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What can break the Wandzura–Wilczek relation?

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We analyze the breaking of the Wandzura–Wilczek relation for the g_2 structure function, emphasizing its connection with transverse momentum dependent parton distribution functions. We find that the relation is broken by two distinct twist-3 terms, and clarify how these can be separated in measurements of double-spin asymmetries in semi-inclusive deep inelastic scattering. Through a quantitative analysis of available g_2 data we also show that the breaking of the Wandzura–Wilczek relation can be as large as 15–40% of the size of g_2.

I. INTRODUCTION

The spin structure of the nucleon remains one of the most challenging and controversial problems in hadronic physics [1, 2, 3]. Experimentally it is now known, through many careful measurements of the nucleon's g_1 structure function, that quarks carry only some 30% of the proton's longitudinal spin, a feature which is now qualitatively understood [4]. Moreover, polarized *pp* scattering observables [5] and open charm production in deep inelastic scattering [6] suggest that gluons carry an even smaller fraction of the longitudinal spin. Presumably, the remainder arises from quark and gluon orbital angular momentum.

Although less attention has been paid to it, there are a number of intriguing questions associated with the transverse spin structure of the nucleon. An example is the study of the g_2 structure function, which only in recent years has been probed experimentally with high precision. Unlike all other inclusive deep-inelastic scattering (DIS) observables, the g_2 structure function is unique in directly revealing information on the long-range quark-gluon correlations in the nucleon. In the language of the operator product expansion (OPE) these are parametrized through matrix elements of higher twist operators, which characterize the strength of nonperturbative multi-parton interactions. (In the OPE "twist" is defined as the mass dimension minus the spin of a local operator.) In other inclusive structure functions higher twist contributions are suppressed by powers of the four-momentum transfer squared Q^2, whereas in g_2 these appear at the same order as the leading twist.

As discussed by Wandzura and Wilczek [7], the leading twist contribution to the g_2 structure function, which is denoted by $g_2^{\rm WW}$, can be expressed in terms of the leading twist (LT) part of the g_1 structure function,

$$g_2^{\rm WW}(x_B) = -g_1^{\rm LT}(x_B) + \int_{x_B}^1 \frac{dy}{y} g_1^{\rm LT}(y) \ , \qquad (1)$$

where x_B is the Bjorken scaling variable, and we suppress the explicit dependence of the structure functions on Q^2. The Wandzura–Wilczek (WW) relation asserts that the total g_2 structure function is given by the leading twist approximation (1),

$$g_2(x_B) \stackrel{?}{=} g_2^{\rm WW}(x_B) \ , \qquad (2)$$

which would be valid in the absence of higher twist contributions. In this case the g_2 structure function would also satisfy the Burkhardt–Cottingham (BC) sum rule [8],

$$\int_0^1 dx_B \ g_2(x_B) = 0 \ . \qquad (3)$$

Its violation would also signal the presence of twist-3 or higher contributions. Unlike the WW relation, however, the validity of the BC sum rule (which is yet to be conclusively demonstrated experimentally [9, 10]) would not necessarily imply that higher twist terms vanish [11, 12].

In this paper we explore the physics that can lead to the breaking of the WW relation in QCD, preliminary results for which have appeared in Ref. [13]. In Sec. II we present a detailed theoretical analysis of quark-quark and quark-gluon-quark correlation functions, and discuss the so-called Lorentz invariance relations and equations of motion relations. From these we show that the WW relation is valid if pure twist-3 and quark mass terms are neglected, in agreement with OPE results. We find that there are *two* distinct contributions with twist 3, denoted by $\widetilde{g}_T$ and $\widehat{g}_T$, which correspond to two different "projections" of the quark-gluon-quark correlator. An explicit demonstration of our findings is made for the case of a point-like quark target, which shows that the twist-3 terms can in principle be as large as the twist-2 terms.

In Sec. III we discuss the phenomenology of the WW relation for both the proton and neutron, and find that the available data from SLAC and Jefferson Lab indicate a breaking of the WW relation at the level of 15–40% of the size of g_2 within the 1-σ confidence level. The two twist-3 terms can be separated by measuring, in addition to g_2, the function $g_{1T}^{(1)}$ in semi-inclusive DIS, as we outline in Sec. IV. There we explain the importance of measuring the two twist-3 functions $\widetilde{g}_T$ and $\widehat{g}_T$ separately, and the insight which this can bring, for example, to understanding the physics of quark-gluon-quark correlations [14], or to determining the QCD evolution kernel for g_2 and the large momentum tails of transverse momentum distributions (TMDs).

Finally, in Sec. V we briefly summarize our findings. Some technical details for the analysis with a non-lightlike Wilson line and the model calculation of parton correlation functions are presented in the appendices.

II. THEORETICAL ANALYSIS

In this section we set forth the framework for our analysis of the WW relation by first defining quark-quark correlation functions and examining their most general Lorentz and Dirac decomposition. This is followed by a discussion of quark-gluon-quark correlators, and of the Lorentz invariance and equations of motion relations from which a generalization of the WW relation is derived.

A. Parton correlation functions

The quark-quark correlator for a quark of momentum k in a nucleon with momentum P and spin S is defined as

$$\Phi^a_{ij}(k,P,S;v) = \int \frac{d^4\xi}{(2\pi)^4}\, e^{ik\cdot\xi} \times \langle P,S\,|\,\overline{\psi}^a_j(0)\,\mathcal{W}^v_{(0,\infty)}\,\mathcal{W}^v_{(\infty,\xi)}\,\psi^a_i(\xi)\,|\,P,S\rangle\,, \qquad (4)$$

where the quark fields ψ^a_i are labeled by the flavor index a and Dirac index i. For ease of notation, the Dirac and flavor indices will be suppressed in the following. The operator $\mathcal{W}^v_{(0,\infty)}$ represents a Wilson line (or gauge link) from the origin to infinity along the direction specified by the vector v, and is necessary to ensure gauge invariance of the correlator. The gauge links contain transverse pieces at infinity [15, 16] and their precise form depends on the process [17, 18]. In a covariant gauge, the dependence of the correlator Φ on v is evident from the presence of the Wilson line in the direction conjugate to v. In light-cone gauges the vector v is orthogonal to the gauge field A, $v\cdot A = 0$, and the dependence on v appears explicitly only in the gauge field propagators.

In tree-level analyses of semi-inclusive DIS (SIDIS) [19, 20] or the Drell-Yan process [21, 22, 23] v is identified with the light-cone vector n_-, where $n_-^2 = 0 = n_+^2$ and $n_-\cdot n_+ = 1$, with n_+ the corresponding orthogonal light-cone vector proportional to P (up to mass corrections). However, factorization theorems beyond tree-level [24, 25, 26, 27] demand a slightly non-lightlike vector v in order to regularize light-cone divergences. We leave a more detailed discussion of the effect of the choice of v to Appendix A and consider $v = n_-$ unless otherwise specified.

The correlator Φ can be parametrized in terms of structures built from the four vectors P, S, k and v. Its full decomposition has been studied in Ref. [28] (and further generalized in Ref. [29]). It contains 12 scalar functions A_i already known from Refs. [19, 30], and 20 scalar functions B_i which are multiplied by factors depending explicitly on v, which were first introduced in Ref. [31] and called parton correlation functions (PCFs) in Ref. [27]. For brevity we consider only those terms of the full decomposition [28] which are necessary for the present analysis,

$$\begin{aligned}\Phi(k,P,S;v) &= M\not{S}\gamma_5 A_6 + \frac{k\cdot S}{M}\not{P}\gamma_5 A_7 \\ &+ \frac{k\cdot S}{M}\not{k}\gamma_5 A_8 + M\frac{(S\cdot v)}{(P\cdot v)}\not{P}\gamma_5 B_{11} + M\frac{(S\cdot v)}{(P\cdot v)}\not{k}\gamma_5 B_{12} \\ &+ M\frac{(k\cdot S)}{(P\cdot v)}\not{v}\gamma_5 B_{13} + M^3\frac{(S\cdot v)}{(P\cdot v)^2}\not{v}\gamma_5 B_{14} + \cdots\ ,\end{aligned} \qquad (5)$$

where the nucleon mass M is explicitly included to ensure that all PCFs have the same mass dimension. (Any other hadronic scale, such as $\Lambda_{\rm QCD}$, can be chosen, but we follow the choice used in the TMD literature [19].)

