{\mu\nu}(p) = p^\mu G^0_{\mu\nu}(p) \qquad (4.138)$$

This means that the longitudinal part of the photon propagator is not renormalized, or in other words, that the self-energy of the photon (vacuum polarization) is transverse. In fact

$$p^\mu G^0_{\mu\nu}(p) = -i\xi \frac{p_\nu}{p^2} \qquad (4.139)$$

or

$$p^\mu = -i\xi \frac{p_\nu}{p^2} G^{-1}_{\nu\mu}(p) = -\xi \frac{p_\nu}{p^2} \Gamma_{\nu\mu}(p) \qquad (4.140)$$

But, in agreement with our conventions, we have

$$\Gamma_{\nu\mu}(p) = -(g_{\nu\mu}p^2 - p_\nu p_\mu) - \frac{1}{\xi} p_\nu p_\mu + \Pi_{\nu\mu}(p^2) \qquad (4.141)$$

and therefore

$$-\xi \frac{p_\nu}{p^2} \Gamma_{\nu\mu}(p) = p_\mu - \frac{1}{\xi}\frac{1}{p^2} p^\nu \Pi_{\nu\mu}(p^2) = p_\mu \tag{4.142}$$

which gives

$$p^\nu \Pi_{\nu\mu}(p^2) = 0 \tag{4.143}$$

that is, the self-energy is transverse.

4.2.2 Identity for the vertex $n = 1$, $p = 0$

We are now interested in the Green function,

$$\langle 0 | T j_\mu(x) \psi_\beta(x_1) \overline{\psi}_\alpha(y_1) | 0 \rangle \tag{4.144}$$

to which corresponds the diagram of Fig. 4.13. This Green function can be related with

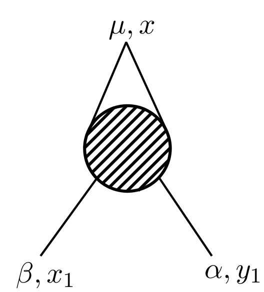


Figure 4.13: Green function $\langle 0 | T j_\mu(x) \psi_\beta(x_1) \overline{\psi}_\alpha(y_1) | 0 \rangle$.

the vertex $\langle 0 | T A_\mu(x) \psi_\beta(x_1) \overline{\psi}_\alpha(y_1) | 0 \rangle$ corresponding to the diagram of Fig. 4.14, through

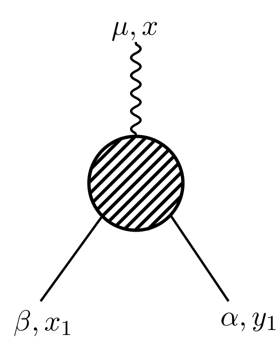
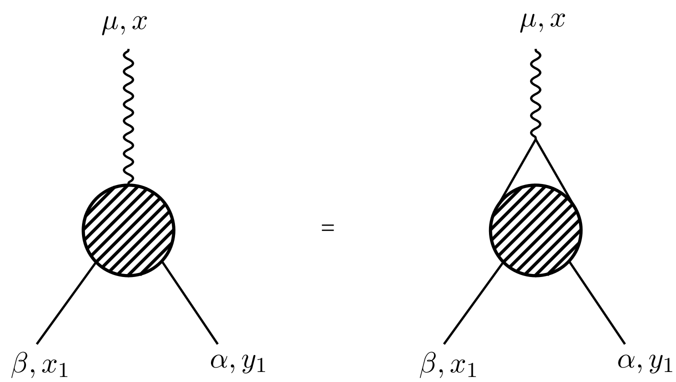


Figure 4.14: Full vertex.

the following diagrammatic equation,



(4.145)

that we can write as,

$$\langle 0|\, T A_\mu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)\,|0\rangle = -i\int d^4x' G^0_{\mu\nu}(x-x')\,\langle 0|\, T j^\nu(x')\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)\,|0\rangle \quad (4.146)$$

Taking the Fourier transform,

$$\begin{aligned}&\int d^4x d^4x_1 d^4y_1 e^{i(p'\cdot x_1 - p\cdot y_1 - q\cdot x)}\,\langle 0|\, T A_\mu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)\,|0\rangle \\ &\qquad = \; -iG^0_{\mu\nu}(q)\int d^4x d^4x_1 d^4y_1 e^{i(p'\cdot x_1 - p\cdot y_1 - q\cdot x)}\,\langle 0|\, T j^\nu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)\,|0\rangle \quad (4.147)\end{aligned}$$

where the direction of the momenta are shown in Fig. 4.15, and the momentum transfered

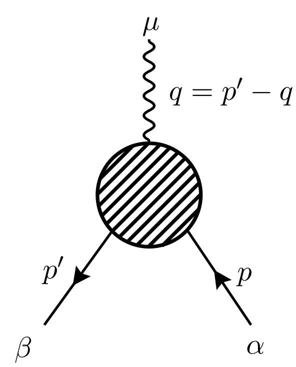


Figure 4.15: Definition of the momenta in the vertex.

is $q = p' - p$.

On the other side, using the definition of Γ_μ, we have,

$$\int d^4x d^4x_1 d^4y_1 e^{i(p'\cdot x_1 - p\cdot y_1 - q\cdot x)}\,\langle 0|\, T A_\mu(x)\psi_\beta(x_1)\overline{\psi}_\alpha(y_1)\,|0\rangle$$

$$= \quad (2\pi)^4\delta^4(p'-p-q)G_{\mu\nu}(q)\big[S(p')i\Gamma_\nu(p',p)S(p)\big]_{\beta\alpha} \tag{4.148}$$

Therefore we get,

$$\begin{aligned}&(2\pi)^4\delta(p'-p-q)G_{\mu\nu}(q)S(p')i\Gamma^\nu(p',p)S(p)\\ &\quad = \quad -iG^0_{\mu\nu}(q)\int d^4xd^4x_1d^4y_1e^{i(p'\cdot x_1-p\cdot y_1-q\cdot x)}\ \langle 0|\,Tj^\nu(x)\psi(x_1)\overline{\psi}(y_1)\,|0\rangle\end{aligned} \tag{4.149}$$

Multiplying by q^μ and using the result,

$$q^\mu G_{\mu\nu}(q) = q^\mu G^0_{\mu\nu}(q) = -i\xi\frac{q_\nu}{q^2} \tag{4.150}$$

we can then write (using the Ward identity for $n=1$, $p=0$)

$$\begin{aligned}&(2\pi)^4\delta(p'-q-p)S(p')q^\nu\Gamma_\nu(p',p)S(p)\\ &\quad = i\int d^4xd^4x_1d^4y_1\partial^\nu_x\,\langle 0|\,Tj_\nu(x)\psi(x_1)\overline{\psi}(y_1)\,|0\rangle\,e^{i(p'\cdot x_1-p\cdot y_1-q\cdot x)}\\ &\quad = ie\int d^4_xd^4x_1d^4y_1e^{i(p'\cdot x_1-p\cdot y_1-q\cdot x)}\,\langle 0|\,T\psi(x_1)\overline{\psi}(y_1)\,|0\rangle\,\big[\delta(x-y_1)-\delta(x-x_1)\big]\\ &\quad = ie(2\pi)^4\delta(p'-p-q)\big[S(p')-S(p)\big]\end{aligned} \tag{4.151}$$

or

$$q^\nu\Gamma_\nu(p',p) = ie\left[S^{-1}(p)-S^{-1}(p')\right] \tag{4.152}$$

As $q^\nu=(p'-p)^\nu$ we get in the limit $p'=p$,

$$\begin{aligned}\Gamma_\nu(p,p) &= -ie\frac{\partial S^{-1}}{\partial p^\nu}\\ &= -e\left(\gamma_\nu-\frac{\partial\Sigma}{\partial p^\nu}\right)\end{aligned} \tag{4.153}$$

Using $\Gamma_\nu=-e(\gamma_\nu+\Lambda_\nu)$ we finally get the Ward identity in the form used before,

$$\Lambda_\nu(p,p) = -\frac{\partial\Sigma}{\partial p^\nu}\ . \tag{4.154}$$

4.3 Counterterms and power counting

All that we have shown in the previous sections can be interpreted as follows. The initial Lagrangian $\mathcal{L}(e,m,\cdots)$ has been obtained from a correspondence between classical and quantum theory. It is then natural that the initial Lagrangian has to be modified by quantum corrections. The total Lagrangian is then given by,

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e,m,\cdots)+\Delta\mathcal{L} \tag{4.155}$$

and

$$\Delta\mathcal{L} = \Delta\mathcal{L}^{(1)} + \Delta\mathcal{L}^{[2]} + \cdots \tag{4.156}$$

where $\Delta\mathcal{L}^{[i]}$ is the $i^{\text{th}} - $ *loops* correction. This also correspond to order $\hbar^i$ as counting in terms of loops is equivalent to counting in terms of $\hbar^{10}$. This interpretation is quite attractive because in the limit $\hbar \to 0$ the total Lagrangian reduces to the classical one. With the Lagrangian $\mathcal{L}_{tot}$ we can then obtain finite results, although $\mathcal{L}_{tot}$ is divergent because of the counter-terms in $\Delta\mathcal{L}$.

With this language the results up to the first order in $\hbar$ can be written as,

$$\begin{aligned}
\mathcal{L}(e,m,\cdots) &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{2}A^\mu A_\mu - \frac{1}{2\xi}(\partial\cdot A)^2 \\
&\quad + i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi - e\overline{\psi}A\!\!\!/\psi & (4.157)\\
\Delta\mathcal{L}^{(1)} &= -\frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu} + (Z_2-1)(i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi) \\
&\quad + Z_2\delta m\overline{\psi}\psi - e(Z_1-1)\overline{\psi}A\!\!\!/\psi & (4.158)
\end{aligned}$$

The Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{total}} &= -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{2}A_\mu A^\mu - \frac{1}{2\xi}(\partial\cdot A)^2 \\
&\quad + Z_2(i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi + \delta m\overline{\psi}\psi) \\
&\quad - eZ_1\overline{\psi}A\!\!\!/\psi & (4.159)
\end{aligned}$$

will give the renormalized Green's functions up the the order $\hbar$.

In fact, we have only shown that the two-point and three-point Green's functions (self-energies and vertex) were finite. It is important to verify that all the Green's functions, with an arbitrary number of external legs are finite, as we have already used all our freedom in the renormalization of those Green's functions. This leads us to the so-called *power counting*.

Let us consider a Feynman diagram G, with L *loops*, I_B bosonic and I_F fermionic internal lines. If there are vertices with derivatives, δ_v is the number of derivatives in that vertex. We define then the *superficial degree of divergence* of the diagram (note that $L = I_B + I_F + 1 - V$) by,

$$\begin{aligned}
\omega(G) &= 4L + \sum_v \delta_v - I_F - 2I_B \\
&= 4 + 3I_F + 2I_B + \sum_v(\delta_v - 4) & (4.160)
\end{aligned}$$

For large values of the momenta the diagram will be divergent as

$$\Lambda^\omega(G) \quad \text{if} \quad \omega(G) > 0 \tag{4.161}$$

[10] $\hbar^{E-1+L} = \hbar^{\frac{E}{2}+\frac{V}{2}}$. We have the following relations $L = I - V + 1$; $3V = E + 2I$ (this only for QED).

and as

$$\ln \Lambda \quad \text{if} \quad \omega(G) = 0 \tag{4.162}$$

where Λ is a *cutoff*. The origin of the different terms can be seen in the following correspondence,

$$\begin{aligned}
&\int \frac{d^4q}{(2\pi)^4} \ (\text{for each loop}) && \to \quad 4L \\
&\partial_\mu \Leftrightarrow k_\mu && \to \quad \delta_v \\
&\frac{i}{\not{q} - m} && \to \quad -I_F \\
&\frac{i}{q^2 - m^2} && \to \quad -2I_B
\end{aligned} \tag{4.163}$$

The expression for $\omega(G)$ is more useful when expressed in terms of the number of external legs and of the dimensionality of the vertices of the theory. Let ω_v be the dimension, in terms of mass, of the vertex v, that is,

$$\omega_v = \delta_v + \#\text{campos bosónicos} \ + \ \frac{3}{2}\#\text{campos fermiónicos} \tag{4.164}$$

Then, if we denote by $f_v(b_v)$ the number of fermionic (bosonic) internal lines that join at the vertex v, we can write,

$$\sum_v \omega_v = \sum_v (\delta_v + \frac{3}{2} f_v + b_v) + \frac{3}{2} E_F + E_B \tag{4.165}$$

where $E_F(E_B)$ are the total number of *external* fermionic (bosonic) lines of the diagram. As we have,

$$\begin{aligned}
I_F &= \frac{1}{2} \sum_v f_v \\
I_B &= \frac{1}{2} \sum_v b_v
\end{aligned} \tag{4.166}$$

we get

$$\sum_v \omega_v = \sum_v \delta_v + 3I_F + 2I_B + \frac{3}{2} E_F + E_B \tag{4.167}$$

Substituting in the expression for $\omega(G)$ we get finally,

$$\begin{aligned}
\omega(G) =& 4 - \frac{3}{2} E_F - E_B + \sum_v (\omega_v - 4) \\
=& 4 - \frac{3}{2} E_F - E_B - \sum_v [g_v]
\end{aligned} \tag{4.168}$$

where $[g_v]$ denotes the dimension in terms of mass of the coupling constant of vertex v, satisfying,

$$\omega_v + [g_v] = 4 \ . \tag{4.169}$$

From the previous expression for the superficial degree of divergence, Eq. (4.168), we can then classify theories in three classes, *i) Non-renormalizable Theories*

They have at least one vertex with $\omega_v > 4$ (or $[g_v] < 0$). The superficial degree of divergence increases with the number of vertices, that is, with the order of perturbation theory. For an order high enough all the Green functions will diverge.

ii) Renormalizable Theories

All the vertices have $\omega_v \leq 4$ and at least one has $\omega_v = 4$. If all vertices have $\omega_v = 4$ then

$$\omega(G) = 4 - \frac{3}{2}E_F - E_B \tag{4.170}$$

and all the diagrams contributing to a given Green function have the same degree of divergence. Only a *finite* number of Green functions are divergent.

iii) Super-Renormalizable Theories

All the vertices have $\omega_v < 4$. Only a finite number of diagrams are divergent.[11]

Coming back to our question of knowing which are the divergent diagrams in QED, we can now summarize the situation in Table 4.1. All the other diagrams are superficially

E_F	E_B	$\omega(G)$	Effective degree of divergence
0	2	2	0 (Current Conservation (CC))
0	3		0 (Furry's Theorem)
0	4	0	Convergent (CC)
2	0	1	0 (Current Conservation)
2	1	0	0

Table 4.1: Superficial and effective degree of divergence in QED.

convergent. We have therefore a situation where there are only a finite number of divergent diagrams, exactly the ones that we considered before. This analysis shows that, up to order $\hbar$, the Lagrangian

$$\begin{aligned}
\mathcal{L}_{\text{total}} &= -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\lambda^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial\cdot A)^2 \\
&\quad + Z_2(i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi + \delta m\overline{\psi}\psi) \\
&\quad - eZ_1\overline{\psi}A\!\!\!/\psi
\end{aligned} \tag{4.171}$$

gives Green functions that are finite and renormalized with an arbitrary number of external legs. It remains to be shown that this Lagrangian is still valid up an arbitrary order in $\hbar$, with the only modification that the renormalization constants Z_1, Z_2, Z_3 e δm are now given by power series,

$$Z_1 = Z_1^{(1)} + Z_1^{(2)} + \cdots \tag{4.172}$$

[11]The effective degree of divergence it is sometimes smaller than the superficial degree because of symmetries of the theory. This is what happens for gauge theories like QED (see Table 4.1).

The previous Lagrangian, Eq. (4.171), allows for another interpretation that it is also useful. The fields $A, \overline{\psi}$ and ψ are the renormalized fields that give the residues equal to 1 for the poles of the propagators and the constants e, m are the physical electric charge and mass of the electron. Let us define the non-renormalized fields $\psi_0, \overline{\psi}_0$ and A_0 and the *bare* (cutoff dependent) μ_0^2, m_0 through the definitions,

$$\begin{aligned}
&\psi_0 = \sqrt{Z_2}\ \psi \qquad && m_0 = m - \delta m \\
&\overline{\psi}_0 = \sqrt{Z_2}\ \overline{\psi} \qquad && \lambda_0^2 = Z_3^{-1}\lambda^2 \\
&A_0 = \sqrt{Z_3}\ A \qquad && e_0 = Z_1 Z_2^{-1}\sqrt{Z_3^{-1}}e = \tfrac{1}{\sqrt{Z_3}}e \\
&\xi_0 = Z_3\xi
\end{aligned} \tag{4.173}$$

Then the Lagrangian written in terms of the bare quantities is identical to the original Lagrangian[12]

$$\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} + \frac{1}{2}\lambda_0^2\, A_{0\mu}A_0^\mu - \frac{1}{2\xi_0}(\partial\cdot A_0)^2 \\
& + i(\overline{\psi}_0 \partial\!\!\!/ \psi_0 - m_0\overline{\psi}_0\psi_0) - e_0\overline{\psi}_0 A\!\!\!/_0\psi_0
\end{aligned} \tag{4.174}$$

Finally we notice that the bare Green functions are related to the renormalized ones by

$$\begin{aligned}
& G_0^{n,\ell}(p_1, \cdots p_{2n}, k_1, \cdots k_\ell, \mu_0, m_0, \ell_0, \xi_0, \Lambda) \\
& \qquad = Z_2^n(\Lambda) Z_3^{\ell/2} G_R^{n,\ell}(p_1, \cdots p_{2n}, k_1 \cdots k_\ell, \mu, m, e, \xi)
\end{aligned} \tag{4.175}$$

where $p_1 \cdots p_{2n}$ ($k_1 \cdots k_\ell$) are the fermion (boson) momenta. We will come back to these relations in the study of the renormalization group, in chapter 7.

4.4 Finite contributions from RC to physical processes

4.4.1 Anomalous electron magnetic moment

We will show here, for the case of the electron anomalous moment, how the finite part of the radiative corrections can be compared with experiment, given credibility to the renormalization program. In fact we will just consider the first order, while to compare with the present experimental limit one has to go to fourth order in QED and to include also the weak and QCD corrections. The electron magnetic moment is given by

$$\vec{\mu} = \frac{e}{2m} g \frac{\vec{\sigma}}{2} \tag{4.176}$$

where $e = -|e|$ for the electron. One of the biggest achievements of the Dirac equation was precisely to predict the value $g = 2$. Experimentally we know that g is close to, but

[12] The terms $\frac{\lambda^2}{2}A^2 = \frac{\lambda_0^2}{2}A_0^2$ and $\frac{1}{2\xi}(\partial\cdot A)^2 = \frac{1}{2\xi_0}(\partial\cdot A_0)^2$ are not renormalized. This a consequence of the Ward-Takashashi identities for `QED`. The Ward identity $Z_1 = Z_2$ is crucial for the equality $e_0 A_0 = eA$ giving a meaning to the electric charge independently of the renormalization scheme.

not exactly, 2. It is usual to define this difference as the anomalous magnetic moment. More precisely,

$$g = 2(1+a) \tag{4.177}$$

or

$$a = \frac{g}{2} - 1 \tag{4.178}$$

Our task is to calculate a from the radiative corrections that we have computed in the previous sections. To do that let us start to show how a value $a \neq 0$ will appear in non relativistic quantum mechanics. Schrödinger's equation for a charged particle in an exterior field is,

$$i\frac{\partial \varphi}{\partial t} = \left[\frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi - \frac{e}{2m}(1+a)\vec{\sigma} \cdot \vec{B} \right] \varphi \tag{4.179}$$

Now we consider that the external field is a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Then keeping only terms first order in e we get

$$\begin{aligned} H &= \frac{p^2}{2m} - e\frac{\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}}{2m} - \frac{e}{2m}(1+a)\vec{\sigma}\cdot\vec{\nabla}\times\vec{A} \\ &\equiv H_0 + H_{int} \end{aligned} \tag{4.180}$$

With this interaction Hamiltonian we calculate the transition amplitude between two electron states of momenta p and p'. We get

$$\begin{aligned} \langle p' | H_{int} | p \rangle &= -\frac{e}{2m} \int \frac{d^3x}{(2\pi)^3} \chi^\dagger e^{-i\vec{p}'\cdot\vec{x}} \big[\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p} + (1+a)\vec{\sigma}\times\vec{\nabla}\cdot\vec{A} \big] e^{i\vec{p}\cdot x}\chi \\ &= -\frac{e}{2m} \int \frac{d^3x}{(2\pi)^3} \chi^\dagger \big[(\vec{p}' + \vec{p})\cdot\vec{A} + i(1+a)\sigma^i \epsilon^{ijk} q^j A^k \big] e^{-i\vec{q}\cdot x}\chi \\ &= -\frac{e}{2m}\chi^\dagger \big[(p'+p)^k + i(1+a)\sigma^i\epsilon^{ijk}q^j \big] A^k(q)\chi \end{aligned} \tag{4.181}$$

This is the result that we want to compare with the non relativistic limit of the renormalized vertex. The amplitude is given by,

$$\begin{aligned} A &= e\overline{u}(p')(\gamma_\mu + \Lambda_\mu^R)u(p)A^\mu(q) \\ &= e\overline{u}(p') \left[\gamma_\mu(1 + F_1(q^2)) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2) \right] u(p)A^\mu(q) \\ &= \frac{e}{2m}\overline{u}(p') \Big\{ (p'+p)_\mu \left[1 + F_1(q^2)\right] + i\sigma_{\mu\nu}q^\nu \left[1 + F_1(q^2) + F_2(q^2)\right] \Big\} u(p)A^\mu(q) \end{aligned} \tag{4.182}$$

where we have used Gordon's identity. For an external magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ and in the limit $q^2 \to 0$ this expression reduces to

$$\begin{aligned} A &= \frac{e}{2m}\overline{u}(p') \left\{ (p'+p)_k \left[1 + F_1(0)\right] + i\sigma_{kj}q^j \left[1 + F_1(0) + F_2(0)\right] \right\} u(p)A^k(q) \\ &= \frac{e}{2m}\overline{u}(p') \left[-(p'+p)^k + i\Sigma^i\epsilon^{kij}q^j \left(1 + \frac{\alpha}{2\pi}\right) \right] u(p)A^k(q) \end{aligned} \tag{4.183}$$

where we have used the results of Eq. (4.123),

$$\begin{cases} F_1(0) = 0 \\ F_2(0) = \dfrac{\alpha}{2\pi} \end{cases} \tag{4.184}$$

Using the explicit form for the spinors u

$$u(p) = \begin{pmatrix} \chi \\ \dfrac{\vec{\sigma}\cdot(\vec{p}-e\vec{A})}{2m}\,\chi \end{pmatrix} \tag{4.185}$$

we can write in the non relativistic limit,

$$A = -\frac{e}{2m}\chi^\dagger\left[(p'+p)^k + i\left(1+\frac{\alpha}{2\pi}\right)\sigma^i\epsilon^{ijk}q^j\right]\chi A^k \tag{4.186}$$

which after comparing with Eq. (4.181) leads to

$$a^e_{th} = \frac{\alpha}{2\pi} \tag{4.187}$$

This result obtained for the first time by Schwinger and experimentally confirmed, was very important in the acceptance of the renormalization program of Feynman, Dyson and Schwinger for QED.

4.4.2 Cancellation of IR divergences in Coulomb scattering

In this section we will show how the IR divergences cancel in physical processes. We will take as an example the Coulomb scattering from a fixed nucleus. This is better done if we start from first principles. Coulomb scattering corresponds to the diagram of Fig. 4.16, which gives the following matrix element for the S matrix,

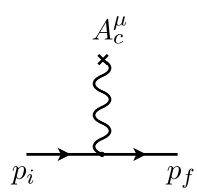


Figure 4.16: Lowest order diagram to Coulomb scattering.

$$S_{fi} = iZe^2(2\pi)\delta(E_i - E_f)\frac{1}{|\vec{q}|^2}\,\overline{u}(p_f)\gamma^0 u(p_i) \tag{4.188}$$

We will now study the radiative corrections to this result in lowest order in perturbation theory. Due to the IR divergences it is convenient to introduce a mass λ for the photon. For a classical field, as we are considering, this means a screening. If we take,

$$A_c^0(x) = Ze\frac{e^{-\lambda|\vec{x}|}}{4\pi|\vec{x}|} \tag{4.189}$$

the Fourier transform will be,

$$A_c^0(q) = Ze\frac{1}{|\vec{q}|^2+\lambda^2} \tag{4.190}$$

that shows that the screening is equivalent to a mass for the photon. With these modifications we have,

$$S_{fi} = iZe^2(2\pi)\delta(E_f-E_i)\ \frac{i}{|\vec{q}|^2+\lambda^2}\ \overline{u}(p_f)\gamma^0 u(p_i) \tag{4.191}$$

We are interested in calculating the corrections up to order e^3 in the amplitude. To this contribute[13] the diagrams of Fig. 4.17. Diagram 1 is of order e^2 while diagrams $2,3,4$

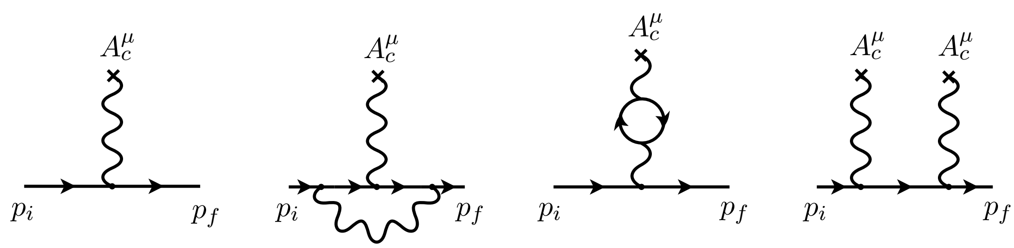


Figure 4.17: Coulomb scattering up to second order.

are of order e^4. Therefore the interference between 1 and $(2+3+4)$ is of order α^3 and should be added to the result of the bremsstrahlung in a Coulomb field. The contribution from $1+2+3$ can be easily obtained by noticing that

$$eA_c^\mu\gamma_\mu \to eA_c^\mu(\gamma_\mu + \Lambda_\mu^R + \Pi_{\mu\nu}^R G^{\nu\rho}\gamma_\rho) \tag{4.192}$$

where Λ_μ^R e $\Pi_{\mu\nu}^R$ have been calculated before. We get

$$\begin{aligned} S_{fi}^{(1+2+3)} &= iZe^2(2\pi)\delta(E_i-E_f)\frac{1}{|\vec{q}|^2+\lambda^2}\overline{u}(p_f)\gamma^0\left\{1+\frac{\alpha}{\pi}\left[-\frac{1}{2}\varphi\tanh\varphi\right.\right. \\ &\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth 2\varphi-1)-2\coth 2\varphi\int_0^\varphi \beta\tanh\beta d\beta \\ &\left.\left.+\left(1-\frac{\coth^2\varphi}{\beta}\right)(\varphi\coth\varphi-1)+\frac{1}{9}\right]-\frac{\not{q}}{2m}\frac{\alpha}{\pi}\frac{\varphi}{\sinh 2\varphi}\right\}u(p_i) \end{aligned} \tag{4.193}$$

where

$$\frac{|\vec{q}|^2}{4m} = \sinh^2\varphi\,. \tag{4.194}$$

Finally the fourth diagram gives

[13] We do not have to consider the self-energies of the external legs of the electron because they are on-shell.

$$
\begin{aligned}
S_{fi}^{(4)} &= (iZe)^2(e)^2 \int \frac{d^4k}{(2\pi)^4}\overline{u}(p_f)\left[\frac{2\pi\delta(E_f-k^0)}{(p_f-k)^2-\lambda^2}\gamma^0 \frac{i}{\not{k}-m+i\varepsilon}\gamma^0 \frac{2\pi\delta(k^0-E_i)}{(k-p_i)^2-\lambda^2}\right] \\
&= -2i\frac{Z^2\alpha^2}{\pi}2\pi\delta(E_f-E_i)\ \overline{u}(p_f)\left[m(I_1-I_2)+\gamma^0E_i(I_1+I_2)\right]u(p_i) \qquad (4.195)
\end{aligned}
$$

with

$$
I_1 = \int d^3k \frac{1}{[(\vec{p}_f-\vec{k})^2+\lambda^2][(\vec{p}_i-\vec{k})^2+\lambda^2][(\vec{p})^2-(\vec{k})^2+i\varepsilon]} \qquad (4.196)
$$

and

$$
\frac{1}{2}(\vec{p}_i+\vec{p}_f)I_2 \equiv \int d^3k \frac{\vec{k}}{[(\vec{p}_f-\vec{k})^2+\lambda^2][(\vec{p}_i-\vec{k})^2+\lambda^2][(\vec{p})^2-(\vec{k})^2+i\varepsilon]}\,. \qquad (4.197)
$$

In the limit $\lambda \to 0$ it can be shown that

$$
\begin{aligned}
I_1 &= \frac{\pi^2}{2ip^3\sin^2\theta/2}\ln\left(\frac{2p\sin(\theta/2)}{\lambda}\right) \qquad (4.198)\\
I_2 &= \frac{\pi^2}{2p^3\cos^2\theta/2}\left\{\frac{\pi}{2}\left[1-\frac{1}{\sin\theta/2}\right]-i\left[\frac{1}{\sin^2\theta/2}\ln\left(\frac{2p\sin\theta/2}{\lambda}\right)+\ln\frac{\lambda}{2p}\right]\right\} \\
& \qquad (4.199)
\end{aligned}
$$

With these expressions we get for the cross section

$$
\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{|\vec{q}|^2}\frac{1}{2}\sum_{pol}|\overline{u}(p_f)\Gamma u(p_i)|^2 \qquad (4.200)
$$

where

$$
\Gamma = \gamma^0(1+A)+\gamma^0\frac{\not{q}}{2m}B+C \qquad (4.201)
$$

and

$$
\begin{aligned}
A &= \frac{\alpha}{\pi}\left[\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth 2\varphi-1)-2\coth 2\varphi\int_0^{\varphi}d\beta\beta\tanh\beta-\frac{\varphi}{2}\tanh\varphi\right.\\
&\left.+\left(1-\frac{1}{3}\coth^2\varphi\right)(\varphi\coth\varphi-1)+\frac{1}{9}\right]-\frac{Z\alpha}{2\pi^2}|\vec{q}|^2E(I_1+I_2) \qquad (4.202)\\
B &= -\frac{\alpha}{\pi}\frac{\varphi}{\sinh 2\varphi} \qquad (4.203)\\
C &= -\frac{Z\alpha}{2\pi^2}m|\vec{q}|^2(I_1-I_2) \qquad (4.204)
\end{aligned}
$$

Therefore

$$\begin{aligned}\frac{1}{4}\sum_{pol}|\overline{u}(p_f)pu(p_i)|^2 &= \frac{1}{4}\mathrm{Tr}[\Gamma(\not p_i+m)\overline{\Gamma}(\not p_f+m)]\\ &= 2E^2(1-\beta^2\sin^2\theta/2)+2E^2 2B\beta^2\sin^2\frac{\theta}{2}\\ &+2E^2 2ReA\left(1-\beta^2\sin^2\frac{\theta}{2}\right)+2ReC(2mE)+O(\alpha^2) \end{aligned} \tag{4.205}$$

Notice that A, B e C are of order α. Therefore the final result is, up to order α^3:

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\left\{1+\frac{2\alpha}{\pi}\left[\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth\varphi-1)-\frac{\varphi}{2}\tanh\varphi\right.\right.\\ &-2\coth 2\varphi\int_0^\varphi d\beta\beta\tanh\beta+\left(-\frac{\coth^2\varphi}{3}\right)(\varphi\coth\varphi-1)+\frac{1}{9}\\ &\left.\left.-\frac{\varphi}{\sinh 2\varphi}\frac{B^2\sin^2\theta/2}{1-\beta^2\sin^2\theta/2}\right]+Z\alpha\pi\frac{\beta\sin\frac{\theta}{2}[1-\sin\theta/2]}{1-\beta^2\sin^2\theta/2}\right\} \end{aligned} \tag{4.206}$$

As we had said before the result is IR divergent in the limit $\lambda\to 0$. This divergence is not physical and can be removed in the following way. The detectors have an energy threshold, below which they can not detect. Therefore in the limit $\omega\to 0$ bremsstrahlung in a Coulomb field and Coulomb scattering can not be distinguished. This means that we have to add both results. If we consider an energy interval ΔE with $\lambda\leq\Delta E\leq E$ we get

$$\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR}=\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\int_{\omega\leq\Delta E}\frac{d^3k}{2\omega(2\pi)^3}e^2\left[\frac{2p_i\cdot p_f}{k_i\cdot p_i k\cdot p_f}-\frac{m^2}{(k\cdot p\cdot)^2}-\frac{m^2}{(k\cdot p_f)^2}\right] \tag{4.207}$$

Giving a mass to the photon (that is $\omega=(|\vec{k}|^2+\lambda^2)^{1/2}$) the integral can be done with the result,

$$\begin{aligned}\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}\frac{2\alpha}{\pi}\left\{(2\varphi\coth 2\varphi-1)\ln\frac{2\Delta E}{\lambda}+\frac{1}{2\beta}\ln\frac{1+\beta}{1-\beta}\right.\\ &\left.-\frac{1}{2}\cosh 2\varphi\frac{1-\beta^2}{\beta\sin\theta/2}\int_{\cos\theta/2}^1 d\xi\frac{1}{(1-\beta^2\xi^2)[\xi-\cos^2\theta/2]^{1/2}}\ln\frac{1+\beta\xi}{1-\beta\xi}\right\} \end{aligned} \tag{4.208}$$

The inclusive cross section can now be written as

$$\frac{d\sigma}{d\Omega}(\Delta E) = \left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}}+\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR}$$

$$= \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} (1 - \delta_R + \delta_B) \tag{4.209}$$

where δ_R and δ_B are complicated expressions that depend on the resolution of the detector ΔE but do not depend on λ that can be finally put to zero. One can show that in QED all the IR divergences can be treated in a similar way. One should note that the final effect of the bremsstrahlung is finite and can be important.

Chapter 5

Functional Methods

5.1 Introduction

In this chapter, called *Functional Methods*, we are going to present the path integral quantization. For systems that are not described by gauge theories this method may seem unnecessary, as the canonical quantization works without problems. However, for non-abelian gauge theories, as we shall see in the next chapter, this is the only known method. Besides this fundamental point, the quantization done using functional methods and the path integral formalism is very elegant and allows us to obtain the results much faster, even for the cases where the canonical quantization works. Examples of this are the Ward-Takahashi identities and the Dyson-Schwinger equations, as we will discuss at the end of the chapter.

We are going to assume that the reader is familiar with the path-integral quantization for systems of N particles in non-relativistic quantum mechanics. Therefore only a brief sumamry of the results will be given. A more detailed account is given in Appendix A. The step from the quantization of a system with N particles to the quantization of a field theory will be done heuristically. A more rigorous treatment will be given in Appendix B.

Before we start, let us clarify some questions related with the notation. Let us assume that we have real scalar field $\phi^a(x)$ where $a = 1, ...N$. In the following we will encounter expressions of the type,

$$I_1 = \int d^4x\, \phi^a(x)\phi^a(x) \tag{5.1}$$

or

$$I_2 = \int d^4x d^4y\, \phi^a(x) M^{ab}(x,y)\phi^b(y) \tag{5.2}$$

where $M^{ab}(x,y)$ is normally a differential operator. According to the rules for functional derivation, we have,

$$\frac{\delta I_1}{\delta \phi^b(y)} = 2\phi^b(y) \tag{5.3}$$

where we used the result

$$\frac{\delta \phi^a(x)}{\delta \phi^b(y)} = \delta^{ab}\delta^4(x-y) \tag{5.4}$$

If we keep all the indices in the previous expressions (and in some much more complicated that we will encounter soon), we will get a very complicated situation with respect to the notation. Therefore it will be useful to make use of a more compact notation. To this end we identify,

$$\phi_i \Longleftrightarrow \phi^a(x) \tag{5.5}$$

that is, the indice i will represent *both* the discrete indice a as weel as the continuous x,

$$i \Longleftrightarrow \{a, x\} \tag{5.6}$$

In the case that the fields have further indices we will assume that i will always reprent them collectively. We also use the Einstein convention meaing a sum for discrete indices and an integration for continuous indices. With these conventions Eq. (5.1) and Eq. (5.4) can be writtem as

$$\begin{aligned} I_1 = \phi_i \phi_i \quad & I_2 = \phi_i M_{ij} \phi_j \\ \frac{\delta I_1}{\delta \phi_j} = 2\phi_j \qquad & \frac{\delta \phi_i}{\delta \phi_j} = \delta_{ij} \end{aligned} \tag{5.7}$$

In the following we will use these conventions, returning to the more usual notation when convenient or in case of a possible confusion.

5.2 Generating functional for Green's functions

5.2.1 Green's functions

Os objectos básicos em Teoria Quântica dos Campos são as *funções de Green*. Para evitar complicações desnecessárias vamos aqui considerar quase exclusivamente campos escalares. As generalizações são no entanto imediatas. Assim a função de Green de ordem n é dada por

$$G^{(n)}(x_1, \ldots, x_n) \equiv \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle \ . \tag{5.8}$$

As funções de Green definidas por 5.8 são, por vezes, designadas *completas* por oposição às chamadas funções de Green *conexas*, *truncadas* e *irredutíveis (ou próprias)* que passamos a definir.

5.2.2 Connected Green's functions

Chamam-se funções de Green *conexas* aquelas em que nenhuma das linhas exteriores passa através do diagrama sem interagir. Por exemplo na Figura 5.1 está representada uma contribuição desconexa para $G^4(x_1, \ldots, x_4)$, enquanto que a Figura 5.2 representa uma contribuição conexa para a mesma função.

Está implícito que as funções de Green são calculadas em teoria das perturbações usando diagramas de Feynman. Assim as funções de Green conexas $G_c^{(4)}(x_1, \ldots, x_n)$, são obtidas somando todos os diagramas conexos. As funções de Green desconexas, correspondendo aos diagramas desconexos, podem ser obtidas a partir de funções de Green conexas de ordem mais baixa, pelo que as quantidades relevantes são as funções de Green conexas $G_c^n(x_1, \ldots, x_n)$. É claro que temos

$$G_c^n(x_1, \ldots, x_n) = G^n(x_1, \ldots, x_n) - \text{partes desconexas}, \tag{5.9}$$

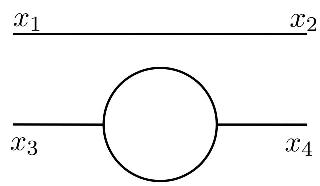


Figure 5.1: Disconneted contribution to $G^4(x_1, \ldots, x_4)$.

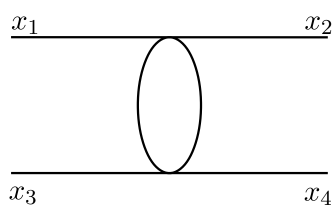


Figure 5.2: Connected contribution to $G^4(x_1, \ldots, x_4)$.

e ainda

$$G_c^2(x_1, x_2) = G^2(x_1, x_2) \ . \tag{5.10}$$

Convencionalmente representamos $G_c^n(x_1, \ldots, x_n)$ pelo diagrama da Figura 5.3.

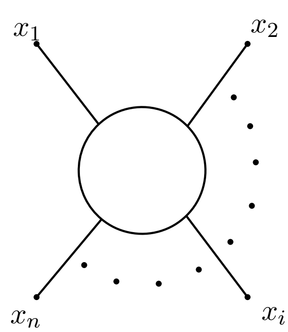


Figure 5.3: Graphical representation for $G_c^n(x_1, \ldots, x_n)$.

Por vezes interessa considerar as funções de Green no espaço dos momentos. Definimos então $G_c^n(p_1, \ldots, p_n)$ através da relação

$$\begin{aligned}
&(2\pi)^4 \delta^4(p_1 + p_2 + \cdots p_n)\ G_c^n(p_1, \ldots, p_n) \\
&\equiv \int d^4x_1 \cdots d^4x_n e^{-i(p_1 \cdot x_1 + \cdots + p_n \cdot x_n)}\ G_c^n(x_1, \ldots, x_n) \ ,
\end{aligned} \tag{5.11}$$

onde os momentos $p_1, \ldots, p_n$ estão a entrar no diagrama (*incoming momenta*), conforme indicado na Figura 5.4. Notar ainda que na definição 5.11 se factorizou a função delta

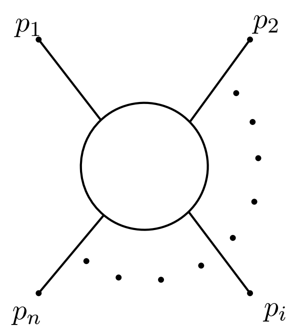


Figure 5.4: Graphical representation for $G_c^n(p_1, \ldots, p_n)$

que assegura a conservação de 4-momento. Com estas convenções $G^2(p, -p) \equiv G^2(p)$ é o propagador completo representado na Figura 5.5.

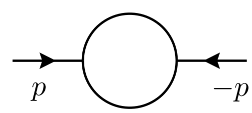


Figure 5.5: Full propagator $G^2(p)$.

5.2.3 Truncated Green's functions

Para $n > 2$ definem-se as funções de Green *truncadas* através da relação

$$G^n_{\text{trunc}}(p_1, \ldots, p_n) = \prod_{k=1}^{n} \left[G^2(p_k)\right]^{-1} G_c^n(p_1, \ldots, p_n) \tag{5.12}$$

isto é, multiplica-se cada linha exterior pelo inverso do propagador completo referente a essa linha. São estas funções que representam um papel fundamental na Teoria pois são elas que estão relacionadas com os elementos da matriz S. De facto a fórmula de redução LSZ para campos escalares é

$$\begin{aligned}
\langle p_1, \ldots, p_n \text{ out} | q_1, \ldots, q_\ell \text{ in}\rangle &= \langle p_1, \ldots, p_n \text{ in} | S | q_1, \ldots, q_\ell \text{ in}\rangle \\
&= \text{termos desconexos} \\
&+ \left(iZ^{-1/2}\right)^{n+\ell} \int d^4y_1 \cdots d^4x_\ell \ \exp\left[i\left(\sum_{k=1}^{n} p_k \cdot y_k - \sum_{k=1}^{\ell} q_k \cdot x_k\right)\right] \\
&\times (\Box_{y_1} + m^2) \cdots (\Box_{x_\ell} + m^2) \left\langle 0 | T\phi(y_1) \cdots \phi(x_\ell) | 0\right\rangle_c
\end{aligned} \tag{5.13}$$

ou seja

$$\langle p_1, \ldots, p_n \text{ out}|q_1, \ldots, q_\ell \text{ in}\rangle = \langle p_1, \ldots, p_n \text{ in}|S|q_1, \ldots, q_\ell \text{ in}\rangle$$

$$= \text{termos desconexos}$$

$$+Z^{-(n+\ell)/2}\ (2\pi)^4\delta\left(\sum p_i - \sum q_j\right)\ G^{n+\ell}_{\text{trunc}}(-p_1, \ldots, -p_n, q_1, \ldots, q_\ell) \quad (5.14)$$

5.2.4 Irreducible diagrams

Vimos na Eq. 5.14 que os elementos da matriz S, relacionados com as secções eficazes, são expressos em termos dos diagramas truncados. De entre os diagramas truncados desempenha um papel importante o subconjunto dos diagramas *próprios* ou *irredutíveis* (em inglês *1-Particle Irreducible*), que são os diagramas truncados que permanecem ligados quando uma linha interna arbitrária é cortada. Por exemplo, o diagrama da Figura 5.6[1] é truncado mas não é próprio enquanto que o diagrama da Figura 5.7 é próprio (na teoria $\lambda\phi^3$). Nestas figuras as barras indicam que as linhas exteriores estão truncadas.

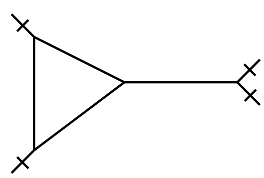


Figure 5.6: Example of a truncated diagram that is not proper.

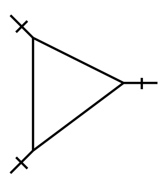


Figure 5.7: Example of a proper diagram.

A razão pela qual os diagramas truncados não irredutíveis não são importantes é que estes se podem escrever sempre em termos de diagramas irredutíveis de ordem mais baixa (recordar a série que conduz à definição de *self-energy*). É conveniente introduzir uma notação para as funções de Green irredutíveis (soma de todos os diagramas irredutíveis para determinado número de pernas exteriores) onde o factor i é introduzido por conveniência. Na Figura 5.8 as pernas externas estão desenhadas para tornar a figura mais clara. Elas estão de facto truncadas. Dentro desta notação pode por vezes ser conveniente

[1] As barras indicam que as linhas exteriores estão truncadas.

$$i\Gamma^{(n)}(x_1,\ldots,x_n) =$$

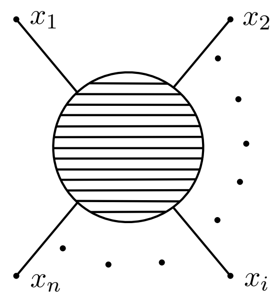


Figure 5.8: Irreducible Green functions.

definir um diagrama para as funções de Green truncadas de ordem n. Este está representado na Figura 5.9 ou doutra forma na Figura 5.10. Diagramas semelhantes podem-se

$$G^{(n)}_{\text{trunc}}(x_1,\ldots,x_n) \equiv$$

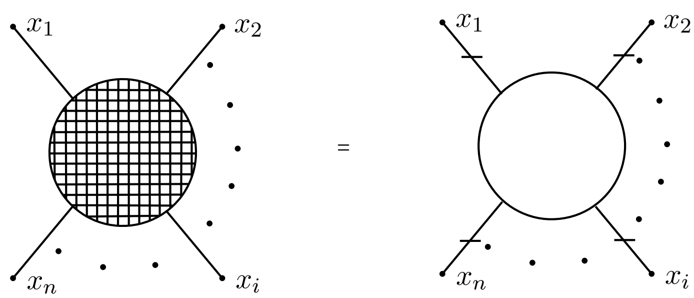


Figure 5.9: Graphical representation of truncated Green functions.

definir no espaço dos momentos.

5.3 Generating functionals for Green's functions

Os *funcionais geradores* (FG) das funções de Green representam um papel muito importante em teoria quântica dos campos. De facto a partir deles, por derivação funcional em relação a fontes exteriores podem-se obter todas as funções de Green. Permitem assim tratar ao mesmo tempo um número infinito de funções de Green. O FG das funções de Green completas é dado por

$$Z(J) \equiv \left\langle 0|Te^{iJ_i\phi_i}|0\right\rangle \ , \tag{5.15}$$

onde estamos a usar a notação condensada explicada na introdução

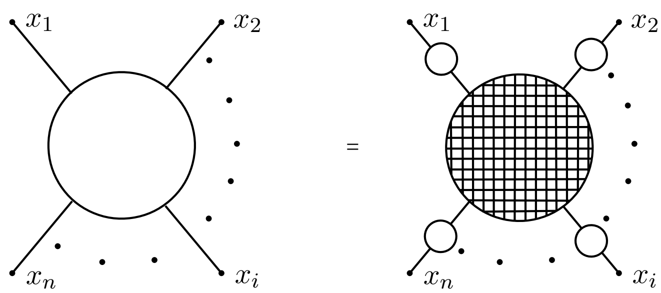


Figure 5.10: Another representation of truncated Green functions.

$$J_i\phi_i \equiv \int d^4x\ J(x)\phi(x)\ . \tag{5.16}$$

É fácil de ver que $Z(J)$ gera todas as funções de Green. Se expandirmos a exponencial em 5.15 obtemos

$$\begin{aligned} Z(J) &= \textstyle\sum_{n=0}^{\infty} \frac{i^n}{n!} J_{i_1}\cdots J_{i_n} \langle 0|T\phi_{i_1}\cdots\phi_{i_n}|0\rangle \\ &= \textstyle\sum_{n=0}^{\infty} \frac{i^n}{n!} J_{i_1}\cdots J_{i_n} G^n_{i_1\cdots i_n} \end{aligned} \tag{5.17}$$

As funções de Green são então dadas por

$$G^n_{i_1\cdots i_n} = \left.\frac{\delta^n Z}{i\delta J_{i_1}\cdots i\delta J_{i_n}}\right|_{J_i=0} \tag{5.18}$$

O FG das funções de Green conexas é definido através da relação

$$Z(J) = e^{iW(J)} \tag{5.19}$$

ou ainda

$$W(J) = -i\ln Z(J)\ . \tag{5.20}$$

As funções de Green conexas são então obtidas por derivação funcional

$$G^n_{c\ i_1\cdots i_n} = i\ \left.\frac{\delta^n W}{i\delta J_{i_1}\cdots i\delta J_{i_n}}\right|_{J_i=0} \tag{5.21}$$

Antes de mostrarmos que isto é de facto verdade, vamos definir o funcional gerador das funções de Green irredutíveis. Este é definido através da primeira transformada da Legendre de $W(J)$, isto é

$$\Gamma(\phi) \equiv W(J) - J_i\phi_i \tag{5.22}$$

onde

$$\left\{ \begin{array}{ll} \phi_i & \equiv \dfrac{\delta W(J)}{\delta J_i} \\ J_i & = -\dfrac{\delta \Gamma(\phi)}{\delta \phi_i} \end{array} \right. \tag{5.23}$$

As funções de Green próprias são então dadas por

$$\Gamma^n_{i_1 \cdots i_n} = \left. \frac{\delta^n \Gamma(\phi)}{\delta \phi_{i_1} \cdots \delta \phi_{i_n}} \right|_{\phi=0} . \tag{5.24}$$

Dadas as definições falta-nos agora mostrar que $W(J)$ e $\Gamma(\phi)$ geram efectivamente as funções de Green conexas e próprias. Comecemos por $W(J)$. A demonstração faz-se calculando $G^n_{c\ i_1 \cdots i_n}$ dada por 5.21 e usando a relação 5.20. Vamos fazer somente para $n = 2$, $n = 3$ e $n = 4$. As generalizações são imediatas.

- $n = 2$

$$\begin{aligned} G^2_{c\ i_1 i_2} &= i \left. \frac{\delta^2 W}{i\delta J_{i_1} i\delta J_{i_2}} \right|_{J_i=0} = \left. \frac{\delta^2 \ln Z}{i\delta J_{i_1} i\delta J_{i_2}} \right|_{J_i=0} = \left. \frac{\delta}{i\delta J_{i_1}} \frac{1}{Z} \frac{\delta Z}{i\delta J_{i_2}} \right|_{J_i=0} \\ &= \frac{1}{Z} \left. \frac{\delta^2 Z}{i\delta J_{i_1} i\delta J_{i_2}} \right|_{J_i=0} - \frac{1}{Z} \frac{\delta Z}{i\delta J_{i_1}} \frac{1}{Z} \left. \frac{\delta Z}{i\delta J_{i_2}} \right|_{J_i=0} \\ &= \left. \frac{\delta^2 Z}{i\delta J_{i_1} i\delta J_{i_2}} \right|_{J_i=0} \end{aligned} \tag{5.25}$$

ou seja

$$G^2_{c\ i_1 i_2} = G^2_{i_1 i_2} \tag{5.26}$$

Para se obter 5.25 fez-se uso dos seguintes resultados

$$\begin{aligned} Z(0) &= 1 \qquad \text{O vácuo está normalizado} \\ \frac{\delta Z}{i\delta J_i} &= \langle 0|T\phi_i|0\rangle = 0 \qquad \text{Não há quebra de simetria} \end{aligned} \tag{5.27}$$

- $n = 3$

$$\begin{aligned} G^3_{c\ i_1 i_2 i_3} = \Bigg[& \frac{1}{Z} \frac{\delta^3 Z}{i\delta J_{i_1} i\delta J_{i_2} i\delta J_{i_3}} - \frac{1}{Z} \frac{\delta^2 Z}{i\delta J_{i_1} i\delta J_{i_2}} \frac{1}{Z} \frac{\delta Z}{i\delta J_{i_3}} \\ & - \frac{1}{Z} \frac{\delta^2 Z}{i\delta J_{i_2} i\delta J_{i_3}} \frac{1}{Z} \frac{\delta Z}{i\delta J_{i_1}} - \frac{1}{Z} \frac{\delta^2 Z}{i\delta J_{i_1} i\delta J_{i_3}} \frac{1}{Z} \frac{\delta Z}{i\delta J_{i_2}} \\ & + 2\, \frac{1}{Z} \frac{\delta Z}{i\delta J_{i_1}} \frac{1}{Z} \frac{\delta Z}{i\delta J_{i_2}} \frac{1}{Z} \frac{\delta Z}{i\delta J_{i_3}} \Bigg]_{|J_i=0} \end{aligned} \tag{5.28}$$

logo

$$G^3_{i_1 i_2 i_3} = G^3_{c\ i_1 i_2 i_3} \tag{5.29}$$

O caso $n = 4$ é deixado como problema (ver Problema 1.1) A extensão a $n > 4$ é imediata. Mostrámos assim que $W(J)$ dado por 5.20 é o funcional gerador das funções de Green conexas. Mostremos agora que $\Gamma(\phi)$ é o funcional gerador das funções de Green próprias, ou irredutíveis. Para isto necessitamos de dois resultados prévios que passamos a demonstrar. O primeiro baseia-se na relação

$$\frac{\delta J_i}{\delta J_k} = \delta_{ik} \tag{5.30}$$

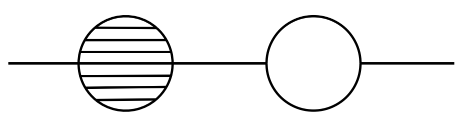
Esta relação é evidente mas podemos obter a partir dela uma relação importante. De facto

$$\frac{\delta J_i}{\delta J_k} = \frac{\delta J_i}{\delta \phi_\ell}\frac{\delta \phi_\ell}{\delta J_k} = -\frac{\delta^2 \Gamma}{\delta\phi_i \delta\phi_\ell}\frac{\delta^2 W}{\delta J_\ell \delta J_k} = -i\Gamma_{i\ell}G_{\ell k} \tag{5.31}$$

ou ainda

$$\Gamma_{i\ell}G_{\ell k} = i\delta_{ik} \tag{5.32}$$

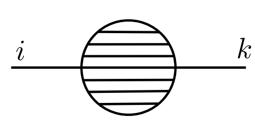
Esta relação fundamental exprime que Γ^2 é o inverso do propagador G^2 (à parte o factor i que tem que ver com convenções). É também útil escrevê-la duma forma diagramática conforme indicado na Figura 5.11.



$= \quad -1$

Figure 5.11: Graphical representation of Eq.5.32.

Notar que

$i\Gamma^{(2)}_{ik} \equiv$ 

(5.33)

o que explica o desaparecimento do i na equação 5.32.
O segundo resultado diz respeito à seguinte derivada funcional

$$\frac{\delta}{i\delta J_i}\ . \tag{5.34}$$

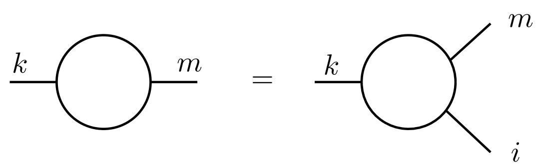
Pretende-se derivar em ordem a J_i quantidades que dependem de J_i indirectamente através de ϕ_k (ver definições 5.23). Obtemos então

$$\frac{\delta}{i\delta J_i} = \frac{\delta \phi_k}{i\delta J_i}\frac{\delta}{\delta \phi_k} = \frac{\delta^2 W}{\delta J_k i\delta J_i}\frac{\delta}{\delta\phi_k} = G^{(2)}_{ik}\frac{\delta}{\delta\phi_k} \tag{5.35}$$

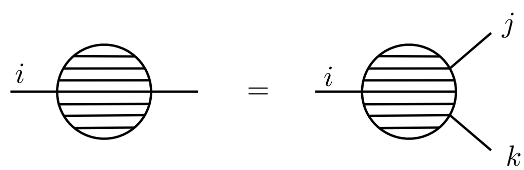
e portanto

$$\frac{\delta}{i\delta J_i} = G_{ik}\frac{\delta}{\delta\phi_k} \tag{5.36}$$

As equações 5.32 e 5.36 permitem obter todas as relações entre as funções de Green próprias e as funções de Green conexas. Esta análise é mais fácil em termos de diagramas desde que se notem as seguintes identidades

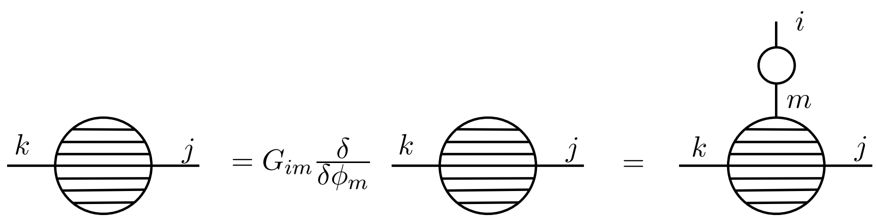
$\dfrac{\delta}{i\delta J_i}$ 

(5.37)

$\dfrac{\delta}{\delta\phi_k}$ 

(5.38)

e

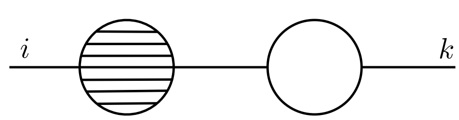
$\dfrac{\delta}{i\delta J_i}$ 

(5.39)

onde se usou 5.36 para estabelecer 5.39. Em todas estas manipulações está subentendido que no final se faz $J = 0$ e $\phi = 0$ nos sítios convenientes. Usemos agora estes métodos para relacionar as funções de Green próprias e conexas para $n = 3$ e $n = 4$.

- $n = 3$

  O ponto de partida é a equação 5.32. Aplicamos $\frac{\delta}{i\delta J_\ell}$ a 5.32 e obtemos

$\dfrac{\delta}{i\delta J_\ell}$ 

$= 0$ (5.40)

Usando as equações 5.37 e 5.39 obtemos então o diagrama da Figura 5.12. Multiplicando à esquerda por $G^{(2)}_{mi}$ e usando 5.32 obtemos

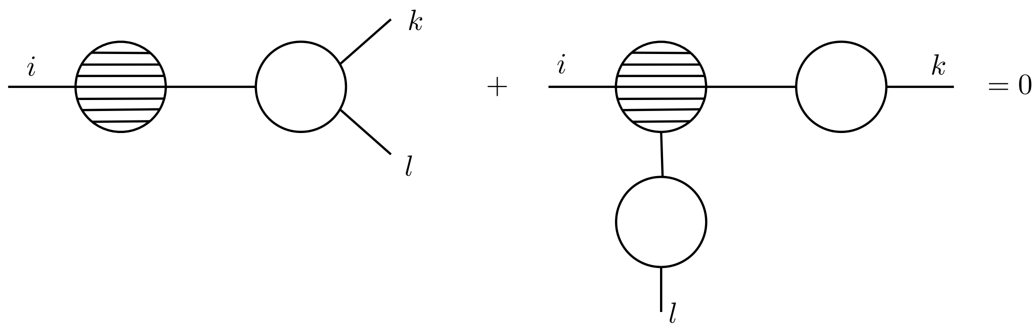
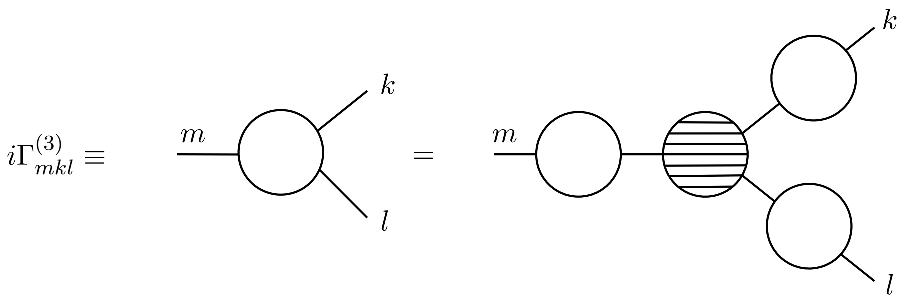


Figure 5.12: Graphical result of Eq.5.40.



(5.41)

o que mostra que $\Gamma^{(3)}_{mkl}$ é de facto a função de Green própria com 3 pernas exteriores porque para 3 pernas as funções próprias e truncadas coincidem. Para se ver que se trata de facto de funções próprias e não somente truncadas, é preciso ir para $n = 4$ pois aí é que começa a diferença.

- $n = 4$

Partimos da equação 5.41 e derivamos em relação a $\frac{\delta}{i\delta J_n}$. Usando os métodos anteriores obtemos a equação representada na Figura 5.13. Se usarmos 5.41 para expressar $G^{(3)}_{kml}$ em termos de $\Gamma^{(3)}_{kml}$ obtemos a equação diagramática da Figura 5.14 que mostra claramente que $\Gamma^{(4)}_{mkln}$ é de facto a função de Green própria ou irredutível de 4 pernas.

- $n > 4$

É agora trivial continuar o processo para $n > 4$. Para um dado n parte-se da relação para $n-1$ e aplicam-se as equações 5.37 e 5.39. Estes resultados mostram de facto que os objectos mais importantes são as funções de Green irredutíveis, todas as outras se podem obter a partir delas. Isto é um resultado importante porque reduz imenso o número de diagramas de Feynman que têm que ser calculados.

5.4 Feynman rules

The formalism of functional generators allows us to obtain the Feynman rules of any theory with all the correct conventions. We have already shown how to get the Feynman rules in

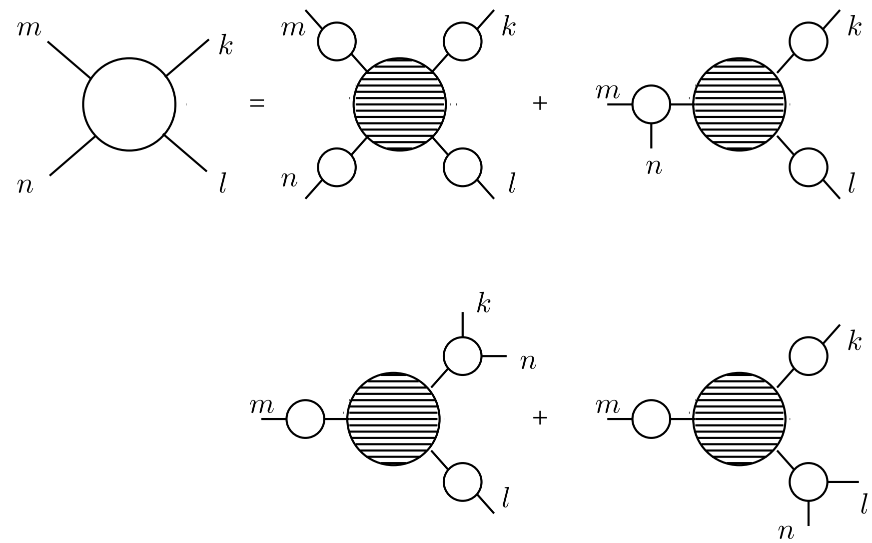


Figure 5.13: Graphical representation of G_{mkln} in terms of Γ_{mkln}.

section 3.7. There we used the result that in lowest order (tree level) we have

$$\Gamma_{\text{tree}}(\phi) = \int d^4x \mathcal{L}[\phi] \equiv \Gamma_0(\phi) \tag{5.42}$$

Here we are going just to show this result. For the interaction terms ($n > 2$) this is clear. For instance for $n = 3$ we have

$$i\Gamma^{(3)} = G^{(3)}_{\text{tree}} \tag{5.43}$$

while for $n = 4$ we get

$$i\Gamma^{(4)} = G^{(4)} - \text{ irreducible parts} \tag{5.44}$$

and it is obvious that $i\Gamma^{(4)}_{\text{tree}}$ generates the vertices.

The i factor is in agreement with the usual conventions for the Feynman rules as it comes from the term $\exp\left(i\int d^4x\mathcal{L}_{int}\right)$ in the calculation of the Green functions. For the quadratic terms we have, Eq. 5.32,

$$\Gamma^{(2)}_{\text{tree}}(p) = p^2 - m^2 \tag{5.45}$$

therefore, doing the inverse Fourier transform,

$$\Gamma^{(2)}_{\text{tree}}(x,y) = \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} e^{i(p_1\cdot x + p_2\cdot y)}(2\pi)^4\delta^4(p_1+p_2)\ (p_1^2 - m^2) \tag{5.46}$$

and

$$\frac{1}{2}\int d^4x d^4y\, \phi(x)\Gamma^{(2)}_{\text{tree}}(x,y)\phi(y) =$$

$$= \frac{1}{2}\int d^4x d^4y\, \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\, e^{i(p_1\cdot x+p_2\cdot y)}(2\pi)^4\delta(p_1+p_2)(p_1^2-m^2)\phi(x)\phi(y)$$

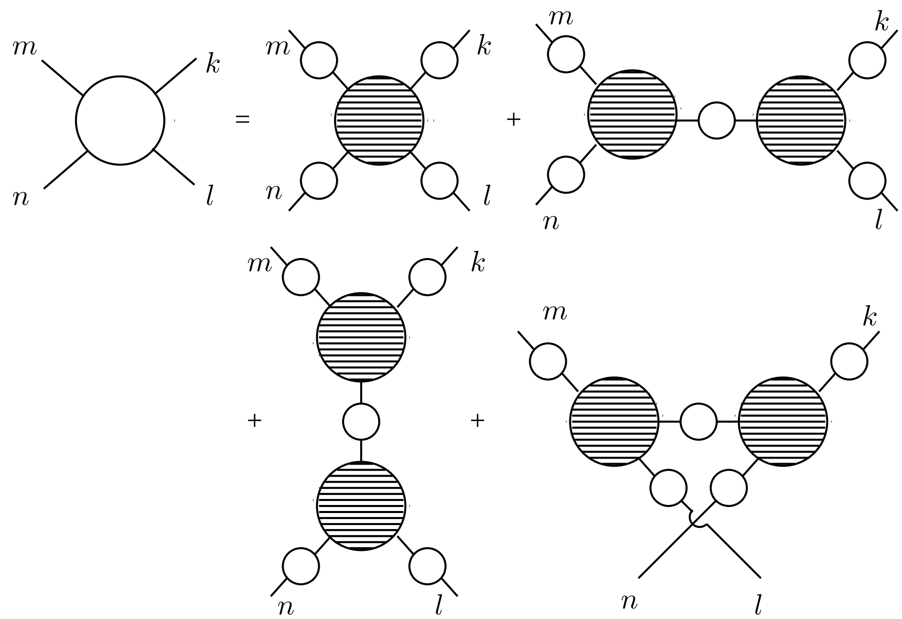


Figure 5.14: Graphical representation of G_{mkln} in terms of Γ_{mkln}.

$$= \frac{1}{2} \int d^4x\, \phi(x)(-\Box - m^2)\phi(x) = \frac{1}{2} \int d^4x \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) \tag{5.47}$$

which shows that $\Gamma_{\rm tree}$ is in fact the action. In getting to Eq. (5.47) we have done an integration by parts and discarded, usual, the boundary term. We refer the reader to Section 3.7 for the actual recipes on how to determine the Feynman rules of any theory.

5.5 Path integral for generating functionals

5.5.1 Quantum mechanics of n degrees of freedom

Comecemos por recordar os resultados conhecidos para sistemas com 1 grau de liberdade. No Apêndice A faz-se uma introdução à quantificação via integral de caminho. Lá poderão ser encontradas as justificações para os resultados que usaremos no seguimento. O resultado fundamental é para a amplitude de transição

$$\left\langle q'; t' | q; t \right\rangle = N \int \mathcal{D}(q) e^{i \int_t^{t'} dt L(q,\dot{q})} = N \int \mathcal{D}(q) e^{iS} \tag{5.48}$$

onde N é um factor de normalização e $\mathcal{D}(q)$ é uma forma simbólica de representar a medida de integração que é de facto um limite complicado (ver Apêndice A). Outro resultado importante diz respeito aos elementos de matriz do produto ordenado no tempo de operadores. Seja

$$O(t_1, \ldots, t_n) = T[O_1^H(t_1) O_2^H(t_2) \ldots O_n^H(t_n)] \tag{5.49}$$

tal que

$$t' \geq (t_1, t_2, \ldots, t_n) \geq t \tag{5.50}$$

Então

$$\langle q'; t' | \, O(t_1, \ldots, t_n) \, | q; t \rangle = N \int \mathcal{D}(q) O_1(q(t_1)) \cdots O_n(q(t_n)) e^{iS} \tag{5.51}$$

onde se admitiu que os operadores O_i são diagonais no espaço das coordenadas. Para a generalização à Teoria do Campo os objectos importantes não são as amplitudes de transição mas as funções de Green e os seus funcionais geradores. Consideremos por exemplo a função de Green

$$G(t_1, t_2) \equiv \langle 0 | \, T(Q^H(t_1) Q^H(t_2)) \, | 0 \rangle \tag{5.52}$$

onde $|0\rangle$ é o estado de base e $Q^H(t)$ é o operador coordenada na representação de Heisenberg. Para escrevermos a Eq. (5.52) em termos dum integral de caminho introduzimos conjuntos completos de estados e escrevemos

$$\begin{aligned} G(t_1, t_2) &= \int dq \; dq' \, \langle 0 | q'; t' \rangle \, \langle q'; t' | \, T(Q^H(t_1) Q^H(t_2)) \, | q; t \rangle \, \langle q; t | 0 \rangle \\ &= \int dq dq' \phi_0(q', t') \phi_0^*(q, t) \int \mathcal{D}(q) q(t_1) q(t_2) e^{i \int_t^{t'} L d\tau} \end{aligned} \tag{5.53}$$

onde

$$\phi_0(q, t) = \langle 0 | q; t \rangle = \phi_0(q) e^{-iE_0 t} \tag{5.54}$$

A presença na expressão 5.53 das funções onda do estado base torna a expressão pouco prática. Podemos removê-los do modo seguinte. Consideremos o elemento de matriz

$$\begin{aligned} &\langle q'; t' | \, O(t_1, t_2) \, | q; t \rangle \\ &= \int dQ \; dQ' \, \langle q'; t' | Q'; T' \rangle \, \langle Q'; T' | \, O(t_1, t_2) \, | Q; T \rangle \, \langle Q; T | q; t \rangle \end{aligned} \tag{5.55}$$

onde

$$\begin{aligned} O(t_1, t_2) &= T(Q^H(t_1) Q^H(t_2)) \\ t' \geq T' \geq (t_1, t_2) &\geq T \geq t \end{aligned} \tag{5.56}$$

Sejam $|n\rangle$ os estados próprios com energia E_n e função de onda $\phi_n(q)$ isto é

$$\begin{aligned} H \, |n\rangle &= E_n \, |n\rangle \\ \langle q | n \rangle &= \phi_n^*(q) \end{aligned} \tag{5.57}$$

Então

$$
\begin{aligned}
\langle q';t'|Q';T'\rangle &= \langle q'|\, e^{-iH(t'-T')}\, |Q'\rangle \\
&= \sum_n \langle q'|n\rangle \langle n|\, e^{-iH(t'-T')}\, |Q'\rangle \\
&= \sum_n \phi_n^*(q')\phi_n(Q')e^{-iE_n(t'-T')}
\end{aligned}
\tag{5.58}
$$

Consideremos agora o limite $t' \to -i\infty$. Então

$$
\lim_{t'\to -i\infty} \langle q';t'|Q';T'\rangle = \phi_0^*(q')\phi_0(Q')e^{-E_0|t'|}e^{iE_0T'}
\tag{5.59}
$$

De modo semelhante

$$
\lim_{t\to i\infty} \langle Q;T|q;t\rangle = \phi_0(q)\phi_0^*(Q)e^{-E_0|t|}e^{-iE_0T}
\tag{5.60}
$$

Aplicando estes limites à Eq. (5.55) obtemos

$$
\begin{aligned}
&\lim_{t'\to -i\infty}\ \lim_{t\to i\infty} \langle q';t'|\, O(t_1t_2)\, |q;t\rangle \\
&\quad = \int dQdQ'\phi_0^*(q')\phi_0(Q')e^{-E_0|t'|}e^{iE_0T'} \\
&\qquad\quad \langle Q';T'|\, O(t_1,t_2)\, |Q;T\rangle\, \phi_0(q)\phi_0^*(Q)e^{-E_0|t|}e^{-iE_0T} \\
&\quad = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|} \\
&\qquad \int dQdQ'\phi_0(Q',T')\phi_0^*(Q,T)\, \langle Q';T'|\, O(t_1,t_2)\, |Q;T\rangle
\end{aligned}
\tag{5.61}
$$

Usando 5.53 obtemos o resultado importante

$$
\lim_{t'\to -i\infty}\ \lim_{t\to i\infty} \langle q';t'|\, O(t_1t_2)\, |q;t\rangle = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}G(t_1,t_2)
\tag{5.62}
$$

Por outro lado

$$
\lim_{t'\to -i\infty}\ \lim_{t\to i\infty} \langle q';t'|q;t\rangle = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{E_0|t|}
\tag{5.63}
$$

pelo que finalmente podemos escrever

$$
G(t_1,t_2) = \lim_{t'\to -i\infty}\ \lim_{t\to i\infty} \left[\frac{\langle q';t'|\, T(Q^H(t_1)Q^H(t_2))\, |q;t\rangle}{\langle q';t'|q;t\rangle} \right]
\tag{5.64}
$$

Usando agora a expressão 5.51 podemos finalmente escrever $G(t_1,t_2)$ em termos dum integral de caminho

$$
G(t_1,t_2) = \lim_{t'\to -i\infty}\ \lim_{t\to i\infty} \frac{1}{\langle q';t'|q;t\rangle} \int \mathcal{D}(q)q(t_1)q(t_2)e^{i\int_t^{t'} Ld\tau}
\tag{5.65}
$$

Este resultado é facilmente generalizado para funções de Green com *n-pontos*,

$$\begin{aligned} G(t_1,\ldots,t_n) &= \langle 0|\,T(q(t_1)\cdots q(t_n))\,|0\rangle \\ &= \lim_{t'\to -i\infty}\ \lim_{t\to i\infty} \frac{1}{\langle q';t'|q;t\rangle}\int \mathcal{D}(q)q(t_1)\cdots q(t_n)e^{i\int_t^{t'} Ld\tau} \end{aligned} \tag{5.66}$$

É agora fácil de ver que todas as funções de Green podem ser obtidos a partir do funcional gerador

$$Z[J] = \lim_{t'\to -i\infty}\ \lim_{t\to i\infty} \frac{1}{\langle q';t'|q;t\rangle}\int \mathcal{D}(q)e^{i\int_t^{t'}[L(q,\dot{q})+Jq]d\tau} \tag{5.67}$$

por derivação funcional

$$G(t_1,\ldots,t_n) = \left.\frac{\delta^n Z[J]}{i\delta J(t_1)\cdots i\delta J(t_n)}\right|_{J=0} \tag{5.68}$$

A expressão 5.67 para o funcional gerador mostra que ele é a amplitude de transição entre o estado base no instante t e o estado base no instante t' na presença duma fonte exterior

$$Z[J] = \langle 0|0\rangle_J \tag{5.69}$$

com a normalização $Z[J=0]=1$.

Para um sistema com N graus de liberdade temos a generalização de 5.67

$$Z[J_1,\ldots,J_n] = \lim_{t'\to -i\infty}\ \lim_{t\to i\infty} N\int \mathcal{D}(q_i)e^{i\int_t^{t'} d\tau[L(q_i,\dot{q}_i)+\sum_{i=1}^N J_iq_i]} \tag{5.70}$$

Notas

- Na equação anterior os limites nos tempos t e t' são imaginários. Isto quer dizer que as funções de Green bem definidas são as funções de Green Euclidianas. Para a teoria de campos isto corresponde à prescrição $m^2 - i\epsilon$.

- Na equação 5.70 não escrevemos explicitamente a normalização. Ela é obviamente escolhida para que $Z[0,\ldots,0]=1$ mas como veremos, para as funções de Green conexas em teoria dos campos a normalização não é relevante pelo que não nos vamos preocupar mais com ela.

5.5.2 Field theory

Para obter o funcional gerador das funções de Green em Teoria dos Campos procedemos da forma heurística usual

$$\begin{aligned} t &\to x^\mu \\ q(t) &\to \phi(x) \\ \mathcal{D}(q) &\to \mathcal{D}(\phi) \end{aligned}$$

$$L(q_i, \dot{q}_i) \quad \rightarrow \quad \int d^3x \mathcal{L}(\phi, \partial_\mu \phi) \tag{5.71}$$

Então obtemos

$$Z[J] = N \int \mathcal{D}(\phi) e^{i \int d^4x [\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x)]} \tag{5.72}$$

O limite em 5.70 recorda-nos que os integrais têm que se continuar analiticamente para o espaço euclidiano ou equivalentemente que se tem que fazer a prescrição $m^2 - i\epsilon$.

A expressão 5.66 é a expressão fundamental que procurávamos para o funcional gerador das funções de Green completas. É fácil de ver que para o funcional gerador das funções de Green conexas

$$W(J) = -i \ln Z(J) \tag{5.73}$$

a normalização é irrelevante (pois não depende de J). Uma demonstração mais rigorosa da expressão 5.72 pode ser encontrada no Apêndice B.

5.5.3 Applications

Uma vez conhecido o funcional gerador $Z(J)$ são conhecidas todas as funções de Green e portanto qualquer problema em Teoria dos Campos. Pode-se então perguntar em que condições é possível calcular 5.72. Como só se sabem fazer exactamente integrais gaussianos a resposta é que só se pode calcular 5.66 em situações triviais, sem interacções no Lagrangeano. Contudo as vantagens de 5.72 resultam de dois aspectos:

- *Manipulações formais*
  Relações entre funções de Green que tenham a ver com propriedades de simetria (identidades de Ward) são muito facilmente deduzidas por manipulações dos funcionais geradores. Aqui a forma 5.66 é particularmente útil como veremos nas secções 4 e 5.

- *Teoria de perturbações*
  A expressão 5.72 permite imediatamente desenvolver a teoria de perturbações.

Como exemplo do ponto *ii)* consideremos o Lagrangeano para um campo escalar que por simplicidade tomaremos real,

$$\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_I(\phi) \tag{5.74}$$

onde $\mathcal{L}_0(\phi)$ é quadrático, isto é

$$\mathcal{L}_0(\phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{5.75}$$

Então podemos escrever

$$Z[J] \quad = \quad N \int \mathcal{D}(\phi) e^{i \int d^4x [\mathcal{L}_0(\phi) + \mathcal{L}_I(\phi) + J\phi]}$$

$$= \quad N \int \mathcal{D}(\phi) e^{i\int d^4x[\mathcal{L}_I(\phi)]} \ e^{i\int d^4x[\mathcal{L}_0(\phi)+J\phi]} \tag{5.76}$$

ou seja

$$Z[J] = \exp\left[i \int d^4x \mathcal{L}_I \left(\frac{\delta}{i\delta J}\right)\right] \ Z_0[J] \tag{5.77}$$

onde

$$Z_0[J] = N \int \mathcal{D}(\phi) e^{i\int d^4x[\mathcal{L}_0 + J\phi]} \ . \tag{5.78}$$

A utilidade desta expressão resulta do facto de que por um lado $Z_0[J]$ pode ser calculado exactamente, porque é quadrático nos campos, e por outro se $\mathcal{L}_I(\phi)$ tiver um parâmetro pequeno a exponencial pode ser desenvolvida em série nesse parâmetro e o funcional gerador $Z[J]$ obtido até à ordem que se pretender em teoria das perturbações.

