Generalized Negative Binomial Processes and the Representation of Cluster Structures

Mingyuan Zhou*

McCombs School of Business, The University of Texas at Austin

October 8, 2013

Abstract

The paper introduces the concept of a cluster structure to define a joint distribution of the sample size and its exchangeable random partitions. The cluster structure allows the probability distribution of the random partitions of a subset of the sample to be dependent on the sample size, a feature not presented in a partition structure. A generalized negative binomial process count-mixture model is proposed to generate a cluster structure, where in the prior the number of clusters is finite and Poisson distributed and the cluster sizes follow a truncated negative binomial distribution. The number and sizes of clusters can be controlled to exhibit distinct asymptotic behaviors. Unique model properties are illustrated with example clustering results using a generalized Pólya urn sampling scheme. The paper provides new methods to generate exchangeable random partitions and to control both the cluster-number and cluster-size distributions.

Keywords: Bayesian nonparametrics, clustering, count-mixture modeling, exchangeable cluster/partition probability functions, generalized Chinese restaurant/gamma/negative binomial processes, partition structure.

1 Introduction

A foundation of contemporary probabilistic clustering algorithms is sampling consistency, which requires the probability distribution of the random partitions of a subset of size j to be the same as that of the random partitions of j elements, uniformly at random subsampled without replacement from a sample of size $m \geq j$. In other words, in the prior, the clustering of a subset with j elements is independent of the number of elements remaining in the data set. In

^{*}Address for correspondence: Department of Information, Risk, and Operations Management, 2110 Speedway Stop B6500, Austin, TX 78712, USA. Email: mingyuan.zhou@mccombs.utexas.edu

practice, however, it is possible that the clustering is directly related to the sample size. E.g., to achieve an 8% projected annual return with minimized risk by investing in Lending Club, an online peer-to-peer lending platform where each note is purchased at 25 dollars, the partition of one thousand dollar total investment into 40 notes and the partition of one thousand dollars from a total investment of one million dollars could be drastically different. It is also common in real applications that the clusters of the current sample may be deleted, split or merged, as additional data points are observed. E.g., the Olinguito, which was originally classified as the Olingos, has been recently identified as a new mammal species and has been further classified into four subspecies (Helgen et al., 2013). Relaxing the constraint of sampling consistency, the paper aims to build a clustering strategy that takes the sample size into careful consideration.

Motivated by the success of the Ewens sampling formula (Ewens, 1972) in population genetics, where a sample is subsampled from a large population, Kingman (1978a,b) introduced the concept of a partition structure. Defining a family of consistent probability distributions for random partitions of positive integers, the partition structure is commonly featured in probabilistic clustering algorithms. For a random partition $\Pi_m = \{A_1, \cdots, A_l\}$ of the set $[m] := \{1, \cdots, m\}$, where $i \in A_k$ indicates that element i belongs to cluster $k = 1, \cdots, l$, the consistency of the probability functions $P(\Pi_1), \cdots, P(\Pi_\infty)$ requires that for a subset of size j in a sample of size $m \geq j$, its partition probability function $P(\Pi_j|m)$, which governs the probability distribution of the random partitions $(\Pi_j|m)$ obtained by marginalizing out m-j elements from Π_m , is the same for any $m \in \{j, j+1, \cdots\}$. Thus in a partition structure, $P(\Pi_j|m)$ is equal to $P(\Pi_j)$ and is independent of the sample size m. As further developed in (Pitman, 1995, 2006), if $P(\Pi_m)$ depends only on the number and sizes of A_k , regardless of their orders, then it is called an exchangeable partition probability function (EPPF) of Π_m , expressed as $P(\Pi_m = \{A_1, \cdots, A_l\}) = p_m(n_1, \cdots, n_l)$, where $n_k = |A_k|$. The sampling consistency amounts to an addition rule (Pitman, 2006, Gnedin et al., 2009) for the EPPF $P(\Pi_m)$ that $p_1(1) = 1$ and

$$p_m(n_1, \dots, n_l) = p_{m+1}(n_1, \dots, n_l, 1) + \sum_{k=1}^l p_{m+1}(n_1, \dots, n_k + 1, \dots, n_l),$$
(1)

with which Π_{m+1} can be constructed from Π_m by assigning element (m+1) to $A_{z_{m+1}}$ based on the prediction rule as $z_{m+1}|\Pi_m \sim \frac{p_{m+1}(n_1,\cdots,n_l,1)}{p_m(n_1,\cdots,n_l)}\delta_{l+1} + \sum_{k=1}^l \frac{p_{m+1}(n_1,\cdots,n_k+1,\cdots,n_l)}{p_m(n_1,\cdots,n_l)}\delta_k$. This paper calls an EPPF $P(\Pi_m)$ that satisfies (violates) the addition rule as a consistent (inconsistent) EPPF, which is referred as an infinite (finite) EPPF in Pitman (2006).

Moving beyond the Ewens sampling formula, various approaches, including the Pitman-Yor process (Perman et al., 1992, Pitman and Yor, 1997), Poisson-Kingman models (Pitman, 2003), species sampling (Pitman, 1996, Lee et al., 2013), stick-braking priors (Ishwaran and James, 2001), and Gibbs-type random partitions (Gnedin and Pitman, 2006), have been proposed to construct more general partition structures. See Müller and Quintana (2004), Lijoi and

Prünster (2010) and Müller and Mitra (2013) for reviews. Among these approaches, there has been increasing interest in normalized random measures with independent increments (NRMIs) (Regazzini et al., 2003, Lijoi et al., 2005, 2007, James et al., 2009), where a completely random measure with a finite and strictly positive total random mass is normalized to construct a random probability measure. For example, the normalized gamma process is a Dirichlet process (Ferguson, 1973), the marginalization of which leads to a variant of the Ewens sampling formula (Antoniak, 1974). More advanced completely random measures, such as the generalized gamma process of Brix (1999), can be employed to produce more general consistent exchangeable random partitions (Pitman, 2003, 2006, Lijoi et al., 2007), but the expressions of the consistent EPPF and its associated prediction rule usually involve integrations that are difficult to calculate. In addition, the construction of the random probability measure is independent of the sample size and due to normalization, the scale and mass parameters of the completely random measure become redundant to each other and the total random mass has to be strictly positive.

Recognizing the need to revise the probability distribution of the random partitions of a subset of size j along with the change of the sample size m, we propose a novel nonparametric Bayesian approach that allows the partition probability function $P(\Pi_j|m)$ to be dependent on m. If $P(\Pi_j|m)$ is exchangeable in its arguments, we refer it as a size-dependent EPPF. We model the sample size m as a Poisson random variable, whose mean is parameterized by the total random mass of a completely random measure G over a measurable space Ω . The total random mass $G(\Omega)$ is used to normalize G to obtain a random probability measure $G/G(\Omega)$, based on which the m points are then clustered. Linking m to $G(\Omega)$ with a Poisson distribution not only introduces dependence between the normalized random probability measure and the sample size, but also makes the scale of G become identifiable. As further discussed, the new model requires $G(\Omega)$ to be finite but not necessarily strictly positive. With G marginalized out, the joint distribution of m and its exchangeable random partition Π_m is called an exchangeable cluster probability function (ECPF). On observing a sample of size m, the EPPF $P(\Pi_m)$ directly comes from the ECPF $P(\Pi_m, m)$ divided by the marginal distribution of m, and the size-dependent EPPF $P(\Pi_j|m)$ is derived by marginalizing m-j elements from $P(\Pi_m)$.

Distinct from a partition structure that requires $P(\Pi_j|m) \equiv P(\Pi_j)$ and hence a consistent EPPF $P(\Pi_m)$ that satisfies the addition rule, the introduced model allows $P(\Pi_j|m)$ to be dependent on the sample size m and permits the generated exchangeable random partitions Π_m to be inconsistent in distribution as m varies. We call the introduced model as a count-mixture model, since as further shown in the paper, the a priori number of data points X(A) on each measurable subset $A \subset \Omega$ follows a Poisson distribution parameterized by G(A), and both a Poisson mixture and a compound Poisson distributions if G(A) is marginalized out. We introduce the concept of a cluster structure to characterize the count-mixture model, which extends Kingman's concept of a partition structure to allow $P(\Pi_j|m) \neq P(\Pi_j)$ for m > j. A

cluster structure specifies in its prior that the number of clusters follows a Poisson distribution with a finite mean, the sizes of these clusters are independently and identically (iid) drawn from a positive discrete distribution, and hence the sample size m follows a compound Poisson distribution. On observing a set of j data points, the model provides a size-dependent EPPF $P(\Pi_j|m)$ that can be modified to reflect the variation of the sample size $m = j, j + 1, \cdots$.