The PCFs A_i and B_i are in principle functions of the scalar products $P\cdot k$, k^2, $P\cdot v$, $k\cdot v$ and v^2. However, because the correlator Φ is invariant under the scale transformation $v\to\lambda v$, where λ is a constant, the PCFs can only depend on ratios of the scalar products, $P\cdot k$, k^2 and $k\cdot v/P\cdot v$. We therefore choose the PCFs to depend on the parton virtuality $\tau \equiv k^2$, on $\sigma \equiv 2P\cdot k$, and on the parton momentum fraction $x = k\cdot n_-/P\cdot n_-$. We emphasize that the explicit dependence on x is induced in general by the v dependence of the correlator Φ.

These considerations apply even when the correlator is integrated over the parton transverse momentum, and in fact the B_i terms give contributions also to standard collinear parton distribution functions (PDFs), such as the helicity distribution — see Eq. (22) below. However, when the correlator is fully integrated over d^4k the B_i no longer contribute; indeed

$$\int d^4k\,\Phi(k,P,S;v) = \langle P,S\,|\,\overline{\psi}(0)\,\psi_i(0)\,|\,P,S\rangle\ , \qquad (6)$$

and the dependence of the integral on v disappears because $\mathcal{W}^v_{(0,\infty)}\mathcal{W}^v_{(\infty,0)} = 1$.

In TMD factorization the relevant objects are the integrals of $\Phi(k,P,S;v)$ over $k^- = k_\mu n_+^\mu$,

$$\Phi(x,\boldsymbol{k}_T) = \int dk^-\,\Phi(k,P,S;v) = \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3}\, e^{ik\cdot\xi}\,\langle P,S\,|\,\overline{\psi}(0)\,\mathcal{W}^v_{(0,\infty)}\,\mathcal{W}^v_{(\infty,\xi)}\,\psi(\xi)\,|\,P,S\rangle\Big|_{\xi^+=0}\,. \qquad (7)$$

It is also useful to define the $\boldsymbol{k}_T$-integrated correlators

$$
\begin{aligned}
\Phi(x) &= \int d^2\boldsymbol{k}_T\, \Phi(x,\boldsymbol{k}_T) = \int \frac{d\xi^-}{2\pi}\, e^{ik\cdot\xi}\, \langle P,S\,|\,\overline{\psi}(0)\, \mathcal{W}^v_{(0,\infty)}\, \mathcal{W}^v_{(\infty,\xi)}\, \psi(\xi)\,|\,P,S\rangle\Big|_{\xi^+=\xi_T=0} \\
&\overset{\text{LC}}{=} \int \frac{d\xi^-}{2\pi}\, e^{ik\cdot\xi}\, \langle P,S\,|\,\overline{\psi}(0)\, \psi(\xi)\,|\,P,S\rangle\Big|_{\xi^+=\xi_T=0}\,, \qquad (8)\\
\Phi^\alpha_\partial(x) &= \int d^2\boldsymbol{k}_T\, k_T^\alpha \Phi(x,\boldsymbol{k}_T) = \int \frac{d\xi^-}{2\pi}\, e^{ik\cdot\xi}\, \langle P,S\,|\,\overline{\psi}(0)\, \mathcal{W}^v_{(0,\infty)}\, i\partial_T^\alpha\, \mathcal{W}^v_{(\infty,\xi)}\, \psi(\xi)\,|\,P,S\rangle\Big|_{\xi^+=\xi_T=0}\,, \\
&\overset{\text{LC}}{=} \int \frac{d\xi^-}{2\pi}\, e^{ik\cdot\xi}\, \langle P,S\,|\,\overline{\psi}(0)\, i\partial_T^\alpha\, \psi(\xi)\,|\,P,S\rangle\Big|_{\xi^+=\xi_T=0}\,. \qquad (9)
\end{aligned}
$$

where LC refers to the correlators in the light-cone gauge. The correlator Φ^α_∂ actually depends on the detailed form of the Wilson line, and changes, for example, between the SIDIS and Drell–Yan processes. However, for our discussion this will not be relevant and we can consider the average between the correlator for SIDIS and Drell–Yan [15].

For any correlator, we can introduce the Dirac projections

$$
\Phi^{[\Gamma]} \equiv \frac{1}{2}\mathrm{Tr}[\Gamma\Phi]\,, \qquad (10)
$$

where Γ is a matrix in Dirac space. The transverse momentum dependent parton distribution functions then appear as terms of the general decomposition of the projections $\Phi^{[\Gamma]}(x,\boldsymbol{k}_T)$, the full list of which can be found in Refs. [20, 28]. Usually a TMD is defined to have "twist" equal to n if in the expansion of the correlator it appears at order $(M/P^+)^{n-2}$, where $P^+ = P_\mu n_-^\mu$. In physical observables, TMDs of twist n appear with a suppression factor $(M/Q)^{n-2}$ compared to twist-2 TMDs. We finally note that at present TMD factorization for SIDIS has been proven for twist-2 TMDs only [24], and problems are known to occur at twist 3, indicating that the formalism may not yet be complete [32, 33].

For the following discussion we shall need the definitions of certain TMDs (note that here and in the following α is restricted to be a transverse index) [20]

$$
\Phi^{[\gamma^+\gamma_5]}(x,\boldsymbol{k}_T) = S_L\, g_{1L}(x,\boldsymbol{k}_T^2) + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M}\, g_{1T}(x,\boldsymbol{k}_T^2)\,, \qquad (11)
$$

$$
\begin{aligned}
\Phi^{[\gamma^\alpha\gamma_5]}(x,\boldsymbol{k}_T) = \frac{M}{P^+}\Bigg[& S_T^\alpha\, g_T(x,\boldsymbol{k}_T^2) + S_L\, \frac{k_T^\alpha}{M}\, g_L^\perp(x,\boldsymbol{k}_T^2) \\
& - \frac{k_T^\alpha\, k_T^\rho + \frac{1}{2}\,\boldsymbol{k}_T^2\, g_T^{\alpha\rho}}{M^2}\, S_{T\rho}\, g_T^\perp(x,\boldsymbol{k}_T^2) \\
& - \frac{\epsilon_T^{\alpha\rho} k_{T\rho}}{M}\, g^\perp(x,\boldsymbol{k}_T^2)\Bigg], \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
\Phi^{[i\sigma^{\alpha+}\gamma_5]}(x,\boldsymbol{k}_T) = {} & S_T^\alpha\, h_1(x,\boldsymbol{k}_T^2) + S_L\, \frac{p_T^\alpha}{M}\, h_{1L}^\perp(x,\boldsymbol{k}_T^2) \\
& - \frac{p_T^\alpha p_T^\rho - \frac{1}{2}\, p_T^2\, g_T^{\alpha\rho}}{M^2}\, S_{T\rho}\, h_{1T}^\perp(x,\boldsymbol{k}_T^2) \\
& - \frac{\epsilon_T^{\alpha\rho} p_{T\rho}}{M}\, h_1^\perp(x,\boldsymbol{k}_T^2)\,, \qquad (13)
\end{aligned}
$$

where $S_L = S^+M/P^+$, and the transverse tensors $g_T^{\alpha\rho}$ and $\epsilon_T^{\alpha\rho}$ are defined as

$$
\begin{aligned}
g_T^{\alpha\rho} &= g^{\alpha\rho} - n_+^\alpha n_-^\rho - n_-^\alpha n_+^\rho\,, \qquad (14)\\
\epsilon_T^{\alpha\rho} &= \epsilon^{\alpha\rho\beta\sigma}(n_+)_\beta (n_-)_\sigma. \qquad (15)
\end{aligned}
$$

For the $\boldsymbol{k}_T$-integrated distributions, we correspondingly have

$$
\begin{aligned}
\Phi^{[\gamma^+\gamma_5]}(x) &= S_L\, g_{1L}(x)\,, \qquad (16)\\
\Phi^{[i\sigma^{\alpha+}\gamma_5]}(x) &= S_T^\alpha\, h_1(x)\,, \qquad (17)\\
\Phi_\partial^{\alpha[\gamma^+\gamma_5]}(x) &= S_T^\alpha M g_{1T}^{(1)}(x)\,, \qquad (18)\\
\Phi^{[\gamma^\alpha\gamma_5]}(x) &= \frac{M}{P^+} S_T^\alpha\, g_T(x)\,, \qquad (19)
\end{aligned}
$$

where for any TMD $f = f(x,\boldsymbol{k}_T^2)$ we define

$$
\begin{aligned}
f^{(1)}(x,\boldsymbol{k}_T^2) &= \frac{\boldsymbol{k}_T^2}{2M^2} f(x,\boldsymbol{k}_T^2)\,, \qquad (20)\\
f^{(1)}(x) &= \int d^2\boldsymbol{k}_T\, f^{(1)}(x,\boldsymbol{k}_T^2)\,. \qquad (21)
\end{aligned}
$$

To avoid confusion with the structure function g_1, here we use the notation g_{1L} also for the helicity-dependent PDF, contrary to what is used in some of the TMD literature [20].