5.5.4 Example: perturbation theory for $\lambda\phi^4$

Para se ver a ligação com os resultados usuais vamos considerar como exemplo a teoria dum campo escalar em que a interacção é

$$\mathcal{L}_I = -\frac{\lambda}{4!} \ \phi^4 \ . \tag{5.79}$$

O funcional gerador $Z[J]$ é dado por

$$Z[J] = \mathcal{N} \exp\left\{(-i\lambda)\frac{1}{4!} \int d^4x \left(\frac{\delta}{i\delta J}\right)^4\right\} Z_0[J] \tag{5.80}$$

onde (ver problema 1.2)

$$Z_0[J] = \exp\left\{-\frac{1}{2} \int d^4x d^4y J(x) \Delta(x,y) J(y)\right\} \tag{5.81}$$

A normalização $\mathcal{N}$ é escolhida para que $Z[0] = 1$, como veremos adiante. Desenvolvemos a exponencial em série na constante de acoplamento:

$$Z[J] = \mathcal{N} Z_0[J] \left\{1 + (-i\lambda) Z_1'[J] + (-i\lambda)^2 Z_2'[J] + \cdots\right\} \tag{5.82}$$

onde

$$Z_1'[J] \equiv Z_0^{-1}[J] \left\{\frac{1}{4!} \int d^4x \left(\frac{\delta}{\delta J}\right)^4\right\} Z^0[J] \tag{5.83}$$

e

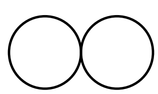
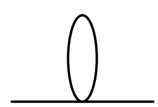
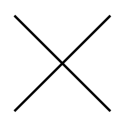
$$Z_2'[J] \quad \equiv \quad \frac{1}{2} Z_0^{-1}[J] \left\{\frac{1}{4!} \int d^4x \left(\frac{\delta}{\delta J}\right)^4\right\}^2 Z^0[J]$$

$$
\begin{aligned}
&= \frac{1}{2} Z_0^{-1}[J] \left\{ \frac{1}{4!} \int d^4x \left(\frac{\delta}{\delta J} \right)^4 \right\} (Z_0 Z_1') \\
&= \frac{1}{2} \left(Z_1'[J] \right)^2 + \frac{1}{2} Z_0^{-1}[J] \ \frac{1}{4!} \int d^4x \left\{ 4 \frac{\delta^3 Z_0}{\delta J^3(x)} \frac{\delta Z_1'}{\delta J(x)} \right. \\
&\qquad \left. +6 \frac{\delta^2 Z_0}{\delta J^2(x)} \frac{\delta^2 Z_1'}{\delta J^2(x)} + 4 \frac{\delta Z_0}{\delta J(x)} \frac{\delta^3 Z_1'}{\delta J^3(x)} + Z_0 \frac{\delta^4 Z_1'}{\delta J^4(x)} \right\}
\end{aligned}
\tag{5.84}
$$

Obtemos

$$
\begin{aligned}
& Z_1'[J] = \\
&= \frac{1}{4!} \int d^4x \left[3\Delta(x,x)\Delta(x,x) - 3!\Delta(x,x) \int d^4y_1 d^4y_2 \Delta(x,y_1)\Delta(x,y_2) J(y_1) J(y_2) \right. \\
&\quad \left. + \int d^4y_1 \cdots d^4y_4 \Delta(x,y_1)\Delta(x,y_2)\Delta(x,y_3)\Delta(x,y_4) J(y_1)J(y_2)J(y_3)J(y_4) \right]
\end{aligned}
\tag{5.85}
$$

Este resultado pode ser representado diagramaticamente na forma seguinte

$Z_1' = \frac{1}{8}$  $- \frac{1}{4}$  $+ \frac{1}{4!}$  (5.86)

Para Z_2' virá

$$
\begin{aligned}
& Z_2'[J] \\
=\ & \frac{1}{2} \left(Z_1'[J] \right)^2 + \frac{1}{2} \left(\frac{1}{4!} \right)^2 4! \int d^4x_1 d^4x_2 \Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,x_2) \\
& + \left(\frac{1}{4!} \right)^2 \left[-72 \int d^4x_2 \int d^4x_1 \Delta(x_1,x_2) \int d^4y_1 \Delta(x_1,y_1) J(y_1) \right. \\
& \qquad \Delta(x_1,x_2)\Delta(x_2,x_2) \int d^4y_2 \Delta(x_2,y_2) J(y_2) \\
& \qquad +24 \int d^4x_2 d^4x_1 \Delta(x_1,x_1) \int d^4y_1 \Delta(x_1,y_1) J(y_1) \Delta(x_1,y_2) \\
& \qquad \int d^4y_2 \Delta(x_2,y_1) J(y_2) \int d^4y_3 \Delta(x_2,y_3) J(y_2) \int d^4y_4 \Delta(x_1,y_4) J(y_4) \\
& \qquad +24 \int d^4x_2 \int d^4x_1 \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 \Delta(x_1,x_2)\Delta(x_1,y_1)
\end{aligned}
$$

$$
\begin{aligned}
&\Delta(x_1,y_2)\Delta(x_1,y_2)\Delta(x_2,x_2)\Delta(x_2,y_4)J(y_1)\cdots J(y_4)\\
-8&\int d^4x^2\int d^4x_1\int d^4y_1\cdots d^4y_6\Delta(x_1,x_2)\Delta(x_1,y_1)\Delta(x_1y_2)\\
&\Delta(x_1,y_3)\Delta(x_2,y_4)\Delta(x_2,y_5)\Delta(x_2,y_6)J(y_1)\cdots J(y_6)\\
+36&\int d^4x_2d^4x_1\Delta(x_1,x_1)\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_2,x_2)\\
-36&\int d^4x_2d^4x_1\Delta(x_1,x_1)\Delta(x_1,x_2)\Delta(x_1,x_2)\int d^4y_1d^4y_2\Delta(x_2,y_1)\\
&\Delta(x_2,y_2)J(y_1)J(y_2)\\
-36&\int d^4x_2d^4x_1d^4y_1d^4y_2\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_2,x_1)\\
&\Delta(x_1,y_1)\Delta(x_1,y_2)J(y_1)J(y_2)\\
+36&\int d^4x_2d^4x_1d^4y_1\cdots d^4y_4\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,y_1)\\
&\Delta(x_1,y_2)\Delta(x_2,y_3)\Delta(x_2,y_4)J(y_1)\cdots J(y_4)\\
-48&\int d^4x_2dy_1d^4y_1d^4y_2\Delta(x_1,x_2)\Delta(x_1,x_2)\Delta(x_1,x_2)\\
&\Delta(x_1,y_1)\Delta(x_2,y_2)J(y_1)J(y_2)\Big] \qquad (5.87)
\end{aligned}
$$

ou seja:

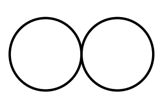
$$
\begin{aligned}
&Z_2'[J]\\
&=\frac{1}{2}\left(Z_1'[J]\right)^2+\frac{1}{2\cdot 4!}\int d^4x_1d^4x_2\Delta^4(x_1,x_2)\\
&+\frac{3}{2\cdot 4!}\int d^4x_1d^4x_2\Delta(x_1,x_1)\Delta^2(x_1,x_2)\Delta(x_2,x_2)\\
&-\frac{1}{2\cdot 3!\cdot 3!}\int d^4x_1d^4x_2d^4y_1\cdots d^4y_6\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta(y_3,x_1)\\
&\qquad\Delta(x_1,x_2)\Delta(x_2,y_4)\Delta(x_2,y_5)\Delta(x_2,y_6)J(y_1)\cdots J(y_6)
\end{aligned}
$$

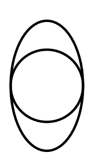
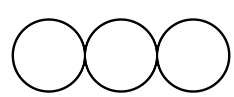
$$
\begin{aligned}
&+\frac{2}{4!}\int d^4x_1 d^4x_2 d^4y_1 \cdots d^4y_4 \Delta(y_1,x_1)\Delta(x_1,x_1)\Delta(x_1,x_2)\\
&\qquad\qquad \Delta(x_2,y_2)\Delta(x_2,y_3)\Delta(x_2,y_4)J(y_1)\cdots J(y_4)\\
&+\frac{3}{2\cdot 4!}\int d^4x_1 d^4x_2 d^4y_1 \cdots d^4y_4 \Delta(y_1,x_1)\Delta(y_2,x_1)\Delta^2(x_1,x_2)\\
&\qquad\qquad \Delta(x_2,y_3)\Delta(x_2,y_4)J(y_1)\cdots J(y_4)\\
&-\frac{1}{8}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \Delta(y_1,x_1)\Delta(x_1,x_1)\Delta(x_1,x_2)\Delta(x_2,x_2)\\
&\qquad\quad \Delta(x_2,y_2)J(y_1)J(y_2)\\
&-\frac{1}{8}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \Delta(y_1,x_1)\Delta^2(x_1,x_2)\Delta(x_2,x_2)\Delta(x_1,y_2)J(y_1)J(y_2)\\
&-\frac{1}{12}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \Delta(y_1,x_1)\Delta^3(x_1,x_2)\Delta(x_2,y_2)J(y_1)J(y_2) \qquad (5.88)
\end{aligned}
$$

Vamos agora calcular a normalização até à segunda ordem em teoria de perturbações. Para isso a condição $Z[0]=1$ dá:

$$1=\mathcal{N}\left[1+(-i\lambda)n_1+(-i\lambda)^2 n_2+\cdots\right] \qquad (5.89)$$

onde

$$n_1 \;=\; \frac{1}{8}$$  (5.90)

$$n_2 \;=\; \frac{1}{2}n_1^2 + \frac{1}{2\cdot 4!}$$  $$+\ \frac{3}{2\cdot 4!}$$  (5.91)

Obtemos portanto

$$
\begin{aligned}
\mathcal{N} &= \frac{1}{1+(-i\lambda)n_1+(-i\lambda)^2 n_2+\cdots}\\
&= 1-(-i\lambda)n_1-(-i\lambda)^2(n_2-n_1^2)+\cdots \qquad (5.92)
\end{aligned}
$$

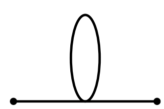
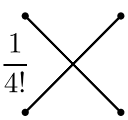
Então

$$
\begin{aligned}
Z[J] &= Z_0[J]\left\{1-(-i\lambda)n_1-(-i\lambda)^2(n_2-n_1^2)+\cdots\right\} \\
&\qquad\left\{1+(-i\lambda)Z_1'+(-i\lambda)^2Z_2'+\cdots\right\} \\
&= Z_0[J]\left\{1+(-i\lambda)(Z_1'-n_1)+(-i\lambda)^2(Z_2'-n_2+n_1^2-n_1Z_1')+\cdots\right\}
\end{aligned} \tag{5.93}
$$

Definindo agora

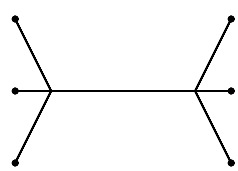
$$
\begin{aligned}
Z_1 &\equiv Z_1'-n_1 \\
Z_2 &\equiv Z_2'-n_2+n_1^2-n_1Z_1' = Z_2'-n_2-n_1Z_1
\end{aligned} \tag{5.94}
$$

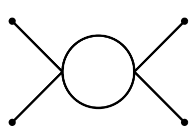
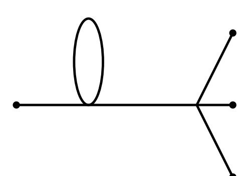
obtemos

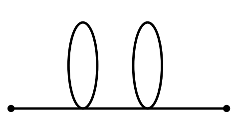
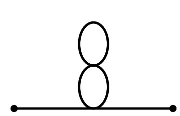
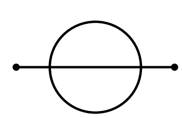
$Z_1[J] = -\frac{1}{4}$  $+\frac{1}{4!}$  (5.95)

e

$Z_2[J]$

$= \frac{1}{2}\left(Z_1[J]\right)^2 + \frac{-1}{2!\,3!\,3!}$ 

$+\frac{3}{2!\,4!}$  $+\ \frac{2}{4!}$ 

$-\frac{1}{8}$  $-\ \frac{1}{8}$  $-\ \frac{1}{12}$  (5.96)

com $Z_1[0]=Z_2[0]=0$. Logo o funcional gerador

$$Z[J] = Z_0[J]\left\{1 + (-i\lambda)Z_1[J] + (-i\lambda)^2 Z_2[J] + \cdots\right\} \tag{5.97}$$

é automaticamente normalizado se desprezarmos todas as amplitudes vácuo-vácuo, designadas por *bubles*. Para verificarmos que a expressão anterior reproduz os resultados da teoria de perturbações usual calculemos como exemplo o propagador até à ordem λ^2. Obtemos

$$\begin{aligned}
\Delta'(x_1,x_2) =& \left.\frac{\delta^2 Z[J]}{i\delta J(x_1) i\delta J(x_2)}\right|_{J=0} \\
=& -\left.\frac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} - (-i\lambda)\left.\frac{\delta^2 Z_1[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} - (-i\lambda)^2\left.\frac{\delta^2 Z_2[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} \\
=& \Delta(x_1,x_2) \ + \ (-i\lambda)\frac{1}{2}\int d^4y \Delta(x_1,y)\Delta(x_2,y)\Delta(y,y) \\
&+(-i\lambda)^2 \int d^4y_1 d^4y_2 \left[\frac{1}{4}\Delta(x_1,y_1)\Delta(y_1,y_1)\Delta(y_1,y_2)\Delta(y_2,y_2)\Delta(y_2,x_2)\right. \\
&\left. + \frac{1}{4}\Delta(x_1,y_1)\Delta^2(y_1,y_2)\Delta(y_2,y_2)\Delta(y_1,x_2) + \frac{1}{6}\Delta(x_1,y_1)\Delta^3(y_1,y_2)\Delta(y_2,x_2)\right]
\end{aligned} \tag{5.98}$$

Em termos diagramáticos temos a situação da Figura 5.15

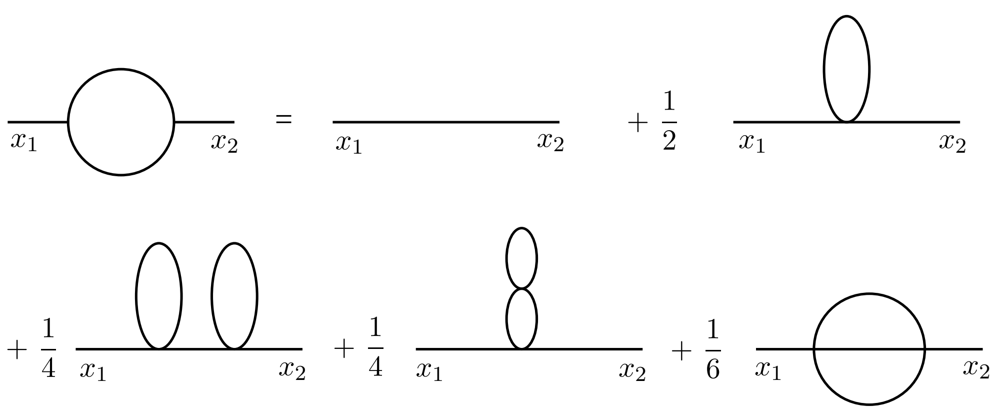


Figure 5.15:

Continuando a estudar o exemplo da teoria $\lambda\phi^4$ passemos a analisar o funcional gerador das funções de Green conexas $W[J]$. É fácil de ver que termos do tipo $Z_1^2[J]$ correspondem a diagramas disconexos contidos em $Z[J]$. Vamos ver que elas desaparecem no funcional $W[J]$. Temos

$$iW[J] \ = \ \ln Z[J] =$$

$$
\begin{aligned}
&= \ln Z_0[J] + \ln\left\{1 + (-i\lambda)Z_1[J] + (-i\lambda)^2 Z_2[J] + \cdots\right\} \\
&= iW_0[J] + (-i\lambda)Z_1[J] - \frac{1}{2}(-i\lambda)^2\left(Z_1[J]\right)^2 + (-i\lambda)^2 Z_2[J] + \cdots \\
&= iW_0[J] + (-i\lambda)Z_1[J] + \left\{-i\lambda)^2(Z_2[J] - \frac{1}{2}\left(Z_1[J]\right)^2\right\} + \cdots \\
&\equiv i\left\{W_0[J] + (-i\lambda)W_1[J] + (-i\lambda)^2 W_2[J] + \cdots\right\} \qquad (5.99)
\end{aligned}
$$

com

$$
\begin{aligned}
iW_1[J] &= Z_1[J] \\
iW_2[J] &= Z_2[J] - \frac{1}{2}\left(Z_1[J]\right)^2 \qquad (5.100)
\end{aligned}
$$

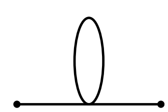
Portanto os diagramas disconexos contidos em $Z_2[J]$ são subtraídos e W_1 e W_2 contém somente diagramas conexos como seria de esperar.

5.5.5 Symmetry factors

Depois de efectuar as derivadas em ordem a J para obter numa dada função de Green os números que resultam são os chamados factores de simetria. Por exemplo para a correcção a *1-loop* ao propagador obtemos

$$
\begin{aligned}
\Delta'(x_1,x_2) &= \left.\frac{\delta^2 Z}{i\delta J(x_1) i\delta J(x_2)}\right|_{J=0} = \\
&= \left.\frac{\delta^2 Z_0}{i\delta J(x_1) i\delta J(x_2)}\right|_{J=0} + (-i\lambda)\left.\frac{\delta^2 Z_1}{i\delta J(x_1) i\delta J(x_2)}\right|_{J=0} + \cdots
\end{aligned}
$$

$$= \Delta(x_1,x_2) + \frac{1}{2}$$



$$+ \cdots \qquad (5.101)$$

O factor $\frac{1}{2}$ é o factor de simetria correspondente àquele diagrama. Como vimos o método de obter as funções de Green a partir do funcional gerador *automaticamente* dá os estes factores correctos. Contudo na maior parte das aplicações é mais fácil aplicar directamente as regras de Feynman e então uma regra para os factores de simetria deve ser fornecida.

Regra para os factores de simetria:

O factor de simetria S dum dado diagrama é dado por

$$S = \frac{N}{D} \qquad (5.102)$$

onde N é o # de maneiras diferentes de formar o diagrama e D é o produto dos factores de simetrias de cada vértice e do número de permutações de vértices iguais.

Como exemplo consideremos o diagrama que contribui para o propagador a *1-loop*, representado na Figura 5.16.

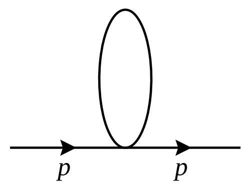


Figure 5.16:

Então de acordo com a regra obtemos

$$S = \frac{4 \times 3}{4!} = \frac{1}{2} \tag{5.103}$$

5.5.6 A comment on the normal ordering

No exemplo anterior diagramas como o da Figura 5.16, designados por *tadpoles*, aparecem enquanto que no formalismo canónico usual estão excluídos devido ao ordenamento normal. Esta diferença deve-se ao facto de não termos sido muito rigorosos na definição do integral de caminho. Se o tivéssemos feito chegaríamos à conclusão que para obter a expressão do Lagrangeano a incluir em $e^{i\int d^4x \mathcal{L}(\phi)}$ teríamos primeiro que ordenar normalmente o Lagrangeano Quântico e só então efectuar a transcrição para os campos clássicos do integral de caminho. Isto faz com que o Lagrangeano $\mathcal{L}(\phi)$ usado no integral de caminho seja diferente do Lagrangeano para a teoria clássica. Vejamos o exemplo de ϕ^4. Vamos usar as relações[2]

$$\hat{\phi}(x)\hat{\phi}(x) =: \hat{\phi}(x)\hat{\phi}(x) : + \langle 0|\, \hat{\phi}(x)\hat{\phi}(x)\, |0\rangle \tag{5.104}$$

ou, mais simbolicamente,

$$: \hat{\phi}^2 := \hat{\phi}^2 - \langle 0|\, \hat{\phi}^2\, |0\rangle \ . \tag{5.105}$$

De modo semelhante

$$\hat{\phi}^4 =: \hat{\phi}^4 : +6 : \hat{\phi}^2 : \ \langle 0|\, \hat{\phi}^2\, |0\rangle + 6\, \langle 0|\, \hat{\phi}^2\, |0\rangle \ \langle 0|\, \hat{\phi}^2\, |0\rangle \tag{5.106}$$

e portanto obtemos

$$: \hat{\phi}^4 := \hat{\phi}^4 - 6 : \hat{\phi}^2(x) : \ \langle 0|\, \hat{\phi}^2(x)\, |0\rangle - 6 \left(\langle 0|\, \hat{\phi}^2\, |0\rangle \right)^2 \tag{5.107}$$

ou ainda

[2]Usaremos nesta secção a notação $\hat{\phi}$ para o campo quântico (operador) para o distinguir do campo clássico ϕ.

$$: \hat{\phi}^4 := \hat{\phi}^4 - 6\ \hat{\phi}^2 \langle 0| \hat{\phi}^2 |0\rangle \tag{5.108}$$

Isto quer dizer que o Lagrangeano quântico se escreve

$$\begin{aligned}\mathcal{L}^Q_{Int} &= -\frac{\lambda}{4!} : \hat{\phi}^4 : \\ &= -\frac{\lambda}{4!}\hat{\phi}^4 + \frac{\lambda}{4}\hat{\phi}^2 I\end{aligned} \tag{5.109}$$

onde

$$\begin{aligned}I &\equiv \langle 0| \hat{\phi}^2(x) |0\rangle \\ &= \int d\tilde{k}_1 d\tilde{k}_2 \langle 0| \left(a^\dagger(k_1)e^{ik_1\cdot x} + a(k_1)e^{-ik_1\cdot x}\right)\left(a^\dagger(k_2)e^{ik_2\cdot x} + a(k_2)e^{-ik_2\cdot x}\right)|0\rangle \\ &= \int d\tilde{k}_1 d\tilde{k}_2 \langle 0| a(k_1)a^\dagger(k_2)|0\rangle\, e^{i(k_2-k_1)\cdot x} \\ &= \int d\tilde{k}_1 = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\end{aligned} \tag{5.110}$$

Na expressão anterior usámos as relações

$$\left[a(k), a^+(k')\right] = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}') \tag{5.111}$$

$$\omega_k = \sqrt{k_0^2 + |\vec{k}|^2} \tag{5.112}$$

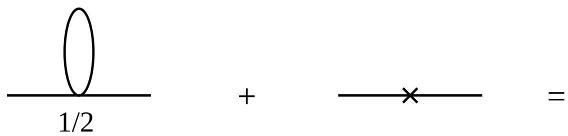
O integral I é divergente e é igual ao integral do *loop* da Figura 5.16. De facto

$$\begin{aligned}&\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon} \\ &= \int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{+\infty} dk_0\ \frac{i}{(k_0-\omega_k)(k_0+\omega_k)} \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} = I\end{aligned} \tag{5.113}$$

Portanto se tivéssemos sido cuidadosos teríamos que incluir o termo $\frac{\lambda}{4}\phi^2 I$ na interacção. O Lagrangeano a introduzir na exponencial do integral de caminho seria então

$$\mathcal{L}^{IC}_{Int} = -\frac{\lambda}{4!}\phi^4 + \frac{\lambda}{4}\phi^2 I \tag{5.114}$$

É fácil de ver que o termo adicional cancela o *tadpole*. De facto temos



$$
\begin{aligned}
&= \frac{i}{p^2 - m^2} \left[-i\frac{\lambda}{2}\, I + i\frac{\lambda}{2}\, I \right] \frac{i}{p^2 - m^2} \\
&= 0
\end{aligned}
\tag{5.115}
$$

e portanto os *tadpoles* não apareceriam. Contudo muitas vezes não nos preocupamos em usar o Lagrangeano correcto e usamos simplesmente o Lagrangeano clássico pois a contribuição do *tadpole* é uma renormalização da massa (infinita) e pode assim ser sempre reabsorvida no processo de renormalização.

Para QED o mesmo se passa, isto é, devíamos usar como Lagrangeano de interacção

$$
\mathcal{L}_{Int}^{IC} = -e\overline{\psi}\gamma^\mu\psi A_\mu + eA_\mu \langle 0| \overline{\psi}\gamma^\mu\psi |0\rangle \tag{5.116}
$$

e o segundo termo removeria o *tadpole* representado na Figura 5.17,

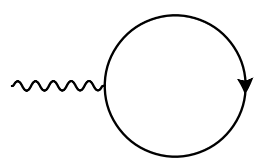


Figure 5.17: Tadpole for `QED`.

No entanto, devido à invariância de Lorentz da teoria, pode-se mostrar que este *tadpole* é zero em todas as ordens e portanto não nos temos que preocupar.

5.5.7 Generating functionals for fermions

Para sistemas de fermiões introduzimos variáveis de Grassman. Estas variáveis anticomutativas são de alguma forma o limite clássico dos campos quânticos fermiónicos. Os detalhes desta construção estão explicados nos Apêndices A e B. Aqui apenas recordamos as nossas convenções. Devido ao carácter anticomutativo é necessário explicitar a ordem da derivação. Assim

- As derivadas são esquerdas

$$
\frac{\delta}{\delta\overline{\eta}(x)} \int d^4y\overline{\eta}(y)\psi(y) = \psi(x)
$$

$$\frac{\delta}{\delta\eta(x)}\int d^4y\overline{\psi}(y)\eta(y) = -\psi(x) \tag{5.117}$$

- Nas funções de Green a ordem de derivação é

$$\begin{aligned} G^{2n}(x_1,\ldots,y_n) &= \langle 0|\, T\psi(x_1)\cdots\psi(x_n)\overline{\psi}(y_1)\cdots\overline{\psi}(y_n)\,|0\rangle \\ &\equiv \frac{\delta^{2n}Z[\eta,\overline{\eta}]}{i\delta\eta(y_n)\cdots i\delta\eta(y_1)i\delta\overline{\eta}(x_n)\cdots i\delta\overline{\eta}(x_1)} \\ &\equiv \frac{\delta}{i\delta\eta(y_n)}\cdots\frac{\delta}{i\delta\overline{\eta}(x_1)}Z[\eta,\overline{\eta}] \end{aligned} \tag{5.118}$$

  onde

$$\begin{aligned} Z[\eta,\overline{\eta}] &= \langle 0|\, Te^{i\int d^4x\left[\overline{\eta}(x)\psi(x)+\overline{\psi}(x)\eta(x)\right]}\,|0\rangle \\ &= \int\mathcal{D}(\psi,\overline{\psi})e^{i\int d^4x\left[\mathcal{L}+\overline{\eta}(x)\psi(x)+\overline{\psi}(x)\eta(x)\right]} \end{aligned} \tag{5.119}$$

Exemplos de aplicação destes resultados serão dados nos problemas.

5.6 Change of variables in path integrals. Applications

5.6.1 Introduction

Uma das grandes vantagens de ter uma expressão para o funcional gerador $Z(J)$ em termos dum integral de caminho é que um grande número de manipulações familiares para integrais usuais (mudança de variáveis de integração, integração por partes ...) podem agora ser aqui aplicadas. Vamos ver as consequências da mudança de variáveis de integração.

Consideremos uma transformação infinitesimal da forma

$$\phi_i \to \phi_i + \varepsilon F_i(\phi) \tag{5.120}$$

onde

$$F_i(\phi) = f_i + f_{ij}\phi_j + \cdots \tag{5.121}$$

Então devemos ter

$$\begin{aligned} \mathcal{D}(\phi) \to&\quad \mathcal{D}(\phi)\det\left|\delta_{ij} + \varepsilon\tfrac{\delta F_i}{\delta\phi_j}\right| \\ =&\quad \mathcal{D}(\phi)\left(1+\varepsilon\tfrac{\delta F_i}{\delta\phi_i}\right) \end{aligned} \tag{5.122}$$

e

$$e^{i(S(\phi)+J_i\phi_i)} \to e^{i(S(\phi)+J_i\phi_i)}\left[1+i\varepsilon\left(\frac{\delta S}{\delta\phi_i}+J_i\right)F_i(\phi)\right] \tag{5.123}$$

Como $Z(J)$ deverá ser independente de transformação de variáveis obtemos

$$0=\int\mathcal{D}(\phi)\left[i\left(\frac{\delta S}{\delta\phi_i}+J_i\right)F_i+\frac{\delta F_i}{\delta\phi_i}\right]e^{i(S[\phi]+J_i\phi_i)} \tag{5.124}$$

Usando $\phi_i \to \frac{\delta}{i\delta J_i}$ obtemos a expressão mais compacta

$$\left\{\left[\frac{\delta S}{\delta\phi_i}\left(\frac{\delta}{i\delta J_i}\right)+J_i\right]F_i\left(\frac{\delta}{i\delta J_i}\right)+\frac{\delta F_i}{\delta\phi_i}\left(\frac{\delta}{i\delta J}\right)\right\}Z(J)=0 \tag{5.125}$$

Esta é a expressão geral que vamos aplicar a dois casos particulares importantes, as equações de Dyson-Schwinger e as identidades de Ward.

5.6.2 Dyson-Schwinger equations

Seja $F_i = f_i$ independente de ϕ_i, isto é uma simples translação dos campos. Então a equação anterior escreve-se

$$\left(\frac{\delta S}{\delta\phi_i}\left[\frac{\delta}{i\delta J}\right]+J_i\right)Z(J)=0 \tag{5.126}$$

que, como veremos, é a expressão da equação de Dyson-Schwinger (DS) para o funcional gerador das funções de Green completas. Assim as equações de DS não são mais do que uma consequência da invariância dos integrais de caminho para translações. A equação 5.126 pode-se ainda escrever

$$J_k=-\frac{1}{Z}E\left[\frac{\delta}{i\delta J_k}\right]Z[J] \tag{5.127}$$

onde o funcional $E[\phi]$ é a equação de movimento,

$$E\left[\phi_k\right]\equiv\frac{\delta S}{\delta\phi_k}\ . \tag{5.128}$$

Para muitas aplicações é mais conveniente escrever as equações de Dyson-Schwinger para as funções de Green conexas e próprias. Para isso temos de escrever a equação equivalente a 5.126 para os funcionais W e Γ

i) Funções de Green conexas

Usando a identidade

$$\frac{1}{Z}\frac{\delta}{i\delta J_k}\left(Z[J]f[J]\right)=\left(\frac{\delta iW}{i\delta J_k}+\frac{\delta}{i\delta J_k}\right)f[J] \tag{5.129}$$

Podemos escrever

$$\frac{1}{Z}E\left[\frac{\delta}{i\delta J_k}\right]Z[J]=E\left[i\frac{\delta W}{i\delta J_k}+\frac{\delta}{i\delta J_k}\right]1 \tag{5.130}$$

Portanto a equação de DS para o funcional gerador das funções de Green conexas escreve-se simplesmente

$$J_k = -E\left[i\frac{\delta W}{i\delta J_k} + \frac{\delta}{i\delta J_k}\right] 1 \tag{5.131}$$

ii) Funções de Green próprias

Mais útil é a equação de DS para as funções de Green próprias ou irredutíveis. Para isso utilizamos as relações

$$\begin{cases} \phi_k = \dfrac{\delta W}{i\delta J_k} & J_k = -\dfrac{\delta \Gamma}{\delta \phi_k} \\ \dfrac{\delta}{i\delta J_k} = G_{km}\dfrac{\delta}{\delta \phi_m} & \dfrac{\delta}{\delta \phi_k} = -i\Gamma_{kr}\dfrac{\delta}{i\delta J_r} \end{cases}$$

e obtemos

$$\frac{\delta \Gamma}{\delta \phi_k} = E\left[\phi_k + G_{km}\frac{\delta}{\delta \phi_m}\right] 1 \tag{5.132}$$

É na forma 5.132 que as equações de DS são mais úteis.

Example : Self-energy in ϕ^3

A acção para esta teoria escreve-se, usando a notação compacta

$$S[\phi] = \phi_k(-\Box - m^2)\delta_{km}\phi_m - \frac{\lambda}{3!}(\phi_k)^3 \tag{5.133}$$

logo

$$E[\phi_k] = (-\Box - m^2)\phi_k - \frac{\lambda}{2}(\phi_k)^2 \tag{5.134}$$

e portanto

$$E\left[\phi_k + G_{km}\frac{\delta}{\delta \phi_m}\right] 1 = -(\Box + m^2)\phi_k - \frac{\lambda}{2}\left(\phi_k + G_{kr}\frac{\delta}{\delta \phi_r}\right)\phi_k \tag{5.135}$$

dá

$$\frac{\delta \Gamma}{\delta \phi_k} = -(\Box + m^2)\phi_k - \frac{\lambda}{2}\left(\phi_k^2 + G_{kr}\delta_{rk}\right) \tag{5.136}$$

Derivando funcionalmente em relação a ϕ_m obtemos as equações de DS para diversas funções de Green. Por exemplo para a self-energy obtemos

$$\frac{\delta^2 \Gamma}{\delta \phi_k \delta \phi_m} = -(\Box + m^2)\delta_{km} - \frac{\lambda}{2}\left(2\phi_k\delta_{km} - i\Gamma_{mn}G_{krn}\delta_{rk}\right) \tag{5.137}$$

Pondo $\phi_k = 0$ obtemos

$$\Gamma_{km} - (-\Box - m^2)\delta_{km} \quad = \quad i\frac{\lambda}{2}\,\Gamma_{mn}G_{krn}\delta_{rk}$$

$$\begin{aligned}
&= \frac{\lambda}{2}\ \Gamma_{mn} G_{krn} \Gamma_{rs} G_{sk} \\
&= i\frac{\lambda}{2}\ \Gamma_{mn}\Gamma_{rs} G_{sk} G_{kk'} G_{rr'} G_{nn'} \Gamma_{k'r'n'} \\
&= -i\frac{\lambda}{2}\ G_{kk'} G_{ks} \Gamma_{k'sm}
\end{aligned} \tag{5.138}$$

onde se usou repetidamente a equação 5.32 e a definição da Figura 5.11. Mas por definição de self-energy

$$\Gamma_{km} - (-\Box - m^2)\delta_{km} \equiv -\Sigma_{km} \tag{5.139}$$

e portanto

$$-i\Sigma_{km} = -i\frac{\lambda}{2}\ G_{kk'} G_{ks} i\Gamma_{k'sm} \tag{5.140}$$

ou em termos de diagramas da Figura 5.18

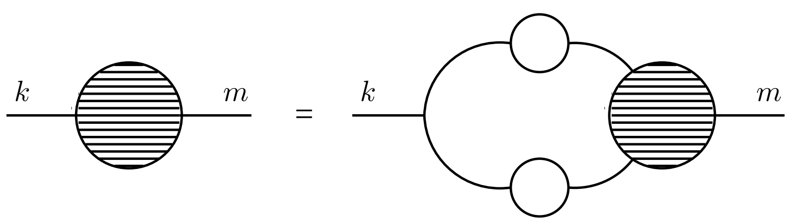


Figure 5.18: Dyson-Schwinger equation for ϕ^3.

Como vemos a equação de DS não é mais que a afirmação que o vértice na teoria é $\frac{\lambda}{3!}\phi^3$.

Example: Self-energy in ϕ^4

Neste caso a acção escreve-se

$$S[\phi] = \phi_k(-\Box - m^2)\delta_{km}\phi_m - \frac{\lambda}{4!}\ (\phi_k)^4\ . \tag{5.141}$$

Logo a equação de movimento é

$$E[\phi] = (-\Box - m^2)\phi_k - \frac{\lambda}{3!}\ (\phi_k)^3\ . \tag{5.142}$$

Portanto

$$\begin{aligned}
E\left[\phi_k + G_{km}\frac{\delta}{\delta\phi_m}\right] 1 &= -(\Box + m^2)\phi_k - \frac{\lambda}{3!}\left(\phi_k + G_{km}\frac{\delta}{\delta\phi_m}\right)\left(\phi_k + G_{kn}\frac{\delta}{\delta\phi_n}\right)\phi_k \\
&= -(\Box + m^2)\phi_k - \frac{\lambda}{3!}\left(\phi_k + G_{km}\frac{\delta}{\delta\phi_m}\right)\left(\phi_k^2 + G_{kn}\delta_{nk}\right) \\
&= -(\Box + m^2)\phi_k - \frac{\lambda}{3!}\left(\phi_k^3 + \phi_k G_{kn}\delta_{nk} + 2G_{km}\phi_k\delta_{km}\right.
\end{aligned}$$

$$- iG_{km}\Gamma_{m\ell}G_{kn\ell}\delta_{nk}\Big) \tag{5.143}$$

e obtemos

$$\frac{\delta\Gamma}{\delta\phi_k} = -(\Box + m^2)\phi_k - \frac{\lambda}{3!}\ \left(\phi_k^3 + \phi_k G_{kn}\delta_{nk} + 2G_{km}\phi_k\delta_{km} - iG_{km}\Gamma_{m\ell}G_{kn\ell}\delta_{nk}\right) \tag{5.144}$$

Para obter a equação de DS para a self-energy derivamos em ordem a ϕ_j e fazemos $\phi_i = 0$ depois de derivar. Obtemos assim

$$\begin{aligned} \Gamma_{kj} \quad - \quad & (-\Box - m^2)\delta_{kj} \\ = \quad & -\frac{\lambda}{3!}\ (G_{kn}\delta_{nk}\delta_{kj} + 2G_{km}\delta_{km}\delta_{kj} - iG_{km}\Gamma_{m\ell j}G_{kn\ell}\delta_{nk} \\ & -G_{kmp}\Gamma_{pj}\Gamma_{m\ell}G_{kn\ell}\delta_{nk} - G_{km}\Gamma_{m\ell}G_{kn\ell p}\Gamma_{pj}\delta_{nk}) \end{aligned} \tag{5.145}$$

logo

$$\begin{aligned} -i\Sigma_{kj} \quad = \quad & -i\frac{\lambda}{2}G_{kk}\delta_{kj} + i\frac{\lambda}{3!}G_{km}i\Gamma_{m\ell j}G_{kk'}G_{nn'}G_{\ell\ell'}i\Gamma_{k'n'\ell'}\delta_{nk} \\ & +i\frac{\lambda}{3!}G_{kk'}G_{mm'}G_{pp'}i\Gamma_{k'm'p'}\Gamma_{pj}\Gamma_{m\ell}G_{kk''}G_{nn'}G_{\ell\ell'}i\Gamma_{k''n'\ell'}\delta_{nk} \\ & +i\frac{\lambda}{3!}G_{km}\Gamma_{m\ell}G_{kn\ell p}\Gamma_{pj}\delta_{nk} \\ = \quad & -i\frac{\lambda}{2}G_{kk}\delta_{kj} + i\frac{\lambda}{3!}\delta_{k\ell}\delta_{nk}G_{kn\ell p}i\Gamma_{pj} \end{aligned} \tag{5.146}$$

Para a teoria ϕ^4 temos $\Gamma_{ijk} = 0$ pelo que

$$G_{kn\ell p} = G_{kk'}G_{nn'}G_{\ell\ell'}G_{pp'}\ i\Gamma_{k'n'\ell'p'} \tag{5.147}$$

e obtemos

$$-i\Sigma_{kj} = -i\frac{\lambda}{2}G_{kk}\delta_{kj} - i\frac{\lambda}{3!}G_{kk'}G_{kn'}G_{k\ell'}\ i\Gamma_{k'n'\ell'j} \tag{5.148}$$

que representamos diagramaticamente na Figura 5.19

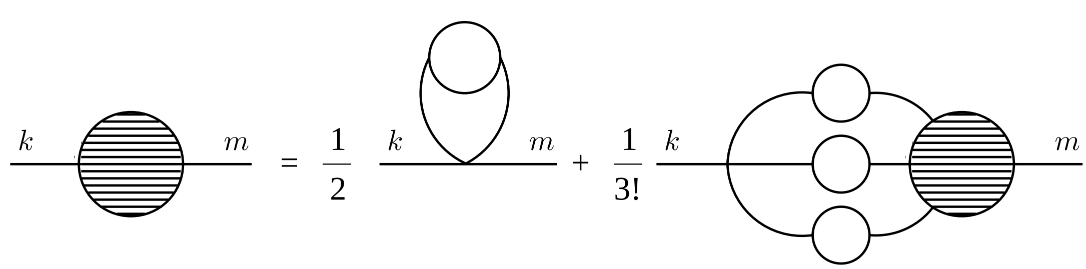


Figure 5.19: Dyson-Schwinger equation for ϕ^4.

5.6.3 Ward identities

Seja uma teoria com uma simetria qualquer. Essa simetria é expressa pela invariância da acção

$$\frac{\delta S[\phi]}{\delta \phi_i} F_i(\phi) = 0 \tag{5.149}$$

onde considerámos uma transformação do tipo 5.120. Se esta expressão deixar invariante a medida $\mathcal{D}(\phi)$ então a expressão geral 5.125 reduz-se a

$$J_i F_i \left[\frac{\delta}{i \delta J} \right] Z(J) = 0 \tag{5.150}$$

Esta expressão é conhecida por *identidade de Ward*. Derivação em ordem às fontes conduz a relações entre as funções de Green que expressam as simetrias da teoria.

Para teorias de gauge a expressão é um pouco mais complicada. A razão é que no processo de quantificação das teorias de gauge é normalmente necessário introduzir termos que quebram a simetria para fixar a gauge. Assim podemos escrever

$$S_{eff} = S_I + S_{NI} \tag{5.151}$$

onde $\frac{\delta S_I}{\delta \phi_i} F_i = 0$ e $\frac{\delta S_{NI}}{\delta \phi_i} F_i \neq 0$. Então se a medida continuar a ser invariante, devemos ter agora a identidade de Ward na forma

$$\left(\frac{\delta S_{NI}}{\delta \phi_i} \left[\frac{\delta}{i \delta J} \right] + J_i \right) F_i \left[\frac{\delta}{i \delta J} \right] Z(J) = 0 \tag{5.152}$$

Na próxima secção vamos aplicar esta expressão para obter as identidades de Ward em QED. Para as teorias de gauge não abelianas a questão da invariância da medida é um pouco mais delicada e será analisada no próximo capítulo depois de mostrarmos como se quantificam estas teorias.

Problems for Chapter 5

5.1 Calcule G_c^4 a partir de 5.21 e mostre que é de facto a função de Green conexa de quatro pernas.

5.2 Mostre que para um campo escalar real temos

$$Z_0[J] = \exp\left\{-\frac{1}{2}\int d^4x d^4y J(x)\Delta^{(0)}(x,y)J(y)\right\} \tag{5.153}$$

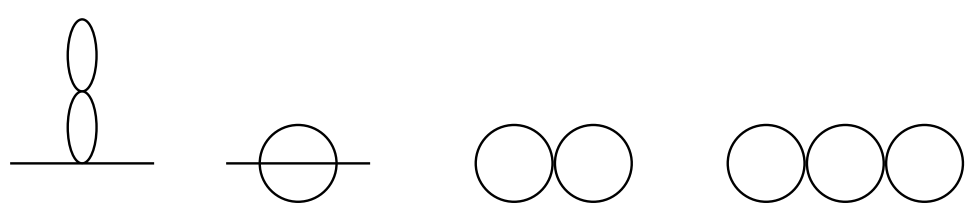
onde

$$\Delta^{(0)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \frac{i}{k^2-m^2+i\epsilon} \tag{5.154}$$

Sugestão: Use a generalização do resultado

$$\int_{-\infty}^{+\infty} dx_1 \cdots dx_N e^{-\frac{1}{2}x_i M_{ij} x_j + b_i x_i} = \pi^{N/2} (\det M)^{-1/2} e^{\frac{1}{2} b_i (M^{-1})_{ij} b_j} \tag{5.155}$$

5.3 Determine os factores de simetria dos diagramas seguintes:



5.4 Considere a teoria ϕ^3, isto é $V(\phi) = \frac{\lambda}{3!}\phi^3$. Usando

$$Z[J] = \exp\left\{-i\int d^4x V\left[\frac{\delta}{i\delta J}\right]\right\} Z_0(J) \tag{5.156}$$

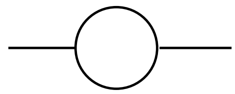
onde

$$Z_0(J) = exp[-\frac{1}{2}\int d^4x d^4x' J(x) G_F^{(0)}(x-x') J(x')] \tag{5.157}$$

e

$$G_F^{(0)}(x-x') = i\int d^4k e^{ik(x-x')} \frac{1}{k^2-m^2+i\varepsilon} \tag{5.158}$$

mostre que o factor de simetria do diagrama



é $S = \frac{1}{2}$.

5.5 Dado o Lagrangeano de Dirac (teoria livre)

$$\mathcal{L}_0 = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \ , \tag{5.159}$$

mostre que o funcional gerador das funções de Green é dado por

$$Z_0[\eta, \overline{\eta}] = e^{-\int d^4x d^4y \ \overline{\eta}(x) S_F^0(x,y)\eta(y)} \tag{5.160}$$

onde

$$\begin{aligned} S^0_{F\alpha\beta}(x,y) &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \left(\frac{i}{\not{p} - m + i\varepsilon}\right)_{\alpha\beta} \\ &= \frac{\delta^2 Z_0}{i\delta\eta_\alpha(y) \ i\delta\overline{\eta}_\beta(x)} \\ &= \langle 0| \, T\psi_\beta(x)\overline{\psi}_\alpha(y) \, |0\rangle \ . \end{aligned}$$

5.6 Como mostraremos no Capítulo 6 o funcional gerador das funções de Green para QED é dado por

$$Z(J_\mu, \eta, \overline{\eta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}) \ e^{i\int d^4x(\mathcal{L}_{QED} + \mathcal{L}_{GF} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta)} \ . \tag{5.161}$$

onde

$$\begin{aligned} &\mathcal{L}_{QED} = -\frac{1}{4} \ F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \\ &\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial \cdot A)^2 \\ &D_\mu = \partial_\mu + ieA_\mu \ . \end{aligned}$$

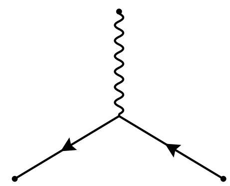
a) Calcule $Z_0[J^\mu, \eta, \overline{\eta}]$
b) Mostre que

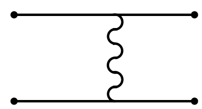
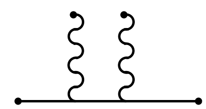
$$Z[J^\mu, \eta, \overline{\eta}] = \exp\left\{(-ie)\int d^4x \frac{\delta}{\delta\eta_\alpha(x)} \, (\gamma^\mu)_{\alpha\beta} \, \frac{\delta}{\delta\overline{\eta}_\beta(x)} \frac{\delta}{\delta J_\mu(x)}\right\} Z_0[J^\mu, \eta, \overline{\eta}] \ . \tag{5.162}$$

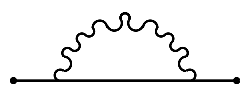
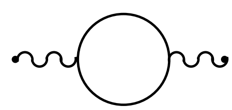
c) Expanda

$$Z = Z_0 \left[1 + (-ie)Z_1 + (-ie)^2 Z_2 + \cdots\right] \tag{5.163}$$

onde se retiraram as amplitudes vácuo-vácuo em Z_i, isto é, $Z_i[0] = 0 \to Z[0] = 1$. Mostre que

$Z_1 = -i$  (5.164)

$Z_2 = \frac{1}{2} Z_1^2 \quad + \quad \frac{1}{2}$  $+$ 

$-$  $+ \quad \frac{1}{2}$  (5.165)

d) Discuta os factores numéricos e os sinais das expressões anteriores.
e) Calcule em ordem mais baixa

$$\langle 0| \, T A^\mu(x) \psi_\beta(y) \overline{\psi}_\alpha(z) \, |0\rangle = \frac{\delta^3 Z}{i\delta\eta_\alpha(z) i\delta\overline{\eta}_\beta(y) i\delta J_\mu(x)} \tag{5.166}$$

e verifique que coincide com as regras de Feynman para o vértice.
f) Calcule a amplitude para o efeito de Compton em ordem mais baixa, isto é

$$\langle 0| \, T A^\mu(x) A^\nu(y) \psi_\beta(z) \overline{\psi}_\alpha(w) \, |0\rangle = \frac{\delta^4 Z}{i\delta\eta_\alpha(w) i\delta\overline{\eta}_\beta(z) i\delta J_\nu(y) i\delta J_\mu} \tag{5.167}$$

e verifique que reproduz o que se obtém usando as regras de Feynman usuais.

5.7 As identidades de Ward para QED deduzidas na secção 1.7 não têm a forma

$$J_i F_i \left[\frac{\partial}{i\partial J} \right] \; Z(J) = 0 \tag{5.168}$$

onde $\delta\phi_i = F_i[\phi]$ pois

$$S_{GF} = \int d^4x \left(-\frac{1}{2\xi} \; (\partial \cdot A)^2 \right) \tag{5.169}$$

não é invariante para transformações de gauge. Introduza o funcional

$$Z'(J_\mu, \eta, \overline{\eta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega}) \; e^{i\int d^4x (\mathcal{L}_{eff} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta)} \tag{5.170}$$

onde

$$\mathcal{L}_{eff} = \mathcal{L}_{QED} + \mathcal{L}_{GF} + \mathcal{L}_G \tag{5.171}$$

e

$$\mathcal{L}_G = -\overline{\omega} \Box \omega \; . \tag{5.172}$$

onde ω e $\overline{\omega}$ são campos escalares anticomutativos.

a) Mostre que

$$Z'(J_\mu, \eta, \overline{\eta}) = \mathcal{N}\ Z(J_\mu, \eta, \overline{\eta}) \tag{5.173}$$

onde $\mathcal{N}$ não depende das fontes nem dos campos. Explique porque é que esta renormalização (infinita) não afecta o cálculo das funções de Green. Assim tanto Z como Z' servem para o cálculo destas. b) Mostre que a medida $\mathcal{D}(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega})$ e $\int d^4x \mathcal{L}_{eff}$ são invariantes para a transformação

$$\begin{aligned} &\delta\psi = -ie\omega\theta\psi \qquad \delta\overline{\psi} = ie\overline{\psi}\omega\theta \\ &\delta A_\mu = \partial_\mu \omega\theta \\ &\delta\overline{\omega} = \tfrac{1}{\xi}(\partial\cdot A)\theta \quad \delta\omega = 0 \end{aligned} \tag{5.174}$$

onde θ é um parâmetro anticomutativo constante (variável de Grassman). c) Introduza fontes anticomutativas para os campos ω e $\overline{\omega}$, isto é

$$\overline{Z}(J_\mu, \eta, \overline{\eta}, \zeta, \overline{\zeta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega}) e^{i\int d^4x(\mathcal{L}_{eff} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta + \overline{\omega}\zeta + \overline{\zeta}\omega)} \tag{5.175}$$

Mostre que

$$\overline{Z}(J_\mu, \eta, \overline{\eta}, \zeta, \overline{\zeta}) = Z_G(\zeta, \overline{\zeta})\ Z(J_\mu, \eta, \overline{\eta}) \tag{5.176}$$

onde

$$Z(J_\mu, \eta, \overline{\eta}) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi})\ e^{i\int d^4x(\mathcal{L}_{QED} + \mathcal{L}_{GF} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta)}\ . \tag{5.177}$$

Considere os funcionais $\overline{W}$, W_G e W e ainda $\overline{\Gamma}$, Γ_G e Γ definidos de maneira semelhante. Qual a relação entre $\overline{W}$, W_G e W e entre $\overline{\Gamma}$, Γ_G e Γ . d) Mostre que a equação de Dyson Schwinger para os campos ω e $\overline{\omega}$ é

$$\frac{\delta\overline{\Gamma}}{\delta\overline{\omega}} = -\Box\omega\ . \tag{5.178}$$

e) Mostre que as identidades de Ward se podem agora escrever na forma

$$J_i F_i[\frac{\delta}{i\delta J}]\overline{Z} = 0\ . \tag{5.179}$$

Escreva as identidades de Ward para $\overline{\Gamma}(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega})$. Mostre que conduzem aos resultados conhecidos f) Mostre que um termo de massa para o fotão, embora quebre a simetria de gauge, não estraga as identidades de Ward desde que os fantasmas ω adquiram massa. Se o termo de massa do fotão for $\frac{1}{2}\ \mu^2 A_\mu A^\mu$ qual a massa dos fantasmas?

Chapter 6

Non Abelian Gauge Theories

6.1 Classical theory

6.1.1 Introduction

Vamos brevemente rever como se constrói a acção clássica para uma teoria de gauge não abeliana (Yang-Mills). Consideremos um grupo compacto G correspondendo a uma simetria interna. Seja ϕ_i, $(i = 1, \cdots, N)$ um conjunto de campos que se transformam de acordo com uma representação de dimensão N de G, isto é

$$\phi(x) \to \phi'(x) = U(g)\phi(x) \tag{6.1}$$

onde $U(g)$ é uma matriz $N \times N$. Numa transformação infinitesimal

$$g = 1 - i\alpha^a t^a \qquad\qquad a = 1, \cdots, r \tag{6.2}$$

onde os parâmetros α^a são infinitesimais e t^a são os geradores do grupo satisfazendo as relações, para a representação fundamental

$$\begin{aligned} \left[t^a, t^b\right] &= i f^{abc} t^c \\ Tr\left(t^a t^b\right) &= \frac{1}{2}\delta^{ab} \end{aligned} \tag{6.3}$$

Exemplos destes geradores são

$$\begin{aligned} &SU(2) \qquad t^a = \frac{\sigma^a}{2} \quad ; \quad a = 1, 2, 3 \\ &SU(3) \qquad t^a = \frac{\lambda^a}{2} \quad ; \quad a = 1, \cdots, 8 \end{aligned} \tag{6.4}$$

onde σ^a e λ^a são as matrizes de Pauli e Gell-Mann respectivamente.

Na representação associada aos campos ϕ, as matrizes T^a de dimensão $(N \times N)$ formam uma representação de álgebra de Lie, isto é,

$$[T^a, T^b] = i f^{abc} T^c \tag{6.5}$$

A sua normalização é dada por

$$Tr(T^a T^b) = \delta^{ab} T(R) \tag{6.6}$$

onde $T(R)$ é um número característico da representação R. Para uma dada representação mostra-se a identidade (ver problema 2.1)

$$T(R)\, r = d(R) C_2(R) \tag{6.7}$$

onde r é a dimensão do Grupo G e $d(R)$ é a dimensão da representação R. Numa transformação infinitesimal

$$\delta\phi = -i\alpha^a T^a \phi \equiv -i\underset{\sim}{\alpha}\phi \tag{6.8}$$

onde introduzimos a notação $\underset{\sim}{\alpha} \equiv \alpha^a T^a$.

6.1.2 Covariant derivative

Para resolver o problema da derivada não se transformar como os campos, isto é,

$$\partial_\mu \phi' \neq U \partial_\mu \phi \tag{6.9}$$

quando os parâmetros dependem de x^α, introduz-se a derivada covariante

$$D_\mu \phi = (\partial_\mu - ig\underset{\sim}{A}_\mu)\phi \qquad ; \qquad \underset{\sim}{A}_\mu = A^a_\mu T^a \tag{6.10}$$

onde A^a_μ são os campos de gauge (tantos quantos os geradores do grupo). As propriedades de transformação de A^a_μ são obtidas exigindo que $D_\mu\phi$ se transforme como ϕ, isto é,

$$\begin{aligned}
(D_\mu\phi)' &= (\partial_\mu - ig\underset{\sim}{A}'_\mu)\phi' = (\partial_\mu - ig\underset{\sim}{A}'_\mu)U\phi \\
&= \partial_\mu U\phi + U\partial_\mu\phi - ig\underset{\sim}{A}'_\mu U\phi \\
&= UD_\mu\phi + (igU\underset{\sim}{A}_\mu - ig\underset{\sim}{A}'_\mu U + \partial_\mu U)\phi
\end{aligned} \tag{6.11}$$

Portanto $(D_\mu\phi)' = U(D_\mu\phi)$ requer

$$\underset{\sim}{A}'_\mu = U\underset{\sim}{A}_\mu U^{-1} - \frac{i}{g}\partial_\mu U U^{-1} \tag{6.12}$$

Infinitesimalmente $U \simeq 1 - i\underset{\sim}{\alpha}$ e obtemos

$$\delta\underset{\sim}{A}_\mu = -i\left[\underset{\sim}{\alpha}, \underset{\sim}{A}_\mu\right] - \frac{1}{g}\partial_\mu\underset{\sim}{\alpha}\ . \tag{6.13}$$

o que se escreve em componentes

$$\begin{aligned}
\delta A^a_\mu &= -\frac{1}{g}\partial_\mu\alpha^a + f^{bca}\alpha^b A^c_\mu \\
&= -\frac{1}{g}(\partial_\mu\alpha^a - gf^{bca}\alpha^b A^c_\mu)
\end{aligned} \tag{6.14}$$

Como na representação adjunta $(T^c)_{ab} = -if^{bca}$ então

$$\delta A^a_\mu = -\frac{1}{g}\left(\partial_\mu \delta_{ab} - ig(T^c)_{ab}A^c_\mu\right)\alpha^b \tag{6.15}$$

ou ainda

$$\delta A^a_\mu = -\frac{1}{g}(D_\mu \alpha)^a \tag{6.16}$$

6.1.3 Tensor $F_{\mu\nu}$

Calculemos o comutador de duas derivadas covariantes

$$\begin{aligned}
[D_\mu, D_\nu]\,\phi &= \left[\partial_\mu - ig\utilde{A}_\mu, \partial_\nu - ig\utilde{A}_\nu\right]\phi \\
&= -ig\left(\partial_\mu \utilde{A}_\nu - \partial_\nu \utilde{A}_\mu - ig\left[\utilde{A}_\mu, \utilde{A}_\nu\right]\right)\phi \\
&\equiv -ig\ \utilde{F}_{\mu\nu}\ \phi
\end{aligned} \tag{6.17}$$

onde se definiu o tensor $\utilde{F}_{\mu\nu} \equiv F^a_{\mu\nu}T^a$ designado por curvatura,

$$\utilde{F}_{\mu\nu} = \partial_\mu \utilde{A}_\nu - \partial_\nu \utilde{A}_\mu - ig\left[\utilde{A}_\mu, \utilde{A}_\nu\right] \tag{6.18}$$

ou em componentes

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{6.19}$$

Vejamos como se transforma $F_{\mu\nu}$ numa transformação de gauge.

$$\begin{aligned}
\utilde{F}'_{\mu\nu} &= \partial_\mu \utilde{A}'_\nu - \partial_\nu \utilde{A}'_\mu - ig\left[\utilde{A}'_\mu, \utilde{A}'_\nu\right] \\
&= \left[\partial_\mu(U\utilde{A}_\nu U^{-1}) - \frac{i}{g}\partial_\mu(\partial_\nu U U^{-1}) - (\mu \leftrightarrow \nu)\right] \\
&\quad -igU[\utilde{A}_\mu, \utilde{A}_\nu]U^{-1} - \left[\partial_\mu U U^{-1}, U\utilde{A}_\nu U^{-1}\right] \\
&\quad -\left[U\utilde{A}_\mu U^{-1}, \partial_\nu U U^{-1}\right] + \frac{i}{g}\left[\partial_\mu U U^{-1}, \partial_\nu U U^{-1}\right]
\end{aligned} \tag{6.20}$$

Usando

$$\partial_\mu U^{-1} = -U^{-1}\partial_\mu U U^{-1} \tag{6.21}$$

obtemos

$$\utilde{F}'_{\mu\nu} = U\utilde{F}_{\mu\nu}U^{-1} \tag{6.22}$$

ou infinitesimalmente

$$\delta \underset{\sim}{F}_{\mu\nu} = -i \left[\underset{\sim}{\alpha}, \underset{\sim}{F}_{\mu\nu} \right] \tag{6.23}$$

É fácil de ver que com o tensor $F_{\mu\nu}$ é possível construir um invariante. De facto a quantidade

$$Tr(\underset{\sim}{F}'_{\mu\nu}\underset{\sim}{F}'^{\mu\nu}) = Tr(\underset{\sim}{F}_{\mu\nu}\underset{\sim}{F}^{\mu\nu}) = \frac{1}{2} F^a_{\mu\nu} F^{a\mu\nu} \tag{6.24}$$

é invariante e pode ser usada para construir uma acção, generalizando a acção de Maxwell para as teorias abelianas.

6.1.4 Choice of gauge

Chama-se *gauge pura* ao potencial $\underset{\sim}{A}^\mu$ tal que $\underset{\sim}{F}_{\mu\nu} = 0$. É fácil de mostrar que

$$F_{\mu\nu} = 0 \Longleftrightarrow \exists U : \underset{\sim}{A}_\mu = \partial_\mu U U^{-1} \tag{6.25}$$

Para evitar a arbitrariedade de gauge é conveniente por vezes impor condições de gauge. Exemplos são a *Gauge Axial* definida por

$$n^\mu A_\mu(x) = 0 \tag{6.26}$$

onde n^μ é um quadrivector constante, e a *Gauge Lorentz*

$$\partial_\mu A^\mu(x) = 0 \tag{6.27}$$

6.1.5 The action and the equations of motion

A acção para a teoria de gauge pura é

$$\begin{aligned} S &= -\frac{1}{2} \int d^4x Tr(\underset{\sim}{F}_{\mu\nu}\underset{\sim}{F}^{\mu\nu}) \\ &= -\frac{1}{4} \int d^4x F^a_{\mu\nu} F^{\mu\nu a} \end{aligned} \tag{6.28}$$

Devido ao resultado anterior sobre $Tr(\underset{\sim}{F}_{\mu\nu}\underset{\sim}{F}^{\mu\nu})$, Eq. (6.24), a acção é invariante para transformações do grupo de gauge G. A equação de Euler-Lagrange

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu A^a_\nu)} - \frac{\delta \mathcal{L}}{\delta A^a_\nu} = 0 \tag{6.29}$$

obtém-se facilmente notando que

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu A^a_\nu)} = \frac{\delta \mathcal{L}}{\delta F^b_{\rho\sigma}} \frac{\delta F^b_{\rho\sigma}}{\delta(\partial_\mu A^a_\nu)} = -F^{a\mu\nu} \tag{6.30}$$

e

$$\frac{\delta \mathcal{L}}{\delta A^a_\nu} = \frac{\delta \mathcal{L}}{\delta F^b_{\rho\sigma}} \frac{\delta F^b_{\rho\sigma}}{\delta A^a_\nu} = g f^{bca} A^b_\mu F^{c\mu\nu} \tag{6.31}$$

Então

$$\partial_\mu F^{\mu\nu a} + g f^{bca} A^b_\mu F^{\mu\nu c} = 0 \tag{6.32}$$

ou ainda atendendo a que na representação adjunta $(T^c)_{ab} = -if^{bca}$

$$(\partial_\mu \delta_{ab} - ig(T^c)_{ab} A^c_\mu) F^{\mu\nu b} = 0 \tag{6.33}$$

isto é

$$D^{ab}_\mu F^{\mu\nu b} = 0 \tag{6.34}$$

6.1.6 Energy–momentum tensor

Como para o Electromagnetismo o tensor canónico não é invariante de gauge. De facto [1]

$$\begin{aligned}
\tilde{\theta}^{\mu\nu} &= -\frac{\delta \mathcal{L}}{\delta(\partial_\mu A^a_\rho)} \partial^\nu A^a_\rho + g^{\mu\nu} \mathcal{L} \\
&= F^{\mu\rho a} \partial^\nu A^a_\rho - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma a} F^a_{\rho\sigma}
\end{aligned} \tag{6.35}$$

Para o tornar invariante de gauge procedemos como no electromagnetismo. Subtraímos a $\tilde{\theta}^{\mu\nu}$ uma quantidade que seja uma divergência para que as leis de conservação não venham alteradas. A quantidade relevante é

$$\begin{aligned}
\Delta\theta^{\mu\nu} &= \partial_\rho (F^{\mu\rho a} A^{\nu a}) \\
&= \partial_\rho F^{\mu\rho a} A^{\nu a} + F^{\mu\rho a} \partial_\rho A^{\nu a} \\
&= g f^{bca} A^b_\rho F^{\rho\mu c} A^{\nu a} + F^{\mu\rho a} \partial_\rho A^{\nu a} \\
&= F^{\mu\rho a} (-F^{\nu a}_\rho + \partial^\nu A^a_\rho)
\end{aligned} \tag{6.36}$$

logo

$$\begin{aligned}
\theta^{\mu\nu} &\equiv \tilde{\theta}^{\mu\nu} - \Delta\theta^{\mu\nu} \\
&= F^{\mu\rho a} F^{\nu a}_\rho - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma a} F^a_{\rho\sigma}
\end{aligned} \tag{6.37}$$

expressão análoga à obtida no electromagnetismo. Introduzindo os análogos dos campos eléctricos e magnéticos

$$E^i_a = F^{i0}_a \ ; \ B^k_a = -\frac{1}{2} \varepsilon_{ijk} F^{ij}_a \qquad i,j,k = 1,2,3 \tag{6.38}$$

[1] One should note an overall sign difference with respect to the general definition of Eq. (1.66). This is to maintain the component θ^{00} with the meaning of a positive energy density. Obvsiously, the overall sign in Eq. (1.66), has no meaning prior to make contact with the model.

obtemos

$$\begin{cases} \theta^{00} & = \frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \\ \theta^{0i} & = (\vec{E}^a \times \vec{B}^a)^i \end{cases} \tag{6.39}$$

com uma interpretação semelhante ao caso do electromagnetismo.

6.1.7 Hamiltonian formalism

Da expressão para θ^{00} é claro que o Hamiltoniano é

$$\begin{aligned} H &= \int d^3x \frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \\ &= \int d^3x \mathcal{H} \end{aligned} \tag{6.40}$$

onde a $\mathcal{H}$ é a densidade Hamiltoniana.

Vamos ver no entanto que devido à invariância de gauge a relação entre o Hamiltoniano e o Lagrangeano não é a usual. Para isso é conveniente partir da acção escrita em formalismo de 1ª ordem.

$$S = \int d^4x \left\{ -\frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu) F^{\mu\nu a} + \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \right\} \tag{6.41}$$

onde A^a_μ e $F^a_{\mu\nu}$ são agora variáveis independentes. É fácil de ver que a variação de S em ordem a $F^a_{\mu\nu}$ dá a sua definição

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\nu + g f^{abc} A^b_\mu A^c_\nu \tag{6.42}$$

e que por sua vez substituindo 6.42 em S obtemos a acção usual. Usando as definições de $\vec{E}^a$ e $\vec{B}^a$ obtemos

$$\begin{aligned} S &= \int d^4x - (\partial^0 \vec{A}^a + \vec{\nabla} A^{0a} - g f^{abc} A^{0b} \vec{A}^c) \cdot \vec{E}^a - \frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \\ &= \int d^4x \left\{ -\partial^0 \vec{A}^a \cdot \vec{E}^a - \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + A^{0a}(\vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c) \right\} \end{aligned} \tag{6.43}$$

A densidade Lagrangeana escreve-se, portanto

$$\mathcal{L} = -E^{ka} \partial^0 A^{ka} - \mathcal{H}(E^{ka}, A^{ka}) + A^{0a} C^a \tag{6.44}$$

onde

$$\begin{cases} \mathcal{H} \equiv \frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) \\ B^{ka} \equiv -\frac{1}{2} \epsilon^{kmn} F^{mna} \\ C^a = \vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c \end{cases} \tag{6.45}$$

As variáveis A_k^a e $-E_k^a$ são as variáveis conjugadas, $\mathcal{H}(E_k^a, A_k^a)$ é a densidade Hamiltoniana. As variáveis A^{0a} desempenham o papel de multiplicadores de Lagrange para as condições,

$$\vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c = 0 \tag{6.46}$$

que não são mais do que as equações de movimento 6.34 para $\nu = 0$.

Se introduzirmos os parêntesis de Poisson a tempo igual

$$\{A^{ia}(x), E^{jb}(y)\}_{x0=y0} = \delta^{ij}\delta^{ab}\delta^3(\vec{x} - \vec{y}) \tag{6.47}$$

é fácil de mostrar que

$$\begin{aligned} &\{C^a(x), C^b(y)\}_{x0=y0} = -g f^{abc} C^c(x) \delta^3(\vec{x} - \vec{y}) \\ &\{\mathcal{H}, C^a(x)\} = 0 \end{aligned} \tag{6.48}$$

Isto mostra que as teorias de gauge (não abelianas neste caso, mas a afirmação é igualmente verdadeira para teorias abelianas) representam um exemplo daquilo a que se chama *Sistemas de Hamilton Generalizados* primeiro introduzidos por Dirac. Para definir estes sistemas consideremos um sistema com as variáveis canónicas (p_i, q_i) que geram o espaço de fase Γ^{2n} $(i = 1, \ldots, n)$. Então a acção destes sistemas é escrita na forma

$$S = \int L(t) dt \tag{6.49}$$

onde

$$L(t) = \sum_{i=1}^{n} p_i \dot{q}_i - h(p,q) - \sum_{\alpha=1}^{m} \lambda^\alpha \varphi_\alpha(p,q) \ . \tag{6.50}$$

As variáveis $\lambda^\alpha(\alpha = 1, ...m)$ são multiplicadores de Lagrange e φ^α são as ligações. Para que este sistema seja um sistema de Hamilton generalizado é necessário que as condições seguintes sejam verificadas

$$\begin{aligned} \{\varphi^\alpha, \varphi^\beta\} &= \sum_\alpha f^{\alpha\beta\gamma}(p,q)\varphi^\gamma \\ \{h, \varphi^\alpha\} &= f^{\alpha\beta}(p,q)\varphi^\beta \end{aligned} \tag{6.51}$$

O caso das teorias de gauge é um caso particular com $f^{\alpha\beta} = 0$. Portanto para a quantificação das teorias de gauge temos primeiro de aprender a quantificar sistemas Hamilton generalizados.

6.2 Quantization

6.2.1 Systems with n degrees of freedom

Consideremos um *sistema de Hamilton generalizado* descrito na última secção. O seu Lagrangeano é

$$L(t) = p_i \dot{q}_i - h(p,q) - \lambda^\alpha \varphi^\alpha(p,q) \tag{6.52}$$

que conduz às equações de movimento

$$\begin{cases} \dot{q}_i = \dfrac{\partial h}{\partial p_i} + \lambda^\alpha \dfrac{\partial \varphi^\alpha}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial h}{\partial q_i} - \lambda^\alpha \dfrac{\partial \varphi^\alpha}{\partial q_i} \\ \varphi^\alpha(p,q) = 0 \qquad \alpha = 1,\ldots,m \end{cases} \tag{6.53}$$

Pode-se mostrar que um sistema de Hamilton generalizado (**SHG**) é equivalente a um sistema de Hamilton usual (**SH**) definido um espaço de fase $\Gamma^{*2(n-m)}$. Isto é um **SHG** é equivalente a um **SH** com $n-m$ graus de liberdade. O **SH** Γ^* pode ser construído da maneira seguinte. Sejam m condições

$$\chi^\alpha(p,q) = 0 \quad ; \quad \alpha = 1,\ldots,m \tag{6.54}$$

que satisfaçam

$$\left\{\chi^\alpha, \chi^\beta\right\} = 0 \tag{6.55}$$

e

$$\det\left|\{\varphi^\alpha, \chi^\beta\}\right| \neq 0 \tag{6.56}$$

Então o subespaço de Γ^{2n} definido pelas condições

$$\left\{ \begin{array}{lcl} \chi^\alpha(p,q) & = & 0 \\ \varphi^\alpha(p,q) & = & 0 \end{array} \right. \qquad \alpha = 1,\ldots,m \tag{6.57}$$

é o espaço $\Gamma^{*2(n-m)}$ pretendido. As variáveis canónicas p^* e q^* em $\Gamma^{*2(n-m)}$ podem ser encontradas da maneira seguinte. Devido à condição 6.55 podemos escolher as variáveis q_i em Γ^{2n} de tal forma que os χ^α coincidam com as primeiras m variáveis do tipo coordenada, isto é,

$$\underbrace{q}_{n} \equiv (\underbrace{\chi^\alpha}_{m}, \underbrace{q^*}_{n-m}) \tag{6.58}$$

Sejam $p = (p^\alpha, p^*)$ os correspondentes momentos conjugados. Nestas variáveis a condição 6.56 toma a forma

$$\det\left|\frac{\partial \varphi^\alpha}{\partial p^\beta}\right| \neq 0 \tag{6.59}$$

portanto as condições $\varphi^\alpha(p,q)=0$ podem ser resolvidas para p^α, isto é

$$p^\alpha = p^\alpha(p^*, q^*) \tag{6.60}$$

O subespaço Γ^* é portanto definido pelas condições

$$\left\{ \begin{array}{lcl} \chi^\alpha & \equiv & q^\alpha = 0 \\ \\ p^\alpha & = & p^\alpha(p^*, q^*) \end{array} \right. \tag{6.61}$$

As variáveis p^* e q^* são canónicas e o Hamiltoniano é dado por

$$h^*(p^*, q^*) = h(p,q)\big|_{(\chi=0\ ;\ \varphi=0)} \tag{6.62}$$

e as equações de movimento são agora

$$\dot{q}^* = \frac{\partial h^*}{\partial p^*} \qquad \dot{p}^* = -\frac{\partial h^*}{\partial q^*} \tag{6.63}$$

num total de $2(n-m)$ equações. O resultado fundamental pode ser enunciado na forma dum teorema.

Teorema 2.1

As duas representações são equivalentes, isto é, conduzem às mesmas equações de movimento.

Dem:

As relações $q^\alpha = 0 \Longrightarrow \dot{q}^\alpha = 0$ ou seja na descrição (p,q)

$$\frac{\partial h}{\partial p_\alpha} + \lambda^\beta \frac{\partial \varphi_\beta}{\partial p_\alpha} = 0 \quad ; \quad \alpha = 1, \ldots, m \tag{6.64}$$

Consideremos agora as equações de movimento para as coordenadas q^* nas duas representações

$$\begin{aligned} \dot{q}^* &= \frac{\partial h}{\partial p^*} + \lambda^\alpha \frac{\partial \varphi_\alpha}{\partial p^*} \\ \dot{q}^* &= \frac{\partial h^*}{\partial p^*} = \frac{\partial h}{\partial p^*} + \frac{\partial h}{\partial p^\alpha}\frac{\partial p_\alpha}{\partial p^*} \end{aligned} \tag{6.65}$$

As duas equações serão equivalentes se

$$\lambda^\alpha \frac{\partial \varphi_\alpha}{\partial p^*} = \frac{\partial h}{\partial p^\alpha}\frac{\partial p_\alpha}{\partial p^*} \tag{6.66}$$

ou seja usando as relações 6.64

$$\lambda^\alpha \left(\frac{\partial \varphi_\alpha}{\partial p^*} + \frac{\partial \varphi_\alpha}{\partial p_\beta}\frac{\partial p_\beta}{\partial p^*} \right) = 0 \tag{6.67}$$

Ora esta relação é verdadeira em virtude da ligação $\varphi_\alpha = 0$. Portanto as duas representações são equivalentes o que demonstra[2] *o teorema.*

Para efectuar a quantificação destes sistemas podemos usar as expressões para o operador evolução em termos dum integral de caminho nas variáveis (p^*, q^*) pois estas formam um sistema Hamiltoniano normal. Temos

$$U(q_f^*, q_i^*) = \int \prod_t \frac{dp^* dq^*}{(2\pi)} e^{i\int [p^*\dot{q}^* - h(p^*, q^*)]dt} \tag{6.68}$$

Embora este seja um modo possível de proceder à quantificação, não é o mais conveniente em muitas situações onde é difícil inverter as relações $\varphi^\alpha = 0$ para obter $p^\alpha = p^\alpha(p^*, q^*)$. Será mais conveniente usar as variáveis (p, q) com restrições apropriadas. Isto pode ser feito facilmente substituindo

$$\prod_t \frac{dp^* dq^*}{(2\pi)} \to \prod_t \frac{dpdq}{2\pi} \prod_t \delta(q^*)\delta(p^\alpha - p^\alpha(p^*, q^*)) \tag{6.69}$$

Então

$$U(q_f, q_i) = \int \prod_t \frac{dpdq}{2\pi} \prod_t \delta(q^\alpha)\delta(p^\alpha - p^\alpha(p^*, q^*)) e^{i\int dt(p\dot{q} - h(p,q))} \tag{6.70}$$

Esta expressão pode ainda ser escrita em termos das ligações se recordarmos que

$$\begin{aligned} \delta(q^\alpha) &= \delta(\chi^\alpha) \\ \delta(p^\alpha - p^\alpha(p^*, q^*)) &= \delta(\varphi^\alpha) \det \left| \frac{\partial \varphi_\alpha}{\partial p_\beta} \right| \end{aligned} \tag{6.71}$$

Então

$$\prod_t \delta(q^\alpha)\delta(p^\alpha - p^\alpha(p^*, q^*)) = \prod_t \delta(\varphi^\alpha)\delta(\chi^\alpha) \det |\{\varphi_\alpha, \chi_\beta\}| \tag{6.72}$$

Finalmente se usarmos a identidade

$$\delta(\varphi^\alpha) = \int \frac{d\lambda}{2\pi} e^{-i\int dt \lambda^\alpha \varphi_\alpha} \tag{6.73}$$

obtemos

$$U(q_f, q_i) = \int \prod_t \frac{dpdq}{2\pi} \frac{d\lambda}{2\pi} \prod_{t,x} \delta(\chi^\alpha) \det |\{\varphi^\alpha, \chi_\beta\}| \, e^{iS(p,q,\lambda)} \tag{6.74}$$

onde

$$S(p, q, \lambda) = \int [p\dot{q} - h(p, q) - \lambda\varphi] dt \tag{6.75}$$

[2] As equações para as variáveis p^* tratavam-se de modo semelhante.

Esta é a expressão que iremos aplicar às teorias de gauge. Notar que a expressão dentro do parêntesis recto é precisamente a do Lagrangeano para sistemas de Hamilton generalizados, 6.52. Pode-se mostrar (ver problema 2.2) que os *resultados físicos* não dependem da escolha das condições auxiliares $\chi^\alpha = 0$. Em teorias de gauge, fala-se da *escolha de gauge*.

6.2.2 QED as a simple example

Consideremos o campo electromagnético acoplado a uma corrente conservada $J^\mu = (\rho, \vec{J})$, $\partial_\mu J^\mu = 0$. O Lagrangeano é

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \tag{6.76}$$

A acção pode ser escrita na seguinte forma equivalente[3]

$$S = \int d^4x \left[-\vec{E} \cdot (\vec{\nabla} A^0 + \dot{\vec{A}}) - \vec{B} \cdot \vec{\nabla} \times \vec{A} + \frac{\vec{B}^2 - \vec{E}^2}{2} - \rho A^0 + \vec{J} \cdot \vec{A} \right] \tag{6.77}$$

As equações do movimento são, variando em ordem a $\vec{E}$ e $\vec{B}$

$$\left\{ \begin{array}{lcl} \vec{E} & = & -(\vec{\nabla} A^0 + \dot{\vec{A}}) \\ \vec{B} & = & \vec{\nabla} \times \vec{A} \end{array} \right. \rightarrow \left\{ \begin{array}{lcl} \vec{\nabla} \cdot \vec{B} & = & 0 \\ \vec{\nabla} \times \vec{E} & = & -\frac{\partial \vec{B}}{\partial t} \end{array} \right. \tag{6.78}$$

e, variando em ordem a A^0 e $\vec{A}$,

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{E} = \rho \\ \vec{\nabla} \times \vec{B} - \dfrac{\partial \vec{E}}{\partial t} = \vec{J} \end{array} \right. \tag{6.79}$$

Se substituirmos $\vec{B} = \vec{\nabla} \times \vec{A}$ obtemos, depois de uma integração por partes,

$$S = \int d^4x \left\{ -\vec{E} \cdot \dot{\vec{A}} - \left(\frac{\vec{E}^2 + (\vec{\nabla} \times A)^2}{2} - \vec{J} \cdot \vec{A} \right) + A^0 (\vec{\nabla} \cdot \vec{E} - \rho) \right\} \tag{6.80}$$

É claro que A^0 desempenha o papel dum multiplicador de Lagrange. As variáveis canónicas são $\vec{A}$ e $\vec{E}$ mas elas não são livres pois existe uma ligação que tem que ser respeitada, $\vec{\nabla} \cdot \vec{E} = \rho$. Esta ligação é linear nos campos. Aqui reside a grande simplificação do electromagnetismo. Isto porque se escolhermos uma condição de gauge também linear então o $\det\{\varphi^\alpha, \chi_\beta\}$ não dependerá de $\vec{E}$ e $\vec{A}$ e será uma constante que apenas afectará a normalização. Uma tal condição de gauge é obtida, por exemplo, da forma seguinte (*gauge de Lorentz*)

$$\chi = \partial_\mu A^\mu - c(\vec{x}, t) \tag{6.81}$$

[3] A acção escrita na forma 6.77 é designada por formalismo de 1ª ordem. Comparar com a equação 6.43.

onde $c(\vec{x}, t)$ é uma função arbitrária. Então é fácil de ver que a expressão para o funcional gerador das funções de Green é (o termo vindo de $\det\{\varphi^\alpha, \chi_\beta\}$ é absorvido na normalização)

$$Z[J^\mu] = \mathcal{N} \int \mathcal{D}(\vec{E}, \vec{A}, A^0) \prod_x \delta(\partial_\mu A^\mu - c(x)) e^{iS} \tag{6.82}$$

onde

$$\begin{aligned} S &= \int d^4x \left\{ -\vec{E} \cdot \dot{\vec{A}} - \left[\frac{E^2 + (\vec{\nabla} \times A)^2}{2} + (\vec{J} \cdot \vec{A}) \right] + A^0 (\vec{\nabla} \cdot \vec{E} - \rho) \right\} \\ &= \int d^4x \left\{ -\frac{E^2}{2} - \vec{E} \cdot (\vec{\nabla} A^0 + \dot{\vec{A}}) - \frac{(\vec{\nabla} \times A)^2}{2} - J_\mu A^\mu \right\} \end{aligned} \tag{6.83}$$

A integração em $\vec{E}$ é gaussiana e pode ser imediatamente efectuada obtendo-se (mantemos a designação $\mathcal{N}$ embora seja diferente depois da integração)

$$Z[J\mu] = \mathcal{N} \int \mathcal{D}(A_\mu) \prod_x \delta(\partial_\mu A^\mu - c(x)) e^{iS} \tag{6.84}$$

onde agora

$$\begin{aligned} S &= \int d^4x \left[-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - J_\mu A^\mu \right] \\ &= \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \right] \end{aligned} \tag{6.85}$$

Como as funções $c(x)$ são arbitrárias podemos integrar sobre elas com um peso

$$\exp \left(-\frac{1}{2\xi} \int d^4x c^2(x) \right) \tag{6.86}$$

obtendo então o resultado familiar

$$Z[J^\mu] = \int \mathcal{D}(A_\mu) e^{i \int d^4x \left[-\frac{1}{4} F^2 - \frac{1}{2\xi} (\partial \cdot A)^2 - J \cdot A \right]} \tag{6.87}$$

Como veremos adiante se tivéssemos escolhido uma condição de gauge não linear, o $\det |\{q, \chi\}|$ já dependeria de $\vec{E}$ ou $\vec{A}$ e não teria sido possível absorvê-lo na normalização (que é irrelevante pois escolhemos sempre a normalização de forma que $Z[0] = 1$) . Nesse caso será necessário utilizar os métodos das teorias de gauge não abelianas que vamos estudar na próxima secção.