We consider a generalized negative binomial (NB) process count-mixture model where G is drawn from a generalized gamma process (Brix, 1999). A draw from the generalized NB process (gNBP) represents a cluster structure with a Poisson distributed finite number of clusters, whose sizes follow a truncated NB distribution. Marginally the sample size follows a generalized NB distribution. The EPPF of the gNBP generally violates the addition rule. A stochastic process with this EPPF is referred as the generalized Chinese restaurant process (CRP), which has a simple prediction rule that is used to develop a generalized Pólya urn sampling scheme. These three count distributions and the prediction rule are determined by a discount, a probability and a mass parameters. The discount parameter controls the asymptotic behaviors of both the number and sizes of clusters, whereas how the mass parameter is modeled determines whether the asymptotic behavior of the cluster number is similar or completely opposite to that of the normalized generalized gamma process mixtures in Lijoi et al. (2007), James et al. (2009).

The remainder of the paper is organized as follows. Section 2 briefly reviews some background materials. Section 3 introduces the concept of a cluster structure and constructs the count-mixture modeling framework. Section 4 introduces the gNBP, discusses how to control its asymptotic behaviors, and shows how its partition probability functions are dependent on the sample size. Section 5 presents a generalized Pólya urn sampling scheme for posterior simulation and Section 6 presents example clustering results.

2 Preliminaries

2.1 Generalized Gamma Process

A completely random measure (Kingman, 1967, 1993, Wolpert et al., 2011) G defined on the product space $\mathbb{R}_+ \times \Omega$, where $\mathbb{R}_+ = \{x : x \geq 0\}$ and Ω is a measurable space, assigns independent infinitely divisible random variables $G(A_i)$ to disjoint Borel sets $A_i \subset \Omega$, with characteristic functions $\mathbb{E}[e^{iuG(A)}] = e^{\int \int_{\mathbb{R}_+ \times A} (e^{iur}-1)\nu(drd\omega)}$. If the Lévy measure $\nu(drd\omega)$ satisfies $\iint_{\mathbb{R}_+ \times A} \min\{1, |r|\}\nu(drd\omega) < \infty$ for any $A \subset \Omega$, then G is well defined, even if the Poisson intensity $\nu^+ = \nu(\mathbb{R}_+ \times \Omega)$ is infinite. For simplicity, we consider G to be a homogenous completely random measure, whose atoms' weights are independent to their locations. We write the Lévy measure of a homogenous G as $\nu(drd\omega) = \rho(dr)G_0(d\omega)$, where G_0 is a continuous base measure over Ω , with a finite total mass $\gamma_0 = G_0(\Omega)$.

The gamma process (Ferguson, 1973, Wolpert and Ickstadt, 1998) and beta process (Hjort, 1990, Thibaux and Jordan, 2007, Zhou et al., 2011, 2012a) are two commonly used completely random measures. The widely used Dirichlet process (Ferguson, 1973, Antoniak, 1974, Escobar and West, 1995) is a normalized gamma process. Another example is the generalized gamma process $G \sim \text{gGaP}(G_0, a, c)$ defined on the product space $\mathbb{R}_+ \times \Omega$, where a < 1 is a discount parameter and 1/c is a scale parameter (Brix, 1999). It assigns independent infinitely divisible generalized gamma random variables $G(A_j) \sim \text{gGamma}(G_0(A_j), a, 1/c)$ to disjoint Borel sets $A_j \subset \Omega$, with Laplace transforms

$$\mathbb{E}[e^{-sG(A)}] = e^{-\frac{G_0(A)}{a}[(c+s)^a - c^a]}.$$
 (2)

The generalized gamma distribution $x \sim \text{gGamma}(\gamma, a, 1/c)$, with $\mathbb{E}[x] = \gamma c^{a-1}$, was independently suggested by Tweedie (1984) and Hougaard (1986) and also studied in Bar-Lev and Enis (1986), Aalen (1992), Jørgensen (1997). As $a \to 0$, since $\lim_{a\to 0} \frac{1-(1-p)^a}{ap^a} = -\ln(1-p)$ and hence $\lim_{a\to 0} \mathbb{E}[e^{-sx}] = (1+s/c)^{-\gamma}$, the generalized gamma distribution becomes a gamma distribution $x \sim \text{Gamma}(\gamma, 1/c)$, with $\mathbb{E}[x] = \gamma/c$. A generalized gamma distribution scaled with a positive constant $\beta > 0$ is distributed as $\beta x \sim \text{gGamma}(\gamma, a, a, a, a)$.

Using (2), the Lévy measure of the generalized gamma process can be expressed as

$$\nu(drd\omega) = \frac{1}{\Gamma(1-a)} r^{-a-1} e^{-cr} dr G_0(d\omega). \tag{3}$$

When $a \to 0$, we recover the gamma process; if 0 < a < 1 and $c \to 0$, we recover the a-stable process (Pitman, 2003, Lijoi et al., 2007); and if a = 1/2, we recover the inverse Gaussian process (Lijoi et al., 2005). A draw from $G \sim \text{gGaP}(G_0, a, c)$ can be expressed as

$$G = \sum_{k=1}^{K} r_k \delta_{\omega_k}, K \sim \text{Pois}(\nu^+), \ (r_k, \omega_k) \stackrel{iid}{\sim} \pi(dr d\omega),$$

where $r_k = G(\omega_k)$ is the weight for atom ω_k and $\pi(drd\omega)\nu^+ \equiv \nu(drd\omega)$. Except where otherwise specified, we consider a < 1 and c > 0 in this paper. If $0 \le a < 1$, since the Poisson intensity $\nu^+ = \nu(\mathbb{R}_+ \times \Omega) = \infty$ (i.e., $K = \infty$) and $\iint_{\mathbb{R}_+ \times \Omega} \min\{1, r\}\nu(drd\omega)$ is finite, a drawn from $\mathrm{gGaP}(G_0, a, c)$ consists of countably infinite atoms. On the other hand, if a < 0, then $\nu^+ = \frac{\gamma_0 c^a}{-a}$ and thus $K \sim \mathrm{Pois}(\frac{\gamma_0 c^a}{-a})$ (i.e., K is finite a.s.) and $r_k \stackrel{iid}{\sim} \mathrm{Gamma}(-a, 1/c)$.

2.2 Normalized Random Measure Mixtures

An NRMI mixture model (Regazzini et al., 2003, Lijoi and Prünster, 2010) emploies a normalized completely random measure $\tilde{G} = G/G(\Omega)$ to model the density of a data point x as f(x|G) =

 $\int_{\Omega} \kappa(x|\omega) d\widetilde{G}(\omega) = \sum_{k=1}^{K} \frac{r_k}{G(\Omega)} \kappa(x|\omega_k), \text{ where } \kappa(x|\omega) \text{ is a density function for } x \text{ with parameter } \omega,$ $G(\Omega) = \sum_{k=1}^{K} r_k \text{ is the total random mass that is required to be finite and strictly positive, and}$ $K = \infty \text{ is the total number of atoms. Note that the strict positivity of } G(\Omega) \text{ implies that } \nu^+ = \infty$ and hence $K = \infty$ (Regazzini et al., 2003, Lijoi and Prünster, 2010). By introducing a categorical latent variable $z|G \sim \sum_{k=1}^{K} \frac{r_k}{G(\Omega)} \delta_k$, we can augument f(x|G) as $f(x,z|G) = f(x|z,G)f(z|G) = \kappa(x|\omega_z) \frac{r_z}{\sum_{k=1}^{K} r_k}$. When a sample of m data points $\{x_i\}_{1,m}$ is observed, we have

$$f(\mathbf{z}|G,m) = \prod_{i=1}^{m} \frac{r_{z_i}}{\sum_{k=1}^{K} r_k} = \frac{1}{(\sum_{k=1}^{K} r_k)^m} \prod_{k=1}^{K} r_k^{n_k},$$
(4)

where $\mathbf{z} = (z_1, \dots, z_m)$ is a sequence of categorical random variables indicating the cluster memberships of data points $\mathbf{z} = (x_1, \dots, x_m)$, $n_k = \sum_{i=1}^m \delta(z_i = k)$ is the number of data points assigned to ω_k and $m = \sum_{k=1}^K n_k$. The ties between z_i define a random partition Π_m of [m].