The connection between the TMDs and the A_i and B_i amplitudes has been worked out in detail in the Appendix of Ref. [34] for $v = n_-$. In Appendix A we extend these results to a non-lightlike vector v. We shall not repeat here the calculations but only quote the results relevant for our discussion, namely

$$
\begin{aligned}
g_{1L}(x,\boldsymbol{k}_T^2) = \int d\sigma d\tau\, & \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{k}_T^2) \\
& \times \Big(-A_6 - B_{11} - xB_{12}) \\
& \quad - \frac{\sigma - 2xM^2}{2M^2}(A_7 + xA_8)\Big)\,, \qquad (22)\\
g_{1T}(x,\boldsymbol{k}_T^2) = \int d\sigma d\tau\, & \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{k}_T^2)
\end{aligned}
$$

$$\times \Big(A_7 + x A_8 \Big) , \tag{23}$$

$$g_T(x, \boldsymbol{k}_T^2) = \int d\sigma d\tau\, \delta(\tau - x\sigma + x^2 M^2 + \boldsymbol{k}_T^2) \times \Big(-A_6 - \frac{\tau - x\sigma + x^2 M^2}{2M^2} A_8 \Big) . \tag{24}$$

As anticipated, we see that B_i terms appear also in the function g_{1L}, which survives if the correlator is integrated over $\boldsymbol{k}_T$.

B. Lorentz invariance relations

From the preceding discussion, using the techniques discussed for example in Ref. [30], it is possible to derive the so-called Lorentz invariance relation (LIR)

$$g_T(x) = g_{1L}(x) + \frac{d}{dx}\, g_{1T}^{(1)}(x) + \widehat{g}_T(x) , \tag{25}$$

where the function $\widehat{g}_T$ is given by

$$\begin{aligned} \widehat{g}_T(x) = & \int d^2\boldsymbol{k}_T d\sigma d\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{k}_T^2) \\ & \times \Big[B_{11} + x B_{12} - \frac{\boldsymbol{k}_T^2}{2M^2} \Big(\frac{\partial A_7}{\partial x} + x \frac{\partial A_8}{\partial x} \Big) \Big] \\ & + \pi \int d\sigma d\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{k}_T^2)\, \boldsymbol{k}_T^2 \\ & \times \frac{\sigma - 2xM^2}{2M^2} \Big(A_7 + x A_8 \Big) \Big|_{\boldsymbol{k}_T^2 \to 0}^{\boldsymbol{k}_T^2 \to \infty} . \end{aligned} \tag{26}$$

The proper operator definition for $\widehat{g}_T$ can be traced back to Ref. [35] (see also [36, 37]), and requires the introduction of the twist-3 quark-gluon-quark correlator

$$\begin{aligned} i\Phi_F^\alpha(x,x') = & \int \frac{d\xi^- d\eta^-}{(2\pi)^2}\, e^{ik\cdot\xi}\, e^{i(k'-k)\cdot\eta}\, \delta_T^{\alpha\rho} \\ & \times \langle P | \overline{\psi}(0)\, \mathcal{W}^v_{(0,\eta)}\, ig\, F^{+\alpha}(\eta)\, \mathcal{W}^v_{(\eta,\xi)}\, \psi(\xi) | P \rangle \Big|_{\substack{\xi^+ = \xi_T = 0 \\ \eta^+ = \eta_T = 0}} \\ \stackrel{\mathrm{LC}}{=} & \int \frac{d\xi^- d\eta^-}{(2\pi)^2}\, e^{ik\cdot\xi}\, e^{i(k'-k)\cdot\eta} \\ & \times \langle P | \overline{\psi}(0)\, ig\, \partial_\eta^+ A_T^\alpha(\eta)\, \psi(\xi) | P \rangle \Big|_{\substack{\xi^+ = \xi_T = 0 \\ \eta^+ = \eta_T = 0}} , \end{aligned} \tag{27}$$

where $F^{+\alpha}$ is the gluon field strength tensor, k' is the gluon momentum, and $x' = k' \cdot n_- / P \cdot n_-$. Note that this correlator has been discussed in slightly different forms in Refs. [15, 38, 39, 40], for example. It can be expanded in terms of four scalar functions G_F, $\widetilde{G}_F$, H_F and E_F according to [39, 40]

$$\begin{aligned} i\Phi_F^\alpha(x,x') = \frac{M}{4} \Big[& G_F(x,x') i\epsilon_T^{\alpha\rho} S_{T\rho} + \widetilde{G}_F(x,x') S_T^\alpha \gamma_5 \\ & + H_F(x,x')\, S_L \gamma_5 \gamma_T^\alpha + E_F(x,x')\, \gamma_T^\alpha \Big] \not{n}_+ . \end{aligned} \tag{28}$$

Hermiticity and parity invariance impose that these functions are real and either odd or even under the interchange of x and x' [40],

$$G_F(x,x') = G_F(x',x) , \quad \widetilde{G}_F(x,x') = -\widetilde{G}_F(x',x) , \tag{29}$$

$$E_F(x,x') = E_F(x',x) , \quad H_F(x,x') = -H_F(x',x) . \tag{30}$$

We can then express the function $\widehat{g}_T$ as

$$\begin{aligned} M S_T^\alpha\, \widehat{g}_T(x) & = - \int dx'\, \frac{i\Phi_F^{\alpha[\gamma^+\gamma_5]}(x',x)}{(x-x')^2} \\ & = M S_T^\alpha\, \mathcal{P} \int dx'\, \frac{\widetilde{G}_F(x,x')/(x-x')}{x - x'} , \end{aligned} \tag{31}$$

where $\mathcal{P}$ denotes the principal value integral. (The need for the principal value was apparently overlooked in Refs. [36, 37].) The imaginary part arising from the pole at $x = x'$ cannot give a contribution to the LIR in Eq. (25), but rather contributes to a LIR involving the functions f_T and $f_{1T}^{\perp(1)}$, which we do not discuss here. We note that $\widehat{g}_T$ is a "pure twist-3" function, being part of the twist-3 correlator of Eq. (27). Since the integrand in Eq. (31) is antisymmetric in $x \leftrightarrow x'$, one obtains the nontrivial property

$$\int_0^1 dx\, \widehat{g}_T(x) = 0 . \tag{32}$$

In some analyses [30, 41] $\widehat{g}_T$ was believed to vanish because (i) the B_i parton correlation functions were not taken into account, (ii) the partial derivatives in Eq. (26) were neglected since an explicit x-dependence of the PCFs is generated only through the additional v-dependence, (iii) the boundary terms like the last terms in (26) were neglected. However, *none* of these assumptions is justified, as we show explicitly in a quark-target perturbative calculation in Appendix B. We can further draw some model-independent conclusions about the boundary terms by comparing them with the expression for g_{1T} in Eq. (23). Positivity bounds imply that $|\boldsymbol{k}_T^2 g_{1T}| \le M |\boldsymbol{k}_T| f_1$ [42], which is sufficient to guarantee that the $\boldsymbol{k}_T^2 = 0$ boundary term indeed vanishes. However, since g_{1T} behaves as $1/\boldsymbol{k}_T^4$ at large $\boldsymbol{k}_T$ [33], the boundary term at $\boldsymbol{k}_T^2 = \infty$ cannot be neglected.