6.2.3 Non abelian gauge theories. Non covariant gauges

Vimos anteriormente que a acção para as teorias de gauge não abelianas (TGNA), se podia escrever na forma

$$S = -2\int d^4x \ \mathrm{Tr}\left[\underset{\sim}{\vec{E}}\cdot\partial^0\underset{\sim}{\vec{A}} + \frac{1}{2}(\underset{\sim}{\vec{E}}^2 + \underset{\sim}{\vec{B}}^2) - \underset{\sim}{A}^0(\vec{\nabla}\cdot\underset{\sim}{\vec{E}} + g[\underset{\sim}{\vec{A}},\underset{\sim}{\vec{E}}])\right] \tag{6.88}$$

$$= \int d^4x \left[-E_k^a\partial^0 A_k^a - \mathcal{H}(E_k, A_k) + A^{0a}C^a\right] \tag{6.89}$$

onde

$$C^a = \vec{\nabla}\cdot\vec{E}^a - gf^{abc}\vec{A}^b\cdot\vec{E}^c \tag{6.90}$$

Se introduzirmos os parêntesis de Poisson a tempo igual

$$\left\{-E_a^i(x), A_b^j(y)\right\}_{x^0=y^0} = \delta^{ij}\delta_{ab}\delta^3(\vec{x}-\vec{y}) \tag{6.91}$$

é fácil mostrar que

$$\begin{aligned} &\left\{C^a(x), C^b(y)\right\}_{x_0=y_0} = -gf^{abc}C^c(x)\delta^3(\vec{x}-\vec{y}) \\ &\{H, C^a(x)\} = 0 \end{aligned} \tag{6.92}$$

onde

$$H = \int d^3x\mathcal{H}(E_k, A_k) = \frac{1}{2}\int d^3x \left[(E^{ka})^2 + (B^{ka})^2\right] \tag{6.93}$$

Assim as teorias de gauge não abelianas representam um exemplo de sistemas da Hamilton generalizados. As variáveis do género coordenada são A_k^a, os momentos conjugados são $-E_k^a$. As varáveis A^{0a} são multiplicadores de Lagrange para garantir as ligações

$$\vec{\nabla}\cdot\vec{E}^a - gf^{abc}\vec{A}^b\cdot\vec{E}^c = 0 \tag{6.94}$$

que são parte das equações de movimento.

Para procedermos à quantificação podemos usar o formalismo da secção 6.2.1 para **SHG**. Temos para isso que impor r condições auxiliares (r é a dimensão do grupo de Lie), isto é, tantas como as condições de ligação $C^a(x) = 0$, $a = 1, \ldots, r$. Escolher estas condições é aquilo que se chama escolher, ou fixar, a gauge. Esta escolha é arbitrária, os resultados físicos não devem depender dela (ver problema 2.2). No entanto expressões intermédias como sejam, por exemplo, as regras de Feynman, dependem fortemente da gauge.

Como vimos no exemplo do electromagnetismo se for possível fixar uma condição de gauge linear nas variáveis dinâmicas, $\vec{A}^a$ e $\vec{E}^a$, então a expressão do integral de caminho simplifica-se bastante devido ao facto do determinante não depender dessas variáveis e poder ser absorvido numa constante de normalização. Uma gauge em que isto é possível é a chamada gauge axial que passaremos a estudar.

• **Gauge Axial**

É sempre possível efectuar uma transformação de gauge tal que a componente de $\vec{A}^a$ segundo uma direcção espacial seja nula em todos os pontos, isto é, escolhendo a direcção segundo o eixo dos zz

$$A^{3a} = 0 \qquad\qquad a = 1, \ldots, r \tag{6.95}$$

Estas r condições constituem as nossas condições auxiliares necessárias para se proceder à quantificação da teoria. A vantagem desta escolha de gauge é a seguinte. Se calcularmos $\{C^a, A^{3b}\}$ obtemos

$$\begin{aligned}
\{C_a(x), A^3_b(y)\} &= \{\partial_k E^k_a(x), A^3_b(y)\} - g f_{adc} A^k_d \{E^k_c(x), A^3_b(y)\} \\
&= -g\ \delta_{ab} \frac{\partial}{\partial x^3} \delta^3(\vec{x} - \vec{y}) = \frac{\delta}{\delta \alpha^a(x)} (\delta A^3_b(y))
\end{aligned} \tag{6.96}$$

onde se usou o facto de $A^3_b = 0$. Vemos assim que $\{C^a, A^3_b\}$ não depende de $\vec{A}_a$ e $\vec{E}_a$ e o determinante que aparece na expressão do integral de caminho pode ser absorvido na normalização. Podemos assim escrever para o funcional gerador das funções de Green

$$Z[J^{\mu a}] = \int \mathcal{D}(\vec{E}, \vec{A}, A^0) \prod_x \delta(A^3) e^{iS(\vec{E}, \vec{A}, A^0, J^\mu)} \tag{6.97}$$

onde

$$S(\vec{E}, \vec{A}, A^0, J^\mu) = \int d^4x \left[-\vec{E}^a \cdot \partial^0 \vec{A}^a - \frac{1}{2} \left[(\vec{E}^a)^2 + (\vec{B}^a)^2 \right] + A^{0a} C^a + A^a \cdot J^a \right] \tag{6.98}$$

e

$$C^a = \vec{\nabla} \cdot \vec{E}^a - g f^{abc} \vec{A}^b \cdot \vec{E}^c \tag{6.99}$$

Como a integração em $\vec{E}$ aparece na forma duma integração gaussiana obtemos facilmente

$$Z_A[J^{\mu a}] = \int \mathcal{D}(A^\mu) \prod_x \delta(A^3) e^{i \int d^4x \left[\mathcal{L}(x) + A^a \cdot J^a \right]} \tag{6.100}$$

onde

$$\mathcal{L} = -\frac{1}{4} (F^a_{\mu\nu} F^{a\mu\nu}) \tag{6.101}$$

O índice A em $Z_A[J^{\mu a}]$ realça o facto deste funcional corresponder à escolha de gauge axial. Embora a expressão para o funcional gerador se escreva facilmente nesta gauge ela tem a desvantagem das regras de Feynman não serem covariantes. Antes de introduzirmos as gauge covariantes mostraremos ainda outra gauge não covariante a chamada gauge de Coulomb:

- **Gauge de Coulomb**

Esta gauge é definida pelas condições auxiliares

$$\vec{\nabla} \cdot \vec{A}_a = 0 \qquad\qquad a = 1, \ldots, r \tag{6.102}$$

Estas condições auxiliares têm um parêntesis de Poisson não trivial com as ligações $C^a(x)$. De facto (ver problema 2.3)

$$\delta\vec{A}_a = -\frac{1}{g}\int d^3y \left\{\vec{A}_a(x), \alpha^b(y)C_b(y)\right\}_{x_0=y_0} \tag{6.103}$$

logo

$$\left\{\vec{A}_a(x), C_b(y)\right\}_{x_0=y_0} = -g\frac{\delta}{\delta\alpha_b(y)}(\delta\vec{A}_a(x)) \tag{6.104}$$

e

$$\left\{\vec{\nabla}\cdot\vec{A}_a(x), C_b(y)\right\}_{x_0=y_0} = -g\frac{\delta}{\delta\alpha_b(y)}\vec{\nabla}\cdot(\delta\vec{A}_a(x)) \tag{6.105}$$

Como

$$\delta\vec{A}_a(x) = \frac{1}{g}\vec{\nabla}\alpha_a(x) + f_{abc}\alpha^b(x)\vec{A}^c(x) \tag{6.106}$$

obtemos (com a condição $\vec{\nabla}\cdot\vec{A}_a = 0$)

$$-g\vec{\nabla}\cdot(\delta\vec{A}_a(x)) = -\nabla_x^2\alpha_a(x) - gf_{abc}\vec{A}_c(x)\cdot\vec{\nabla}\alpha_b(x) \tag{6.107}$$

e finalmente

$$\begin{aligned}\left\{\vec{\nabla}\cdot\vec{A}_a(x), C_b(y)\right\} &= \left[-\nabla_x^2\delta_{ab} - gf_{abc}\vec{A}_c(x)\cdot\vec{\nabla}_x\right]\delta^3(\vec{x}-\vec{y}) \\ &\equiv \mathcal{M}^c_{ab}(x,y)\end{aligned} \tag{6.108}$$

Como $\det\mathcal{M}$ embora dependendo de $\vec{A}$ não depende de $\vec{E}$, a integração gaussiana em $\vec{E}$ pode ainda ser feita e obtemos

$$Z_C[J^{\mu a}] = \int \mathcal{D}(A_\mu)\prod_x[\det\mathcal{M}_C\prod_x\delta(\vec{\nabla}\cdot\vec{A})]e^{i\int d^4x[\mathcal{L}+A^a\cdot J^a]} \tag{6.109}$$

Agora não é possível absorver $\det\mathcal{M}$ na normalização. As regras de Feynman que podem ser obtidas a partir de $Z_C[J^\mu]$ também não são covariantes.

6.2.4 Non abelian gauge theories in covariant gauges

As condições de gauge escolhidas até aqui (gauge axial e gauge de Coulomb) conduzem a regras de Feynman onde a covariância de Lorentz é perdida. Claro que os resultados finais não devem depender desta escolha, mas a não covariância dos cálculos intermédios é normalmente uma complicação. Vamos aqui generalizar os resultados anteriores a gauges covariantes. O método a seguir surgirá como um subproduto da resposta a um outro problema: Como mostrar a equivalência das gauges axial e de Coulomb?

Para o argumento que se segue é conveniente trabalhar com quantidades invariantes de gauge. Assim em vez do funcional $Z_A[J^\mu]$ vamos por agora considerar o integral $Z_A[J=0]$ que como vimos tem o significado duma amplitude transição vácuo $\rightarrow$ vácuo na ausência das fontes exteriores,

$$Z_A[0] = \int \mathcal{D}(A_\mu) \prod_{x,a} \delta(A^{3a}(x)) \exp\{iS[A_\mu]\} \tag{6.110}$$

onde $S[A_\mu]$ é a acção. Numa transformação de gauge

$$\underset{\sim}{A}_\mu \to \underset{\sim}{A}'_\mu = {}^g\!\underset{\sim}{A}_\mu = U(g)\underset{\sim}{A}_\mu U^{-1}(g) - \frac{i}{g}\partial_\mu U U^{-1} \tag{6.111}$$

A acção $S[A_\mu]$ e a medida $\mathcal{D}(A_\mu)$ são invariantes, pelo que obtemos

$$Z_A(J=0) = \int \mathcal{D}(A_\mu) \prod_{x,a} \delta({}^g\!A^{3a}(x)) \exp\{iS[A_\mu]\}\ . \tag{6.112}$$

Definimos agora o funcional $\Delta_C[A_\mu]$ através da relação

$$\Delta_C^{-1}(A_\mu) = \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla} \cdot {}^g\!\vec{A}^a) \tag{6.113}$$

onde $\mathcal{D}(g)$ representa o produto infinito de medidas invariantes para o grupo G em cada ponto, isto é

$$\mathcal{D}(g) = \prod_x dg(x)\ . \tag{6.114}$$

A invariância da medida da integração do grupo G, $dg' = d(gg')$ tem como consequência que Δ_C é invariante de gauge. De facto

$$\begin{aligned}
\Delta_C^{-1}({}^g\!A_\mu) &= \int \mathcal{D}(g') \prod_{x,a} \delta(\vec{\nabla} \cdot {}^{g'g}\!\vec{A}^a) \\
&= \int \mathcal{D}(g'g) \prod_{x,a} \delta(\vec{\nabla} \cdot {}^{g'g}\!\vec{A}^a) \\
&= \Delta_C^{-1}[A_\mu]
\end{aligned} \tag{6.115}$$

Introduzimos agora na expressão de $Z_A[J=0]$ a identidade

$$1 = \Delta_C[A_\mu] \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla} \cdot {}^g\vec{A}^a) \tag{6.116}$$

Obtemos então

$$\begin{aligned}
Z_A(J=0) &= \int \mathcal{D}A_\mu e^{iS[A_\mu]} \prod_{x,a} \delta\left(A^{3a}(x)\right) \Delta_C[A_\mu] \int \mathcal{D}(g) \prod_{y,b} \delta(\vec{\nabla} \cdot {}^g\!\vec{A}^b) \\
&= \int \mathcal{D}A_\mu e^{iS[A_\mu]} \Delta_C[A_\mu] \prod_{y,b} \delta\left(\vec{\nabla} \cdot \vec{A}^b\right) \int \mathcal{D}(g) \prod_{x,a} \delta({}^{g^{-1}}\!A^{3a})
\end{aligned} \tag{6.117}$$

onde usámos a invariância de $\mathcal{D}$, $S[A_\mu]$ e $\Delta_C[A_\mu]$. Como a medida é invariante podemos escrever no último integral $g^{-1} \to g g_0$. Então

$$\int \mathcal{D}(g) \prod_{x,a} \delta\left({}^{g^{-1}}\!A^{3a}(x)\right) = \int \mathcal{D}(g) \prod_{x,a} \delta\left({}^{gg_0}\!A^{3a}(x)\right) \tag{6.118}$$

onde g_0 é a transformação de gauge que leva da gauge $\vec{\nabla} \cdot \vec{A} = 0$ para a gauge $A'^3 = 0$, isto é

$$A'^3 = {}^{g_0}\!A^3 = 0 \tag{6.119}$$

com $\vec{\nabla} \cdot \vec{A} = 0$. Falta-nos portanto calcular o integral sobre o grupo que agora se escreve

$$\int \mathcal{D}(g) \prod_{x,a} \delta\left({}^{g}\!A'^{3a}(x)\right) \tag{6.120}$$

com $A'^{3a} = 0$. Como $A'^{3a} = 0$ basta considerar transformações infinitesimais, na vizinhança da identidade,

$$g(x) = e - i\underset{\sim}{\alpha}(x) = e - i\alpha^a(x) t^a \tag{6.121}$$

onde $\alpha^a(x)$ são infinitesimais. Nestas condições a medida de integração $dg(x)$ vem dada por

$$dg(x) = \prod_a d\alpha^a(x) \tag{6.122}$$

Por outro lado em primeira ordem em α^a temos

$${}^{g}\!A'^{3a}(x) = \frac{1}{g} \frac{\partial \alpha^a}{\partial x^3} \tag{6.123}$$

pelo que o integral virá

$$\int \mathcal{D}(g) \prod_{x,a} \delta\left({}^{g}\!A'^{3a}(x)\right) = \int \mathcal{D}(\alpha) \prod_{x,a} \delta\left(\frac{1}{g}\frac{\partial \alpha^a}{\partial x^3}\right) = \mathcal{N} \tag{6.124}$$

O integral é independente de A_μ pelo que pode ser absorvido na normalização. Obtemos assim

$$Z_A[J=0] = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_C[A_\mu] \prod_x \delta(\vec{\nabla} \cdot \vec{A}) e^{iS[A_\mu]} \tag{6.125}$$

Na secção anterior obtivemos uma expressão para $Z_C[J=0]$, que é

$$Z_C[J=0] = \int \mathcal{D}(A_\mu) \prod_x \det \mathcal{M}_C \prod_x \delta(\vec{\nabla} \cdot \vec{A}) e^{iS[A_\mu]} \tag{6.126}$$

Portanto para mostrar a equivalência dos integrais representando a amplitude *vácuo → vácuo* na ausência de fontes exteriores nas duas gauges consideradas falta-nos mostrar que $\Delta_C[A_\mu] = \det \mathcal{M}_C$. De facto

$$
\begin{aligned}
\Delta_f^{-1}[A_\mu] &= \int \mathcal{D}(g) \prod_{x,a} \delta\left(\vec{\nabla} \cdot {}^g\!\vec{A}^a\right) \\
&= \int \mathcal{D}(\alpha) \prod_{x,a} \delta\left[\vec{\nabla} \cdot \left(\frac{1}{g}\vec{\nabla}\alpha^a(x) + f^{abc}\alpha^b \vec{A}^c\right)\right] \\
&= \int \mathcal{D}(\alpha) \prod_{x,a} \delta\left(\frac{1}{g}\nabla_x^2\alpha^a(x) + f^{abc}\vec{\nabla}\alpha^b \cdot \vec{A}^c\right) \\
&\propto \det{}^{-1}\mathcal{M}_C
\end{aligned} \tag{6.127}
$$

onde

$$
\begin{aligned}
\mathcal{M}_f^{ab}(x,y) &= -g\frac{\delta}{\delta\alpha^b(y)}\left(\vec{\nabla} \cdot {}^g\vec{A}^a\right)_{\alpha=0} \\
&= \left(-\nabla_x^2\delta_{ab} - gf_{abc}\vec{A}_c \cdot \vec{\nabla}_x\right)\delta^3(\vec{x}-\vec{y})
\end{aligned} \tag{6.128}
$$

Portanto $\Delta_C[A_\mu] \propto \det \mathcal{M}_C$ e à parte uma normalização irrelevante $Z_A[0] = Z_C[0]$.

A maneira como se demonstrou a equivalência entre as gauges de Coulomb e axial sugere a forma de definir a amplitude *vácuo* $\rightarrow$ *vácuo* para uma gauge arbitrária definida pela condição

$$
F^a[A_\mu] = 0 \qquad\qquad a = 1, \ldots, r \tag{6.129}
$$

Para isso definimos $\Delta_F[A_\mu]$ pela expressão

$$
\Delta_F^{-1}[A_\mu] = \int \mathcal{D}(g) \prod_{x,a} \delta\left(F^a[{}^g\!A_\mu]\right) \tag{6.130}
$$

e como anteriormente introduzimos

$$
1 = \Delta_F[A_\mu] \int \mathcal{D}(g) \prod_{x,a} \delta\left(F^a[{}^g\!A_\mu]\right) \tag{6.131}
$$

na expressão para $Z_A[J=0]$. Obtemos

$$
\begin{aligned}
Z_A[J=0] &= \int \mathcal{D}(A_\mu) \prod_{x,a} \delta\left(A^{3a}(x)\right) e^{iS[A_\mu]}\Delta_F[A_\mu] \int \mathcal{D}(g) \prod_{y,b} \delta\left(F^b[{}^g A_\mu]\right) \\
&= \int \mathcal{D}(A_\mu) \prod_{y,b} \delta\left(F^b[A_\mu]\right) \Delta_F[A_\mu] e^{iS[A_\mu]} \int \mathcal{D}(g) \prod_{x,a} \delta\left({}^{g^{-1}}A^{3a}(x)\right) \\
&= \mathcal{N} \int \mathcal{D}(A_\mu)\Delta_F[A_\mu] \prod_{x,a} \delta\left(F^b[A_\mu]\right) e^{iS[A_\mu]}
\end{aligned}
$$

$$= \mathcal{N} Z_F[J=0] \tag{6.132}$$

mostrando que as gauges axial e as gauges do tipo F são equivalentes. A amplitude *vácuo* $\rightarrow$ *vácuo* na gauge $F^a = 0$ é portanto dada por

$$Z_F[J=0] = \int \mathcal{D}(A_\mu)\Delta_F[A_\mu]\prod_{x,a}\delta\left(F^a[A_\mu]\right)e^{iS[A_\mu]} \tag{6.133}$$

Falta-nos calcular $\Delta_F[A_\mu]$. Como na definição anterior $\Delta_F[A_\mu]$ aparece multiplicado por $\prod \delta\left(F^a[A_\mu]\right)$, basta-nos conhecer $\Delta_F[A_\mu]$ para A_μ que satisfaz $F^a[A_\mu]=0$. Então para g perto da identidade obtemos

$$\begin{aligned} F^a[{}^g\!A^b_\mu] &= F^a[A^b_\mu] + \frac{\delta F^a}{\delta A^b_\mu}\delta A^b_\mu \\ &= -\frac{1}{g}\,\frac{\delta F^a}{\delta A^b_\mu}\left(D_\mu\alpha\right)^b \end{aligned} \tag{6.134}$$

onde se usou $F^a[A^b_\mu]=0$ e $\delta A^b_\mu = -\frac{1}{g}(D_\mu\alpha)^b$. Calculemos então Δ_F. Obtemos

$$\begin{aligned} \Delta_F^{-1}[A_\mu] &= \int \mathcal{D}(g)\prod_{x,a}\delta\left(F^a[{}^g\!A^b_\mu]\right) \\ &= \int \mathcal{D}(\alpha)\prod_{x,a}\delta\left(-\frac{1}{g}\frac{\delta F^a}{\delta A^b_\mu}(D_\mu\alpha)^b\right) \\ &\propto \det{}^{-1}\mathcal{M}_F \end{aligned} \tag{6.135}$$

onde

$$\mathcal{M}_F^{ab}(x,y) = \frac{\delta F^a}{\delta A^c_\mu(x)}D^{cb}_\mu\delta^4(x-y) = -g\frac{\delta F^a[{}^g A(x)]}{\delta\alpha^b(y)} \tag{6.136}$$

e portanto

$$\Delta_F[A_\mu] = \det\mathcal{M}_F = \det\left(-g\frac{\delta F^a(x)}{\delta(\alpha^b(y))}\right) \tag{6.137}$$

Já sabemos como escrever a amplitude *vácuo* $\rightarrow$ *vácuo* na ausência de fontes exteriores para uma gauge arbitrária. De facto não é esta quantidade a mais interessante, mas sim a amplitude *vácuo* $\rightarrow$ *vácuo* na presença de fontes, $Z_F[J]$ pois será esta que gera as funções de Green. Em toda a discussão até aqui se fizeram as fontes exteriores nulas. A razão é que o termo das fontes, $\int d^4x J^a_\mu A^{\mu a}$, não é invariante de gauge. Se definirmos $Z_F[J^a_\mu]$ pela relação

$$Z_F[J^a_\mu] \equiv \int \mathcal{D}(A_\mu)\Delta_F[A_\mu]\prod_{x,a}\delta(F^a[A^b_\mu(x)])e^{i(S[A_\mu]+\int d^4x J^a_\mu A^{\mu a})} \tag{6.138}$$

então é claro que os funcionais Z_F não serão equivalentes para diferentes escolhas de $F^a = 0$. Isto quer dizer que as funções de Green calculadas a partir de $Z_F[J^\mu]$ vão depender de gauge $F^a = 0$. Na secção 2.2.5 mostraremos que embora as funções de Green dependam da escolha de gauge, este não é um problema importante porque os resultados fisicamente relevantes (mensuráveis) estão relacionados com os elementos da matriz S renormalizada e esta é independente da gauge conforme aí mostraremos.

Antes de acabarmos esta secção façamos uma transformação no funcional $Z_F[J^a_\mu]$ para nos vermos livres da função δ que aí intervém. Para os cálculos é de toda a conveniência exponenciar$\prod \delta(F^a[A_\mu])$. Isto pode fazer-se do seguinte modo. Definamos uma condição de gauge mais geral

$$F^a[A^b_\mu] - c^a(x) = 0 \tag{6.139}$$

onde $c^a(x)$ são funções arbitrárias do espaço-tempo, mas não dependem dos campos. Então $\Delta_F[A]$ não vem alterado e escrevemos

$$Z_F[J^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu)\Delta_F[A_\mu] \prod \delta(F^a[A_\mu] - c^a) e^{i(S[A_\mu] + \int d^4x J^a_\mu A^\mu)} \tag{6.140}$$

O lado esquerdo da equação não depende de $c^a(x)$ pelo que podemos integrar em $c^a(x)$ com um peso conveniente, especificamente com

$$\exp\left\{-\frac{i}{2}\int d^4x\ c^2_a(x)\right\} \tag{6.141}$$

onde x é um parâmetro real. Obtemos então

$$\begin{aligned}
Z_F[J^a_\mu] &= \mathcal{N} \int \mathcal{D}(A_\mu)\Delta_F[A_\mu] e^{i\left(S[A_\mu] + \int d^4x \left(-\frac{1}{2}F^2_a + J^{\mu a} A^a_\mu\right)\right)} \\
&= \mathcal{N} \int \mathcal{D}(A_\mu)\Delta_F[A_\mu] e^{i\int d^4x[\mathcal{L}(x) - \frac{1}{2}F^2_a + J^{\mu a} A^a_\alpha]}
\end{aligned} \tag{6.142}$$

Esta expressão é o ponto de partida para o cálculo das funções de Green numa gauge arbitrária definida pela função F^a. Para sermos capazes de estabelecer as regras de Feynman para esta teoria teremos ainda de exponenciar $\Delta_F[A_\mu]$. Isto será feito numa das secções seguintes com a introdução dos chamados *fantasmas* de Fadeev-Popov.

6.2.5 Gauge invariance of the S matrix

Na secção anterior definimos o funcional gerador das funções de Green $Z_F[J^a_\mu]$, para uma gauge dada pela função $F^a[A^b_\mu]$, através da relação

$$Z_F[J^a_\mu] \equiv \mathcal{N} \int \mathcal{D}(A_\mu)\Delta_F[A] \prod_{x,a} \delta(F^a[A^b_\mu(x)]) e^{i(S[A_\mu] + \int d^4x J^a_\mu A^{\mu a})} \tag{6.143}$$

e mostrámos a equivalência das diferentes gauges quando as fontes eram nulas. Vamos agora mostrar o que acontece quando $J^a_\mu \neq 0$. Para isso vamos refazer a demonstração da equivalência entre duas gauges na presença das fontes. Escolhemos para esta demonstração as gauges de Coulomb e Lorentz[4] definidas por

[4]Depois de estudarmos as identidades de Ward-Takahashi faremos uma demonstração geral (ver secção 6.3.4).

$$\begin{cases} F^a = \vec{\nabla} \cdot \vec{A}^a & \textit{gauge de Coulomb} \\ \\ F^a = \partial_\mu A^{\mu a} & \textit{gauge de Lorentz} \end{cases} \tag{6.144}$$

Definimos então os funcionais geradores $Z_C[j^a_\mu]$ e $Z_L[J^a_\mu]$ por

$$Z_C[j^a_\mu] \equiv \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_c[A] \prod_{x,a} \delta(\vec{\nabla} \cdot \vec{A}^a) e^{i(S[A] + \int d^4x j^a_\mu A^{\mu a})} \tag{6.145}$$

e

$$Z_L[J^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_L[A] \prod_{x,a} \delta(\partial_\mu A^{\mu a}) e^{i(S[A] + \int d^4x J^a_\mu A^{\mu a})} \tag{6.146}$$

Vamos mostrar a relação entre eles. Seguindo os métodos da secção anterior introduzimos em $Z_C[j^a_\mu]$ a identidade dada por

$$1 = \Delta_L[A] \int \mathcal{D}(g) \prod_{x,a} (\partial_\mu {}^g A^{\mu a}) \tag{6.147}$$

e obtemos

$$\begin{aligned} & Z_C[j^a_\mu] \\ & = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_C[A] \textstyle\prod_{x,a} \delta(\vec{\nabla} \cdot \vec{A}^a) e^{i(S[A] + \int d^4x j^a_\mu A^{\mu a})} \Delta_L[A] \int \mathcal{D}(g) \prod_{y,b} \delta(\partial_\mu {}^g A^{\mu a}) \\ & = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_L[A] \textstyle\prod_{y,b} \delta(\partial_\mu A^{\mu b}) e^{iS[A]} \Delta_C[A] \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla} \cdot {}^{g^{-1}}\vec{A}^a) e^{i \int d^4x j^a_\mu {}^{g^{-1}}A^{\mu a}} \\ & = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_L[A] \textstyle\prod_{y,b} \delta(\partial_\mu A^{\mu b}) e^{iS[A]} \Delta_C[A] \int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla} \cdot {}^{gg^0}\vec{A}^a) e^{i \int d^4x j^a_\mu {}^{gg^0}A^{\mu a}} \end{aligned} \tag{6.148}$$

onde g^0 é a transformação de gauge que leva de gauge $\partial_\mu A^{\mu a} = 0$ para a gauge $\vec{\nabla} \cdot \vec{A}'^a = 0$, $\vec{A}'^a = {}^g\vec{A}$ e é portanto obtida resolvendo a equação

$$\vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \left[U(g^0) \vec{A} U^{-1}(g^0) - \frac{i}{g} \vec{\nabla} U(g^0) U^{-1}(g^0) \right] = 0 \tag{6.149}$$

onde $\partial_\mu A^{\mu a} = 0$. Devido ao factor $\prod_x \delta(\vec{\nabla} \cdot {}^g\vec{A}')$ só nos interessam as transformações g infinitesimais pelo que

$$Z_C[j^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_L[A] \prod_{y,b} \delta(\partial_\mu A^{\mu b}) e^{iS[A]} e^{i \int d^4x j^a_\mu {}^{g^0}A^\mu} \tag{6.150}$$

onde se usou o resultado

$$\int \mathcal{D}(g) \prod_{x,a} \delta(\vec{\nabla} \cdot {}^g\vec{A}') = \Delta_C^{-1}[A] \ . \tag{6.151}$$

Para comparar com $Z_L[J^a_\mu]$ é necessário escrever ${}^{g^0}A^\mu$ em função de A^μ, resolvendo a equação para g^0. Isto pode ser feito formalmente em série de potenciais de A^μ. É fácil de ver que devemos ter

$$A'_i = \left(\delta_{ij} - \nabla_i \frac{1}{\nabla^2} \nabla_j \right) A_j + O(A_\lambda^2) \tag{6.152}$$

Se restringirmos a fonte na gauge de Coulomb a ser transversal $j^0 = 0$ e $\vec{\nabla} \cdot \vec{j} = 0$ podemos então escrever

$$Z_C[j^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu) \Delta_L[A] \prod_{y,b} \delta(\partial_\mu A^{\mu b}) e^{iS[A] + \int d^4x F^a_\mu j^{\mu a}} \tag{6.153}$$

onde $F^C_\mu[A] = A^a_\mu + O(A^2_\lambda)$. Comparando com a expressão de $Z_L[J^a_\mu]$ obtemos finalmente

$$Z_C[j^a_\mu] = \exp \left\{ i \int d^4x j^a_\mu F^{\mu a} \left[\frac{\delta}{i\delta J^b} \right] \right\} Z_L[J^a_\mu] \tag{6.154}$$

Esta é a expressão que relaciona Z_C com Z_L. Como $F_\mu[A]$ é um funcional complicado é fácil de ver que as funções de Green nas duas gauges vão ser diferentes. Mas o que tem significado físico (comparável com a experiência) são os elementos de matriz S renormalizada. O teorema da equivalência que a seguir demonstramos mostra que a matriz S renormalizada é invariante de gauge. Por simplicidade demonstraremos o teorema para a teoria $\lambda\phi^4$ mas o raciocínio é análogo para o caso das teorias de gauge.

Teorema 2.2

Se dois funcionais geradores Z e $\tilde{Z}$ diferem somente nos termos das fontes exteriores então eles conduzem à mesma matriz S renormalizada.

Dem. *Consideremos o funcional gerador das funções de Green*

$$Z[J] = \mathcal{N} \int \mathcal{D}(\phi) e^{i(S[\phi] + \int d^4x J\phi)} \tag{6.155}$$

onde

$$S[\phi] + \int d^4x J\phi = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + J\phi \right] \ . \tag{6.156}$$

Que acontece se acoplarmos a fonte exterior a $\phi + \phi^3$ em vez de ser somente a ϕ ? Podemos escrever o funcional gerador $\tilde{Z}[j]$ dado por

$$\tilde{Z}[j] = \mathcal{N} \int \mathcal{D}(\phi) e^{i[S[\phi] + \int d^4x j(\phi + \phi^3)]} \tag{6.157}$$

e podemos escrever $\tilde{Z}[j]$ em termos de $Z[J]$,

$$\tilde{Z}[j] = \exp \left\{ i \int d^4x j(x) F \left[\frac{\delta}{i\delta J} \right] \right\} Z[J] \tag{6.158}$$

onde $F[\phi] = \phi + \phi^3$. Consideremos a função de 4- pontos, $\tilde{G}(1,2,3,4)$ gerada por $\tilde{Z}[j]$

$$\tilde{G}(1,2,3,4) = (-i)^4 \frac{\delta^4 \tilde{Z}[j]}{\delta j(1)\delta j(2)\delta j(3)\delta j(4)} \tag{6.159}$$

Um diagrama típico que contribui para $\tilde{G}(1,2,3,4)$ é o da Figura 6.1, onde a parte do diagrama dentro do quadrado a tracejado é uma função de Green gerada por $Z[J]$.

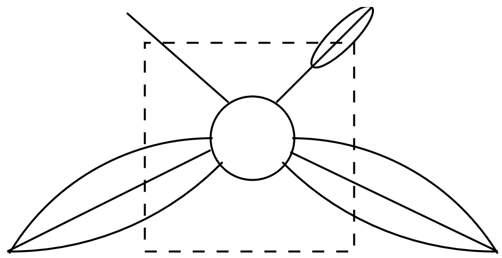
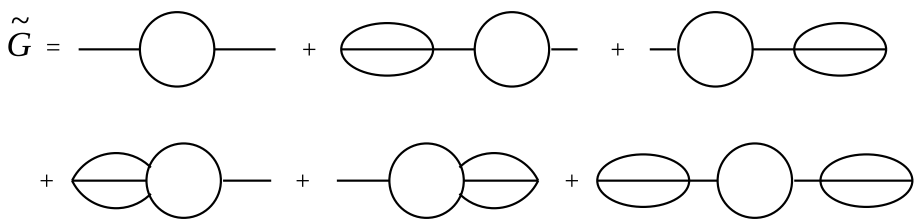


Figure 6.1: Green functions generated by $Z[J]$ and $\tilde{Z}[j]$.

Consideremos agora os propagadores $\tilde{G}(1,2)$ e $G(1,2)$ gerados por $\tilde{Z}[j]$ e $Z[J]$ respectivamente. Obtemos a seguinte expansão de $\tilde{G}(1,2)$ em termos de $G(1,2)$

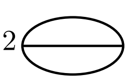
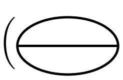


(6.160)

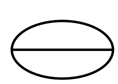
Se examinarmos os propagadores junto do pólo na massa física, obtemos (Z_2 e $\tilde{Z}_2$ são as constantes de renormalização nos dois esquemas)

$$\lim_{p^2 \to m_R^2} \tilde{G} = \frac{i\tilde{Z}_2}{p^2 - m_R^2} \qquad ; \qquad \lim_{p^2 \to m_R^2} G = \frac{iZ_2}{p^2 - m_R^2} \ . \tag{6.161}$$

Então multiplicando a expansão da equação 6.160 por $p^2 - m_R^2$ e tomando o limite $p^2 \to m_R^2$ obtemos

$\tilde{Z}_2 = Z_2 \Big[1 + 2$  $+ \Big($  $\Big)^2 + \cdots \Big]$ (6.162)

ou seja

$\sigma \equiv \left(\frac{\tilde{Z}_2}{Z_2} \right)^{1/2} = 1 +$  $+ \cdots$ (6.163)

A matriz S não renormalizada é dada por

$$S^{\mathrm{NR}}(k_1,\ldots,k_n)=\prod_{i=1}^{n}\lim_{k_i^2\to m_R^2}(k_i^2-m_R^2)G(k_1,\ldots,k_n) \tag{6.164}$$

para as funções de Green calculadas a partir de $Z[J]$. Definimos de igual modo

$$\tilde{S}^{\mathrm{NR}}(k_1,\ldots,k_n)=\prod_{i=1}^{n}\lim_{k_i^2\to m_R^2}(k_i^2-m_R^2)\tilde{G}(k_1,\ldots,k_n) \tag{6.165}$$

para as funções de Green calculadas a partir do funcional $\tilde{Z}[j]$, sendo n o número de partículas exteriores. Do argumento usado para relacionar $\lim(k^2-m_R^2)\tilde{G}(k_1,\ldots,k_n)$ com $\lim(k^2-m_R^2)G(k_1,\ldots,k_n)$ é fácil de ver que para relacionar $\prod\lim(k_i^2-m_R^2)\tilde{G}$ com $\prod\lim(k_i^2-m_R^2)G$ somente contribuem os diagramas que tiverem pólos em todas as variáveis k_i^2. Então

$$\begin{aligned}\prod_{i=1}^{n}\lim_{k_i^2\to m_R^2}(k_i^2-m_R^2)\tilde{G} &= \left(\frac{Z}{\tilde{Z}}\right)^{-n}\prod_{i=1}^{n}\lim_{k_i^2\to m_R^2}(k_i^2-m_R^2)G\\ &= \sigma^{\frac{n}{2}}\prod_{i=1}^{n}\lim_{k_i^2\to m_R^2}(k_i^2-m_R^2)G\end{aligned} \tag{6.166}$$

Portanto obtemos uma relação entre os elementos da matriz S não normalizada

$$\tilde{S}^{\ \mathrm{NR}}=\sigma^{\frac{n}{2}}S^{\ \mathrm{NR}} \tag{6.167}$$

ou ainda

$$\frac{1}{\tilde{Z}^{\frac{n}{2}}}\tilde{S}^{\mathrm{NR}}(k_1,\ldots,k_n)=\frac{1}{Z^{\frac{n}{2}}}S^{\mathrm{NR}}(k_1,\ldots,k_n) \tag{6.168}$$

Mas $\frac{1}{Z^{\frac{n}{2}}}S^{\mathrm{NR}}(k_1,\ldots,k_n)$ é precisamente a definição da matriz S renormalizada pelo que

$$\tilde{S}^{\mathrm{R}}=S^{\mathrm{R}}\ . \tag{6.169}$$

Concluímos assim que dois funcionais geradores que defiram pelo acoplamento à fonte exterior produzem os mesmos elementos de matriz da matriz S renormalizada. Isto completa a demonstração do teorema da equivalência.

A aplicação deste resultado ao nosso caso é agora imediata pois

$$Z_C[j_\mu^a]=\exp\left\{i\int d^4x j_\mu^a F^{\mu a}\left[\frac{\delta}{i\delta J_x}\right]\right\}Z_L[J_\mu^c] \tag{6.170}$$

onde $F_\mu^a[A]=A_\mu^a+O(A_\lambda^2)$. A diferença entre $Z_C[j_\mu]$ e $Z_L[J_\mu]$ reside no acoplamento à fonte exterior, pelo que embora as funções de Green dependam da gauge, a matriz S renormalizada deverá ser invariante.

6.2.6 Fadeev-Popov ghosts

Tendo demonstrado a invariância de gauge de matriz S renormalizada, voltemos ao funcional gerador numa gauge arbitrária definida pela condição $F^a[A^b_\mu]$. No final de secção 6.2.4 tínhamos escrito este funcional na forma

$$Z_F[J^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu)\Delta_F[A] e^{i\int d^4x[\mathcal{L}(x) - \frac{1}{2\xi}(F^a)^2 + J^a_\mu A^{\mu a}]} \tag{6.171}$$

onde

$$\Delta_F[A] = \det \mathcal{M}_F = \det \left(-g\frac{\delta F^a(x)}{\delta \alpha^b(y)} \right) \tag{6.172}$$

Nesta forma as regras de Feynman são complicadas porque o $\det \mathcal{M}_F$ conduz a interacções não locais entre os campos de gauge. Se de alguma forma pudéssemos exponenciar $det\mathcal{M}_F$ e metê-lo numa acção efectiva teríamos o nosso problema resolvido.

Ora no nosso estudo dos integrais gaussianos sobre variáveis de Grassman obtivemos o resultado

$$\int \mathcal{D}(\overline{\omega}, \omega) e^{-\int d^4x \overline{\omega} \mathcal{M}_F \omega} = \det \mathcal{M}_F \tag{6.173}$$

usando este resultado e mudando por conveniência $\mathcal{M}_F \to i\mathcal{M}_F$ (uma mudança na normalização irrelevante) obtemos

$$Z_F[J^a_\mu] = \mathcal{N} \int \mathcal{D}(A_\mu, \overline{\omega}, \omega) e^{i\int d^4x[\mathcal{L}_{eff} + J^a_\mu A^{\mu a}]} \tag{6.174}$$

onde $\overline{\omega}$ e ω são campos escalares anticomutativos e o $\mathcal{L}_{eff}$ é definido por

$$\mathcal{L}_{eff} = \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_G \tag{6.175}$$

onde

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \\ \mathcal{L}_{GF} &= -\frac{1}{2\xi}(F^a)^2 \\ \mathcal{L}_G &= -\overline{\omega}^a \mathcal{M}^{ab}_F \omega^b \end{aligned} \tag{6.176}$$

Os campos ω e $\overline{\omega}$ são campos auxiliares não físicos e chamam-se *fantasmas de Fadeev-Popov*. Como não são físicos não há problema com o teorema que relaciona o spin com a estatística[5].

Calculemos agora duma forma mais explícita o Lagrangeano do fantasmas. Como

$$\mathcal{M}^{ab}_F(x,y) = -g\frac{\delta F^a(x)}{\delta \alpha^b(y)} = \frac{\delta F^a[A(x)]}{\delta A^c_\mu(y)} D^{cb}_\mu \tag{6.177}$$

[5] Campos físicos com spin inteiro são bosões (comutativos) e campos físicos com spin semi- inteiro são fermiões (anticomutativos).

obtemos

$$\int d^4x d^4y \overline{\omega}^a(x) \mathcal{M}_F^{ab}(x,y) \omega^b(y) = \int d^4x \int d^4y \; \overline{\omega}^a(x) \frac{\delta F^a(x)}{\delta A_\mu^c(y)} D_\mu^{cb} \omega_b(y) \tag{6.178}$$

ou seja

$$\mathcal{L}_G(x) = -\int d^4y \; \overline{\omega}^a(x) \frac{\delta F^a(x)}{\delta A_\mu^b(y)} D_\mu^{bc} \omega_c(y) \tag{6.179}$$

Para termos uma forma mais explícita temos que especificar a gauge. Na gauge de Lorentz $F^a = \partial_\mu A^{\mu a}$ e portanto

$$\begin{aligned} \mathcal{L}_G(x) &= -\int d^4y \overline{\omega}^a(x) \partial_x^\mu \left[\delta^4(x-y)\right] D_\mu^{ab} \omega^b(y) \\ &= \partial^\mu \overline{\omega}^a(x) D_\mu^{ab} \omega^b(x) \end{aligned} \tag{6.180}$$

onde se efectuou uma integração por partes e a derivada covariante na representação adjunta[6] é

$$D_\mu^{ab} = \partial_\mu \delta^{ab} - g f^{abc} A_\mu^c \tag{6.181}$$

6.2.7 Feynman rules in the Lorenz gauge

Estamos agora em posição de escrever as regras de Feynman para calcular, em teoria das perturbações, qualquer processo que envolva partículas cujas interacções possam ser descritas por uma teoria de gauge não abeliana referente a um dado grupo de simetria. Todo o nosso trabalho até aqui se pode resumir na procura do Lagrangeano efectivo a partir do qual as regras de Feynman podem ser obtidas como se se tratasse duma teoria normal sem graus de liberdade a mais. O nosso Lagrangeano efectivo é, como vimos, dado por

$$\mathcal{L}_{eff} = \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_G \tag{6.182}$$

onde

$$\begin{aligned} &\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \qquad ; \qquad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\nu^a + g f^{bca} A_\mu^b A_\nu^c \\ &\mathcal{L}_{GF} = -\frac{1}{2\xi} (F_a)^2 \\ &\mathcal{L}_G = -\overline{\omega}^a \int d^4y \frac{\delta F^a}{\delta A_\mu^b} D_\mu^{bc} \omega_c \end{aligned} \tag{6.183}$$

As constantes f^{abc} são definidas pela comutação dos geradores do grupo, sendo as nossas convenções

[6] Os fantasmas, tal como os campos de gauge estão na representação adjunta do grupo G.

$$[t^a, t^b] = i f^{abc} t^c$$
$$Tr(t^a t^b) = \frac{1}{2}\delta^{ab} \tag{6.184}$$

Para fixar ideias vamos considerar a gauge de Lorentz definida por

$$F^a[A] = \partial_\mu A^{\mu a}(x)\ . \tag{6.185}$$

Obtemos então

$$\mathcal{L}_{eff} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 + \partial^\mu \overline{\omega}^a D^{ab}_\mu \omega^b \tag{6.186}$$

onde

$$D^{ab}_\mu \omega^b = (\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu)\omega^b \tag{6.187}$$

e usámos o facto de que os fantasmas se encontram na representação adjunta pelo que

$$(D_\mu \omega)^a = \left(\partial_\mu \delta^{ab} - i g A^c_\mu (T^c)^{ab}\right)\omega^b \tag{6.188}$$

com

$$(T^c)^{ab} \equiv -i f^{bca} = -i f^{abc} \tag{6.189}$$

Podemos escrever portanto

$$\mathcal{L}_{eff} = \mathcal{L}_{cin} + \mathcal{L}_{int} \tag{6.190}$$

Onde

$$\begin{aligned}
\mathcal{L}_{cin} &= -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 + \partial_\mu \overline{\omega}^a \partial^\mu \omega^a \\
&= \frac{1}{2} A^{\mu a}\left[\Box g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial_\mu \partial_\nu\right]\delta^{ab} A^{\nu b} - \overline{\omega}^a \Box\, \delta^{ab}\omega^b
\end{aligned} \tag{6.191}$$

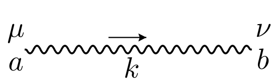
onde se desprezaram divergências totais. O Lagrangeano de interacção é

$$\mathcal{L}_{int} = -g f^{abc}\partial_\mu A^a_\nu A^{\mu b} A^{\nu c} - \frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} + g f^{abc}\partial^\mu \overline{\omega}^a A^b_\mu \omega^c\ . \tag{6.192}$$

Com as convenções usuais (ver capítulo 5) obtemos as seguintes regras de Feynman

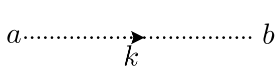
• **Propagadores:**

i) Campos de gauge



$$-i\delta_{ab}\left[\frac{g^{\mu\nu}}{k^2 + i\epsilon} - (1-\xi)\frac{k^\mu k^\nu}{(k^2+i\epsilon)^2}\right] \tag{6.193}$$

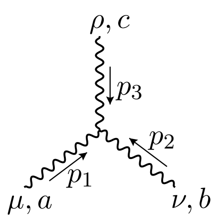
ii) Fantasmas



$$\frac{i}{k^2+i\epsilon}\,\delta_{ab} \tag{6.194}$$

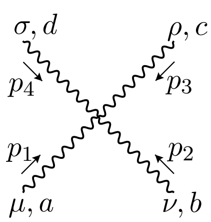
- **Vértices:**

i) Vértice triplo dos bosões de gauge



$$\begin{aligned} -gf^{abc}[\quad & g^{\mu\nu}(p_1-p_2)^\rho + g^{\nu\rho}(p_2-p_3)^\mu \\ & +g^{\rho\mu}(p_3-p_1)^\nu] \\ & p_1+p_2+p_3=0 \end{aligned} \tag{6.195}$$

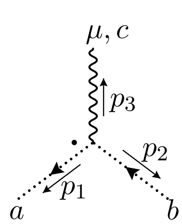
ii) Vértice quártico dos bosões de gauge



$$\begin{aligned} -ig^2\Big[\quad & f_{eab}f_{ecd}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}) \\ & +f_{eac}f_{edb}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\nu}g_{\rho\sigma}) \\ & +f_{ead}f_{ebc}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\rho}g_{\nu\sigma})\Big] \end{aligned} \tag{6.196}$$

$$p_1+p_2+p_3+p_4=0$$

iii) Interacção Fantasmas-Bosões de Gauge



$$\begin{aligned} & g\ f^{abc}p_1^\mu \\ & p_1 \quad +p_2+p_3=0 \end{aligned} \tag{6.197}$$

Notas:

1. O ponto no vértice entre os fantasmas e os bosões de gauge refere-se à perna que tem a derivada e que corresponde à linha de saída (as linhas dos fantasmas são orientadas)

2. As outras regras são as usuais não esquecendo o sinal $-$ por cada *loop* de fantasmas.

6.2.8 Feynamn rules for the interaction with matter

Na secção anterior vimos as regras de Feynman para a teoria de gauge pura, sem interacção com a matéria. A interacção com a matéria faz-se da forma habitual passando as derivadas usuais a derivadas covariantes. Em geral a matéria é descrita por partículas escalares

$$\phi_i \quad ; \quad i = 1, ...M \tag{6.198}$$

e partículas spinoriais

$$\psi_j \quad ; \quad j = 1, ...N \tag{6.199}$$

pertencendo a representações de dimensão M e N, respectivamente. O Lagrangeano será dado por

$$\begin{aligned}
\mathcal{L}_{\text{matéria}} &= (D_\mu\phi)^\dagger D^\mu\phi - m_\phi^2\phi^\dagger\phi - V(\phi) \\
&\quad + i\overline{\psi}D^\mu\gamma_\mu\psi - m_\psi\overline{\psi}\psi \\
&\equiv \mathcal{L}_{cin} + \mathcal{L}_{int} \ .
\end{aligned} \tag{6.200}$$

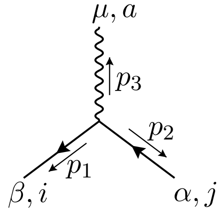
O Lagrangeano de interacção entre a matéria e os campos de gauge obtém-se facilmente a partir da derivada covariante

$$D^\mu_{ij} = \partial_\mu\delta_{ij} - igA^a_\mu T^a_{ij} \tag{6.201}$$

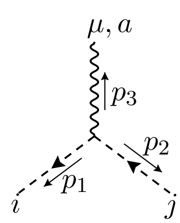
onde T^a_{ij} são os geradores nas representações adequadas para os campos ϕ e ψ. Assim obtemos

$$\begin{aligned}
\mathcal{L}_{int} &= ig\phi_i^*(\overrightarrow{\partial} - \overleftarrow{\partial})^\mu\phi_j T^a_{ij}A_{\mu a} + g^2\phi_i^* T^a_{ij}T^b_{jk}\phi_k A^a_\mu A^{\mu b} \\
&\quad + g\overline{\psi}_i\gamma^\mu\psi_j T^a_{ij}A^a_\mu
\end{aligned} \tag{6.202}$$

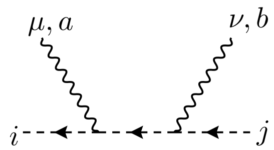
o que conduz aos seguintes vértices



$$ig(\gamma^\mu)_{\beta\alpha}T^a_{ij} \tag{6.203}$$



$$ig(p_1 - p_2)^\mu T^a_{ij} \tag{6.204}$$



$$ig^2 g_{\mu\nu}\{T^a, T^b\}_{ij} \tag{6.205}$$

• *Factores do Grupo*

Os factores f^{abc} e T^a_{ij} que aparecem nos vértices não precisam de facto de ser conhecidos. Nos cálculos aparecem, como veremos, combinações daqueles factores que podem ser expressas em termos de quantidades invariantes que caracterizam o grupo e a representação. Por conveniência resumimos aqui os resultados mais usados.

Os nossos geradores são hermitícos ($T^{a+} = +T^c$) satisfazendo as relações

$$[T^a, T^b] = if^{abc}T^c$$
$$Tr(T^aT^b) = \delta^{ab}T(R) \tag{6.206}$$

onde $T(R)$ é um número caracterizando a representação R. Outra quantidade frequentemente usada é o operador de Casimir da representação definido por

$$\sum_{a,k} T^a_{ik}T^a_{kj} = \delta_{ij}C_2(R) \tag{6.207}$$

Para a representação adjunta obtemos

$$f^{acd}f^{bcd} = \delta^{ab}C_2(G) \ . \tag{6.208}$$

$T(R)$ e $C_2(R)$ não são independentes obedecendo à relação

$$T(R)r = d(R)C_2(R) \tag{6.209}$$

onde r é a dimensão do grupo G e $d(R)$ é a dimensão da representação R.
Em muitas aplicações estamos interessados em grupos $SU(N)$. Para estes temos os seguintes resultados

$$r = N^2 - 1 \ ; \ d(N) = N \ ; \ d(adj) \equiv d(G) = r \tag{6.210}$$

$$T(N) = \frac{1}{2} \ ; \ C_2(N) = \frac{N^2-1}{2N} \tag{6.211}$$

$$T(G) = C_2(G) = N \tag{6.212}$$

• *Factores de Simetria*

Para o cálculo de diagramas envolvendo partículas idênticas é necessário multiplicar o resultado de aplicar as regras de Feynman por um factor de simetria conveniente. Estes factores de simetria foram discutidos na secção 5.5.5 . Por conveniência reproduzimos aqui a regra lá deduzida. O *factor de simetria* é o # de maneiras diferentes em que as linhas podem ser ligadas com o mesmo resultado topológico a dividir pelos factores permutacionais dos vértices e pelo número de permutações de pontos com vértices iguais.

6.3 Ward identities

6.3.1 BRS transformation

Vamos aqui deduzir as *identidades de Ward*[7] para as teorias de gauge não abelianas. O método mais conveniente é o chamado método das transformações de Becchi, Rouet e Stora (BRS) que já introduzimos no capítulo 5 para o caso de QED. As transformações de BRS são uma generalização das transformações de gauge que tornam invariante a acção efectiva.

Como vimos para uma teoria de gauge não abeliana a acção efectiva é dada por ($A =$ campos de gauge ; $\phi =$ campos de matéria)

$$S_{eff}[A,\phi] = S[A,\phi] - \frac{1}{2\xi}\int d^4x F_a^2[A,\phi] - \int d^4x \overline{\omega}^a \mathcal{M}_{ab}\omega^b \tag{6.213}$$

onde $S[A,\phi]$ é a acção clássica, invariante para transformações de gauge

$$\begin{aligned} \delta A_\mu^a &= -\frac{1}{g} D_\mu^{ab}\alpha^b \\ \delta\phi_i &= -i(T^a)_{ij}\phi_j\alpha^a \ , \end{aligned} \tag{6.214}$$

$F_a[A,\phi]$ são as condições de gauge e o operador $\mathcal{M}_{ab}$ é tal que

$$\mathcal{M}_{ab}\omega^b = \frac{\delta F_a}{\delta A_\mu^c} D_\mu^{cb}\omega^b + \frac{\delta F_a}{\delta\phi_i} ig(T^b)_{ij}\phi_j\omega^b \ . \tag{6.215}$$

S_{eff} não é invariante para as transformações de gauge devido à não invariância do termo que fixa a gauge e do Lagrangeano dos fantasmas. Esta não invariância pode desaparecer se escolhermos transformações apropriadas para os fantasmas para compensar a não invariância do termo $\int d^4xF_a^2$. Estas transformações são

$$\left\{ \begin{aligned} \delta_{\rm BRS} A_\mu^a &= D_\mu^{ab}\omega^b\theta \\ \delta_{\rm BRS}\phi_i &= ig(T^b)_{ij}\phi_j\omega^b\theta \\ \delta_{\rm BRS}\overline{\omega}^a &= \tfrac{1}{\xi}F_a[A,\phi]\theta \\ \delta_{\rm BRS}\omega^a &= \tfrac{1}{2} g f^{abc}\omega^b\omega^c\theta \end{aligned} \right. \tag{6.216}$$

onde θ é um parâmetro anticomutativo independente do ponto do espaço-tempo *(variável de Grassman)*. Vemos que as transformações BRS para os campos A_μ^a e ϕ_i são transformações de gauge com parâmetro $\alpha^a(x) = -g\omega^a(x)\theta$. Notar que o carácter anticomutativo de θ é necessário para que o produto $\omega^a\theta$ tenha um carácter bosónico (comutativo).

Para demonstrar a invariância de $S_{eff}[A,\phi]$ vamos demonstrar uma série de teoremas que são necessários para a prova geral. Antes é no entanto conveniente introduzir o operador de Slavnov s, definido pelas relações seguintes,

[7] Designamos pelo nome genérico de identidades de Ward as identidades que foram descobertas por Ward, Takahashi, Slavnov e Taylor.

$$\begin{aligned} \delta_{\mathrm{BRS}} A_\mu^a &= s A_\mu^a \theta \qquad & \delta_{\mathrm{BRS}} \omega^a &= s\omega^a \theta \\ \delta_{\mathrm{BRS}} \phi_i &= s\phi_i \theta \qquad & \delta_{\mathrm{BRS}} \overline{\omega}^a &= s\overline{\omega}^a \theta \end{aligned} \tag{6.217}$$

Este operador é distributivo em relação à multiplicação verificando-se as relações seguintes

$$\begin{aligned} &s(B_1 B_2) = sB_1 B_2 + B_1 s B_2 \\ &s(F_1 B_2) = sF_1 B_2 + F_1 s B_2 \\ &s(B_1 F_2) = -sB_1 F_2 + B_1 s F_2 \\ &s(F_1 F_2) = -sF_1 F_2 + F_1 s F_2 \end{aligned} \tag{6.218}$$

que podem ser demonstradas a partir da definição.

Teorema 2.3

O operador s é nilpotente nos campos A_μ^a, ϕ_i e ω^a, isto é $s^2 A_\mu^a = s^2\phi_i = s^2\omega^a = 0$.

Dem.

Demonstremos para cada um dos casos. Obtemos

a) $s^2 A_\mu^a = 0$

$$\begin{aligned} s^2 A_\mu^a &= s(D_\mu^{ab}\omega^b) = -\frac{\delta D_\mu^{ab}}{\delta A_\nu^c} s A_\nu^c \omega^b + D_\mu^{ab} s\omega^b \\ &= -\delta_\mu^\nu(-gf^{abc}) D_\nu^{cd}\omega^d\omega^b + \frac{1}{2} g f^{bcd} D_\mu^{ab}(\omega^c\omega^d) \\ &= \left[g f^{abc}\partial_\mu\omega^c\omega^b + \frac{1}{2} g f^{acd}\partial_\mu\omega^c\omega^d + \frac{1}{2} g f^{acd}\omega^c\partial_\mu\omega^d \right] \\ &\quad + \left[g f^{abc}(-g) f^{cde} A_\mu^e\omega^d\omega^b + \frac{1}{2} g(-g) f^{bcd} f^{abe} A_\mu^e\omega^c\omega^d \right] \\ &= (g f^{abc}\partial_\mu\omega^c\omega^b - g f^{abc}\partial_\mu\omega^c\omega^b) \\ &\quad -\frac{1}{2} g^2 (f^{abc} f^{cde} - f^{adc} f^{cbe} + f^{cdb} f^{ace}) A_\mu^e\omega^d\omega^b \\ &= 0 \end{aligned} \tag{6.219}$$

onde se usou a identidade de Jacobi e a antisimetria das constantes de estrutura do grupo.

b) $s^2\phi_i = 0$

$$s^2\phi_i = s\left[ig(T^a)_{ij}\phi_j\omega^a \right]$$

$$
\begin{aligned}
&= -ig(T^a)_{ij} s\phi_j \omega^a + ig(T^a)_{ij}\phi_j s\omega^a \\
&= g^2(T^a)_{ij}(T^b)_{jk}\phi_k\omega^b\omega^a + ig(T^a)_{ij}\phi_j \frac{1}{2} g f^{abc}\omega^b\omega^c \\
&= \frac{1}{2} g^2 [T^c, T^b]_{ik}\omega^b\omega^c\phi_k + \frac{i}{2} g^2 (T^a)_{ij}\phi_j f^{abc}\omega^b\omega^c \\
&= \frac{i}{2} g^2 (T^a)_{ij}\phi_j (f^{acb} + f^{abc})\omega^b\omega^c \\
&= 0
\end{aligned} \tag{6.220}
$$

c) $s^2\omega^a = 0$

$$
\begin{aligned}
s^2\omega^a &= s\left(\frac{1}{2} g f^{abc}\omega^b\omega^c\right) \\
&= -\frac{1}{2} g f^{abc} s\omega^b\omega^c + \frac{1}{2} g f^{abc}\omega^b s\omega^c \\
&= -g f^{abc} s\omega^b\omega^c \\
&= -\frac{1}{2} g^2 f^{abc} f^{bef}\omega^e\omega^t\omega^c \\
&= -\frac{1}{6} g^2 (f^{abc} f^{bef} + f^{abe} f^{bfc} + f^{abf} f^{bce})\omega^e\omega^t\omega^c \\
&= 0
\end{aligned} \tag{6.221}
$$

onde se usou a anticomutatividade dos fantasmas e a identidade de Jacobi.

Fica assim demonstrado o teorema 2.3. Para gauges lineares pode-se demonstrar um resultado importante a que daremos a forma de teorema.

Teorema 2.4

Para gauges lineares o operador de Slavnov verifica a relação

$$
s(\mathcal{M}_{ab}\omega^b) = 0 \tag{6.222}
$$

Dem:

Vimos anteriormente que

$$
\mathcal{M}_{ab}\omega^b(x) = \int d^4y \left[\frac{\delta F_a(x)}{\delta A^c_\mu(y)} D^{cb}_\mu \omega^b(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} ig(T^b)_{ij}\phi_j\omega^b(y)\right] \tag{6.223}
$$

Se usarmos as definições de $\delta_{\rm BRS}$ *e do operador de Slavnov podemos escrever*

$$
\mathcal{M}_{ab}\omega^b(x) = \int d^4y \left[\frac{\delta F_a(x)}{\delta A^c_\mu(y)} sA^c_\mu(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} s\phi_i(y)\right] \tag{6.224}
$$

Se a gauge for linear $\frac{\delta F_a}{\delta A_\mu^c}$ *e* $\frac{\delta F_a}{\delta \phi_i}$ *não dependem dos campos e portanto*

$$s\left[\mathcal{M}_{ab}\omega^b(x)\right] = \int d^4y \left[\frac{\delta F_a(x)}{\delta A_\mu(y)} s^2 A_\mu^c(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} s^2 \phi_i(y)\right] = 0 \tag{6.225}$$

onde se usaram os resultados do teorema 2.3.

Usando os teoremas 2.3 e 2.4 podemos agora mostrar que a acção efectiva é invariante para transformações de BRS. Vamos apresentar este resultado também sobre a forma de teorema.

Teorema 2.5

A acção S_{eff} *é invariante para as transformações de BRS.*

Dem.

A acção efectiva é

$$S_{eff}[A,\phi] = S[A,\phi] + \int d^4x \left[-\frac{1}{2\xi} F_a^2[A,\phi] - \overline{\omega}^a \mathcal{M}_{ab}\omega^b\right] \tag{6.226}$$

Como a acção clássica é invariante para transformações de gauge devemos ter

$$s\left(S[A,\phi]\right) = 0 \; . \tag{6.227}$$

Para os outros termos obtemos

$$s\left(-\frac{1}{2\xi}F_a^2 - \overline{\omega}^a \mathcal{M}_{ab}\omega^b\right) = -\frac{1}{\xi} F_a s F_a + s\overline{\omega}^a \mathcal{M}_{ab}\omega^b - \overline{\omega}^a s(\mathcal{M}_{ab}\omega^b) \tag{6.228}$$

Mas

$$sF_a(x) = \int d^4y \left[\frac{\delta F_a}{\delta A_\mu^b(y)} s A_\mu^b(y) + \frac{\delta F_a}{\delta \phi_i(y)} s\phi_i(y)\right] = \mathcal{M}_{ab}\omega^b(x) \tag{6.229}$$

e pelo teorema 2.4

$$s(\mathcal{M}_{ab}\omega^b) = 0 \tag{6.230}$$

logo

$$s\left(-\frac{1}{2\xi}F_a^2 - \overline{\omega}^a \mathcal{M}_{ab}\omega^b\right) = \left(-\frac{1}{\xi}F_a + s\overline{\omega}^a\right)\mathcal{M}_{ab}\omega^b = 0 \tag{6.231}$$

onde se usou $s\overline{\omega}^a = \frac{1}{\xi}F_a$*. Pondo tudo junto obtemos portanto*

$$sS_{eff}[A,\phi] = 0 \; . \tag{6.232}$$

Para o seguimento é ainda importante um outro teorema,

Teorema 2.6

A medida $\mathcal{D}(A_\mu, \phi_i, \overline{\omega}^a, \omega^b)$ é invariante para transformações BRS.

Dem:

Cálculos simples conduzem às seguintes relações:

$$
\begin{aligned}
&\frac{\delta(sA^a_\mu)}{\delta A^a_\mu} = -g f^{aba} \delta^\mu_\mu \omega^b = 0 \\
&\frac{\delta(s\phi_i)}{\delta\phi_i} = ig(T^a)_{ii}\omega^a = 0 \qquad ; \qquad (Tr(T^a) = 0) \\
&\frac{\delta(s\omega^a)}{\delta\omega^a} = g f^{aac}\omega^c = 0 \\
&\frac{\delta(s\overline{\omega}^a)}{\delta\overline{\omega}^a} = 0
\end{aligned}
\tag{6.233}
$$

Como vimos no capítulo 5 estas relações implicam que a medida é invariante, o que demonstra o teorema.

6.3.2 Ward-Takahashi-Slavnov-Taylor identities

Vamos aqui deduzir a generalização das identidades de Ward-Takahashi para as teorias de gauge não abelianas. Esse trabalho foi feito, entre outros, por Slavnov e Taylor mas usaremos com frequência o nome de identidades de Ward mesmo para as teorias não abelianas. Duma forma genérica, as identidades de Ward são relações entre as funções de Green que resultam da simetria de gauge da teoria. De acordo com o que vimos no capítulo 5 a maneira mais conveniente das expressar é usar os funcionais geradores das funções de Green.

Consideremos então uma teoria de gauge não abeliana. Por simplicidade consideramos que a matéria é constituída por campos escalares ϕ_i. A introdução de fermiões é imediata. O funcional gerador das funções de Green é então

$$
Z[J^a_\mu, J_i, \eta^a, \overline{\eta}^a] = \int \mathcal{D}(A_\mu, \phi_i, \overline{\omega}, \omega) e^{i\int d^4x[\mathcal{L}_{eff} + J^a_\mu A^{\mu a} + J_i\phi_i + \overline{\eta}^a\omega^a + \overline{\omega}^a\eta^a]}
\tag{6.234}
$$

onde introduzimos também fontes para os fantasmas. Uma transformação de BRS é uma mudança de variável no integral. O valor do integral não deve ser alterado por essa mudança de variáveis. Como S_{eff} e a medida são invariantes devemos ter o seguinte teorema:

Teorema 2.7

Dada uma função de Green qualquer

$$
G(x_1, ..., y_1, ..., z_1, ..., w_1, ...)
$$

$$= \langle 0|\, T A^a_{\mu_1}(x_1)\cdots\phi_{i_1}(y_1)\cdots\overline{\omega}^a(z_1)\cdots\omega^b(w_1)\cdots|0\rangle \tag{6.235}$$

temos as relações

$$\begin{aligned}
&i) \qquad s\,\langle 0|\, T A^a_\mu(x_1)\cdots\phi_{i_1}(y_1)\cdots\overline{\omega}^a(z_1)\cdots\omega^{b1}(w_1)\,|0\rangle = 0\\
&ii) \qquad 0 = \langle 0|\, T sA^a_\mu(x_1)\cdots|0\rangle + \cdots + \langle 0|\, T\cdots s\phi_i\cdots|0\rangle + \cdots\\
&\qquad\qquad + \langle 0|\, T\cdots s\overline{\omega}^a\cdots|0\rangle\cdots + \langle 0|\, T\cdots s\omega^a\cdots|0\rangle
\end{aligned} \tag{6.236}$$

Dem:

A demonstração é imediata se escrevermos

$$\begin{aligned}
&\langle 0|\, T A^a_\mu(x_1)\cdots\phi_{i_1}(y_1)\cdots\overline{\omega}^a(z_1)\cdots\omega^b(w_1)\,|0\rangle =\\
&= \int \mathcal{D}(A_\mu,\phi_i,\omega,\overline{\omega})A^a_\mu(x_1)\cdots\phi_{i1}(y_1)\cdots\overline{\omega}^a(z_1)\cdots\omega^b(w_1)e^{iS_{eff}}
\end{aligned} \tag{6.237}$$

Então a transformação de BRS deve deixar o valor do integral invariante pelo que a primeira relação é imediata. A segunda relação resulta da primeira e da invariância de medida e da acção efectiva.

Este teorema constitui uma forma expedita de estabelecer relações entre as funções de Green para casos particulares e é muito útil em cálculos práticos, como veremos no seguimento. Contudo para estabelecer resultados gerais sobre a renormalização e invariância de gauge da matriz S interessa-nos as identidades de Ward expressas em termos dos funcionais geradores. Usando a invariância dum integral numa mudança de variáveis, a invariância da medida $\mathcal{D}$ e de S_{eff} obtemos a identidade de Ward para o funcional gerador Z

$$0 = \int \mathcal{D}(A_\mu,\phi_i,\omega,\overline{\omega})\int d^4x(J^{\mu a}sA^a_\mu + J_i s\phi_i + \overline{\eta}^a s\omega^a - s\overline{\omega}^a\eta^a)e^{i(S_{eff}+\text{fontes})} \tag{6.238}$$

Como vimos em QED as identidades de Ward mais úteis são para o funcional Γ. A expressão anterior não permite passar para o funcional Γ porque sA^a_μ, $s\phi_i$ e $s\omega^a$ são não lineares nos campos. Para resolver este problema introduzimos fontes para estes operadores não lineares. Generalizamos assim a acção efectiva definindo uma nova quantidade Σ tal que

$$\begin{aligned}
&\Sigma[A^a_\mu,\phi_i,\overline{\omega}^a,\omega^a,K^a_\mu,K_i,L^a]\\
&\qquad \equiv S_{eff}[A^a_\mu,\phi_i,\overline{\omega}^a,\omega^a] + \int d^4x(K^{a\mu}sA^a_\mu + K^i s\phi_i + L^a s\omega^a)
\end{aligned} \tag{6.239}$$

onde $K^{a\mu}$, K_i e L^a são fontes para os operadores compostos sA^a_μ, $s\phi_i$ e $s\omega^a$ respectivamente. Usando os teoremas 2.3 e 2.5 é imediato mostrar que Σ é invariante para transformações BRS, isto é,

$$s\Sigma = 0\ . \tag{6.240}$$

Consideremos agora que o funcional gerador das funções de Green na presença das fontes $J^a_\mu, J_i, \eta^a, \overline{\eta}^a, K^{\mu a}, K^i$ e L^a, isto é

$$Z[J^a_\mu, J_i, \eta, \overline{\eta}, K_\mu, K_i, L] = \int \mathcal{D}(A_\mu, \phi_i, \overline{\omega}, \omega) e^{i\left[\Sigma + \int d^4x (J^a_\mu A^{\mu a} + J_i\phi_i + \overline{\eta}\omega + \overline{\omega}\eta)\right]} \tag{6.241}$$

Podemos agora repetir o raciocínio da invariância para as transformações de BRS. Como anteriormente obtemos (recordar que $s\Sigma = 0$)

$$0 = \int \mathcal{D}(\cdots) \int d^4x [J^\mu_a sA^a_\mu + J^i s\phi_i + \overline{\eta}^a s\omega^a - s\overline{\omega}^a \eta^a] e^{i(\Sigma + \text{fontes})}\ , \tag{6.242}$$

mas agora temos operadores compostos $sA, s\phi$ e $s\omega$, isto é

$$\begin{aligned} sA^a_\mu &= \frac{\delta\Sigma}{\delta K^{\mu a}} & \qquad s\phi_i &= \frac{\delta\Sigma}{\delta K_i} \\ s\omega^a &= \frac{\delta\Sigma}{\delta L^a} & \qquad s\overline{\omega}^a &= \frac{1}{\xi} F^a\ . \end{aligned} \tag{6.243}$$

Obtemos então

$$\int \mathcal{D}(\cdots) \int d^4x \left[J^\mu_a \frac{\delta\Sigma}{\delta K^{\mu a}} + J^i \frac{\delta\Sigma}{\delta K^i} + \overline{\eta}^a \frac{\delta\Sigma}{\delta L^a} - \frac{1}{\xi} F^a \eta^a \right] e^{i(\Sigma + \text{fontes})} = 0 \tag{6.244}$$

ou ainda

$$\int d^4x \left[J^\mu_a \frac{\delta}{i\delta K^{\mu a}} + J^i \frac{\delta}{i\delta K^i} + \overline{\eta}^a \frac{\delta}{i\delta L^a} - \frac{1}{\xi} F^a \left[\frac{\delta}{i\delta J_\mu}, \frac{\delta}{i\delta J_i} \right] \eta^a \right] e^{iW[J^a_\mu, J_i, \eta, \overline{\eta}, K_\mu, K_i, L]} = 0 \tag{6.245}$$

Para uma condição de gauge, linear todos os operadores diferenciais dentro do parêntesis recto são de 1^a ordem e portanto podemos escrever

$$\int d^4x \left[J^\mu_a \frac{\delta}{\delta K^{\mu a}} + J^i \frac{\delta}{\delta K_i} + \overline{\eta}^a \frac{\delta}{\delta L^a} - \frac{1}{\xi} F_a \eta^a \right] W = 0\ . \tag{6.246}$$

Esta é a expressão da identidade de Ward para o funcional gerador das funções de Green conexas. Normalmente as identidades de Ward são mais úteis para o funcional gerador das funções de Green irredutíveis que é definido por

$$\Gamma[A_\mu, \phi_i, \overline{\omega}, \omega, K_\mu, K_i, L] \equiv W[J_\mu, J_i, \eta, \overline{\eta}, K_\mu, K_i, L] - \int d^4x [J^a_\mu A^{a\mu} + J_i\phi_i + \overline{\eta}\omega + \overline{\omega}\eta] \tag{6.247}$$

com as relações habituais

$$\begin{aligned} \phi_i &= \frac{\delta W}{\delta J_i} & \omega^a &= \frac{\delta W}{\delta \overline{\eta}^a} \\ A^a_\mu &= \frac{\delta W}{\delta J^{\mu a}} & \overline{\omega}^a &= -\frac{\delta W}{\delta \eta^a} \end{aligned} \tag{6.248}$$

e as relações inversas

$$\begin{aligned} J_i &= -\frac{\delta \Gamma}{\delta \phi_i} & \overline{\eta}^a &= \frac{\delta \Gamma}{\delta \omega^a} \\ J^a_\mu &= -\frac{\delta \Gamma}{\delta A^{\mu a}} & \eta^a &= -\frac{\delta \Gamma}{\delta \overline{\omega}^a} \end{aligned} \tag{6.249}$$

Como a transformada de Legendre deixa inertes as fontes K^a_μ, K_i e L^a devemos ter

$$\frac{\delta W}{\delta K^a_\mu} = \frac{\delta \Gamma}{\delta K^a_\mu} \quad ; \quad \frac{\delta W}{\delta K_i} = \frac{\delta \Gamma}{\delta K_i} \quad ; \quad \frac{\delta W}{\delta L^a} = \frac{\delta \Gamma}{\delta L^a} \tag{6.250}$$

Obtemos então facilmente

$$\int d^4x \left[\frac{\delta \Gamma}{\delta K^a_\mu(x)} \frac{\delta \Gamma}{\delta A^{\mu a}(x)} + \frac{\delta \Gamma}{\delta K_i(x)} \frac{\delta \Gamma}{\delta \phi_i(x)} - \frac{\delta \Gamma}{\delta L^a(x)} \frac{\delta \Gamma}{\delta \omega^a(x)} - \frac{1}{\xi} F^a \frac{\delta \Gamma}{\delta \overline{\omega}^a(x)} \right] = 0 \tag{6.251}$$

Esta equação é o funcional gerador das identidades de Ward para uma teoria de gauge não abeliana numa gauge linear. As identidades de Ward para funções de Green específicas obtém-se por derivação funcional em ordem aos campos apropriados.

Na prática a equação anterior usa-se em ligação com outra identidade funcional, a equação de movimento (ou de Dyson- Schwinger) para os fantasmas. Esta pode ser obtida fazendo a seguinte mudança de variáveis no integral funcional (ver capítulo 5),

$$\left\{ \begin{aligned} \delta A^a_\mu &= \delta \phi_i = \delta \omega^a = 0 \\ \delta \overline{\omega}^a &= f^a = \text{constante infinitesimal} \end{aligned} \right. \tag{6.252}$$

Então

$$\delta Z = 0 = \int \mathcal{D}(\cdots) \left(i \frac{\delta \Sigma}{\delta \overline{\omega}^a} + i \eta^a \right) f^a e^{i(\Sigma + \text{fontes})} \tag{6.253}$$

mas

$$\begin{aligned} \frac{\delta \Sigma}{\delta \overline{\omega}^a(x)} &= -\mathcal{M}_{ab} \omega^b(x) = -s F_a(x) \\ &= -\int d^4y \left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)} s A^b_\mu(y) + \frac{\delta F_a(x)}{\delta \phi_i(y)} s \phi_i(y) \right] \\ &= -\int d^4y \left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)} \frac{\delta \Sigma}{\delta K^{b\mu}(y)} + \frac{\delta F_a(x)}{\delta \phi_i(y)} \frac{\delta \Sigma}{\delta K_i(y)} \right] \end{aligned} \tag{6.254}$$

e portanto obtemos

$$\begin{aligned}
0 &= \int \mathcal{D}(\cdots)\left\{-i\int d^4y\left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)}\frac{\delta\Sigma}{\delta K^{b\mu}(y)}+\frac{\delta F_a(x)}{\delta\phi_i(y)}\frac{\delta\Sigma}{\delta K_i(y)}\right]+i\eta^a(x)\right\}e^{i(\Sigma+\text{fontes})} \\
&= \left\{-\int d^4y\left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)}\frac{\delta}{\delta K^{b\mu}(y)}+\frac{\delta F_a(x)}{\delta\phi_i(y)}\frac{\delta}{\delta K_i(y)}\right]+i\eta^a(x)\right\}e^{iW}\ . \qquad (6.255)
\end{aligned}$$

Usando agora

$$\eta^a = -\frac{\delta\Gamma}{\delta\overline{\omega}^a} \qquad (6.256)$$

obtemos finalmente (para gauges lineares)

$$\int d^4y\left[\frac{\delta F_a(x)}{\delta A^b_\mu(y)}\frac{\delta\Gamma}{\delta K^{\mu b}(y)}+\frac{\delta F_a(x)}{\delta\phi_i(y)}\frac{\delta\Gamma}{\delta K_i(y)}\right]=-\frac{\delta\Gamma}{\delta\overline{\omega}^a(x)} \qquad (6.257)$$

que é o funcional gerador das equações de Dyson-Schwinger para os fantasmas.