Posterior simulation of G based on (4) is usually challenging as the term $(\sum_{k=1}^{K} r_k)^{-1}$ is not in a factorized form and the scale of G lacks identifiability. Following James et al. (2009), a specific auxiliary variable $\beta > 0$, with $(\beta|m, G(\Omega)) \sim \text{Gamma}(m, 1/G(\Omega))$ and thus $\mathbb{E}[\beta|m, G(\Omega)] = m/G(\Omega)$, can be introduced to yield a likelihood as

$$f(\mathbf{z}, \beta | G, m) = \frac{\beta^{m-1}}{(m-1)!} e^{-\beta \sum_{k=1}^{K} r_k} \prod_{k=1}^{K} r_k^{n_k}.$$
 (5)

Marginalizing out G leads to

$$f(\boldsymbol{z},\beta|m,\gamma_0,\rho) = \frac{\beta^{m-1}}{(m-1)!} e^{\gamma_0 \int_0^\infty (e^{-\beta r} - 1)\rho(dr)} \prod_{k:\omega_k \in \mathcal{D}_m} \gamma_0 \int_0^\infty r^{n_k} e^{-\beta r} \rho(dr), \tag{6}$$

where $\mathcal{D}_m = \{\omega_k\}_{k:n_k>0}$ includes all the points of discontinuity occupied by the m data points. Further marginalizing out β yields a consistent EPPF (Pitman, 2003, 2006), expressed as $f(\mathbf{z}|m,\gamma_0,\rho) = \int_0^\infty f(\mathbf{z},\beta|m,\gamma_0,\rho)d\beta$, where the integral is usually not analytic. For $G \sim \text{gGaP}(a,c,G_0)$, Lijoi et al. (2007) derived the analytic expressions of the EPPF and the associated prediction rule. These analytical expressions, however, are still not easy to calculate. To simplify the computation, by conditioning on the auxiliary variable β , James et al. (2009) further developed a generalized Blackwell-MacQueen sampling scheme by exploiting the fully factorized form of (6). Conditioning on the auxiliary variable, the EPPF of an NRMI mixture model is usually inconsistent but often becomes more amenable to posterior simulation (James et al., 2009), stimulating the development of a number of posterior simulation algorithms including Griffin and Walker (2011), Barrios et al. (2012), Favaro and Teh (2013). For the proposed count-mixture models, we show below that inconsistent exchangeable random partitions can be constructed without the need of an auxiliary variable.

3 Count-Mixture Modeling and Cluster Structures

Distinct from a partition structure, a cluster structure defines a probability distribution of all possible exchangeable random partitions, which are not constrained to be consistent in distribution as the sample size varies. Without imposing sampling consistency, there is a need to define a new mechanism to generate a sample of arbitrary size; and as discussed in Section 2.2, the completely random measure G in an NRMI mixture model is independent of the sample size and $G(\Omega)$ has to be strictly positive. To resolve these issues, we propose a "two birds with one stone" solution where we link the sample size m to the total random mass of G with a Poisson distribution as $m \sim \text{Pois}(\beta G(\Omega))$, i.e., $\mathbb{E}[m|\beta, G(\Omega)] = \beta G(\Omega)$, where β is a positive scale parameter. Since the m data points are further clustered based on the normalized random probability measure $G/G(\Omega) = \sum_{k=1}^K \frac{r_k}{\sum_{k=1}^K r_k} \delta_{\omega_k}$, equivalently we have

$$m = \sum_{k=1}^{K} n_k, \ n_k \sim \text{Pois}(\beta r_k). \tag{7}$$

Therefore, letting $m \sim \operatorname{Pois}(\beta G(\Omega))$ directly links the cluster sizes n_k to weights r_k with Poisson distributions (Zhou et al., 2012b). The mechanism to generate a sample of arbitrary size is now well defined and G can no longer be scaled freely. The new construction also allows $G(\Omega) = 0$, for which $m \equiv 0$ and there is no more need to calculate $G/G(\Omega)$. Allowing $G(\Omega) = 0$ with a nonzero probability relaxes the requirement of $\nu^+ = \infty$ (i.e., $K = \infty$). We call a model constructed in this way as a count-mixture model.

3.1 Count-Mixture Modeling

We construct a cluster structure via a count-mixture model as

$$x_i \sim \kappa(\omega_{z_i}), \ \omega_k \sim g_0, \ z_i \sim \sum_{k=1}^K \frac{r_k}{G(\Omega)} \delta_k, \ m \sim \text{Pois}(\beta G(\Omega)),$$
 (8)

where $i=1,\dots,m,\ g_0(d\omega):=G_0(d\omega)/\gamma_0$ is the base distribution and $K\leq\infty$. Using (4), the joint conditional likelihood of $\boldsymbol{z}=(z_1,\dots,z_m)$ and m becomes

$$f(\boldsymbol{z}, m | \beta, G) = f(\boldsymbol{z} | G, m) \operatorname{Pois}(m; \beta G(\Omega)) = \frac{\beta^m}{m!} e^{-\beta \sum_{k=1}^K r_k} \prod_{k=1}^K r_k^{n_k}$$
(9)

and hence the conditional likelihood of $n_{1:K} = (n_1, \dots, n_K)$ and m can be expressed as

$$f(n_{1:K}, m|\beta, G) = \frac{m!}{\prod_{k=1}^{K} n_k!} f(\mathbf{z}, m|\beta, G) = \delta_{\sum_{k=1}^{K} n_k}(m) \prod_{k=1}^{K} \text{Pois}(n_k; \beta r_k),$$
(10)

where $\delta_n(m)$ is a unit point mass at m = n. The conditional likelihoods in (9) and (10) are fully factorized by construction, a highly desired property for posterior simulation. Distinct from James et al. (2009), here an auxiliary variable is not needed and $K < \infty$ is permitted.

Note that β in (5) is an auxiliary variable that has to be sampled or marginalized out and hence shall never be fixed. Whereas β in (9) is a scale parameter that can be absorbed into the parameters of G and hence can be fixed at any positive constants. Without loss of generality, we first fix $\beta \equiv 1$ in count-mixture modeling in the following discussion. We will further show how the likelihood of a count-mixture model is related to that of an NRMI mixture by fixing the scale parameter of G but treating β as a random variable.

3.2 Cluster Structures

Below we show that a count-mixture model is characterized by a compound Poisson process, under which the a priori cluster structure becomes evident. As in (10), with $\beta \equiv 1$, we have $f(n_{1:K}|G) = \prod_{k=1}^K \operatorname{Pois}(n_k; r_k)$. With $X := \sum_{k=1}^K n_k \delta_{\omega_k}$, we obtain a Poisson process $X \sim \operatorname{PP}(G)$ such that $X(A) \sim \operatorname{Pois}(G(A))$ for each $A \subset \Omega$. With G marginalized out, X becomes a G mixed Poisson process, with characteristic functions $\mathbb{E}[e^{iuX(A)}] = \mathbb{E}[e^{G(A)(e^{iu}-1)}] = \exp\left\{\iint_{\mathbb{R}_+ \times A} (e^{(e^{iu}-1)r}-1)\nu(drd\omega)\right\}$. Since $e^{(e^{iu}-1)r}-1 = \sum_{n=1}^{\infty} \frac{r^n e^{-r}}{n!}(e^{iun}-1)$, we have

$$\mathbb{E}[e^{iuX(A)}] = \exp\left\{G_0(A) \sum_{n=1}^{\infty} (e^{iun} - 1) \int_0^{\infty} \frac{r^n e^{-r}}{n!} \rho(dr)\right\}$$

$$= \exp\left\{G_0(A) \int_0^{\infty} (1 - e^{-r}) \rho(dr) \left(\sum_{n=1}^{\infty} e^{iun} \frac{\int_0^{\infty} r^n e^{-r} \rho(dr)}{n! \int_0^{\infty} (1 - e^{-r}) \rho(dr)} - 1\right)\right\}.$$
 (12)