If $\widehat{g}_T$ is nonetheless neglected, it is possible to express the twist-3 function g_T in terms of the twist-2 functions g_{1L} and g_{1T} [19, 30]. Relations of this kind have been often mistakenly called Lorentz invariance relations [19, 30, 43], but should *not* be confused with the correct Lorentz invariance relations such as in Eq. (25).

In the literature, model calculations have been used to argue that the pure twist-3 terms are not necessarily small [11, 44]. For example, $\widehat{g}_T$ can be computed perturbatively in the quark-target model of Refs. [37, 44]. Using Eqs. (38), (40) and (42) of Ref. [37] one finds

$$g_T(x) - g_{1L}(x) = \frac{\alpha_s}{2\pi} C_F \ln \frac{Q^2}{\mu^2} \big[2x - \delta(1-x) \big] , \tag{33}$$

$$g_{1T}^{(1)}(x) = -\frac{\alpha_s}{2\pi} C_F \ln \frac{Q^2}{\mu^2}\, x(1-x) , \tag{34}$$

where $C_F = 4/3$, μ is an infrared cutoff, and from Eq. (25) one has

$$\widehat{g}_T(x) = \frac{\alpha_s}{2\pi} C_F \ln\frac{Q^2}{\mu^2}\left[1-\delta(1-x)\right] . \tag{35}$$

From this calculation one can see that $\widehat{g}_T$ is comparable to the size of the other twist-2 functions. Moreover, its lowest moment vanishes, so that the nontrivial requirement of Eq. (32) is fulfilled. In Appendix C we confirm the above result (for $x < 1$ only) starting directly from the definition in Eq. (31).

C. Equations of motion relations

The equations of motion (EOM) for quarks, $\not{D}\psi = m\psi$, imply further relations between twist-2 and pure twist-3 functions (namely, between qq and qgq matrix elements). They are referred to as "equations of motion relations", and for the case of interest here are expressed as

$$g_{1T}^{(1)}(x) = x g_T(x) - x\widetilde{g}_T(x) - \frac{m}{M}\, h_1(x) \ , \tag{36}$$

where

$$\begin{aligned} xMS_T^\sigma\, \widetilde{g}_T(x) &= \mathcal{P}\int dx'\, \frac{i\Phi_{F\rho}^{[\gamma^+\gamma_T^\sigma\gamma_T^\rho\gamma_5]}(x',x)}{x-x'} \\ &= MS_T^\sigma\left(\mathcal{P}\int dx'\,\frac{G_F(x,x')}{2(x'-x)} + \int dx'\,\frac{\widetilde{G}_F(x,x')}{2(x'-x)}\right) . \end{aligned} \tag{37}$$

The full list of EOM relations can be found in Ref. [20].

Using Eq. (36) to eliminate $g_{1T}^{(1)}(x)$ in Eq. (25), one finds the differential equation

$$x\frac{d}{dx}\left(g_T - \widetilde{g}_T - \frac{m}{M}\frac{h_1}{x}\right) + g_{1L} - \widetilde{g}_T - \frac{m}{M}\frac{h_1}{x} + \widehat{g}_T = 0 \ . \tag{38}$$

Assuming that the relevant functions are integrable by $\int_x^1 (dy/y)$ and solving for g_T one finds

$$\begin{aligned} g_T(x) = &\int_x^1 \frac{dy}{y}\Big(g_{1L}(y) + \widehat{g}_T(y)\Big) \\ &+ \widetilde{g}_T^\star(x) + \frac{m}{M}(h_1/x)^\star(x) \ , \end{aligned} \tag{39}$$

where we have introduced the shorthand notation

$$f^\star(x) \equiv f(x) - \int_x^1 \frac{dy}{y} f(y) = -\int_x^1 \frac{dy}{y}\frac{d}{dy}\left[y f(y)\right] \ . \tag{40}$$

Note that if the integrals over x and y can be exchanged, the function f satisfies

$$\int_0^1 dx\, f^\star(x) = 0 \ . \tag{41}$$

In general, however, this is not necessarily true, as stressed in Refs. [11, 12].

In DIS on a quark-target, $\widetilde{g}_T$ can be computed using Eqs. (38) and (43) of Ref. [37], giving

$$\begin{aligned} x g_T(x) - \frac{m}{M} h_1(x) = &\frac{\alpha_s}{2\pi} C_F \ln\frac{Q^2}{\mu^2} \\ &\times\left[-x(1-x) + \frac{\delta(1-x)}{2}\right] , \end{aligned} \tag{42}$$

and using Eq. (36) we obtain

$$\widetilde{g}_T(x) = \frac{\alpha_s}{2\pi} C_F \ln\frac{Q^2}{\mu^2}\,\frac{\delta(1-x)}{2} . \tag{43}$$

Again we see that the twist-3 function $\widetilde{g}_T$ has a size comparable to that of the other twist-2 functions.

D. Breaking of the Wandzura–Wilczek relation

The hadronic tensor relevant for spin-dependent DIS structure functions is given by the standard Lorentz decomposition

$$\begin{aligned} W^{\mu\nu}(P,q) = &\frac{1}{P\cdot q}\varepsilon^{\mu\nu\rho\sigma} q_\rho \\ &\times\left[S_\sigma g_1(x_B,Q^2) + \Big(S_\sigma - \frac{S\cdot q}{P\cdot q}\,p_\sigma\Big) g_2(x_B,Q^2)\right] , \end{aligned} \tag{44}$$

where q is the momentum of the exchanged photon and $x_B = Q^2/(2P\cdot q)$ is the Bjorken variable. In general the structure functions g_1 and g_2 in Eq. (44) are functions of the physical (external) variable x_B and are given by convolutions of the hard γ^*–parton scattering coefficient functions and the relevant PDFs. At *leading order* in α_s, and including terms up to twist 3, they can be expressed in terms of the distributions g_{1L}^a and g_T^a (where we now explicitly include the flavor index a) introduced above as [20]

$$g_1(x) = \frac{1}{2}\sum_a e_a^2\, g_{1L}^a(x) \ , \tag{45}$$

$$g_1(x) + g_2(x) = \frac{1}{2}\sum_a e_a^2\, g_T^a(x) , \tag{46}$$

where for simplicity we have suppressed the Q^2 dependence. This then enables the difference between the full g_2 structure function and the WW approximation (1) to be written as

$$\begin{aligned} &g_2(x) - g_2^{\rm WW}(x) \\ &\quad = \frac{1}{2}\sum_a e_a^2\left(\widetilde{g}_T^{a\star}(x) + \frac{m}{M}(h_1^a/x)^\star(x) + \int_x^1 \frac{dy}{y}\widehat{g}_T^a(y)\right) , \end{aligned} \tag{47}$$

which represents the breaking of the WW relation. Note that the right-hand-side of Eq. (47) contains a quark mass term and *two* pure twist-3 terms. This is the main result of our analysis.

From Eq. (41) the x integral of the pure twist-3 function containing $\widetilde{g}_T^a$ and the mass term vanish. Using Eq. (32), and assuming that $\widehat{g}_T^a$ is regular enough to exchange the x and y integrals, we see that the $\widehat{g}_T^a$ term also vanishes. This implies that the above expression for g_2 satisfies the Burkhardt–Cottingham sum rule, Eq. (3), which is not in general guaranteed in the OPE [11, 12].

To obtain the WW relation one must neglect quark mass terms compared to the hadron mass (which can be reasonably done for light quarks), and either neglect both of the pure twist-3 terms, or assume that they cancel each other. The explicit quark-target perturbative calculations show that such a cancellation does not take place in general, and that the size of the WW breaking term can be comparable to the size of $g_2^{\rm WW}$,

$$g_2^{\rm WW}(x) = 1 - \delta(1-x) - \frac{\alpha_s}{2\pi} C_F \ln\frac{Q^2}{\mu^2} \times \Big[-\log\frac{(1-x)^2}{x} + \frac{3}{2}\,\delta(1-x) + \frac{2x^2}{(1-x)_+} + \frac{1}{2} \Big], \tag{48}$$

$$g_2(x) - g_2^{\rm WW}(x) = \delta(1-x) - 1 + \frac{\alpha_s}{2\pi} C_F \ln\frac{Q^2}{\mu^2} \times \Big[-\log\frac{(1-x)^2}{x} + \frac{1}{2}\,\delta(1-x) + \frac{2}{(1-x)_+} - \frac{3}{2} \Big]. \tag{49}$$

To obtain the above expressions we again made use of the results in Ref. [37]. Note that both $g_2^{\rm WW}$ and the total g_2 structure function in the quark-target model respect the BC sum rule.