6.3.3 Example: Transversality of vacuum polarization

Vamos aqui dar um exemplo de aplicação das identidades de Ward mostrando que a polarização do vácuo é transversal. Como a teoria de gauge pura já é não trivial vamos somente considerar este caso, as generalizações são imediatas. Para mostrar os detalhes dos cálculos vamos fazer este exemplo usando dois métodos. O primeiro, que chamaremos *método formal*, consiste na aplicação das identidades de Ward para o funcional Γ que acabámos de mostrar, o segundo é o *método prático* que resulta da aplicação dos resultados do teorema 2.7. A comparação dos dois métodos será importante para a compreensão das expressões.

i) Método formal

Como estamos a considerar uma teoria de gauge pura a expressão para o funcional gerador das identidades de Ward para o funcional Γ é

$$\int d^4x\left[\frac{\delta\Gamma}{\delta K^a_\mu(x)}\frac{\delta\Gamma}{\delta A_{\mu a}(x)}-\frac{\delta\Gamma}{\delta L^a(x)}\frac{\delta\Gamma}{\delta\omega^a(x)}-\frac{1}{\xi}F^a(x)\frac{\delta\Gamma}{\delta\overline{\omega}^a(x)}\right]=0 \qquad (6.258)$$

onde vamos escolher a gauge covariante

$$F^a(x)=\partial_\mu A^{a\mu}(x) \qquad (6.259)$$

Para prosseguir é necessário saber o que representam $\frac{\delta\Gamma}{\delta K^a_\mu}$ e $\frac{\delta\Gamma}{\delta L^a}$. Da forma como foram introduzidos temos

$$\begin{aligned}
\frac{\delta\Gamma}{\delta K^a_\mu(x)} &= \frac{\delta W}{\delta K^a_\mu}=\frac{\delta}{i\delta K^a_\mu}\ln Z=\frac{1}{Z}\frac{\delta Z}{i\delta K^a_\mu(x)} \\
&= \frac{1}{Z}\int\mathcal{D}(\cdots)sA^a_\mu(x)e^{i(\Sigma+\text{fontes})} \qquad (6.260)
\end{aligned}$$

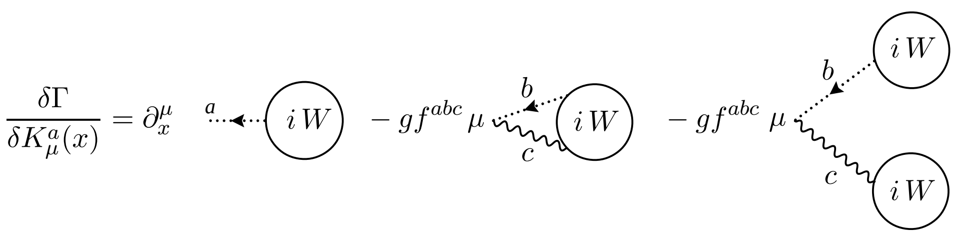
Como $sA_\mu^a(x) = D_\mu^{ab}\omega^b = \partial_\mu\omega^a(x) - gf^{abc}\omega^b(x)A_\mu^c(x)$, obtemos então

$$\frac{\delta\Gamma}{\delta K_\mu^a(x)} = \partial_\mu^x \frac{1}{Z}\frac{\delta Z}{i\delta\overline{\eta}^a(x)} - gf^{abc}\frac{1}{Z}\frac{\delta^2 Z}{i\delta J_\mu^c(x) i\delta\overline{\eta}^b(x)} \tag{6.261}$$

Introduzindo $Z \equiv \exp(iW)$, a equação 6.261 escreve-se

$$\frac{\delta\Gamma}{\delta K_\mu^a(x)} = \partial_x^\mu \frac{\delta(iW)}{i\delta\overline{\eta}^a(x)} - gf^{abc}\left[\frac{\delta^2 iW}{i\delta J_\mu^c(x) i\delta\overline{\eta}^b(x)} + \frac{\delta iW}{i\delta J_\mu^c(x)}\frac{\delta iW}{i\delta\overline{\eta}^b(x)}\right] \tag{6.262}$$

que tem a seguinte representação diagramática:

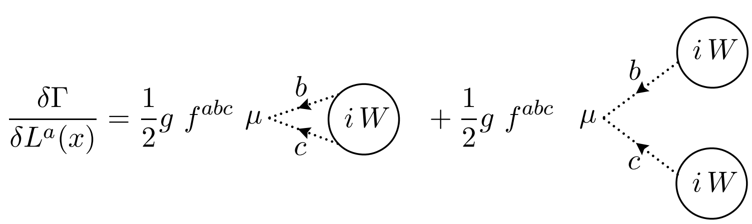


(6.263)

onde W é o funcional gerador das funções de Green conexas. De igual modo se pode mostrar

$$\begin{aligned} \frac{\delta\Gamma}{\delta L^a(x)} &= \frac{1}{2}g\ f^{abc}\ \frac{1}{Z}\ \frac{\delta^2 Z}{i\delta\overline{\eta}^c(x) i\delta\overline{\eta}^b(x)} \\ &= \frac{1}{2}g\ f^{abc}\left[\frac{\delta^2(iW)}{i\delta\overline{\eta}^c(x) i\delta\overline{\eta}^b(x)} + \frac{\delta(iW)}{i\delta\overline{\eta}^c(x)}\frac{\delta(iW)}{i\delta\overline{\eta}^b(x)}\right] \end{aligned} \tag{6.264}$$

ou diagramaticamente



(6.265)

Posto isto voltemos ao problema de provar a transversabilidade do vácuo. Olhando para a expressão inicial é fácil de ver que temos que aplicar $\dfrac{\delta^2}{\delta\omega^b(y)\delta A_\nu^c(z)}$ à equação de partida. Temos sucessivamente

$$\frac{\delta^2}{\delta\omega^b(y)\delta A_\nu^c(z)}\left(\frac{\delta\Gamma}{\delta K_\mu^a(x)}\frac{\delta\Gamma}{\delta A^{\mu a}(x)}\right)\Bigg|_{=0} = \frac{\delta^2\Gamma}{\delta\omega^b(y)\delta K_\mu^a(x)}\Bigg|_{=0}\ \frac{\delta^2\Gamma}{\delta A_\nu^c(z)\delta A^{\mu a}(x)}\Bigg|_{=0} \tag{6.266}$$

mas, usando a Eq. (6.249),

$$
\begin{aligned}
&\left.\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta K^a_\mu(x)}\right|_{=0} = \\
&= \int d^4w \left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right)\left.\left(\frac{\delta^2\Gamma}{i\delta\eta^f(w)\delta K^a_\mu(x)}\right)\right|_{=0} \\
&= \partial^\mu_x \int d^4w \left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right)\left.\left(\frac{\delta^2(iW)}{i\delta\eta^f(w)i\delta\overline{\eta}^a(x)}\right)\right|_{=0} \\
&-g\ f^{ab'c}\int d^4w \left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right)\left.\left(\frac{\delta^3 iW}{i\delta\eta^f(w)i\delta\overline{\eta}^{b'}(x)i\delta J^c_\mu(x)}\right)\right|_{=0} \\
&= \partial^\mu_x\delta^4(x-y)\delta^{ab} - gf^{ab'c}\int d^4w\left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right) \\
&\qquad\qquad \left.\left(\frac{\delta^3 iW}{i\delta\eta^f(w)i\delta\overline{\eta}^{b'}(x)i\delta J^c_\mu(x)}\right)\right|_{=0}
\end{aligned} \tag{6.267}
$$

De modo semelhante

$$
\frac{\delta^2}{\delta\omega^b(y)\delta A^c_\nu(z)}\left.\left(\frac{\delta\Gamma}{\delta L^a}\frac{\delta\Gamma}{\delta\omega^a}\right)\right|_{=0} = 0 \tag{6.268}
$$

e

$$
\frac{\delta^2}{\delta\omega^b(y)\delta A^c_\nu(z)}\left.\left(\frac{1}{\xi}\partial_\rho A^{\rho a}(x)\frac{\delta\Gamma}{\delta\overline{\omega}^a(x)}\right)\right|_{=0} = \frac{1}{\xi}\partial^\nu_x\delta^4(x-z)\left.\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^a(x)}\right|_{=0} \tag{6.269}
$$

Usando estes resultados obtemos

$$
\begin{aligned}
&-\partial^y_\mu\frac{\delta^2\Gamma}{\delta A^b_\mu(y)\delta A^c_\nu(z)} - gf^{ade}\int d^4x d^4w\left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right) \\
&\left(\frac{\delta^3 iW}{i\delta\eta^f(w)i\delta\overline{\eta}^d(x)i\delta J^e_\mu(x)}\right)\left(\frac{\delta^2\Gamma}{\delta A^a_\mu(x)\delta A^c_\nu(z)}\right) + \frac{1}{\xi}\partial^\nu_z\ \frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^c(z)} = 0
\end{aligned} \tag{6.270}
$$

Aplicando transformadas de Fourier, com a convenção da Figura 6.2, obtemos

$$
-ip^\mu(i)G^{-1}{}^{cb}_{\nu\mu}(p) - gf^{ade}iG^{-1}{}^{ca}_{\nu\mu}(p)\Delta^{-1fb}X^{\mu def} + (-ip^\nu)\frac{i}{\xi}\Delta^{-1cb}(p) = 0 \tag{6.271}
$$

ou ainda

$$
p^\mu G^{-1}{}^{cb}_{\nu\mu} = -\frac{1}{\xi}\Delta^{-1cb}p_\nu + ig\ f^{ade}G^{-1}{}^{ca}_{\nu\mu}(p)\ \Delta^{-1fb}X^{\mu def} \tag{6.272}
$$

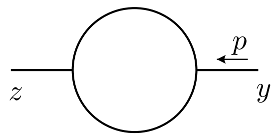


Figure 6.2: Momentum definition for the Fourier Transform.

onde

$$X^{\mu def} = TF\left[<0|T\omega^d(x)\overline{\omega}^f(w)A^{\mu e}(x)|0>_c \right]$$

$$\equiv \tag{6.273}$$

Para demonstrar a transversabilidade precisamos ainda da equação de movimento para os fantasmas que é, para o nosso caso,

$$\frac{\delta\Gamma}{\delta\overline{\omega}^a(z)} = -\partial_z^\mu \frac{\delta\Gamma}{\delta K^{\mu a}(z)} \tag{6.274}$$

Aplicando o operador $\dfrac{\delta}{\delta\omega^b(y)}$, obtemos

$$\begin{aligned} \frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^a(z)} &= -\Box\,\delta^{ab}\delta^4(y-z) \\ &+ gf^{adc}\int d^4w \left(-i\frac{\delta^2\Gamma}{\delta\omega^b(y)\delta\overline{\omega}^f(w)}\right)\partial_z^\mu\left(\frac{\delta^3 iW}{i\delta J_\mu^c(z)i\delta\eta^f(w)i\delta\overline{\eta}^d(z)}\right) \end{aligned} \tag{6.275}$$

Aplicando a transformada de Fourier, obtemos

$$i\Delta^{-1ab} = p^2\delta^{ab} + gf^{adc}(-ip^\mu)X_\mu^{dcf}\Delta^{-1fb} \tag{6.276}$$

As equações 6.272 e 6.276 permitem mostrar a transversabilidade do vácuo. Para isso escrevamos

$$G^{-1ab}_{\ \ \mu\nu} = G_{T\ \mu\nu}^{-1ab} + i\frac{a}{\xi}\delta^{ab}p_\mu p_\nu \tag{6.277}$$

onde $p^\mu G_{T\ \mu\nu}^{-1ab} = 0$. Para o propagador livre $a = 1$. Para mostrar que a polarização do vácuo é transversal basta mostrar que a parte longitudinal não é renormalizada e portanto que o valor de a continuar a ser $a = 1$. Usando

$$p^\mu G^{-1ab}_{\mu\nu} = i\frac{a}{\xi}\delta^{ab}p^2 p_\nu \tag{6.278}$$

e multiplicando a equação 6.272 por p^ν obtemos

$$i\frac{a}{\xi}p^4\delta^{cb} = -\frac{1}{\xi}p^2\Delta^{-1cb} - \frac{a}{\xi}p^2 g\ f^{cde}p_\mu X^{\mu def}\Delta^{-1fb} \tag{6.279}$$

Usando agora a equação 6.276 obtemos depois de alguma álgebra trivial

$$0 = -\frac{1}{\xi}p^2\Delta^{-1cb} + \frac{a}{\xi}p^2\Delta^{-1cb} \tag{6.280}$$

o que implica

$$a = 1 \tag{6.281}$$

como queríamos mostrar.

ii) Método prático

Vamos agora mostrar a transversabilidade da polarização do vácuo usando o método prático baseado nos resultados do Teorema 2.7. Como

$$s\overline{\omega}^b(x) = \frac{1}{\xi}\partial_\mu A^{\mu b}(x) \tag{6.282}$$

e

$$sA^a_\nu = \partial_\nu\omega^a - gf^{adc}\omega^d A^c_\nu \tag{6.283}$$

é fácil de ver que a função de Green de partida deverá ser $< 0|TA^a_\nu(x)\overline{\omega}^b(y)|0 >$. Então o teorema diz-nos que

$$s < 0|TA^a_\mu(x)\overline{\omega}^b(y)|0 >= 0 \tag{6.284}$$

ou seja

$$\begin{aligned} \frac{1}{\xi} < 0|TA^a_\nu(x)\partial_\mu A^{\mu b}(y)|0 > \;\; &= \;\; < 0|T\partial_\nu\omega^a(x)\overline{\omega}^b(y)|0 > \\ &\quad -gf^{adc} < 0|T\omega^d(x)A^c_\nu(x)\overline{\omega}^b(y)|0 > \end{aligned} \tag{6.285}$$

Aplicando a transformação de Fourier obtemos

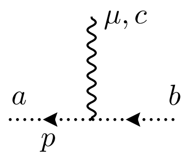
$$\frac{i}{\xi}p^\rho G^{ab}_{\nu\rho}(p) = -ip_\nu\Delta^{ab}(p) - gf^{adc}X^{dcb}_\nu \tag{6.286}$$

onde X^{dcb}_ν foi definido anteriormente. Multiplicando por $G^{-1\nu\mu}\Delta^{-1}$ obtemos

$$p^\mu G^{-1ac}_{\nu\mu} = -\frac{1}{\xi}p_\nu\Delta^{-1ac} + igf^{fde}\chi^{deb}_\nu\Delta^{-1bc}G^{-i\nu\mu af} \tag{6.287}$$

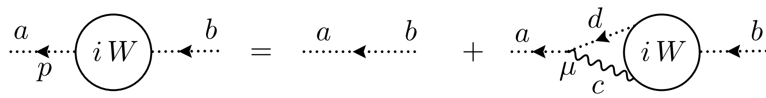
que é precisamente a equação 6.272.

A equação 6.276 pode ser obtida facilmente sabendo que o único vértice dos fantasmas é



$$g f^{abc} p^{\mu} \tag{6.288}$$

Então



$$\tag{6.289}$$

ou seja

$$\Delta^{ab}(p) = \frac{i}{p^2}\delta^{ab} + \frac{i}{p^2} g f^{adc} p^{\mu} X_{\mu}^{dcb} \tag{6.290}$$

ou ainda

$$i\Delta^{-1ab} = p^2\delta^{ab} - igf^{adc}p^{\mu}X_{\mu}^{dcb'}\Delta^{-1b'b} \tag{6.291}$$

que é precisamente a equação 6.276. A demonstração da transversabilidade é agora igual a i).

6.3.4 Gauge invariance of the S matrix

Mostrámos na secção 6.2.5 a invariância da gauge da matriz S, usando o teorema da equivalência e o facto que os funcionais geradores correspondentes a condições de gauge diferentes diferiam somente no termo das fontes. A demonstração que fizemos usava as propriedades específicas de gauge de Coulomb e podia levantar alguma dúvida quanto à sua validade geral.

Vamos aqui mostrar, usando as identidades de Ward, que os funcionais Z_F e $Z_{F+\Delta F}$ correspondentes às condições de gauge F e $F+\Delta F$, respectivamente, diferem somente no termo das fontes. Como F e ΔF são arbitrários, e demonstração é geral. Temos

$$Z_F[J_{\mu}^a] = \int D(\cdots) e^{i[S_{eff} + \int d^4x (J_{\mu}^a A^{\mu a} + J_i \phi_i)]} \tag{6.292}$$

Então

$$\begin{aligned} Z_{F+\Delta F} - Z_F \;\; = & \;\; \int D(\cdots) \int d^4x \; i \left[-\frac{1}{\xi} F^a \Delta F^a - \overline{\omega}^a \int d^4y \frac{\delta \Delta F^a_{(x)}}{\delta A_{\mu}^b(y)} s A^{b\mu}(y) \right. \\ & \left. -\overline{\omega}^a \int d^4y \frac{\delta \Delta F^a(x)}{\delta \phi_i(y)} s\phi_i(y) \right] e^{i(S_{eff} + \text{fontes})} \end{aligned} \tag{6.293}$$

Usamos agora as identidades de Ward na forma correspondente ao funcional Z, isto é

$$0 = \int D(\cdots) \int d^4x [J^{\mu a} s A^a_\mu + J^i s\phi_i + \overline{\eta} s\omega - s\overline{\omega}\eta] \; e^{\{i(S_{eff} + J^a_\mu A^{\mu a} + J_i\phi_i + \overline{\omega}\eta + \overline{\eta}\omega)\}} \tag{6.294}$$

Derivando em ordem a $\eta^a(x)$ e pondo as fontes dos fantasmas nulas obtemos

$$0 = \int D(\cdots) \left[\frac{1}{\xi} F^a(x) + i\overline{\omega}^a(x) \int d^4y [J^{\mu b} s A^b_\mu + J^i s\phi_i] \right] e^{i[S_{eff} + \int d^4x (J^a_\mu A^{\mu i} + J_i\phi_i)]} \tag{6.295}$$

ou ainda

$$\begin{aligned} -\frac{1}{\xi} F^a \left[\frac{\delta}{i\delta J}\right] & \int D(\cdots) e^{i(S_{eff} + \text{fontes})} = \\ & = \int D(\cdots) i\overline{\omega}^a(x) \int d^4y [J^{\mu b} s A^b_\mu + J_i s\phi_i] e^{i(S_{eff} + \text{fontes})} \end{aligned} \tag{6.296}$$

Então

$$\begin{aligned} \int & \mathcal{D}(\cdots) \left(-\frac{1}{\xi} F^a \Delta F^a\right) e^{i(S_{eff} + \text{fontes})} = \\ = & \Delta F^a \left[\frac{\delta}{i\delta J}\right] \left(-\frac{1}{\xi} F^a \left[\frac{\delta}{i\delta J}\right]\right) \int D(\cdots) e^{i(S_{eff} + \text{fontes})} \\ = & \Delta F^a \left[\frac{\delta}{i\delta J}\right] \int D(\cdots) i\overline{\omega}^a(x) \int d^4y [J^{\mu b} s A^b_\mu + J^i s\phi_i] e^{i(S_{eff} + \text{fontes})} \\ = & \int \mathcal{D}(\cdots) \left\{ \overline{\omega}^a(x) \int d^4y \left[\frac{\delta \Delta F^a(x)}{\delta A^b_\mu(y)} s A^b_\mu(y) + \frac{\delta \Delta F^a(x)}{\delta \phi_i(y)} s\phi_i(y) \right] \right. \\ & \left. + i\overline{\omega}^a(x) \Delta F^a(x) \int d^4y [J^{\mu b} s A^b_\mu + J^i s\phi_i] \right\} e^{i(S_{eff} + \text{fontes})} \end{aligned} \tag{6.297}$$

Portanto

$$\begin{aligned} & \int \mathcal{D}(\cdots) \left(-\frac{1}{\xi} F^a \Delta F^a - \overline{\omega}^a(x) \int d^4y \left[\frac{\delta \Delta F^a(x)}{\delta A^b_\mu(y)} s A^b_\mu(\eta) + \frac{\delta \Delta F^a}{\delta \phi_i(y)} s\phi_i(y) \right] \right) e^{i(S_{eff} + \text{fontes})} \\ & = \int \mathcal{D}(\cdots) i\overline{\omega}^a(x) \Delta F^a(x) \int d^4y \left[J^{\mu b} s A^b_\mu + J^i s\phi_i \right] e^{i(S_{eff} + \text{fontes})} \end{aligned} \tag{6.298}$$

Podemos então escrever

$$\begin{aligned}
Z_{F+\Delta F} \quad & -Z_F \\
&= \int \mathcal{D}(\cdots) i \int d^4x \Big[i\overline{\varpi}^a(x)\Delta F^a(x) \int d^4y (J^{\mu b} s A^b_\mu + J_i s\phi_i)\Big] e^{i(S_{eff}+\text{fontes})} \\
&= \int \mathcal{D}(\cdots) e^{i\left\{S_{eff}+\int d^4y [J^a_\mu(y)\mathcal{A}^{\mu a}(y) + J_i\Phi_i(y)]\right\}}
\end{aligned} \tag{6.299}$$

onde

$$\Phi_i(y) \equiv \phi_i(y) + i \int d^4x [\overline{\varpi}^a(x)\Delta F^a(x) s\phi_i(y)] \tag{6.300}$$

e

$$\mathcal{A}^a_\mu(y) \equiv A^a_\mu(y) + i \int d^4x [\overline{\varpi}^b(x)\Delta F^b(x) s A^a_\mu(y)] \tag{6.301}$$

A diferença entre os funcionais geradores $Z_{F+\Delta F}$ e Z_F é apenas na forma funcional dos termos das fontes. Podemos portanto usar o teorema da equivalência para mostrar que as matrizes S renormalizadas são iguais nos dois casos

$$S^R_{F+\Delta F} = S^R_F \ . \tag{6.302}$$

6.4 Ward-Takahashi identities in QED

6.4.1 Ward-Takahashi identities for the funcional $Z[J]$

Vamos aqui tornar a derivar as identidades de Ward para QED já encontradas no estudo da renormalização, mas utilizando agora os métodos funcionais.

Pode-se mostrar que para o funcional gerador das funções de Green completas se pode escrever, numa gauge linear,

$$Z(J_\mu, \overline{\eta}, \eta) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}) e^{i(S_{eff} + \int d^4x (J_\mu A^\mu + \overline{\eta}\psi + \overline{\psi}\eta)} \tag{6.303}$$

onde J_μ, $\overline{\eta}$ e η são as fontes associadas a A_μ, ψ e $\overline{\psi}$ respectivamente. A acção efectiva é dada por

$$S_{eff} = \int d^4x \left[\mathcal{L}_{QED} - \frac{1}{2\xi}(\partial\cdot A)^2\right] = S_{QED} + S_{GF} \tag{6.304}$$

onde

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \ . \tag{6.305}$$

S_{QED} é invariante para as transformações de gauge do grupo $U(1)$

$$\begin{cases} \delta A_\mu = \partial_\mu \Lambda \\ \delta\psi = -ie\Lambda\psi \\ \delta\overline{\psi} = ie\Lambda\overline{\psi} \end{cases}$$

enquanto que S_{eff} contém a parte de *gauge fixing* que não é invariante nas transformações 6.306. Portanto as identidades de Ward tomam aqui a forma

$$\left(\frac{\delta S_{GF}}{\delta\phi_i}\left[\frac{\delta}{i\delta J}\right]+J_i\right)F_i\left[\frac{\delta}{i\delta J}\right]Z(J)=0 \tag{6.306}$$

o que se escreve no nosso caso, reintroduzido as integrações,,

$$0=\int d^4x\left[\frac{1}{\xi}\partial^\mu\partial_\nu\left(\frac{\delta}{i\delta J_\nu}\right)\partial_\mu\Lambda+J^\mu\partial_\mu\Lambda-ie\Lambda\overline{\eta}\frac{\delta}{i\delta\overline{\eta}}+ie\Lambda\eta\frac{\delta}{i\delta\eta}\right]Z(J^\mu,\overline{\eta},\eta) \tag{6.307}$$

ou seja, integrando por partes,

$$\int d^4x\ \Lambda\left[-\frac{1}{\xi}\Box\partial_\nu\left(\frac{\delta}{i\delta J_\nu}\right)-\partial_\mu J^\mu-ie\overline{\eta}\frac{\delta}{i\delta\overline{\eta}}+ie\eta\frac{\delta}{i\delta\eta}\right]Z(J^\mu,\overline{\eta},\eta)=0 \tag{6.308}$$

Podemos ainda escrever

$$\left[\frac{1}{\xi}\Box\partial_\mu\left(\frac{\delta}{i\delta J_\mu}\right)+\partial_\mu J^\mu+ie\overline{\eta}\left(\frac{\delta}{i\delta\overline{\eta}}\right)-ie\eta\left(\frac{\delta}{i\delta\eta}\right)\right]Z(J,\overline{\eta},\eta)=0 \tag{6.309}$$

6.4.2 Ward-Takahashi identities for the functionals W and Γ

Do ponto de vista das aplicações é mais útil a identidade de Ward em termos do funcional gerador das funções de Green próprias. Este problema para as Teorias de Gauge não abelianas é bastante difícil de resolver, conforme veremos no próximo capítulo. Aqui o problema é simples pois a equação acima é linear nas derivadas funcionais em relação às diversas fontes (note-se que se se tivesse escolhido uma condição de gauge *não linear* isto já não seria verdade, mesmo em QED). Esta linearidade permite escrever imediatamente

$$\partial_\mu J^\mu+\left[\frac{1}{\xi}\Box\partial_\mu\left(\frac{\delta}{i\delta J_\mu}\right)+ie\overline{\eta}\frac{\delta}{i\delta\overline{\eta}}-ie\eta\frac{\delta}{i\delta\eta}\right]W(J,\overline{\eta},\eta)=0 \tag{6.310}$$

onde W é o funcional gerador das funções de Green conexas

$$Z(J^\mu,\overline{\eta},\eta)\equiv e^{iW(J^\mu,\overline{\eta},\eta)} \tag{6.311}$$

Como vimos, o funcional gerador das funções de Green próprias é

$$\Gamma(A_\mu,\psi,\overline{\psi})=W(J_\mu,\overline{\eta},\eta)-\int d^4x[J^\mu A_\mu+\overline{\eta}\psi+\overline{\psi}\eta] \tag{6.312}$$

onde

$$A_\mu=\frac{\delta W}{i\delta J^\mu}\ ;\ \psi=\frac{\delta W}{i\delta\overline{\eta}}\ ;\ \overline{\psi}=-\frac{\delta W}{i\delta\eta} \tag{6.313}$$

e

$$J_\mu=-\frac{\delta\Gamma}{\delta A^\mu}\ ;\ \eta=-\frac{\delta\Gamma}{\delta\overline{\psi}}\ ;\ \overline{\eta}=\frac{\delta\Gamma}{\delta\psi} \tag{6.314}$$

onde, como habitualmente, as derivadas fermiónicas são derivadas esquerdas. Então a equação 6.310 pode ser escrita

$$\frac{1}{\xi}\Box\partial_\mu A^\mu - \partial_\mu \frac{\delta\Gamma}{\delta A_\mu} + ie\frac{\delta\Gamma}{\delta\psi}\psi + ie\overline{\psi}\frac{\delta\Gamma}{\delta\overline{\psi}} = 0 \tag{6.315}$$

Esta equação é o ponto de partida para gerar todas as identidades de Ward em QED. A sua aplicação é muito mais fácil do que a expressão equivalente usada no estudo da renormalização e que foi demonstrada utilizando o formalismo canónico. Os métodos funcionais tornam estas expressões particularmente simples.

6.4.3 Example: Ward identity for the QED vertex

Para nos convencermos que a equação 6.315 conduz às identidades de Ward já nossas conhecidas, vamos obter a identidade de Ward para o vértice em QED. Apliquemos $\frac{\delta^2}{\delta\psi_\alpha(y)\delta\overline{\psi}_\beta(z)}$ a 6.315. Obtemos então

$$\begin{aligned}\partial_x^\mu \quad & \frac{\delta^3\Gamma}{\delta\psi_\alpha(y)\delta\overline{\psi}_\beta(z)\delta A^\mu(x)} \\ = \quad & ie\left[\frac{\delta^2\Gamma}{\delta\psi_\alpha(y)\delta\overline{\psi}_\beta(x)}\delta^4(z-x) - \frac{\delta^2\Gamma}{\delta\psi_\alpha(x)\delta\overline{\psi}_\beta(z)}\delta^4(y-x)\right]\end{aligned} \tag{6.316}$$

ou seja

$$\partial_x^\mu \Gamma_{\mu\beta\alpha}(x,z,y) = ie\left[\Gamma_{\beta\alpha}(x,y)\delta^4(z-x) - \Gamma_{\beta\alpha}(z,x)\delta^4(y-x)\right] \tag{6.317}$$

Aplicando agora a transformada de Fourier a ambos os membros, com os momentos definidos de acordo com a Figura 6.3

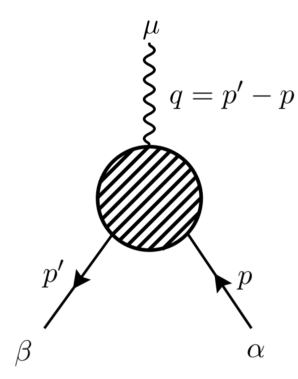


Figure 6.3: Definition of the momenta for the vertex.

obtemos, omitindo os índices spinoriais,

$$q^\mu \Gamma_\mu(p',p) = ie[S^{-1}(p) - S^{-1}(p')] \tag{6.318}$$

que é exactamente a identidade pretendida.

6.4.4 Ghosts in QED

Anteriormente dissemos que para QED o funcional gerador é dado por

$$Z(J_\mu, \overline{\eta}, \eta) = \int \mathcal{D}(A_\mu, \psi, \overline{\psi}) e^{i \int d^4x [\mathcal{L}_{QED} + \mathcal{L}_{GF} + J_\mu A^\mu + \overline{\eta}\psi + \overline{\psi}\eta]} \tag{6.319}$$

onde $\mathcal{L}_{QED}$ é o Lagrangeano usual de QED e o termo que fixa a gauge é dado por

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial \cdot A)^2 \ . \tag{6.320}$$

De facto isto não é estritamente verdade. Como veremos o funcional gerador que obteríamos se usássemos a prescrição para teorias de gauge seria

$$\tilde{Z}(J_\mu, \eta, \overline{\eta}, \zeta, \overline{\zeta}) = \int D(A_\mu, \psi, \overline{\psi}, \omega, \overline{\omega}) e^{i \int d^4x [\mathcal{L}_{eff} + J^\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta + \overline{\omega}\zeta + \overline{\zeta}\omega]} \tag{6.321}$$

onde ω e $\overline{\omega}$ são campos escalares anticomutativos. Estas partículas fictícias são designadas por *fantasmas* de Fadeev-Popov e desempenham um papel fulcral em teorias de gauge não abelianas. Embora não apareçam como estados finais em processos físicos, a introdução de fontes para elas é conveniente para discutir as identidades de Ward. No Lagrangeano anterior $\mathcal{L}_{eff}$ é dado por

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QED}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{G}} \tag{6.322}$$

onde

$$\mathcal{L}_{\text{G}} = -\overline{\omega}\, \Box\, \omega \tag{6.323}$$

A razão pela qual em QED podemos trabalhar com o funcional gerador Z e não $\tilde{Z}$ tem a ver com o facto de os fantasmas em QED não terem acoplamentos com as partículas físicas e poderem ser omitidos completamente da teoria (ver problema 1.6).

Vamos introduzir agora as transformações de Becchi Rouet e Stora (BRS). O objectivo destas transformações é fazer com que $\mathcal{L}_{eff}$ seja invariante. É fácil de ver que para QED obtemos esse resultado com as transformações

$$\begin{cases} \delta\psi = -ie\omega\theta\psi \\ \delta\overline{\psi} = ie\overline{\psi}\omega\theta \\ \delta A_\mu = \delta_\mu \omega\theta \\ \delta\overline{\omega} = \frac{1}{\xi}(\partial \cdot A)\theta \\ \delta\omega = 0 \end{cases} \tag{6.324}$$

onde o parâmetro θ é anticomutativo (*variável de Grassman*). As transformações nos campos físicos são transformações de gauge de parâmetro $\Lambda = \omega\theta$ pelo que $\mathcal{L}_{QED}$ é invariante. As transformações em ω e $\overline{\omega}$ são tais que a variação de $\mathcal{L}_{GF}$ cancela a de $\mathcal{L}_G$. A invariância da medida de integração e de S_{eff} permite imediatamente escrever as identidades de Ward, para os funcionais geradores (ver problema 1.6).

As transformações BRS permitem obter as identidades de Ward duma forma expedita sem ter que recorrer à derivação funcional de $\tilde{\Gamma}$. Este método baseia-se no facto de que o operador $\delta_{\rm BRS}$ aplicado a qualquer função de Green é zero (ver problema 1.6), isto é

$$\delta_{\rm BRS} \langle 0| T A_{\mu_1} \cdots \overline{\omega} \cdots \omega \cdots \psi \cdots \overline{\psi} \cdots |0\rangle = 0 \tag{6.325}$$

Vejamos duas aplicações simples do método:

i) A parte longitudinal do propagador do fotão não é renormalizada

Este resultado é equivalente, como é sabido, a dizer que a polarização do vácuo é transversal. Prova-se facilmente partindo da função de Green $\langle 0| T A_\mu \overline{\omega} |0\rangle$ e fazendo uso de 6.325.

$$\delta_{\rm BRS} \langle 0| T A_\mu \overline{\omega} |0\rangle = 0 \tag{6.326}$$

ou seja

$$\frac{1}{\xi} \langle 0| |T A_\mu \partial^\nu A_\nu |0\rangle \theta - \langle 0| T \partial_\mu \omega \overline{\omega} |0\rangle \theta = 0 \tag{6.327}$$

Portanto

$$\frac{1}{\xi} k^\mu G_{\mu\nu}(k) = -k_\nu \Delta(k) \tag{6.328}$$

usando para propagador dos fantasmas

$$\Delta(k) = \frac{i}{k^2} \tag{6.329}$$

pois o fantasma não tem interacções. Multiplicando pelo inverso do propagador do fotão obtemos

$$\frac{1}{\xi} k^\mu = -i \frac{k_\nu}{k^2} G^{-1\nu\mu}(k) \tag{6.330}$$

e portanto

$$k_\nu G^{-1\nu\mu}(k) = \frac{i}{\xi} k^\mu k^2 = k_\nu G^{-1\nu\mu}_{(0)}(k) \tag{6.331}$$

o que mostra que a parte longitudinal não é renormalizada.

ii) Identidade de Ward para o vértice

Partimos de

$$\delta_{\rm BRS} \langle 0| T \overline{\omega} \psi \overline{\psi} |0\rangle = 0 \tag{6.332}$$

Então

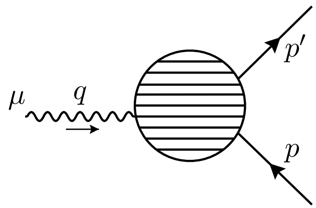
$$\frac{1}{\xi} \langle 0| T \partial^\mu A_\mu \psi \overline{\psi} |0\rangle = ie \langle 0| T \overline{\omega} \omega \psi \overline{\psi} |0\rangle - ie \langle 0| T \overline{\omega} \psi \overline{\psi} \omega |0\rangle \tag{6.333}$$

ou ainda

$$\frac{i}{\xi} q^\mu T_\mu = T \tag{6.334}$$

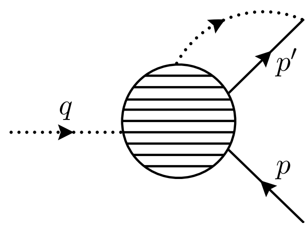
onde

$$iT_\mu =$$

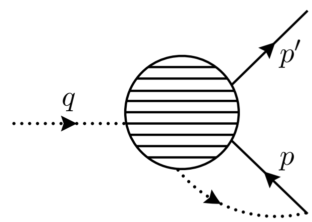


$$= G_{\mu\nu}(q)S(p')i\Gamma^\nu S(p) \tag{6.335}$$

$$iT \;=\; -ie$$



$$+\,ie$$



$$= \; -ie\Delta(q)S(p) + ie\Delta(q)S(p') \tag{6.336}$$

A última igualdade resulta dos fantasmas não terem interacções em QED numa gauge linear. Pondo tudo junto obtemos

$$\frac{i}{\xi}q^\mu G_{\mu\nu}(q)S(p')i\Gamma^\nu S(p) = -ie\Delta(q)S(p) + ie\Delta(q)S(p') \tag{6.337}$$

Usando

$$\frac{1}{\xi}k^\mu G_{\mu\nu}(k) = -k_\nu \Delta(k) \tag{6.338}$$

e multiplicando pelos inversos dos propagadores dos fermiões obtemos finalmente a identidade pretendida

$$q_\mu \Gamma^\mu(p', p) = ie\left[S^{-1}(p) - S^{-1}(p')\right] \tag{6.339}$$

6.5 Unitarity and Ward identities

6.5.1 Optical theorem

A matriz S pode-se escrever na forma

$$S = 1 + iT \tag{6.340}$$

Então a unitariedade, $SS^\dagger = 1$ implica

$$2\,\mathrm{Im}\ T = TT^{\dagger} \tag{6.341}$$

Se inserirmos esta relação entre o mesmo estado inicial e final obtemos

$$\begin{aligned} 2\,\mathrm{Im} < i|T|i> &= < i|TT^{\dagger}|i> \\ &= \sum_f |< f|T|i>|^2 \end{aligned} \tag{6.342}$$

onde introduzimos um conjunto completo de estados. Esta relação pode ainda escrever-se na forma

$$\sigma_{\text{total}} = 2\,\mathrm{Im}\ T^{\text{elástica}}_{\text{frente}} \tag{6.343}$$

conhecida por teorema óptico. O que chamamos aqui σ_{total} não é exactamente a secção eficaz porque faltam os factores de fluxo. É rigorosamente a quantidade definida por

$$\sigma_{\text{total}} \equiv \sum_f |< f|T|i>|^2 \tag{6.344}$$

A unitariedade estabelece portanto uma relação entre a secção eficaz total e a parte imaginária da amplitude elástica na direcção frontal (o estado inicial e final têm que ser o mesmo).

6.5.2 Cutkosky rules

Para mostrar que a unitariedade é respeitada num dado processo é necessário saber calcular a parte imaginária de diagramas. Claro que há sempre a possibilidade de fazer as contas explícitas até ao fim e ver qual foi a parte imaginária que ficou, mas este processo não é muito conveniente para diagramas complicados.

Assim existem regras, chamadas regras de Cutkosky que nos dão simplesmente a parte imaginária duma amplitude qualquer. Estas regras são:

Regra 1:

A parte imaginária duma amplitude obtém-se através da expressão

$$2\,\mathrm{Im}\ T = -\sum_{\text{cortes}} T \tag{6.345}$$

Regra 2:

O corte obtém-se escrevendo a amplitude $iT = \cdots$ e substituindo nesta expressão os propagadores das linhas cortadas pelas seguintes expressões:

i) Campos Escalares

$$\Delta(p) \Longrightarrow 2\pi\theta(p^o)\delta(p^2 - m^2) \tag{6.346}$$

ii) Campos Spinoriais

$$S(p) \Rightarrow (\not p + m)2\pi\theta(p^o)\delta(p^2 - m^2) \tag{6.347}$$

iii) Campos Spin 1 (na gauge de Feynman)

$$G_{\mu\nu}(p) \Rightarrow -g_{\mu\nu}2\pi\theta(p^0)\delta(p^2 - m^2) \tag{6.348}$$

Nestas expressões as funções θ asseguram o fluxo da energia. As regras de Cutkosky são um pouco complicados de mostrar em geral[8] mas nós vamos aqui mostrar dois casos e verificá-las explicitamente

Exemplo 2.1 Propagador livre

A amplitude é

$$iT = \frac{i}{p^2 - m^2 + i\varepsilon} \tag{6.349}$$

A parte imaginária obtém-se explicitamente usando

$$\frac{1}{x + i\varepsilon} = P\left(\frac{1}{x}\right) - i\pi\delta(x) \tag{6.350}$$

logo

$$T = P\left(\frac{1}{p^2 - m^2}\right) - i\pi\delta(p^2 - m^2) \tag{6.351}$$

e portanto

$$2\,\mathrm{Im}\,T = -2\pi\delta(p^2 - m^2) \tag{6.352}$$

Pela regra de Cutkosky obtemos imediatamente

$$2\,\mathrm{Im}\,T = -2\pi\delta(p^2 - m^2)\theta(p^0) \tag{6.353}$$

que é o mesmo resultado. A função $\theta(p^0)$ assegura que o fluxo da energia é da esquerda para a direita.

Exemplo 2.2: Self-energy em $\frac{\lambda}{3!}\phi^3$

Consideremos a self-energy na teoria dada por

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3 \tag{6.354}$$

O diagrama de self-energy é o representado na Figura 6.4. A amplitude correspondente é

$$iT = (i\lambda)^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} \quad \frac{i}{(p-k)^2 - m^2 + i\varepsilon} \tag{6.355}$$

Calculemos a parte imaginária de T por dois métodos, primeiro explicitamente e depois usando a regra de Cutkosky.

[8]Para um tratamento mais completo ver G. 't Hooft, "Diagrammar", CERN Report 1972.

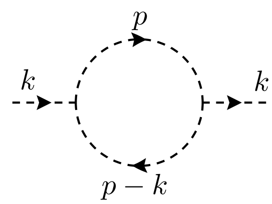


Figure 6.4: Self-energy

i) Cálculo explícito

$$\begin{aligned}
iT &= \lambda^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2-m^2+i\varepsilon)[(p-k)^2-m^2+i\varepsilon]} \\
&= \lambda^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{1}{(p^2+2p\cdot P - M^2 + i\varepsilon)^2} \\
&= \lambda^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{1}{[(p+P)^2-\Delta]^2}
\end{aligned} \tag{6.356}$$

onde

$$\begin{cases} P &= -x\,k \\ \Delta &= P^2+M^2 = m^2 - k^2x(1-x) - i\varepsilon \end{cases} \tag{6.357}$$

Então

$$iT = \lambda^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{1}{(p^2-\Delta)^2} \tag{6.358}$$

O integral é divergente. Fazendo regularização dimensional obtemos finalmente

$$T = \frac{\lambda}{16\pi^2}\mu^\varepsilon \Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx \Delta^{-\left(2-\frac{d}{2}\right)} \tag{6.359}$$

Para prosseguir temos que impor um esquema de renormalização. Fazendo renormalização on-shell, $T_R(k^2=m^2)=0$, obtemos

$$\begin{aligned}
T_R &= T - T(k^2=m^2) \\
&= \frac{\lambda^2}{16\pi^2}\Gamma\left(\frac{\varepsilon}{2}\right)\int_0^1 dx \left[\left(\frac{\Delta(k^2)}{\mu^2}\right)^{-\frac{\varepsilon}{2}} - \left(\frac{\Delta(k^2=m^2)}{\mu^2}\right)^{-\frac{\varepsilon}{2}}\right] \\
&= \frac{\lambda^2}{16\pi^2}\left(\frac{2}{\varepsilon} - C + O(\varepsilon)\right)\int_0^1 dx \left[1 - 1 - \frac{\varepsilon}{2}\ln\frac{m^2-k^2x(1-x)-i\varepsilon}{m^2-m^2x(1-x)-i\varepsilon}\right] \\
&= -\frac{\lambda^2}{16\pi^2}\int_0^1 dx \ln\left[\frac{1-\beta x(1-x)-i\varepsilon}{1-x(1-x)-i\varepsilon}\right]
\end{aligned}$$

$$= -\frac{\lambda^2}{16\pi^2}\left[L(\beta) - L(1)\right] \tag{6.360}$$

onde $\beta = \frac{k^2}{m^2}$ e a função Ł(β) é definida por

$$L(\beta) \equiv \int_0^1 dx \ln\left[1 - \beta(1-x)x - i\varepsilon\right] \tag{6.361}$$

e satisfaz

$$\operatorname{Im} L(\beta) = -\pi\sqrt{1 - \frac{4}{\beta}}\ \theta(\beta - 4) \tag{6.362}$$

Então

$$\operatorname{Im} T = -\frac{\lambda^2}{16\pi^2}\left[\operatorname{Im} L(\beta) - \operatorname{Im} L(1)\right] \tag{6.363}$$

e obtemos finalmente

$$\operatorname{Im} T = \frac{\lambda^2}{16\pi}\sqrt{1 - \frac{4m^2}{k^2}}\ \theta\left(1 - \frac{4m^2}{k^2}\right) \tag{6.364}$$

A função θ assegura que só há parte imaginária quando o estado intermédio puder ser final (produção de 2 partículas de massa m).

ii) Cálculo usando as Regras de Cutkosky

Usando as regras obtemos

$$\begin{aligned}
2\operatorname{Im} T &= -(i\lambda)^2 \int \frac{d^4p}{(2\pi)^4}(2\pi)^2\theta(p^0)\theta(k^0 - p^0)\delta(p^2 - m^2)\delta((p^2 - k^2) - m^2) \\
&= \lambda^2 \int \frac{d^4p}{(2\pi)^4} d^4p'(2\pi)^2\theta(p^0)\theta(k^0 - p^0)\delta(p^2 - m^2)\delta(p'^2 - m^2)\delta^4(p' - k + p)
\end{aligned} \tag{6.365}$$

Usando agora

$$\int d^4p\theta(p^0)\delta(p^2 - m^2) = \int d^3p\frac{1}{2p^0} \tag{6.366}$$

obtemos

$$2\operatorname{Im} T = \lambda^2 \int \frac{d^3p}{(2\pi)^3} d^3p' \frac{1}{2p^0}\frac{1}{2p'^0} 2\pi\delta^4(p' - k + p) \tag{6.367}$$

ou ainda

$$2\operatorname{Im} T = \lambda^2 \int \frac{d^3p}{(2\pi)^3}\frac{1}{2p^0}\frac{1}{2p'^0} 2\pi\delta(k^0 - p^0 - p'^0) \tag{6.368}$$

No referencial do centro de massa

$$k = (\sqrt{s}, \vec{0}) \ ; \ p = (\sqrt{|\vec{p}|^2 + m^2}, \vec{p}) \ ; \ p' = (\sqrt{|\vec{p}'|^2 + m^2}, -\vec{p}) \tag{6.369}$$

e portanto

$$\begin{aligned}
2\,\mathrm{Im}\,T &= \lambda^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{4(|\vec{p}|^2 + m^2)} 2\pi\delta(\sqrt{s} - 2\sqrt{|\vec{p}|^2 + m^2}) \\
&= \frac{\lambda^2}{4\pi} \int d|\vec{p}| \frac{|\vec{p}|^2}{|\vec{p}|^2 + m^2} \frac{\delta(|\vec{p}| - \sqrt{\frac{s}{4} - m^2})}{\frac{2|\vec{p}|}{\sqrt{|\vec{p}|^2 + m^2}}} \theta\left(1 - \frac{4m^2}{s}\right) \\
&= \frac{\lambda^2}{8\pi} \sqrt{1 - \frac{4m^2}{s}}\ \theta\left(1 - \frac{4m^2}{s}\right)
\end{aligned} \tag{6.370}$$

Logo usando $k^2 = s$ obtemos

$$\mathrm{Im}\,T = \frac{\lambda^2}{16\pi} \sqrt{1 - \frac{4m^2}{k^2}}\ \theta\left(1 - \frac{4m^2}{k^2}\right) \tag{6.371}$$

que é de facto o mesmo resultado que 6.364.

6.5.3 Example of Unitariedade: scalars and fermions

Consideremos a teoria descrita pelo seguinte Lagrangeano

$$\mathcal{L} = \overline{i\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2 + g\overline{\psi}\psi\phi \tag{6.372}$$

Vamos mostrar a unitariedade em 2 casos em que as linhas cortadas são fermiónicas

i) Self-energy dos escalares

A self energy dos escalares e dada pelo diagrama da Figura 6.5, a que corresponde a amplitude

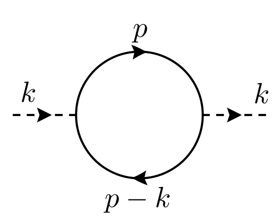


Figure 6.5: Fermion contribution to the scalar self-energy.

$$iT = g^2 \int \frac{d^4p}{(2\pi)^4} Tr\left[\frac{i}{p\!\!\!/ - m + i\varepsilon} \frac{i}{p\!\!\!/ - k\!\!\!/ - m + i\varepsilon}\right] \tag{6.373}$$

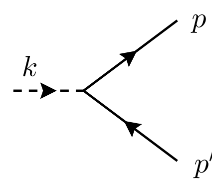
logo

$$2\,\mathrm{Im}\,T = -\sum_{\mathrm{cuts}} T$$

$$
\begin{aligned}
&= -g^2 \int \frac{d^4p}{(2\pi)^4} Tr[(\not{p}+m)(\not{p}-\not{k}+m)](2\pi)\theta(p^0)\delta(p^2-m^2)\cdot \\
&\qquad (2\pi)\theta(k^0-p^0)\delta((p-k)^2-m^2)
\end{aligned}
\tag{6.374}
$$

Para mostrarmos a unitariedade calculemos

$$\sigma = \sum_f \Bigg| \qquad \Bigg|^2 \tag{6.375}$$



ou seja

$$\sigma = \sum_f |ig\overline{u}(p)v(p')|^2 = -g^2 \sum_f Tr[(\not{p}+m)(-\not{p}'+m)] \tag{6.376}$$

onde se usou $\sum_{\text{spins}} v(p')\overline{v}(p) = -(-\not{p}'+m)$ e $\sum_{\text{spins}} u(p)\overline{u}(p') = \not{p}+m$. Logo

$$\sigma = -g^2 \int d\rho_2 \text{Tr}[(\not{p}+m)(-\not{p}'+m)] \tag{6.377}$$

onde $d\rho_2$ é o espaço de fase de duas partículas, isto é

$$
\begin{aligned}
\int d\rho_2 &\equiv \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{2p^0} \frac{1}{2p'^0} (2\pi)^4 \delta^4(k-p-p') \\
&= \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} (2\pi)\theta(p^0)\delta(p^2-m^2)(2\pi)\theta(p'^0)\delta(p'^2-m^2)(2\pi)^4\delta^4(k-p-p')
\end{aligned}
\tag{6.378}
$$

Daqui se conclui que

$$\sigma = -g^2 \int \frac{d^4p}{(2\pi)^4} (2\pi)\theta(p^0)\delta(p^2-m^2)(2\pi)\theta(k^0-p^0)\delta((p-k)^2-m^2)\text{Tr}[(\not{p}+m)(\not{p}-\not{k}+m)] \tag{6.379}$$

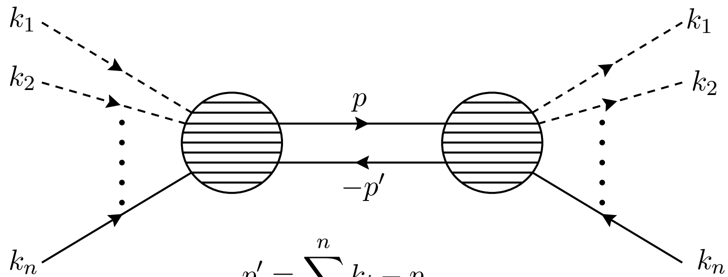
e obtemos portanto finalmente

$$2\,\text{Im}\,T = \sigma \tag{6.380}$$

como queríamos mostrar.

ii) Caso geral

Consideremos o caso geral com 2 linhas internas de fermiões. A amplitude iT é representada pelo seguinte diagrama



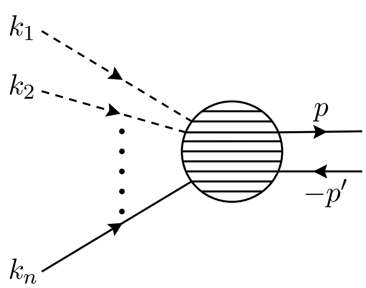
$$\equiv iT$$

$$p' = \sum_{i=1}^{n} k_i - p \tag{6.381}$$

A amplitude iT escreve-se

$$iT = -\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\Big[\overline{T'} S(p) T' S(-p')\Big] \tag{6.382}$$

onde a amplitude iT' é, por sua vez, definida pelo diagrama seguinte



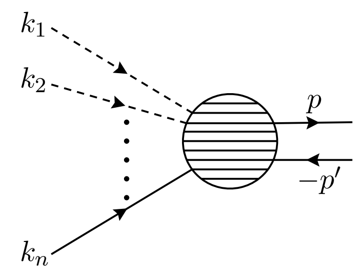
$$\equiv \overline{u}(p) iT' v(p') \tag{6.383}$$

Então

$$\begin{aligned}
2\,\mathrm{Im}\,T &= -\int \frac{d^4p}{(2\pi)^4} (2\pi)^2 \delta(p^2 - m^2)\theta(p^0)\delta(p'^2 - m^2)\theta(p'^0) \cdot \\
&\qquad \mathrm{Tr}\Big[\overline{T'}(\not{p} + m) T' (-\not{p}' + m)\Big] \\
&= -\int d\rho_2 \mathrm{Tr}\Big[\overline{T'}(\not{p} + m) T' (-\not{p}' + m)\Big]
\end{aligned} \tag{6.384}$$

Por outro lado

$$\sigma = \sum_f \left| \vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right|^2$$

$$
\begin{aligned}
&= \sum_f |\overline{u}(p) T' v(p')|^2 \\
&= -\int d\rho_2 \mathrm{Tr}\Big[(\not{p} + m) T' (-\not{p}' + m) \overline{T'}\Big] \qquad (6.385)
\end{aligned}
$$

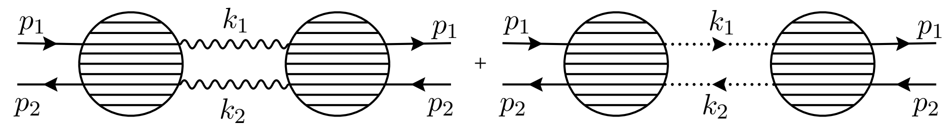
e portanto

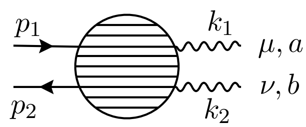
$$\sigma = 2\,\mathrm{Im}\,T \qquad (6.386)$$

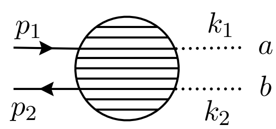
Se as linhas cortadas fossem de escalares em vez de fermiões o resultado seria o mesmo, não haveria o sinal menos do loop mas também não haveria o sinal menos da soma dos spins (ver problema 2.8).

6.5.4 Unitarity and gauge fields

- Na secção 6.5.3 demonstrou-se a unitariedade das teorias com escalares e spinores. Vamos aqui ver que a demonstração da unitariedade para o caso dos campos de gauge é mais complicada e exige o uso das identidades de Ward. O problema reside no facto que os campos de gauge em linhas internas podem ter polarizações não físicas enquanto que no estado final o não podem. Esta diferença levaria a uma violação da unitariedade se as linhas internas não pudessem ser também de fantasmas que compensam os graus de liberdade a mais. Consideremos as seguintes amplitudes

$iT =$ 

$iT^{ab}_{\mu\nu} =$ 

$iT^{ab} =$ 

(6.387)

onde

$$k_2 = p_1 + p_2 - k_1 \tag{6.388}$$

Então a amplitude escreve-se[9]

$$iT = \int \frac{d^4k_1}{(2\pi)^4} \left\{ \frac{1}{2} T^{ab}_{\mu\nu} G^{aa'}_{\mu\mu'}(k_1) G^{bb'}_{\nu\nu'}(k_2) T^{*a'b'\mu'\nu'} - T^{ab} \Delta^{aa'}(k_1) \Delta^{bb'}(k_2) T^{*a'b'} \right\} \tag{6.389}$$

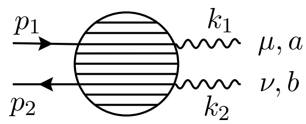
Aplicando as regras de Cutkosky a parte imaginária é

$$\begin{aligned} 2\,\mathrm{Im}\,T &= \int \frac{d^4k_1}{(2\pi)^4} (2\pi)^2 \theta(k_1^0)\theta(k_2^0)\delta(k_1^2)\delta(k_2^2) \left\{ \frac{1}{2} T^{ab}_{\mu\nu} T^{*ab\mu\nu} - T^{ab} T^{*ab} \right\} \\ &\equiv \int d\rho_2 \left[\frac{1}{2} T^{ab}_{\mu\nu} T^{*ab\mu\nu} - T^{ab} T^{*ab} \right] \end{aligned} \tag{6.390}$$

Calculemos agora σ_{tot}. Como os fantasmas não são físicos teremos

[9] O factor $1/2$ é o factor de simetria dum loop com escalares. O sinal $-$ é devido ao loop de fantasmas.

$$\sigma \quad = \quad \sum \Bigg|$$



$$\Bigg|^2$$

$$= \quad \frac{1}{2}\int d\rho_2 \sum_{Pol} \left|\varepsilon^\mu(k_1)\varepsilon^\nu(k_2)T^{ab}_{\mu\nu}\right|^2 \tag{6.391}$$

onde o factor 1/2 se deve agora a haver partículas idênticas no estado final. Pondo

$$\sum_{Pol} \varepsilon^\mu(k_1)\varepsilon^{\mu'*}(k_1) = P^{\mu\mu'}(k_1) \tag{6.392}$$

obtemos

$$\sigma = \int d\rho_2 \frac{1}{2} T^{ab}_{\mu\nu} T^{*ab}_{\mu'\nu'} P^{\mu\mu'}(k_1) P^{\nu\nu'}(k_2) \ . \tag{6.393}$$

Usando agora o resultado do problema 2.10

$$P^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu\eta^\nu + k^\nu\eta^\mu}{k\cdot\eta} \tag{6.394}$$

onde η^μ é um 4-vector que satisfaz $\eta \cdot \varepsilon$ e $\eta^2 = 0$, obtemos

$$\begin{aligned}
&\frac{1}{2}T^{ab}_{\mu\nu}T^{*ab}_{\mu'\nu'}P^{\mu\mu'}(k_1)P^{\nu\nu'}(k_2) = \\
= \quad &\frac{1}{2}T^{ab}_{\mu\nu}T^{*ab\mu\nu} - \frac{1}{2}(T^{ab}\cdot k_2)\cdot(T^{*ab}\cdot\eta)\frac{1}{k_2\cdot\eta} \\
&-\frac{1}{2}(T^{ab}\cdot\eta)\cdot(T^{*ab}\cdot k_2)\frac{1}{k_2\cdot\eta} - \frac{1}{2}(k_1\cdot T^{ab})\cdot(\eta\cdot T^{*ab})\frac{1}{k_1\cdot\eta} \\
&-\frac{1}{2}(\eta\cdot T^{ab})\cdot(k_1\cdot T^{*ab})\frac{1}{k_1\cdot\eta} + \left[\frac{1}{2}(k_1\cdot T^{ab}\cdot\eta)(\eta\cdot T^{*ab}\cdot k_2) + \right. \\
&+\frac{1}{2}(k_1\cdot T^{ab}\cdot k_2)(\eta\cdot T^{*ab}\cdot\eta) + \frac{1}{2}(\eta\cdot T^{ab}\cdot\eta)(k_1 T^{*ab}\cdot k_2) \\
&\left. +\frac{1}{2}(\eta\cdot T^{ab}\cdot k_2)(k_1\cdot T^{*ab}\cdot\eta)\right]\frac{1}{(k_1\cdot\eta)(k_2\cdot\eta)}
\end{aligned} \tag{6.395}$$

Fazendo uso das identidades de Ward (ver problema 2.11),

$$\begin{aligned}
k_1^\mu T^{ab}_{\mu\nu} &= k_{2\nu}T^{ab} \\
k_2^\mu T^{ab}_{\mu\nu} &= k_{1\nu}T^{ab}
\end{aligned} \qquad \Longrightarrow \qquad k_1\cdot T^{ab}\cdot k_2 = 0 \tag{6.396}$$

obtemos

$$
\begin{aligned}
&\frac{1}{2}T^{ab}_{\mu\nu}T^{*ab}_{\mu'\nu'}P^{\mu\mu'}(k_1)P^{\nu\nu'}(k_2) = \\
&\quad = \frac{1}{2}T^{ab}_{\mu\nu}T^{*ab\mu\nu} - \frac{1}{2}T^{ab}(k_1 \cdot T^{*ab} \cdot \eta)\frac{1}{k_2 \cdot \eta} \\
&\quad - \frac{1}{2}T^{*ab}(k_1 \cdot T^{ab} \cdot \eta)\frac{1}{k_2 \cdot \eta} - \frac{1}{2}T^{ab}(\eta \cdot T^{*ab} \cdot k_2)\frac{1}{k_1 \cdot \eta} \\
&\quad - \frac{1}{2}(\eta \cdot T^{ab} \cdot k_2)T^{*ab}\frac{1}{k_1 \cdot \eta} + \frac{1}{2}T^{ab}T^{*ab} + \frac{1}{2}T^{ab}T^{*ab} \\
&\quad = \frac{1}{2}T^{ab}_{\mu\nu}T^{*ab\mu\nu} - T^{ab}T^{*ab}
\end{aligned}
\tag{6.397}
$$

e portanto

$$
\sigma = \int d\rho_2 \left[\frac{1}{2}T^{ab}_{\mu\nu}T^{*ab\mu\nu} - T^{ab}T^{*ab}\right] \tag{6.398}
$$

o que comparando com 6.390 dá

$$
\sigma = 2\,\mathrm{Im}\,T \tag{6.399}
$$

como queríamos mostrar.

Problems for Chapter 6

2.1 Mostre que $T(R)$ está relacionado com o operador de Casimir da representação R $C_2(R)$ através de

$$T(R)r = d(R)C_2(R) \tag{6.400}$$

onde r é a dimensão do Grupo G e $d(R)$ é a dimensão da representação R. O Casimir $C_2(R)$ é definido por

$$\sum_{a,k} T^a_{ik} T^a_{kj} = \delta_{ij} ... C_2(R) \ . \tag{6.401}$$

2.2 Mostrar que numa escolha diferente de condições auxiliares $\chi^{i\alpha} = 0$ conduz ao mesmo resultado.

Sugestão: considere uma variação infinitesimal

$$\chi^\alpha + \delta\chi^\alpha = 0 \qquad\qquad \alpha = 1, ...m \tag{6.402}$$

Mostre então que

$$\pi_\alpha \delta(\varphi_\alpha)\delta(\chi_a) \det(\{\varphi, \chi\}) \to \pi_\alpha \delta(\varphi_\alpha \delta(\chi_\alpha + \delta\chi_\alpha) \det(\{\varphi, \chi + \delta\chi\}) \ . \tag{6.403}$$

2.3 Mostrar que para transformações infinitesimais

$$\begin{aligned} \delta\vec{E}_a(x) &= -\frac{1}{y}\int_{x_o=y_o} d^3y\{\vec{E}_a(x), \alpha^b(y)C_b(y)\} \\ \delta\vec{A}_a(x) &= -\frac{1}{y}\int_{x_o=y_o} d^3y\{\vec{A}_a(x), \alpha^b(y)C_b(y)\} \end{aligned} \tag{6.404}$$

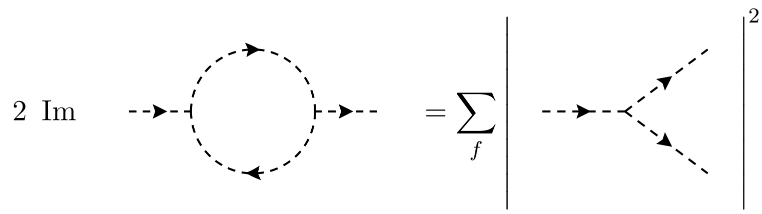
isto é, as ligações C_a são os geradores infinitesimais das transformações de gauge independentes do tempo.

2.4 Mostre que é sempre possível encontrar uma gauge onde $A^3_a = 0 \qquad a = 1, ...r$.

2.5 Mostre os resultados expressos na equação 6.92.

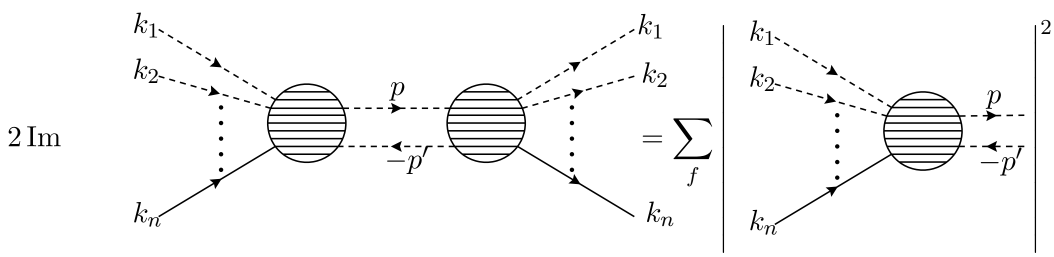
2.6 Mostre que a parte imaginária não depende do esquema de renormalização, calculando-a em $\overline{MS}$ e MS para o exemplo 2.2, isto é para a teoria descrita pelo Lagrangeano da equação 6.354.

2.7 Considere a teoria $\frac{\lambda}{3!}\phi^3$ do problema 2.6. Faça a demonstração da unitariedade para a self-energy dessa teoria, isto é



(6.405)

2.8 Considere a teoria descrita pelo Lagrangeano da Eq. (6.372). Refaça a demonstração geral da unitariedade no caso dos estados intermédios serem escalares, isto é



(6.406)

2.9 Mostre que o integral que resulta de cortar n linhas internas é igual ao integral do espaço de fase de n partículas. Use este resultado para fazer uma demonstração geral da unitariedade.

2.10 Mostre que

$$P^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu \eta^\nu + k^\nu \eta^\mu}{k \cdot \eta} \tag{6.407}$$

onde $k^\mu, \varepsilon^\nu(k,1), \varepsilon^\rho(k,2)$ e η^σ são quatro 4-vectores independentes e satisfazendo

$$\begin{aligned}
&\eta \cdot \varepsilon(k,\sigma) = 0 \qquad \sigma = 1,2 \\
&\varepsilon(k,1) \cdot \varepsilon(k,2) = 0 \\
&k \cdot \varepsilon(k,\sigma) = 0 \qquad \sigma = 1,2 \\
&k^2 = 0 \\
&\eta^2 = 0 \qquad \text{(escolha conveniente)} \\
&\varepsilon^2(k,\sigma) = -1 \qquad \sigma = 1,2
\end{aligned} \tag{6.408}$$

Sugestão: A expressão mais geral para $P^{\mu\nu}$ é

$$P^{\mu\nu} = ag^{\mu\nu} + bk^\mu k^\nu + c\eta^\mu\eta^\nu + d(k^\mu\eta^\nu + k^\nu\eta^\mu) \ . \tag{6.409}$$

Use as relações anteriores para determinar a, b, c, d.

2.11 Demonstrar as identidades de Ward,

$$\begin{aligned} k_1^\mu T^{ab}_{\mu\nu} &= k_{2\nu} T^{ab} \\ k_2^\mu T^{ab}_{\mu\nu} &= k_{1\nu} T^{ab} \end{aligned} \qquad \Longrightarrow \qquad k_1 \cdot T^{ab} \cdot k_2 = 0 \tag{6.410}$$

onde $T^{ab}_{\mu\nu}$ e T^{ab} são definidas em 6.387.

2.12 Mostre que o tensor $F^a_{\mu\nu}$ dos campos de Yang-Mills satisfaz as identidades de Bianchi:

$$D^{ab}_\mu F^b_{\rho\sigma} + D^{ab}_\rho F^b_{\sigma\mu} + D^{ab}_\sigma F^b_{\mu\rho} = 0 \tag{6.411}$$

ou

$$D^{ab}_\mu {}^*F^{\mu\nu\ b} = 0 \tag{6.412}$$

onde

$${}^*F^{\mu\nu\ a} = \frac{1}{2}\, \varepsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma} \tag{6.413}$$

2.13 Explique o significado geométrico da Identidades de Bianchi.

Sugestão: Veja o artigo de R.P. Feynman em *Les Houches, Session XXIX, 1976*, North Holland, 1977, Pags: 135-140.

2.14 Considere a teoria de Yang-Mills (YM) sem matéria.

a) Mostre que as eqs. de YM sem matéria se podem escrever na forma

$$\begin{cases} \vec{\nabla} \cdot \vec{E}^a = \rho^a \\ \vec{\nabla} \cdot \vec{B}^a = {}^*\rho^a \\ \vec{\nabla} \times \vec{E}^a = -\frac{\partial \vec{B}^a}{\partial t} + \vec{J}^a \\ \vec{\nabla} \times \vec{B}^a = -\frac{\partial \vec{E}^a}{\partial t} + {}^*\vec{J}^a \end{cases} \tag{6.414}$$

calcule ρ^a, ${}^*\rho^a$, $\vec{J}^a$ e ${}^*\vec{J}^a$.

b) Mostre que as 4-correntes $j^a_\mu \equiv (\rho^a, \vec{J}^a)$ e ${}^*j^a_\mu \equiv ({}^*\rho^a, {}^*\vec{J}^a)$ são conservadas.

2.15 Mostre que $\mathrm{Tr}\,({}^*F_{\mu\nu}F^{\mu\nu})$ é uma 4-divergência. Comente sobre a sua inclusão na acção.

2.16 Mostre que o seguinte *Ansatze* (S. Coleman, Phys. Lett70B (77), 59)

$$\begin{aligned} A^{1a} &= A^{2a} = 0 \\ A^{0a} &= -A^{3a} = x^1 f^a(x^0 + x^3) + x^2 g^a(x^0 + x^3) \end{aligned} \tag{6.415}$$

onde f^a e g^a são funções arbitrárias, são soluções das equações de YM sem matéria. Discuta esta solução.

2.17 Considere o *Ansatze* de Wu-Yang para soluções estáticas em SU(2) YM.

$$A^{0a} = x^a \, \frac{G(r)}{r^2} \qquad\qquad A^{ia} = \varepsilon^{aij} \, x^j \frac{F(r)}{r^2} \tag{6.416}$$

a) Deduza as equações a que F e G devem obedecer.

b) Mostre que elas são satisfeitas para $F = -1/g$ e $G =$ constante. Mostre que estas soluções correspondem a $\rho^a = {}^*\rho^a = 0$ e $\vec{J}^a = {}^*\vec{J}^a = 0$. ($\rho^a$, ... são definidos no problema 2.14).

c) Para as soluções da alínea b) descreva o potencial e os campos e calcule a energia.

2.18 Considere QED com a condição de gauge não linear

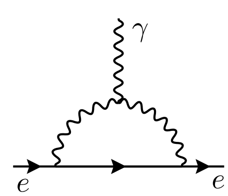
$$F = \partial_\mu A^\mu + \frac{\lambda}{2} \, A_\mu A^\mu \ . \tag{6.417}$$

a) Escreva $\mathcal{L}_{eff}$ e mostre que $s\mathcal{L}_{eff} = 0$, onde s é o operador de Slavnov.

b) Calcule a polarização do vácuo a *1-loop*. Discuta o programa de renormalização dando especial atenção aos vértices proporcionais a λ. Pode aqui considerar a teoria sem fermiões.

c) Demonstre a invariância da matriz S renormalizada em relação ao parâmetro λ.

d) Verifique o resultado anterior mostrando que o diagrama da figura junta, potencialmente perigoso para o momento magnético anómalo do electrão, não dá contribuição (seria proporcional a λ).



e) Deduza as identidades de Ward desta teoria para os funcionais Z e Γ. Escreva o funcional gerador das equações de Dyson-Schwinger para os fantasmas, isto é,

$$\frac{\delta\Gamma}{\delta\overline{\omega}} = \cdots \tag{6.418}$$

f) Calcule ao nível árvore $\gamma + \gamma \to \gamma + \gamma$. Compare com o resultado na gauge linear.

g) Calcule ao nível árvore a amplitude $T^{\mu\nu}$ para $e^+ + e^- \to \gamma + \gamma$. Verifique que $k_{1\mu} T^{\mu\nu} \neq 0$ e $k_{2\mu} T^{\mu\nu} \neq 0$ onde k_1 e k_2 são os 4- momentos dos fotões. Utilize as identidades de Ward para verificar os resultados. Há algum problema com este resultado?

2.19 Considere a teoria que descreve as interacções dos quarks com os gluões (Cromodinâmica Quântica) descrita pelo Lagrangeano

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \sum_{\alpha=1}^{n} \overline{\psi}^\alpha_i (i\not{D} - m_\alpha)_{ij} \psi^\alpha_j \tag{6.419}$$

onde

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ (D_\mu)_{ij} &= \delta_{ij}\partial_\mu - ig \left(\frac{\lambda^a}{2}\right)_{ij} A^a_\mu \ . \end{aligned} \tag{6.420}$$

O índice $\alpha = 1, 2, \ldots, n$ indica os diferentes *sabores* de quarks (*up*, *down*, $\cdots$, *top*). Para quantificar a teoria considere a condição de gauge

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} \ (\partial_\mu A^{\mu a})^2 \ , \tag{6.421}$$

para a qual resulta o Lagrangeano dos fantasmas

$$\mathcal{L}_G = \partial_\mu \overline{\omega}^a \partial^\mu \omega^a + g f^{abc} \partial^\mu \overline{\omega}^a A^b_\mu \omega^c \ . \tag{6.422}$$

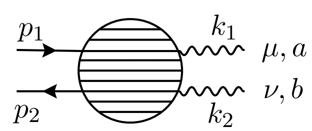
Para renormalizar a teoria necessitamos do seguinte Lagrangeano de contratermos:

$$\begin{aligned} \Delta\mathcal{L} \ = \ & -\frac{1}{4}(Z_3 - 1)\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - (Z_4 - 1) g f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c} \\ & -\frac{1}{4} g^2 (Z_5 - 1) f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} + \sum_\alpha (Z_2 - 1) i \overline{\psi}^\alpha_i \gamma^\mu \partial_\mu \psi^\alpha_i \\ & - \sum_\alpha m_\alpha (Z_{m_\alpha} - 1) \overline{\psi}^\alpha_i \psi^\alpha_i + (Z_1 - 1) g \sum_\alpha \overline{\psi}^\alpha_i \gamma^\mu \left(\frac{\lambda^a}{2}\right)_{ij} \psi^\alpha_j A^a_\mu \\ & + (Z_6 - 1) \partial_\mu \overline{\omega}^a \partial^\mu \omega^a + (Z_7 - 1) g f^{abc} \partial^\mu \overline{\omega}^a A^b_\mu \omega^c \ . \end{aligned} \tag{6.423}$$

a) Verifique a expressão para $\mathcal{L}_G$.

b) Considere a amplitude

$$iT^{ab}_{\mu\nu} \equiv$$



(6.424)

Calcule ao nível árvore $T^{ab}_{\mu\nu}$. Verifique que $k^\mu_1 T^{ab}_{\mu\nu} \neq 0$.

c) Verifique as contas da alínea anterior calculando $k^\mu_1 T^{ab}_{\mu\nu}$ através das identidades de Ward.

d) Supondo que os gluões possam ser estados finais, a amplitude para o processo físico $q + q \to g + g$ onde g é o gluão, é dada pela expressão

$$\mathcal{M} = \varepsilon^{\mu}(k_1) s^a T^{ab}_{\mu\nu} \varepsilon^{\nu}(k_2) s^b \ , \tag{6.425}$$

onde $\varepsilon^{\mu}(k_1)$ e s^a são os vectores de polarização de spin e de cor, respectivamente (o mesmo para $\varepsilon^{\nu}(k_2)$ e s^b). Sabe-se que num processo físico, $\mathcal{M}$ se deve anular quando se faz a substituição $\varepsilon^{\mu}(k) \to k^{\mu}$. Como é que este resultado é compatível com as alíneas anteriores?

e) Mostre que se devem verificar as relações

$$\frac{Z_1}{Z_2} = \frac{Z_4}{Z_3} = \frac{Z_7}{Z_6} = \frac{\sqrt{Z_5}}{\sqrt{Z_3}} \tag{6.426}$$

f) Calcule Z_1, Z_2, Z_3, Z_6 e Z_7, usando subtracção mínima (MS), (isto é, calcule só a parte divergente dos diagramas) e verifique explicitamente que $Z_1 Z_6 = Z_2 Z_7$.

g) Calcule a contribuição dos fermiões para Z_4 e Z_5 e verifique que também obedecem às relações da alínea a).

h) Calcule as funções do Grupo de Renormalização β, γ_A e γ_F.

Chapter 7

Renormalization Group

7.1 Callan -Symanzik equation

7.1.1 Renormalization scheme with momentum subtraction

Em teoria quântica dos campos um esquema de renormalização tem duas componentes. Primeiro há um processo de *regularização* que isola os infinitos que aparecem nos diagramas de Feynman. A regularização é arbitrária desde que mantenha as simetrias da teoria. Para teorias sem campos de gauge há muitos processos alternativos. Para teorias de gauge o melhor processo parece ser a regularização dimensional.

Depois de regularizada a teoria teremos que especificar um método sistemático para remover as divergências e definir os parâmetros renormalizados de teoria. A este processo chamamos esquema de renormalização. Há uma grande arbitrariedade na escolha do processo de subtração. A física contudo não pode depender desta escolha. Este é o conteúdo do grupo de renormalização: *O conteúdo fí sico de teoria deve ser invariante para transformações que apenas mudem as condições de normalização.* Esta afirmação trivial põe no entanto, como veremos, constrangimentos altamente não triviais no comportamento assimptótico da teoria.

Vamos começar por estudar os chamados esquemas com subtração de momento. Conforme o ponto no espaço dos momentos externos que serve de definição às funções de Green irredutíveis, podemos ter várias formas deste esquema. Vamos exemplificar com a teoria $\lambda\phi^4$.

On–shell renormalization

Isto corresponde a uma série de Taylor para os momentos exteriores on - shell. Para a self-energy isto dá

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + \widetilde{\Sigma}(p^2) \tag{7.1}$$

com as condições

$$\begin{cases} \widetilde{\Sigma}(m^2) = 0 \\ \left.\dfrac{\partial\widetilde{\Sigma}(p^2)}{\partial p^2}\right|_{p^2=m^2} = 0 \end{cases} \tag{7.2}$$

Em termos de $\Gamma_R^{(2)}(p^2)$ dado por

$$\Gamma_R^2(p) = p^2 - m^2 - \widetilde{\Sigma}(p^2) \tag{7.3}$$

temos

$$\left\{ \begin{array}{l} \Gamma_R^{(2)}(m^2) = 0 \\ \left. \dfrac{\partial \Gamma_R^{(2)}}{\partial p^2} \right|_{p^2=m^2} = 1 \end{array} \right. \tag{7.4}$$

Para $\Gamma_R^{(4)}$ uma escolha conveniente é

$$\Gamma_R^{(4)}(p_1, p_2, p_3) = -\lambda \qquad \text{para} \qquad \left\{ \begin{array}{l} p_i^2 = m^2 \\ s = t = u = \dfrac{4m^2}{3} \end{array} \right. \tag{7.5}$$

Neste caso os parâmetros m^2 e λ são a massa física e, a menos de factores cinemáticos, a secção eficaz para $s = t = u = \frac{4}{3}m^2$ respectivamente.

Intermediate renormalization

Este esquema corresponde a uma expansão de Taylor em torno de momentos nulos.

$$\Sigma(p^2) = \Sigma(0) + \Sigma'(0)p^2 + \widetilde{\Sigma}(p^2) \tag{7.6}$$

A parte finita $\widetilde{\Sigma}(p^2)$ obdece às condições

$$\left\{ \begin{array}{l} \widetilde{\Sigma}(0) = 0 \\ \left. \dfrac{\partial \widetilde{\Sigma}}{\partial p^2} \right|_{p^2=0} = 0 \end{array} \right. \tag{7.7}$$

que traduzidas em termos de $\Gamma_R^{(2)}$ se escrevem

$$\left\{ \begin{array}{l} \Gamma_R^{(2)}(0) = m^2 \\ \dfrac{\partial \Gamma_R^{(2)}}{\partial p^2} = 1 \end{array} \right. \tag{7.8}$$

Para $\Gamma_R^{(4)}$ a condição é

$$\Gamma_R^{(4)}(p_1, p_2, p_3) = -\lambda \qquad \text{para} \qquad p_1 = p_2 = p_3 = 0 \tag{7.9}$$

Neste esquema m^2 não é a massa física e λ não é nenhuma quantidade mensurável pois os pontos $p_i = 0$ não pertencem à região física. Podemos no entanto exprimir todas as quantidades mensuráveis em termos destes dois parâmetros, como veremos na secç ão 3.3.

General case

Os dois exemplos anteriores são casos particulares do esquema geral onde as condições de normalização podem ser funções de vários *momentos de referência* ξ_1, ξ_2... tais que

$$\begin{cases} \Gamma_R^{(2)}(\xi_1^2) = m^2 \\ \left.\dfrac{\partial \Gamma_R^{(2)}}{\partial p^2}\right|_{p^2=\xi_2^2} = 1 \\ \Gamma_R^{(4)}(\xi_3, \xi_4, \xi_5) = -\lambda \end{cases} \tag{7.10}$$

7.1.2 Renormalization group

Consideremos dois esquemas de renormalização R e R'. Como ambos partem do mesmo Lagrangeano não renormalizado

$$\mathcal{L} = \mathcal{L}_R + \Delta\mathcal{L}_R = \mathcal{L}_{R'} + \Delta\mathcal{L}_{R'} \tag{7.11}$$

devemos ter

$$\phi_R = Z_\phi^{-1/2}(R)\phi_0 \qquad ; \qquad \phi'_R = Z_\phi^{-1/2}(R')\phi_0 \ . \tag{7.12}$$

Logo

$$\phi'_R = Z_\phi^{-1/2}(R', R)\phi_R \tag{7.13}$$

onde

$$Z_\phi(R', R) = \frac{Z_\phi(R')}{Z_\phi(R)} \tag{7.14}$$

Estas relações indicam que os campos renormalizados em diferentes esquemas estão relacionados por uma constante multiplicativa. Esta constante é finita pois tanto $\phi_{R'}$ como ϕ_R são finitos. De modo semelhante

$$\begin{aligned} \lambda_{R'} &= Z_\lambda^{-1}(R', R) Z_\phi^2(R', R)\lambda_R \\ m_{R'}^2 &= m_R^2 + \delta m^2(R', R) \end{aligned} \tag{7.15}$$

onde

$$\begin{aligned} Z_\lambda(R', R) &= \frac{Z_\lambda(R')}{Z_\lambda(R)} \\ \delta m^2(R', R) &= \delta m^2(R') - \delta m^2(R) \end{aligned} \tag{7.16}$$

são quantidades finitas. A operação que leva as quantidades num esquema de renormalização R para outro esquema R' pode ser vista como uma transformação de R em R'. O conjunto de todas estas transformaçoes forma o *Grupo de Renormalização.*

7.1.3 Callan - Symanzik equation

Vamos agora ver como dar uma expressão analítica à invariância para transformações do grupo de renormalização. A forma da equação do grupo de renormalização depende do esquema de renormalização utilizado. Vamos aqui obter as equações do GR para o esquema com substração de momento, a chamada equação de Callan - Symanzik.