Theorem 1 (Poisson Mixture Process). $X := \sum_{k=1}^{K} n_k \delta_{\omega_k}$ can be considered as a draw from a G mixed Poisson process, with characteristic functions as in (11) and Lévy measure

$$\tilde{\nu}(dnd\omega) = \sum_{j=1}^{\infty} \int_0^{\infty} \frac{r^j e^{-r}}{j!} \rho(dr) \delta_j(dn) G_0(d\omega).$$

Theorem 2 (Compound Poisson Process). It is evident from (12) that the G mixed Poisson process is also a compound Poisson process, a random draw of which can be expressed as

$$X = \sum_{k=1}^{l} n_k \delta_{\omega_k}, \ l \sim \operatorname{Pois}\left(\gamma_0 \int_0^\infty (1 - e^{-r}) \rho(dr)\right), \ n_k \stackrel{iid}{\sim} \sum_{j=1}^\infty \frac{\int_0^\infty r^j e^{-r} \rho(dr)}{j! \int_0^\infty (1 - e^{-r}) \rho(dr)} \delta_j,$$

where $\int_0^\infty (1 - e^{-r}) \rho(dr) < \infty$ by definition and $\omega_k \stackrel{iid}{\sim} g_0$.

The compound Poisson representation dictates the count-mixture model to have a Poisson

distributed finite number of clusters, whose sizes follow a positive discrete distribution as $P(n = j | \rho) = \frac{\int_0^\infty r^j e^{-r} \rho(dr)}{j! \int_0^\infty (1 - e^{-r}) \rho(dr)}$, $j = 1, 2, \cdots$. The mass parameter γ_0 has a linear relationship with the expected number of clusters, but has no direct impact on the cluster-size distribution. Since the cluster indexes are unordered and exchangeable, without loss of generality, in the following discussion, we relabel the atoms in \mathcal{D}_m in order of appearance from 1 to $l := |\mathcal{D}_m|$ and then $z_i \in \{1, \dots, l\}$ for $i = 1, \dots, m, n_k > 0$ if and only if $1 \le k \le l$ and $n_k \equiv 0$ if $l < k \le K$.

Theorem 3 (Exchangeable Cluster/Partition Probability Functions). The count-mixture model has a fully factorized exchangeable cluster probability function (ECPF) as

$$f(\boldsymbol{z}, m | \gamma_0, \rho) = \mathbb{E}_G[f(\boldsymbol{z}, m | G)] = \frac{\gamma_0^l}{m!} e^{\gamma_0 \int_0^\infty (e^{-r} - 1)\rho(dr)} \prod_{k=1}^l \int_0^\infty r^{n_k} e^{-r} \rho(dr),$$

a marginal distribution for the sample size $m = X(\Omega)$ with probability generating function (PGF) $\mathbb{E}[t^m|\gamma_0,\rho] = e^{\gamma_0 \int_0^\infty (e^{-(1-t)r}-1)\rho(dr)}$ and probability mass function (PMF) $f_M(m|\gamma_0,\rho) = \frac{d^m(\mathbb{E}[t^m|\gamma_0,\rho])}{dt^m}|_{t=0}$, and an exchangeable partition probability function (EPPF) as

$$f(\boldsymbol{z}|m,\gamma_0,\rho) = f(\boldsymbol{z},m|\gamma_0,\rho)/f_M(m|\gamma_0,\rho).$$

With l^{-i} representing the number of clusters in $\mathbf{z}^{-i} := \mathbf{z} \setminus z_i$ and $n_k^{-i} := \sum_{j \neq i} \delta(z_j = k)$, we may reexpress the ECPF as

$$f(z_i, \boldsymbol{z}^{-i}, m | \gamma_0, \rho) = \frac{f(\boldsymbol{z}^{-i}, m - 1 | \gamma_0, \rho)}{m} \Big(\gamma_0 \int_0^\infty r e^{-r} \rho(dr) \delta(z_i = l^{-i} + 1) + \sum_{k=1}^{l^{-i}} \frac{\int_0^\infty r^{n_k^{-i} + 1} e^{-r} \rho(dr)}{\int_0^\infty r^{n_k^{-i}} e^{-r} \rho(dr)} \delta(z_i = k) \Big).$$

Marginalized out z_i with $f(\boldsymbol{z}^{-i}, m|\gamma_0, \rho) = \sum_{z_i=1}^{l^{-i}+1} f(z_i, \boldsymbol{z}^{-i}, m|\gamma_0, \rho)$, we have

$$f(\boldsymbol{z}^{-i}, m | \gamma_0, \rho) = \frac{f(\boldsymbol{z}^{-i}, m-1 | \gamma_0, \rho)}{m} \Big(\gamma_0 \int_0^\infty r e^{-r} \rho(dr) + \sum_{k=1}^{l-i} \frac{\int_0^\infty r^{n_k^{-i}+1} e^{-r} \rho(dr)}{\int_0^\infty r^{n_k^{-i}} e^{-r} \rho(dr)} \Big).$$

Theorem 4 (Prediction Rule). Since $f(z_i|\mathbf{z}^{-i}, m, \gamma_0, \rho) = \frac{f(z_i, \mathbf{z}^{-i}, m|\gamma_0, \rho)}{f(\mathbf{z}^{-i}, m|\gamma_0, \rho)}$, we can express the prediction rule of the count-mixture model as

$$P(z_i = k | \boldsymbol{z}^{-i}, m, \gamma_0, \rho) \propto \begin{cases} \frac{\int_0^\infty r^{n_k^{-i} + 1} e^{-r} \rho(dr)}{\int_0^\infty r^{n_k^{-i}} e^{-r} \rho(dr)}, & \text{for } k = 1, \dots, l^{-i}; \\ \gamma_0 \int_0^\infty r e^{-r} \rho(dr), & \text{if } k = l^{-i} + 1. \end{cases}$$

We may use a Gibbs sampler to simulate the EPPF $f(\boldsymbol{z}|m, \gamma_0, \rho)$: initializing $\boldsymbol{z} = (z_1, \dots, z_m)$ at random, in each Gibbs sampling iteration we remove elements one by one from their current clusters and immediately reassign the removed element to a cluster based on the prediction rule in Theorem 4; as the Markov chain converges, the ties between the elements of \boldsymbol{z} in each

iteration would define an exchangeable random partition Π_m of [m], and the ties between a subset of j elements uniformly at random subsampled without replacement from z would define a size-dependent exchangeable random partition $(\Pi_j|m)$.

Corollary 5 (Size-Dependent EPPF). Adding an element to a sample of size m, the EPPF of $\mathbf{z} = (z_1, \dots, z_m)$ in a sample of size m+1 is revised from the original EPPF $f(\mathbf{z}|m, \gamma_0, \rho)$ as

$$f(\boldsymbol{z}|m+1,\gamma_0,\rho) = f(\boldsymbol{z}|m,\gamma_0,\rho) \frac{f_M(m|\gamma_0,\rho)}{(m+1)f_{M+1}(m+1|\gamma_0,\rho)} \left(\gamma_0 \int_0^\infty r e^{-r} \rho(dr) + \sum_{k=1}^l \frac{\int_0^\infty r^{n_k+1} e^{-r} \rho(dr)}{\int_0^\infty r^{n_k} e^{-r} \rho(dr)}\right)$$

and the new element is clustered with the prediction rule $f(z_{m+1}|\boldsymbol{z}, m+1, \gamma_0, \rho) = \frac{f(\boldsymbol{z}, z_{m+1}, m+1|\gamma_0, \rho)}{f(\boldsymbol{z}, m+1|\gamma_0, \rho)}$.