III. CONSTRAINTS FROM DATA

It is often stated in the literature (see *e.g.* Ref. [45]) that the WW relation holds experimentally to a good accuracy. While there are certainly indications that this may indeed be so [9, 10], it is important to quantify the degree to which this relation holds and place limits on the size of its violation. This is the focus of this section.

We define the experimental WW breaking term $\Delta_{\rm ex}(x_B)$ as the difference between the experimental data and $g_2^{\rm WW}$,

$$\Delta_{\rm ex}(x_B, Q^2) = g_2^{\rm ex}(x_B, Q^2) - g_2^{\rm WW}(x_B, Q^2)\ , \tag{50}$$

with the Wandzura–Wilczek term computed using the LSS2006 (set 1) fit of the g_1 structure function [46]. The fit was performed including a phenomenological higher-twist term and target mass corrections in order to extract the pure twist-2 contribution, $g_1^{\rm LT}$. Using parametrizations of g_1 which do not account for the $1/Q$ power corrections [47, 48] would risk inadvertantly including spurious higher twist contributions when computing the WW approximation. We will demonstrate the impact of this difference by comparing our $g_2^{\rm WW}$ with $(g_2^{\rm WW})'$ computed using the total g_1 instead of $g_1^{\rm LT}$ in Eq. (1).

For proton targets we consider data from the SLAC E142 [49] and E155x [9] experiments, while for the neutron only the high-precision data sets from the SLAC E155x [9], and Jefferson Lab E99-117 [50] and E01-012 [51] experiments, obtained using ^{2}H or ^{3}He targets, are included. We checked explicitly that including the lower-precision data sets from Refs. [49, 52, 53] does not alter the fit results, except for artificially lowering the χ^2 values due to the much larger errors compared to the higher-precision data sets. In total, there are 52 data points for the proton and 18 points for the neutron, which are used separately to fit the WW breaking term Δ. Systematic errors, when quoted, were added in quadrature. For the shape of Δ we choose the form

$$\Delta(x_B, \alpha, \beta) = \alpha(1-x_B)^\beta\big((\beta+2)x_B - 1\big)\ , \tag{51}$$

which vanishes at $x_B = 1$, has no divergences at $x_B = 0$, fulfills the BC sum rule, and only has a single node. We do not consider its Q^2 QCD evolution. The evolution of g_2 has been studied numerically in Ref. [54] in the limit of a large number of colors. Most of the data considered lie in the range 1 GeV$^2 \le Q^2 \le 10$ GeV^2 where the effect of QCD evolution is rather mild, as indicated also by the results of the E01-012 experiment [51].

The goodness of the fit is estimated using the χ^2 function

$$\chi^2 = \sum_{i=1}^{N} \frac{\big[\Delta(x_{Bi}) - \Delta_{\rm ex}(x_{Bi})\big]^2}{\sigma_{\rm ex}^2(x_{Bi})}\ . \tag{52}$$

To quantify the size of the breaking term Δ compared to $g_2^{\rm WW}$ we define, for any interval $[x_B^{\rm min}, x_B^{\rm max}]$, the ratio of their quadratic integrals

$$r^2 = \frac{\int_{y^{\rm min}}^{y^{\rm max}} dy\, x_B^2 \Delta^2(x_B)}{\int_{y^{\rm min}}^{y^{\rm max}} dy\, x_B^2 g_2^2(x_B)}\ , \tag{53}$$

with $y = \log(x_B)$. The value of r is a good indicator of the relative magnitude of Δ and g_2, which change sign as a function of x_B. In practice we compute r at the average kinematics of the E155 experiment [9]. For the proton, we consider three intervals: the entire measured x_B range, [0.02,1]; the low-x_B region, [0.02,0.15]; and the high-x_B region, [0.15,1]. For the neutron, due to the limited statistical significance of the low-x_B data, we limit ourselves to quoting the value of r for the large-x_B region, [0.15,1].

The results of the fits are presented in Table I and Figure 1. The proton fit displays a positive WW breaking at large-x_B and a negative breaking at small-x_B. The size of the breaking term is typically 15–35% of the size of g_2 (see the r values in Table I). The neutron fit is completely dominated by the high-precision E01-012 data, which are concentrated on a very limited x_B range; it clearly indicates a 18–40% breaking of the WW relation at high x_B,

	proton	χ^2/d.o.f.	$r_{\rm tot}$	$r_{\rm low}$	$r_{\rm hi}$
(I)	$\Delta = 0$	1.22			
(II)	$\Delta = \alpha(1-x_B)^\beta\big((\beta+2)x_B - 1\big)$				
	$\alpha = 0.13 \pm 0.05$				
	$\beta = 4.4 \pm 1.0$	1.05	15–32%	18–36%	14–31%
	neutron				
(I)	$\Delta = 0$	1.66			
(II)	$\Delta = \alpha(1-x_B)^\beta\big((\beta+2)x_B - 1\big)$				
	$\alpha = 0.64 \pm 0.92$				
	$\beta = 24 \pm 10$	1.11			18–40%

TABLE I: Results of the 1-parameter fits of the WW breaking term Δ for different choices of its functional form. The value r of the relative size of the breaking term is computed for three regions of x_B: the entire measured x_B range, [0.02,1]; the low-x_B region, [0.02,0.15]; and the high-x_B region, [0.15,1]. See text for further details.

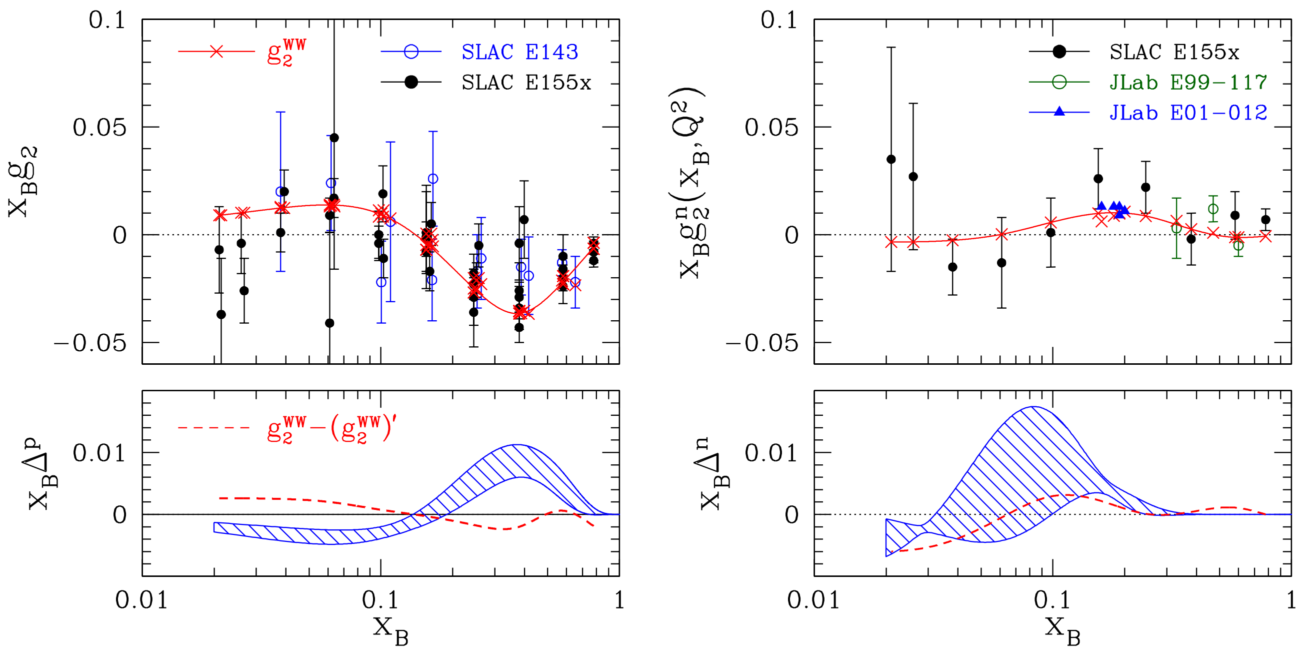

FIG. 1: *Top panels*: Experimental proton and neutron g_2 structure functions compared to $g_2^{\rm WW}$. The crosses represent $g_2^{\rm WW}$ computed at the experimental kinematics, while the solid lines are $g_2^{\rm WW}$ computed at the average Q^2 of the E155x experiment. Data points for the proton target [9, 49] have been slightly shifted in x_B for clarity. For the neutron only the high-precision data from [9, 50, 51] are included. *Bottom panels*: The WW-breaking term Δ fitted to $\Delta_{\rm ex}$ computed using the LSS2006 $g_1^{\rm LT}$ (hashed region). The dashed line represents $g_2^{\rm WW} - (g_2^{\rm WW})'$, the spurious HT contribution to Δ that would be obtained using the total g_1 to compute $\Delta_{\rm ex}$.