Notemos primeiro que

$$\frac{\partial}{\partial m_0^2}\left(\frac{i}{p^2-m_0^2+i\varepsilon}\right)=\frac{i}{p^2-m_0^2+i\varepsilon}(-i)\frac{i}{p^2-m_0^2+i\varepsilon} \tag{7.17}$$

isto é, a derivação duma função de Green não renormalizada em relação à massa despida é equivalente à inserção dum operador composto $\frac{1}{2}\phi^2$ levando momento zero, isto é

$$\frac{\partial\Gamma^{(n)}(p_i)}{\partial m_0^2}=-i\Gamma^{(n)}_{\phi^2}(0,p_i) \tag{7.18}$$

As funções Green irredutíveis renormalizadas são dadas por

$$\begin{cases}\Gamma_R^{(n)}(p_i;\lambda;m)=Z_\phi^{(n/2)}\Gamma^{(n)}(p_i;\lambda_0;m_0)\\[2ex] \Gamma^{(n)}_{\phi^2 R}(p;p_i;\lambda;m)=Z_{\phi^2}^{-1}Z_\phi^{n/2}\Gamma^{(n)}_{\phi^2}(p;p_i;\lambda_0;m_0)\end{cases} \tag{7.19}$$

Então a equação anterior escreve-se

$$\frac{\partial}{\partial m_0^2}\left[Z_\phi^{-n/2}\Gamma_R^{(n)}(p_i,\lambda,m)\right]=-iZ_{\phi^2}Z^{-n/2}\Gamma^{(n)}_{\phi^2 R}(0,p_i,\lambda,m) \tag{7.20}$$

e portanto

$$-\frac{n}{2}Z_\phi^{-1}\frac{\partial Z_\phi}{\partial m_0^2}Z_\phi^{-n/2}\Gamma_R^{(n)}+Z_\phi^{-n/2}\frac{\partial}{\partial m_0^2}\Gamma_R^{(n)}=-iZ_{\phi^2}Z_\phi^{-n/2}\Gamma^{(n)}_{\phi^2 R}(0,p_i,\lambda,m) \tag{7.21}$$

ou seja

$$\begin{aligned}&\left[\frac{\partial}{\partial m_0^2}-\frac{n}{2}\frac{\partial\ln Z_\phi}{\partial m_0^2}\right]\Gamma_R^{(n)}=-i\ Z_{\phi^2}\Gamma^{(n)}_{\phi^2 R}\\ &\left[\frac{\partial m^2}{\partial m_0^2}\frac{\partial m}{\partial m^2}\frac{\partial}{\partial m}+\frac{\partial\lambda}{\partial m_0^2}\frac{\partial}{\partial\lambda}-\frac{n}{2}\frac{\partial\ln Z_\phi}{\partial m_0^2}\right]\Gamma_R^{(n)}=-iZ_{\phi^2}\Gamma^{(n)}_{\phi^2 R}\end{aligned} \tag{7.22}$$

ou ainda

$$\left[m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial\lambda}-n\gamma\right]\Gamma_R^{(n)}=-im^2\alpha\Gamma^{(n)}_{\phi^2 R} \tag{7.23}$$

que é a equação de Callan - Symanzik para a teoria ϕ^4, onde α,β e γ são funções sem dimensões

$$\beta = 2m^2 \frac{\dfrac{\partial\lambda}{\partial m_0^2}}{\dfrac{\partial m^2}{\partial m_0^2}} \tag{7.24}$$

$$\gamma = m^2 \frac{\dfrac{\partial \ln Z_\phi}{\partial m_0^2}}{\dfrac{\partial m^2}{\partial m_0^2}} \tag{7.25}$$

$$\alpha = 2\frac{Z_{\phi^2}}{\dfrac{\partial m}{\partial m_0^2}} \tag{7.26}$$

A função α está relacionada com γ. De facto se escolhermos as condições de normalização a $p_i = 0$

$$\begin{cases} \Gamma_R^{(2)}(0, \lambda, m) = -m^2 \\ \\ \Gamma_{\phi^2 R}^{(2)}(0, 0, \lambda, m) = i \end{cases} \tag{7.27}$$

obtemos

$$\alpha = 2(\gamma - 1) \tag{7.28}$$

Como as quantidades $\Gamma_R^{(n)}$ e $\Gamma_{\phi^2 R}^{(n)}$ não dependem do cut - off, esperamos também que α, β e γ sejam independentes do cut - off. Para vermos isso pomos $n = 2$ e diferenciamos em ordem a p^2

$$\left[m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial\lambda} - 2\gamma\right]\frac{\partial}{\partial p^2}\Gamma_R^{(2)}(p, \lambda, m) = -im^2\alpha\frac{\partial}{\partial p^2}\Gamma_{\phi^2 R}^{(2)}(0, p, \lambda, m) \tag{7.29}$$

Pondo $p^2 = 0$ e usando

$$\left.\frac{\partial\Gamma_R^{(2)}}{\partial p^2}\right|_{p^2=0} = 1 \tag{7.30}$$

obtemos

$$\gamma = im^2(\gamma - 1)\left[\frac{\partial}{\partial p^2}\Gamma_{\phi^2 R}^{(2)}(0, p, \lambda, m)\right]_{p^2=0} \tag{7.31}$$

o que demonstra que γ é independente do cut - off. Então $\alpha = 2(\gamma - 1)$ também o é e todas as funções excepto β são agora independentes do cut - off. Portanto β também o é. Como α, β e γ são sem dimensões e não dependem do cut - off, então são somente funções da constante de acoplamento que também não tem dimensões, isto é

$$\alpha = \alpha(\lambda)$$

$$\begin{aligned} \beta &= \beta(\lambda) \\ \gamma &= \gamma(\lambda) \end{aligned} \tag{7.32}$$

Nós vamos sobretudo estar interessados no esquema de subtracção mí nima, por isso não vamos agora calcular as funções α, β e γ para todas as teorias, faremos isso na secção 3.3. Indicaremos no entanto um método expedito para o seu cálculo. Seja por exemplo a função $\beta(\lambda)$. Notando que

$$\begin{aligned} \frac{\partial \lambda}{\partial m_0^2}(\lambda_0, \Lambda/m) &= \frac{\partial m^2}{\partial m_0^2}\frac{\partial}{\partial m^2}\lambda(\lambda_0, \Lambda/m) \\ &= \frac{\partial m^2}{\partial m_0^2}\frac{1}{2m}\frac{\partial}{\partial m}\lambda(\lambda_0, \Lambda/m) \end{aligned} \tag{7.33}$$

obtemos da definição 7.24

$$\beta = m\frac{\partial}{\partial m}\lambda(\lambda_0, \Lambda/m) = m\frac{\partial}{\partial m}[\overline{Z}(\lambda_0, \Lambda/m)\lambda_0] = -\lambda_0\Lambda\frac{\partial}{\partial \Lambda}[\overline{Z}(\lambda_0, \Lambda/m)] \tag{7.34}$$

ou

$$\beta = -\lambda\frac{\partial}{\partial \ln \Lambda}[\ln \overline{Z}(\lambda_0, \Lambda/m)] \tag{7.35}$$

onde[1] $\overline{Z} = Z_\lambda^{-1}Z_\phi^2$. O resultado de 1 - loop dá

$$\begin{aligned} Z_\lambda &= 1 + \frac{3\lambda_0}{32\pi^2}\ln\frac{\Lambda^2}{m^2} + O(\lambda_0^2) \\ Z_\phi &= 1 + O(\lambda_0^2) \end{aligned} \tag{7.36}$$

logo

$$\overline{Z} = 1 - \frac{3\lambda_0}{32\pi^2}\ln\frac{\Lambda^2}{m^2} + ... \tag{7.37}$$

e

$$\ln \overline{Z} = \frac{3\lambda_0}{16\pi^2}\ln\frac{\Lambda}{m} + \cdots \tag{7.38}$$

Portanto para ϕ^4

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)\ . \tag{7.39}$$

[1] Por definição $\lambda = \overline{Z}\lambda_0$.

7.1.4 Weinberg's theorem and the solution of the RG equations

O teorema de Weinberg diz respeito ao comportamento assimptótico das funções de Green $1PI$ na região Euclediana ($p_i^2 < 0$) e para valores não excepcionais dos momentos (nenhuma soma parcial é nula).

Teorema 3.1

Se os momentos não forem excepcionais e se os parametrizarmos com $p_i = \sigma k_i$ as funções de Green irredutí veis de e partícula $\Gamma_R^{(n)}$ comportam-se na região euclediana profunda ($\sigma \to \infty$ e k_i fixos, $p_i^2 < 0$) do modo seguinte

$$\lim_{\sigma\to\infty} \Gamma^{(n)}(\sigma k_i, \lambda, m) = \sigma^{4-n}[a_0(\ln\sigma)^{b_0} + a_1(\ln\sigma)^{b_1} + \cdots] \tag{7.40}$$

e

$$\lim_{\sigma\to\infty} \Gamma_{\phi^2}^{(n)}(\sigma k_i, \lambda, m) = \sigma^{2-n}[a_0'(\ln\sigma)^{b_0'} + a_1'(\ln\sigma)^{b_1'} + \cdots] \ . \tag{7.41}$$

Não faremos a demonstração (ver Bjorken and Drell) mas notemos que as potências de σ são as dimensões canónicas das funções de Green (em termos da massa). Se este comportamento é o verificado assimptoticamente depende da soma da série dos logaritmos. Se esta somar para uma potência de σ, por exemplo $\sigma^{-\gamma}$, então assimptóticamente o comportamento canónico σ^{4-n} é modificado para $\sigma^{4-n-\gamma}$.γ é chamada a *dimensão anómala*. Como vamos ver o GR vai efectuar esta soma de logaritmos e dar-nos qual a dimensão anómala.

7.1.5 Asymptotic solution of the RG equations

Do teorema de Weinberg temos que $\Gamma_R^{(n)} \gg \Gamma_{\phi^2 R}^{(n)}$ para qualquer ordem (finita) em λ na região euclediana profunda ($\sigma \to \infty$). Se admitirmos que isto continua verdade mesmo depois de somar todas as ordens de teoria de perturbações, então podemos desprezar o segundo membro da equação de Callan-Symanzik e obtemos uma equação diferencial homogénea

$$\left[m\frac{\partial}{\partial m} + \beta(\lambda)\frac{\partial}{\partial\lambda} - n\gamma(\lambda)\right]\Gamma_{as}^{(n)}(p_i, \lambda, m) = 0 \tag{7.42}$$

onde $\Gamma_{as}^{(n)}$ é a forma assimptótica de $\Gamma_R^{(n)}$. O significado desta equação é que nesta região assimptótica, uma mudança no parâmetro de massa pode ser sempre compensada por mudanças apropriadas do acoplamento e da escala dos campos.

Para resolver esta equação começamos por definir uma quantidade $\overline{\Gamma}_R^{(n)}$ sem dimensões, usando análise dimensional

$$\Gamma_{as}^{(n)}(p_i, \lambda, m) = m^{4-n}\overline{\Gamma}_R^{(n)}(p_i/m, \lambda) \ . \tag{7.43}$$

$\overline{\Gamma}_R^{(n)}$ satisfaz

$$\left(m\frac{\partial}{\partial m}+\sigma\frac{\partial}{\partial\sigma}\right)\overline{\Gamma}_R^{(n)}\left(\sigma\frac{p_i}{m},\lambda\right)=0\ . \tag{7.44}$$

Então

$$\left(m\frac{\partial}{\partial m}+\sigma\frac{\partial}{\partial\sigma}\right)m^{n-4}\Gamma_{as}^{(n)}(\sigma p_i,\lambda,m)=0 \tag{7.45}$$

ou seja

$$\left[m\frac{\partial}{\partial m}+\sigma\frac{\partial}{\partial\sigma}+(n-4)\right]\Gamma_{as}^{(n)}(\sigma p_i,\lambda,m)=0 \tag{7.46}$$

Usando esta equação podemos trocar a derivação em ordem à massa pela derivação em ordem à escala na equação de Callan-Symanzik para obter

$$\left[\sigma\frac{\partial}{\partial\sigma}-\beta(\lambda)\frac{\partial}{\partial\lambda}+n\gamma(\lambda)+(n-4)\right]\Gamma_{as}^{(n)}(\sigma p_i,\lambda,m)=0 \tag{7.47}$$

Para resolver esta equação removemos os termos sem derivadas com a transformação

$$\Gamma_{as}^{(n)}(\sigma p_i,\lambda,m)=\sigma^{4-n}e^{n\int_0^\lambda\frac{\gamma(x)}{\beta(x)}dx}F^{(n)}(\sigma p_i,\lambda,m)\ . \tag{7.48}$$

Substituindo na equação diferencial vemos que os termos sem derivadas desaparecem e obtemos uma equação diferencial para $F^{(n)}$

$$\left[\sigma\frac{\partial}{\partial\sigma}-\beta(\lambda)\frac{\partial}{\partial\lambda}\right]F^{(n)}(\sigma p,\lambda,m)=0 \tag{7.49}$$

Introduzindo $t=\ln\sigma$ podemos escrever

$$\left[\frac{\partial}{\partial t}-\beta(\lambda)\frac{\partial}{\partial\lambda}\right]F^{(n)}(e^t p,\lambda,m)=0 \tag{7.50}$$

Para resolver esta equação introduzimos a constante de acoplamento efectiva $\overline{\lambda}(t,\lambda)$ como solução da equação

$$\frac{\partial\overline{\lambda}(t,\lambda)}{\partial t}=\beta(\overline{\lambda}) \tag{7.51}$$

com a condição fronteira $\overline{\lambda}(0,\lambda)=\lambda$. Para vermos que esta definição nos vai dar a solução, escrevemos

$$t=\int_\lambda^{\overline{\lambda}(t,\lambda)}\frac{dx}{\beta(x)} \tag{7.52}$$

e diferenciamos em ordem a λ

$$0=\frac{1}{\beta(\overline{\lambda})}\frac{\partial\overline{\lambda}}{\partial\lambda}-\frac{1}{\beta(\lambda)} \tag{7.53}$$

ou ainda

$$\beta(\overline{\lambda}) - \beta(\lambda)\frac{\partial\overline{\lambda}}{\partial\lambda} = 0 \tag{7.54}$$

Usando agora a definição de $\overline{\lambda}$ obtemos

$$\left[\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial\lambda}\right]\overline{\lambda}(t,\lambda) = 0 \tag{7.55}$$

O operador diferencial é exactamente o mesmo da equação para $F^{(n)}(e^t p, \lambda, m)$. Portanto $F^{(n)}$ satisfaz aquela equação se depender da t e λ através de $\overline{\lambda}(t,\lambda)$. Portanto a solução geral de $\Gamma_{as}^{(n)}$ é

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, m) = \sigma^{4-n} e^{n\int_0^\lambda \frac{\gamma(x)}{\beta(x)}dx} F^{(n)}(p_i, \overline{\lambda}(t,\lambda), m) \tag{7.56}$$

Para se obter uma interpretação física desta solução notemos que

$$\begin{aligned} e^{n\int_0^\lambda \frac{\gamma(x)}{\beta(x)}dx} &= e^{n\int_0^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} e^{n\int_{\overline{\lambda}}^{\lambda} \frac{\lambda(x)}{\beta(x)}dx} \\ &= e^{n\int_0^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} e^{-n\int_{\lambda}^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} \\ &= e^{n\int_0^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} e^{-n\int_0^t \gamma(\overline{\lambda}(t',\lambda))dt'} \end{aligned} \tag{7.57}$$

Portanto

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, m) = \sigma^{4-n} e^{-n\int_0^t \gamma(\overline{\lambda}(t',\lambda))dt'} e^{-n\int^{\overline{\lambda}} \frac{\gamma(x)}{\beta(x)}dx} F^{(n)}(p_i, \overline{\lambda}(t,\lambda), m) \tag{7.58}$$

Se pusermos $\sigma = 1 (t = 0)$, vemos que $e^{n\int_0^{\overline{\lambda}} \frac{\gamma}{\beta}dx} F^{(n)}$ é $\Gamma_{as}^{(n)}$. Então obtemos finalmente a solução da equação do GR.

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, m) = \sigma^{4-n} e^{-n\int_0^t \gamma(\overline{\lambda}(t',\lambda))dt'} \Gamma_{as}^{(n)}(p_i, \overline{\lambda}(t,\lambda), m) \tag{7.59}$$

Nesta forma a solução tem uma interpretação simples. O efeito de efectuar uma mudança de escala nos momentos p_i nas funções de Green $\Gamma_R^{(n)}$ é equivalente a substituir a constante de acoplamento λ, pela constante de acoplamento efectiva $\overline{\lambda}$ à parte factores multiplicativos. O primeiro é simplesmente resultante do facto de $\Gamma_R^{(n)}$ ter dimensão canónica $4 - n$ em unidades de massa. O factor exponencial é o termo da dimensão anómala que resultou de somar todos os logaritmos em teoria de perturbações. Este factor é controlado por γ, a dimensão anómala. Veremos à frente como calcular a dimensão anómala, numa teoria qualquer.

7.2 Minimal subtraction (MS) scheme

7.2.1 Renormalization group equations for MS

Vamos agora ver outras formas que pode tomar a equação do grupo de renormalização. A afirmação que a renormalização é multiplicativa pode ser escrita na forma

$$\Gamma^{(n)}(p_i, \lambda_0, m_0) = Z_\phi^{-n/2} \Gamma_R^{(n)}(p_i, \lambda, m, \mu) \tag{7.60}$$

onde μ é a escala usada para definir a normalização das funções de Green. O lado esquerdo da equação não depende de μ, mas o lado direito depende explicitamente e implicitamente através de λ e m. Então temos

$$\mu \frac{\partial}{\partial \mu} \left[Z_\phi^{-n/2} \Gamma_R^{(n)}(p_i, \lambda, m, \mu) \right] = 0 \tag{7.61}$$

ou seja

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + \gamma_m m \frac{\partial}{\partial m} - n\gamma \right) \Gamma_R^{(n)} = 0 \tag{7.62}$$

com

$$\begin{aligned} \beta\left(\lambda, \frac{m}{\mu}\right) &= \mu \frac{\partial \lambda}{\partial \mu} \\ \gamma_m\left(\lambda, \frac{m}{\mu}\right) &= \mu \frac{\partial \ln m}{\partial \mu} \\ \gamma\left(\lambda, \frac{m}{\mu}\right) &= \frac{1}{2} \mu \frac{\partial \ln Z_\phi}{\partial \mu} \end{aligned} \tag{7.63}$$

Esta equação tem a vantagem sobre a equação de Callan - Symanzik de ser homogénea. A dificuldade reside nas funções β e γ dependerem de duas variáveis λ e $\frac{m}{\mu}$ e portanto a equação ser de difícil resolução. Existe contudo um esquema de renormalização em que a dependência em $\frac{m}{\mu}$ desaparece e portanto a equação é simples de resolver. É o chamado esquema de subtracção mínima que passamos a expôr.

7.2.2 Minimal subtraction scheme

O esquema de subtracção mínima (MS) está relacionado com o método de regularização dimensional. As divergências dos integrais aparecem neste método como pólos em $\frac{1}{\varepsilon}$ onde $\varepsilon = 4 - d$. O esquema de subtracção mínima consiste em escolher os contratermos para cancelar *somente* os pólos.

Vamos exemplificar com a *self-energy* em $\lambda\phi^4$, a que corresponde o diagrama da Fig. 7.1

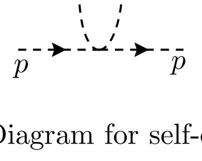


Figure 7.1: Diagram for self-energy in ϕ^4.

Temos

$$\begin{aligned} -i\Sigma(p) &= (-i\lambda)\mu^\varepsilon \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{p^2-m^2+i\varepsilon} \\ &= -i\lambda \frac{1}{32\pi^2}\mu^\varepsilon \frac{\Gamma(1-d/2)}{m^{2-d}} 2^\varepsilon \pi^{\varepsilon/2} \end{aligned} \tag{7.64}$$

onde $\varepsilon = 4-d$. Então

$$\begin{aligned} \Sigma(p^2) &= \lambda \frac{1}{32\pi^2}\mu^\varepsilon \frac{\Gamma(-1+\varepsilon/2)}{m^{-2+\varepsilon}} \cdot (2\sqrt{\pi})^\varepsilon \\ &= \lambda \frac{m^2}{32\pi^2}\left(\frac{\mu}{m}\right)^\varepsilon \Gamma(-1+\varepsilon/2)\cdot(2\sqrt{\pi})^\varepsilon \end{aligned} \tag{7.65}$$

Usando[2]

$$\Gamma\left(-1+\frac{\varepsilon}{2}\right) = -\left[\frac{2}{\varepsilon} + \overbrace{1-\gamma}^{\psi(2)} + O(\varepsilon)\right] \tag{7.66}$$

e

$$\left(\frac{\mu}{m}\right)^\varepsilon = 1 + \varepsilon \ln\left(\frac{\mu}{m}\right) \tag{7.67}$$

obtemos

$$\Sigma(p^2) = -\frac{\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon} + \psi(2) + 2\ln(\mu/m) + 2\ln 2\sqrt{\pi} + O(\varepsilon)\right] \tag{7.68}$$

Portanto no esquema de subtracção mínima devemos adicionar um contratermo

$$\Delta\mathcal{L}^{MS}_{\phi^2} = -\frac{\lambda m^2}{32\pi^2}\frac{1}{\varepsilon}\phi^2 \tag{7.69}$$

Se tivéssemos feito subtracção de momento à escala μ, isto é $\Sigma_R(p^2=\mu^2)=0$ teríamos o contratermo

$$\Delta\mathcal{L}^{MOM}_{\phi^2} = -\frac{\lambda m^2}{32\pi^2}\left[\frac{1}{\varepsilon} + \frac{1}{2}\psi(2) + \ln(\mu/m) + \ln 2\sqrt{\pi}\right]\phi^2 \tag{7.70}$$

Vemos assim que o Lagrangeano de contratermos no esquema de subtracção mínima quando expandido em série de Laurent em ε contém só termos divergentes. Portanto

$$\begin{aligned} \phi_0 &= \sqrt{Z_\phi}\phi \\ m_0 &= Z_m m \end{aligned}$$

[2] γ é a constante de Euler e $\psi(x)$ a derivada logaritmica da função Γ. Ver o Apêndice da Mecânica Quântica Relativista.

$$\lambda_0 = \mu^{\varepsilon} Z_\lambda \lambda \tag{7.71}$$

tendo as constantes de renormalização Z_ϕ, Z_m e Z_λ a forma

$$\begin{aligned} Z_\lambda &= 1 + \sum_{r=1}^{\infty} a_r(\lambda)/\varepsilon^r \\ Z_m &= 1 + \sum_{r=1}^{\infty} b_r(\lambda)/\varepsilon^r \\ Z_\phi &= 1 + \sum_{r=1}^{\infty} c_r(\lambda)/\varepsilon^r \end{aligned} \tag{7.72}$$

Assim os coeficientes da equação do grupo de renormalização são independentes de μ, e como são adimensionais, também são independentes de m dependendo somente da constante de acoplamento. Isto simplifica a solução da equação do grupo de renormalização

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \gamma_m m\frac{\partial}{\partial m} - n\gamma\right)\Gamma_R^{(n)} = 0 \tag{7.73}$$

Usando análise dimensional

$$\left[m\frac{\partial}{\partial m} + (n-4) + \mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma}\right]\Gamma_R(\sigma p, m, \lambda, \mu) = 0 \tag{7.74}$$

e podemos escrever

$$\left[\sigma\frac{\partial}{\partial\sigma} - \beta\frac{\partial}{\partial\lambda} - (\gamma_m - 1)\, m\frac{\partial}{\partial m} + n\gamma + (n-4)\right]\Gamma_R(\sigma p, m, \lambda, \mu) = 0 \tag{7.75}$$

que tem a solução

$$\Gamma_R(\sigma p_i, m, \lambda, \mu) = \sigma^{4-n} e^{-n\int_0^t \gamma(\overline{\lambda}(t'))dt'}\Gamma_R^{(n)}(p_i, \overline{m}(t), \overline{\lambda}(t), \mu) \tag{7.76}$$

onde se introduziram a massa efectiva $\overline{m}(t)$ e a constante de acoplamento efectiva $\overline{\lambda}(t)$ definidas por

$$\left\{ \begin{array}{lll} \dfrac{d\overline{\lambda}}{dt} = \beta(\overline{\lambda}) & ; & \overline{\lambda}(t=0) = \lambda \\[2ex] \dfrac{d\overline{m}(t)}{dt} = \Big[\gamma_m(\lambda) - 1\Big]\overline{m}(t) & ; & \overline{m}(t=0) = m \end{array} \right. \tag{7.77}$$

A solução desta equação é

$$\begin{aligned} \overline{m}(t) &= m\; e^{\int_0^t [\gamma_m(\overline{\lambda}(t'))-1]dt'} \\ &= m\; e^{-t} e^{\int_0^t \gamma_m(\overline{\lambda}(t'))dt'} \\ &= m\; e^{-t} e^{\int_\lambda^{\overline{\lambda}(t)} dx \frac{\gamma_m(x)}{\beta(x)}} \end{aligned} \tag{7.78}$$

7.2.3 Physical parameters

Os parâmetros definidos pelo esquema de subtracção mínima não são parâmetros físicos. Os parâmetros físicos podem no entanto ser calculados em função deles. Por parâmetro físico entendemos um elemento de matriz S ou a posição do pólo no propagador. Para eles é válido o teorema seguinte,

Teorema 3.2:

Qualquer parâmetro físico $P(\lambda, m, \mu)$ satisfaz a seguinte equação do grupo de renormalização

$$\mathcal{D}P(\lambda, m, \mu) \equiv \left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma_m m\frac{\partial}{\partial m}\right] P(\lambda, m, \mu) = 0 \tag{7.79}$$

Dem: *Consideremos primeiro o propagador escalar $\Delta(p^2)$ que satisfaz a equação do grupo de renormalização*

$$[\mathcal{D} + 2\gamma]\Delta(p^2, \lambda, m, \mu) = 0 \tag{7.80}$$

Podemos escrever uma série de Laurent em voltado pólo em $p^2 = m_p^2$

$$\Delta(p^2, \lambda, m, \mu) = \frac{R^2}{p^2 - m_p^2} + \widetilde{\Delta} \tag{7.81}$$

A posição do pólo $m_p(\lambda, m, \mu)$ e o resíduo $R^2(\lambda, m, \mu)$ satisfazem as equações do grupo de renormalização que podem ser obtidas por aplicação do operador $(\mathcal{D} + 2\gamma)$ à equação anterior. Igualando os resíduos dos pólos obtemos

$$\mathcal{D}m_p(\lambda, m, \mu) = 0 \tag{7.82}$$

$$[\mathcal{D} + \gamma(\lambda)]R(\lambda, m, \mu) = 0 \tag{7.83}$$

Demonstrámos portanto o teorema para a massa física. Para um elemento da matriz S temos ($S_R = R^n\Gamma^{(n)}$)

$$\begin{aligned}\mathcal{D} \lim_{p_i^2\to m_p^2} R^n\Gamma^{(n)} &= \lim_{p_i^2\to m_p^2} \mathcal{D}(R^n\Gamma^n)\\ &= \lim_{p_i^2\to m_p^2}[n\mathcal{D}RR^{n-1}\Gamma^n + R^n\mathcal{D}\Gamma^n]\\ &= \lim_{p_i^2\to m_p^2}[-n\gamma + n\gamma]R^n\Gamma^n = 0\end{aligned} \tag{7.84}$$

o que completa a demonstração.

Veremos à frente como estes resultados podem ser usados para relacionar os parâmetros físicos com os parâmetros da teoria.

7.2.4 Renomalization group functions in minimal subtraction

Vimos anteriormente que

$$\begin{cases} \phi_0 & = \sqrt{Z_\phi}\phi \\ m_0 & = Z_m m \\ \\ \lambda_0 & = \mu^\varepsilon Z_\lambda \lambda \end{cases} \tag{7.85}$$

e que as constantes de renormalização têm a forma.

$$\begin{cases} Z_\lambda & = 1 + \sum_{r=1}^\infty a_r(\lambda)/\varepsilon^r \\ \\ Z_m & = 1 + \sum_{r=1}^\infty b_r(\lambda)/\varepsilon^r \\ \\ Z_\phi & = 1 + \sum_{r=1}^\infty c_r(\lambda)/\varepsilon^r \ . \end{cases} \tag{7.86}$$

Vejamos como se calculam β, γ_m e γ.

i) Cálculo de $\beta(\lambda)$

Por definição

$$\beta(\lambda) = \mu \frac{\partial \lambda}{\partial \mu} \tag{7.87}$$

Esta quantidade é finita no limite $\varepsilon \to 0$. Isto quer dizer que antes de fazermos $\varepsilon \to 0$ deve ser uma função analítica em ε. É então conveniente definir

$$\beta(\lambda) = \hat{\beta}(\lambda, \varepsilon = 0) = d_0 \tag{7.88}$$

onde

$$\hat{\beta}(\lambda, \varepsilon) = d_0 + d_1\varepsilon + d_2\varepsilon^2 + \cdots \tag{7.89}$$

com coeficientes d_r a determinar. Posto isto, usamos o facto de λ_0 não depender da escala μ. Então

$$\begin{aligned} 0 &= \mu \frac{\partial}{\partial \mu}(\mu^\varepsilon Z_\lambda \lambda) \\ &= \varepsilon \mu^\varepsilon Z_\lambda \lambda + \mu^\varepsilon \hat{\beta}(\lambda, \varepsilon) \lambda \frac{\partial Z_\lambda}{\partial \lambda} + \mu^\varepsilon Z_\lambda \hat{\beta}(\lambda, \varepsilon) \end{aligned} \tag{7.90}$$

Então

$$\varepsilon \lambda Z_\lambda + \hat{\beta}(\lambda, \varepsilon) \left(Z_\lambda + \lambda \frac{\partial Z_\lambda}{\partial \lambda} \right) = 0 \tag{7.91}$$

Usando as expressões de Z_λ e $\hat{\beta}$ obtemos

$$\varepsilon\lambda + a_1\lambda + \lambda \sum_{r=1}^\infty \frac{a_{r+1}}{\varepsilon^r} + (d_0 + d_1\varepsilon + d_2\varepsilon^2 + \cdots) \left[1 + \sum_{r=1}^\infty \frac{1}{\varepsilon^r} \left(a_r + \lambda \frac{da_r}{d\lambda} \right) \right] = 0 \tag{7.92}$$

Então $d_r = 0$ para $r > 1$ e

$$\begin{aligned} \varepsilon(\lambda + d_1) &+ \left[a_1\lambda + d_0 + d_1\left(a_1 + \lambda\frac{da_1}{d\lambda}\right)\right] + \sum_r \frac{1}{\varepsilon^r}\left[a_{r+1}\lambda + d_0\left(a_r + \lambda\frac{da_r}{d\lambda}\right)\right. \\ &+ \left. d_1\left(a_{r+1} + \lambda\frac{da_{r+1}}{d\lambda}\right)\right] = 0 \end{aligned} \tag{7.93}$$

logo

$$\begin{aligned} &\lambda + d_1 = 0 \\ &a_1\lambda + d_0 + d_1\left(a_1 + \lambda\frac{da_1}{d\lambda}\right) = 0 \\ &a_{r+1}\lambda + d_0\left(a_r + \lambda\frac{da_r}{d\lambda}\right) + d_1\left(a_{r+1} + \lambda\frac{da_{r+1}}{d\lambda}\right) = 0 \end{aligned} \tag{7.94}$$

Estes cálculos dão

$$d_1 = -\lambda \tag{7.95}$$

$$\begin{aligned} \beta(\lambda) &= d_0 = \lambda^2\frac{da_1}{d\lambda} \\ \lambda^2\frac{d}{d\lambda}(a_{r+1}) &= \beta(\lambda)\frac{d}{d\lambda}(\lambda a_r) \end{aligned} \tag{7.96}$$

Portanto a função $\beta(\lambda)$ depende somente do coeficiente em $\frac{1}{\varepsilon}$ de Z_λ que se calcula fácilmente em teoria de perturbações. Além disso vemos que os resíduos dos pólos de ordem superior se podem calcular em termos do resíduo do pólo simples. Para $\lambda\phi^4$ é fácil de ver que

$$Z_\lambda = 1 + \frac{3\lambda}{16\pi^2}\frac{1}{\varepsilon} + \cdots \tag{7.97}$$

e portanto

$$\beta(\lambda) = \lambda^2\frac{da_1}{d\lambda} = \lambda^2\frac{d}{d\lambda}\left(\frac{3\lambda}{16\pi^2}\right) = \frac{3\lambda^2}{16\pi^2} \tag{7.98}$$

como tínhamos obtido anteriormente. Para teorias de gauge há uma pequena modificação pois $g_0 = \mu^{\varepsilon/2}Z_g g$. Um cálculo trivial dá neste caso

$$d_1 = -g/2 \tag{7.99}$$

e

$$\beta(g) = \frac{1}{2}g^2\frac{da_1}{dg}$$

$$\frac{1}{2}g^2\frac{da_{r+1}}{dg} = \beta(g)\frac{d}{dg}(ga_r) \tag{7.100}$$

onde, como anteriormente

$$Z_g = 1 + \sum_{r=1}^{\infty} a_r(g)/\varepsilon^r \ . \tag{7.101}$$

ii) Cálculo de $\gamma_m(\lambda)$

Partimos de $m_0 = Z_m m$. Aplicando $\mu\frac{\partial}{\partial\mu}$ obtemos

$$\begin{aligned} 0 &= \mu\frac{\partial Z_m}{\partial\mu}m + Z_m\mu\frac{\partial m}{\partial\mu} \\ &= \hat{\beta}(\lambda,\varepsilon)\frac{\partial Z_m}{\partial\lambda}m + mZ_m\mu\frac{\partial \ln m}{\partial\mu} \end{aligned} \tag{7.102}$$

Como $\mu\frac{\partial \ln m}{\partial\mu} = \gamma_m$, obtemos a equação

$$\left[\hat{\beta}(\lambda,\varepsilon)\frac{\partial}{\partial\lambda} + \gamma_m\right] Z_m = 0 \tag{7.103}$$

ou seja

$$\left(\gamma_m + d_1\frac{db_1}{d\lambda}\right) + \sum_{r=1}^{\infty}\frac{1}{\varepsilon_r}\left[d_0\frac{db_r}{d\lambda} + \gamma_m b_r + d_1\frac{db_{r+1}}{d\lambda}\right] = 0 \tag{7.104}$$

Portanto

$$\gamma_m = -d_1\frac{db_1}{d\lambda} \tag{7.105}$$

$$-d_1\frac{db_{r+1}}{d\lambda} = \beta(\lambda)\frac{db_r}{d\lambda} + \gamma_m b_r \tag{7.106}$$

onde

$$d_1 = \begin{cases} -\lambda \qquad \text{teoria} \qquad \lambda\phi^4 \\ \\ -\,g/2 \qquad \text{teorias de gauge} \end{cases} \tag{7.107}$$

Mais uma vez γ_m depende somente do resíduo do pólo simples.

iii) Cálculo de $\gamma(\lambda)$

Aqui é mais fácil partir da definição de $\gamma(\lambda)$

$$\gamma(\lambda) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_\phi = \frac{1}{2}\mu\frac{\partial}{\partial\mu}Z_\phi\frac{1}{Z_\phi} \tag{7.108}$$

logo

$$\left[\hat{\beta}(\lambda,\varepsilon)\frac{\partial}{\partial\lambda} - 2\gamma(\lambda)\right] Z_\phi = 0 \tag{7.109}$$

o que dá

$$-2\gamma(\lambda) + d_1 \frac{dc_1}{d\lambda} + \sum_{r=1}^{\infty} \frac{1}{\varepsilon^r} \left[d_0 \frac{dc_r}{d\lambda} - 2\gamma c_r + d_1 \frac{dc_{r+1}}{d\lambda} \right] = 0 \tag{7.110}$$

Então

$$\gamma(\lambda) = \frac{1}{2} d_1 \frac{dc_1}{d\lambda} \tag{7.111}$$

$$-d_1 \frac{dc_{r+1}}{d\lambda} = \beta(\lambda) \frac{dc_r}{d\lambda} - 2\gamma c_r \tag{7.112}$$

sendo o coeficiente d_1 dado anteriormente, Eq. (7.107).

Podemos concluir dizendo que o coeficiente do pólo simples nas constantes de renormalização, determina univocamente as funções β, γ_m e γ e também os valores dos pólos de ordem superior.

7.2.5 β and γ properties

Nós adoptamos um esquema particular de renormalização. Se tivéssemos adoptado outro esquema teríamos outra definição dos parâmetros da teoria e funções β, γ_m e γ diferentes. Vamos aqui discutir os aspectos do grupo de renormalização que são independentes do esquema usado.

Consideremos então dois esquemas (ambos independentes da massa). Então

$$\begin{aligned} g' &= g F_g(g) & F_g(g) &= 1 + O(g^2) \\ Z'_m(g') &= Z_m(g) F_m(g) & F_m(g) &= 1 + O(g^2) \\ Z'_\phi(g') &= Z_\phi(g) F_\phi(g) & F_\phi(g) &= 1 + O(g') \end{aligned} \tag{7.113}$$

O 1 nas funções F expressa o facto que ao nível árvore não há ambiguidades. Usando as relações acima podemos ver como estão relacionadas as funções β, γ_m e γ em dois esquemas. Obtemos (estamos a considerar o caso duma teoria de gauge)

$$\begin{aligned} \beta'(g') &= \mu \frac{\partial}{\partial \mu} g' = \mu \frac{\partial}{\partial \mu} (g F_g(g)) = \beta(g) \left(F_g + g \frac{\partial F_g}{\partial g} \right) \\ \gamma'_m(g') &= \mu \frac{\partial}{\partial \mu} \ln m' = \mu \frac{\partial \ln}{\partial \mu} (F_m^{-1}(g) m) = \gamma_m(g) - \beta(g) \frac{\partial}{\partial g} \ln F_m \\ \gamma'(g') &= \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z'_\phi(g') = \gamma(g) + \frac{1}{2} \beta(g) \frac{\partial}{\partial g} \ln F_\phi \end{aligned} \tag{7.114}$$

As funções β, γ_m e γ só coincidirão se os esquemas forem o mesmo, isto é $F_g = F_m = F_\phi = 1$. Contudo as propriedades seguintes são ainda independentes do esquema.

i) A existência de um zero de $\beta(g)$.

Se $\beta(g_0) = 0$ então $\beta'(g'_0) = 0$ para $g'_0 = g_0 F_g(g_0)$. Notar que em geral g_0 depende do esquema, isto é $g_0 \neq g'_0$.

ii) A primeira derivada de $\beta(g)$ no zero.
Seja $\beta(g_0) = 0$. Então

$$\begin{aligned}
\frac{\partial \beta'(g'_0)}{\partial g'} &= \left\{ \frac{\partial g}{\partial g'} \frac{\partial}{\partial g} \left[\beta(g) \left(F_g + g \frac{\partial F_g}{\partial g} \right) \right] \right\}_{g_0} \\
&= \left[F_g + g \frac{\partial F_g}{\partial g} + g \frac{\partial \beta}{\partial g} + \beta(g) \frac{1}{F_g + g \frac{\partial F_g}{\partial g}} \frac{\partial \left(F_g + g \frac{\partial F_g}{\partial g} \right)}{\partial g} \right]_{g_0} \\
&= \frac{\partial \beta}{\partial g}(g_0) \cdot
\end{aligned} \tag{7.115}$$

iii) Os primeiros dois termos em $\beta(g)$.
Seja $\beta(g) = b_0 g^3 + b_1 g^5 + O(g^7)$, e

$$F_g(g) = 1 + a g^2 + O(g^4) \cdot \tag{7.116}$$

Então

$$g' = g + a g^3 + O(g^5) \tag{7.117}$$

e

$$g = g' - a g'^3 + O(g^5) \tag{7.118}$$

Portanto

$$\begin{aligned}
\beta'(g') &= \beta(g) \frac{\partial}{\partial g}(g F_g) = (b_0 g^3 + b_1 g^5 + O(g^7))(1 + 3 a g^2 + O(g^4)) \\
&= b_0 g^3 + (3 a b_0 + b_1) g^5 + O(g^7) \\
&= b_0 (g'^3 - 3 a g'^5 + O(g'^7) + (3 a b_0 + b_1)(g'^5 + O(g'^7)) \\
&= b_0 g'^3 + b_1 g'^5 + O(g'^7)
\end{aligned} \tag{7.119}$$

iv) O primeiro termo em $\gamma(g)$ e $\gamma_m(g)$.
Seja

$$\begin{aligned}
\gamma(g) &= c g^2 + O(g^4) \\
\gamma_m(g) &= d g^2 + O(g^4)
\end{aligned} \tag{7.120}$$

Então como $\beta(g) = O(g^3)$ é evidente que

$$\gamma'(g') = c g'^2 + O(g'^4)$$

$$\gamma'_m(g') \quad = \quad dg'^2 + O(g'^4) \ . \tag{7.121}$$

v) O valor de $\gamma(g_0)$ *e* $\gamma_m(g_0)$ *se* $\beta(g_0) = 0$.

Este resultado é imediato. Como veremos na secção seguinte todos estes resultados são necessários pois eles controlam resultados físicos e estes não podem depender do esquema de renormalização.

7.2.6 Gauge independence of β and γ_m in MS

A equação do grupo de renormalização em MS foi escrita para a teoria $\lambda\phi^4$. Vamos agora considerar teorias da gauge (abelianas ou não abelianas). Para a quantificação destas teorias é necessário introduzir um termo que fixe a gauge

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial \cdot A)^2 \tag{7.122}$$

onde escolhemos as gauges do tipo de Lorentz. Como não há correcções radiativas para a parte longitudinal do propagador, não é necessário nenhum contratermo para o termo que fixa a gauge. Portanto se pusermos, como habitualmente,

$$A^\mu = Z_A^{-1/2} A_0^\mu \tag{7.123}$$

obtemos

$$\frac{1}{2\xi}(\partial \cdot A)^2 = \frac{1}{2\xi Z_A}(\partial \cdot A_0)^2 = \frac{1}{2\xi_0}(\partial \cdot A_0)^2 \tag{7.124}$$

o que quer dizer que o parâmetro de gauge é renormalização de acordo com

$$\xi_0 = Z_A \xi \ . \tag{7.125}$$

As funções de Green irredutíveis renormalizadas, dependem em geral de ξ, isto é

$$\Gamma_R^{(n)}(g, m, \xi, \mu) = Z_A^{n/2} \Gamma_0^{(n)}(g_0, m_0, \xi_0, \varepsilon) \tag{7.126}$$

A equação do grupo de renormalização é então

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g,\xi)\frac{\partial}{\partial g} + \gamma_m(g,\xi) m \frac{\partial}{\partial m} + \delta(g,\xi)\frac{\partial}{\partial \xi} - \gamma_A(g,\xi)\right] \Gamma_R^{(n)}(g, m, \xi, \mu) = 0 \tag{7.127}$$

onde

$$\begin{aligned}
\delta(g,\xi) \quad &= \quad \mu \frac{\partial}{\partial \mu}\xi = \mu \frac{\partial}{\partial \mu}(Z_A^{-1}\xi_0) = \\
&= \quad -\xi_0 \frac{1}{Z_A^2}\frac{\partial}{\partial \mu} Z_A \\
&= \quad -2\xi \gamma_A(g,\xi)
\end{aligned} \tag{7.128}$$

e se admitiu a possibilidade de β, γ_m e γ_A dependerem do parâmetro ξ. Contudo a dependência em ξ não é arbitrária, obedece a certos constrangimentos. Para vermos isso consideremos uma função de Green sem dimensões e correspondendo a operadores invariantes de gauge. Então

$$\frac{\partial}{\partial \xi_0} G_0(g_0, m_0, \xi_0, \varepsilon) = 0 \qquad \text{(independente de gauge)} \tag{7.129}$$

e

$$G_0(g_0, m_0, \xi_0, \varepsilon) = G(g, m, \xi, \mu) \qquad \text{(sem dimensões} \tag{7.130}$$

e portanto

$$\frac{\partial}{\partial \xi} G = 0 \tag{7.131}$$

ou seja

$$\mathcal{D}_G G \equiv \left[\frac{\partial}{\partial \xi} + \rho(g, \xi) \frac{\partial}{\partial g} + \sigma(g, \xi) m \frac{\partial}{\partial m} \right] G(g, m, \xi, \mu) = 0 \tag{7.132}$$

onde

$$\rho(g, \xi) = \frac{\partial g}{\partial \xi} \qquad ; \qquad \sigma(g, \xi) = \frac{\partial}{\partial \xi} \ln m \tag{7.133}$$

Mas G obedece à equação do grupo de renormalização

$$\mathcal{D} G \equiv \left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + \gamma_m m \frac{\partial}{\partial m} + \delta \frac{\partial}{\partial \xi} \right] G = 0 \tag{7.134}$$

Usando a equação para $\mathcal{D}_G G = 0$ podemos substituir a derivada em ordem a ξ por derivadas em ordem aos outros parâmetros, obtendo uma equação do grupo de renormalização semelhante à das teorias que não têm campos de gauge, isto é

$$\left[\mu \frac{\partial}{\partial \mu} + \overline{\beta} \frac{\partial}{\partial g} + \overline{\gamma}_m m \frac{\partial}{\partial m} \right] G = 0 \tag{7.135}$$

onde

$$\overline{\beta} \equiv \beta - \rho \delta \qquad \qquad \overline{\gamma}_m = \gamma_m - \sigma \delta \tag{7.136}$$

Calculemos agora o comutador $[\mathcal{D}_G, \mathcal{D}] G = 0$. Obtemos

$$\begin{aligned} &\left\{ \left[\frac{\partial \beta}{\partial \xi} + \beta \frac{\partial \beta}{\partial g} - \beta \frac{\partial \rho}{\partial g} - \delta \frac{\partial \rho}{\partial \xi} \right] \frac{\partial}{\partial g} + \left[\frac{\partial \delta}{\partial \xi} + \rho \frac{\partial \delta}{\partial g} \right] \frac{\partial}{\partial \xi} \right. \\ &\left. + \left[\frac{\partial \gamma_m}{\partial \xi} + \rho \frac{\partial \gamma_m}{\partial g} - \beta \frac{\partial \sigma}{\partial g} - \delta \frac{\partial \sigma}{\partial \xi} \right] m \frac{\partial}{\partial m} \right\} G = 0 \end{aligned} \tag{7.137}$$

Introduzindo as funções $\overline{\beta}$ e $\overline{\gamma}_m$ e o operador

$$\overline{\mathcal{D}} \equiv \frac{\partial}{\partial \xi} + \rho \frac{\partial}{\partial g} \tag{7.138}$$

a equação anterior escreve-se

$$\begin{aligned} &\left[(\overline{\mathcal{D}}\delta) \frac{\partial}{\partial \xi} + \left(\overline{\mathcal{D}}\,\overline{\beta} + \overline{\mathcal{D}}(\rho\delta) - \overline{\beta}\frac{\partial \rho}{\partial g} - \delta \overline{\mathcal{D}}\rho \right) \frac{\partial}{\partial g} \right. \\ &\left. + \left(\overline{\mathcal{D}}\overline{\gamma}_m + \overline{\mathcal{D}}(\sigma\delta) - \overline{\beta} - \frac{\partial \sigma}{\partial g} - \delta \overline{\mathcal{D}}\sigma \right) m \frac{\partial}{\partial m} \right] G = 0 \end{aligned} \tag{7.139}$$

Multiplicando a equação $\mathcal{D}_G G = 0$ por $(\overline{\mathcal{D}}\delta)$ obtemos

$$\left[(\overline{\mathcal{D}}\delta)\frac{\partial}{\partial \xi} + \rho (\overline{\mathcal{D}}\delta) \frac{\partial}{\partial g} + \sigma (\overline{\mathcal{D}}\delta) m \frac{\partial}{\partial m} \right] G = 0 \tag{7.140}$$

Comparando as duas equações vemos que

$$\overline{\mathcal{D}}\,\overline{\beta} = \overline{\beta} \frac{\partial \rho}{\partial g} \qquad \text{e} \qquad \overline{\mathcal{D}}\,\overline{\gamma}_m = \overline{\beta}\frac{\partial \sigma}{\partial g} \tag{7.141}$$

Estas equações asseguram que resultados físicos sejam independentes de gauge. Assim $\overline{\beta} = 0$ tem consequências físicas. Então $\overline{\mathcal{D}}\,\overline{\beta} = 0$ e $\overline{\mathcal{D}}\,\overline{\gamma}_m = 0$ dizendo que a existência dos zeros de $\overline{\beta}$ e a dimensão anómala da massa $\overline{\gamma}_m$ são independentes de gauge. Também se $\overline{\beta} = 0$ obtemos

$$\begin{aligned} \overline{\mathcal{D}} \left(\frac{\partial \overline{\beta}}{\partial g} \right) &= \frac{\partial}{\partial g} \overline{\mathcal{D}}\,\overline{\beta} + \left[\overline{\mathcal{D}}, \frac{\partial}{\partial g} \right] \overline{\beta} \\ &= \frac{\partial}{\partial g} \overline{\mathcal{D}}\,\overline{\beta} - \frac{\partial \rho}{\partial g} \frac{\partial \overline{\beta}}{\partial g} = 0 \end{aligned} \tag{7.142}$$

e portanto a primeira derivada de $\overline{\beta}$ no zero é independente da gauge. Finalmente como $\rho = O(g^3)$ e $\delta = O(g^2)$ temos então

$$\overline{\beta} = \beta + O(g^5) \; . \tag{7.143}$$

Estes resultados não dependem de se ter adoptado subtracção miníma ou não. Se adoptarmos subtracção mínima temos então

Teorema 3.3

No esquema de subtracção mínima temos $\rho = \sigma = 0$ e portanto

$$\overline{\mathcal{D}} = \frac{\partial}{\partial \xi} \quad ; \quad \overline{\beta} = \beta \quad e \quad \overline{\gamma}_m = \gamma_m \tag{7.144}$$

e β e γ_m são independentes da gauge em todas as ordens.

Dem: *Demonstramos só para ρ, para σ é igual.*

$$\rho = g\frac{\partial}{\partial\xi}\ln g = -\frac{g}{Z_g}\frac{\partial Z_g}{\partial\xi} \tag{7.145}$$

Então

$$\begin{aligned} 0 &= Z_g\rho + g\frac{\partial}{\partial\xi}\left(1+\frac{a_1}{\varepsilon}+\frac{a_2}{\varepsilon^2}+\cdots\right) \\ &= \rho + \frac{1}{\varepsilon}\left(\rho a_1 + g\frac{\partial a_1}{\partial\xi}\right) + O(1/\varepsilon^2) \end{aligned} \tag{7.146}$$

obtemos portanto

$$\rho = 0 \ . \tag{7.147}$$

7.3 Effective gauge couplings

7.3.1 Fixed points

Como vimos na secção anterior, o comportamento assimptótico das funções de Green irredutíveis depende do comportamento assimptótico das soluções das equações para a constante de acoplamento efectivo $\overline{\lambda}(t)$ e para a massa efectiva, que como vimos são

$$\left\{ \begin{aligned} \frac{d\overline{\lambda}}{dt} &= \beta(\overline{\lambda}) \qquad ; \qquad \overline{\lambda}(0) = \lambda \\ \frac{d\overline{m}}{dt} &= \left[\gamma_m(\overline{\lambda}) - 1\right]\overline{m}(t) \qquad ; \qquad \overline{m}(0) = m \end{aligned} \right. \tag{7.148}$$

Destas equações resulta que as variações da constante de acoplamento efectiva e da massa efectiva com uma variação de escala de energia são controladas pelas funções β e γ_m, respectivamente. Para estudar o comportamento assimptótico de λ vamos admitir que $\beta(\lambda)$ tem a forma da figura 7.2.

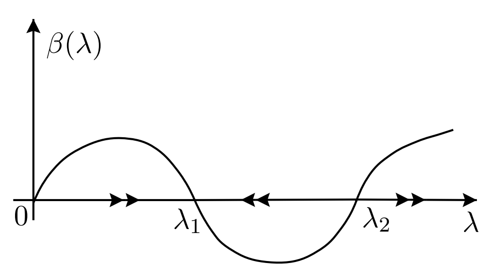


Figure 7.2: $\beta(\lambda)$ as function pf λ.

Os pontos, $0, \lambda_1$ e λ_2 onde $\beta(\lambda)$ se anula são chamadas *pontos fixos*, pois se $\overline{\lambda}$ se encontra num desses pontos em $t=0$ então ficará aí para todos os valores do momento $\left(\frac{d\overline{\lambda}}{dt}=0\right)$. Os pontos fixos podem ser de dois tipos:

i) Ponto fixo estável ultravioleta(UV)

São aqueles em que $\beta'(\lambda) < 0$. É o caso do ponto λ_1 na figura 7.2. Neste caso $\beta(\lambda) > 0$ para $\lambda < \lambda_1$ e $\beta(\lambda) < 0$ para $\lambda > \lambda_1$. Então se para $t = 0 \quad 0 < \lambda < \lambda_1$ então quando $t \to \infty \quad \overline{\lambda} \to \lambda_1$. Por outro lado se $\lambda_1 < \lambda < \lambda_2$ quando $t \to \infty$ também $\overline{\lambda} \to \lambda_1$. Poratnto no intervalo $0 < \lambda < \lambda_2$ a constante de acoplamento é sempre conduzida para λ_1 quanto $t \to \infty$, isto é, para momentos grandes.

ii) Ponto fixo estável infravermelho(IR)

São aqueles em que $\beta'(\lambda) > 0$. É o caso dos pontos 0 e λ_2 da figura. É fácil de ver que quando $t \to \infty$ a constante de acoplamento se afasta de 0 e λ_2, mas que no limite $t \to 0$ se aproxima deles.

Podemos agora estudar o comportamento assimptótico das soluções do grupo de renormalização. Supomos, por exemplo $0 < \lambda < \lambda_2$. Então (ver figura 7.2)

$$\lim_{t \to 0} \overline{\lambda}(t, \lambda) = \lambda_1 \tag{7.149}$$

A maneira como tende para λ_1 depende da primeira derivada de $\beta(\lambda)$. Suponhamos que na vizinhança de λ_1 temos

$$\begin{aligned} &\beta(\lambda) = a(\lambda_1 - \lambda) \quad ; \quad a > 0 \\ &\beta'(\lambda_1) = -a < 0 \end{aligned} \tag{7.150}$$

Então

$$\overline{\lambda}(t, \lambda) = \lambda_1 + (\lambda - \lambda_1) e^{-at} \tag{7.151}$$

isto é, a aproximação do ponto fixo é exponencial na variável t. Será tanto maior quanto maior for $|\beta'(\lambda_1)| = a$. Vimos anteriormente que a solução da equação da massa efectiva era

$$\overline{m}(t) = m e^{-t} e^{\int_0^t \gamma_m(\overline{\lambda}) dt'} \tag{7.152}$$

Se $\lim\limits_{t \to \infty} \overline{\lambda} = \lambda_1$ então temos para $t \to \infty$

$$\overline{m} = m e^{-t(1 - \gamma_m(\lambda_1))} \tag{7.153}$$

o que mostra que se $\gamma_m(\lambda_1) < 1$ então $m(t) \to 0$ quando $t \to \infty$. Na mesma aproximação

$$\int_0^t \gamma(\overline{\lambda}(t')) dt' \simeq \gamma(\lambda_1) t \tag{7.154}$$

e portanto a solução assimptótica é

$$\lim_{\sigma \to \infty} \Gamma^n(\sigma p_i, m, \lambda, \mu) = \sigma^{4 - n[1 + \gamma(\lambda_1)]} \Gamma^{(n)}(p_i, \overline{m}, \lambda_1, \mu) \tag{7.155}$$

o que mostra que a dimensão dos campos não é 1 mas $1 + \gamma(\lambda_1)$. Daí o nome de dimensão anómala para $\gamma(\lambda)$.

Em geral é difícil calcular os zeros da função β, pois requere normalmente resultados para além da teoria de perturbações. Contudo $\beta(\lambda), \gamma_m(\lambda)$ e $\gamma(\lambda)$ têm um zero trivial na origem. Se acontecer que a origem seja um ponto fixo estável UV então quer dizer que quando a escala da energia aumenta a constante de acoplamento diminui. No limite $t \to \infty, \overline{\lambda} \to 0$ e por isso se diz destas teorias que são *assimptoticamente livres*. É fácil de ver que isso acontece se $\beta'(0) < 0$. Na secção seguinte vamos ver quais as teorias em que isso pode acontecer.

7.3.2 β function for theories with scalars, fermions and gauge fields

Vamos nesta secção mostrar que só as teorias de gauge não abelianas podem ser assimptoticamente livres, isto é, só estas verificam a propriedade $\beta'(0) < 0$.

i) Teorias com escalares

Já vimos anteriormente que para a teoria escalar mais simples, $\lambda\phi^4$, temos

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^4) \tag{7.156}$$

e portanto não é assimptoticamente livre. Consideramos agora a teoria escalar mais geral com campos ϕ_i e acoplamento

$$\mathcal{L}_I = -\lambda_{ijk\ell}\phi_i\phi_j\phi_k\phi_\ell \tag{7.157}$$

onde se somam os índices repetidos. Então

$$\beta_{ijk\ell} = \frac{d\overline{\lambda}_{ijk\ell}(t)}{dt} = A(\overline{\lambda}_{i\ell mn}\overline{\lambda}_{kjmn} + \overline{\lambda}_{ijmn}\overline{\lambda}_{k\ell mn} + \overline{\lambda}_{ikmn}\overline{\lambda}_{j\ell mn}) \tag{7.158}$$

com $A > 0$. A teoria não é assimptoticamente livre pois há sempre funções β com derivadas positivas. Por exemplo

$$\frac{d\overline{\lambda}_{1111}}{dt} = \beta_{1111} = 3A|\overline{\lambda}_{11mn}|^2 > 0 \qquad ; \qquad \forall t \tag{7.159}$$

ii) Teorias com escalares + fermiões + acoplamentos de Yukawa

O termo da interacção mais geral para uma teoria com escalares e fermiões é

$$\mathcal{L}_I = -\sum_{i,j,k,\ell}\lambda_{ijk\ell}\phi_i\phi_j\phi_k\phi_\ell + \sum_{a,b,k}\overline{\psi}^a(A^k_{ab} + iB^k_{ab}\gamma_5)\overline{\psi}^b\phi_k \tag{7.160}$$

onde A e B são matrizes reais. Agora já não é possível mostrar que $\frac{d\overline{\lambda}_{iiii}}{dt} > 0$ por causa do loop de fermiões de ordem A^2 ou B^2 com um sinal negativo. Se definirmos $(g^i)_{ab} \equiv A^i_{ab} + iB^i_{ab}$, obtemos

$$\begin{aligned}16\pi^2\frac{dg^i}{dt} &= (\mathrm{Tr} g^i g^{j\dagger})g^j + \mathrm{Tr}(g^{i\dagger}g^j)g^j + M^{ij}g^j \\ &\quad + \frac{1}{2}g^i g^{\dagger j}g^j + \frac{1}{2}g^j g^{\dagger j}g^i + 2g^j g^{\dagger i}g^j \end{aligned}\tag{7.161}$$

onde $M^{ij} \equiv \frac{1}{4}\lambda_{ik\ell m}\lambda_{jk\ell m}$. Usando este resultado é possível demonstrar o teorema seguinte:

Teorema 3.4

A teoria mais geral com escalares e fermiões não é assimptoticamente livre pois $\frac{d}{dt}\mathrm{Tr}(g^{i\dagger}g^i) > 0$ *e portanto não é possivel* $g_i \to 0$ *quando* $t \to \infty$.

Dem:

$$\begin{aligned}
8\pi^2 \frac{d}{dt}\mathrm{Tr}(g^{i\dagger}g^i) &= 8\pi^2 \frac{d}{dt}\sum_{a,b,i}|g^i_{ab}|^2 \\
&= \mathrm{Tr}(g^i g^{j\dagger})\mathrm{Tr}(g^{i\dagger}g^j) + \mathrm{Tr}(g^i g^{j\dagger})(\mathrm{Tr} g^i g^{j\dagger}) \\
&\quad + \frac{1}{2}\mathrm{Tr}(g^i g^{i\dagger} g^j g^{j\dagger}) + \frac{1}{2}\mathrm{Tr}(g^{i\dagger} g^i g^{j\dagger} g^j) \\
&\quad + 2\mathrm{Tr}(g^i g^{j\dagger} g^i g^{j\dagger}) + M^{ij}\mathrm{Tr}(g^{i\dagger}g^j)
\end{aligned} \tag{7.162}$$

Agora o último termo é positivo, assim como o terceiro e o quarto. O primeiro é maior que o segundo e portanto

$$8\pi^2 \frac{d}{dt}\mathrm{Tr}(g^{i\dagger}g^i) \geq 2\Big[\mathrm{Tr}(g^i g^{j\dagger})\mathrm{Tr}(g^i g^{j\dagger}) + \mathrm{Tr}(g^i g^{j\dagger} g^i g^{j\dagger})\Big] \tag{7.163}$$

e o segundo membro é positivo pois pode ser escrito

$$8\pi^2 \frac{d}{dt}\mathrm{Tr}(g^{i\dagger}g^i) \geq (g^i_{ab}g^i_{cd} + g^i_{ad}g^i_{cd})(g^{j\dagger}_{ba}g^{j\dagger}_{dc} + g^{j\dagger}_{da}g^{j\dagger}_{bc}) \geq 0 \tag{7.164}$$

como queríamos demonstrar.

iii) Teorias gauge abelianas

Consideremos o caso de QED. Temos

$$Z_e = Z_1 Z_2^{-1} Z_3^{-1/2} = Z_3^{-1/2} \tag{7.165}$$

Z_3 pode ser calculado do diagrama de polarização do vácuo representado na Fig. 7.3, e o resultado é

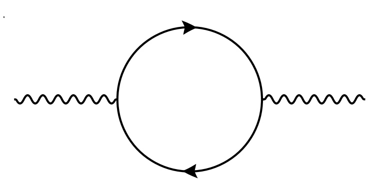


Figure 7.3: Vacuum polarization in QED.

$$Z_3^{-1/2} = 1 + \frac{e^2}{12\pi^2}\frac{1}{\varepsilon} + \cdots \tag{7.166}$$

logo

$$\beta(e) = \frac{1}{2} e^2 \frac{da_1}{de} = \frac{e^3}{12\pi^2} > 0 \tag{7.167}$$

Se tivéssemos electrodinâmica escalar Z_3 seria obtido a partir dos diagramas da figura 7.4, e o resultado seria

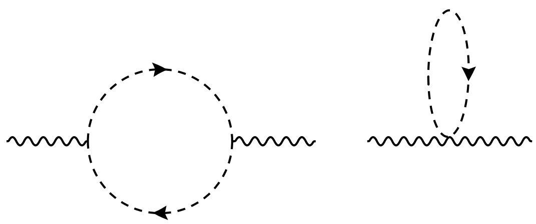


Figure 7.4: Vacuum polarization in scalar electrodynamics.

$$Z_3^{-1/2} = 1 + \frac{e^2}{48\pi^2} \frac{1}{\varepsilon} \tag{7.168}$$

o que dá neste caso $\beta(e) = \frac{e^3}{48\pi^2} > 0$. Portanto as teorias de gauge abelianas não são livres assimptoticamente.

iv) Teorias de gauge não abelianas

Comecemos pela teoria de gauge pura definida no capítulo 2. A renormalização da função de onda para os campos de gauge é obtida a partir dos diagramas da figura 7.5. Em subtracção mínima obtemos,

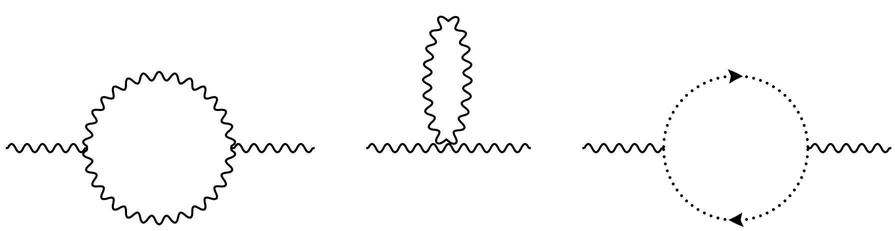


Figure 7.5: Vacuum polarization in a pure non-abelian gauge theory.

$$Z_A = 1 + \frac{g^2}{16\pi^2} \left(\frac{13}{3} - \xi \right) C_2(V) \frac{1}{\varepsilon} \tag{7.169}$$

onde $C_2(V)$ é o operador de Casimir definido no capítulo 2 para a representação adjunta, a que pertencem os campos da gauge (vectores). A constante de renormalização do vértice triplo, Z_1, é obtida a partir dos diagramas da figura 7.6.

Obtemos

$$Z_1 = 1 + \frac{g^2}{16\pi^2} \left(\frac{17}{6} - \frac{3\xi}{2} \right) C_2(V) \frac{1}{\varepsilon} + \cdots \tag{7.170}$$

Então

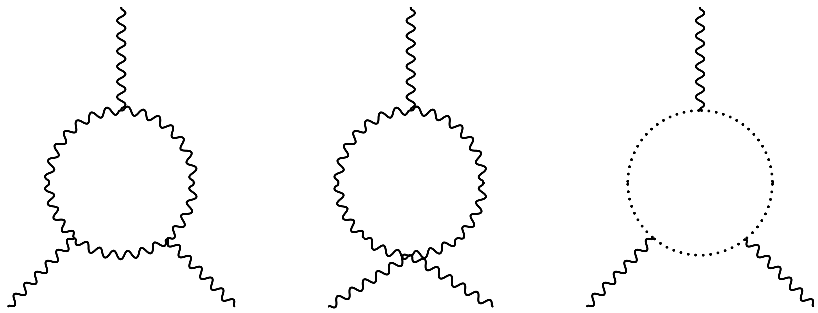


Figure 7.6: Vertex in a pure non-abelian gauge theory.

$$Z_g \equiv Z_1 Z_A^{-3/2} = 1 - \frac{g^2}{16\pi^2}\left(\frac{11}{3}C_2(V)\right)\frac{1}{\varepsilon} + \cdots \tag{7.171}$$

Usando Z_A e Z_g e as definições de β e γ obtemos

$$\beta = -\frac{g^3}{16\pi^2}\frac{11}{3}C_2(V) < 0 \tag{7.172}$$

e

$$\gamma_A = -\frac{g^2}{16\pi^2}\frac{1}{2}\left(\frac{13}{3} - \xi\right)C_2(V) \tag{7.173}$$

Portanto as teorias de gauge não abelianas sem campos de matéria são assimptoticamente livres. Notar que a dependência da gauge (em ξ) desapareceu de β de acordo com o resultado demonstrado anteriormente.

A inclusão de fermiões e escalares acoplados mínimamente é agora trivial. O lagrangeano de interacção é

$$\begin{aligned}\mathcal{L}_{int} &= g\overline{\psi}_i\gamma^\mu\psi_j T^a_{Fij}A^a_\mu \\ &+ig\phi^*_i\overleftrightarrow{\partial}_\mu\phi_j T^a_{Sij}A^{\mu a} \\ &+g^2\phi^*_i T^a_{Sij}T^b_{Sjk}\phi_k A^a_\mu A^{\mu b}\end{aligned} \tag{7.174}$$

onde T^a_F e T^a_S são os geradores na representação em que se encontram os fermiões e os escalares respectivamente. Para se encontrar a contribuição destas partículas para a função β temos que calcular a contribuição delas para Z_g. O mais fácil é usar os resultados de QED e electrodinâmica escalar que dizem que

$$Z_g = Z_A^{-1/2} \tag{7.175}$$

e calcular a contribuição para Z_A dos fermiões e escalares devido aos diagramas da figura 7.7. O resultado é

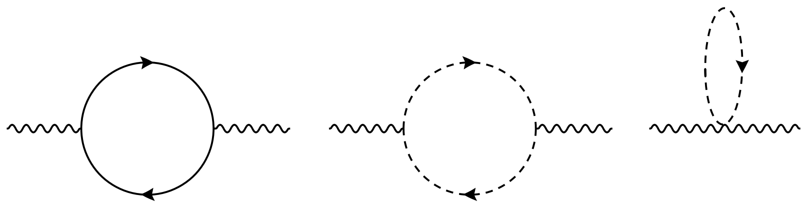


Figure 7.7: Contribution from fermions and scalars to vacuum polarization.

$$Z_g(\text{fermiões} + \text{escalares}) = 1 + \frac{g^2}{16\pi^2}\left[\frac{4}{3}T(R_F) + \frac{1}{3}T(R_S)\right]\frac{1}{\varepsilon} + \cdots \tag{7.176}$$

pelo que

$$\beta(\text{fermiões}) = \frac{g^3}{16\pi^2}\frac{4}{3}T(R_F) \tag{7.177}$$

e

$$\beta(\text{escalares}) = \frac{g^3}{16\pi^2}\frac{1}{3}T(R_S) \tag{7.178}$$

Pondo tudo junto obtemos

$$\beta = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}C_2(V) + \frac{4}{3}T(R_F) + \frac{1}{3}T(R_s)\right] \tag{7.179}$$

onde as quantidades $T(R)$ são definidas para uma dada representação por

$$\text{Tr}(T^aT^b) = T(R)\delta^{ab} \tag{7.180}$$

Se a teoria contém fermiões de Majorana (ou spinores de Weyl) ou campos escalares, os coeficientes em frente de $T(R_F)$ e $T(R_S)$ são multiplicados por um factor adicional de $1/2$. Consideremos agora um exemplo simples:

Exemplo 3.1:

QCD ($SU(3)$) com as três familias de quarks.

Para $SU(N)$ temos

$$C_2(V) = N \tag{7.181}$$

e como os quarks se encontram na representação fundamental

$$T(R_F) = \frac{1}{2} \tag{7.182}$$

Então

$$\beta = \frac{g}{16\pi^2}\left[-\frac{33}{3} + \frac{4}{3} \times \frac{1}{2} \times 2N_g\right] \tag{7.183}$$

ou seja (N_g = número de gerações ou famílias)

$$\beta = \frac{g^3}{16\pi^2}\left[-\frac{33 - 4N_g}{3}\right] \tag{7.184}$$

Portanto $SU(3)$ será assimptoticamente livre se

$$33 - 4N_g > 0 \tag{7.185}$$

ou ainda

$$N_g < \frac{33}{4} \rightarrow N_g \leq 8 \tag{7.186}$$

São portanto permitidas 8 famílias ou seja 16 tripletos de $SU(3)$.

7.3.3 The vacuum of a $NAGT$ as a paramagnetic medium ($\mu > 1$)

Um argumento recente (Nielsen 1981, Hughs 1981) permite compreender melhor o que se passa de diferente nas teorias de gauge não abelianas para que elas tenham liberdade assimptotica. Primeiro o facto de a carga diminuir a curta distância pode ser interpretado como um *anti - shielding* do vácuo, isto é

$$\varepsilon < 1 \tag{7.187}$$

O problema em compreender o que se passa resulta do facto de não conhecermos substâncias[3] com $\varepsilon < 1$. Contudo o vácuo deve ser invariante relativista e portanto deve ter uma permeabilidade μ tal que (estamos a fazer $c = 1$)

$$\mu\varepsilon = 1 \tag{7.188}$$

Assim o *antiscreening* corresponde a $\mu > 1$. Portanto o vácuo duma teoria de gauge não abeliana é um *paramagnético* e este conceito pode ser compreendido mais facilmente.

A permeabilidade magnética pode ser calculada calculando a densidade de energia do vácuo num campo exterior

$$u_0 = \frac{1}{2\mu} B_{ext}^2 \tag{7.189}$$

Nielsen e Hughes mostraram que $\mu = 1 + \chi$ onde a susceptibilidade χ é dada por

$$\chi \sim (-1)^{2s} q^2 \sum_{s_3} \left(-\frac{1}{3} + \gamma^2 s_3^2\right) \tag{7.190}$$

[3] Em QED a carga aumenta a curta distância e portanto o vácuo é um dieléctrico normal $\varepsilon>1$.

onde s é o spin, q a carga, γ a razão giromagnética e s_3 a projecção de spin na direcção do campo magnético externo. Assim para escalares, fermiões e campos de gauge obtemos

Escalares

$$\chi_S \sim -\frac{1}{3}q_S^2 < 0 \qquad \text{(diamagnético)} \tag{7.191}$$

Fermiões ($\gamma_F = 2$)

$$\chi_F \sim (-1)q_F^2 2\left(-\frac{1}{3}+1\right) = -\frac{4}{3}q_F^2 \qquad \text{(diamagnético)} \tag{7.192}$$

Bosões de gauge ($\gamma_V = 2$)

$$\chi_V \sim q_V^2 2\left(-\frac{1}{3}+4\right) = \frac{22}{3}q_V^2 \qquad \text{(paramagnético)} \tag{7.193}$$

e portanto

$$\chi_{\text{Total}} \sim \frac{22}{3}q_V^2 - \frac{4}{3}q_F^2 - \frac{1}{3}q_S^2 \tag{7.194}$$

Comparando com a função β podemos fazer a correspondência

$$\begin{aligned} q_V^2 &\to \frac{1}{2}C_2(V) \\ q_F^2 &\to T(R_F) \\ q_S^2 &\to T(R_S) \end{aligned} \tag{7.195}$$

o que permite compreender o vácuo das teorias de gauge não abelianas como um meio paramagnético.

7.4 Renormalization group applications

We consider the Grand Unified Theory (GUT) with the gauge group $SU(5)$, that is

$$SU(5) \supset SU_c(3) \times SU_L(2) \times U_Y(1) \ . \tag{7.196}$$

The unification takes place at the GUT scale M_X. Using the renormalization group equations and the low energy data on the coupling constants, it is possible to determine the scale M_X as well as other predictions for the theory at the low scale, which we take to be the scale M_Z. For this we need to know how the different coupling constants evolve with the scale.

7.4.1 Scale M_X

We start by writing the covariant derivatives for the unified theory and for the theory with the broken symmetry.

$$SU(5): D_\mu = \partial_\mu + i g_5 \sum_{a=0}^{23} A_\mu^a \frac{\lambda^a}{2} \tag{7.197}$$

$$\begin{aligned} SU(3)\times SU(2)\times U(1): D_\mu = \partial_\mu + i g_3 \sum_{\alpha}^{8} G_\mu^a \frac{\lambda}{2}^a \\ + i g_2 \sum_{\alpha}^{3} A_\mu^a \frac{\sigma}{2}^a + i g' \frac{Y}{2} B_\mu \end{aligned} \tag{7.198}$$

At the scale M_X where the unification takes place we have

$$g_5 = g_3 = g_2 = g_1 \tag{7.199}$$

where g_1 is the coupling constant of the abelian subgroup of $SU(5)$. However for the abelian groups there are no constraints in the normalization of the generators, and therefore the the generator λ^0 of that $U(1)$ can be normalized in a different way from the hypercharge. We must have

$$g_1 \lambda^0 = g' Y \tag{7.200}$$

As λ^0 is a generator of $SU(5)$ it is normalized according to

$$T_F(\lambda^a \lambda^b) = 2\delta^{ab} \tag{7.201}$$

that is, for the fundamental representation we must have

$$\lambda^0 = \frac{1}{\sqrt{15}} \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{bmatrix} \tag{7.202}$$

Now, for the fundamental representation, we have

$$5 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ \nu_e^c \end{pmatrix}_R \tag{7.203}$$

and the hypercharge[4] can be read directly. We obtain,

$$Y = \begin{bmatrix} -2/3 & & & & \\ & -2/3 & & & \\ & & -2/3 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \tag{7.204}$$

Therefore $Y = -\sqrt{\frac{5}{3}}\lambda^0$ and $g' = -\sqrt{\frac{3}{5}}g_1$. This allows to determine $\sin^2\theta_W$ at the GUT scale M_X,

$$\sin^2\theta_W(M_X) = \frac{g'^2}{g^2 + g'^2} = \frac{\frac{3}{5}g_1}{g_2 + \frac{3}{5}g_1} = \frac{3}{8} \tag{7.205}$$

Also, for future reference, we note that

$$g'^2 = \frac{3}{5}\, g_1^2 \ . \tag{7.206}$$

7.4.2 Scale M_Z

Let us look now at what happens at the scale M_Z. The evolution of the coupling constants is governed by the RGE equations for the three gauge groups in the broken phase

$$\frac{dg_i}{dt} = \beta_i \tag{7.207}$$

These β functions are given by

$$\beta_i = \frac{g_i^3}{16\pi^2}\left[-\frac{11}{3}C_2(V) + \sum_j \frac{4}{3}T(R_{F_j}) + \sum_k \frac{1}{3}T(R_{S_k})\right] \tag{7.208}$$

where the sums are over all the fermion and scalar physical states of the theory at a given scale. Given the form of Eq. (7.208), it is usual to define

$$\beta_i \equiv \frac{1}{16\pi^2}\, b_i g_i^3 \tag{7.209}$$

and therefore the b_i are defined by the bracket in Eq. (7.208). Before we evaluate them let us introduce Eq. (7.209) into Eq. (7.207). We get

$$\frac{dg_i}{dt} = \frac{b_i}{16\pi^2}\, g_i^3 \tag{7.210}$$

Let us solve this equations before we evaluate the beta function coefficients b_i. For that it is usual to introduce the generalization of the fine structure constant, that is, we define

$$\alpha_i \equiv \frac{g_i^2}{4\pi} \tag{7.211}$$

[4] Remember that our convention is such that $Q = T_3 + \dfrac{Y}{2}$.

Multiplying both sides of Eq. (7.210) by g_i and doing some trivial algebra we get,

$$\frac{d\alpha_i}{dt} = \frac{b_i}{2\pi}\,\alpha_i^2 \tag{7.212}$$

Rearranging and integrating between some initial (μ_i), and final scale (μ_f), we get

$$\int_i^f \frac{d\alpha_i}{\alpha_i^2} = \frac{b_i}{2\pi}\int_i^f dt \tag{7.213}$$

or[5]

$$\left[-\frac{1}{\alpha_i}\right]_i^f = \frac{b_i}{2\pi}\,(t_f - t_i) = \frac{b_i}{2\pi}\ln\left(\frac{\mu_f}{\mu_i}\right) \tag{7.214}$$

and finally

$$\alpha_i^{-1}(\mu_f) = \alpha_i^{-1}(\mu_i) - \frac{b_i}{4\pi}\ln\left(\frac{\mu_f^2}{\mu_i^2}\right) \tag{7.215}$$

As at the unification scale M_X we have, by definition (see Eq. (7.199)), that

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 \tag{7.216}$$

where α_5 is the $SU(5)$ unified value, and we can write the final solution

$$\alpha_i^{-1}(\mu) = \alpha_5^{-1} + \frac{b_i}{4\pi}\ln\left(\frac{M_X^2}{\mu^2}\right), \quad i = 1,2,3 \tag{7.217}$$

We can rewrite these equations in terms of electromagnetic fine structure constant $\alpha(\mu)$ and of the strong coupling equivalent $\alpha_s\mu)$, that are measured at the weak scale, to obtain

$$\left\{\begin{array}{l} \alpha_s^{-1}(\mu) = \alpha_5^{-1} + \dfrac{b_3}{4\pi}\ln\left(\dfrac{M_X^2}{\mu^2}\right) \\[2ex] \alpha^{-1}(\mu)\sin^2\theta_W(\mu) = \alpha_5^{-1} + \dfrac{b_2}{4\pi}\ln\left(\dfrac{M_X^2}{\mu^2}\right) \\[2ex] \dfrac{3}{5}\cos^2\theta_W(\mu)\alpha^{-1}(\mu) = \alpha_5^{-1} + \dfrac{b_1}{4\pi}\ln\left(\dfrac{M_X^2}{\mu^2}\right) \end{array}\right. \tag{7.218}$$

From these equations we obtain,

$$\ln\frac{M_X^2}{\mu^2} = \frac{12\pi}{-8b_3 + 3b_2 + 5b_1}\left[\frac{1}{\alpha(\mu)} - \frac{8}{3}\frac{1}{\alpha_s(\mu)}\right] \tag{7.219}$$

That allows to determine M_X, once $\alpha(\mu)$ and $\alpha_s(\mu)$ are known, at a given scale μ, and

$$\sin^2\theta_W(\mu) = \frac{3(b_2 - b_3)}{5b_1 + 3b_2 - 8b_3} + \frac{5(b_1 - b_2)}{5b_1 + 3b_2 - 8b_3}\frac{\alpha(\mu)}{\alpha_S(\mu)} \tag{7.220}$$

which allows to determine $\sin^2\theta_W$ at the scale $\mu = M_Z$, once $\alpha(M_Z)$ and $\alpha_s(M_Z)$ are known. Finally we can also solve for the value of α_5^{-1}. We get

$$\alpha_5^{-1} = \alpha^{-1}(\mu)\frac{1}{5b_1 + 3b_2 - 8b_3}\left[-3b_3 + (5b_1 + 3b_2)\,\frac{\alpha(\mu)}{\alpha_S(\mu)}\right] \tag{7.221}$$

Now we turn to the evaluation of the coefficients b_i first in the Standard Model (SM) and the in the Minimal Supersymmetric Standard Model (MSSM).

[5]Remember that $t = \ln(\mu)$.

Standard Model

In the SM we have the gauge fields, $N_g = 3$ families of leptons, $N_F = 2N_g = 6$ quark flavours and one Higgs. With this information we can find the coefficients b_i for the SM using the definition

$$b_i = -\frac{11}{3}C_2(V_i) + \sum_j \frac{2}{3}T(R_{F_j}) + \sum_k \frac{1}{3}T(R_{S_k}) \tag{7.222}$$

where we have modified Eq. (7.208), as the sum in the fermions is done separately for each quirality. This is important for the SM as the model is described in terms of left and right-handed fermions.

- $SU(3)$

For $SU(3)$, we have $C_2(V_3) = 3$ and the quarks are in the fundamental representation, therefore $T(R_{F_j}) = 1/2$. Then the counting goes as follows,

$$b_3 = \underbrace{-\frac{11}{3}\times 3}_{\text{Gauge}} + N_g \times \left[\underbrace{\frac{2}{3}\times\frac{1}{2}\times(2+1+1)}_{\text{quarks}}\right] = -7 \tag{7.223}$$

where the meaning of $(2+1+1)$ is that we count the up and down components of each $(SU(2)_L)$ doublet and then the corresponding right-handed quarks for each generation.

- $SU(2)$

For the $SU(2)$ we get

$$b_2 = \underbrace{-\frac{11}{3}\times 2}_{\text{Gauge}} + N_g \times \left(\underbrace{N_c \times \frac{2}{3}\times\frac{1}{2}}_{\text{quarks}_\text{L}} + \underbrace{\frac{2}{3}\times\frac{1}{2}}_{\text{leptons}}\right) + \underbrace{\frac{1}{3}\times\frac{1}{2}}_{\text{Higgs}} = -\frac{19}{6} \tag{7.224}$$

where $N_c = 3$ is the number of colours.

- $U(1)$

Finally for the $U(1)$ part, with the correct normalization, we have

$$b_1 = \frac{3}{5}\times\left[\frac{2}{3}\times\sum_{f_L,f_R}\left(\frac{Y}{2}\right)^2 + \frac{1}{3}\times\sum_{\text{scalars}}\left(\frac{Y}{2}\right)^2\right] \tag{7.225}$$

and therefore,

$$b_1 = \frac{3}{5}\times\left[N_g \times \left(\underbrace{\frac{2}{3}\times\left(-\frac{1}{2}\right)^2\times 2}_{\text{Leptons}_\text{L}} + \underbrace{\frac{2}{3}\times(-1)^2}_{\text{Leptons}_\text{R}} + \underbrace{N_c \times \frac{2}{3}\times\left(\frac{1}{6}\right)^2\times 2}_{\text{Quarks}_\text{L}} + \underbrace{N_c \times \frac{2}{3}\times\left(\frac{2}{3}\right)^2}_{\text{Up-Quarks}_\text{R}}\right.\right.$$

$$
\left. + N_c \times \underbrace{\frac{2}{3} \times \left(\frac{1}{3}\right)^2}_{\text{Down-Quarks}_{\text{R}}} \right) + \underbrace{\frac{1}{3} \times \left(\frac{1}{2}\right)^2 \times 2}_{\text{Higgs}} \Bigg]
$$

$$
= 4 + \underbrace{\frac{1}{10}}_{\text{Higgs}} = \frac{41}{10} \tag{7.226}
$$

So in summary we have for the SM,

$$
b_1 = \frac{41}{10}, \quad b_2 = -\frac{19}{6}, \quad b_3 = -7 \tag{7.227}
$$

Now let us look to see what are the results for M_X, $\sin^2\theta_W(M_Z)$ and α_5^{-1}. We will use the current values from the Particle Data Group. These are (without worrying about errors)[6]

$$
\alpha^{-1}(M_Z) = 127.916, \quad \alpha_s(M_Z) = 0.118, \quad M_Z = 91.1896 \text{ GeV} \tag{7.228}
$$

we get

$$
M_X = 6.7 \times 10^{14} \text{ GeV}, \quad \sin^2\theta_W(M_Z) = 0.208, \quad \alpha_5^{-1} = 41.48 \tag{7.229}
$$

At the time that this GUT model was proposed by the first time, the constants were not known so precisely as today. Also the bound on the lifetime of the proton was much lower than today. So at that time the model was completely consistent. However after many years of dedicated experiments for find the decay of the proton, the lower limit was substantially improved and also after LEP the coupling constants are known with greater precision. So today the value for M_X is too low, the same being true for the value of $\sin^2\theta_W(M_Z)$ (the best value today is around $\sin^2\theta_W(M_Z) = 0.230$ [7]).