Corollary 6 (Equivalent Construction). If both the marginal distribution $f_M(m|\gamma_0, \rho)$ and the EPPF $f(\mathbf{z}|m, \gamma_0, \rho)$ are known, then the count-mixture model can also be constructed as

$$x_i \sim \kappa(\omega_{z_i}), \ \omega_k \sim g_0, \ \boldsymbol{z} \sim f(\boldsymbol{z}|m, \gamma_0, \rho), \ m \sim f_M(m|\gamma_0, \rho).$$

There are several notable distinctions differing a count-mixture model from an NRMI mixture model. First, the model introduces a new mechanism to generate a sample of arbitrary size and specifies a prior distribution on all possible exchangeable random partitions. Second, both the conditional likelihood and the ECPF are fully factorized by construction and hence naturally amenable to posterior simulation. Third, G no longer scales freely, removing the redundancy between its parameters, and $K = \infty$ is no longer mandatory. Fourth, $f(z|m, \gamma_0, \rho) = f(z|m + 1, \gamma_0, \rho)$ is not required and the EPPF may not satisfy the addition rule in (1). Fifth, the ECPF, EPPF and prediction rule are all straightforward to calculate, without the need of introducing an auxiliary variable. Last but not least, the count-mixture model is characterized by a compound Poisson process, which clearly specifies the *a priori* cluster-number and cluster-size distributions.

4 Generalized Negative Binomial Processes

The previous section shows that every completely random measure G with a finite total random mass can generate a cluster structure, with the a priori cluster-number and cluster-size distributions determined by the Lévy measure. In the following discussion, we study the generalized NB process (gNBP) count-mixture model where $G \sim \text{gGaP}(G_0, a, (1-p)/p)$ with a < 0, a = 0 or 0 < a < 1, for which both $\int_0^\infty r^j e^{-r} \rho(dr)$ and $\int_0^\infty (1-e^{-r}) \rho(dr)$ are convenient to calculate.

4.1 Generalized and Truncated Negative Binomial Distributions

Marginalizing out λ in $m \sim \text{Pois}(\lambda)$, $\lambda \sim \text{gGamma}(\gamma_0, a, p/(1-p))$ leads to a generalized NB distribution $m \sim \text{gNB}(\gamma_0, a, p)$, with shape parameter γ_0 , discount parameter a < 1 and prob-

ability parameter p, whose PGF is $\mathbb{E}[t^m] = \mathbb{E}[\mathbb{E}[t^m|\lambda]] = e^{-\frac{\gamma_0((1-pt)^a-(1-p)^a)}{ap^a}}$, mean is $\gamma_0\left(\frac{p}{1-p}\right)^{1-a}$ and variance is $\gamma_0\left(\frac{p}{1-p}\right)^{1-a}\frac{1-ap}{1-p}$. The PGF of this distribution was presented in Willmot (1988), Gerber (1992). With the PGF reexpressed as $\mathbb{E}[t^m] = e^{\gamma_0\frac{(1-p)^a}{ap^a}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-\gamma_0(1-pt)^a}{ap^a}\right)^k = e^{\gamma_0\frac{(1-p)^a}{ap^a}}$ $\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-\gamma_0}{ap^a}\right)^k\sum_{j=0}^{\infty}\binom{ak}{j}(-pt)^j$, we derive the PMF as

$$f_M(m|\gamma_0, a, p) = \frac{p^m}{m!} e^{\gamma_0 \frac{(1-p)^a}{ap^a}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{\gamma_0}{ap^a} \right)^k \frac{\Gamma(m-ak)}{\Gamma(-ak)}, \ m = 0, 1, \dots$$
 (13)

Since the PGF can be reexpressed as $\mathbb{E}[t^m] = e^{\frac{\gamma 0(1-(1-p)^a)}{ap^a}\left(\frac{1-(1-pt)^a}{1-(1-p)^a}-1\right)}$, the generalized NB distribution $m \sim \text{gNB}(\gamma_0, a, p)$ can also be generated from a compound Poisson distribution as $m = \sum_{i=1}^l u_i, u_i \stackrel{iid}{\sim} \text{TNB}(a, p), \ l \sim \text{Pois}\left(\frac{\gamma_0(1-(1-p)^a)}{ap^a}\right)$, where $u \sim \text{TNB}(a, p)$ is a truncated NB distribution, with PGF $\mathbb{E}[t^u] = \frac{1-(1-pt)^a}{1-(1-p)^a}$ and thus $\mathbb{E}[u] = \frac{a(1-p)^a}{1-(1-p)^a}\frac{p}{1-p}$ and PMF

$$f_U(u|a,p) = \frac{\Gamma(u-a)}{u!\Gamma(-a)} \frac{p^u(1-p)^{-a}}{1-(1-p)^{-a}}, \ u = 1, 2, \cdots.$$
 (14)

Note that as $a \to 0$, $u \sim \text{TNB}(a, p)$ becomes a logarithmic distribution (Quenouille, 1949) $u \sim \text{Log}(p)$ with PMF $f_U(u|p) = \frac{-1}{\ln(1-p)} \frac{p^u}{u}$, $u = 1, 2, \dots$, and $m \sim \text{gNB}(\gamma_0, a, p)$ becomes a NB distribution $m \sim \text{NB}(\gamma_0, p)$. The truncated NB distribution with 0 < a < 1 is the extended NB distribution introduced in Engen (1974).

Denote $\sum_{k=1}^{l} n_k = m$ as the summation over all sets of positive integers (n_1, \dots, n_l) with $\sum_{k=1}^{l} n_k = m$. Using both (14) and $\left[\frac{1-(1-pt)^a}{1-(1-p)^a}\right]^l = \frac{\sum_{k=0}^{l} \binom{l}{k}(-1)^k \sum_{j=0}^{\infty} \binom{ak}{j}(-pt)^j}{[1-(1-p)^a]^l}$, we may express the PMF of the sum-truncated NB distribution $m = \sum_{i=1}^{l} u_i, u_i \stackrel{iid}{\sim} \text{TNB}(a, p)$ as $f_M(m|l, a, p) = \sum_{k=1}^{l} n_k = m \prod_{k=1}^{l} \frac{\Gamma(n_k - a)}{n_k!\Gamma(-a)} \frac{p^{n_k}(1-p)^{-a}}{1-(1-p)^{-a}} = \frac{p^m}{[1-(1-p)^a]^l} \sum_{k=0}^{l} (-1)^k \binom{l}{k} \frac{\Gamma(m-ak)}{m!\Gamma(-ak)}$, leading to identity

$$S_a(m,l) := \frac{m!}{l!} \sum_{\sum_{k=1}^l n_k = m} \prod_{k=1}^l \frac{\Gamma(n_k - a)}{n_k! \Gamma(1 - a)} = \frac{1}{l!a^l} \sum_{k=0}^l (-1)^k \binom{l}{k} \frac{\Gamma(m - ak)}{\Gamma(-ak)}, \tag{15}$$

where $S_a(m,l)$ are generalized Stirling numbers of the first kind, which can be recursively calculated with $S_a(m,1) = \frac{\Gamma(m-a)}{\Gamma(1-a)}$, $S_a(m,m) = 1$ and $S_a(m+1,l) = (m-al)S_a(m,l) + S_a(m,l-1)$ (Charalambides, 2005, Pitman, 2006). When $a \to 0$, the sum-truncated NB distribution becomes the sum-logarithmic distribution (Zhou and Carin, 2013), and (15) becomes $\frac{m!}{l!} \sum_{\sum_{k=1}^{l} n_k = m} \prod_{k=1}^{l} \frac{1}{n_k} = |s(m,l)|$, where $|s(m,l)| = S_0(m,l)$ are unsigned Stirling numbers of the first kind (Johnson et al., 2005).