but cannot be used to conclude much at lower x_B values. A striking feature of the proton WW-breaking term in Fig. 1 is that it is comparable in size and *opposite* in sign to $g_2^{\rm WW} - (g_2^{\rm WW})'$. It is essential, therefore, to use fits of g_1 that subtract higher twist terms, which would otherwise largely cancel the proton WW-breaking term and obscure the violation of the WW relation. In the case of the neutron one would generally obtain an enhancement of the WW-breaking term, although the experimental uncertainties there are considerably larger.

In summary, we have found that the experimental data are consistent with a substantial breaking of the WW relation (2). Previous analyses have verified the WW relation only qualitatively, and using parametrizations which do not subtract higher twist terms in g_1. The present analysis clearly demonstrates that this can give the misleading impression that the WW relation holds to much better accuracy than it does in more complete analyses where the higher twist corrections have been consistently taken into account. More data are certainly needed to pin down the breaking of the WW relation to higher precision. New data are expected soon from the HERMES Collaboration and from the d2n (E06-014) and SANE (E07-003) experiments at Jefferson Lab [55, 56].

IV. TOWARD A DEEPER UNDERSTANDING OF QUARK-GLUON-QUARK CORRELATIONS

In the past, since the LIR-breaking $\widehat{g}_T$ term was not considered in Eq. (47) and the quark-mass term with h_1 was neglected, the breaking of the WW relation was considered to be a direct measurement of the pure twist-3 term $\widetilde{g}_T$. The presumed experimental validity of the WW relation was therefore taken as evidence that $\widetilde{g}_T$ is small. This observation was then generalized to assume that all pure twist-3 terms are small. In contrast, the present analysis shows that, precisely due to the presence of $\widehat{g}_T$, the measurement of the breaking of the WW relation does *not* provide information on a single pure twist-3 matrix element. Even if in future the WW relation were to be found to be satisfied to greater accuracy than the present data suggest, one could only conclude that the sum of the terms in (47) is small,

$$\sum_a e_a^2 \biggl(-\widetilde{g}_T^a(x) + \int_x^1 \frac{dy}{y} \Bigl(\widehat{g}_T^a(y) + \widetilde{g}_T^a(y) \Bigr) \biggr) \approx 0 \ . \quad (54)$$

This can occur either because $\widehat{g}_T^a$ and $\widetilde{g}_T^a$ are both small, or because they (accidentally) cancel each other. No information can be obtained on the size of the twist-3 quark-gluon-quark term $\widetilde{g}_T$ from the experimental data on g_2 alone. Note that these results were essentially already obtained in Ref. [34]. In that work, however, the authors considered the WW breaking to be small and *assumed* that $\widetilde{g}_T^a$ was small (which we argue is not necessarily the case), concluding that $\widehat{g}_T^a$ is also small.

Of course it is desirable to test our conclusions empirically. A reliable way to investigate $\widetilde{g}_T$ experimentally is through measurement of the function $g_{1T}^{(1)}$. This function is accessible in semi-inclusive deep inelastic scattering with transversely polarized targets and longitudinally polarized lepton beams (see, *e.g.*, the second line of Tab. IV in Ref. [41]). Preliminary data related to this function have been presented by the COMPASS Collaboration [57] and more are expected from the HERMES Collaboration and from the E06-010 experiment at Jefferson Lab [58]. Using the EOM relation (36) and assuming $m = 0$, one obtains

$$x\widetilde{g}_T(x) = x g_T(x) - g_{1T}^{(1)}(x) \ . \quad (55)$$

In combination with the measurement of the WW breaking, this can be used to determine the size of twist-3 function $\widehat{g}_T$. (Alternatively, one can use the LIR (25).)

The importance of separately studying $\widetilde{g}_T$ and $\widehat{g}_T$ resides in the fact that these are projections of different combinations of the twist-3 functions $G_F(x,x')$ and $\widetilde{G}_F(x,x')$. As with all other terms in the decomposition of the quark-gluon-quark correlator in Eq. (28), these functions are involved in the evolution equation of twist-3 collinear PDFs [59, 60], in the evolution of the transverse moments of the TMDs [61, 62], in the calculation of processes at high transverse momentum [38], and in the calculation of the high transverse momentum tails of TMDs [63, 64]. Ultimately, through a global study of all of these observables, one could simultaneously obtain better knowledge of twist-3 collinear functions and twist-2 TMDs, and at the same time test the validity of the formalism. Gathering as much information as one can on the quark-gluon-quark correlator is essential to reach this goal. The separation of the functions $\widetilde{g}_T$ and $\widehat{g}_T$ is an important first step in this direction.

V. CONCLUSIONS

In this analysis we have shown that the Wandzura–Wilczek relation for the g_2 structure function is violated by a quark mass term, and two distinct pure twist-3 contributions, containing the parton distribution functions $\widehat{g}_T$ and $\widetilde{g}_T$. As evident from their definitions in Eqs. (31) and (37) respectively, these correspond to two different projections of the general quark-gluon-quark correlator in Eq. (27). Their measurement can give unique and complementary information on twist-3 physics.

The two twist-3 functions have some interesting connections with the formalism of transverse momentum distributions. One of them is involved in the equation-of-motion relation expressed in Eq. (36), while the other is involved in the Lorentz invariance relation in Eq. (25). Both relations contain the same moment of the transverse momentum distribution g_{1T}. From the theoretical point of view, this is another intriguing example of the interplay between transverse momentum distributions and (collinear) twist-3 distributions. From the phenomenological point of view, this means that a measurement of the function g_{1T} in semi-inclusive DIS in principle allows one to separately measure $\widehat{g}_T$ and $\widetilde{g}_T$.

Although the Wandzura–Wilczek relation is often used to simplify the treatment of twist-3 and TMD physics, we stress that there are no compelling theoretical or phenomenological grounds supporting its validity. In fact, using the experimental information currently available, we were able to provide a quantitative assessment of the violation of the Wandzura–Wilczek relation. Assuming a simple functional form for the WW-breaking term, we found that it can be as large as 15–40% at the 1-σ confidence level.

As new data become available, it should be possible to better pin down the violation of the Wandzura–Wilczek relation and measure the transverse momentum distribution g_{1T} in semi-inclusive DIS. This will offer us a deeper look into the physics of quark-gluon-quark correlations and its connection to transverse momentum distributions.

Acknowledgments

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APPENDIX A: TMDS WITH A NON-LIGHTLIKE WILSON LINE DIRECTION

Factorization theorems beyond tree-level [24, 26, 27, 65, 66] demand a slightly non-lightlike vector v in order to regularize the lightcone (or rapidity) divergences [67, 68]. In Ref. [24] the Wilson line vector is chosen to be timelike and a parameter $\zeta^2 = 4(P\cdot v)^2/v^2$ is used as a regulator, with the requirement that $\zeta^2 \gg M^2, \boldsymbol{k}_T^2$. In other articles in the literature v has been chosen to be spacelike [25].