This can be seen very clearly if we use Eq. (7.215), with $\mu_i = M_Z$ and plot the α_i^{-1} as a function of $\ln(\mu^2/M_Z^2)$. This is shown in Fig. 7.8. We clearly see that the agreement is quite poor with today's values.

Minimal Supersymmetric Standard Model

Let us now turn to the MSSM. Below the GUT scale the gauge group is the same as in the SM, but the particle content is larger, more than duplicated in relation to the SM. We summarize in the Table 7.1 the particle content and their quantum numbers under $G = SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$.

With the values in Table 7.1 we can calculate the contribution of the various particles to the b_i coefficients. We will do it in succession for the three groups and for the different supermultiplets.

- $SU(3)$

[6] Not only errors but also the difference between different renormalization schemes. Also this discussion is at one-loop level.

[7] Again, without discussing the very small errors and the dependence on the renormalization scheme.

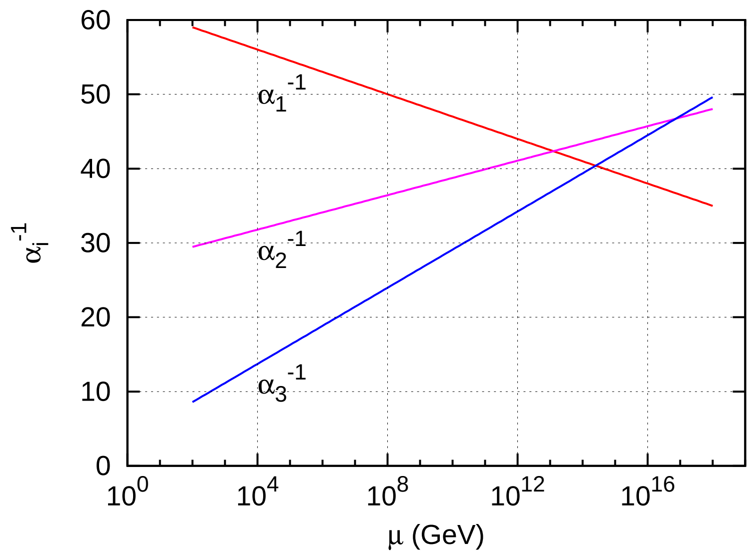


Figure 7.8: Evolution of α_i as function of the scale μ in the Standard Model for a $SU(5)$ minimal GUT theory.

– Gauge Supermultiplet

We first do it in general for any gauge group and then apply it to the cases of interest. The gauge multiplet has a gauge boson contributing with

$$b^{\text{gauge boson}} = -\frac{11}{3}\, C_2(V) \tag{7.230}$$

and the left-handed gauginos in the adjoint representation of the gauge group. These therefore contribute

$$b^{\text{gauginos}} = \frac{2}{3}\, C_2(V) \tag{7.231}$$

and therefore

$$b^{\text{gauge SM}} = -3\, C_2(V) \tag{7.232}$$

where SM stands here for *super-multiplet*. Applying now to $SU(3)$ we get

$$b_3^{\text{gauge SM}} = -9 \tag{7.233}$$

– Left-handed Lepton Supermultiplet

$$b_3^{\text{Leptons}_\text{L}\ \text{SM}} = 0 \tag{7.234}$$

– Right-handed Lepton Supermultiplet

$$b_3^{\text{Leptons}_\text{R}\ \text{SM}} = 0 \tag{7.235}$$

Supermultiplet	$SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$ Quantum Numbers
$\widehat{V}_1 \equiv (\lambda', W_1^{\mu})$	$(1,1,1)$
$\widehat{V}_2 \equiv (\lambda^a, W_2^{\mu a})$	$(1,3,0)$
$V_3 \equiv (\widetilde{g}^b, W_3^{\mu b})$	$(8,1,0)$
$\widehat{L}_i \equiv (\widetilde{L}, L)_i$	$(1,2,-1)$
$\widehat{R}_i \equiv (\widetilde{\ell}_R, \ell_L^c)_i$	$(1,1,2)$
$\widehat{Q}_i \equiv (\widetilde{Q}, Q)_i$	$(3,2,\frac{1}{3})$
$\widehat{D}_i \equiv (\widetilde{d}_R, d_L^c)_i$	$(3,1,\frac{2}{3})$
$\widehat{U}_i \equiv (\widetilde{u}_R, u_L^c)_i$	$(3,1,-\frac{4}{3})$
$\widehat{H}_d \equiv (H_d, \widetilde{H}_d)$	$(1,2,-1)$
$\widehat{H}_u \equiv (H_u, \widetilde{H}_u)$	$(1,2,1)$

Table 7.1: Particle content of the MSSM. Note that $Q = T_3 + Y/2$.

– Left-handed Quark Supermultiplet

$$b_3^{\mathrm{Quarks_L\,SM}} = \underbrace{\frac{2}{3} \times \frac{1}{2} \times 2}_{\mathrm{Quarks_L}} + \underbrace{\frac{1}{3} \times \frac{1}{2} \times 2}_{\mathrm{Squarks_L}} = 1 \tag{7.236}$$

– Right-handed Up-Quark Supermultiplet

$$b_3^{\mathrm{Up-Quark_R\,SM}} = \underbrace{\frac{2}{3} \times \frac{1}{2}}_{\mathrm{Up-Quarks_R}} + \underbrace{\frac{1}{3} \times \frac{1}{2}}_{\mathrm{Up-Squarks_R}} = \frac{1}{2} \tag{7.237}$$

– Right-handed Down-Quark Supermultiplet

$$b_3^{\mathrm{Down-Quark_R\,SM}} = \underbrace{\frac{2}{3} \times \frac{1}{2}}_{\mathrm{Down-Quarks_R}} + \underbrace{\frac{1}{3} \times \frac{1}{2}}_{\mathrm{Down-Squarks_R}} = \frac{1}{2} \tag{7.238}$$

– Up type Higgs Supermultiplet

$$b_3^{\mathrm{Up-Higgs\,SM}} = 0 \tag{7.239}$$

– Down type Higgs Supermultiplet

$$b_3^{\mathrm{Down-Higgs\,SM}} = 0 \tag{7.240}$$

• $SU(2)$

Gauge Supermultiplet

We get

$$b_2^{\text{gauge SM}} = -6 \tag{7.241}$$

– **Left-handed Lepton Supermultiplet**

$$b_2^{\text{Leptons}_\text{L}\text{ SM}} = \underbrace{\frac{2}{3}\times\frac{1}{2}}_{\text{Leptons}_\text{L}} + \underbrace{\frac{1}{3}\times\frac{1}{2}}_{\text{Sleptons}_\text{L}} = \frac{1}{2} \tag{7.242}$$

– **Right-handed Lepton Supermultiplet**

$$b_2^{\text{Leptons}_\text{R}\text{ SM}} = 0 \tag{7.243}$$

– **Left-handed Quark Supermultiplet**

$$b_2^{\text{Quarks}_\text{L}\text{ SM}} = N_c\underbrace{\frac{2}{3}\times\frac{1}{2}}_{\text{Quarks}_\text{L}} + N_c\underbrace{\frac{1}{3}\times\frac{1}{2}}_{\text{Squarks}_\text{L}} = N_c\,\frac{1}{2} = \frac{3}{2} \tag{7.244}$$

– **Right-handed Up-Quark Supermultiplet**

$$b_2^{\text{Up}-\text{Quark}_\text{R}\text{ SM}} = 0 \tag{7.245}$$

– **Right-handed Down-Quark Supermultiplet**

$$b_2^{\text{Down}-\text{Quark}_\text{R}\text{ SM}} = 0 \tag{7.246}$$

– **Up type Higgs Supermultiplet**

$$b_2^{\text{Up}-\text{Higgs SM}} = \underbrace{\frac{1}{3}\times\frac{1}{2}}_{\text{Higgs}_\text{u}} + \underbrace{\frac{2}{3}\times\frac{1}{2}}_{\text{Higgsino}_\text{u}} = \frac{1}{2} \tag{7.247}$$

– **Down type Higgs Supermultiplet**

$$b_2^{\text{Down}-\text{Higgs SM}} = \underbrace{\frac{1}{3}\times\frac{1}{2}}_{\text{Higgs}_\text{d}} + \underbrace{\frac{2}{3}\times\frac{1}{2}}_{\text{Higgsino}_\text{d}} = \frac{1}{2} \tag{7.248}$$

• $U(1)$

Gauge Supermultiplet

We get

$$b_1^{\text{gauge SM}} = 0 \tag{7.249}$$

– Left-handed Lepton Supermultiplet

$$b_1^{\text{Leptons}_\text{L}\,\text{SM}} = \frac{3}{5} \times \left[\underbrace{\frac{2}{3} \times \left(-\frac{1}{2}\right)^2 \times 2}_{\text{Leptons}_\text{L}} + \underbrace{\frac{1}{3} \times \left(-\frac{1}{2}\right)^2 \times 2}_{\text{Sleptons}_\text{L}} \right] = \frac{3}{10} \tag{7.250}$$

– Right-handed Lepton Supermultiplet

$$b_1^{\text{Leptons}_\text{R}\,\text{SM}} = \frac{3}{5} \times \left[\underbrace{\frac{2}{3} \times (-1)^2}_{\text{Leptons}_\text{R}} + \underbrace{\frac{1}{3} \times (-1)^2}_{\text{Sleptons}_\text{R}} \right] = \frac{3}{5} \tag{7.251}$$

– Left-handed Quark Supermultiplet

$$b_1^{\text{Quarks}_\text{L}\,\text{SM}} = \frac{3}{5} \times N_c \times \left[\underbrace{\frac{2}{3} \times \left(\frac{1}{6}\right)^2 \times 2}_{\text{Quarks}_\text{L}} + \underbrace{\frac{1}{3} \times \left(\frac{1}{6}\right)^2 \times 2}_{\text{Squarks}_\text{L}} \right] = N_c \times \frac{3}{5} \times \frac{1}{18} = \frac{1}{10} \tag{7.252}$$

– Right-handed Up-Quark Supermultiplet

$$b_1^{\text{Up}-\text{Quarks}_\text{R}\,\text{SM}} = \frac{3}{5} \times N_c \times \left[\underbrace{\frac{2}{3} \times \left(\frac{2}{3}\right)^2}_{\text{Up}-\text{Quarks}_\text{R}} + \underbrace{\frac{1}{3} \times \left(\frac{2}{3}\right)^2}_{\text{Up}-\text{Squarks}_\text{R}} \right] = N_c \times \frac{3}{5} \times \frac{4}{9} = \frac{4}{5} \tag{7.253}$$

– Right-handed Down-Quark Supermultiplet

$$b_1^{\text{Down}-\text{Quarks}_\text{R}\,\text{SM}} = \frac{3}{5} \times N_c \times \left[\underbrace{\frac{2}{3} \times \left(-\frac{1}{3}\right)^2}_{\text{Down}-\text{Quarks}_\text{R}} + \underbrace{\frac{1}{3} \times \left(-\frac{1}{3}\right)^2}_{\text{Down}-\text{Squarks}_\text{R}} \right] = N_c \times \frac{3}{5} \times \frac{1}{9} = \frac{1}{5} \tag{7.254}$$

– Up type Higgs Supermultiplet

$$b_1^{\text{Higgs}_\text{u}\,\text{SM}} = \frac{3}{5} \times \left[\underbrace{\frac{1}{3} \times \left(\frac{1}{2}\right)^2 \times 2}_{\text{Higgs}_\text{u}} + \underbrace{\frac{2}{3} \times \left(\frac{1}{2}\right)^2 \times 2}_{\text{Higgs}_\text{u}} \right] = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \tag{7.255}$$

– Down type Higgs Supermultiplet

$$b_1^{\text{Higgs}_\text{d}\,\text{SM}} = \frac{3}{5} \times \left[\underbrace{\frac{1}{3} \times \left(-\frac{1}{2}\right)^2 \times 2}_{\text{Higgs}_\text{d}} + \underbrace{\frac{2}{3} \times \left(-\frac{1}{2}\right)^2 \times 2}_{\text{Higgs}_\text{d}} \right] = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10} \tag{7.256}$$

Now we put everything together to obtain fro the MSSM,

$$\begin{aligned}
b_1 =& N_g \times \left(\frac{3}{10} + \frac{3}{5} + \frac{1}{10} + \frac{4}{5} + \frac{1}{5}\right) + \frac{3}{10} + \frac{3}{10} = 3 \times 2 + \frac{3}{5} = \frac{33}{5} \\
b_2 =& -6 + N_g \times \left(\frac{1}{2} + \frac{3}{2}\right) + \frac{1}{2} + \frac{1}{2} = 1 \\
b_3 =& -9 + N_g \times \left(1 + \frac{1}{2} + \frac{1}{2}\right) = -3
\end{aligned} \tag{7.257}$$

Now let us look to see what are the results for M_X, $\sin^2\theta_W(M_Z)$ and α_5^{-1} in the MSSM. Using the same inputs as for the SM, Eq. (7.228) [8] , we get

$$M_X = 2.1 \times 10^{16} \text{ GeV}, \quad \sin^2\theta_W(M_Z) = 0.231, \quad \alpha_5^{-1} = 24.27 \tag{7.258}$$

we immediately see that these values are quite good. This can be seen very clearly if we use Eq. (7.215), with $\mu_i = M_Z$ and plot the α_i^{-1} as a function of $\ln(\mu^2/M_Z^2)$. This is shown in Fig. 7.9 and the agreement is excellent.

We can still go a step further. We know that supersymmetry must be broken above the electroweak scale, so what we have done in Fig. 7.9 is not quite correct because we are running with the MSSM content down to the weak scale. Of course each particle will decouple at its mass, but assuming that their masses are not much different we can assume that there will a scale M_{SUSY}, below which we will have the SM RGEs. We can redo the calculation taking now the evolved SM values at M_{SUSY} as the boundary conditions for the MSSM evolution. In Fig. and Fig. the results are shown for various values of the SUSY scale. We see from these results that if the SUSY scale is much higher than, say

[8]Again we do not take into account errors and the difference between different renormalization schemes. Also this discussion is at one-loop level and the effects of the supersymmetric particles not decoupling at the same scale (thresholds) are not taken in account.

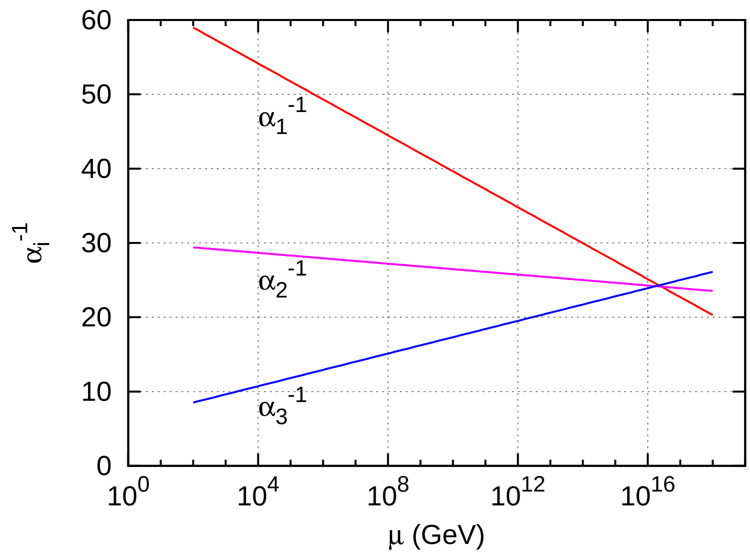


Figure 7.9: Evolution of α_i as function of the scale μ in the MSSM for a $SU(5)$ minimal GUT theory.

1 TeV, the good agreement starts do disappear. Before we end we should emphasize that these are one loop results, without many fine details, like thresholds (talking in account that not all the supersymmetric particles decouple at the same scale) and the important two-loop effects.

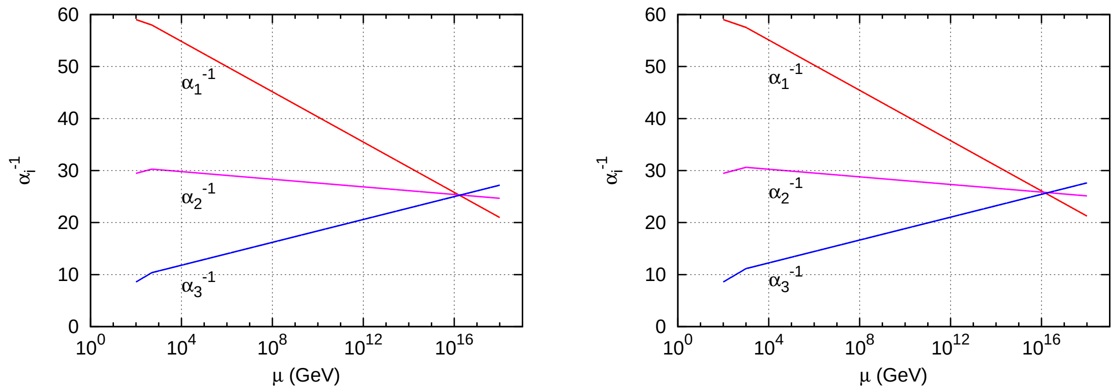


Figure 7.10: Evolution of α_i as function of the scale in the MSSM for a $SU(5)$ minimal GUT theory. On the left panel $M_{\rm SUSY} = 500$ GeV and on the right panel $M_{\rm SUSY} = 1000$ GeV.

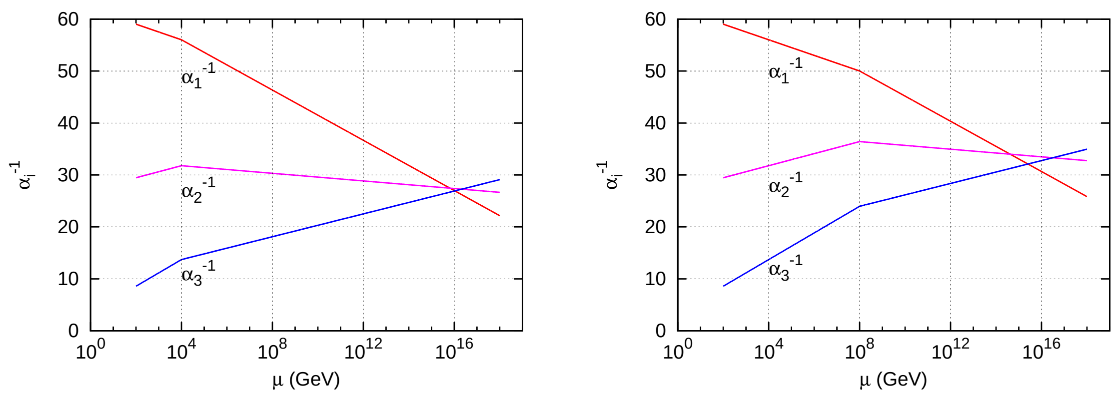
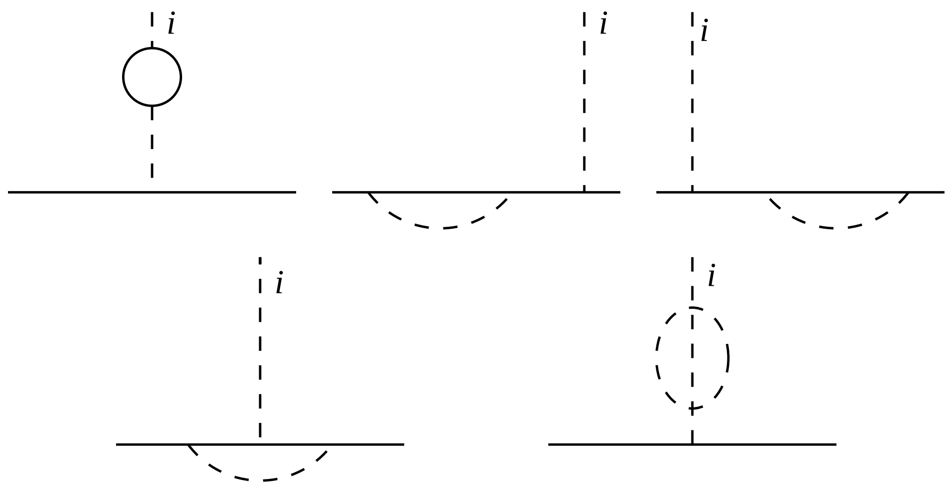


Figure 7.11: Evolution of α_i as function of the scale in the MSSM for a $SU(5)$ minimal GUT theory. On the left panel $M_{\rm SUSY} = 10^4$ GeV and on the right panel $M_{\rm SUSY} = 10^8$ GeV.

Problems for Chapter 7

3.1 Verifique a equação 7.161. Para isso note que $\beta^i = \frac{dg^i}{dt}$ onde β^i é calculada a partir dos diagramas seguintes



3.2 Calcule em subtração mínima (MS) a constante de renormalização Z_3 para QED, equação 7.166.

3.3 Calcule em MS a constante de renormalização Z_3 em electrodinâmica escalar, equação 7.168.

3.4 Considere uma teoria não abeliana com grupo de simetria G e sem matéria. Calcule as constantes de renormalização do propagador Z_A, e do vértice triplo Z_1.

3.5 Considere uma teoria não abeliana em interacção com campos escalares e fermiónicos. Calcule a contribuição destes campos para Z_A e Z_1. Utilize estes resultados juntamente com os resultados do problema 3.4 para determinar a função β do grupo de renormalização para essa teoria.

3.6 Considere o modelo padrão das interacções electrofracas e fortes. Considerando todos os campos do modelo (incluindo os Higgs) calcule os coeficientes b_1, b_2 e b_3 definidos na equação (tc2-3.201).

3.7 Considere agora o modelo padrão supersimétrico mínimo (MSSM). Calcule os coeficientes b_1, b_2 e b_3 definidos na equação (tc2-3.201). Refaça a análise da convergência das constantes de acoplamento no quadro duma teoria de grande unificação supersimétrica com o grupo $SU(5)$.

Appendix A

Path Integral in Quantum Mechanics

A.1 Introduction

A formulação usual é dada pela equação de Schrödinger

$$i\hbar \frac{\partial}{\partial t} |a(t)\rangle = H |a(t)\rangle \tag{A.1}$$

onde

$$H = \frac{P^2}{2m} + V(Q) \tag{A.2}$$

e

$$[Q, P] = i\hbar \tag{A.3}$$

Esta formulação é equivalente a uma outra definida em termos de integrais de caminho. Para isso baseamo-nos na observação que em mecânica quântica sabemos responder a qualquer pergunta sobre o sistema se soubermos calcular as amplitudes de transição

$$\langle b(t') | a(t) \rangle = \langle b| e^{-iH(t'-t)} |a\rangle \tag{A.4}$$

São estas amplitudes de transição que são definidas em termos de integrais de caminho. Conforme a representação escolhida para os estados $|a\rangle$ e $|b\rangle$ as expressões para o integral de caminho vêm diferentes. Assim vamos analisar separadamente os casos das representações no espaço das configurações (coordenadas), no espaço de fase e por meio de estados coerentes (espaço de Bargmann-Fock).

A.2 Configuration space

Introduzimos os estados $|q\rangle$ e $|p\rangle$ tais que

$$
\begin{aligned}
&Q\left|q\right\rangle = q\left|q\right\rangle \qquad &;\qquad &P\left|p\right\rangle = p\left|p\right\rangle \\
&\langle q'|q\rangle = \delta(q'-q) \qquad &;\qquad &\langle p'|p\rangle = \delta(p'-p) \\
&\langle q|p\rangle = \langle p|q\rangle^* = \tfrac{1}{\sqrt{2\pi}}e^{ipq} & & \qquad\qquad (A.5)
\end{aligned}
$$

Então

$$\langle q_f, t_f | q_i, t_i\rangle = \int \mathcal{D}(q) e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt L(q,\dot{q})} \tag{A.6}$$

onde $\mathcal{D}$ é uma medida de integração definida pelo limite

$$\mathcal{D}(q) = \lim_{n\to\infty} \prod_{1}^{n-1} dq_p \left[\frac{nme^{-i\pi/2}}{2\pi(t_f - t_i)}\right]^{\frac{n}{2}} \tag{A.7}$$

sendo n o número de intervalos em que se fez a partição do intervalo (t_i, t_f). O limite $n \to \infty$ é bastante complicado e só existe prova matemática para certas classes de potenciais. A Eq. (A.6) permite uma interpretação da mecânica clássica como limite da mecânica quântica. De facto quando $\hbar \to 0$ a maior contribuição para a amplitude vem das trajectórias que minimizam a ac cão, isto é, as trajectórias clássicas. A mecânica quântica é então vista como o estudo das flutuações à volta da trajectória clássica.

A.2.1 Matrix elements of operators

Usando a propriedade dos integrais de caminho

$$\int \mathcal{D}(q)e^{iS(f,i)} = \int dq(t) \int \mathcal{D}(q)e^{iS(f,t)} \int \mathcal{D}(q)e^{iS(t,i)} \tag{A.8}$$

onde $t_i < t < t_f$, é fácil mostrar que

$$
\begin{aligned}
\langle q_f, t_f|\, \mathcal{O}(t)\, |q_i, t_i\rangle \;&=\; \int dq' dq'' \int \mathcal{D}(q) e^{iS(q_f,t_f;q'',t)} \\
&\qquad \left\langle q''\right| \mathcal{O} \left|q'\right\rangle \int \mathcal{D}(q) e^{iS(q',t;q_i,t_i)}
\end{aligned} \tag{A.9}
$$

Então se $\mathcal{O}$ for diagonal no espaço das coordenadas, isto é, se

$$\left\langle q''\right| \mathcal{O} \left|q'\right\rangle = \mathcal{O}(q')\delta(q'-q'') \tag{A.10}$$

obtemos

$$\langle q_f, t_f|\, \mathcal{O}\, |q_i, t_i\rangle = \int \mathcal{D}(q)\; e^{iS(f,i)}\mathcal{O}(q(t)) \tag{A.11}$$

A.2.2 Time ordered product of operators

Seja $\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)$ com $t_1 \geq t_2 \geq \cdots \geq t_n$. Então é fácil de mostrar que a ordenação no tempo é automática no integral de caminho, isto é,

$$\langle q_f, t_f|\, \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\cdots\mathcal{O}_n(t_n)\, |q_i, t_i\rangle = \int \mathcal{D}(q) e^{iS(f,i)} \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \tag{A.12}$$

Este resultado é particularmente importante, pois permitirá escrever as funções de Green de produtos de operadores ordenados no tempo como simples integrais de caminho de produtos dos equivalentes clássicos desses operadores.

A.2.3 Exact results I: harmonic oscillator

Para alguns potenciais é possível calcular exactamente o limite introduzido em (A.5). Para esses casos o integral de caminho é portanto perfeitamente bem definido. Esses potenciais não são muitos, mas são particularmente importantes. Para o seguimento interessa-nos discutir dois deles. O primeiro é o oscilador harmónico definido pelo potencial

$$V(Q) = m\frac{\omega^2}{2}Q^2 \tag{A.13}$$

Para este caso obtém-se, (os integrais são gaussianos e por isso podem ser explicitamente calculados)

$$\langle f|i\rangle = \left(\frac{m\omega e^{-i\pi/2}}{2\pi \sin\omega t}\right)^{\frac{1}{2}} \exp\left\{i\frac{m\omega}{2}\left[(q_f^2+q_i^2)\cot\omega t - \frac{2q_f q_i}{\sin\omega t}\right]\right\} \tag{A.14}$$

Este resultado vai ser útil adiante.

A.2.4 Exact results II: external force

Consideremos agora uma força exterior tal que o potencial é dado por

$$V(Q) = -QF(t) \tag{A.15}$$

Neste caso obtemos

$$\langle f|i\rangle_F = \left[\frac{me^{-i\pi/2}}{2\pi(t_f-t_i)}\right]^{\frac{1}{2}} e^{iS(f,i)} \tag{A.16}$$

onde $S(f,i)$ é a acção calculada ao longo da trajectória clássica,

$$\begin{aligned} S(f,i) &= \frac{m}{2}\frac{(q_f-q_i)^2}{t_f-t_i} + \int_{t_i}^{t_f} dtF(t)\left(q_f\frac{t-t_i}{t_f-t_i}+q_i\frac{t_f-t}{t_f-t_i}\right) \\ &\quad +\frac{1}{2m}\int_{t_i}^{t_f}\int_{t_i}^{t_f} dt'dt''F(t')G(t',t'')F(t'') \end{aligned} \tag{A.17}$$

onde $G(t',t'') = \frac{t't''}{T} - \text{Inf}(t',t'')$ é a função de Green simétrica para o problema $\ddot{q} = F(t)/m$ com as condições na fronteira $G(0,t'') = G(t',0) = 0$.

A.2.5 Perturbation theory

A importância do resultado exacto para a força exterior deve-se ao facto que usando esse resultado podemos formalmente resolver o problema dum potencial qualquer. Para isso notemos que a derivação funcional em realação à fonte $F(t)$ faz baixar $Q(t)$. Mais explicitamente

$$\langle f|\, Q(t)\, |i\rangle_F = \frac{\delta}{i\delta F(t)}\ \langle f|i\rangle_F \tag{A.18}$$

onde $\langle|\rangle_F$ significa calculado na presença da fonte exterior (i.e. para o hamiltoniano $H = P^2/2m - QF(t)$). Então para um potencial arbitrário $V(q)$ temos

$$\begin{aligned}\langle f|i\rangle &= \int \mathcal{D}(q) e^{i\int_{t_i}^{t_f} dt\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]} \\ &= \exp\left\{-i\int_{t_i}^{t_f} dt V\left[\frac{\delta}{i\delta F(t)}\right]\right\}\langle f|i\rangle_F\Big|_{F=0}\end{aligned} \tag{A.19}$$

Esta expressão formal torna-se muito útil quando a exponencial é expandida em série. Então todos os integrais são do tipo gaussiano e podem ser exactamente executados. Obtemos assim a teoria das perturbações. Claro que só terá significado se houver um parâmetro *pequeno* no potencial. É importante notar que enquanto se faça teoria das perturbações não há qualquer problema com a indefinição matemática do integral de caminho, pois todas as integrações são gaussianas.

A.3 Phase space formulation

Para este caso obtemos

$$\langle q_f, t_f|q_i, t_i\rangle = \int \mathcal{D}(p,q) e^{i\int_{t_i}^{t_f} dt\left[p\dot{q} - h(p,q)\right]} \tag{A.20}$$

onde $h(p,q)$ é o hamiltoniano clássico e a medida é dada pelo limite

$$\mathcal{D}(p,q) = \lim_{n\to\infty} \frac{\prod_1^n dp_s \prod_1^{n-1} dq_r}{(2\pi)^n} \tag{A.21}$$

A fase da exponencial é novamente a acção clássica expressa nas variáveis canónicas p e q. Se $h(p,q)$ depender quadraticamente de p como é usual, pode-se fazer a integração gaussiana em p e a expressão reduz-se à do caso anterior, equação (A.4).

A.4 Bargmann-Fock space (coherent states)

Nesta representação usamos funções analíticas de variável complexa para descrevermos os operadores a e $a^\dagger$ ($[a, a^\dagger] = 1$). A correspondência é feita do modo seguinte. As funções analíticas geram um despaço de Hilbert com o produto interno definido por

$$\langle g|f\rangle \equiv \int \frac{dzd\overline{z}}{2\pi i}\; e^{-z\overline{z}}\; \overline{g}(z) f(\overline{z}) \tag{A.22}$$

Os operadores a e $a^\dagger$ são representados neste espaço por

$$\begin{aligned} a &\to \frac{\partial}{\partial z} \\ a^\dagger &\to \overline{z} \end{aligned} \tag{A.23}$$

Dado um estado $|f\rangle$, representado pela função $f(\overline{z})$, a acção do operador A em $|f\rangle$ produz outro estado que também pode ser representado por funções analíticas. Se designarmos por $g(\overline{z})$ essa função temos

$$g(\overline{z}) \equiv \langle \overline{z}|\, A\, |f\rangle \equiv \int \frac{d\xi d\overline{\xi}}{2\pi i}\; e^{-\xi\overline{\xi}} A(\overline{z},\xi) f(\overline{\xi}) \tag{A.24}$$

onde $A(\overline{z},\xi)$ é o kernel do operador A. Uma representação explicita para o kernel é fácil de obter. Para isso introduzimos os estados $|n\rangle$, definidos por

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}\, |0\rangle \tag{A.25}$$

É fácil de verificar (ver Problema A.2) que com a definição de produto interno acima introduzida estes estados são ortonormados, isto é $\langle f_m|f_n\rangle = \delta_{mn}$.
Então

$$\begin{aligned} \langle \overline{z}|\, A\, |f\rangle &= \sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}} \langle n|\, A\, |m\rangle \langle m|f\rangle \\ &= \sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}}\; A_{n,m} \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\xi\overline{\xi}} \frac{\xi^m}{\sqrt{m!}} f(\overline{\xi}) \\ &= \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\xi\overline{\xi}} \left[\sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}} A_{n,m} \frac{\xi^m}{\sqrt{m!}} \right] f(\overline{\xi}) \end{aligned} \tag{A.26}$$

Portanto

$$A(\overline{z},\xi) \equiv \sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}} A_{n,m} \frac{\xi^m}{\sqrt{m!}} \tag{A.27}$$

O kernel de qualquer operador é assim obtido desde que se conheçam os seus elementos de matriz na base $|n\rangle$.

Já vimos como se representam estados e operadores. Vamos ver como representar produtos de operadores. Sejam dois operadores A_1 e A_2 e um estado $|f\rangle$. Seja ainda

$$g(\overline{\eta}) = \langle \overline{\eta}|\, A_2\, |f\rangle = \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\xi\overline{\xi}} A_2(\overline{\eta},\xi) f(\overline{\xi}) \tag{A.28}$$

Então

$$
\begin{aligned}
\langle\overline{z}| A_1 |g\rangle &= \int \frac{d\xi d\overline{\xi}}{2\pi i}\ e^{-\xi\overline{\xi}} A_1(\overline{z},\eta)g(\overline{\eta}) \\
&= \int \frac{d\eta d\overline{\eta}}{2\pi i}\frac{d\xi d\overline{\xi}}{2\pi i}\ e^{-\xi\overline{\xi}}e^{-\eta\overline{\eta}}\ A_1(\overline{z},\eta)A_2(\overline{\eta},\xi)f(\overline{\xi}) \\
&= \int \frac{d\xi d\overline{\xi}}{2\pi i}\ e^{-\xi\overline{\xi}}\ \left[\int \frac{d\eta d\overline{\eta}}{2\pi i}\ e^{-\eta\overline{\eta}}\ A_1(\overline{z},\eta)A_2(\overline{\eta},\xi)\right] f(\overline{\xi}) \\
&= \int \frac{d\xi d\overline{\xi}}{2\pi i}\ e^{-\xi\overline{\xi}}\ A_3(\overline{z},\xi)f(\overline{\xi}) \qquad \text{(A.29)}
\end{aligned}
$$

Portanto o kernel do operador $A_3 = A_1A_2$ é obtido por convolução dos kernéis de A_1 e A_2, isto é

$$
A_3(\overline{z},\eta) = \int \frac{d\eta d\overline{\eta}}{2\pi i}\ e^{-\eta\overline{\eta}}\ A_1(\overline{z},\eta)A_2(\overline{\eta},\xi) \qquad \text{(A.30)}
$$

A.4.1 Normal form for an operator

Como já sabemos representar estados, operadores e produtos de operadores já temos toos os ingredientes para fazer mecânica quântica neste espaço. Há contudo um outro assunto que é importante tendo em atenção que pretendemos aplicar este formalismo em teoria quântica dos campos. Trata-se da *forma normal* dum operador[1]. O operador A na sua forma normal é definido por

$$
A = \sum_{n,m} A^N_{n,m}\ \frac{a^{\dagger n}a^m}{\sqrt{n!}\sqrt{m!}} \qquad \text{(A.31)}
$$

isto é, os operadores de destruição estão à direita dos operadores de criação. O *kernel normal* é definido por

$$
A^N(\overline{z},z) \equiv \sum_{n,m} \frac{\overline{z}^n}{\sqrt{n!}}\ A^N_{n,m}\ \frac{z^m}{\sqrt{m!}} \qquad \text{(A.32)}
$$

isto é, é obtido por subtituição directa dos operadores de destruição por z e dos de criação por $\overline{z}$. Para um operador dado na sua forma normal este é o kernel imediato de obter. É contudo diferente do kernel atrás definido. Para ver a relação entre eles notemos a seguinte relação

$$
\begin{aligned}
f(\overline{z}) &= \sum_n \frac{\overline{z}}{\sqrt{n!}}\ \langle n|f\rangle \\
&= \sum_n \int \frac{d\xi d\overline{\xi}}{2\pi i}\ e^{-\xi\overline{\xi}}\ \frac{\overline{z}^n\xi^n}{n!} f(\overline{\xi})
\end{aligned}
$$

[1]Notar que em teoria quântica dos campos tem que se proceder ao ordenamento normal do hamiltoniano para definir o zero da energia.

$$= \int \frac{d\xi d\overline{\xi}}{2\pi i} \; e^{-\xi\overline{\xi}} \; e^{\overline{z}\xi} \; f(\overline{\xi}) \tag{A.33}$$

O kernel $e^{\overline{z}\xi}$ é portanto uma função delta neste espaço. Usando este resultado obtemos

$$\begin{aligned}
\langle \overline{z} | \, A \, | f \rangle &= \sum_{n,m} \frac{A^N_{n,m}}{\sqrt{n!}\sqrt{m!}} \; \overline{z}^n \; \frac{d^m}{d\overline{z}^m} \; f(\overline{z}) \\
&= \sum_{n,m} \frac{A^N_{n,m}}{\sqrt{n!}\sqrt{m!}} \int \frac{d\xi d\overline{\xi}}{2\pi i} e^{-\xi\overline{\xi}} \; e^{\overline{z}\xi} \; \overline{z}^n \xi^m f(\overline{\xi}) \\
&= \int \frac{d\xi d\overline{\xi}}{2\pi i} \; e^{-\xi\overline{\xi}} \; e^{\overline{z}\xi} \; A^N(\overline{z}, \xi) f(\overline{\xi})
\end{aligned} \tag{A.34}$$

donde resulta

$$A(\overline{z}, \xi) = e^{\overline{z}\xi} \; A^N(\overline{z}, \xi) \tag{A.35}$$

Esta relação é muito importante pois permite imediatamente escrever o kernel dum operador qualquer uma vez que seja conhecida a sua forma normal. Isto é particularmente útil em teoria quântica dos campos onde o hamiltoniano é dado na sua forma normal.

A.4.2 Evolution operator

Podemos obter agora a expressão para o operador de evolução nesta representação. De acordo com aquilo que acabámos de dizer, para um intervalo infinitesimal, devemos ter para o kernel de U

$$U(\overline{z}, \xi, \Delta t) = e^{\overline{z}\xi} \; e^{-i\Delta t \; h(\overline{z}, \xi)} \tag{A.36}$$

onde $h(\overline{z}, \xi)$ é o kernel normal obtido por substituição directa dos operadores $a^\dagger$ e a pelas variáveis complexas $\overline{z}$ e ξ. Notar que quando $\Delta t \to 0$ o kernel do operador de evolução se reduz ao kernel da identidade, $e^{\overline{z}\xi}$, que como vimos é a função δ neste espaço.

Para um intervalode tempo finito $t = t_f - t_i$, dividimos o intervalo em n intervalos

$$z_0 \qquad z_1 \qquad z_2 \qquad \cdots \qquad z_{n-1} \qquad z_n \qquad\qquad \Delta t = \frac{t}{n} \tag{A.37}$$

Então

$$\begin{aligned}
U(\overline{z}_1, z_0) \quad &\simeq \quad e^{\overline{z}_1 z_0 - i\Delta t h(\overline{z}_1, z_0)} \\
U(\overline{z}_2, z_1) \quad &\simeq \quad e^{\overline{z}_2 z_1 - i\Delta t h(\overline{z}_2, z_1)} \\
\vdots \qquad &\vdots \qquad\quad \vdots \\
U(\overline{z}_n, z_{n-1}) \quad &\simeq \quad e^{\overline{z}_n z_{n-1} - i\Delta t h(\overline{z}_n, z_{n-1})}
\end{aligned} \tag{A.38}$$

Aplicando agora a regra de multiplicação dos kernéis obtemos

$$\begin{aligned} U(\overline{z}_f, t_f; z_i, t_i) &= \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{dz_k d\overline{z}_k}{2\pi i} \exp\Big[\sum_{k=1}^{n} \overline{z}_k\ z_{k-1} \\ &\qquad - \sum_{k=1}^{n-1} \overline{z}_k z_k - i \sum_{k=1}^{n} h(\overline{z}_k, z_{k-1})\Delta t\Big] \end{aligned} \tag{A.39}$$

ou seja

$$U(\overline{z}_f, t_f; z_i, t_i) \equiv \int \mathcal{D}(z,\overline{z})\ e^{\frac{1}{2}\ (\overline{z}_f z_f + \overline{z}_i z_i) + i\int_{t_i}^{t_f} \left[\frac{1}{2i}(\dot{\overline{z}} z - \overline{z}\dot{z}) - h(\overline{z},z)\right] dt} \tag{A.40}$$

Nesta expressão $\overline{z}_f(t_f)$ e $z_i(t_i)$ são fixados pelas condições fronteiras mas $\overline{z}_f(t_i)$ e $z_i(t_f)$ são arbitrários. A fase da exponencial é novamente a acção, agora escrita nas variáveis complexas z e $\overline{z}$. Para ver isso basta lembrar que

$$\frac{1}{2}(pdq + qdp) = \frac{1}{2i}(zd\overline{z} - \overline{z}dz) \tag{A.41}$$

A.4.3 Exact results I: harmonic oscillator

Também aqui vamos analisar os casos importantes em que há resultados exactos, nomeadamente o oscilador hamónico e o caso das fontes externas. Comecemos pelo oscilador harmónico. O hamiltoniano é dado por

$$H_0 = \omega\ a^\dagger a \tag{A.42}$$

Trata-se portanto dum caso em que o hamiltoniano é dado na forma normal. Este problema pode ser resolvido exactamente. Temos

$$\begin{aligned} U(\overline{z}_f, z_i, t) &= \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{dz_k d\overline{z}_k}{2\pi i} \exp\Big[\sum_{k=1}^{n} \overline{z}_k\ z_{k-1} \\ &\qquad - \sum_{k=1}^{n-1} \overline{z}_k z_k - i\omega \frac{t}{n} \sum_{k=1}^{n} \overline{z}_k z_{k-1}\Big] \\ &= \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{dz_k d\overline{z}_k}{2\pi i}\ e^{\left[-\overline{X}AX + \overline{X}B + \overline{B}X\right]} \end{aligned} \tag{A.43}$$

onde

$$X = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{bmatrix} \qquad ; \qquad \overline{X} = (\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_{n-1}) \tag{A.44}$$

e

$$B = \begin{bmatrix} z_0 a \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad ; \qquad \overline{B} = (0, 0, \cdots, 0, z_n a) \tag{A.45}$$

com $z_0 = z_i$ e $z_n = z_f$. A matriz A de dimensão $(n-1) \times (n-1)$ é dada por

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ -a & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & -a & 1 & 0 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & -a & 1 & 0 \\ \cdots & \cdots & \cdots & 0 & -a & 1 \end{bmatrix} \tag{A.46}$$

onde se definiu

$$a \equiv 1 - i\hbar\omega \frac{t}{n} \tag{A.47}$$

As $(n-1)$ integrações gaussianas podem ser facilmente feitas usando o resultado (ver Problema A.3),

$$\int \prod \frac{dz_k d\overline{z}_k}{2\pi i} \; e^{-\overline{z}Az + \overline{u}z + \overline{z}u} = (\det A)^{-1} \; e^{\overline{u}A^{-1}u} \tag{A.48}$$

obtemos então

$$\begin{aligned} U_0(\overline{z}_f, z_i; t) &= \lim_{n\to\infty} \left[(\det A)^{-1} \; e^{\overline{B}A^{-1}B} \right] \\ &= \lim_{n\to\infty} \left[(\det A)^{-1} \; e^{\overline{z}_f z_i a^2 (A^{-1})_{n-1,1}} \right] \end{aligned} \tag{A.49}$$

É fácil de verificar que para a matriz A se tem

$$(A^{-1})_{k,m} = \begin{cases} a^{k-m} & \text{se } k \geq m \\ \\ 0 & \text{se } k < m \end{cases} \tag{A.50}$$

e portanto

$$\det A = 1 \tag{A.51}$$

e

$$A^{-1}_{n-1,1} = (-1)^n (-a)^{n-2} \tag{A.52}$$

donde se conclui que

$$\lim_{n\to\infty} a^2 (A_{-1})_{n-1,1} = \lim_{n\to\infty} \left(1 - \frac{i\omega t}{n} \right)^n = e^{-i\omega t} \tag{A.53}$$

Obtemos então finalmente

$$U_0(\overline{z}_f, z_i; t) = \exp\left\{\overline{z}_f z_i e^{-i\omega t}\right\} \tag{A.54}$$

Podemos verificar que este resultado é da forma e^{iS} onde S é a acção calculada ao longo da trajectória clássica. De facto a estacionaridade do expoente da exponencial dá

$$\begin{aligned}
&\delta\left\{\frac{1}{2}(\overline{z}_f z(t_f) + \overline{z}(t_i)z_i) \ + \int_{t_i}^{t_f}\left[\frac{\dot{\overline{z}}z - \overline{z}\dot{z}}{2} - i\omega\overline{z}z\right]dt\right\} \\
&\quad = \frac{1}{2}\overline{z}_f\delta z(t_f) + \frac{1}{2}z_i\delta\overline{z}(t_i) - \frac{1}{2}\overline{z}_f\delta z(t_f) - \frac{1}{2}z_i\delta\overline{z}(t_i) \\
&\quad + \int_{t_i}^{t_f}\left[\delta z(\dot{\overline{z}} - i\omega\overline{z}) - \delta\overline{z}(\dot{z} + i\omega z)\right]dt
\end{aligned} \tag{A.55}$$

pois $\delta\overline{z}_f = \delta z_i = 0$. As equações de movimento são portanto

$$\left\{\begin{array}{l} \dot{\overline{z}} - i\omega\overline{z} = 0 \\ \\ \dot{z} + i\omega z = 0 \end{array}\right. \qquad \text{com} \qquad \left\{\begin{array}{l} \overline{z}(t_f) = z_f \\ \\ z(t_i) = z_i \end{array}\right. \tag{A.56}$$

que têm como solução

$$\left\{\begin{array}{l} z(t) = z_i \ e^{i\omega(t_i - t)} \\ \\ \overline{z}(t) = \overline{z}_f \ e^{i\omega t - t_f)} \end{array}\right. \tag{A.57}$$

Substituindo estas soluções no expoente obtemos

$$\begin{aligned}
\frac{1}{2}\left[\overline{z}_f z(t_f) + z_i\overline{z}(t_i)\right] \quad + &\quad \int_{t_i}^{t_f}\left[\frac{1}{2}(\dot{\overline{z}}z - \overline{z}\dot{z}) - i\omega\overline{z}z\right]dt \\
= &\quad \overline{z}_f z_i \ e^{i\omega(t_i - t_f)} \\
= &\quad \overline{z}_f z_i \ e^{-i\omega t}
\end{aligned} \tag{A.58}$$

para $t \equiv t_f - t_i$, como queríamos mostrar.

Um outro resultado importante do oscilador harmónico é que a evolução dum estado sob a acção de $H_0 = \omega a^\dagger a$ é particularmente simples neste espaço das funções de variável complexa. Seja $f(\overline{z})$ a representação do estado $|f\rangle$. A evolução debaixo de H_0 é dada por

$$\begin{aligned}
U_0(t)f(\overline{z}) \quad &= \quad \int \frac{d\xi d\overline{\xi}}{2\pi i} \ e^{-\xi\overline{\xi}} \ e^{\overline{z}\xi e^{-i\omega t}} \ f(\overline{\xi}) \\
&= \quad f(\overline{z}e^{-i\omega t})
\end{aligned} \tag{A.59}$$

isto é, é reduzida à multiplicação por $e^{-i\omega t}$

$$\overline{z} \to \overline{z}\ e^{-i\omega t} \tag{A.60}$$

Isto é importante para descrever a matriz S, em que os estados assimptóticos evoluem de acordo com o hamiltoniano livre.

A.4.4 Exact results II: external force

Seja o hamiltoniano

$$H = \omega a^\dagger a - f(t)a^\dagger - \overline{f}(t)a \tag{A.61}$$

Este hamiltoniano também conduz a um resultado exacto. Usando os mesmos métodos que foram utilizados para o oscilador harmónico pode-se mostrar que neste caso também temos

$$U(\overline{z}_f, z_i; t) = e^{iS(f,i)} \tag{A.62}$$

onde $S(f,i)$ é a acção calculada ao longo das trajectórias clássicas (ver Problema A.1).

A.5 Fermion systems

Vamos generalizar os resultados anteriores ao caso de sistemas de fermiões. Começamos com sistemas com dois níveis com os operadores $a^\dagger$ e a tais que

$$\left\{a^\dagger, a\right\} = 1 \qquad ; \qquad a^2 = a^{2\dagger} = 0 \tag{A.63}$$

Para efectuar a construção anterior vamos tentar representar estes operadores num espaço de Hilbert de funções analíticas. Isto é possível se considerarmos funções (de facto polinómios) com coeficientes complexos em duas variáveis que anticomutam η e $\overline{\eta}$, designadas por variáveis de Grassmann e que obedecem a

$$\eta\overline{\eta} + \overline{\eta}\eta = 0 \qquad ; \qquad \overline{\eta}^2 = \eta^2 = 0 \tag{A.64}$$

Então qualquer função $P(\eta, \overline{\eta})$ terá a forma

$$P(\eta, \overline{\eta}) = p_0 + p_1\overline{\eta} + \widetilde{p}_1\eta + p_{12}\eta\overline{\eta} \tag{A.65}$$

A.5.1 Derivatives

Neste espaço a derivação é definida por (as derivadas são esquerdas)

$$\begin{aligned} \frac{\partial P}{\partial \eta} &= \widetilde{p}_1 + p_{12}\overline{\eta} \\ \frac{\partial P}{\partial \overline{\eta}} &= p_1 - p_{12}\eta \end{aligned} \tag{A.66}$$

De entre todas as funções nas variáveis η e $\overline{\eta}$ definimos o subconjunto das *funções analíticas* tais que

$$\frac{\partial}{\partial\eta} f = 0 \tag{A.67}$$

isto é as funções analíticas têm a forma

$$f = f_0 + f_1 \overline{\eta} \tag{A.68}$$

A.5.2 Dot product

No espaço das funções analíticas define-se o produto interno

$$(g, f) = \overline{g}_0 \; f_0 + \overline{g}_1 \; f_1 \tag{A.69}$$

este produto interno pode ser representado por um integral desde que definamos a integração convenientemente (ver equação (A.74)) .

A.5.3 Integration

A integração nas variáveis de Grassmann é definida pelas relações

$$\begin{aligned} \int d\eta \; \eta = \int d\overline{\eta} \; \overline{\eta} = 1 \\ \int d\eta \; 1 = \int d\overline{\eta} \; 1 = 0 \end{aligned} \tag{A.70}$$

Notar que a integração assim definida é semelhante à derivação. De facto[2]

$$\begin{aligned} &\int d\eta \; P = \partial P \qquad ; \qquad \int d\overline{\eta} \; P = \overline{\partial} P \\ &\int d\overline{\eta} d\eta \; P = \overline{\partial}\partial P \end{aligned} \tag{A.72}$$

Devido à forma da equação A.62 é claro que se tem

$$\overline{\partial}^2 = \partial^2 = 0 \tag{A.73}$$

e que portanto o integral duma derivada é zero. Consideremos agora a mudança de variáveis nos integrais. Seja

$$\begin{pmatrix} \eta \\ \overline{\eta} \end{pmatrix} = A \begin{pmatrix} \xi \\ \overline{\xi} \end{pmatrix} \tag{A.74}$$

Então obtemos

[2] Estamos a usar a notação compacta

$$\partial \equiv \frac{\partial}{\partial\eta} \qquad ; \qquad \overline{\partial} \equiv \frac{\partial}{\partial\overline{\eta}} \tag{A.71}$$

$$
\begin{aligned}
\eta\overline{\eta} &= (A_{11}\xi + A_{12}\overline{\xi})(A_{21}\xi + A_{22}\overline{\xi}) \\
&= (A_{11}A_{22} - A_{12}A_{21})\xi\overline{\xi} \\
&= \det A \; \xi\overline{\xi} \qquad \text{(A.75)}
\end{aligned}
$$

Pelo que

$$\int d\overline{\eta}d\eta P(\eta,\overline{\eta}) = \int d\overline{\xi}d\xi(\det A)^{-1}Q(\xi,\overline{\xi}) \qquad \text{(A.76)}$$

onde $Q(\xi,\overline{\xi})$ é o polinómio que se obtém de $P(\eta,\overline{\eta})$ por substituição de η e $\overline{\eta}$ por ξ e $\overline{\xi}$. Finalmente notemos que se definirmos a conjugação complexa de f por

$$\overline{f} = \overline{f}_0 + \overline{f}_1\eta \qquad \text{(A.77)}$$

então podemos encontrar uma representação integral para o produto interno dada por

$$(g,f) \equiv \int d\overline{\eta}d\eta \; e^{-\overline{\eta}\eta} \; \overline{g} \; f \qquad \text{(A.78)}$$

Para vermos isso calculemos o integral. Obtemos

$$
\begin{aligned}
&\int d\overline{\eta}d\eta \; e^{-\overline{\eta}\eta} \; \overline{g} \; f \\
&= \int d\overline{\eta}d\eta(1 - \overline{\eta}\eta)(\overline{g}_0 + \overline{g}_1\eta)(f_0 + f_1\overline{\eta}) \\
&= \overline{g}_0 f_0 + \overline{g}_1 f_1 \\
&= (g,f) \qquad \text{(A.79)}
\end{aligned}
$$

A.5.4 Representation of operators

Os operadores a e $a^\dagger$ podem ser representados por

$$
\begin{aligned}
&a \rightarrow \overline{\partial} \\
&a^\dagger \rightarrow \overline{\eta} \qquad \text{(A.80)}
\end{aligned}
$$

É fácil de ver que com estas definições temos $a^2 = a^{\dagger 2} = 0$ e $aa^\dagger + a^\dagger a = 1$.

Consideremos agora os estados $|0\rangle$ e $|1\rangle = a^\dagger |0\rangle$ a que correspondem as funções 1 e $\overline{\eta}$. Então podemos encontrar o kernel de qualquer operador

$$A = \sum_{n,m} |n\rangle \, A_{n,m} \, \langle m| \qquad \text{(A.81)}$$

De facto

$$\begin{aligned}
(Af)\overline{\eta} &= \sum_{n,m} \overline{\eta}^n A_{n,m} \langle m|f\rangle \\
&= \int d\overline{\xi} d\xi \; e^{\overline{\xi}\xi} \sum_{n,m} \overline{\eta}^n \; A_{n,m} \; \xi^m \; f(\overline{\xi}) \\
&= \int d\overline{\xi} d\xi \; e^{-\overline{\xi}\xi} \; A(\overline{\eta},\xi) \; f(\overline{\xi}) \qquad \text{(A.82)}
\end{aligned}$$

onde

$$A(\overline{\eta},\xi) \equiv \sum_{n,m} \overline{\eta}^n \; A_{n,m} \; \xi^m \qquad ; \qquad n,m=0,1 \qquad \text{(A.83)}$$

Para o produto de operadores é fácil de ver que temos como anteriormente

$$A_1 A_2(\overline{\eta},\eta) = \int d\overline{\xi} d\xi \; e^{-\overline{\xi}\xi} \; A_1(\overline{\eta},\xi) \; A_2(\overline{\xi},\eta) \qquad \text{(A.84)}$$

A.5.5 Normal form for operators

Seja um operador definido por

$$A = \sum_{n,m} |n\rangle \, A_{n,m} \, \langle m| = \sum_{n,m} a^{\dagger n} \, |0\rangle \, \langle 0| \, a^m \; A_{n,m} \qquad \text{(A.85)}$$

O projector do estado base é

$$|0\rangle \, \langle 0| =: e^{-a^\dagger a} : \qquad \text{(A.86)}$$

logo

$$\begin{aligned}
A &= \sum_{n,m} A_{n,m} : a^{\dagger n} \; e^{-a^\dagger a} \; a^m : \\
&\equiv \sum_{n,m} A^N_{n,m} \; a^{\dagger n} a^m \qquad \text{(A.87)}
\end{aligned}$$

O *kernel normal* é então definido pela substituição $a^\dagger \to \overline{\eta}$ e $a \to \eta$, isto é

$$A^N(\overline{\eta},\eta) = \sum_{n,m} A^N_{n,m} \; \overline{\eta}^n \eta^m \qquad \text{(A.88)}$$

O kernel da identidade é $e^{\overline{\eta}\eta}$, isto é

$$f(\overline{\eta}) = \int d\overline{\xi} d\xi \; e^{-\overline{\xi}\xi + \overline{\eta}\xi} \; f(\overline{\xi}) \qquad \text{(A.89)}$$

o que permite obter a relação entre o kernel usual e kernel normal. De facto

$$\begin{aligned}\left[a^{\dagger n} a^m f\right] \overline{\eta} &= \overline{\eta}^n \frac{\partial^m}{\partial \overline{\eta}^m} f(\overline{\eta}) \\ &= \int d\overline{\xi} d\xi \ e^{-\overline{\xi}\xi} \ e^{\overline{\eta}\xi} \ \overline{\eta}^n \xi^m \ f(\overline{\xi}) \end{aligned} \tag{A.90}$$

o que permite escrever a relação procurada

$$A(\overline{\eta}, \eta) = e^{\overline{\eta}\eta} \ A^N(\overline{\eta}, \eta) \tag{A.91}$$

Finalmente seguindo um raciocínio análogo ao do sistema de bosões é fácil obter o kernel do operador de evolução

$$U(\overline{\eta}_f, t_f; \eta_i, t_i) = \int \mathcal{D}(\overline{\eta}, \eta) \ e^{\frac{1}{2}(\overline{\eta}_f \eta_f + \overline{\eta}_i \eta_i)} \ e^{i \int_{t_i}^{t_f} dt \left[\frac{1}{2i}(\overline{\eta}\dot{\eta} - \dot{\overline{\eta}}) - h(\overline{\eta}, \eta)\right]} \tag{A.92}$$

Problemas do Apêndice A

Problems for Appendix A

A.1 Mostre o resultado da equação A.59. Em particular mostre que

$$
\begin{aligned}
iS(f,i) &= \overline{z}_f\ e^{-i\omega(t_f-t_i)}\ z_i + i\int_{t_i}^{t_f} dt \left[\overline{z}_f\ e^{-i\omega(t_f-t)}\ f(t) + \overline{f}(t)\ e^{-i\omega(t-t_i)}\ z_i\right] \\
&\quad - \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt'\ \overline{f}(t)\ e^{-i\omega(t-t')}\ f(t')\theta(t-t')
\end{aligned}
\tag{A.93}
$$

Este resultado é útil em muitas aplicações (ver equação B.9).

A.2 Mostre que os representantes dos estados $|n\rangle$ e $|m\rangle$, $\frac{\overline{z}^n}{\sqrt{n!}}$ e $\frac{\overline{z}^m}{\sqrt{m!}}$, respectivamente, são ortonormados, isto é, $\langle f_n|f_m\rangle = \delta_{n,m}$.

A.3 Mostrar que para integrais gaussianos se tem

$$
\int \prod \frac{dz_k d\overline{z}_k}{2\pi i}\ e^{-\overline{z}Az+\overline{u}z+\overline{z}u} = (\det A)^{-1}\ e^{\overline{u}A^{-1}u}
\tag{A.94}
$$

Notar que o expoente é o valor no ponto de estacionaridade.

A.4 Mostrar que para integrais gaussianos se obtém

$$
\begin{aligned}
&\int \prod_1^n d\overline{\eta}_k d\eta_k\ e^{\sum \overline{\eta}_k A_{k\ell}\eta_\ell + \sum(\overline{\eta}_k\xi_k+\overline{\xi}_k\eta_k)} \\
&\qquad = \det A\ e^{\sum \overline{\xi}_k (A^{-1})_{k\ell}\xi_\ell}
\end{aligned}
\tag{A.95}
$$

Comparar com o resultado do problema A.3.

Appendix B

Path Integral in Quantum Field Theory

B.1 Path integral quantization

Vamos aqui generalizar os resultados do apêndice A para o caso de sistemas com um número infinito de graus de liberdade que são os que interessam em teoria quântica dos campos. Para evitar complicações com índices e com problemas decorrentes da invariância de gauge vamos estudar o caso do campo escalar cuja acção clássica em presença duma fonte exterior é

$$S(\phi, J) = S_0(\phi, J) + \int d^4x \; V(x) \tag{B.1}$$

onde

$$S_0(\phi, J) = \int d^4x \; \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\; m^2\phi^2 + J\phi\right] \tag{B.2}$$

é a acção do campo escalar livre acoplada a uma fonte exterior. Vamos primeiro estudar este caso, isto é, supor que $V = 0$. O caso geral é fácil, de obter a partir deste, como veremos mais à frente. O hamiltoniano é dado por

$$H = \int d^3x \left[\frac{1}{2}\pi^2_{\rm op} + \frac{1}{2}\left(\nabla\phi_{\rm op}\right)^2 + \frac{1}{2}m^2\phi^2_{\rm op} - J\phi_{\rm op}\right] \tag{B.3}$$

e podemos introduzir os operadores $a(k)$ e $a^\dagger(k)$ tais que num certo instante

$$\phi_{\rm op} = \int \tilde{d}k \left[a(k)\; e^{i\vec{k}\cdot\vec{x}} + a^\dagger(k)\; e^{-i\vec{k}\cdot\vec{x}}\right] \tag{B.4}$$

e

$$\pi_{\rm op} = -i\int \tilde{d}k \left[a(k)\; e^{i\vec{k}\cdot\vec{x}} - a^\dagger(k)\; e^{-i\vec{k}\cdot\vec{x}}\right]\omega(k) \tag{B.5}$$

então

$$H = \int \tilde{d}k \left[\omega(k)a^\dagger(k)a(k) - f(t,\vec{k})a^\dagger(k) - \overline{f}(t,\vec{k})a(k)\right] \tag{B.6}$$

onde introduzimos a transformada de fourier espacial da fonte,

$$f(t,\vec{k}) = \int d^3x \; e^{-i\vec{k}\cdot\vec{x}} \; j(t,\vec{x}) \tag{B.7}$$

e onde definimos

$$\tilde{d}k \equiv \frac{d^3k}{(2\pi)^3 2\omega_k} = \frac{d^4k}{(2\pi)^4} \; 2\pi\delta(k^2-m^2)\theta(k^0) \tag{B.8}$$

usando os resultados do problema A.1 podemos escrever imediatamente o kernel do operador de evolução

$$\begin{aligned} U(\overline{z}_f,t_f;z_i,t_i) \;\; = \;\; & \exp\Bigg\{ \int \tilde{d}k \Bigg[\overline{z}_f(k)\; e^{-i\omega(k)(t_f-t_i)}\; z_i(k) \\ & +i\int_{t_i}^{t_f} dt \left[\overline{z}_f(k)\; e^{-i\omega(k)(t_f-t)}\; f(t,\vec{k}) + \overline{f}(t,\vec{k})\; e^{-i\omega(k)(t-t_i)}\; z_i(k)\right] \\ & -\frac{1}{2}\int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt'\; \overline{f}(t,\vec{k})\; e^{-i\omega(k)(t-t')}\; f(t',\vec{k})\Bigg]\Bigg\} \end{aligned} \tag{B.9}$$

A matriz S é então definida como o limite

$$\lim_{-t_i,t_f\to\infty} e^{it_f H_0}\; U(t_f,t_i)\; e^{-it_i H_0} \tag{B.10}$$

onde H_0 é obtido a partir de H fazendo $J=0$. Na representação que estamos a usar a acção de e^{-itH_0} é uma simples multiplicação (ver eq. A.60).

$$\overline{z} \to \overline{z}\; e^{-i\omega t} \tag{B.11}$$

Portanto o kernel da matriz S é

$$\begin{aligned} S(\overline{z}_f,z_i) \;\; = \;\; & \lim_{-t_i,t_f\to\infty} \exp\left[\int \tilde{d}k\overline{z}_f(k)z_i(k)\right]\; \exp\Bigg\{\int \tilde{d}k \Bigg[\\ & i\int_{t_i}^{t_f} \left[\overline{z}_f(k)\; e^{i\omega(k)t}\; f(t,\vec{k}) + \overline{f}(t,\vec{k})\; e^{-i\omega(k)t} z_i(k)\right] \\ & -\frac{1}{2}\int_{t_i}^{t_f}\int_{t_i}^{t_f} dtdt'\overline{f}(t,\vec{k})\; e^{-i\omega(k)(t-t')}\; f(t',\vec{k})\Bigg]\Bigg\} \end{aligned} \tag{B.12}$$

O primeiro factor é aquilo que é necessário para passar do kernel usual para o kernel normal. O restante pode ser interpretado se definirmos

$$\phi_{\rm as} \equiv \int \tilde{d}k \left[z_i(k)\; e^{-ik\cdot x} + \overline{z}_f(k)\; e^{ik\cdot x}\right] \tag{B.13}$$

Como $\overline{z}_f$ não é o complexo conjugado de z_i então $\phi_{\rm as}$ é dado em termos de condições na fronteira com frequências positivas para $t \to -\infty$ e frequências negativas para $t \to \infty$. Estas são precisamente as condições na fronteira de Feynman. Com estas convenções e notações obtemos para o primeiro termo

$$\begin{aligned}
&\int \tilde{d}k \int_{t_i}^{t_f} dt \left[\overline{z}_f(k)\ e^{i\omega(k)t}\ f(t,\vec{k}) + \overline{f}(t,\vec{k})\ e^{-i\omega(k)t}\ z_i(k)\right] \\
&= \int d^4x \int \tilde{d}k J(x) \left[\overline{z}_f e^{i\omega(k)t - i\vec{k}\cdot\vec{x}} + z_i(k)\ e^{-i\omega(k)t + i\vec{k}\cdot\vec{x}}\right] \\
&= \int d^4x J(x) \phi_{\rm as}(x)
\end{aligned} \tag{B.14}$$

e para o segundo

$$\begin{aligned}
&\int \tilde{d}k \int dt \int dt'\ \overline{f}(t,\vec{k})\ e^{-i\omega(k)(t-t')}\ f(t',\vec{k}) \\
&= \int d^4x d^4x' J(x) J(x') \int \tilde{d}k\ e^{-i\omega(k)(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')} \\
&= \int d^4x d^4x'\ J(x) G_F(x-x') J(x')
\end{aligned} \tag{B.15}$$

pois

$$\begin{aligned}
&\int \tilde{d}k\ e^{-i\omega(k)(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')} \\
&\qquad = \int \tilde{d}k\ e^{-i\omega(k)(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')}\ \theta(t-t') \\
&\qquad\quad + \int \tilde{d}k\ e^{i\omega(k)(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')}\ \theta(t'-t) \\
&\qquad = i \int d^4k\ e^{-ik\cdot(x-x')}\ \frac{1}{k^2 - m^2 + i\varepsilon} \\
&\qquad = G_F(x-x')
\end{aligned} \tag{B.16}$$

Notar que as condições na fronteira mistas conduzem ao propagador de Feynman. Podemos portanto finalmente escrever o *kernel normal* da matriz S na presença da fonte J,

$$S^N(\overline{z}_f, z_i)\Big|_J = e^{i\int d^4x\ j(x)\phi_{\rm as}(x)}\ e^{-\frac{1}{2}\int d^4x d^4x'\ J(x) G_F^0(x-x') J(x')} \tag{B.17}$$

Para se obter o operador S substituimos $\phi_{\rm as}$ por $\phi_{\rm op}$ e fazemos o ordenamento normal, isto é

$$S_0(J) =: e^{i\int d^4x\ J(x)\phi_{\rm op}(x)}:\ e^{-\frac{1}{2}\int d^4x d^4x'\ J(x) G_F^0(x-x') J(x')} \tag{B.18}$$

Como o funcional gerador das funções de green é $\langle 0|S_0(J)|0\rangle$ obtemos imediatamente

$$Z_0(J) = e^{-\frac{1}{2}\int d^4x d^4x' J(x) G_F^0(x-x') J(x')} \tag{B.19}$$

Este resultado permite resolver o problema de qualquer potencial $V(x)$. De facto é fácil de mostrar que no caso geral os kernéis estão relacionadoss por

$$S^N = \exp\left[-i\int d^4x V\left(\frac{\delta}{i\delta J(x)}\right)\right]\ S^N(J)\big|_{J=0} \tag{B.20}$$

e o operador S é

$$S =: e^{\int d^4x J(x)\phi_{\rm op}(x)} :\ \exp\left[-i\int d^4x\ V\left(\frac{\delta}{i\delta J(x)}\right)\right]\ Z_0(J)\big|_{J=0} \tag{B.21}$$

ou seja

$$Z(J) = \exp\left[-i\int d^4x\ V\left(\frac{\delta}{i\delta J(x)}\right)\right]\ Z_0(J) \tag{B.22}$$

com

$$Z_0(J) = e^{-\frac{1}{2}\int d^4x d^4x'\ J(x) G_F^0(x-x') J(x')} \tag{B.23}$$

Estas expressões permitem calcular qualquer função de green com as regras usuais da teoria das perturbações. A quantificação usando os integrais de caminho conduziu aos mesmos resultados (em teoria das perturbações) que a quantificação canónica. As expressões para os funcionais geradores embora dêem resultados perturbativos duma forma imediata não são as mais úteis quando estamos interessados em encontrar resultados válidos para além da teoria das perturbações. Para esses casos (identidades de Ward, etc) é mais útil ter uma expressão formal em termos dum integral de caminho. É isso que vamos agora estudar.

B.2 Path integral for generating functionals

O ponto de partida é a expressão para o kernel da matriz S,

$$\begin{aligned} S(\overline{z}_f, z_i) \quad = \quad & \lim_{-t_i, t_f\to\infty} \int \mathcal{D}(\overline{z}, z)\ e^{\frac{1}{2}\int \tilde{d}k[\overline{z}(k,t_f)z(k,t_f)+\overline{z}(k,t_i)z(k,t_i)]} \\ & \exp\left\{i\int_{t_i}^{t_f} dt\int \tilde{d}k\left[\frac{1}{2i}(\dot{\overline{z}}(k,t)z(k,t) - \overline{z}(k,t)\dot{z}(k,t))\right.\right. \\ & \left.\left. -\omega(k)\overline{z}(k,t)z(k,t) - V(\overline{z}, z)\right]\right\} \end{aligned} \tag{B.24}$$

com as condições na fronteira[1]

[1]Não há restrições em $\overline{z}(k,t_i)$ e $z(k,t_f)$.

$$\begin{cases} \overline{z}(k,t_f) = \overline{z}_f(k)\ e^{i\omega t_f} \\ \\ z(k,t_i) = z_i(k)\ e^{-i\omega t_i} \end{cases} \tag{B.25}$$

Em vez das variáveis $z(k,t)$ e $\overline{z}(k,t)$ vamos introduzir os campos clássicos $\phi(\vec{x},t)$ e $\pi(\vec{x},t)$ definidos por

$$\phi(\vec{x},t) = \int \tilde{d}k \left[z(k,t)\ e^{i\vec{k}\cdot\vec{x}} + \overline{z}(k,t)\ e^{-i\vec{k}\cdot\vec{x}} \right] \tag{B.26}$$

e

$$\pi(\vec{x},t) = -i \int \tilde{d}k\ \omega(k) \left[z(k,t)\ e^{i\vec{k}\cdot\vec{x}} - \overline{z}(k,t)\ e^{-i\vec{k}\cdot\vec{x}} \right] \tag{B.27}$$

Estas fórmulas são obviamente sugeridas pelas relações entre ϕ_{op}, π_{op} e $a(k)$, $a^\dagger(k)$ expressas nas equações B.4 e B.5, só que aqui não se trata de operadores mas sim de campos clássicos. Comecemos por escrever a acção em termos das novas variáveis,

$$\begin{aligned} &\int_{t_i}^{t_f} dt \int \tilde{d}k \left[\frac{1}{2i}(\dot{\overline{z}}(k,t)z(k,t) - \overline{z}(k,t)\dot{z}(k,t)) \right. \\ &\qquad\qquad \left. -\omega(k)\overline{z}(k,t)z(k,t) - V(\overline{z},z) \right] \\ &= \int d^3x \int_{t_i}^{t_f} dt \left[\frac{1}{2}(\pi\partial_0\phi - \partial_0\pi\phi) - \frac{1}{2}\pi^2 \right. \\ &\qquad\qquad \left. -\frac{1}{2}(\partial_k\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \right] \end{aligned} \tag{B.28}$$

Introduzimos agora novas variáveis $\phi_1(\vec{x},t)$ e $\pi_1(\vec{x},t)$ definidas do modo seguinte

$$\begin{cases} \phi(\vec{x},t) \equiv \phi_{\text{as}}(\vec{x},t) + \phi_1(\vec{x},t) \\ \\ \pi(\vec{x},t) \equiv \partial_0\phi(\vec{x},t) + \phi_1(\vec{x},t) \end{cases} \tag{B.29}$$

onde

$$\phi_{\text{as,in}}(\vec{x},t) = \int \tilde{d}k \left[\overline{z}_{\text{in}}(k)\ e^{ik\cdot x} + z_{\text{in}}(k)\ e^{-ik\cdot x} \right] \tag{B.30}$$

e

$$\phi_{\text{as,out}}(\vec{x},t) = \int \tilde{d}k \left[\overline{z}_{\text{out}}(k)\ e^{ik\cdot x} + z_{\text{out}}(k)\ e^{-ik\cdot x} \right] \tag{B.31}$$

onde $in \equiv t \to -\infty$ e $out \equiv t \to +\infty$ com

$$\begin{cases} \overline{z}_{\text{out}}(k) \equiv \overline{z}_f(k) \\ \\ z_{\text{in}}(k) \equiv z_i(k) \end{cases} \tag{B.32}$$

O campo $\phi_{\rm as}$ tem portanto as condições fronteira apropriadas para o problema e satisfaz à equação de Klein-Gordon

$$(\Box + m^2)\phi_{\rm as} = 0 \tag{B.33}$$

Escrevemos a acção nas novas variáveis

$$\begin{aligned}
&\int d^3x \int_{t_i}^{t_f} dt \left[\frac{1}{2}(\pi\partial_0\phi - \partial_0\pi\phi) - \frac{1}{2}\pi^2 - \frac{1}{2}(\partial_k\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)\right]\\
&\quad = \int d^3x \left[-\frac{1}{2}\pi\phi\right]_{t_i}^{t_f}\\
&\quad + \int d^3x \int_{t_i}^{t_f} dt \left[\pi\partial_0\phi - \frac{1}{2}(\pi_1^2 + 2\pi\partial_0\phi - (\partial_0\phi)^2) - \frac{1}{2}(\partial_k\phi_{\rm as})^2 - \frac{1}{2}(\partial_k\phi_1)^2\right.\\
&\qquad\qquad \left. -\partial_k\phi_{\rm as}\partial_k\phi_1 - \frac{1}{2}m^2\phi_{\rm as}^2 - \frac{1}{2}m^2\phi_1^2 - m^2\phi_{\rm as}\phi_1 - V(\phi)\right]\\
= &\quad \int d^3x \left[-\frac{1}{2}\pi\phi\right]_{t_i}^{t_f}\\
&\quad + \int d^3x \int_{t_i}^{t_f} dt \left[-\frac{1}{2}\pi_1^2 + \frac{1}{2}(\partial_0\phi_{\rm as})^2 + \partial_0\phi_{\rm as}\partial_0\phi_1 + \frac{1}{2}(\partial_0\phi_1)^2 - \frac{1}{2}(\partial_k\phi_{\rm as})^2\right.\\
&\qquad\qquad \left. -\frac{1}{2}(\partial_k\phi_1)^2 - \partial_k\phi_{\rm as}\partial_k\phi_1 - \frac{1}{2}m^2\phi_{\rm as}^2 - \frac{1}{2}m^2\phi_1^2 - m^2\phi_{\rm as}\phi_1 - V(\phi)\right]\\
= &\quad \int d^3x \left[\frac{1}{2}\partial_0\phi_{\rm as}\phi_{\rm as} + \partial_0\phi_{\rm as}\phi_1 - \frac{1}{2}\pi\phi\right]_{t_i}^{t_f}\\
&\quad + \int d^3x \int_{t_i}^{t_f} dt \left[-\frac{1}{2}\pi_1^2 + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 - V(\phi)\right]\\
= &\quad \int d^3x \left[\partial_0\phi_{\rm as}\phi - \frac{1}{2}\partial_0\phi_{\rm as}\phi_{\rm as} - \frac{1}{2}\pi\phi\right]_{t_i}^{t_f}\\
&\quad + \int d^3x \int_{t_i}^{t_f} dt \left[-\frac{1}{2}\pi_1^2 + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 - V(\phi)\right]
\end{aligned} \tag{B.34}$$

Vemos que no segundo termo as variáveis ϕ_1 e π_1 estão separadas e π_1 aparece quadraticamente. Isto permitirá eliminar π_1 como veremos no seguimento. Analisemos contudo primeiro o termo que tem as condições na fronteira. Usando as definições de $\phi_{\rm as}$, ϕ e π podemos escrever

$$\begin{aligned}
&i\int d^3x \left[\partial_0\phi_{\rm as}\phi - \frac{1}{2}\partial_0\phi_{\rm as}\phi_{\rm as} - \frac{1}{2}\pi\phi\right]_{t_i}^{t_f}\\
&= \int \tilde{d}k \left\{\overline{z}_f(k)z_i(k) - \frac{1}{2}\left[\overline{z}(k,t_f)z(k,t_f) + \overline{z}(k,t_i)z(k,t_i)\right]\right.
\end{aligned}$$

$$-\frac{1}{4}\big[z(k,t_f) - z_i(k)\ e^{-i\omega t_f}\big]^2 - \frac{1}{4}\big[\overline{z}(k,t_i) - \overline{z}_f(k)\ e^{i\omega t_i}\big]^2 \Big\} \quad \text{(B.35)}$$

Nesta expressão o primeiro termo dá a passagem do kernel usual para o kernel normal, o segundo cancela exactamente o termo na fronteira na definição inicial de $S(\overline{z}_f, z_i)$ e os últimos têm que ser estudados em detalhe. Reunindo tudo até este ponto a expressão do *kernel normal* da matriz S é

$$\begin{aligned} S^N(\phi_{\text{as}}) \quad = \quad & \lim_{-t_i,t_f\to\infty} \int \mathcal{D}(\phi,\pi) \exp\left\{ -\frac{1}{4}\int \tilde{d}k\, \left[\left(z(k,t_f) - z_i(k)\ e^{-i\omega t_f}\right)^2 \right.\right. \\ & \left.\left. + \left(\overline{z}(k,t_i) - z_f(k)\ e^{i\omega t_i}\right)^2 \right] \right\} \\ & \exp\left\{ \int d^3x \int_{t_i}^{t_f} dt \left[-\pi_1^2 + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}m^2\phi_1^2 - V(\phi) \right] \right\} \quad \text{(B.36)} \end{aligned}$$

Esta expressão já está próxima do resultado final. Falta só mostrar que os termos dentro da primeira exponencial tendem para zero quando $-t_i, t_f \to \infty$. Esta é a parte mais delicada do argumento. Vamos expô-lo por passos:

i) Funções rapidamente decrescentes

Queremos que $I(t) = \int d^3x\ \pi_1^2(\vec{x},t)$ seja integrável. Dizemos então que funções como $\pi_1(\vec{x},t)$ são rapidamente decrescentes (RD) quando $|t| \to \infty$.

ii) Informação sobre $\overline{z}_{1,\text{out}}(k,t)$ e $z_{1,\text{in}}(k,t)$

Da definição $\phi = \phi_{\text{as}} + \phi_1$ resultam as definições

$$\left\{ \begin{array}{l} z(k,t) = z_i(k)\ e^{-i\omega t} + z_1(k,t) \\ \\ \overline{z}(k,t) = \overline{z}_f(k)\ e^{i\omega t} + \overline{z}_1(k,t) \end{array} \right. \quad \text{(B.37)}$$

As condições na fronteira dizem-nos que $\overline{z}_{1,\text{out}}(k,t)$ e $z_{1,\text{in}}(k,t)$ são funções RD quando $t \to +\infty$ e $t \to -\infty$ respectivamente, mas não nos dizem nada sobre $\overline{z}_{1,\text{in}}$ e $z_{1,\text{out}}$, que são precisamente os limites que precisamos.

iii) Informação sobre os limites $z_{1,\text{out}}$ e $\overline{z}_{1,\text{in}}$

Informação sobre os limites $z_{1,\text{out}}$ e $\overline{z}_{1,\text{in}}$ obtém-se a partir do seguinte raciocínio,

$$\begin{aligned} \pi_1 \quad &= \quad \pi - \partial_0\phi \\ &= \quad \int \tilde{d}k \Big\{ [i\omega(k)\overline{z}(k,t) - \partial_0\overline{z}(k,t)]\ e^{-i\vec{k}\cdot\vec{x}} \\ & \qquad - [i\omega(k)z(k,t) + \partial_0 z(k,t)]\ e^{i\vec{k}\cdot\vec{x}} \Big\} \\ &\equiv \quad -\int \tilde{d}k \left[\overline{z}_2(k,t)\ e^{-i\vec{k}\cdot\vec{x}} + z_2(k,t)\ e^{i\vec{k}\cdot\vec{x}} \right] \end{aligned} \quad \text{(B.38)}$$

Para que π_1 seja do tipo RD quando $|t| \to \infty$ também teremos que ter $z_2(k,t)$ e $\overline{z}_2(k,t)$ RD nesses limites. Vejamos qual a informação contida neste resultado.