4.2Model Properties

Marginalizing out G in the generalized gamma process mixed Poisson process

$$X \sim PP(G), G \sim gGaP(G_0, a, (1-p)/p)$$
 (16)

leads to a generalized NB process $X \sim \text{gNBP}(G_0, a, p)$, such that for each $A \subset \Omega$, $X(A) \sim \text{gNB}(G_0(A), a, p)$. Since $\rho(dr) = \frac{r^{-a-1}}{\Gamma(1-a)} e^{-\frac{1-p}{p}r} dr$, we have $\int_0^\infty r^n e^{-r} \rho(dr) = \frac{\Gamma(n-a)}{\Gamma(1-a)} p^{n-a}$ and $\int_0^\infty (1 - e^{-r}) \rho(dr) = \frac{1-(1-p)^a}{ap^a}$. Based on the analysis in Section 3.2, the Lévy measure of Xcan be expressed as $\tilde{\nu}(dnd\omega) = \sum_{m=1}^{\infty} \frac{\Gamma(m-a)}{m!\Gamma(1-a)} p^{m-a} \delta_m(dn) G_0(d\omega)$, and the gNBP count-mixture model shown in (8), with $\beta = 1$ and $G \sim \text{gGaP}(G_0, a, (1-p)/p)$, is characterized by a compound Poisson process as $X = \sum_{k=1}^{l} n_k \delta_{\omega_k}$, $l \sim \text{Pois}\left(\gamma_0 \frac{(1-(1-p)^a)}{ap^a}\right)$, $n_k \stackrel{iid}{\sim} \text{TNB}(a,p)$, $\omega_k \stackrel{iid}{\sim} g_0$. The ECPF of the gNBP count-mixture model can be expressed as

$$f(\boldsymbol{z}, m | \gamma_0, a, p) = \frac{1}{m!} e^{-\frac{\gamma_0(1 - (1 - p)^a)}{ap^a}} \gamma_0^l p^{m - al} \prod_{k=1}^l \frac{\Gamma(n_k - a)}{\Gamma(1 - a)}.$$
 (17)

Under the fully factorized ECPF, it is clear that $\gamma_0 \sim \text{Gamma}(e_0, 1/f_0)$ is a conjugate prior and the other two univariate parameters a and p can also be conveniently inferred. The EPPF is the ECPF in (17) divided by the marginal distribution of m in (13), expressed as

$$f(\boldsymbol{z}|m,\gamma_0,a,p) = p_m(n_1,\dots,n_l) = \frac{1}{e^{\frac{\gamma_0}{ap^a}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{\gamma_0}{ap^a}\right)^k \frac{\Gamma(m-ak)}{\Gamma(-ak)}} \gamma_0^l p^{-al} \prod_{k=1}^l \frac{\Gamma(n_k-a)}{\Gamma(1-a)}.$$
 (18)

If $a \to 0$, we recover from (18) the Ewens sampling formula $f(\boldsymbol{z}|\gamma_0, m) = \frac{\gamma_0^l \Gamma(\gamma_0)}{\Gamma(m+\gamma_0)} \prod_{k=1}^l \Gamma(n_k)$, which is the EPPF of the CRP (Aldous, 1983). We define the stochastic process with the EPPF in (18) as the generalized CRP, expressed as $X \sim \text{gCRP}(m, \gamma_0, a, p)$, whose prediction rule is

$$P(z_{i} = k | \boldsymbol{z}^{-i}, m, \gamma_{0}, a, p) \propto \begin{cases} n_{k}^{-i} - a, & \text{for } k = 1, \dots, l^{-i}; \\ \gamma_{0} p^{-a}, & \text{if } k = l^{-i} + 1; \end{cases}$$
(19)

based on which we can construct a Gibbs sampler to draw exchangeable random partitions Π_m of [m] and hence size-dependent exchangeable random partitions $(\Pi_i|m)$, for $1 \leq j \leq m-1$. Using $[z_i, z_j]$ to denote ties that $z_i = z_j$, with (18), we have $f(z_1, z_2|2) = \frac{\gamma_0 p^{-a}}{(1-a)+\gamma_0 p^{-a}}$, $f(z_1, [z_2, z_3]|3) = f([z_1, z_3], z_2|3) = \frac{(1-a)\gamma_0 p^{-a}}{(2-a)(1-a)+3(1-a)\gamma_0 p^{-a}+\gamma_0^2 p^{-2a}}$ and $f(z_1, z_2, z_3|3) = \frac{\gamma_0^2 p^{-2a}}{(2-a)(1-a)+3(1-a)\gamma_0 p^{-a}+\gamma_0^2 p^{-2a}}$. If $a \neq 0$, then the EPPF violates the addition rule in (1) since $f(z_1, z_2|2) \neq f(z_1, z_2|3)$, where $f(z_1, z_2|3) = f(z_1, z_2, z_3|3) + f(z_1, [z_2, z_3]|3) + f([z_1, z_3], z_2|3).$

With Corollary 6, the gNBP count-mixture model in (8) can also be constructed as

The joint distribution of the number of nonempty tables and the number of coustomers at these tables are equivalent:

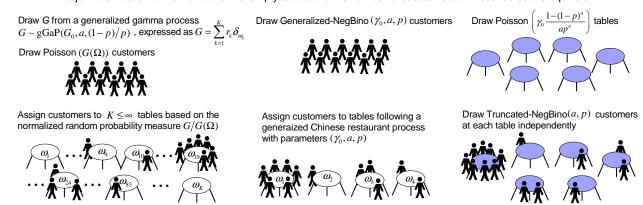


Figure 1: The generalized NB process count-mixture model can be either constructed by assigning $\operatorname{Pois}(G(\Omega))$ number of customers to tables following a normalized generalized gamma process $G/G(\Omega)$, where $G \sim \operatorname{gGaP}(G_0, a, (1-p)/p)$, or constructed by assigning $\operatorname{gNB}(\gamma_0, a, p)$ number of customers to tables following a generalized Chinese restaurant process $X \sim \operatorname{gCRP}(\gamma_0, a, p)$, where $\gamma_0 = G_0(\Omega)$. The joint distribution of the number of clusters and the sizes of unordered clusters can be equivalently generated by first drawing $\operatorname{Pois}(\gamma_0 \frac{(1-(1-p)^a)}{ap^a})$ number of tables, and then drawing $\operatorname{TNB}(a,p)$ number of customers independently at each table.

$$x_i \sim \kappa(\omega_{z_i}), \ \omega_k \sim q_0, \ \boldsymbol{z} \sim \text{gCRP}(m, \gamma_0, a, p), \ m \sim \text{gNB}(\gamma_0, a, p).$$
 (20)

As shown in Figure 1, they both lead to the same cluster structure where the probability for a collection of unordered cluster sizes $\{n_k\}_{1,l}$ can be expressed as

$$f(\{n_k\}_{1,l}, m | \gamma_0, a, p) = \delta_{\sum_{k=1}^l n_k}(m) \frac{1}{l!} \frac{m!}{\prod_{k=1}^l n_k} f(\boldsymbol{z}, m | \gamma_0, a, p)$$

$$= \delta_{\sum_{k=1}^l n_k}(m) \operatorname{Pois}\left(l; \gamma_0 \frac{(1 - (1 - p)^a)}{ap^a}\right) \prod_{k=1}^l \operatorname{TNB}(n_k; a, p). \tag{21}$$

Using the EPPF in (18) and the identity in (15), the conditional distribution of the number of clusters l in a sample of size m can be expressed as

$$f_L(l|m,\gamma_0,a,p) = \frac{1}{l!} \sum_{\sum_{k=1}^l n_k = m} \frac{m!}{\prod_{k=1}^l n_k} f(\boldsymbol{z}|m,\gamma_0,a,p) = \frac{\gamma_0^l p^{-al} S_a(m,l)}{e^{\frac{\gamma_0}{ap^a}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\gamma_0}{ap^a}\right)^k \frac{\Gamma(m-ak)}{\Gamma(-ak)}}, \quad (22)$$

which, since $\sum_{l=0}^{m} f_L(l|m, \gamma_0, a, p) = 1$, further leads to identity

$$e^{\frac{\gamma_0}{ap^a}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-\gamma_0}{ap^a}\right)^k \frac{\Gamma(m-ak)}{\Gamma(-ak)} = \sum_{l=0}^m \gamma_0^l p^{-al} S_a(m,l).$$

Thus we may simplify (13) as
$$f_M(m|\gamma_0, a, p) = \frac{p^m}{m!} e^{-\gamma_0 \frac{1-(1-p)^a}{ap^a}} \sum_{l=0}^m \gamma_0^l p^{-al} S_a(m, l)$$
, (18) as $f(\boldsymbol{z}|m, \gamma_0, a, p) = \frac{\gamma_0^l p^{-al}}{\sum_{l=0}^m \gamma_0^l p^{-al} S_a(m, l)} \prod_{k=1}^l \frac{\Gamma(n_k - a)}{\Gamma(1-a)}$ and (22) as $f(l|m, \gamma_0, a, p) = \frac{\gamma_0^l p^{-al} S_a(m, l)}{\sum_{l=0}^m \gamma_0^l p^{-al} S_a(m, l)}$.