In addition to $k\cdot P$, k^2, $P\cdot v$ and $k\cdot v$, the PCFs A_i and B_i can now in principle depend also on v^2. We can derive the following relation between the invariants

$$\frac{k\cdot v}{P\cdot v} = ax + \frac{2\sigma}{\zeta^2(1+a)}\,, \tag{A1}$$

with $a = \sqrt{1-4M^2/\zeta^2}$. Neglecting terms of order M^2/ζ^2 and σ/ζ^2, the above expression reduces to x. We therefore conclude that the PCFs depend on σ, τ, x and additionally on ζ^2. To be precise, the definition of parton correlation functions in [27] involves an additional soft factor which is not included in the correlator Φ. The inclusion of the soft factor leads to an additional dependence on a gluon rapidity parameter. However, we leave this soft factor aside since it plays no role in our subsequent discussion.

The expressions for the TMDs in Eqs. (22), (23) and (24) then become

$$\begin{aligned} g_{1L}(x,\boldsymbol{k}_T^2,\zeta^2) = &\int d\sigma d\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{k}_T^2) \\ &\times \Big[-A_6 - a\Big(B_{11} + xB_{12} + \frac{4M^2}{\zeta^2(1+a)}B_{14}\Big) \\ &- \frac{\sigma - 2xM^2}{2M^2}\Big(A_7 + xA_8 + \frac{4M^2}{\zeta^2(1+a)}B_{13}\Big)\Big], \end{aligned} \tag{A2}$$

$$g_{1T}(x,\boldsymbol{k}_T^2,\zeta^2) = \int d\sigma d\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{k}_T^2) \times \Big[A_7 + xA_8 + \frac{4M^2}{\zeta^2(1+a)}B_{13}\Big], \tag{A3}$$

$$g_T(x,\boldsymbol{k}_T^2,\zeta^2) = \int d\sigma d\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{k}_T^2) \times \Big[-A_6 - \frac{\tau - x\sigma + x^2M^2}{2M^2}A_8\Big], \tag{A4}$$

The full expression for $\widehat{g}_T$ which generalizes Eq. (26) then becomes

$$\begin{aligned} \widehat{g}_T(x) = &\int d^2\boldsymbol{k}_T\, d\sigma d\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{k}_T^2) \\ &\times \Big[B_{11} + xB_{12} + \frac{4M^2}{\zeta^2(1+a)}B_{14} \\ &- \frac{\boldsymbol{k}_T^2}{2M^2}\Big(\frac{\partial A_7}{\partial x} + x\frac{\partial A_8}{\partial x} + \frac{4M^2}{\zeta^2(1+a)}\frac{\partial B_{13}}{\partial x}\Big)\Big] \\ &+ \pi\int d\sigma d\tau\, \delta(\tau - x\sigma + x^2M^2 + \boldsymbol{k}_T^2)\, \boldsymbol{k}_T^2 \\ &\times \frac{\sigma - 2xM^2}{2M^2}\Big(A_7 + xA_8 + \frac{4M^2}{\zeta^2(1+a)}B_{13})\Big)\Big|_{\boldsymbol{k}_T^2\to 0}^{\boldsymbol{k}_T^2\to\infty}. \end{aligned} \tag{A5}$$

APPENDIX B: PARTON CORRELATION FUNCTIONS FOR A QUARK TARGET

In this Appendix we compute the parton correlation functions relevant for our discussion of the WW relation for the case of a point-like quark target. The calculations are performed in the first non-trivial order in perturbative QCD (i.e., at order α_s) [37, 44]. To this end we insert a complete set of intermediate states into Eq. (4). To order α_s, only the vacuum state and a one-gluon state are relevant. The involved Feynman diagrams are shown in Fig. 2 (real gluon contributions) and Fig. 3 (virtual gluon contributions).

The correlator may be written as

$$\begin{aligned} \Phi_{ij}(k,P,S;v) = &\,\delta^{(4)}(P-k)\Phi_{ij}^{\rm vir}(m^2,\lambda^2,\zeta^2,\mu_R^2) \\ &+ \Phi_{ij}^{\rm real}(k,P,S;v)\,, \end{aligned} \tag{B1}$$

where $\Phi^{\rm vir}$ denotes the contributions from the vacuum intermediate state. Its kinematics is totally determined by the four-dimensional delta-function $\delta^{(4)}(P-k)$ and depends only on the quark mass m, with a small gluon mass λ serving here as an infrared regulator, and the parameter $\zeta^2 = 4(P\cdot v)^2/v^2$ which regulates lightcone divergences. By applying a renormalization procedure we can subtract ultra-violet divergences in $\Phi^{\rm vir}$, which introduces a dependence on the renormalization point μ_R^2. The virtual corrections can be written as

$$\begin{aligned} \Phi_{ij}^{\rm vir}(k,P,S;v) = &\,\delta^{(4)}(P-k)\langle P,S,d|\,\bar\psi_j(0)\,\mathcal{W}^v_{(0,\infty)}\,|0\rangle \\ &\times\langle 0|\,\mathcal{W}^v_{(\infty,0)}\,\psi_i(0)\,|P,S,d\rangle\,, \end{aligned} \tag{B2}$$

where the incoming on-shell quark is described by the state $|P,S,d\rangle$, with d a color index of the quark in the fundamental SU(3) representation. For the sake of brevity we will omit the explicit dependence on and summation over the color indices in the following. Since we work in Feynman gauge, possible contributions from gauge links at lightcone infinity are irrelevant [16].

The second contribution in Eq. (B1) is generated by one gluon in the intermediate state. To order α_s it is

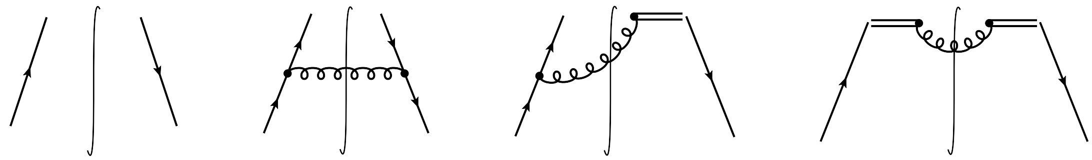

FIG. 2: Diagrams in the quark-target calculation involving only real gluons. The Hermitean conjugate diagrams, which are not shown, are also taken into account in the calculation.

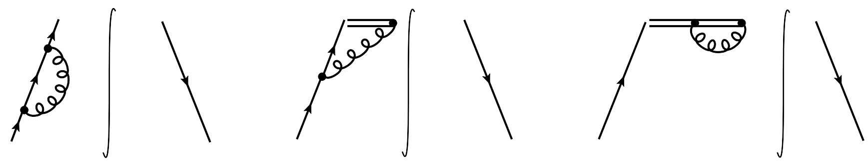

FIG. 3: As in Fig. 2 but for diagrams involving virtual gluons.

given by

$$\Phi_{ij}^{\rm real}(k,P,S;v) = \frac{1}{(2\pi)3}\sum_{\sigma,\beta}\delta^+((P-k)^2-\lambda^2) \\ \times \overline{M}_j^{\sigma,\beta}(k,P,S;v)\, M_i^{\sigma,\beta}(k,P,S;v)\,, \tag{B3}$$

with $\overline{M} \equiv M^\dagger\gamma^0$, $\delta^+(a^2) \equiv \delta(a^2)\Theta(a^0)$, σ denotes the polarization of the gluon in the intermediate state, and β is its color index in the adjoint representation of SU(3). The matrix element M is then represented by

$$M_i^{\sigma,\beta}(k,P,S;v) = \langle P-k,\sigma,\beta|\psi_i(0)|P,S,d\rangle \\ + ig\int_0^\infty d\lambda\,\langle P-k,\sigma,\beta|v\cdot A(\lambda v)\,\psi_i(0)|P,S,d\rangle\,, \tag{B4}$$

where $|P-k,\sigma,\beta\rangle$ denotes the intermediate gluon state with a color index β. The leading perturbative contribution in α_s to the matrix element M gives

$$M_i^{\sigma,\beta}(k,P,S;v) = -gt^\beta\Bigg(\frac{(\not{k}+m)\not{\varepsilon}^*_\sigma(P-k)}{[k^2-m^2+i\epsilon]} \\ + \frac{v\cdot\varepsilon^*_\sigma(P-k)}{[v\cdot(P-k)+i\epsilon]}\Bigg)_{il} u_l(P,S)\,, \tag{B5}$$

where $\varepsilon(P-k)$ denotes the gluon polarization vector and u is the quark spinor. The color flow is given by the color matrix t^β in the fundamental representation. Inserting (B5) into (B3) then yields

$$\Phi_{ij}^{\rm real}(k,P,S;v) = -\frac{\alpha_s}{(2\pi)^2}C_F\delta^+((P-k)^2-\lambda^2) \\ \times\Bigg[\frac{(\not{k}+m)\gamma_\mu(\not{P}+m)(1+\gamma_5\not{S})\gamma^\mu(\not{k}+m)}{[k^2-m^2+i\epsilon][k^2-m^2-i\epsilon]} \\ + \frac{(\not{P}+m)(1+\gamma_5\not{S})\not{v}(\not{k}+m)}{[k^2-m^2-i\epsilon][v\cdot(P-k)+i\epsilon]} \\ + \frac{(\not{k}+m)\not{v}(\not{P}+m)(1+\gamma_5\not{S})}{[k^2-m^2+i\epsilon][v\cdot(P-k)-i\epsilon]} \\ + \frac{v^2(\not{P}+m)(1+\gamma_5\not{S})}{[v\cdot(P-k)+i\epsilon][v\cdot(P-k)-i\epsilon]}\Bigg]_{ij}. \tag{B6}$$