• $t \to +\infty$
Obtemos então que a função

$$z_2(k,t) \equiv \partial_0 z(k,t) + i\omega(k) z(k,t) \tag{B.39}$$

é RD quando $t \to +\infty$. A informação sobre $\overline{z}_2(k,t)$ não trás nada de novo já que está contida nas condições fronteiras. De facto

$$\begin{aligned}
\lim_{t\to+\infty} \overline{z}_2(k,t) \quad &\equiv \quad \lim_{t\to+\infty} \left[\partial_0 \overline{z}(k,t) - i\omega(k)\overline{z}(k,t)\right] \\
&= \quad i\omega(k)\overline{z}_f(k)\ e^{i\omega t} + \partial_0 \overline{z}_{1,\text{out}}(k,t) \\
&\quad\ -i\omega(k)\overline{z}_f(k)\ e^{i\omega t} - i\omega(k)\overline{z}_{1,\text{out}}(k,t) \\
&= \quad \text{RD} \qquad t \to +\infty
\end{aligned} \tag{B.40}$$

• $t \to -\infty$
A informação contida nas condições na fronteira é

$$\overline{z}_2(k,t) \equiv \partial_0 \overline{z}(k,t) - i\omega(k)\overline{z}(k,t) = \text{RD} \qquad t \to -\infty \tag{B.41}$$

iv) Demonstração que $z_{1,\text{out}}$ *e* $\overline{z}_{1,\text{in}}$ *são* RD

Da definição

$$\phi(\vec{x},t) = \phi_{\text{as}} + \phi_1 \tag{B.42}$$

resulta

$$\begin{cases}
\phi(\vec{x},t) = \phi_{\text{as,in}}(\vec{x},t) + \phi_{1,\text{in}}(\vec{x},t) & t \to -\infty \\[2ex]
\phi(\vec{x},t) = \phi_{\text{as,out}}(\vec{x},t) + \phi_{1,\text{out}}(\vec{x},t) & t \to +\infty
\end{cases} \tag{B.43}$$

ou seja

$$\begin{cases}
\overline{z}(k,t) = \overline{z}_{\text{in}}(k)\ e^{i\omega t} + \overline{z}_{1,\text{in}}(k,t) & t \to -\infty \\[2ex]
z(k,t) = z_{\text{out}}(k)\ e^{-i\omega t} + z_{1,\text{out}}(k,t) & t \to +\infty
\end{cases} \tag{B.44}$$

Mas usando os resultados anteriores

$$\begin{aligned}
\partial_0 \overline{z}(k,t) \quad &- \quad i\omega(k)\overline{z}(k,t) = \text{RD} \qquad t \to -\infty \\
&= \quad i\omega(k)\overline{z}_{\text{in}}(k)\ e^{i\omega t} + \partial_0 \overline{z}_{1,\text{in}}(k,t) \\
&\quad\ -i\omega(k)\overline{z}_{\text{in}}(k)\ e^{i\omega t} - i\omega(k)\overline{z}_{1,\text{in}}(k,t)
\end{aligned} \tag{B.45}$$

ou seja

$$\overline{z}_{1,\text{in}}(k,t) = \text{RD} \qquad t \to -\infty \tag{B.46}$$

e igualmente

$$z_{1,\text{out}}(k,t) = \text{RD} \qquad t \to +\infty \tag{B.47}$$

Isto quer dizer que

$$\begin{cases} \phi_{1,\text{in}} = \text{RD} & t \to -\infty \\ \\ \phi_{1,\text{out}} = \text{RD} & t \to +\infty \end{cases} \tag{B.48}$$

isto é, assimptoticamente

$$\phi = \phi_{\text{as}} + \text{RD} \tag{B.49}$$

v) Resultado final

Estamos agora em condições de atacar o nosso problema. Temos

$$\begin{aligned} & \lim_{t_i \to -\infty} \int \tilde{d}k \, \left[\overline{z}(k,t_i) - z_f(k) \; e^{i\omega t_i} \right]^2 \\ = \; & \lim_{t_i \to -\infty} \int \tilde{d}k \, \left[(\overline{z}_{\text{in}}(k) - \overline{z}_f(k)) \; e^{i\omega t_i} + \overline{z}_{1,\text{in}}(k,t_i) \right]^2 \\ = \; & \lim_{t_i \to -\infty} \int \tilde{d}k \, \Big[(\overline{z}_{\text{in}}(k) - \overline{z}_f(k))^2 \;\; e^{2i\omega t_i} + 2 \, (\overline{z}_{\text{in}}(k) - \overline{z}_f(k)) \;\; e^{i\omega t_i} \overline{z}_{1,\text{in}}(k,t_i) \\ & \qquad\qquad + \overline{z}^2_{1,\text{in}}(k,t_i) \Big] \\ = \; & 0 \end{aligned} \tag{B.50}$$

Para o outro termo obter-se-ia o mesmo resultado. Chegamos portanto ao resultado

$$\begin{aligned} S^N(\phi_{\text{as}}) \;\; = \;\; & \int \mathcal{D}(\phi,\pi) \;\; \exp\left\{ -\frac{i}{2} \int d^4x \; \pi_1^2 \right\} \\ & \times \exp\left\{ i \int d^4x \left[\frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1^2 - V(\phi) \right] \right\} \end{aligned} \tag{B.51}$$

onde a integração é feita sobre os campos $\phi = \phi_{\text{as}} + \phi_1$ com as condições fronteiras apropriadas. Fazendo a integração sobre π_1 obtemos (a menos duma normalização)

$$\begin{aligned} S^N(\phi_{\text{as}}) \;\; = \;\; & \int \mathcal{D}(\phi) \;\; e^{i \int d^4x \left[\frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1^2 - V(\phi) \right]} \\ = \;\; & \int_{\phi = \phi_{\text{as}} + \phi_1} \mathcal{D}(\phi) \;\; e^{i \int d^4x [\mathcal{L}(\phi_1) - (V(\phi) - V(\phi_1))]} \end{aligned} \tag{B.52}$$

Na presença de fontes exteriores obtemos

$$S^N(\phi_{\rm as}, J) = \int_{\phi=\phi_{\rm as}+\phi_1} \mathcal{D}(\phi)\ e^{i\int d^4x[\mathcal{L}(\phi_1)-(V(\phi)-V(\phi_1))+J\phi]} \tag{B.53}$$

Normalmente não estamos interessados na matriz S mas no funcional gerador das funções de Green. Por definição

$$Z(J) \equiv S(\phi_{\rm as}, J)\Big|_{\phi_{\rm as}=0} \tag{B.54}$$

Obtemos portanto a expressão fundamental

$$Z(J) = \int \mathcal{D}(\phi)\ e^{i\int d^4x[\mathcal{L}(\phi)+J\phi]} \tag{B.55}$$

B.3 Fermion systems

O uso de variáveis de Grassmann permite escrever expressões de integrais de caminho para a matriz S e para o funcional gerador das funções de Green Z para este caso. Não vamos aqui repetir os cálculos que fizémos para os sistemas de bosões, mas antes apresentar somente os resultados deixando as demonstrações para os problemas.

O ponto de partida é a definição do funcional gerador das funções de Green em presença das fontes exteriores fermiónicas. Este é dado por[2]

$$Z[\eta,\overline{\eta}] = \langle 0|\, T\exp\left[i\int d^4x\left(\overline{\eta}(x)\psi(x)+\overline{\psi}(x)\eta(x)\right)\right]|0\rangle \tag{B.56}$$

Então as funções de Green

$$G^{2n}(x_1,\ldots,y_n) \equiv \langle 0|\, T\psi(x_1)\cdots\psi(x_n)\overline{\psi}(y_1)\cdots\overline{\psi}(y_n)\,|0\rangle \tag{B.57}$$

são dadas por

$$G^{2n}(x_1,\ldots,y_n) = \frac{\delta^{2n}Z}{i\delta\eta(y_n)\cdots i\delta\eta(y_1)i\delta\overline{\eta}(x_n)\cdots i\delta\overline{\eta}(x_1)} \tag{B.58}$$

onde as derivadas são esquerdas, isto é

$$\begin{aligned}
\frac{\delta}{\delta\overline{\eta}(x)}\ \int d^4y\overline{\eta}(y)\psi(y) &= \psi(x) \\
\frac{\delta}{\delta\eta(x)}\ \int d^4y\overline{\psi}(y)\eta(y) &= -\overline{\psi}(x)
\end{aligned} \tag{B.59}$$

e por convenção a ordem da derivação é a indicada, isto é

$$\frac{\delta}{i\delta\eta(y_n)}\cdots\frac{\delta}{i\delta\overline{\eta}(x_1)} \tag{B.60}$$

Consideramos agora o lagrangeano de Dirac livre

[2]Comparar com a definição do caso bosónico, equação 5.15.

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi \tag{B.61}$$

Pode-se mostrar (ver problema B.2) que o funcional gerador é neste caso dado por

$$Z_0[\eta, \overline{\eta}] = e^{-\int d^4x d^4y \ \overline{\eta}(x) S_F^0(x-y)\eta(y)} \tag{B.62}$$

onde $S_F^0(x-y)$ é o propagador de Feynman para a teoria de Dirac livre, dado por

$$S_F^0(x-y) = \int \frac{d^4p}{(2\pi)^4} \ e^{-ip\cdot(x-y)} \ \frac{i}{p\!\!\!/ - m + i\varepsilon} \tag{B.63}$$

Seguindo métodos semelhantes ao do caso bosónico podemos também mostrar que este funcional gerador pode ser representado pelo integral de caminho,

$$Z_0[\eta, \overline{\eta}] = \int \mathcal{D}(\psi, \overline{\psi}) \ e^{i\int d^4x \ \left[\mathcal{L}(x) + \overline{\eta}\psi + \overline{\psi}\eta\right]} \tag{B.64}$$

Tendo o funcional gerador para a teoria livre podemos formalmente escrever o funcional gerador para qualquer teoria fermiónica com interacções. Um exemplo é dado no Problema B.4.

Problems Appendix B

B.1 Mostre que as funções de Green

$$G^{2n}(x_1, \ldots, y_n) \equiv \langle 0| \, T\psi(x_1) \cdots \psi(x_n)\overline{\psi}(y_1) \cdots \overline{\psi}(y_n) \, |0\rangle \tag{B.65}$$

são dadas por

$$G^{2n}(x_1, \ldots, y_n) = \frac{\delta^{2n} Z}{i\delta\eta(y_n) \cdots i\delta\eta(y_1) i\delta\overline{\eta}(x_n) \cdots i\delta\overline{\eta}(x_1)} \tag{B.66}$$

B.2 Mostre que o funcional gerador das funções de Green para a teoria de Dirac livre é dado por

$$Z_0[\eta, \overline{\eta}] = e^{-\int d^4x d^4y \; \overline{\eta}(x) S_F^0(x-y)\eta(y)} \tag{B.67}$$

B.3 Mostre que o funcional gerador das funções de Green para a teoria de Dirac livre se pode representar pelo seguinte integral de caminho

$$Z_0[\eta, \overline{\eta}] = \int \mathcal{D}(\psi, \overline{\psi}) \; e^{i \int d^4x \; \left[\mathcal{L}(x) + \overline{\eta}\psi + \overline{\psi}\eta\right]} \tag{B.68}$$

B.4 Considere o lagrangiano seguinte,

$$\begin{aligned} \mathcal{L}(x) &= \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m\phi^2 \\ &\quad -g\overline{\psi}(x)\psi(x)\phi(x) \end{aligned} \tag{B.69}$$

que descreve a interacção dum campo de Dirac com um campo escalar.

a) Mostre que

$$Z[\eta,\overline{\eta},J] = \exp\left\{-ig\int d^4x \left(\frac{\delta}{i\delta\overline{\eta}}\right)\left(\frac{\delta}{i\delta\eta}\right)\left(\frac{\delta}{i\delta J}\right)\right\}\ Z_0[\eta,\overline{\eta},J] \tag{B.70}$$

onde

$$Z_0[\eta,\overline{\eta},J] = e^{-\int d^4x d^4y \left[\overline{\eta}(x)S_F^0(x-y)\eta(y)+\frac{1}{2}J(x)\Delta_F(x-y)J(y)\right]} \tag{B.71}$$

e Δ_F é o propagador livre do campo escalar.

b) Mostre que $Z[\eta,\overline{\eta},J]$ se pode exprimir por meio do integral de caminho

$$Z[\eta,\overline{\eta},J] = \int \mathcal{D}(\psi,\overline{\psi},\phi)\ e^{i\int d^4x\left[\mathcal{L}(x)+J\phi+\overline{\eta}\psi+\overline{\psi}\eta\right]} \tag{B.72}$$

Appendix C

Useful techniques for renormalization

C.1 μ parameter

The reason for the μ parameter introduced in section 4.1.1 is the following. In dimension $d = 4 - \epsilon$, the fields A_μ and ψ have dimensions given by the kinetic terms in the action,

$$\int d^d x \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \overline{\psi}\gamma \cdot \partial\psi \right] \tag{C.1}$$

We have therefore

$$\begin{aligned} 0 &= -d + 2 + 2[A_\mu] &\Rightarrow [A_\mu] &= \tfrac{1}{2}(d-2) = 1 - \tfrac{\epsilon}{2} \\ 0 &= -d + 1 + 2[\psi] &\Rightarrow [\psi] &= \tfrac{1}{2}(d-1) = \tfrac{3}{2} - \tfrac{\epsilon}{2} \end{aligned} \tag{C.2}$$

Using these dimensions in the interaction term

$$S_I = \int d^d x \; e\overline{\psi}\gamma_\mu \psi A^\mu \tag{C.3}$$

we get

$$\begin{aligned} [S_I] &= -d + [e] + 2[\psi] + [A] \\ &= -4 + \epsilon + [e] + 3 - \epsilon + 1 - \frac{\epsilon}{2} \\ &= [e] - \frac{\epsilon}{2} \end{aligned} \tag{C.4}$$

Therefore, if we want the action to be dimensionless (remember that we use the system where $\hbar = c = 1$), we have to set

$$[e] = \frac{\epsilon}{2} \tag{C.5}$$

We see then that in dimensions $d \neq 4$ the coupling constant has dimensions. As it is more convenient to work with a dimensionless coupling constant we introduce a parameter μ with dimensions of a mass and in $d \neq 4$ we will make the substitution

$$e \to e\mu^{\frac{\epsilon}{2}} \qquad (\epsilon = 4 - d) \tag{C.6}$$

while keeping e dimensionless.

C.2 Feynman parametrization

The most general form for a *1–loop* is [1]

$$\hat{T}_n^{\mu_1 \cdots \mu_p} \equiv \int \frac{d^d k}{(2\pi)^d} \, \frac{k^{\mu_1} \cdots k^{\mu_p}}{D_0 D_1 \cdots D_{n-1}} \tag{C.7}$$

where

$$D_i = (k + r_i)^2 - m_i^2 + i\epsilon \tag{C.8}$$

and the momenta r_i are related with the external momenta (all taken to be incoming) through the relations,

$$\begin{aligned} r_j &= \sum_{i=1}^{j} p_i \quad ; \quad j = 1, \ldots, n-1 \\ r_0 &= \sum_{i=1}^{n} p_i = 0 \end{aligned} \tag{C.9}$$

as indicated in Fig. (C.1). In these expressions there appear in the denominators products

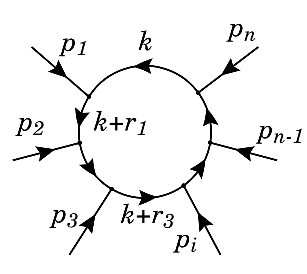


Figure C.1:

of the denominators of the propagators of the particles in the loop. It is convenient to combine these products in just one common denominator. This is achieved by a technique due to Feynman. Let us exemplify with two denominators.

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \tag{C.10}$$

[1] We introduce here the notation $\hat{T}$ to distinguish from a more standard notation that will be explained in subsection C.9.

The proof is trivial. In fact

$$\int dx \, \frac{1}{[ax+b(1-x)]^2} = \frac{x}{b\,[(a-b)x+b]} \tag{C.11}$$

and therefore Eq. (C.10) immediately follows. Taking successive derivatives with respect to a and b we get

$$\frac{1}{a^p b^q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 dx \, \frac{x^{p-1}(1-x)^{q-1}}{[ax+b(1-x)]^{p+q}} \tag{C.12}$$

and using induction we obtain a general formula

$$\begin{aligned} \frac{1}{a_1 a_2 \cdots a_n} = \Gamma(n) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \\ \int_0^{1-x_1-\cdots-x_{n-2}} \frac{dx_{n-1}}{[a_1 x_1 + a_2 x_2 + \cdots + a_n(1-x_1-\cdots-x_{n-1})]^n} \end{aligned} \tag{C.13}$$

Before closing the section let us give an example that will be useful in the self-energy case. Consider the situation with the kinematics described in Fig. (C.2).

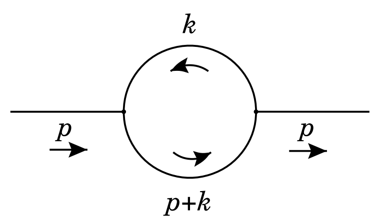


Figure C.2:

We get

$$\begin{aligned} I &= \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{[(k+p)^2 - m_1^2 + i\epsilon]\,[k^2 - m_2^2 + i\epsilon]} \\ &= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{[k^2 + 2p\cdot k\, x + p^2\, x - m_1^2\, x - m_2^2\,(1-x) + i\epsilon]^2} \\ &= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{[k^2 + 2P\cdot k - M^2 + i\epsilon]^2} \\ &= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \, \frac{1}{[(k+P)^2 - P^2 - M^2 + i\epsilon]^2} \end{aligned} \tag{C.14}$$

where in the last line we have completed the square in the term with the loop momenta k. The quantities P and M^2 are, in this case, defined by

$$P = xp \tag{C.15}$$

and

$$M^2 = -x\,p^2 + m_1^2\,x + m_2^2\,(1-x) \tag{C.16}$$

They depend on the masses, external momenta and Feynman parameters, but not in the loop momenta. Now changing variables $k \to k - P$ we get rid of the linear terms in k and finally obtain

$$I = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \, \frac{1}{\left[k^2 - C + i\epsilon\right]^2} \tag{C.17}$$

where C is independent of the loop momenta k and it is given by

$$C = P^2 + M^2 \tag{C.18}$$

Notice that the $i\epsilon$ factors will add correctly and can all be put as in Eq. (C.17).

C.3 Wick Rotation

From the example of the last section we can conclude that all the scalar integrals can be reduced to the form

$$I_{r,m} = \int \frac{d^dk}{(2\pi)^d} \, \frac{k^{2^r}}{\left[k^2 - C + i\epsilon\right]^m} \tag{C.19}$$

It is also easy to realize that also all the tensor integrals can be obtained from the scalar integrals. For instance

$$\begin{aligned} &\int \frac{d^dk}{(2\pi)^d} \, \frac{k^\mu}{\left[k^2 - C + i\epsilon\right]^m} = 0 \\ &\int \frac{d^dk}{(2\pi)^d} \, \frac{k^\mu k^\nu}{\left[k^2 - C + i\epsilon\right]^m} = \frac{1}{d}\, g^{\mu\nu} \int \frac{d^dk}{(2\pi)^d} \, \frac{k^2}{\left[k^2 - C + i\epsilon\right]^m} \end{aligned} \tag{C.20}$$

and so on. Therefore the integrals $I_{r,m}$ are the important quantities to evaluate. We will consider that $C > 0$. The case $C < 0$ can be done by analytical continuation of the final formula for $C > 0$.

To evaluate the integral $I_{r,m}$ we will use integration in the complex plane of the variable k^0 as described in Fig. C.3. We can then write

$$I_{r,m} = \int \frac{d^{d-1}k}{(2\pi)^d} \int dk^0 \, \frac{k^{2^r}}{\left[k_0^2 - |\vec{k}|^2 - C + i\epsilon\right]^m} \tag{C.21}$$

The function under the integral has poles for

$$k^0 = \pm \left(\sqrt{|\vec{k}|^2 + C} - i\epsilon \right) \tag{C.22}$$

as shown in Fig. C.3. Using the properties of functions of complex variables (Cauchy theorem) we can deform the contour, changing the integration from the real to the imaginary axis plus the two arcs at infinity. This can be done because in deforming the contour we do not cross any pole. Notice the importance of the $i\epsilon$ prescription to be able to do this. The contribution from the arcs at infinity vanishes in dimension sufficiently low for the

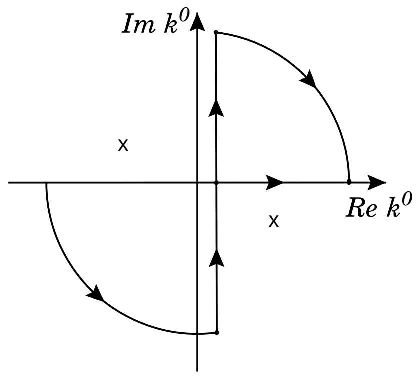


Figure C.3:

integral to converge, as we assume in dimensional regularization. We have then changed the integration along the real axis into an integration along the imaginary axis in the plane of the complex variable k^0. If we write

$$k^0 = ik_E^0 \qquad \text{com} \qquad \int_{-\infty}^{+\infty} dk^0 \to i \int_{-\infty}^{+\infty} dk_E^0 \tag{C.23}$$

and $k^2 = (k^0)^2 - |\vec{k}|^2 = -(k_E^0)^2 - |\vec{k}|^2 \equiv -k_E^2$, where $k_E = (k_E^0, \vec{k})$ is an euclidean vector. By this we mean that we calculate the scalar product using the euclidean metric diag$(+,+,+,+)$,

$$k_E^2 = (k_E^0)^2 + |\vec{k}|^2 \tag{C.24}$$

We can them write

$$I_{r,m} = i(-1)^{r-m} \int \frac{d^d k_E}{(2\pi)^d} \, \frac{k_E^{2r}}{\left[k_E^2 + C\right]^m} \tag{C.25}$$

where we do not need the $i\epsilon$ because the denominator is strictly positive ($C > 0$). This procedure is known as *Wick Rotation*. We note that the Feynman prescription for the propagators that originated the $i\epsilon$ rule for the denominators is crucial for the Wick rotation to be possible.

C.4 Scalar integrals in dimensional regularization

We have seen in the last section that the scalar integrals to be calculated with dimensional regularization had the general form of Eq. (C.25). We are now going to find a general formula for $I_{r,m}$. We begin by writing

$$\int d^d k_E = \int d\overline{k} \; \overline{k}^{\;d-1} \, d\Omega_{d-1} \tag{C.26}$$

where $\overline{k} = \sqrt{(k_E^0)^2 + |\vec{k}|^2}$ is the length of the vector k_E in the euclidean space in d dimensions and $d\Omega_{d-1}$ is the solid angle that generalizes spherical coordinates in that euclidean space. The angles are defined by

$$k_E = \overline{k}(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \ldots, \sin\theta_1\cdots\sin\theta_{d-1}) \tag{C.27}$$

We can then write

$$\int d\Omega_{d-1} = \int_0^{\pi} \sin\theta_1^{d-2}\, d\theta_1 \cdots \int_0^{2\pi} d\theta_{d-1} \tag{C.28}$$

Using now

$$\int_0^{\pi} \sin\theta^m\, d\theta = \sqrt{\pi}\, \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})} \tag{C.29}$$

where $\Gamma(z)$ is the gamma function (see section C.6) we get

$$\int d\Omega_{d-1} = 2\, \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{C.30}$$

The integration in $\overline{k}$ is done using the result

$$\int_0^{\infty} dx\, \frac{x^p}{(x^n + a^n)^q} = \pi(-1)^{q-1}\, a^{p+1-nq}\, \frac{\Gamma(\frac{p+1}{n})}{n \sin(\pi\, \frac{p+1}{n})\, \Gamma(\frac{p+1}{2} - q + 1)} \tag{C.31}$$

and we finally get

$$I_{r,m} = i C^{r-m+\frac{d}{2}}\, \frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}}\, \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})}\, \frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)} \tag{C.32}$$

Before ending the section we note that the integral representation for $I_{r,m}$, Eq. (C.19), is valid only for $d < 2(m-r)$ to ensure convergence when $\overline{k} \to \infty$. However the final form in Eq. (C.32) can be analytically continued for all values of d except for those where the function $\Gamma(m-r-d/2)$ has poles, that is for (see section C.6),

$$m - r - \frac{d}{2} \neq 0, -1, -2, \ldots \tag{C.33}$$

For the application in dimensional regularization it is convenient to rewrite Eq. (C.32) using the relation $d = 4 - \epsilon$. we get

$$I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^2} \left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}} C^{2+r-m}\, \frac{\Gamma(2+r-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\, \frac{\Gamma(m-r-2+\frac{\epsilon}{2})}{\Gamma(m)} \tag{C.34}$$

C.5 Tensor integrals in dimensional regularization

We are frequently faced with the task of evaluating the tensor integrals of the form of Eq. (C.7),

$$\hat{T}_n^{\mu_1\cdots\mu_p} \equiv \int \frac{d^dk}{(2\pi)^d}\, \frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0 D_1 \cdots D_{n-1}} \tag{C.35}$$

The first step is to reduce to one common denominator using the Feynman parameterization technique. The result is,

$$\hat{T}_n^{\mu_1\cdots\mu_p} \quad = \quad \Gamma(n) \int_0^1 dx_1 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1} \int \frac{d^dk}{(2\pi)^d}\, \frac{k^{\mu_1}\cdots k^{\mu_p}}{[k^2 + 2k\cdot P - M^2 + i\epsilon]^n}$$

$$= \Gamma(n) \int_0^1 dx_1 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1}\, I_n^{\mu_1\cdots\mu_p} \tag{C.36}$$

where we have defined

$$I_n^{\mu_1\cdots\mu_p} \equiv \int \frac{d^dk}{(2\pi)^d} \, \frac{k^{\mu_1}\cdots k^{\mu_p}}{[k^2+2k\cdot P - M^2 + i\epsilon]^n} \tag{C.37}$$

that we call, from now on, the tensor integral. In principle all these integrals can be written in terms of scalar integrals. It is however convenient to have a general formula for them. This formula can be obtained by noticing that

$$\frac{\partial}{\partial P^\mu} \, \frac{1}{[k^2+2k\cdot P - M^2 + i\epsilon]^n} = -n \, \frac{2k_\mu}{[k^2+2k\cdot P - M^2 + i\epsilon]^{n+1}} \tag{C.38}$$

Using the last equation one can write the final result

$$I_n^{\mu_1\cdots\mu_p} = \frac{i}{16\pi^2} \frac{(4\pi)^{\epsilon/2}}{\Gamma(n)} \int_0^\infty \frac{dt}{(2t)^p} \, t^{n-3+\epsilon/2} \, \frac{\partial}{\partial P_{\mu_1}} \cdots \frac{\partial}{\partial P_{\mu_p}} \, e^{-t\,C} \tag{C.39}$$

where $C = P^2 + M^2$. After doing the derivatives the remaining integrals can be done using the properties of the Γ function (see section C.6). Notice that P, M^2 and therefore also C depend not only in the Feynman parameters but also in the exterior momenta. The advantage of having a general formula is that it can be programmed [9] and all the integrals can then be obtained automatically.

C.6 Γ function and useful relations

The Γ function is defined by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{C.40}$$

or equivalently

$$\int_0^\infty t^{z-1} e^{-\mu t} dt = \mu^{-z}\Gamma(z) \tag{C.41}$$

The function $\Gamma(z)$ has the following important properties

$$\begin{aligned} \Gamma(z+1) &= z\Gamma(z) \\ \Gamma(n+1) &= n! \end{aligned} \tag{C.42}$$

Another related function is the logarithmic derivative of the Γ function, with the properties,

$$\psi(z) = \frac{d}{dz} \ln\Gamma(z) \tag{C.43}$$

$$\psi(1) = -\gamma \tag{C.44}$$

$$\psi(z+1) = \psi(z) + \frac{1}{z} \tag{C.45}$$

where γ is the Euler constant. The function $\Gamma(z)$ has poles for $z = 0, -1, -2, \cdots$. Near the pole $z = -m$ we have

$$\Gamma(z) = \frac{(-1)^m}{m!}\frac{1}{m+z} + \frac{(-1)^m}{m!}\psi(m+1) + O(z+m) \tag{C.46}$$

From this we conclude that when $\epsilon \to 0$

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} + \psi(1) + O(\epsilon) \qquad \Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{2}{\epsilon} + \psi(n+1) + 1\right] \tag{C.47}$$

and

$$\Gamma(1+\epsilon) = 1 - \gamma\epsilon + \left(\gamma^2 + \frac{\pi^2}{6}\right)\frac{\epsilon^2}{2!} + \cdots \quad , \quad \epsilon \to 0 \tag{C.48}$$

Using these results we can expand our integrals in powers of ϵ and separate the divergent and finite parts. For instance for the one of the integrals of the self-energy,

$$\begin{aligned} I_{0,2} &= \frac{i}{(4\pi)^2}\left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}} \frac{2\,\Gamma(1+\frac{\epsilon}{2})}{\epsilon} \\ &= \frac{i}{16\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln C + O(\epsilon)\right] \\ &= \frac{i}{16\pi^2}\left[\Delta_\epsilon - \ln C + O(\epsilon)\right] \end{aligned} \tag{C.49}$$

where we have introduced the notation

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi \tag{C.50}$$

for a combination that will appear in all expressions.

C.7 Explicit formulæ for the *1–loop* integrals

Although we have presented in the previous sections the general formulæ for all the integrals that appear in *1–loop*, Eqs. (C.34) and (C.39), in pratice it is convenient to have expressions for the most important cases with the expansion on the ϵ already done. The results presented below were generated with the `Mathematica` package `OneLoop` [9] from the general expressions. In these results the integration on the Feynman parameters has still to be done. This is in general a difficult problem and we will present in section C.9 an alternative way of expressing these integrals more convenient for a numerical evaluation.

C.7.1 *Tadpole* integrals

With the definitions of Eqs. (C.34) and (C.39) we get

$$\begin{aligned} I_{0,1} &= \frac{i}{16\pi^2} C(1+\Delta_\epsilon - \ln C) \\ I_1^\mu &= 0 \\ I_1^{\mu\nu} &= \frac{i}{16\pi^2}\frac{1}{8} C^2 g^{\mu\nu}(3+2\Delta_\epsilon - 2\ln C) \end{aligned} \tag{C.51}$$

where for the *tadpole* integrals

$$P = 0 \quad ; \quad C = m^2 \tag{C.52}$$

because there are no Feynman parameters and there is only one mass. In this case the above results are final.

C.7.2 *Self–Energy* integrals

For the integrals with two denominators we get,

$$\begin{aligned} I_{0,2} &= \frac{i}{16\pi^2}\,(\Delta_\epsilon - \ln C) \\ I_2^\mu &= \frac{i}{16\pi^2}(-\Delta_\epsilon + \ln C)P^\mu \\ I_2^{\mu\nu} &= \frac{i}{16\pi^2}\frac{1}{2}\Big[Cg^{\mu\nu}(1+\Delta_\epsilon - \ln C) + 2(\Delta_\epsilon - \ln C)P^\mu P^\nu\Big] \\ I_2^{\mu\nu\alpha} &= \frac{i}{16\pi^2}\frac{1}{2}\Big[-Cg^{\mu\nu}(1+\Delta_\epsilon - \ln C)P^\alpha - Cg^{\nu\alpha}(1+\Delta_\epsilon-\ln C)P^\mu \\ &\quad - Cg^{\mu\alpha}(1+\Delta_\epsilon - \ln C)P^\nu - (2\Delta_\epsilon P^\alpha P^\mu - 2\ln C P^\alpha P^\mu)P^\nu\Big] \end{aligned} \tag{C.53}$$

where, with the notation and conventions of Fig. (C.1), we have

$$P^\mu = x\, r_1^\mu \quad ; \quad C = x^2\, r_1^2 + (1-x)\, m_2^2 + x\, m_1^2 - x\, r_1^2 \tag{C.54}$$

C.7.3 *Triangle* integrals

For the integrals with three denominators we get,

$$\begin{aligned} I_{0,3} &= \frac{i}{16\pi^2}\frac{-1}{2C} \\ I_3^\mu &= \frac{i}{16\pi^2}\frac{1}{2C}\;P^\mu \end{aligned}$$

$$
\begin{aligned}
I_3^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{1}{4C}\,\Big[Cg^{\mu\nu}(\Delta_\epsilon - \ln C) - 2P^\mu P^\nu\Big] \\
I_3^{\mu\nu\alpha} &= \frac{i}{16\pi^2}\,\frac{1}{4C}\,\Big[Cg^{\mu\nu}(-\Delta_\epsilon + \ln C)P^\alpha + Cg^{\nu\alpha}(-\Delta_\epsilon + \ln C)P^\mu \\
&\qquad + Cg^{\mu\alpha}(-\Delta_\epsilon + \ln C)P^\nu + 2P^\alpha P^\mu P^\nu\Big] \\
I_3^{\mu\nu\alpha\beta} &= \frac{i}{16\pi^2}\,\frac{1}{8C}\,\Big[C^2\,(1 + \Delta_\epsilon - \ln C)\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} + g^{\alpha\beta}g^{\mu\nu}\right) \\
&\quad + 2C\,(\Delta_\epsilon - \ln C)\Big(g^{\mu\nu}P^\alpha P^\beta + g^{\nu\beta}P^\alpha P^\mu + g^{\nu\alpha}P^\beta P^\mu + g^{\mu\alpha}P^\beta P^\nu \\
&\quad + g^{\mu\beta}P^\alpha P^\nu + g^{\alpha\beta}P^\mu P^\nu\Big) - 4P^\alpha P^\beta P^\mu P^\nu\Big] \qquad \text{(C.55)}
\end{aligned}
$$

where

$$
\begin{aligned}
P^\mu &= x_1\,r_1^\mu + x_2\,r_2^\mu \\
C &= x_1^2\,r_1^2 + x_2^2\,r_2^2 + 2x_1\,x_2\,r_1\cdot r_2 + x_1\,m_1^2 + x_2\,m_2^2 \\
&\quad + (1 - x_1 - x_2)\,m_3^2 - x_1\,r_1^2 - x_2\,r_2^2 \qquad \text{(C.56)}
\end{aligned}
$$

C.7.4 *Box* integrals

$$
\begin{aligned}
I_{0,4} &= \frac{i}{16\pi^2}\,\frac{1}{6C^2} \\
I_4^\mu &= \frac{i}{16\pi^2}\,\frac{-1}{6C^2}\,P^\mu \\
I_4^{\mu\nu} &= \frac{i}{16\pi^2}\,\frac{-1}{12C^2}\,\Big[Cg^{\mu\nu} - 2P^\mu P^\nu\Big] \\
I_4^{\mu\nu\alpha} &= \frac{i}{16\pi^2}\,\frac{1}{12C^2}\,\Big[C\,(g^{\mu\nu}P^\alpha + g^{\nu\alpha}P^\mu + g^{\mu\alpha}P^\nu) - 2P^\alpha P^\mu P^\nu\Big] \\
I_4^{\mu\nu\alpha\beta} &= \frac{i}{16\pi^2}\,\frac{1}{24C^2}\,\Big[C^2\,(\Delta_\epsilon - \ln C)\left(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} + g^{\alpha\beta}g^{\mu\nu}\right) \\
&\qquad - 2C\,\Big(g^{\mu\nu}P^\alpha P^\beta + g^{\nu\beta}P^\alpha P^\mu + g^{\nu\alpha}P^\beta P^\mu + g^{\mu\alpha}P^\beta P^\nu \\
&\qquad + g^{\mu\beta}P^\alpha P^\nu + g^{\alpha\beta}P^\mu P^\nu\Big) + 4P^\alpha P^\beta P^\mu P^\nu\Big] \qquad \text{(C.57)}
\end{aligned}
$$

where

$$
\begin{aligned}
P^\mu &= x_1\, r_1^\mu + x_2\, r_2^\mu + x_3\, r_3^\mu \\
C &= x_1^2\, r_1^2 + x_2^2\, r_2^2 + x_3^2\, r_3^2 + 2x_1\, x_2\, r_1 \cdot r_2 + 2x_1\, x_3\, r_1 \cdot r_3 + 2x_2\, x_3\, r_2 \cdot r_3 \\
&\quad + x_1\, m_1^2 + x_2\, m_2^2 + x_3\, m_3^2 + (1 - x_1 - x_2 - x_3)\, m_4^2 \\
&\quad - x_1\, r_1^2 - x_2\, r_2^2 - x_3\, r_3^2
\end{aligned}
\tag{C.58}
$$

C.8 Divergent part of *1–loop* integrals

When we want to study the renomalization of a given theory it is often convenient to have expressions for the divergent part of the one-loop integrals, with the integration on the Feynman parameters already done. We present here the results for the most important cases. These divergent parts were calculated with the help of the package `OneLoop` [9]. The results are for the functions $\hat{T}_n^{\mu,\mu_2,\cdots\mu_n}$ defined in Eq. (C.35).

C.8.1 *Tadpole* integrals

$$
\begin{aligned}
\text{Div}\left[\hat{T}_1\right] &= \frac{i}{16\pi^2}\, \Delta_\epsilon\, m^2 \\
\text{Div}\left[\hat{T}_1^\mu\right] &= 0 \\
\text{Div}\left[\hat{T}_1^{\mu\nu}\right] &= \frac{i}{16\pi^2}\, \frac{1}{4}\, \Delta_\epsilon\, m^4\, g^{\mu\nu}
\end{aligned}
\tag{C.59}
$$

C.8.2 *Self–Energy* integrals

$$
\begin{aligned}
\text{Div}\left[\hat{T}_2\right] &= \frac{i}{16\pi^2}\, \Delta_\epsilon \\
\text{Div}\left[\hat{T}_2^\mu\right] &= \frac{i}{16\pi^2} \left(-\frac{1}{2}\right) \Delta_\epsilon\, r_1^\mu \\
\text{Div}\left[\hat{T}_2^{\mu\nu}\right] &= \frac{i}{16\pi^2}\, \frac{1}{12}\, \Delta_\epsilon \left[(3m_1^2 + 3m_2^2 - r_1^2) g^{\mu\nu} + 4 r_1^\mu r_1^\nu\right] \\
\text{Div}\left[\hat{T}_2^{\mu\nu\alpha}\right] &= \frac{i}{16\pi^2} \left(-\frac{1}{24}\right) \Delta_\epsilon \Big[(4m_1^2 + 2m_2^2 - r_1^2)\, (g^{\mu\nu} r_1^\alpha + g^{\nu\alpha} r_1^\mu + g^{\mu\alpha} r_1^\nu) \\
&\qquad + 6\, r_1^\alpha r_1^\mu r_1^\nu\Big]
\end{aligned}
\tag{C.60}
$$

C.8.3 *Triangle* integrals

$$
\text{Div}\left[\hat{T}_3\right] = 0
$$

$$
\begin{aligned}
\text{Div}\left[\hat{T}_3^{\mu}\right] &= 0 \\
\text{Div}\left[\hat{T}_3^{\mu\nu}\right] &= \frac{i}{16\pi^2}\,\frac{1}{4}\,\Delta_\epsilon\, g^{\mu\nu} \\
\text{Div}\left[\hat{T}_3^{\mu\nu\alpha}\right] &= \frac{i}{16\pi^2}\left(-\frac{1}{12}\right)\Delta_\epsilon\left[g^{\mu\nu}(r_1^\alpha+r_2^\alpha)+g^{\nu\alpha}(r_1^\mu+r_2^\mu)+g^{\mu\alpha}(r_1^\nu+r_2^\nu)\right] \\
\text{Div}\left[\hat{T}_3^{\mu\nu\alpha\beta}\right] &= \frac{i}{16\pi^2}\,\frac{1}{48}\,\Delta_\epsilon\left[(2m_1^2+2m_2^2+2m_3^2)\left(g^{\mu\alpha}g^{\nu\beta}+g^{\alpha\beta}g^{\mu\nu}+g^{\mu\beta}g^{\nu\alpha}\right)\right. \\
&\quad +g^{\alpha\beta}\left[2r_1^\mu r_1^\nu+r_1^\mu r_2^\nu+(r_1\leftrightarrow r_2)\right]+g^{\mu\beta}\left[2r_1^\alpha r_1^\nu+r_1^\alpha r_2^\nu+(r_1\leftrightarrow r_2)\right] \\
&\quad +g^{\nu\beta}\left[2r_1^\alpha r_1^\mu+r_1^\alpha r_2^\mu+(r_1\leftrightarrow r_2)\right]+g^{\mu\nu}\left[2r_1^\alpha r_1^\beta+r_1^\alpha r_2^\beta+(r_1\leftrightarrow r_2)\right] \\
&\quad +g^{\mu\alpha}\left[2r_1^\beta r_1^\nu+r_1^\beta r_2^\nu+(r_1\leftrightarrow r_2)\right]+g^{\nu\alpha}\left[2r_1^\beta r_1^\mu+r_1^\beta r_2^\mu+(r_1\leftrightarrow r_2)\right] \\
&\quad \left.+\left(-r_1^2+r_1\cdot r_2-r_2^2\right)\left(g^{\mu\alpha}g^{\nu\beta}+g^{\alpha\beta}g^{\mu\nu}+g^{\mu\beta}g^{\nu\alpha}\right)\right] \qquad \text{(C.61)}
\end{aligned}
$$

C.8.4 *Box* integrals

$$
\begin{aligned}
\text{Div}\left[\hat{T}_4\right] &= \text{Div}\left[\hat{T}_4^{\mu}\right]=\text{Div}\left[\hat{T}_4^{\mu\nu}\right]=\text{Div}\left[\hat{T}_4^{\mu\nu\alpha}\right]=0 \\
\text{Div}\left[\hat{T}_4^{\mu\nu\alpha\beta}\right] &= \frac{i}{16\pi^2}\,\frac{1}{24}\Delta_\epsilon\left[g^{\mu\nu}g^{\alpha\beta}+g^{\mu\beta}g^{\alpha\nu}+g^{\mu\alpha}g^{\nu\beta}\right] \qquad \text{(C.62)}
\end{aligned}
$$

C.9 Passarino-Veltman Integrals

C.9.1 The general definition

The description of the previous sections works well if one just wants to calculate the divergent part of a diagram or to show the cancellation of divergences in a set of diagrams. If one actually wants to numerically calculate the integrals the task is normally quite complicated. Except for the *self-energy* type of diagrams the integration over the Feynman parameters is normally quite difficult.

To overcome this problem a scheme was first proposed by Passarino and Veltman [10]. These scheme with the conventions of [11, 12] was latter implemented in the `Mathematica` package `FeynCalc` [13] and, for numerical evaluation, in the `LoopTools` package [14, 15]. The numerical evaluation follows the code developed earlier by van Oldenborgh [16].

We will now describe this scheme. We will write the generic one-loop tensor integral as

$$
T_n^{\mu_1\cdots\mu_p} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\,\frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0D_1D_2\cdots D_{n-1}} \qquad \text{(C.63)}
$$

where we follow for the momenta the conventions of section C.2 and Fig. C.1 and defined $D_0 \equiv D_n$ and $m_n = m_0$ so that $D_0 = k^2 - m_0^2$ (remember that $r_n \equiv r_0 = 0$. The main difference between this definition and the previous one Eq. (C.7) is that a factor of $\frac{i}{16\pi^2}$ is taken out. This is because, as we have seen in section C.3 these integrals always give that prefactor. So with our new convention that prefactor **has** to included in the end. Factoring out the i has also the convenience of dealing with real functions in many cases.[2] From all those integrals in Eq. (C.63) the scalar integrals are, has we have seen, of particular importance and deserve a special notation. It can be shown that there are only four independent such integrals, namelly $(4-d=\epsilon)$

$$A_0(m_0^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\,\frac{1}{k^2-m_0^2} \tag{C.64}$$

$$B_0(r_{10}^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\,\prod_{i=0}^{1}\frac{1}{[(k+r_i)^2-m_i^2]} \tag{C.65}$$

$$C_0(r_{10}^2, r_{12}^2, r_{20}^2, m_1^2, m_2^2, m_3^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\,\prod_{i=0}^{2}\frac{1}{[(k+r_i)^2-m_i^2]} \tag{C.66}$$

$$D_0(r_{10}^2, r_{12}^2, r_{23}^2, r_{30}^2, r_{20}^2, r_{13}^2, m_1^2, \ldots, m_3^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2}\int d^dk\,\prod_{i=0}^{3}\frac{1}{[(k+r_i)^2-m_i^2]} \tag{C.67}$$

where

$$r_{ij}^2 = (r_i - r_j)^2 \quad ; \quad \forall\, i,j = (0, n-1) \tag{C.68}$$

Remember that with our conventions $r_0 = 0$ so $r_{i0}^2 = r_i^2$. In all these expressions the $i\epsilon$ part of the denominator factors is supressed. The general one-loop tensor integrals are not independent. Their decomposition is not unique. We follow the conventions of [13, 15] to write

$$B^\mu \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\,k^\mu\prod_{i=0}^{1}\frac{1}{[(k+r_i)^2-m_i^2]} \tag{C.69}$$

$$B^{\mu\nu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\,k^\mu k^\nu\prod_{i=0}^{1}\frac{1}{[(k+r_i)^2-m_i^2]} \tag{C.70}$$

$$C^\mu \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\,k^\mu\prod_{i=0}^{2}\frac{1}{[(k+r_i)^2-m_i^2]} \tag{C.71}$$

$$C^{\mu\nu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\,k^\mu k^\nu\prod_{i=0}^{2}\frac{1}{[(k+r_i)^2-m_i^2]} \tag{C.72}$$

$$C^{\mu\nu\rho} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\,k^\mu k^\nu k^\rho\prod_{i=0}^{2}\frac{1}{[(k+r_i)^2-m_i^2]} \tag{C.73}$$

$$D^\mu \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\,k^\mu\prod_{i=0}^{3}\frac{1}{[(k+r_i)^2-m_i^2]} \tag{C.74}$$

[2] The one loop functions are in general complex, but in some cases they can be real. These cases correspond to the situation where cutting the diagram does not corresponding to a kinematically allowed process.

$$D^{\mu\nu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu k^\nu \prod_{i=0}^{3} \frac{1}{\left[(k+r_i)^2 - m_i^2\right]} \tag{C.75}$$

$$D^{\mu\nu\rho} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu k^\nu k^\rho \prod_{i=0}^{3} \frac{1}{\left[(k+r_i)^2 - m_i^2\right]} \tag{C.76}$$

$$D^{\mu\nu\rho\sigma} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^\mu k^\nu k^\rho k^\sigma \prod_{i=0}^{3} \frac{1}{\left[(k+r_i)^2 - m_i^2\right]} \tag{C.77}$$

These integrals can be decomposed in terms of (reducible) functions in the following way:

$$B^\mu = r_1^\mu\, B_1 \tag{C.78}$$

$$B^{\mu\nu} = g^{\mu\nu}\, B_{00} + r_1^\mu r_1^\nu\, B_{11} \tag{C.79}$$

$$C^\mu = r_1^\mu\, C_1 + r_2^\mu\, C_2 \tag{C.80}$$

$$C^{\mu\nu} = g^{\mu\nu}\, C_{00} + \sum_{i=1}^{2} r_i^\mu r_j^\nu\, C_{ij} \tag{C.81}$$

$$C^{\mu\nu\rho} = \sum_{i=1}^{2} \left(g^{\mu\nu} r_i^\rho + g^{\nu\rho} r_i^\mu + g^{\rho\mu} r_i^\nu\right)\, C_{00i} + \sum_{i,j,k=1}^{2} r_i^\mu r_j^\nu r_k^\rho\, C_{ijk} \tag{C.82}$$

$$D^\mu = \sum_{i=1}^{3} r_i^\mu\, D_i \tag{C.83}$$

$$D^{\mu\nu} = g^{\mu\nu}\, D_{00} + \sum_{i=1}^{3} r_i^\mu r_j^\nu\, D_{ij} \tag{C.84}$$

$$D^{\mu\nu\rho} = \sum_{i=1}^{3} \left(g^{\mu\nu} r_i^\rho + g^{\nu\rho} r_i^\mu + g^{\rho\mu} r_i^\nu\right)\, D_{00i} + \sum_{i,j,k=1}^{2} r_i^\mu r_j^\nu r_k^\rho\, D_{ijk} \tag{C.85}$$

$$\begin{aligned} D^{\mu\nu\rho\sigma} = {} & \left(g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}\right)\, D_{0000} \\ & + \sum_{i,j=1}^{3} \Big(g^{\mu\nu} r_i^\rho r_j^\sigma + g^{\nu\rho} r_i^\mu r_j^\sigma + g^{\mu\rho} r_i^\nu r_j^\sigma + g^{\mu\sigma} r_i^\nu r_j^\rho \qquad \text{(C.86)} \\ & \qquad + g^{\nu\sigma} r_i^\mu r_j^\rho + g^{\rho\sigma} r_i^\mu r_j^\nu\Big)\, D_{00ij} \\ & + \sum_{i,j,k,l=1}^{3} r_i^\mu r_j^\nu r_k^\rho r_l^\sigma\, C_{ijkl} \qquad \text{(C.87)} \end{aligned}$$

All coefficient functions have the same arguments as the corresponding scalar functions and are totally symmetric in their indices. In the `FeynCalc` [13] package one generic notation is used,

$$\mathrm{PaVe}\left[\mathrm{i}, \mathrm{j}, \ldots, \{\mathrm{r}_{10}^2, \mathrm{r}_{12}^2, \ldots\}, \{\mathrm{m}_0^2, \mathrm{m}_1^2, \ldots\}\right] \tag{C.88}$$

for instance

$$B_{11}(r_{10}^2, m_0^2, m_1^2) = \mathrm{PaVe}\left[1, 1, \{\mathrm{r}_{10}^2\}, \{\mathrm{m}_0^2, \mathrm{m}_1^2\}\right] \tag{C.89}$$

All these coefficient functions are not independent and can be reduced to the scalar functions. `FeynCalc` provides the command `PaVeREduce[...]` to acomplish that. This is very useful if one wants to check for cancellation of divergences or for gauge invariance where a number of diagrams have to cancel.

C.9.2 The divergences

The package `LoopTools` provides ways to numerically check for the cancellation of divergences. However it is useful to know the divergent part of the Passarino-Veltman integrals. Only a small number of these integrals are divergent. They are

$$\begin{aligned}
\text{Div}\left[\mathrm{A_0(m_0^2)}\right] &= \Delta_\epsilon\, m_0^2 && \text{(C.90)}\\
\text{Div}\left[\mathrm{B_0(r_{10}^2, m_0^2, m_1^2)}\right] &= \Delta_\epsilon && \text{(C.91)}\\
\text{Div}\left[\mathrm{B_1(r_{10}^2, m_0^2, m_1^2)}\right] &= -\frac{1}{2}\,\Delta_\epsilon && \text{(C.92)}\\
\text{Div}\left[\mathrm{B_{00}(r_{10}^2, m_0^2, m_1^2)}\right] &= \frac{1}{12}\,\Delta_\epsilon\, \left(3m_0^2 + 3m_1^2 - r_{10}^2\right) && \text{(C.93)}\\
\text{Div}\left[\mathrm{B_{11}(r_{10}^2, m_0^2, m_1^2)}\right] &= \frac{1}{3}\,\Delta_\epsilon && \text{(C.94)}\\
\text{Div}\left[\mathrm{C_{00}(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2)}\right] &= \frac{1}{4}\,\Delta_\epsilon && \text{(C.95)}\\
\text{Div}\left[\mathrm{C_{001}(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2)}\right] &= -\frac{1}{12}\,\Delta_\epsilon && \text{(C.96)}\\
\text{Div}\left[\mathrm{C_{002}(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2)}\right] &= -\frac{1}{12}\,\Delta_\epsilon && \text{(C.97)}\\
\text{Div}\left[\mathrm{D_{0000}(r_{10}^2, \ldots, m_0^2, \ldots)}\right] &= \frac{1}{24}\,\Delta_\epsilon && \text{(C.98)}\\
& && \text{(C.99)}
\end{aligned}$$

These results were obtained with the package LoopTools, after reducing to the scalar integrals with the command `PaVeReduce`, but they can be verified by comparing with our results of section C.8, after factoring out the $i/(16\pi^2)$.

C.9.3 Useful results for PV integrals

Although the PV approach is intended primarily to be used numerically there are situations where one wants to have explicit results. These can be useful to check cancellation of divergences or because in some simple cases the integrals can be done analytically. We note that as our conventions for the momenta are the same in sections C.9 and C.7 one can read immediately the integral representation of the PV in terms of the Feynman parameters just by comparing both expressions, not forgetting to take out the $i/(16\pi^2)$ factor. For instance, from Eq. (C.81) for $C^{\mu\nu}$ and Eq. (C.55) for $I_3^{\mu\nu}$ we get

$$C_{12}(r_1^2, r_{12}^2, r_2^2, m_0^2, m_1^2, m_2^2) = -\Gamma(3)\frac{2}{4}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{\mathrm{C}} \tag{C.100}$$

with

$$\begin{aligned}
C &= x_1^2\, r_1^2 + x_2^2\, r_2^2 + x_1\, x_2\, (r_1^2 + r_2^2 - r_{12}^2) + x_1\, m_1^2 + x_2\, m_2^2 \\
&\quad + (1 - x_1 - x_2)\, m_0^2 - x_1\, r_1^2 - x_2\, r_2^2 && \text{(C.101)}
\end{aligned}$$

Explicit expression for A_0

This integral is trivial. There is no Feynman parameter and the integral can be read from Eq. (C.51). We get, after factoring out the $i/(16\pi^2)$,

$$A_0(m^2) = m^2 \left(\Delta_\epsilon + 1 - \ln \frac{m^2}{\mu^2} \right) \tag{C.102}$$

Explicit expressions for the B functions

Function B_0

The general form of the integral $B_0(p^2, m_1^2, m_2^2)$ can be read from Eq. (C.53). We obtain

$$B_0(p^2, m_0^2, m_1^2) = \Delta_\epsilon - \int_0^1 dx \ln \left[\frac{-x(1-x)p^2 + x m_1^2 + (1-x) m_0^2}{\mu^2} \right] \tag{C.103}$$

From this expression one can easily get the following results,

$$B_0(0, m_0^2, m_1^2) = \Delta_\epsilon + 1 - \frac{m_0^2 \ln \frac{m_0^2}{\mu^2} - m_1^2 \ln \frac{m_1^2}{\mu^2}}{m_0^2 - m_1^2} \tag{C.104}$$

$$B_0(0, m_0^2, m_1^2) = \frac{A_0(m_0^2) - A_0(m_1^2)}{m_0^2 - m_1^2} \tag{C.105}$$

$$B_0(0, m^2, m^2) = \Delta_\epsilon - \ln \frac{m^2}{\mu^2} = \frac{A_0(m^2)}{m^2} - 1 \tag{C.106}$$

$$B_0(m^2, 0, m^2) = \Delta_\epsilon + 2 - \ln \frac{m^2}{\mu^2} = \frac{A_0(m^2)}{m^2} + 1 \tag{C.107}$$

$$B_0(0, 0, m^2) = \Delta_\epsilon + 1 - \ln \frac{m^2}{\mu^2} \tag{C.108}$$

Function B_0'

The derivative of the B_0 function with respect to p^2 appears many times. From Eq. (C.103) one can derive an integral representation,

$$B_0'(p^2, m_0^2, m_1^2) = -\int_0^1 dx \, \frac{x(1-x)}{-p^2 x(1-x) + x m_1^2 + (1-x) m_0^2} \tag{C.109}$$

An important particular case corresponds to $B_0'(m^2, m_0^2, m^2)$ that appears in the self-energy of the electron. In this case m is the electron mass and $m_0 = \lambda$ is the photon mass that one has to introduce to regularize the IR divergent integral. The integral in this case reduces to

$$B_0'(m^2, \lambda^2, m^2) = -\int_0^1 dx \, \frac{x(1-x)}{m^2 x^2 + (1-x)\lambda^2}$$

$$= \frac{1}{m^2} + \frac{1}{2m^2} \ln \frac{\lambda^2}{m^2} \tag{C.110}$$

It is clear that in the limit $\lambda \to 0$ this integral diverges.

Function B_1

The explicit expression can be read from Eq. (C.53). We have

$$B_1(p^2, m_0^2, m_1^2) = -\frac{1}{2}\Delta_\epsilon + \int_0^1 dx x \ln\left[\frac{-x(1-x)p^2 + x m_1^2 + (1-x)m_0^2}{\mu^2}\right] \tag{C.111}$$

For $p^2 = 0$ this integral can be easily evaluated to give

$$B_1(0, m_0^2, m_1^2) = -\frac{1}{2}\Delta_\epsilon + \frac{1}{2}\ln\left(\frac{m_0^2}{\mu^2}\right) + \frac{-3 + 4t - t^2 - 4t\ln t + 2t^2 \ln t}{4(-1+t)^2} \tag{C.112}$$

where we defined

$$t = \frac{m_1^2}{m_0^2} \tag{C.113}$$

From Eq. (C.112) one can shown that even for $p^2 = 0$ B_1 is **not** a symmetric function of the masses,

$$B_1(p^2, m_0^2, m_1^2) \neq B_1(p^2, m_1^2, m_0^2) \tag{C.114}$$

As this might appear strange let us show with one example how the coefficient functions are tied to our conventions about the order of the momenta and Feynman parameters. Let us consider the contribution to the self-energy of a fermion of mass m_f of the exchange of a scalar with mass m_s. We can consider the two choices in Fig. C.4,

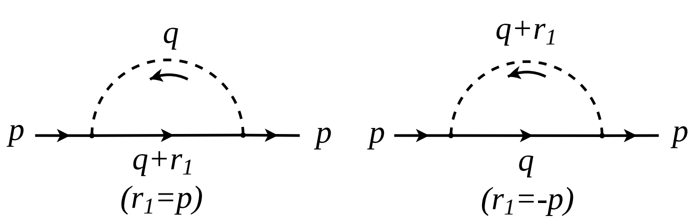


Figure C.4:

Now with the first choice (diagram on the left of Fig. C.4) we have

$$\begin{aligned} -i\Sigma_1 &= \frac{i}{16\pi^2}\left[(\not{p} + m_f)B_0(p^2, m_s^2, m_f^2) + \not{p} B_1(p^2, m_s^2, m_f^2)\right] \\ &= \frac{i}{16\pi^2}\left[\not{p}\left(B_0(p^2, m_s^2, m_f^2) + B_1(p^2, m_s^2, m_f^2)\right) + m_f B_0(p^2, m_s^2, m_f^2)\right] \end{aligned} \tag{C.115}$$

while with the second choice we have

$$-i\Sigma_2 = \frac{i}{16\pi^2}\left[-\not{p} B_1(p^2, m_f^2, m_s^2) + m_f B_0(p^2, m_f^2, m_s^2)\right] \tag{C.116}$$

How can these two expressions be equal? The reason has precisely to do with the non symmetry of B_1 with respect to the mass entries. In fact from Eq. (C.111) we have

$$\begin{aligned}
B_1(p^2, m_0^2, m_1^2) &= -\frac{1}{2}\Delta_\epsilon + \int_0^1 dx x \ln\left[\frac{-x(1-x)p^2 + x m_1^2 + (1-x)m_0^2}{\mu^2}\right] \\
&= -\frac{1}{2}\Delta_\epsilon + \int_0^1 dx(1-x) \ln\left[\frac{-x(1-x)p^2 + (1-x)m_1^2 + x m_0^2}{\mu^2}\right] \\
&= -\frac{1}{2}\Delta_\epsilon + \left(\Delta_\epsilon - B_0(p^2, m_1^2, m_0^2)\right) - \left(\frac{1}{2}\Delta_\epsilon + B_1(p^2, m_1^2, m_0^2)\right) \\
&= -\left(B_0(p^2, m_1^2, m_0^2) + B_1(p^2, m_1^2, m_0^2)\right) \qquad \text{(C.117)}
\end{aligned}$$

where we have changed variables ($x \to 1-x$) in the integral and used the definitions of B_0 and B_1. We have then, remembering that $B_0(p^2, m_s^2, m_f^2) = B_0(p^2, m_f^2, m_s^2)$,

$$B_1(p^2, m_f^2, m_s^2) = -\left(B_0(p^2, m_s^2, m_f^2) + B_1(p^2, m_s^2, m_f^2)\right) \qquad \text{(C.118)}$$

and therefore Eqs. (C.115) and (C.116) are equivalent.

Explicit expressions for the C functions

In Eq. (C.100) we have already given the general form of C_{12}. The other functions are very similar. In the following we just present the results for the particular case of $p^2 = 0$. This case is important in many situations where it is a good approximation to neglect the external momenta in comparison with the masses of the particles in the loop. We also warn the reader that the coefficient functions C_i, C_{ij} obtained from `LoopTools` are not well defined in this limit. Hence there is some utility in given them here.

Function C_0

$$\begin{aligned}
C_0(0,0,0,m_0^2,m_1^2,m_2^2) &= -\Gamma(3)\frac{1}{2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 m_1^2 + x_2 m_2^2 + (1-x_1-x_2)m_0^2} \\
&= -\frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \\
&= -\frac{1}{m_0^2}\frac{-t_1 \ln t_1 + t_1 t_2 \ln t_1 + t_2 \ln t_2 - t_1 t_2 \ln t_2}{(-1+t_1)(t_1-t_2)(-1+t_2)} \qquad \text{(C.119)}
\end{aligned}$$

where

$$t_1 = \frac{m_1^2}{m_0^2} \quad ; \quad t_2 = \frac{m_2^2}{m_0^2} \qquad \text{(C.120)}$$

Using the properties of the logarithms one can show that in this limit C_0 is a symmetric function of the masses. This expression is further simplified when two of the masses are equal, as it happens in the $\mu \to e\gamma$ problem. Then $t = t_1 = t_2$,

$$C_0(0,0,0,m_0^2,m_1^2,m_1^2) = -\frac{1}{m_0^2}\frac{-1+t-\ln t}{(-1+t)^2} \qquad \text{(C.121)}$$

in agreement with Eq.(20) of [17]. In the case of equal masses for all the loop particles we have

$$C_0(0,0,0,m_0^2,m_0^2,m_0^2) = -\frac{1}{2m_0^2} \tag{C.122}$$

Before we close this section on C_0 there is another particular case when it is useful to have an explicit case for it. This in the case when it is IR divergent as in the QED vertex. The functions needed is $C_0(m^2,m^2,0,m^2,\lambda^2,m^2)$. Using the definition we have

$$\begin{aligned}
C_0(m^2,m^2,0,m^2,\lambda^2,m^2) &= -\int_0^1 dx_1 \int_0^{1-x1} dx_2 \frac{1}{m^2(1-2x_1+x_1^2)+x_1\lambda^2} \\
&= -\int_0^1 dx_1 \frac{1-x_1}{m^2(1-x_1)^2+(1-x_1)\lambda^2} \\
&= -\int_0^1 dx \frac{x}{m^2x^2+x\lambda^2} \\
&= \frac{1}{2m^2}\ln\frac{\lambda^2}{m^2} = B_0'(m^2,\lambda^2,m^2) - \frac{1}{m^2}
\end{aligned} \tag{C.123}$$

Function C_{00}

$$\begin{aligned}
C_{00}(0,0,0,m_0^2,m_1^2,m_2^2) &= \Gamma(3)\frac{1}{4}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[\Delta_\epsilon - \ln\left(\frac{C}{\mu^2}\right)\right] \\
&= \frac{1}{4}\Delta_\epsilon - \frac{1}{2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ln\left[\frac{x_1m_1^2+x_2m_2^2+(1-x_1-x_2)m_0^2}{\mu^2}\right] \\
&= \frac{1}{4}\left(\Delta_\epsilon - \ln\frac{m_0^2}{\mu^2}\right) + \frac{3}{8} - \frac{t_1^2}{4(t_1-1)(t_1-t_2)}\ln t_1 \\
&\quad + \frac{t_2^2}{4(t_2-1)(t_1-t_2)}\ln t_2
\end{aligned} \tag{C.124}$$

where, as before

$$t_1 = \frac{m_1^2}{m_0^2} \quad ; \quad t_2 = \frac{m_2^2}{m_0^2} \tag{C.125}$$

Using the properties of the logarithms one can show that in this limit C_{00} is a symmetric function of the masses. This expression is further simplified when two of the masses are equal. Then $t = t_1 = t_2$,

$$\begin{aligned}
C_{00}(0,0,0,m_0^2,m_1^2,m_1^2) &= \frac{1}{4}\left(\Delta_\epsilon - \ln\frac{m_0^2}{\mu^2}\right) - \frac{-3+4t-t^2-4t\ln t+2t^2\ln t}{8(t-1)^2} \\
&= -\frac{1}{2}B_1(0,m_0^2,m_1^2)
\end{aligned} \tag{C.126}$$

Functions C_i and C_{ij}

We recall that the definition of the coefficient functions is not unique, it is tied to a particular convention for assigning the loop momenta and Feynman parameters, as shown in Fig. C.1. For the particular case of the C functions we show our conventions in Fig. C.5.

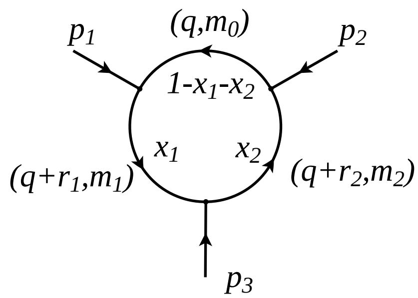


Figure C.5:

With the same techniques we obtain,

$$
\begin{aligned}
C_1(0,0,0,m_0^2,m_1^2,m_2^2) &= \frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \\
&= -\frac{1}{m_0^2}\left[\frac{t_1}{2(-1+t_1)(t_1-t_2)} - \frac{t_1(t_1-2t_2+t_1t_2)}{2(-1+t_1)^2(t_1-t_2)^2}\ln t_1 \right. \\
&\qquad \left. + \frac{{t_2}^2 - 2t_1{t_2}^2 + {t_1}^2{t_2}^2}{2(-1+t_1)^2(t_1-t_2)^2(-1+t_2)}\ln t_2\right] \qquad \text{(C.127)}
\end{aligned}
$$

$$
\begin{aligned}
C_2(0,0,0,m_0^2,m_1^2,m_2^2) &= \frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_2}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \\
&= -\frac{1}{m_0^2}\left[-\frac{t_2}{2(t_1-t_2)(-1+t_2)} + \frac{\ln t_1}{2(-1+t_1)(-1+t_2)^2} \right. \\
&\qquad \left. + \frac{2t_1t_2 - 2{t_1}^2t_2 - {t_2}^2 + {t_1}^2{t_2}^2}{2(-1+t_1)(t_1-t_2)^2(-1+t_2)^2}\ln\left(\frac{t_1}{t_2}\right)\right] \qquad \text{(C.128)}
\end{aligned}
$$

$$
C_{ij}(0,0,0,m_0^2,m_1^2,m_2^2) = -\frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_i x_j}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \qquad \text{(C.129)}
$$

where we have not written explicitly the C_{ij} for $i,j=1,2$ because they are rather lengthy. However a simple `Fortran` program can be developed [9] to calculate all the three point functions in the zero external limit case. This is useful because in this case some of the functions from `LoopTools` will fail. Notice that the C_i and C_{ij} functions are not symmetric in their arguments. This a consequence of their non-uniqueness, they are tied to a particular convention. This is very important when ones compares with other results. However using their definition one can get some relations. For instance we can show

$$C_1(0,0,0,m_0^2,m_1^2,m_2^2) = C_1(0,0,0,m_2^2,m_1^2,m_0^2) \tag{C.130}$$

$$\begin{aligned} C_2(0,0,0,m_0^2,m_1^2,m_2^2) &= -C_0(0,0,0,m_2^2,m_1^2,m_0^2) - C_1(0,0,0,m_2^2,m_1^2,m_0^2) \\ &\quad -C_2(0,0,0,m_2^2,m_1^2,m_0^2) \end{aligned} \tag{C.131}$$

In the limit $m_1 = m_2$ we get the simple expressions,

$$\begin{aligned} C_1(0,0,0,m_0^2,m_1^2,m_1^2) &= C_2(0,0,0,m_0^2,m_1^2,m_1^2) \\ &= -\frac{1}{m_0^2}\frac{3-4t+t^2+2\ln t}{4(-1+t)^3} \end{aligned} \tag{C.132}$$

$$\begin{aligned} C_{11}(0,0,0,m_0^2,m_1^2,m_1^2) &= C_{22}(0,0,0,m_0^2,m_1^2,m_1^2) = 2\ C_{12}(0,0,0,m_0^2,m_1^2,m_1^2) \\ &= -\frac{1}{m_0^2}\frac{-11+18t-9t^2+2t^3-6\ln t}{18(-1+t)^4} \end{aligned} \tag{C.133}$$

in agreement with Eqs. (21-22) of [17]. The case of masses equal gives

$$C_1(0,0,0,m_0^2,m_0^2,m_0^2) = C_2(0,0,0,m_0^2,m_0^2,m_0^2) = \frac{1}{6m_0^2} \tag{C.134}$$

$$C_{11}(0,0,0,m_0^2,m_0^2,m_0^2) = C_{22}(0,0,0,m_0^2,m_0^2,m_0^2) = -\frac{1}{12m_0^2} \tag{C.135}$$

$$C_{12}(0,0,0,m_0^2,m_0^2,m_0^2) = -\frac{1}{24m_0^2} \tag{C.136}$$

The package PVzem

As we said before, in many situations it is a good approximation to neglect the external momenta. In this case, the loop functions are easier to evaluate and one approach is for each problem to evaluate them. However our approach here is more in the direction of automatically evaluating the one-loop amplitudes. If one does that with the use of `FeynCalc`, has we have been doing, then the result is given in terms of standard functions that can be numerically evaluated with the package `LoopTools`. However this package has problems with this limit. This is because this limit is unphysical. Let us illustrate this point calculating the functions $C_1(m^2,0,0,m_S^2,m_F^2,m_F^2)$ and $C_2(m^2,0,0,m_S^2,m_F^2,m_F^2)$ for $m_B = 100$ GeV, $m_F = 80$ GeV and m_2 ranging from 10^{-6} to 100 GeV. To better illustrate our point we show two plots with different scales on the axis.
In these plots, $C_i^{\rm Ex}$ are the exact C_i functions calculated with `LoopTools` and $C_i^{\rm Ap}$ are the C_i calculated in the zero momenta limit. We can see that only for external momenta (in this case corresponding to the mass m_2) close enough to the masses of the particles in the loop, the exact result deviates from the approximate one. However for very small

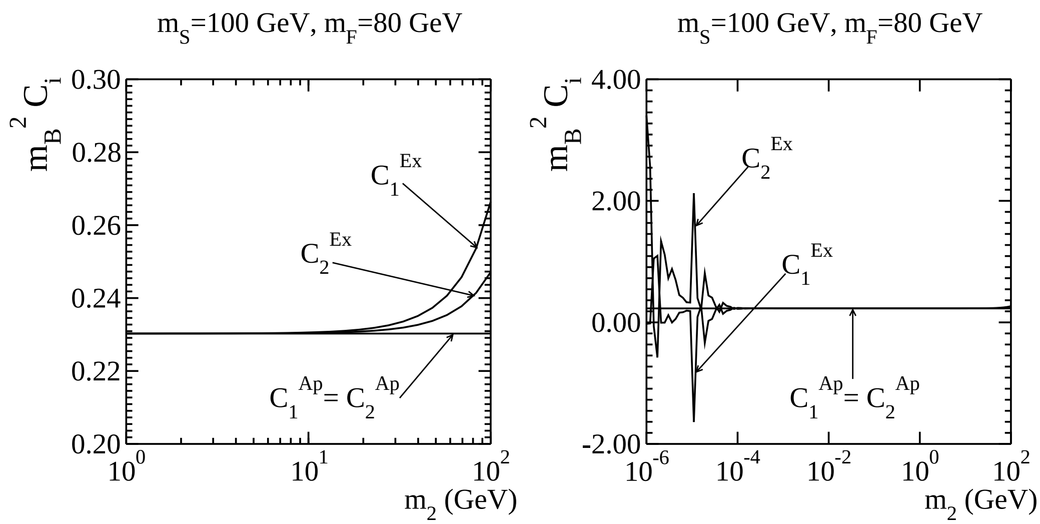


Figure C.6:

values of the external momenta, `LoopTools` has numerical problems as shown in the right panel of Fig. C.6. To overcome this problem I have developed a `Fortran` package that evaluates all the C functions in the zero external momenta limit. There are no restrictions on the masses being equal or different and the conventions are the same as in `FeynCalc` and `LoopTools`, for instance,

$$c12zem(m02, m12, m22) = c0i(cc12, 0, 0, 0, m02, m12, m22) \tag{C.137}$$

where $c0i(cc12, \cdots)$ is the `LoopTools` notation and $c12zem(\cdots)$ is the notation of my package, called `PVzem`. It can be obtained from the address indicated in Ref.[9]. The approximate functions shown in Fig. C.6 were calculated using that package. We include here the `Fortran` code used to produce that figure.

```
************************************************************
*                                                          *
*              Program LoopToolsExample                    *
*                                                          *
*   This program calculates the values used in the plots   *
*   of Figure 20. For the exact results the LoopTools      *
*   package was used. The package PVzem was used for the   *
*   approximate results.                                   *
*                                                          *
*              Version of  14/05/2012                      *
*                                                          *
*     Author: Jorge C. Romao                               *
*     e-mail: jorge.romao@ist.utl.pt                       *
************************************************************
```

```
      program LoopToolsExample
      implicit none

*
*  LoopTools has to be used with FORTRAN programs with the
*  extension .F in order to have the header file "looptools.h"
*  preprocessed. This file includes all the definitions used
*  by LoopTools.
*
*  Functions c1zem and c2zem are provided by the package PVzem.
*

#include "looptools.h"

      integer i
      real*8 m2,mF2,mS2,m
      real*8 lgmmin,lgmmax,lgm,step
      real*8 rc1,rc2
      real*8 c1zem,c2zem

      mS2=100.d0**2
      mF2=80.d0**2

*
*  Initialize LoopTools. See the LoopTools manual for further
*  details. There you can also learn how to set the scale MU
*  and how to handle the UR and IR divergences.
*

      call ltini

      lgmmax=log10(100.d0)
      lgmmin=log10(1.d-6)
      step=(lgmmax-lgmmin)/100.d0
      lgm=lgmmin-step

      open(10,file='plot.dat',status='unknown')

      do i=1,101
         lgm=lgm+step
         m=10.d0**lgm
         m2=m**2
```

```
*
*  In LoopTools the c0i(...) are complex functions. For the
*  kinematics chosen here they are real, so we take the real
*  part for comparison.
*

      rc1=dble(c0i(cc1,m2,0.d0,0.d0,mS2,mF2,mF2))
      rc2=dble(c0i(cc2,m2,0.d0,0.d0,mS2,mF2,mF2))
      write(10,100)m,rc1*mS2,rc2*mS2,c1zem(mS2,mF2,mF2)*mS2,
     &            c2zem(mS2,mF2,mF2)*mS2
      enddo

 100  format(5(e22.14))

      call ltexi

      end
************** End of Program LoopToolsExample.F ************
```

When the above program is compiled, the location of the header file `looptools.h` must be known by the compiler. This is best achieved by using a `Makefile`. We give below, as an example, the one that was used with the above program. Depending on the installation details of `LoopTools` the paths might be different.

```
FC         =
LT         = /usr/local/lib/LoopTools
FFLAGS     =  -c -O -I$(LT)/include
LDFLAGS    =
LINKER     = $(FC)

LIB        = -L$(LT)/lib
LIBS       = -looptools

.f.o:
        $(FC) $(FFLAGS) $*.F

files   = LoopToolsExample.o PVzem.o

all:    $(files)
        $(LINKER) $(LDFLAGS) -o Example $(files) $(LIB) $(LIBS)
```

Explicit expressions for the D functions

Function D_0

The various D functions can be calculated in a similar way. However they are rather lengthy and have to handled numerically [9]. Here we just give D_0 for the equal masses case.

$$\begin{aligned}
D_0(0,\cdots,0,m^2,m^2,m^2,m^2) &= \Gamma(4)\frac{1}{6}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3\ \frac{1}{(m^2)^2} \\
&= \frac{1}{m^4}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \\
&= \frac{1}{6m^4} \qquad \text{(C.138)}
\end{aligned}$$

C.10 Examples of *1-loop* calculations with PV functions

In this section we will work out in detail a few examples of one-loop calculations using the `FeynCalc` package and the Passarino-Veltman scheme.

C.10.1 Vaccum Polarization in QED

We have done this example in section 4.1.1 using the techniques described in sections C.3, C.4 and C.5. Now we will use `FeynCalc`. The first step is to write the `Matematica` program. We list it below:

```
(*********************** Program VacPol.m *************************)

(* First input FeynCalc *)

<< FeynCalc.m

(* These are some shorthands for the FeynCalc notation *)

dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(* Now write the numerator of the Feynman diagram. We define the
   constant

       C=alpha/(4 pi)
*)
```

```
num:= - C Tr[dm[mu] . (ds[q] + m) . dm[nu] .  (ds[q]+ds[k]+m)]


(* Tell FeynCalc to evaluate the integral in dimension D *)


SetOptions[OneLoop,Dimension->D]


(* Define the amplitude *)

amp:=num * FeynAmpDenominator[PropagatorDenominator[q+k,m], \
                              PropagatorDenominator[q,m]]

(* Calculate the result *)

res:=(-I / Pi^2) OneLoop[q,amp]

(***************** End of Program VacPol.m ********************)
```

The output from `Mathematica` is:

```
                   2       2         2   2          2       2       2   2   2
 Out[4]= (4  C  (k  + 6 m  B0[0, m , m ] - 3 (k  + 2 m ) B0[k , m , m ])
       2                                 2
    (k  g[mu, nu] - k[mu] k[nu])) / (9 k )
```

Now remembering that,

$$C = \frac{\alpha}{4\pi} \tag{C.139}$$

and

$$i\,\Pi_{\mu\nu}(k,\varepsilon) = -i\,k^2 P^T_{\mu\nu}\Pi(k,\varepsilon) \tag{C.140}$$

we get

$$\Pi(k,\varepsilon) = \frac{\alpha}{4\pi}\left[-\frac{4}{9} - \frac{8}{3}\frac{m^2}{k^2}\,B_0(0,m^2,m^2) + \frac{4}{3}\left(1+\frac{2m^2}{k^2}\right)\,B_0(k^2,m^2,m^2)\right] \tag{C.141}$$

To obtain the renormalized vacuum polarization one needs to know the value of $\Pi(0,\varepsilon)$. To do that one has to take the limit $k \to 0$ in Eq. (C.141). For that one uses the derivative of the B_0 function

$$B_0'(p^2,m_1^2,m_2^2) \equiv \frac{\partial}{\partial p^2}\,B_0(p^2,m_1^2,m_2^2) \tag{C.142}$$

to obtain

$$\Pi(0,\varepsilon) = \frac{\alpha}{4\pi}\left[-\frac{4}{9} + \frac{4}{3}B_0(0,m^2,m^2) + \frac{8}{3}m^2 B_0'(0,m^2,m^2)\right] \tag{C.143}$$

Using

$$B_0'(0, m^2, m^2) = \frac{1}{6m^2} \tag{C.144}$$

we finally get

$$\Pi(0, \varepsilon) = -\delta Z_3 = \frac{\alpha}{4\pi}\left[\frac{4}{3}\, B_0(0, m^2, m^2)\right] \tag{C.145}$$

and the final result for the renormalized vertex is:

$$\Pi^R(k) = \frac{\alpha}{3\pi}\left[-\frac{1}{3} + \left(1 + \frac{2m^2}{k^2}\right)\left(B_0(k^2, m^2, m^2) - B_0(0, m^2, m^2)\right)\right] \tag{C.146}$$

If we want to compare with our earlier analytical results we need to know that

$$B_0(0, m^2, m^2) = \Delta_\varepsilon - \ln\frac{m^2}{\mu^2} \tag{C.147}$$

Then Eq. (C.146) reproduces the result of Eq. (4.54). The comparison between Eq. (C.146) and Eq. (4.56) can be done numerically using the package `LoopTools`[15].