It is evident from (21) that the gNBP can adjust both a and p to fit the cluster sizes with $n_k \stackrel{iid}{\sim} \text{TNB}(a,p)$, and adjust γ_0 , a and p to fit the number of clusters l with a Poisson distribution and the sample size with $m \sim \text{gNB}(\gamma_0, a, p)$. The way the number and sizes of clusters are modeled in the gNBP is unique, clearly explaining the roles of the three free model parameters on controlling the cluster structure.

4.2.1 Reparameterized Generalized Negative Binomial Process

Since if a generalized gamma process $G \sim \text{gGap}(G_0, a, (1-p)/p)$ is normalized to construct a random probability measure, its scale and mass parameters are redundant to each other and hence it is standard to fix the scale parameter as p/(1-p) = 1. In the gNBP, β is fixed as one and p serves as the probability parameter of the cluster-size distribution. To make connections to the approaches where p could be fixed, including the normalized generalized gamma process mixture model, we modify the count-mixture model in (8) with $m \sim \text{Pois}(\beta H(\Omega))$, $H \sim \text{gGaP}(H_0, a, 1)$, where $\beta^a H_0 = G_0$, $h_0 = H_0(\Omega) = \gamma_0 \beta^{-a}$, $\beta := \frac{p}{1-p}$ and 0 . The generalized gamma process mixed Poisson process in (16) is now reparameterized as

$$X \sim \text{PP}(\beta H), \ H \sim \text{gGaP}(H_0, a, 1),$$
 (23)

in which the marginalization of $\beta H \sim \text{gGaP}\left(\left(\frac{p}{1-p}\right)^a H_0, a, \frac{1-p}{p}\right)$ leads to $X \sim \text{gNBP}\left(\left(\frac{p}{1-p}\right)^a H_0, a, p\right)$. The reparameterized gNBP count-mixture model has reparameterized ECPF and EPPF

$$f(\boldsymbol{z}, m | h_0, a, p) = \frac{p^m}{m!} e^{-\frac{h_0(1 - (1-p)^a)}{a(1-p)^a}} h_0^l (1-p)^{-al} \prod_{k=1}^l \frac{\Gamma(n_k - a)}{\Gamma(1-a)},$$
(24)

$$f(\boldsymbol{z}|m, h_0, a, p) = \frac{h_0^l (1-p)^{-al}}{\sum_{l=0}^m h_0^l (1-p)^{-al} S_a(m, l)} \prod_{k=1}^l \frac{\Gamma(n_k - a)}{\Gamma(1-a)},$$
 (25)

which are the same as the ECPF in (17) and EPPF in (18), respectively, except that γ_0 is replaced with $h_0(\frac{p}{1-p})^a$. As the reparameterized EPPF matches that of $X \sim \text{gCRP}(m, h_0, a, 1-p)$, the reparameterized gNBP count-mixture model can also be constructed as

$$x_i \sim \kappa(\omega_{z_i}), \ \omega_k \sim g_0, \ \boldsymbol{z} \sim \text{gCRP}(m, h_0, a, 1 - p), \ m \sim \text{gNB}((\frac{p}{1-p})^a h_0, a, p),$$
 (26)

whose prediction rule is

$$P(z_i = k | \mathbf{z}^{-i}, m, h_0, a, p) \propto \begin{cases} n_k^{-i} - a, & \text{for } k = 1, \dots, l^{-i}; \\ h_0 (1 - p)^{-a}, & \text{if } k = l^{-i} + 1. \end{cases}$$
 (27)

For the reparameterized gNBP, the number of clusters l in a sample of size m has PMF

$$f_L(l|m, h_0, a, p) = \frac{h_0^l(1-p)^{-al}S_a(m, l)}{\sum_{l=0}^m h_0^l(1-p)^{-al}S_a(m, l)}, \ l = 0, \dots, m.$$
 (28)

Although $\gamma_0 p^{-a} \equiv h_0 (1-p)^{-a}$ and neither γ_0 nor h_0 directly influences the cluster sizes $n_k \sim \text{TNB}(a, p)$, below we show that whether γ_0 or h_0 is regularized with a prior distribution substantially impacts the asymptotic behaviors of both the number and sizes of clusters.

4.2.2 Connections to Normalized Generalized Gamma Process

In a normalized generalized gamma process, it is standard to fix the scale parameter as one. We let $G = H \sim \text{gGaP}(H_0, a, 1)$ in (5), where $0 \le a < 1$ is required to ensure $G(\Omega) > 0$. With $h_0 := H_0(\Omega)$ and $\beta := \frac{p}{1-p}$, we can express (6), the joint marginal distribution of z and the auxiliary variable β , in terms of the ECPF of the reparameterized gNBP in (24) as

$$f\left(\boldsymbol{z},\beta\middle|m,h_0,\rho(dr)=\frac{r^{-a-1}}{\Gamma(1-a)}e^{-r}dr\right)=m\beta^{-1}f(\boldsymbol{z},m|h_0,a,p).$$
(29)

As in Lijoi et al. (2007), the EPPF $f(\boldsymbol{z}|m,h_0,a) = \int_0^\infty f(\boldsymbol{z},\beta|m,h_0,\rho(dr)) = \frac{r^{-a-1}}{\Gamma(1-a)}e^{-r}drds$ is

$$f(\boldsymbol{z}|m, h_0, a) = \frac{1}{\Gamma(m)} a^{l-1} e^{\frac{h_0}{a}} \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^i \left(\frac{h_0}{a}\right)^{i/a} \Gamma\left(l - \frac{i}{a}, \frac{h_0}{a}\right) \prod_{k=1}^l \frac{\Gamma(n_k - a)}{\Gamma(1 - a)},$$

where $\Gamma(a,x) = \int_{x}^{\infty} t^{a-1}e^{-t}dt$ is the incomplete gamma function.

The EPPF $f(\mathbf{z}|m, h_0, a)$ of the NRMI mixture is consistent (Pitman, 2003, 2006, Lijoi et al., 2007) and β is an auxiliary variable that can be neither regularized nor fixed, whereas the EPPF of the (reparameterized) gNBP is generally an inconsistent EPPF that violates the addition rule and β is a model parameter that can be either fixed or regularized with a prior distribution. Despite the distinctions on the interpretations of β , given the similarity between the likelihood functions (29) and (24), significant differences on posterior simulation and hence performance seem to be unlikely between the normalized generalized gamma process using the auxiliary variable β and the reparameterized gNBP. In fact, we observe similar clustering results using (29) and (24) for $0 \le a < 1$ and 0 . However, there are clear differences, in both theory and experiments, on the asymptotic behaviors of both the number and sizes of clusters between the gNBP using (17) and reparameterized gNBP using (24), as discussed below.

4.3 Asymptotics for the Number and Sizes of Clusters

In a normalized generalized gamma process mixture model, Lijoi et al. (2007) showed that with a positive discount parameter 0 < a < 1, the number of clusters increases at a power-law as a function of the sample size, and with a discount parameter a close to one, there is a reinforcement mechanism to favor a few large-size clusters in the posterior. Although intriguing, there is lack of precise description about the distribution of the cluster sizes; in addition, a < 0 is not allowed. Given similar likelihoods $f(\mathbf{z}, \beta | m, a, h_0) = m\beta^{-1}f(\mathbf{z} | m, h_0, a, p)$ as in (29), we expect in the prior for $0 \le a < 1$, the distributions and asymptotic behaviors of the cluster number and sizes to be similar between the normalized generalized gamma process mixture model using the auxiliary variable and the reparameterized gNBP count-mixture model. Thus the analysis of the reparameterized gNBP could also help better understand how the cluster number and sizes are controlled by model parameters in the normalized generalized gamma process.