The various parton correlation functions in Eq. (5) can be extracted from Eq. (B6) by decomposing the numerators in terms of the basis matrices 1, γ_5, γ^μ, $\gamma^\mu\gamma_5$ and $\sigma^{\mu\nu}$. In this way we obtain expressions for parton correlation functions at leading order in α_s for a quark target. In the following we list only the PCFs A_{6-8} and B_{11-14} which are relevant for the discussion of the Wandzura–Wilczek relation, *cf.* Eqs. (22)–(24). Setting $a=\sqrt{1-4m^2/\zeta^2}$, we find (to order α_s)

$$A_6^{\rm real}(\tau,\sigma,x,\zeta^2) = \frac{C_F\alpha_s}{2\pi^2}\delta^+(\tau-\sigma+m^2-\lambda^2) \\ \times\Bigg[\frac{\tau+m^2}{\big(\tau-m^2\big)^2} + \frac{(1+a)(1+ax)+2\sigma/\zeta^2}{\big[\tau-m^2\big]\big[(1+a)(1-ax)-2\sigma/\zeta^2\big]} \\ + \frac{2(1+a)^2}{\big[(1-ax)^2(1+a)^2\zeta^2-4\sigma(1-ax)(1+a)+4\sigma^2/\zeta^2\big]}\Bigg], \tag{B7}$$

$$A_7^{\rm real}(\tau,\sigma,x,\zeta^2) = 0, \tag{B8}$$

$$A_8^{\rm real}(\tau,\sigma,x,\zeta^2) = \frac{C_F\alpha_s}{2\pi^2}\delta^+(\tau-\sigma+m^2-\lambda^2) \\ \times\Bigg[\frac{-2m^2}{\big(\tau-m^2\big)^2}\Bigg], \tag{B9}$$

$$B_{11}^{\rm real}(\tau,\sigma,x,\zeta^2) = \frac{C_F\alpha_s}{2\pi^2}\delta^+(\tau-\sigma+m^2-\lambda^2) \times\left[\frac{-(1+a)}{\big[\tau-m^2\big]\big[(1+a)(1-ax)-2\sigma/\zeta^2\big]}\right], \quad \text{(B10)}$$

$$B_{12}^{\rm real}(\tau,\sigma,x,\zeta^2) = \frac{C_F\alpha_s}{2\pi^2}\delta^+(\tau-\sigma+m^2-\lambda^2) \times\left[\frac{(1+a)}{\big[\tau-m^2\big]\big[(1+a)(1-ax)-2\sigma/\zeta^2\big]}\right], \quad \text{(B11)}$$

$$B_{13}^{\rm real}(\tau,\sigma,x,\zeta^2) = \frac{C_F\alpha_s}{2\pi^2}\delta^+(\tau-\sigma+m^2-\lambda^2) \times\left[\frac{-(1+a)}{\big[\tau-m^2\big]\big[(1+a)(1-ax)-2\sigma/\zeta^2\big]}\right], \quad \text{(B12)}$$

$$B_{14}^{\rm real}(\tau,\sigma,x,\zeta^2) = 0\,. \quad \text{(B13)}$$

These results demonstrate that all terms in Eq. (A5) contribute to generate a nonzero $\widehat{g}_T$ since (i) the B_i terms are nonzero, (ii) the PCFs can depend explicitly on x, and (iii) the boundary term at $\boldsymbol{k}_T^2=\infty$ cannot be neglected.

APPENDIX C: QUARK TARGET TMDS AND PDFS AT $x<1$

We are now in a position to calculate the TMDs for a quark target defined in Eqs. (A2)–(A4), their $\boldsymbol{k}_T$-integrals appearing in the LIR of Eq. (25), and the function $\widehat{g}_T$ as defined in Eq. (A5). Similar calculations have been performed in [24, 37, 44, 69, 70]. Without entering into details, we note that the light-cone divergences occurring for $\zeta\to\infty$ can be moved to $x=1$, introducing the well-known "plus" distribution [24, 33]. If we restrict ourselves to the region $x<1$, the results are free of light-cone divergences and do not depend on ζ. In this region we can use either Eqs. (A2)–(A4) or (22)–(24). The resulting functions are then given by

$$g_{1L}(x<1,\boldsymbol{k}_T^2) = \frac{2C_F\alpha_s}{(2\pi)^2}\frac{1}{\boldsymbol{k}_T^2+x\lambda^2+(1-x)^2m^2} \times\left[1-x-\frac{2(1-x)(1-x(1-x))m^2}{\boldsymbol{k}_T^2+x\lambda^2+(1-x)^2m^2}+\frac{2x}{(1-x)_+}\right], \quad \text{(C1)}$$

$$g_{1T}(x<1,\boldsymbol{k}_T^2) = -\frac{2C_F\alpha_s}{(2\pi)^2}\frac{2x(1-x)m^2}{(\boldsymbol{k}_T^2+x\lambda^2+(1-x)^2m^2)^2}, \quad \text{(C2)}$$

$$g_T(x<1,\boldsymbol{k}_T^2) = \frac{2C_F\alpha_s}{(2\pi)^2}\frac{1}{\boldsymbol{k}_T^2+x\lambda^2+(1-x)^2m^2} \times\left[x-\frac{(1-x)^2(1+x)m^2}{\boldsymbol{k}_T^2+x\lambda^2+(1-x)^2m^2}+\frac{1+x}{(1-x)_+}\right]. \quad \text{(C3)}$$

When working with non-lightlike Wilson lines, it is not clear how to obtain the collinear parton distribution functions upon integration over the transverse momentum [24]. However, at the one-loop level these subtleties are relevant only at $x=1$. Since we restrict ourselves to the region $x<1$, we can safely compute collinear PDFs through $\boldsymbol{k}_T$-integration. For simplicity we choose an upper boundary Q for the $\boldsymbol{k}_T$-integration, and shift quark mass effects into the finite part by introducing an arbitrary infrared cutoff μ in order to obtain agreement with the results of Refs. [37, 44]. The divergent parts of the parton distributions, *i.e.*, the terms including the upper cutoff Q, are given by

$$g_{1L}(x<1) = \frac{\alpha_s C_F}{2\pi}\frac{1+x^2}{(1-x)_+}\ln\frac{Q^2}{\mu^2}, \quad \text{(C4)}$$

$$g_T(x<1) = \frac{\alpha_s C_F}{2\pi}\frac{1+2x-x^2}{(1-x)_+}\ln\frac{Q^2}{\mu^2}, \quad \text{(C5)}$$

$$g_{1T}^{(1)}(x<1) = -\frac{\alpha_s C_F}{2\pi}x(1-x)\ln\frac{Q^2}{\mu^2}. \quad \text{(C6)}$$

These results have appeared earlier in Refs. [24, 37, 44, 69, 70], but have been derived here for the first time starting from the PCFs.

For $\widehat{g}_T$ at $x<1$, using either Eq. (A5) or Eq. (26) we obtain

$$\widehat{g}_T(x<1) = \frac{\alpha_s C_F}{2\pi}\,\ln\frac{Q^2}{\mu^2}\,, \quad \text{(C7)}$$

confirming the result in Eq. (35), which was not obtained directly but rather using the LIR relation Eq. (25).

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