C.10.2 Electron Self-Energy in QED

In this section we repeat the calculation of section 4.1.2 using the Passarino-Veltman scheme. We start with the `Mathematica` program,

```
(******************** Program SelfEnergy.m **********************)

(* First input FeynCalc *)

<< FeynCalc.m

(* These are some shorthands for the FeynCalc notation *)
dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(* Tell FeynCalc to reduce the result to scalar functions *)

SetOptions[{B0,B1,B00,B11},BReduce->True]
```

```
(* Now write the numerator of the Feynman diagram. We define the
   constant

       C= - alpha/(4 pi)

  The minus sign comes from the photon propagator. The factor
  i/(16 pi^2) is already included in this definition.


*)


num:= C dm[mu] .  (ds[p]+ds[k]+m) . dm[mu]


(* Tell FeynCalc to evaluate the one-loop integral in dimension D *)

SetOptions[OneLoop,Dimension->D]

(* Define the amplitude *)

amp:= num \
FeynAmpDenominator[PropagatorDenominator[p+k,m], \
                   PropagatorDenominator[k]]


(* Calculate the result *)

res:=(-I / Pi^2) OneLoop[k,amp]

ans=-res;

(*
The minus sign in ans comes from the fact that -i \Sigma = diagram
*)

(* Calculate the functions A(p^2) and B(p^2) *)

A=Coefficient[ans,DiracSlash[p],0];
B=Coefficient[ans,DiracSlash[p],1];

(* Calculate deltm *)

delm=A + m B /. p->m

(* Calculate delZ2 *)
```

```
Ap2 = A /. ScalarProduct[p,p]->p2
Bp2 = B /. ScalarProduct[p,p]->p2

aux=2 m D[Ap2,p2] + Bp2 \
    + 2 m^2 D[Bp2,p2] /. D[B0[p2,0,m^2],p2]->DB0[p2,0,m^2]

aux2= aux /. p2->m^2

aux3= aux2 /. A0[m^2]->m^2 (B0[m^2,0,m^2] -1)

delZ2=Simplify[aux3]

(***************** End of Program SelfEnergy.m ********************)
```

The output from `Mathematica` is:

```
                                  2       2
     A =  -(C (-2 m + 4 m B0[p , 0, m ]))


                2       2       2    2      2      2
         C (p  + A0[m ] - (m  + p ) B0[p , 0, m ])
     B = -(-----------------------------------------)
                                   2
                                  p


             2       2       2     2      2
        C (m  - A0[m ] - 2 m  B0[m , 0, m ])
  delm = ------------------------------------
                             m


                        2      2        2      2      2
 delZ2= C (-2 + B0[m , 0, m ] - 4 m  DB0[m , 0, m ])
```

We therefore get

$$A = \frac{\alpha m}{\pi} \left[-\frac{1}{2} + B_0(p^2, 0, m^2) \right] \tag{C.148}$$

$$B = \frac{\alpha}{4\pi} \left[1 + \frac{1}{p^2} A_0(m^2) - \left(1 + \frac{m^2}{p^2} \right) B_0(p^2, 0, m^2) \right] \tag{C.149}$$

$$\delta_m = \frac{3\alpha m}{4\pi}\left[-\frac{1}{3}+\frac{1}{3m^2}A_0(m^2)+\frac{2}{3}B_0(m^2,0,m^2)\right] \tag{C.150}$$

One can check that Eq. (C.150) is in agreement with Eq. (4.80). For that one needs the following relations,

$$A_0(m^2) = m^2\ \left(B_0(m^2,0,m^2)-1\right) \tag{C.151}$$

$$B_0(m^2,0,m^2) = \Delta_\varepsilon + 2 - \ln\frac{m^2}{\mu^2} \tag{C.152}$$

$$\int_0^1 dx(1+x)\ln\frac{m^2x^2}{\mu^2} = -\frac{5}{2}+\frac{3}{2}\ln\frac{m^2}{\mu^2} \tag{C.153}$$

For δZ_2 we get

$$\delta Z_2 = \frac{\alpha}{4\pi}\left[2-B_0(m^2,0,m^2)-4m^2B_0'(m^2,\lambda^2,m^2)\right] \tag{C.154}$$

This expression can be shown to be equal to Eq. (4.83) although this is not trivial. The reason is that B_0' is IR divergent, hence the parameter λ that controls the divergence.

C.10.3 QED Vertex

In this section we repeat the calculation of section 4.1.3 for the QED vertex using the Passarino-Veltman scheme. The `Mathematica` program should by now be easy to understand. We just list it here,

```
(******************** Program QEDVertex.m *********************)

(* First input FeynCalc *)

<< FeynCalc.m

(* These are some shorthands for the FeynCalc notation *)
dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]
```

```
(* Tell FeynCalc to reduce the result to scalar functions *)

SetOptions[{B1,B00,B11},BReduce->True]

(* Now write the numerator of the Feynman diagram. We define the
   constant

       C=  alpha/(4 pi)

   The kinematics is: q = p1 -p2 and the internal momenta is k.

*)

num:=Spinor[p1,m] . dm[ro] . (ds[p1]-ds[k]+m) . ds[a] \
      . (ds[p2]-ds[k]+m) . dm[ro] . Spinor[p2,m]

SetOptions[OneLoop,Dimension->D]

amp:=C num \
FeynAmpDenominator[PropagatorDenominator[k,lbd], \
                   PropagatorDenominator[k-p1,m], \
                   PropagatorDenominator[k-p2,m]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m^2,ScalarProduct[p2,p2]->m^2, \
         ScalarProduct[p1,p2]->m^2-q2/2}

(* Define the divergent part of the relevant PV functions*)

div={B0[0,0,m^2]->Div,B0[0,m^2,m^2]->Div, \
     B0[m^2,0,m^2]->Div,B0[m^2,lbd^2,m^2]->Div,\
     B0[q2,m^2,m^2]->Div,B0[0,lbd^2,m^2]->Div}

res1:=(-I / Pi^2) OneLoop[k,amp]

res:=res1 /. onshell

auxV1:= res /.onshell
auxV2:= PaVeReduce[auxV1]
auxV3:= auxV2 /. div
divV:=Simplify[Div*Coefficient[auxV3,Div]]
```

```
(* Check  that the divergences do not cancel *)

testdiv:=Simplify[divV]

ans1=res;
var=Select[Variables[ans1],(Head[#]===StandardMatrixElement)&]
Set @@ {var, {ME[1],ME[2]}}

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 2 Standard Matrix
Elements. To have a simpler result we substitute these elements
by simpler expressions (ME[1],ME[2]).

 {StandardMatrixElement[u[p1, m1] . u[p2, m2]],

 StandardMatrixElement[u[p1, m1] . ga[mu] . u[p2, m2]]}

*)

ans2=Simplify[PaVeReduce[ans1]]

CE11=Coefficient[ans2,ScalarProduct[a,p1] ME[1]]
CE12=Coefficient[ans2,ScalarProduct[a,p2] ME[1]]
CE2=Coefficient[ans2, ME[2]]

ans3=CE11 (ScalarProduct[a,p1]+ScalarProduct[a,p2]) ME[1] + \
     CE2 ME[2]

test1:=Simplify[CE11-CE12]
test2:=Simplify[ans2-ans3]

(* ME[2] is \overline{u}(p')\gamma_{\mu}u(p) and ME[3] is
   \frac{i}{2m} \overline{u}(p')\sigma_{\mu\nu} q^{\u} u(p)
*)

ans4= ans3 /. {(ScalarProduct[a,p1]+ScalarProduct[a,p2]) \
              ME[1] -> 2 m ME[2] -2 m  ME[3]}

CGamma:=Coefficient[ans4,ME[2]]
CSigmaAux:=Coefficient[ans4,ME[3]]

test3:=Simplify[ans4-CGamma ME[2] -CSigmaAux ME[3]]
```

```
F2:=Simplify[CSigmaAux /. lbd->0]

delZ1aux= - CGamma /. q2->0

delZ1:= delZ1aux /. lbd->0

F1:=CGamma + delZ1 /. lbd->0

(**************** End of Program QEDVertex.m *******************)
```

From this program we can obtain first the value of δZ_1. We get

```
                          2                2   2           2       2
  delZ1= C(1 - B0[0, 0, m ] + 2 B0[0, m , m ] - 2 B0[m , 0, m ] -

     2      2   2       2       2
  4 m  C0[m , m , 0, m , 0, m ])
```

which can be written as

$$
\begin{aligned}
\delta Z_1 &= \frac{\alpha}{2\pi} \left[1 - B_0(0,0,m^2) + 2B_0(0,m^2,m^2) - 2B_0(m^2,m^2) \right. \\
&\quad \left. -4m^2 C_0(m^2,m^2,0,m^2,\lambda^2,m^2) \right] \qquad \text{(C.155)}
\end{aligned}
$$

where we have introduced a small mass for the photon in the function $C_0(m^2,m^2,0,m^2,\lambda^2,m^2)$ because it is IR divergent when $\lambda \to 0$ (see Eq. (C.123)). Using the results of Eqs. (C.106), (C.107), (C.108) and Eq. (C.123) we can show the important result

$$
\delta Z_1 = \delta Z_2 \qquad \text{(C.156)}
$$

where δZ_2 was defined in Eq. (C.154). After performing the renormalization the coefficient $F_1(k^2)$ is finite and given by

```
                                            2                 2   2
               q2         q2 B0[0, 0, m ]   2 q2 B0[0, m , m ]
F1 =   C (-(---------) - --------------- + ------------------ -
                    2                2                  2
           q2 - 4 m        q2 - 4 m           q2 - 4 m

   2        2   2                    2   2        2         2   2
 8 m  B0[0, m , m ]   3 q2 B0[q2, m , m ]   8 m  B0[q2, m , m ]
 ------------------ - ------------------- + -------------------
            2                   2                   2
   q2 - 4 m             q2 - 4 m            q2 - 4 m
```

```
            2      2          2     2   2      2      2
   2 q2 B0[m , 0, m ]   4 q2 m  C0[m , m , 0, m , 0, m ]
+  ------------------ - -------------------------------- +
               2                            2
       q2 - 4 m                     q2 - 4 m

      4     2   2      2      2        2     2   2       2      2
  16 m  C0[m , m , 0, m , 0, m ]   2 q2  C0[m , m , q2, m , 0, m ]
  ------------------------------ - ------------------------------- +
                  2                                  2
            q2 - 4 m                          q2 - 4 m

         2     2   2       2      2        4     2   2       2      2
  12 q2 m  C0[m , m , q2, m , 0, m ]   16 m  C0[m , m , q2, m , 0, m ]
  ---------------------------------- - ------------------------------- )
                    2                                    2
              q2 - 4 m                           q2 - 4 m

In[5]:= F1 /. q2->0

Out[5]= 0
```

while the coefficient $F_2(q^2)$ does not need renormalization and it is given by,

```
           2                 2            2   2          2      2
        4 m  (1 + B0[0, 0, m ] + B0[q2, m , m ] - 2 B0[m , 0, m ])
 F2= C ----------------------------------------------------------
                                       2
                               q2 - 4 m
```

and for $F_2(0)$ we get

```
                           2            2   2          2      2
 F2[0]= -(C (1 + B0[0, 0, m ] + B0[0, m , m ] - 2 B0[m , 0, m ]))
```

Using the results of the Appendix we can show that,

$$F_2(0) = \frac{\alpha}{2\pi} \tag{C.157}$$

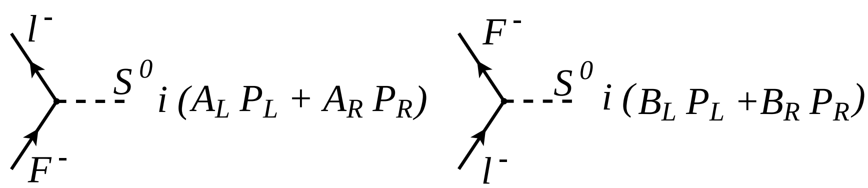
a well known result, first obtained by Schwinger even before the renormalization program was fully understood ($F_2(q^2)$ is finite).

C.11 Modern techniques in a real problem: $\mu \to e\gamma$

In the previous sections we have redone most of the QED standard textbook examples using the PV decomposition and automatic tools. Here we want to present a more complex example, the calculation of the partial width $\mu \to e\gamma$ in an arbitrary theory where the charged leptons couple to scalars and fermions, charged or neutral. This has been done in Ref.[17] for fermions and bosons of arbitrary charge Q_F and Q_B, but for simplicity I will consider here separately the cases of neutral and charged scalars.

C.11.1 Neutral scalar charged fermion loop

We will consider a theory with the following interactions,



where F^- is a fermion with mass m_F and S^0 a neutral scalar with mass m_S. In fact $B_{L,R}$ are not independent of $A_{L,R}$ but it is easier for our programming to consider them completely general. The Feynman rule for the coupling of the photon with the lepton is $-i\,e\,Q_\ell\,\gamma^\mu$ where e is the positron charge (for an electron $Q_\ell = -1$). ℓ_i^- can be any of the leptons but we will omit all indices in the program, the lepton being identified by its mass and from the assumed kinematics

$$\ell_2(p_2) \to \ell_1(p_1) + \gamma(k) \tag{C.158}$$

The diagrams contributing to the process are given in Fig. C.7,

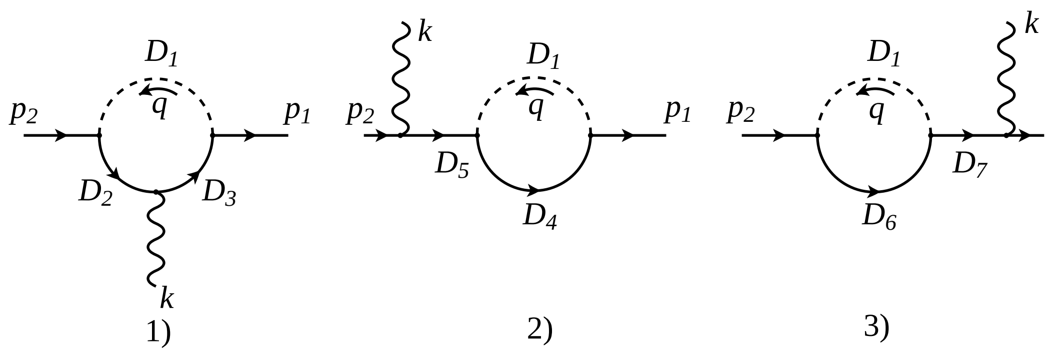


Figure C.7:

where

$$D_1 \;=\; q^2 - m_S^2 \quad ; \quad D_2 = (p_2 + q)^2 - m_F^2 \quad ; \quad D_3 = (q + p2 - k)^2 - m_F^2 \tag{C.159}$$

$$D_4 = D_3 \quad ; \quad D_6 = D_2 \quad ; \quad D_5 = (p_2-k)^2 - m_2^2 = -2p_2 \cdot k \tag{C.160}$$

$$D_7 = (p_1+k)^2 - m_1^2 = 2p_1 \cdot k = -D_5 \tag{C.161}$$

The amplitudes are

$$\begin{aligned} iM_1 &= \frac{e\,Q_\ell}{D_1 D_2 D_3} \overline{u}(p_1)\,(A_L P_L + A_R P_R)\,(\not{q} + \not{p}_2 - \not{k} + m_F)\,\gamma^\mu\,(\not{q} + \not{p}_2 + m_F) \\ &\qquad (B_L P_L + B_R P_R)\,u(p_2)\,\varepsilon_\mu(k) \end{aligned} \tag{C.162}$$

$$\begin{aligned} iM_2 &= \frac{e\,Q_\ell}{D_1 D_4 D_5} \overline{u}(p_1)\,(A_L P_L + A_R P_R)\,(\not{q} + \not{p}_2 - \not{k} + m_F)\,(B_L P_L + B_R P_R) \\ &\qquad (\not{p}_2 - \not{k}_2 + m_2)\,\gamma^\mu u(p_2)\,\varepsilon_\mu(k) \end{aligned} \tag{C.163}$$

$$\begin{aligned} iM_3 &= \frac{e\,Q_\ell}{D_1 D_6 D_7} \overline{u}(p_1)\gamma^\mu\,(\not{p}_1 + \not{k} + m_F)\,(A_L P_L + A_R P_R)\,(\not{q} + \not{p}_2 + m_1) \\ &\qquad (B_L P_L + B_R P_R)\,u(p_2)\,\varepsilon_\mu(k) \end{aligned} \tag{C.164}$$

On-shell the amplitude will take the form (we have $p_1 \cdot k = p_2 \cdot k$)

$$\begin{aligned} iM &= 2p_2 \cdot \varepsilon(k) \Big[C_L \overline{u}(p_1) P_L u(p_2) + C_R \overline{u}(p_1) P_R u(p_2) \Big] \\ &\qquad + D_L u(p_1) \not{\varepsilon} P_L u(p_2) + D_R u(p_1) \not{\varepsilon} P_R u(p_2) \end{aligned} \tag{C.165}$$

If we write the amplitude as

$$M = M_\mu\, \varepsilon^\mu(k) \tag{C.166}$$

then gauge invariance implies

$$M_\mu k^\mu = 0 \tag{C.167}$$

Imposing this condition on Eq. (C.165) we get the relations

$$D_L = -m_2 C_R - m_1 C_L \tag{C.168}$$

$$D_R = -m_1 C_R - m_2 C_L \tag{C.169}$$

Assuming these relations the amplitude can be written as

$$\begin{aligned} iM = {} & C_L \left[2p_2 \cdot \varepsilon(k) \overline{u}(p_1) P_L u(p_2) - m_1 \overline{u}(p_1) \not{\varepsilon}(k) P_L u(p_2) - m_2 \overline{u}(p_1) \not{\varepsilon}(k) P_R u(p_2)\right] \\ & + C_R \left[2p_2 \cdot \varepsilon(k) \overline{u}(p_1) P_R u(p_2) - m_2 \overline{u}(p_1) \not{\varepsilon}(k) P_L u(p_2) - m_1 \overline{u}(p_1) \not{\varepsilon}(k) P_R u(p_2)\right] \end{aligned} \tag{C.170}$$

and the decay width will be

$$\Gamma = \frac{1}{16\pi m_2^3}\,\left(m_2^2 - m_1^2\right)^3 \left(|C_L|^2 + |C_R|^2\right) \tag{C.171}$$

As the coefficient of $p_2 \cdot \varepsilon(k)$ only comes from the 3-point function (amplitude M_1) this justifies the usual procedure of just calculating that coefficient and forgetting about the self-energies (amplitudes M_2 and M_3). However these amplitudes are crucial for the cancellation of divergences and for gauge invariance. Now we will show the power of the automatic `FeynCalc` [13] program and calculate both the coefficients $C_{L,R}$ and $D_{L,R}$, showing the cancellation of the divergences and that the relations, Eqs. (C.168) and (C.169) needed for gauge invariance are satisfied. We start by writing the `mathematica` program:

```
(************************ Program mueg-ns.m *************************)
(*
This program calculates the COMPLETE (both the 3 point amplitude and
the two self energy type on each external line) amplitudes for
\mu -> e \gamma when the fermion line in the loop is charged and the
neutral line is a scalar. The \mu has momentum p2 and mass m2, the
electron (p1,m1) and the photon momentum k. The momentum in the loop
is q.

The assumed vertices are,

1) Electron-Scalar-Fermion:

  Spinor[p1,m1] (AL P_L + AR P_R) Spinor [pf,mf]

2) Fermion-Scalar-Muon:

   Spinor[pf,mf] (BL P_L + BR P_R) Spinor [p2,m2]
*)

dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(*
SetOptions[{B0,B1,B00,B11},BReduce->True]
 *)
```

```
gA:= AL DiracMatrix[7] + AR DiracMatrix[6]
gB:= BL DiracMatrix[7] + BR DiracMatrix[6]


num1:=Spinor[p1,m1] . gA . (ds[q]+ds[p2]-ds[k]+mf) . ds[Polarization[k]]\
      . (ds[q]+ds[p2]+mf) . gB . Spinor[p2,m2]


num2:=Spinor[p1,m1] . gA . (ds[q]+ds[p1]+mf) . gB . (ds[p1]+m2) . \
       ds[Polarization[k]] . Spinor[p2,m2]


num3:=Spinor[p1,m1] . ds[Polarization[k]] . (ds[p2]+m1) . gA . \
      (ds[q]+ds[p2]+mf) . gB . Spinor[p2,m2]



SetOptions[OneLoop,Dimension->D]

amp1:=num1 \
FeynAmpDenominator[PropagatorDenominator[q+p2-k,mf], \
                   PropagatorDenominator[q+p2,mf], \
                   PropagatorDenominator[q,ms]]
amp2:=num2 \
FeynAmpDenominator[PropagatorDenominator[q+p1,mf], \
                   PropagatorDenominator[p2-k,m2],  \
                   PropagatorDenominator[q,ms]]
amp3:=num3 \
FeynAmpDenominator[PropagatorDenominator[p1+k,m1], \
                   PropagatorDenominator[q+p2,mf],  \
                   PropagatorDenominator[q,ms]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m1^2,ScalarProduct[p2,p2]->m2^2, \
        ScalarProduct[k,k]->0,ScalarProduct[p1,k]->(m2^2-m1^2)/2,\
        ScalarProduct[p2,k]->(m2^2-m1^2)/2, \
        ScalarProduct[p2,Polarization[k]]->p2epk, \
        ScalarProduct[p1,Polarization[k]]->p2epk}
```

```
(* Define the divergent part of the relevant PV functions*)
div={B0[m1^2,mf^2,ms^2]->Div,B0[m2^2,mf^2,ms^2]->Div, \
  B0[0,mf^2,ms^2]->Div,B0[0,mf^2,mf^2]->Div,B0[0,ms^2,ms^2]->Div}

res1:=(-I / Pi^2) OneLoop[q,amp1]
res2:=(-I / Pi^2) OneLoop[q,amp2]
res3:=(-I / Pi^2) OneLoop[q,amp3]
res:=res1+res2+res3 /. onshell

auxT1:= res1 /.onshell
auxT2:= PaVeReduce[auxT1]
auxT3:= auxT2 /. div
divT:=Simplify[Div*Coefficient[auxT3,Div]]

auxS1:= res2 + res3 /.onshell
auxS2:= PaVeReduce[auxS1]
auxS3:= auxS2 /. div
divS:=Simplify[Div*Coefficient[auxS3,Div]]

(* Check cancellation of divergences
   testdiv should be zero because divT=-divS  *)

testdiv:=Simplify[divT + divS]

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 8 Standard Matrix Elements.
To have a simpler result we substitute these elements by simpler
expressions (ME[1],...ME[8]). But they are not all independent. The
final result can just be written in terms of 4 Matrix Elements.

{StandardMatrixElement[p2epk u[p1,m1] . ga[6] . u[p2,m2]],

StandardMatrixElement[p2epk u[p1,m1] . ga[7] . u[p2,m2]],

StandardMatrixElement[p2epk u[p1,m1] . gs[k] . ga[6] . u[p2,m2]],

StandardMatrixElement[p2epk u[p1,m1] . gs[k] . ga[7] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[ep[k]] . ga[6] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[ep[k]] . ga[7] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[k] . gs[ep[k]] . ga[6] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[k] . gs[ep[k]] . ga[7]. u[p2,m2]]} *)
```

```
ans1=res;
var=Select[Variables[ans1],(Head[#]===StandardMatrixElement)&]
Set @@ {var, {ME[1],ME[2],ME[3],ME[4],ME[5],ME[6],ME[7],ME[8]}}
identities={ME[3]->-m1 ME[1] + m2 ME[2], ME[4]->-m1 ME[2] + m2 ME[1],
            ME[7]->-m1 ME[5] - m2 ME[6] + 2  ME[1],
            ME[8]->-m1 ME[6] - m2 ME[5] + 2  ME[2]}

ans2 =ans1 /. identities ;
ans=Simplify[ans2];

CR=Coefficient[ans,ME[1]]/2;
CL=Coefficient[ans,ME[2]]/2;
DR=Coefficient[ans,ME[5]];
DL=Coefficient[ans,ME[6]];

(* Test to see if we did not forget any term *)

test1:=Simplify[ans-2 CR*ME[1]-2 CL*ME[2]-DR*ME[5]-DL*ME[6]]


(* Test that the divergences cancel term by term *)

auxCL=PaVeReduce[CL] /. div ;
testdivCL:=Simplify[Coefficient[auxCL,Div]]

auxCR=PaVeReduce[CR] /. div ;
testdivCR:=Simplify[Coefficient[auxCR,Div]]

auxDL=PaVeReduce[DL] /. div ;
testdivDL:=Simplify[Coefficient[auxDL,Div]]

auxDR=PaVeReduce[DR] /. div ;
testdivDR:=Simplify[Coefficient[auxDR,Div]]

(* Test the gauge invariance relations *)
testGI1:=Simplify[PaVeReduce[(m2^2-m1^2)*CR - DR*m1 + DL*m2]]

testGI2:=Simplify[PaVeReduce[(m2^2-m1^2)*CL + DR*m2 - DL*m1]]

(********************* End Program mueg-ns.m  ************************)
```

We first do the tests. The output of `mathematica` is

```
(*********************  Mathematica output   ***********************)
In[3]:= << FeynCalc.m

FeynCalc4.1.0.3b Type ?FeynCalc for help or visit
http://www.feyncalc.org

In[4]:= << mueg-ns.m

In[5]:= test1

Out[5]= 0

In[6]:= testdiv

Out[6]= 0

In[7]:= testdivCL

Out[7]= 0

In[8]:= testdivCR

Out[8]= 0

In[9]:= testdivDL

Out[9]= 0

In[10]:= testdivDR

Out[10]= 0

In[11]:= testGI1

Out[11]= 0

In[12]:= testGI2

Out[12]= 0
(******************  End of Mathematica output   *******************)
```

Now we obtain the results for C_L

```
(*********************  Mathematica output   ************************)
In[13]:= CL
                                 2    2    2    2    2
Out[13]= (-4 AL BL mf C0[0, m2 , m1 , mf , mf , ms ] +

                                2    2      2    2    2
    4 AL BR m2 PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                                2    2      2    2    2
    4 AL BL mf PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                                   2    2      2    2    2
    4 AR BL m1 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                   2    2      2    2    2
    4 AL BR m2 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                   2    2      2    2    2
    4 AL BR m2 PaVe[2, 2, {0, m1 , m2 }, {mf , mf , ms }]) / 4
```

and for C_R

```
In[15]:= CR
                                 2    2    2    2    2
Out[15]= (-4 AR BR mf C0[0, m2 , m1 , mf , mf , ms ] +

                                2    2      2    2    2
    4 AR BL m2 PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                                2    2      2    2    2
    4 AR BR mf PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                                   2    2      2    2    2
    4 AL BR m1 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                   2    2      2    2    2
    4 AR BL m2 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                   2    2      2    2    2
    4 AR BL m2 PaVe[2, 2, {0, m1 , m2 }, {mf , mf , ms }]) / 4
(****************** End of  Mathematica output  *********************)
```

The expressions for $D_{L,R}$ are quite complicated. They are not normally calculated because they can be related to $C_{L,R}$ by gauge invariance. However the power of this automatic program can be illustrated by asking for these functions. As they are very long we calculate them by pieces. We just calculate D_L because one can easily check that $D_R = D_L(L \leftrightarrow R)$.

```
(*********************  Mathematica output   ***********************)

In[12]:= Coefficient[PaVeReduce[DL],AL BL]

                    2    2    2               2    2    2
         m1 mf B0[m1 , mf , ms ]   m1 mf B0[m2 , mf , ms ]
Out[12]= ----------------------- - ----------------------- +
                  2     2                   2     2
                m1  - m2                  m1  - m2

            2    2       2    2    2
  m1 mf C0[m1 , m2 , 0, mf , ms , mf ]

In[13]:= Coefficient[PaVeReduce[DL],AL BR]

            2     2           2    2
         (mf  - ms ) B0[0, mf , ms ]
Out[13]= --------------------------- -
                   2 m1 m2

     2           2         2       2    2    2
  (m1  m2 - m2 mf  + m2 ms ) B0[m1 , mf , ms ]
  -------------------------------------------- +
                        2     2
              2 m1 (m1  - m2 )

        2        2         2       2    2    2
  (m1 m2  - m1 mf  + m1 ms ) B0[m2 , mf , ms ]
  --------------------------------------------
                        2     2
              2 m2 (m1  - m2 )

In[14]:= Coefficient[PaVeReduce[DL],AR BL]

                      2          2       2    2    2
         1   (-2 m1 mf  + 2 m1 ms ) B0[m1 , mf , ms ]
Out[14]= - - ---------------------------------------- +
         2                      2     2
                       2 m1 (m1  - m2 )

          2          2       2    2    2
  (-2 m2 mf  + 2 m2 ms ) B0[m2 , mf , ms ]
  ----------------------------------------
                      2     2
             2 m2 (m1  - m2 )
```

```
     2      2    2       2    2    2
  + mf  C0[m1 , m2 , 0, mf , ms , mf ]

In[15]:= Coefficient[PaVeReduce[DL],AR BR]

                    2    2    2                 2    2    2
         m2 mf B0[m1 , mf , ms ]   m2 mf B0[m2 , mf , ms ]
Out[15]= ----------------------- - -----------------------
                  2     2                    2     2
                m1  - m2                   m1  - m2

              2    2       2    2    2
  + m2 mf C0[m1 , m2 , 0, mf , ms , mf ]

(****************** End of  Mathematica output  ********************)
```

From these expressions one can immediately verify that the divergences cancel in $D_{L,R}$ and that they are not present in $C_{L,R}$. To finish this section we just rewrite the $C_{L,R}$ in our usual notation. We get

$$
\begin{aligned}
C_L &= \frac{e\,Q_\ell}{16\pi^2}\Big[A_L B_L m_F\left(-C_0(0,m_2^2,m_1^2,m_F^2,m_F^2,m_S^2)-C_2(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\right)\\
&\qquad +A_L B_R m_2\Big(C_2(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)+C_{12}(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\\
&\qquad\qquad +C_{22}(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\Big)\\
&\qquad + A_R B_L m_1\, C_{12}(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\Big] \qquad \text{(C.172)}\\
C_R &= C_L(L\leftrightarrow R) \qquad \text{(C.173)}
\end{aligned}
$$

These equations are in agreement with Eqs. (32-34) and Eqs. (38-39) of Ref. [17], although some work has to be done in order to verify that[3]. This has to do with the fact that the PV decomposition functions are not independent (see the Appendix for further details on this point). We can however use the power of `FeynCalc` to verify this. We list below a simple program to accomplish that.

```
(*******************  Program lavoura-ns.m  ************************)
(*
This program tests the results of my program mueg-ns.m against the
results obtained by L. Lavoura (hepph/0302221).
*)

(* First load FeynCalc.m and mueg-ns.m *)
```

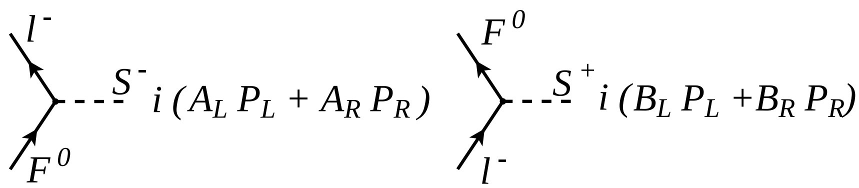
[3] An important difference between our conventions and those of Ref.[17] is that p_1 and p_2 (and obviously m_1 and m_2) are interchanged.

```
<< FeynCalc.m
<< mueg-ns.m

(*
Now write Lavoura integrals in the notation of FeynCalc. Be careful
with the order of the entries.
*)

c1:=PaVe[1,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
c2:=PaVe[2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]

d1:=PaVe[1,1,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
d2:=PaVe[2,2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]

f:=PaVe[1,2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]

(* Write Eqs. (32)-(34) of hepph/0302221 in our notation *)

k1:=PaVeReduce[m2*(c1+d1+f)]
k2:=PaVeReduce[m1*(c2+d2+f)]
k3:=PaVeReduce[mf*(c1+c2)]

(*
Now test the results. For this we should use the equivalences:

\rho    -> AL BR
\lambda -> AR BL
\xi     -> AR BR
\nu     -> AL BL

*)

testCLALBR:=Simplify[PaVeReduce[Coefficient[CL, AL BR]-k1]]
testCLARBL:=Simplify[PaVeReduce[Coefficient[CL, AR BL]-k2]]
testCLALBL:=Simplify[PaVeReduce[Coefficient[CL, AL BL]-k3]]

testCRALBR:=Simplify[PaVeReduce[Coefficient[CR, AL BR]-k2]]
testCRARBL:=Simplify[PaVeReduce[Coefficient[CR, AR BL]-k1]]
testCRARBR:=Simplify[PaVeReduce[Coefficient[CR, AR BR]-k3]]

(***************** End of Program lavoura-ns.m  *********************)
```

One can easily check that the output of the six tests is zero, showing the equivalence between our results. And all this is done in a few seconds.

C.11.2 Charged scalar neutral fermion loop

We consider now the case of the scalar being charged and the scalar neutral. The general case of both charged [17] can also be easily implemented, but for simplicity we do not consider it here. The couplings are now



and the diagrams contributing to the process are given in Fig. C.8, where all the denomi-

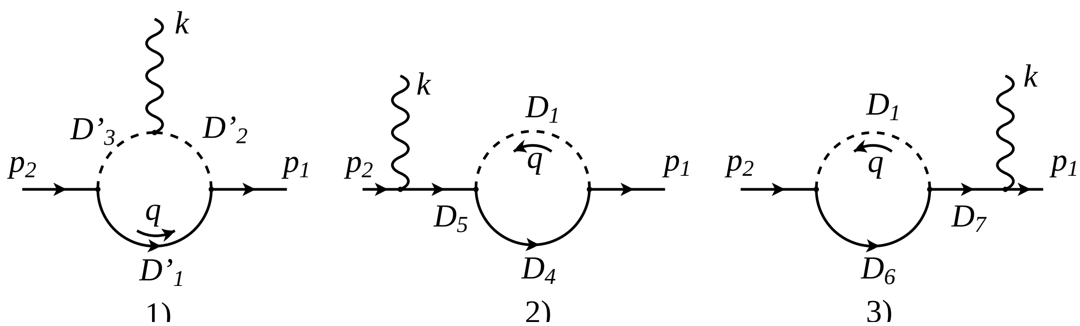


Figure C.8:

nators are as in Eqs. (C.159)- (C.161) except that

$$D_1' = q^2 - m_F^2 \quad ; \quad D_2' = (q-p_1)^2 - m_S^2 \quad ; \quad D_3' = (q-p_1-k)^2 - m_S^2 \tag{C.174}$$

Also the coupling of the photon to the charged scalar is, in our notation,

$$-ie\,Q_\ell\,(-2q+p_1+p_2)^\mu \tag{C.175}$$

The procedure is very similar to the neutral scalar case and we just present here the `mathematica` program and the final result. All the checks of finiteness and gauge invariance can be done as before.

```
(************************ Program mueg-cs.m **************************)
(*
This program calculates the COMPLETE (both the 3 point amplitude and
the two self energy type on each external line) amplitudes for
\mu -> e \gamma when the fermion line in the loop is neutral and the
charged line is a scalar. The \mu has momentum p2 and mass m2, the
electron (p1,m1) and the photon momentum k. The momentum in the loop
is q.
```

```
The assumed vertices are,

1) Electron-Scalar-Fermion:

  Spinor[p1,m1] (AL P_L + AR P_R) Spinor [pf,mf]

2) Fermion-Scalar-Muon:

   Spinor[pf,mf] (BL P_L + BR P_R) Spinor [p2,m2]
*)

dm[mu_]:=DiracMatrix[mu,Dimension->4]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(*
SetOptions[{B0,B1,B00,B11},BReduce->True]
 *)

gA:= AL DiracMatrix[7] + AR DiracMatrix[6]
gB:= BL DiracMatrix[7] + BR DiracMatrix[6]

num1:= Spinor[p1,m1] . gA . (ds[q]+mf) . gB . Spinor[p2,m2]  \
PolarizationVector[k,mu] ( - 2 fv[q,mu] + fv[p1,mu] + fv[p2,mu] )

num11:=DiracSimplify[num1];

num2:=Spinor[p1,m1] . gA . (ds[q]+ds[p1]+mf) . gB . (ds[p1]+m2) . \
       ds[Polarization[k]] . Spinor[p2,m2]


num3:=Spinor[p1,m1] . ds[Polarization[k]] . (ds[p2]+m1) . gA . \
      (ds[q]+ds[p2]+mf) . gB . Spinor[p2,m2]




SetOptions[OneLoop,Dimension->D]
```

```
amp1:=num1 \
FeynAmpDenominator[PropagatorDenominator[q,mf],\
                   PropagatorDenominator[q-p1,ms],\
                   PropagatorDenominator[q-p1-k,ms]]

amp2:=num2 \
FeynAmpDenominator[PropagatorDenominator[q+p1,mf], \
                   PropagatorDenominator[p2-k,m2],  \
                   PropagatorDenominator[q,ms]]
amp3:=num3 \
FeynAmpDenominator[PropagatorDenominator[p1+k,m1], \
                   PropagatorDenominator[q+p2,mf],  \
                   PropagatorDenominator[q,ms]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m1^2,ScalarProduct[p2,p2]->m2^2, \
         ScalarProduct[k,k]->0,ScalarProduct[p1,k]->(m2^2-m1^2)/2, \
         ScalarProduct[p2,k]->(m2^2-m1^2)/2, \
         ScalarProduct[p2,Polarization[k]]->p2epk, \
         ScalarProduct[p1,Polarization[k]]->p2epk}

(* Define the divergent part of the relevant PV functions*)

div={B0[m1^2,mf^2,ms^2]->Div,B0[m2^2,mf^2,ms^2]->Div, \
     B0[0,mf^2,ms^2]->Div,B0[0,mf^2,mf^2]->Div,B0[0,ms^2,ms^2]->Div}

res1:=(-I / Pi^2) OneLoop[q,amp1]
res2:=(-I / Pi^2) OneLoop[q,amp2]
res3:=(-I / Pi^2) OneLoop[q,amp3]

res:=res1+res2+res3 /. onshell


auxT1:= res1 /.onshell
auxT2:= PaVeReduce[auxT1]
auxT3:= auxT2 /. div
divT:=Simplify[Div*Coefficient[auxT3,Div]]

auxS1:= res2 + res3 /.onshell
auxS2:= PaVeReduce[auxS1]
auxS3:= auxS2 /. div
divS:=Simplify[Div*Coefficient[auxS3,Div]]
```

```
(* Check cancellation of divergences

   testdiv should be zero because divT=-divS

*)

testdiv:=Simplify[divT + divS]

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 6 Standard Matrix Elements.
To have a simpler result we substitute these elements by simpler
expressions (ME[1],...ME[6]). Not all are independent.

{StandardMatrixElement[p2epk u[p1, m1] . ga[6] . u[p2, m2]],

StandardMatrixElement[p2epk u[p1, m1] . ga[7] . u[p2, m2]],

StandardMatrixElement[p2epk u[p1, m1] . gs[k] . ga[6] . u[p2, m2]],

StandardMatrixElement[p2epk u[p1, m1] . gs[k] . ga[7] . u[p2, m2]],

StandardMatrixElement[u[p1, m1] . gs[ep[k]] . ga[6] . u[p2, m2]],

StandardMatrixElement[u[p1, m1] . gs[ep[k]] . ga[7] . u[p2, m2]]}
*)

ans1=res;
var=Select[Variables[ans1],(Head[#]===StandardMatrixElement)&]

Set @@ {var, {ME[1],ME[2],ME[3],ME[4],ME[5],ME[6]}}
identities={ME[3]->-m1 ME[1] + m2 ME[2],ME[4]->-m1 ME[2] + m2 ME[1]}

ans2 =ans1 /. identities ;
ans=Simplify[ans2];


CR=Coefficient[ans,ME[1]]/2;
CL=Coefficient[ans,ME[2]]/2;
DR=Coefficient[ans,ME[5]];
DL=Coefficient[ans,ME[6]];

(* Test to see if we did not forget any term *)

test1:=Simplify[ans-2*CR*ME[1]-2*CL*ME[2]-DR*ME[5]-DL*ME[6]]
```

```
(* Test that the divergences cancel term by term *)

auxCL:=PaVeReduce[CL] /. div ;
testdivCL:=Simplify[Coefficient[auxCL,Div]]

auxCR:=PaVeReduce[CR] /. div ;
testdivCR:=Simplify[Coefficient[auxCR,Div]]

auxDL:=PaVeReduce[DL] /. div ;
testdivDL:=Simplify[Coefficient[auxDL,Div]]

auxDR:=PaVeReduce[DR] /. div ;
testdivDR:=Simplify[Coefficient[auxDR,Div]]

(* Test the gauge invariance relations *)
testGI1:=PaVeReduce[(m2^2-m1^2)*CR - DR*m1 + DL*m2]

testGI2:=PaVeReduce[(m2^2-m1^2)*CL + DR*m2 - DL*m1]

(********************* End Program mueg-cs.m  **********************)
```

Note that although these programs look large, in fact they are very simple. Most of it are comments and tests. The output of this program gives,

```
(********************  Mathematica output   **********************)
In[3]:= CL

                              2    2    2    2    2
Out[3]= (-2 AR BL m1 C0[0, m1 , m2 , ms , ms , mf ] -

                         2       2      2    2    2
    2 AR BL m1 PaVe[1, {m1 , 0, m2 }, {mf , ms , ms }] -

                         2    2         2    2    2
    4 AR BL m1 PaVe[1, {m1 , m2 , 0}, {ms , mf , ms }] -

                         2    2         2    2    2
    2 AL BL mf PaVe[1, {m1 , m2 , 0}, {ms , mf , ms }] -

                         2       2      2    2    2
    2 AL BR m2 PaVe[2, {m1 , 0, m2 }, {mf , ms , ms }] -

                         2    2         2    2    2
    2 AR BL m1 PaVe[2, {m1 , m2 , 0}, {ms , mf , ms }] +
```

```
                           2    2          2    2    2
    2 AL BR m2 PaVe[2, {m1 , m2 , 0}, {ms , mf , ms }] -

                              2    2          2    2    2
    2 AR BL m1 PaVe[1, 1, {m1 , m2 , 0}, {ms , mf , ms }] -

                              2    2          2    2    2
    2 AR BL m1 PaVe[1, 2, {m1 , m2 , 0}, {ms , mf , ms }] +

                              2    2          2    2    2
    2 AL BR m2 PaVe[1, 2, {m1 , m2 , 0}, {ms , mf , ms }]) / 2

(****************** End of  Mathematica output  *******************)
```

To finish this section we just rewrite the $C_{L,R}$ in our usual notation. We get

$$
\begin{aligned}
C_L \;=\; & \frac{e\,Q_\ell}{16\pi^2}\Big[A_L B_L m_F\left(-C_1(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)\right) \\
& +A_L B_R m_2\Big(-C_2(m_1^2,0,m_2^2,m_F^2,m_S^2,m_S^2)+C_2(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2) \\
& \qquad +C_{12}(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)\Big) \\
& +A_R B_L m_1\left(-C_0(0,m_1^2,m_2^2,m_S^2,m_S^2,m_F^2)-C_1(m_1^2,0,m_2^2,m_F^2,m_S^2,m_S^2)\right. \\
& \qquad -2C_1(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)-C_2(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2) \\
& \qquad \left.-C_{11}(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)-C_{12}(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)\right)\Big] \\
C_R \;=\; & C_L(L\leftrightarrow R) \qquad\qquad (C.176)
\end{aligned}
$$

It is left as an exercise to write a `mathematica` program that proves that these equations are in agreement with Eqs. (35-37) and Eqs. (38-39) of Ref. [17].

Appendix D

Feynman Rules for the Standard Model

D.1 Introduction

To do actual calculations it is very important to have all the Feynman rules with consistent conventions. In this Appendix we will give the complete Feynman rules for the Standard Model in the general R_ξ gauge.

D.2 The Standard Model

One of the most difficult problems in having a consistent set of of Feynman rules are the conventions. We give here those that are important for building the SM. We will separate them by gauge group.

D.2.1 Gauge Group $SU(3)_c$

Here the important conventions are for the field strengths and the covariant derivatives. We have

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu, \quad a = 1, \ldots, 8 \tag{D.1}$$

where f^{abc} are the group structure constants, satisfying

$$\left[T^a, T^b\right] = i f^{abc} T^c \tag{D.2}$$

and T^a are the generators of the group. The covariant derivative of a (quark) field q in some representation T^a of the gauge group is given by

$$D_\mu q = \left(\partial_\mu - i\, g\, G^a_\mu T^a\right) q \tag{D.3}$$

In QCD the quarks are in the fundamental representation and $T^a = \lambda^a/2$ where λ^a are the Gell-Mann matrices. A gauge transformation is given by a matrix

$$U = e^{-iT^a\alpha^a} \tag{D.4}$$

and the fields transform as

$$
\begin{aligned}
&q \to e^{-iT^a\alpha^a} q && \delta q = -iT^a\alpha^a q \\
&G^a_\mu T^a \to U G^a_\mu T^a U^{-1} - \frac{i}{g}\partial_\mu U U^{-1} && \delta G^a_\mu = -\frac{1}{g}\partial_\mu \alpha^a + f^{abc}\alpha^b G^c_\mu
\end{aligned}
\tag{D.5}
$$

where the second column is for infinitesimal transformations. With these definitions one can verify that the covariant derivative transforms like the field itself,

$$
\delta(D_\mu q) = -i\, T^a \alpha^a (D_\mu q) \tag{D.6}
$$

ensuring the gauge invariance of the Lagrangian.

D.2.2 Gauge Group $SU(2)_L$

This is similar to the previous case. We have

$$
W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu, \quad a = 1, \ldots, 3 \tag{D.7}
$$

where, for the fundamental representation of $SU(2)_L$ we have $T^a = \sigma^a/2$ and ϵ^{abc} is the completely anti-symmetric tensor in 3 dimensions. The covariant derivative for any field ψ_L transforming non-trivially under this group is,

$$
D_\mu \psi_L = \left(\partial_\mu - i\, g\, W^a_\mu T^a\right) \psi_L \tag{D.8}
$$

D.2.3 Gauge Group $U(1)_Y$

In this case the group is abelian and we have

$$
B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{D.9}
$$

with the covariant derivative given by

$$
D_\mu \psi_R = \left(\partial_\mu + i\, g'\, Y\, B_\mu\right) \psi_R \tag{D.10}
$$

where Y is the hypercharge of the field. Notice the different sign convention between Eq. (D.8) and Eq. (D.9). This is to have the usual definition[1]

$$
Q = T_3 + Y\ . \tag{D.12}
$$

It is useful to write the covariant derivative in terms of the mass eigenstates A_μ and Z_μ. These are defined by the relations,

$$
\left\{
\begin{array}{l}
W^3_\mu = Z_\mu \cos\theta_W - A_\mu \sin\theta_W \\
B_\mu = Z_\mu \sin\theta_W + A_\mu \cos\theta_W
\end{array}
\right. , \quad
\left\{
\begin{array}{l}
Z_\mu = W^3_\mu \cos\theta_W + B_\mu \sin\theta_W \\
A_\mu = -W^3_\mu \sin\theta_W + B_\mu \cos\theta_W
\end{array}
\right. . \tag{D.13}
$$

[1] For this to be consistent one must also have, under hypercharge transformations, for a field of hypercharge Y,

$$
\psi' = e^{+iY\alpha_Y}\psi, \quad B'_\mu = B_\mu - \frac{1}{g'}\partial_\mu \alpha_Y\ . \tag{D.11}
$$

This is important when finding the ghost interactions. It would have been possible to have a minus sign in Eq. (D.10), with a definition $\theta_W \to \theta_W + \pi$. This would also mean reversing the sign in the exponent of the hypercharge transformation in Eq. (D.11) maintaining the similarity with Eq. (D.5).

Field	ℓ_L	ℓ_R	ν_L	u_L	d_L	u_R	d_R	ϕ^+	ϕ^0
T_3	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
Y	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
Q	-1	-1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	1	0

Table D.1: Values of T_3^f, Q and Y for the SM particles.

For a field ψ_L, with hypercharge Y, we get,

$$\begin{aligned} D_\mu \psi_L &= \left[\partial_\mu - i\frac{g}{\sqrt{2}}\left(\tau^+ W^+_\mu + \tau^- W^-_\mu\right) - i\frac{g}{2}\tau_3 W^3_\mu + ig'YB_\mu\right]\psi_L \qquad \text{(D.14)} \\ &= \left[\partial_\mu - i\frac{g}{\sqrt{2}}\left(\tau^+ W^+_\mu + \tau^- W^-_\mu\right) + ie\,Q\,A_\mu - i\frac{g}{\cos\theta_W}\left(\frac{\tau_3}{2} - Q\,\sin^2\theta_W\right)Z_\mu\right]\psi_L \end{aligned}$$

where, as usual, $\tau^\pm = (\tau_1 \pm i\tau_2)/2$ and the charge operator is defined by

$$Q = \begin{bmatrix} \frac{1}{2} + Y & 0 \\ 0 & -\frac{1}{2} + Y \end{bmatrix}, \qquad \text{(D.15)}$$

and we have used the relations,

$$e = g\sin\theta_W = g'\cos\theta_W\,, \qquad \text{(D.16)}$$

and the usual definition,

$$W^\pm_\mu = \frac{W^1_\mu \mp i\,W^2_\mu}{\sqrt{2}}\,. \qquad \text{(D.17)}$$

For a singlet of $SU(2)_L$, ψ_R we have,

$$\begin{aligned} D_\mu\psi_R &= \left[\partial_\mu + ig'YB_\mu\right]\psi_R \\ &= \left[\partial_\mu + ie\,Q\,A_\mu + i\frac{g}{\cos\theta_W}Q\,\sin^2\theta_W Z_\mu\right]\psi_R\,. \qquad \text{(D.18)} \end{aligned}$$

We collect in Table D.1 the quantum number of the SM particles.

D.2.4 The Gauge Field Lagrangian

For completeness we write the gauge field Lagrangian. We have

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \qquad \text{(D.19)}$$

where the field strengths are given in Eqs. (D.1), and (D.9).

D.2.5 The Fermion Fields Lagrangian

Here we give the kinetic part and gauge interaction, leaving the Yukawa interaction for a next section. We have

$$\mathcal{L}_{\text{Fermion}} = \sum_{\text{quarks}} i\overline{q}\gamma^\mu D_\mu q + \sum_{\psi_L} i\overline{\psi_L}\gamma^\mu D_\mu \psi_L + \sum_{\psi_R} i\overline{\psi_R}\gamma^\mu D_\mu \psi_R \tag{D.20}$$

where the covariant derivatives are obtained with the rules in Eqs. (D.3), (D.14) and (D.18).

D.2.6 The Higgs Lagrangian

In the SM we use an Higgs doublet with the following assignments,

$$\Phi = \begin{bmatrix} \phi^+ \\ \dfrac{v + H + i\varphi_Z}{\sqrt{2}} \end{bmatrix} \tag{D.21}$$

The hypercharge of this doublet is 1/2 and therefore the covariant derivative reads

$$\begin{aligned} D_\mu \Phi &= \left[\partial_\mu - i\frac{g}{\sqrt{2}}\left(\tau^+ W^+_\mu + \tau^- W^-_\mu\right) - i\frac{g}{2}\tau_3 W^3_\mu + i\frac{g'}{2}B_\mu\right]\Phi \qquad \text{(D.22)} \\ &= \left[\partial_\mu - i\frac{g}{\sqrt{2}}\left(\tau^+ W^+_\mu \tau^- W^-_\mu\right) + ie\,Q\,A_\mu - i\frac{g}{\cos\theta_W}\left(\frac{\tau_3}{2} - Q\,\sin^2\theta_W\right)Z_\mu\right]\Phi \end{aligned}$$

The Higgs Lagrangian is then

$$\mathcal{L}_{\text{Higgs}} = \left(D_\mu\Phi\right)^\dagger D_\mu\Phi + \mu^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 \tag{D.23}$$

If we expand this Lagrangian we find the following terms

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} = \cdots &+ \frac{1}{8}g^2v^2 W^3_\mu W^{\mu 3} + \frac{1}{8}g'^2v^2 B_\mu B^\mu + \frac{1}{4}gg'v^2 W^3_\mu B^\mu + \frac{1}{4}g^2v^2 W^+_\mu W^{-\mu} \\ &+ \frac{1}{2}v\,\partial^\mu\varphi_Z\left(g'B_\mu + gW^3_\mu\right) + \frac{i}{2}gvW^-_\mu\partial^\mu\varphi^+ - \frac{i}{2}gvW^+_\mu\partial^\mu\varphi^- \qquad \text{(D.24)} \end{aligned}$$

The first three terms give, after diagonalization, a massless field, the photon, and a massive one, the Z, with the relations given in Eq. (D.13), while the fourth gives the mass to the charged $W^\pm_\mu$ boson. Using Eq. (D.13) we get,

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} = \cdots &+ \frac{1}{2}M_Z^2 Z_\mu Z^\mu + M_W^2 W^+_\mu W^{-\mu} \\ &+ M_Z Z_\mu \partial^\mu\varphi_Z + iM_W\left(W^-_\mu\partial^\mu\varphi^+ - W^+_\mu\partial^\mu\varphi^-\right) \qquad \text{(D.25)} \end{aligned}$$

where

$$M_W = \frac{1}{2}gv, \quad M_Z = \frac{1}{\cos\theta_W}\frac{1}{2}gv = \frac{1}{\cos\theta_W}M_W \tag{D.26}$$

By looking at Eq. (D.25) we realize that besides finding a realistic spectra for the gauge bosons, we also got a problem. In fact the terms in the last line are quadratic in the fields and complicate the definition of the propagators. We now see how one can use the needed gauge fixing to solve also this problem.

D.2.7 The Yukawa Lagrangian

Now we have to spell out the interaction between the fermions and the Higgs doublet that after spontaneous symmetry breaking gives masses to the elementary fermions. We have,

$$\mathcal{L}_{\text{Yukawa}} = -\, Y_l\, \overline{L}\, \Phi\ \ell_R - Y_d\, \overline{Q}\, \Phi\ d_R - Y_u\, \overline{Q}\, \widetilde{\Phi}\ u_R + \text{h.c.} \tag{D.27}$$

where sum is implied over generations, L (Q) are the lepton (quark) doublets and,

$$\widetilde{\Phi} = i\,\sigma_2 \Phi^* = \begin{bmatrix} \dfrac{v + H - i\varphi_Z}{\sqrt{2}} \\ -\varphi^- \end{bmatrix} \tag{D.28}$$

D.2.8 The Gauge Fixing

As it is well known, we have to gauge fix the gauge part of the Lagrangian to be able to define the propagators. We will use a generalization of the class of Lorenz gauges, the so-called R_ξ gauges. With this choice the gauge fixing Lagrangian reads

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi} F_G^2 - \frac{1}{2\xi} F_A^2 - \frac{1}{2\xi} F_Z^2 - \frac{1}{\xi} F_- F_+ \tag{D.29}$$

where

$$\begin{aligned} F_G^a =& \partial^\mu G_\mu^a, \quad F_A = \partial^\mu A_\mu, \quad F_Z = \partial^\mu Z_\mu - \xi M_Z \varphi_Z \\ F_+ =& \partial^\mu W_\mu^+ - i\xi M_W \varphi^+, \quad F_- = \partial^\mu W_\mu^- + i\xi M_W \varphi^- \end{aligned} \tag{D.30}$$

One can easily verify that with these definitions we cancel the quadratic terms in Eq. (D.25).

D.2.9 The Ghost Lagrangian

The last piece in writing the SM Lagrangian is the ghost Lagrangian. As it is well known, this is given by the Fadeev-Popov prescription,

$$\begin{aligned} \mathcal{L}_{\text{Ghost}} =& \sum_{i=1}^{4} \left[\overline{c}_+ \frac{\partial(\delta F_+)}{\partial \alpha^i} + \overline{c}_- \frac{\partial(\delta F_+)}{\partial \alpha^i} + \overline{c}_Z \frac{\partial(\delta F_Z)}{\partial \alpha^i} + \overline{c}_A \frac{\partial(\delta F_A)}{\partial \alpha^i} \right] c_i \\ & + \sum_{a,b=1}^{8} \overline{\omega}^a \frac{\partial(\delta F_G^a)}{\partial \beta^b} \omega^b \end{aligned} \tag{D.31}$$

where we have denoted by ω^a the ghosts associated with the $SU(3)_c$ transformations defined by,

$$U = e^{-iT^a \beta^a}, \quad a = 1, \ldots, 8 \tag{D.32}$$

and by $c_\pm, c_A, c_Z$ the electroweak ghosts associated with the gauge transformations,

$$U = e^{-iT^a \alpha^a}, \quad a = 1, \ldots, 3, \quad U = e^{iY \alpha^4} \tag{D.33}$$

For completeness we write here the gauge transformations of the gauge fixing terms needed to find the Lagrangian in Eq. (D.31). It is convenient to redefine the parameters as

$$\begin{aligned}
\alpha^{\pm} =& \frac{\alpha^1 \mp \alpha^2}{\sqrt{2}} \\
\alpha_Z =& \alpha^3 \cos\theta_W + \alpha^4 \sin\theta_W \\
\alpha_A =& -\alpha^3 \sin\theta_W + \alpha^4 \cos\theta_W
\end{aligned} \tag{D.34}$$

We then get

$$\begin{aligned}
\delta F_G^a =& -\partial_\mu \beta^a + g_s f^{abc} \beta^b G_\mu^c \\
\delta F_A =& -\partial_\mu \alpha_A \\
\delta F_Z =& \partial_\mu (\delta Z^\mu) - M_Z \delta\varphi_Z \\
\delta F_+ =& \partial_\mu (\delta W_\mu^+) - i M_W \delta\varphi^+ \\
\delta F_- =& \partial_\mu (\delta W_\mu^-) + i M_W \delta\varphi^-
\end{aligned} \tag{D.35}$$

Using the explicit form of the gauge transformations we can finally find the missing pieces,

$$\delta Z_\mu = -\partial_\mu \alpha_Z + ig\cos\theta_W \left(W_\mu^+ \alpha^- - W_\mu^- \alpha^+ \right) \tag{D.36}$$

$$\delta W_\mu^+ = -\partial_\mu \alpha^+ + ig\left[\alpha^+ \left(Z_\mu \cos\theta_W - A_\mu \sin\theta_w\right) - \left(\alpha_Z \cos\theta_w - \alpha_A \sin\theta_W\right) W_\mu^+\right]$$

$$\delta W_\mu^- = -\partial_\mu \alpha^- - ig\left[\alpha^- \left(Z_\mu \cos\theta_W - A_\mu \sin\theta_w\right) - \left(\alpha_Z \cos\theta_w - \alpha_A \sin\theta_W\right) W_\mu^-\right]$$

and

$$\begin{aligned}
\delta\varphi_Z =& -\frac{1}{2} g \left(\alpha^- \varphi^+ + \alpha^+ \varphi^-\right) + \frac{g}{2\cos\theta_W} \alpha_Z (v+H) \\
\delta\varphi^+ =& -i\,\frac{g}{2}(v + H + i\varphi_Z)\alpha^+ - i\,\frac{g}{2}\frac{\cos 2\theta_W}{\cos\theta_W}\varphi^+ \alpha_Z + ie\,\varphi^+ \alpha_A \\
\delta\varphi^- =& i\,\frac{g}{2}(v + H - i\varphi_Z)\alpha^- + i\,\frac{g}{2}\frac{\cos 2\theta_W}{\cos\theta_W}\varphi^- \alpha_Z - ie\,\varphi^- \alpha_A
\end{aligned} \tag{D.37}$$

D.2.10 The Complete SM Lagrangian

Finally the complete Lagrangian for the Standard Model is obtained putting together all the pieces. We have,

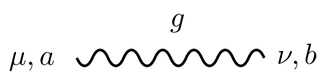
$$\mathcal{L}_{\rm SM} = \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm Fermion} + \mathcal{L}_{\rm Higgs} + \mathcal{L}_{\rm Yukawa} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm Ghost} \tag{D.38}$$

where the different terms were given in Eqs. (D.19), (D.20), (D.23), (D.27), (D.29), (D.31).

D.3 The Feynman Rules for QCD

We give separately the Feynman Rules for QCD and the electroweak part of the Standard Model.

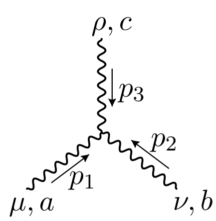
D.3.1 Propagators



$$-i\delta_{ab}\left[\frac{g_{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\frac{k_\mu k_\nu}{(k^2)^2}\right] \tag{D.39}$$

$$\delta_{ab}\,\frac{i}{k^2+i\epsilon} \tag{D.40}$$

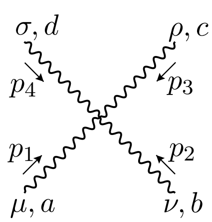
D.3.2 Triple Gauge Interactions



$$\begin{aligned} g f^{abc}[\ & g^{\mu\nu}(p_1-p_2)^\rho + g^{\nu\rho}(p_2-p_3)^\mu \\ & + g^{\rho\mu}(p_3-p_1)^\nu] \\ & p_1+p_2+p_3=0 \end{aligned} \tag{D.41}$$

D.3.3 Quartic Gauge Interactions

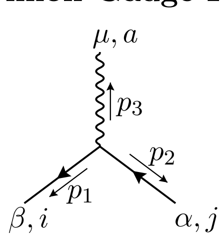
ii) Vértice quártico dos bosões de gauge



$$\begin{aligned} -ig^2\Big[\ & f_{eab}f_{ecd}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}) \\ & + f_{eac}f_{edb}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\nu}g_{\rho\sigma}) \\ & + f_{ead}f_{ebc}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\rho}g_{\nu\sigma})\Big] \end{aligned} \tag{D.42}$$

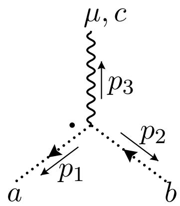
$$p_1+p_2+p_3+p_4=0$$

D.3.4 Fermion Gauge Interactions



$$ig(\gamma^\mu)_{\beta\alpha}T^a_{ij} \tag{D.43}$$

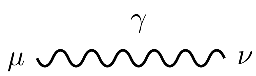
D.3.5 Ghost Interactions



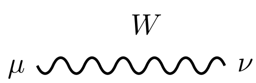
$$g\ C^{abc} p_1^{\mu} \qquad p_1 + p_2 + p_3 = 0 \tag{D.44}$$

D.4 The Feynman Rules for the Electroweak Theory

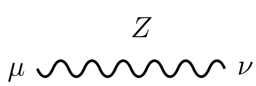
D.4.1 Propagators



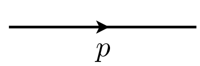
$$-i\left[\frac{g_{\mu\nu}}{k^2+i\epsilon} - (1-\xi)\frac{k_\mu k_\nu}{(k^2)^2}\right] \tag{D.45}$$



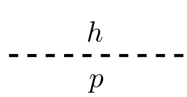
$$\frac{-ig_{\mu\nu}}{k^2 - M_W^2 + i\epsilon} \tag{D.46}$$



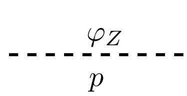
$$\frac{-ig_{\mu\nu}}{k^2 - M_Z^2 + i\epsilon} \tag{D.47}$$



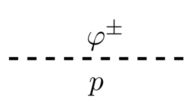
$$\frac{i(\not{p} + m_f)}{p^2 - m_f^2 + i\epsilon} \tag{D.48}$$



$$\frac{i}{p^2 - M_h^2 + i\epsilon} \tag{D.49}$$

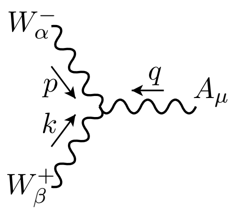


$$\frac{i}{p^2 - \xi m_Z^2 + i\epsilon} \tag{D.50}$$

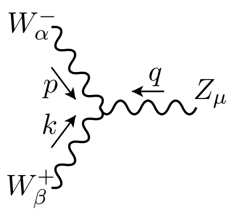


$$\frac{i}{p^2 - \xi m_W^2 + i\epsilon} \tag{D.51}$$

D.4.2 Triple Gauge Interactions

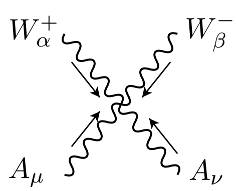


$$-ie\left[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta\right] \tag{D.52}$$

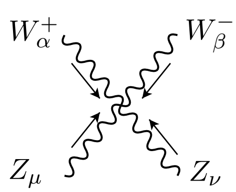


$$ig\cos\theta_W\left[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta\right] \tag{D.53}$$

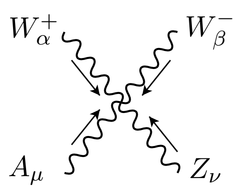
D.4.3 Quartic Gauge Interactions



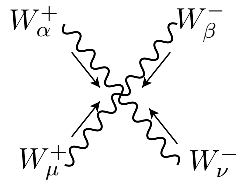
$$-ie^2\left[2g_{\alpha\beta}g_{\mu\mu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \tag{D.54}$$



$$-ig^2\ \cos^2\theta_W\left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \tag{D.55}$$

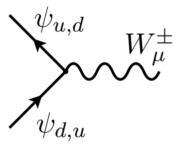


$$ieg\cos\theta_W\left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \tag{D.56}$$



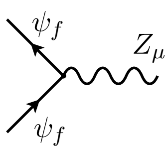
$$ig^2\left[2g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \tag{D.57}$$

D.4.4 Charged Current Interaction

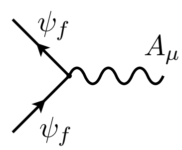


$$i\frac{g}{\sqrt{2}}\gamma_\mu\frac{1-\gamma_5}{2} \tag{D.58}$$

D.4.5 Neutral Current Interaction



$$i\frac{g}{\cos\theta_W}\gamma_\mu\left(g_V^f - g_A^f\gamma_5\right)$$

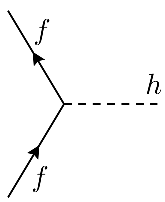


$$-ieQ_f\gamma_\mu \tag{D.59}$$

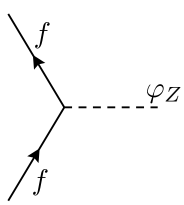
where

$$g_V^f = \frac{1}{2}T_f^3 - Q_f\sin^2\theta_W, \quad g_A^f = \frac{1}{2}T_f^3\ . \tag{D.60}$$

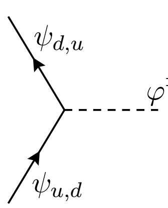
D.4.6 Fermion-Higgs and Fermion-Goldstone Interactions



$$-i\,\frac{g}{2}\,\frac{m_f}{m_W} \tag{D.61}$$

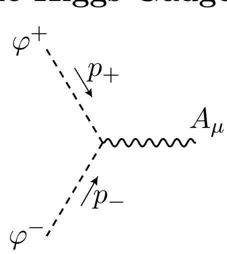


$$-g\,T_f^3\,\frac{m_f}{m_W}\,\gamma_5 \tag{D.62}$$

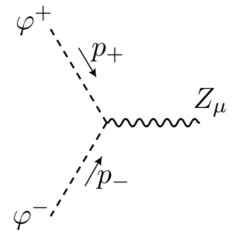


$$i\frac{g}{\sqrt{2}}\left(\frac{m_u}{m_W}P_{R,L} - \frac{m_d}{m_W}P_{L,R}\right) \tag{D.63}$$

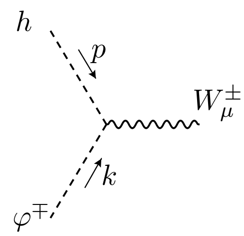
D.4.7 Triple Higgs-Gauge and Goldstone-Gauge Interactions



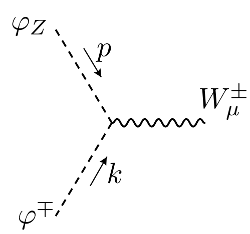
$$-i\,e\,(p_+ - p_-)_\mu \tag{D.64}$$



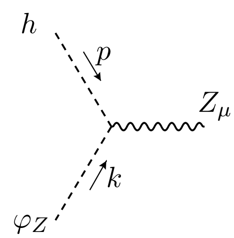
$$i\,g\,\frac{\cos 2\theta_W}{2\cos\theta_W}\,(p_+ - p_-)_\mu \tag{D.65}$$



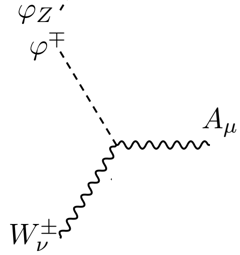
$$\mp\frac{i}{2}\,g\,\left(k-p\right)_\mu \tag{D.66}$$



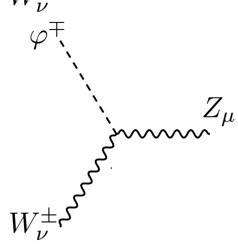
$$\frac{g}{2}\,\left(k-p\right)_\mu \tag{D.67}$$



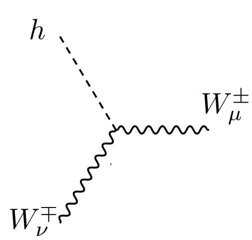
$$\frac{g}{2\cos\theta}\,\left(k-p\right)_\mu \tag{D.68}$$



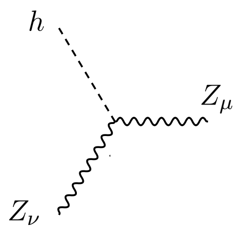
$$-ie\,m_W\,g_{\mu\nu} \tag{D.69}$$



$$-ig\,m_Z\,\sin^2\theta_W\,g_{\mu\nu} \tag{D.70}$$

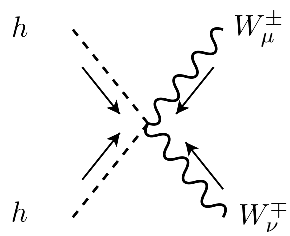


$$ig\,m_W\,g_{\mu\nu} \tag{D.71}$$

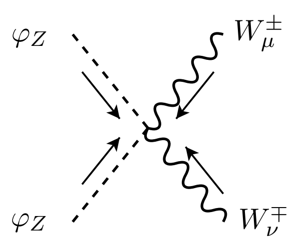


$$i\,\frac{g}{\cos\theta_W}\,m_Z\,g_{\mu\nu} \tag{D.72}$$

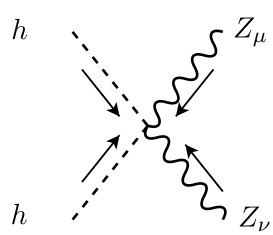
D.4.8 Quartic Higgs-Gauge and Goldstone-Gauge Interactions



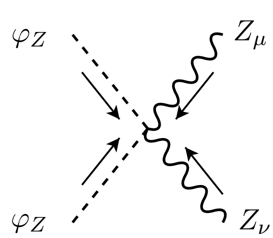
$$\frac{i}{2}\,g^2\,g_{\mu\nu} \tag{D.73}$$



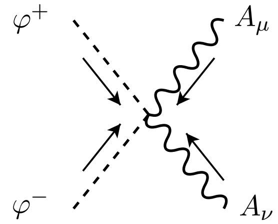
$$\frac{i}{2}\,g^2\,g_{\mu\nu} \tag{D.74}$$



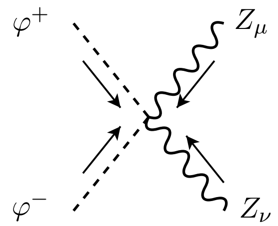
$$\frac{i}{2}\frac{g^2}{\cos^2\theta_W}g_{\mu\nu} \tag{D.75}$$



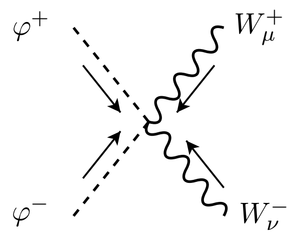
$$\frac{i}{2}\frac{g^2}{\cos^2\theta_W}g_{\mu\nu} \tag{D.76}$$



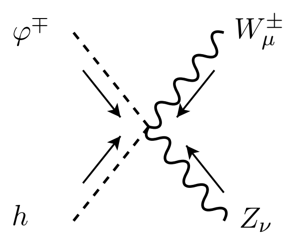
$$2i\, e^2\, g_{\mu\nu} \tag{D.77}$$



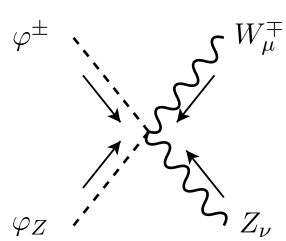
$$\frac{i}{2}\left(\frac{g\cos 2\theta_W}{\cos\theta_W}\right)^2 g_{\mu\nu} \tag{D.78}$$



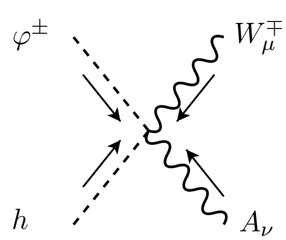
$$\frac{i}{2}\, g^2\, g_{\mu\nu} \tag{D.79}$$



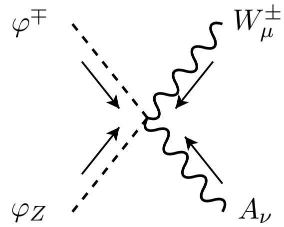
$$-i\, g^2\, \frac{\sin^2\theta_W}{2\cos\theta_W}\, g_{\mu\nu} \tag{D.80}$$



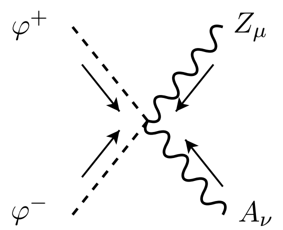
$$\mp\, g^2\, \frac{\sin^2\theta_W}{2\cos\theta_W}\, g_{\mu\nu} \tag{D.81}$$



$$-\frac{i}{2}\, eg\, g_{\mu\nu} \tag{D.82}$$

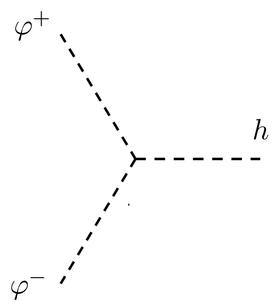


$$\pm\frac{1}{2}\,eg\,g_{\mu\nu} \tag{D.83}$$

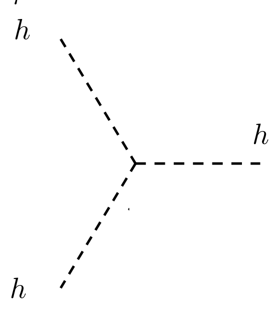


$$-i\,eg\,\frac{\cos 2\theta_W}{\cos\theta_W}g_{\mu\nu} \tag{D.84}$$

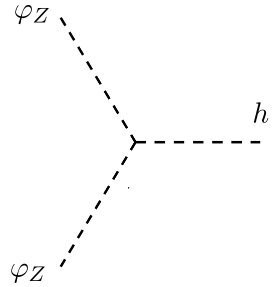
D.4.9 Triple Higgs and Goldstone Interactions



$$-\frac{i}{2}\,g\,\frac{m_h^2}{m_W} \tag{D.85}$$

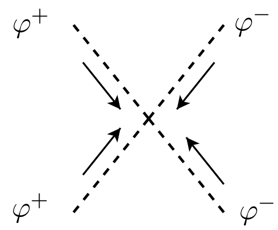


$$-\frac{3}{2}\,i\,g\frac{m_h^2}{m_W^2} \tag{D.86}$$

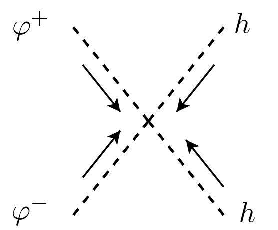


$$-\frac{i}{2}\,g\frac{m_h^2}{m_W^2} \tag{D.87}$$

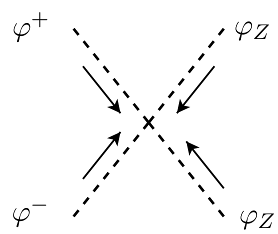
D.4.10 Quartic Higgs and Goldstone Interactions



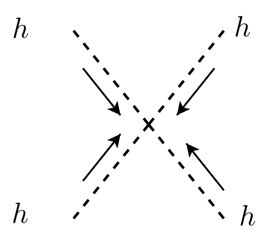
$$-\frac{i}{2}\, g^2\, \frac{m_h^2}{m_W^2} \tag{D.88}$$



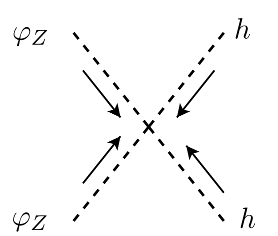
$$-\frac{i}{4}\, g^2\, \frac{m_h^2}{m_W^2} \tag{D.89}$$



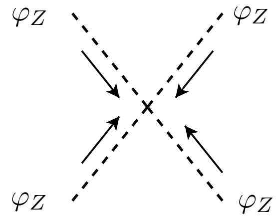
$$-\frac{i}{4}\, g^2\, \frac{m_h^2}{m_W^2} \tag{D.90}$$



$$-\frac{3}{4}\, i\, g^2\, \frac{m_h^2}{m_W^2} \tag{D.91}$$



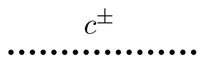
$$-\frac{i}{4}\, g^2\, \frac{m_h^2}{m_W^2} \tag{D.92}$$



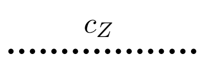
$$-\frac{3}{4}\, i\, g^2 \frac{m_h^2}{m_W^2} \tag{D.93}$$

D.4.11 Ghost Propagators

$$\frac{i}{k^2 + i\epsilon} \tag{D.94}$$

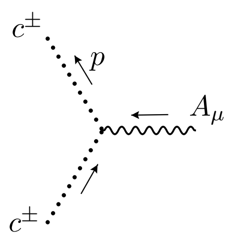


$$\frac{i}{k^2 - \xi m_W^2 + i\epsilon} \tag{D.95}$$

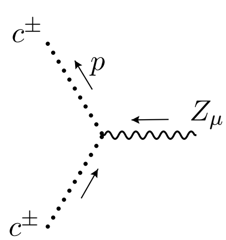


$$\frac{i}{k^2 - \xi m_Z^2 + i\epsilon} \tag{D.96}$$

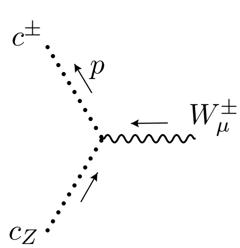
D.4.12 Ghost Gauge Interactions



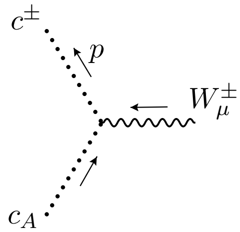
$$\mp ie\, p_\mu \tag{D.97}$$



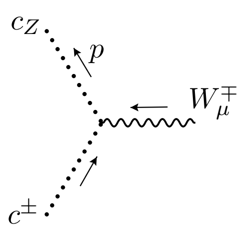
$$\pm ig\, \cos\theta_W\, p_\mu \tag{D.98}$$



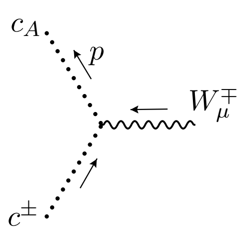
$$\mp ig\, \cos\theta_W\, p_\mu \tag{D.99}$$



$$\pm\, ie\, p_\mu \tag{D.100}$$

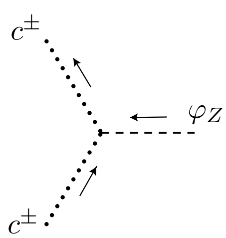


$$\mp\, ig\, \cos\theta_W\, p_\mu \tag{D.101}$$

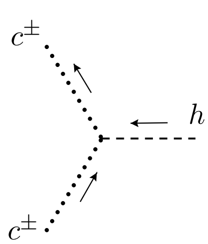


$$\pm\, ie\, p_\mu \tag{D.102}$$

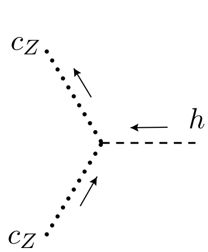
D.4.13 Ghost Higgs and Ghost Goldstone Interactions



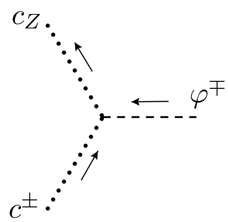
$$\pm\,\frac{g}{2}\,\xi\, m_W \tag{D.103}$$



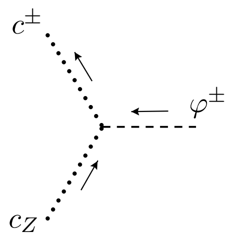
$$-\frac{i}{2}\, g\, \xi\, m_W \tag{D.104}$$



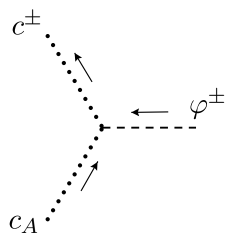
$$-\frac{ig}{2\cos\theta_W}\,\xi\, m_Z \tag{D.105}$$



$$\frac{i}{2}\, g\, \xi\, m_Z \tag{D.106}$$



$$-ig\,\frac{\cos 2\theta_W}{2\cos\theta_W}\,\xi\, m_W \tag{D.107}$$



$$ie\,\xi\, m_W \tag{D.108}$$

Bibliography

[1] J. Bjorken and S. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1962).

[2] S. N. Gupta, Proc. Phys. Soc. **63**, 681 (1950).

[3] K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

[4] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Dover Publications, 1962).

[5] J. C. Romão, *Introdução à Teoria do Campo* (IST, 2012), Available online at http://porthos.ist.utl.pt/ftp/textos/itc.pdf.

[6] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, N.Y., 1964).

[7] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[8] H. Lehmann, K. Symanzik and W. Zimmerman, Nuovo Cimento **1**, 2005 (1955).

[9] J. C. Romão, *Utilities for One Loop Calculations* (IST, 2004), Available online at http://porthos.ist.utl.pt/OneLoop/.

[10] G. Passarino and M. J. G. Veltman, Nucl. Phys. **B160**, 151 (1979).

[11] A. Denner, Fortschr. Phys. **41**, 307 (1993).

[12] R. Mertig, M. Bohm and A. Denner, Comput. Phys. Commun. **64**, 345 (1991).

[13] R. Mertig, http://www.feyncalc.org .

[14] T. Hahn and M. Perez-Victoria, Comput. Phys. Commun. **118**, 153 (1999), [hep-ph/9807565].

[15] T. Hahn, http://www.feynarts.de/looptools .

[16] G. J. van Oldenborgh, Comput. Phys. Commun. **66**, 1 (1991).

[17] L. Lavoura, Eur. Phys. J. **C29**, 191 (2003), [hep-ph/0302221].