For both the gNBP and reparameterized gNBP, we can precisely describe the cluster-size distributions as $n_k \sim \text{TNB}(a, p)$. However, the gNBP will be shown to have asymptotic behaviors on both the number and average size of clusters that are completely opposite to that of the reparameterized gNBP. Recall that $m \sim \text{gNB}(\gamma_0, a, p)$ is used to model the sample size, we have $\mathbb{E}[m] = \gamma_0 \left(\frac{p}{1-p}\right)^{1-a} = h_0 \frac{p}{1-p}$. If $\gamma_0 = G_0(\Omega)$ is assumed to be finite, then the increase of m towards infinity would drive p towards one and hence $h_0 = \gamma_0 \left(\frac{1-p}{p}\right)^a$ towards zero for 0 < a < 1 and towards infinity for a < 0, making $H \sim \text{gGaP}(H_0, a, 1)$ not well defined. Whereas if $h_0 = H_0(\Omega)$ is assumed to be finite, then the increase of m towards infinity would drive $\gamma_0 = h_0 \left(\frac{p}{1-p}\right)^a$ towards infinity for 0 < a < 1 and towards zero for a < 0, making $G \sim \text{gGaP}(G_0, a, (1-p)/p)$ not well defined. Therefore, the gNBP from (16) and reparameterized gNBP from (23), despite their close connections, shall be treated differently when $a \neq 0$.

4.3.1 Number of Clusters

To visualize the differences between the cluster-number distributions, we show in Figure 2 the PMF $f_L(l|m, \gamma_0, a, p)$ of the gNBP as in (22) and the PMF $f_L(l|m, h_0, a, p)$ of the reparameterized gNBP as in (28), assuming $\gamma_0 = 1$, $h_0 = 1$ and a finite sample size m = 100, where $m = \gamma_0 \left(\frac{p}{1-p}\right)^{1-a}$ and $m = h_0 \frac{p}{1-p}$ are used to calculate the probability parameter p in the gNBP and reparameterized gNBP, respectively. It is evident that if the models behave in the way we expect in the prior, for the gNBP, the increase of a would drive the high density region of the PMF towards left, encouraging fewer clusters, whereas for the reparameterized gNBP, the increase of a would drive the high density region of the PMF towards right, encouraging more clusters.

For the gNBP, with (21), the expected number of clusters and size of each cluster are $\mathbb{E}[l] = \gamma_0 \frac{(1-(1-p)^a)}{ap^a}$ and $\mathbb{E}[n_k] = \frac{a(1-p)^a}{1-(1-p)^a} \frac{p}{1-p}$, respectively. With $\mathbb{E}[m] = \mathbb{E}[l]\mathbb{E}[n_k] = \gamma_0 \left(\frac{p}{1-p}\right)^{1-a}$, we have $p = 1 - (1 + (\frac{\mathbb{E}[m]}{\gamma_0})^{\frac{1}{1-a}})^{-1}$ and hence for $a \neq 0$, we have $\mathbb{E}[l] = \frac{\gamma_0}{a} \left(\left(\frac{\gamma_0}{\mathbb{E}[m]}\right)^{\frac{1}{1-a}} + 1\right)^a - \left(\frac{\gamma_0}{\mathbb{E}[m]}\right)^{\frac{a}{1-a}}\right)$.

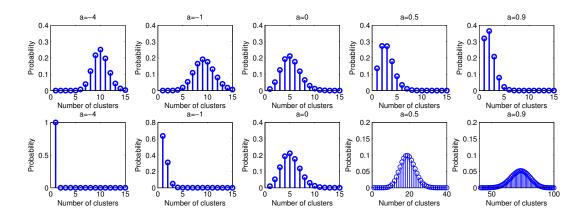


Figure 2: The *a priori* probability mass functions (PMFs) of the number of clusters l in a sample of size m=100 for a=-4, -1, 0, 0.5 and 0.9. The first and second rows show the results for the gNBP and reparameterized gNBP, respectively. The mass parameters are set as $\gamma_0=1$ for the gNBP and $h_0=1$ for the reparameterized gNBP. The probability parameters p for the gNBP and the reparameterized gNBP are calculated based on $m=\gamma_0\left(\frac{p}{1-p}\right)^{1-a}$ and $m=h_0\frac{p}{1-p}$, respectively.

When a = 0, we have $\mathbb{E}[l] = -\gamma_0 \ln(1 - p) = \gamma_0 (\ln(\mathbb{E}[m] + \gamma_0) - \ln \gamma_0)$. Thus as $\mathbb{E}[m]$ increases towards infinity, if the model behaves in the way we expect, we have

$$\mathbb{E}[l] \sim \begin{cases} \frac{\gamma_0^{\frac{1}{1-a}}}{-a} (\mathbb{E}[m])^{\frac{-a}{1-a}}, & \text{if } a < 0; \\ \gamma_0 \ln(\mathbb{E}[m]), & \text{if } a = 0; \\ \gamma_0/a, & \text{if } 0 < a < 1. \end{cases}$$
(30)

For the gNBP count-mixture model, the larger the discount parameter a is, the more slowly the number of clusters increases: $\mathbb{E}[l]$ increases at a power-law rate for a < 0, at a logarithmic rate for a = 0, and towards a fixed value for 0 < a < 1. The mass parameter γ_0 influences the asymptotic behaviors only with scaling.

For the reparameterized gNBP, substituting γ_0 with $h_0\left(\frac{p}{1-p}\right)^a$, we have $\mathbb{E}[l] = h_0\frac{(1-(1-p)^a)}{a(1-p)^a}$, $\mathbb{E}[n_k] = \frac{a(1-p)^a}{1-(1-p)^a}\frac{p}{1-p}$ and $\mathbb{E}[m] = h_0\left(\frac{p}{1-p}\right)$. With $p = \frac{\mathbb{E}[m]}{h_0+\mathbb{E}[m]}$, for $a \neq 0$, we have $\mathbb{E}[l] = \frac{h_0}{a}\left(\left(\frac{h_0+\mathbb{E}[m]}{h_0}\right)^a-1\right)$. When a=0, we have $\mathbb{E}[l] = -h_0\ln(1-p) = h_0(\ln(\mathbb{E}[m]+h_0)-\ln(h_0)$. Thus as $\mathbb{E}[m]$ increases towards infinity, if the model behaves in the way we expect, we have

$$\mathbb{E}[l] \sim \begin{cases} h_0/(-a), & \text{if } a < 0; \\ h_0 \ln(\mathbb{E}[m]), & \text{if } a = 0; \\ \frac{h_0^{1-a}}{a} (\mathbb{E}[m])^a, & \text{if } 0 < a < 1. \end{cases}$$
 (31)

Apparently, reparameterizing (16) as (23) leads to completely opposite asymptotic behaviors that the larger the discount parameter a is, the more rapidly the number of clusters increases: $\mathbb{E}[l]$ increases towards a fixed value for a < 0, at a logarithmic rate for a = 0, and at a power-law

Table 1: Asymptotic behaviors of the average size of clusters.

Model	a < 0	a = 0	0 < a < 1
gNBP, $\mathbb{E}[n_k]$	$\left \frac{\frac{-a}{\frac{1}{1-a}}}{\gamma_0^{\frac{1}{1-a}}} (\mathbb{E}[m])^{1-\frac{-a}{1-a}} \right $	$\frac{1}{\gamma_0} \frac{\mathbb{E}[m]}{\ln(\mathbb{E}[m])}$	$rac{a}{\gamma_0}\mathbb{E}[m]$
Reparameterized gNBP, $\mathbb{E}[n_k]$	$\frac{-a}{h_0}\mathbb{E}[m]$	$\frac{1}{h_0} \frac{\mathbb{E}[m]}{\ln(\mathbb{E}[m])}$	$\frac{\frac{a}{h_0^{1-a}}(\mathbb{E}[m])^{1-a}}$

rate for 0 < a < 1. Given the similarity between the likelihoods, it is unsurprising that when $0 \le a < 1$, the reparameterized gNBP count-mixture model has similar asymptotic behaviors as the normalized generalized gamma process mixture model, whose prior number of clusters is shown in Lijoi et al. (2007) to increase as a power function of the sample size for 0 < a < 1.

4.3.2 Sizes of Clusters

The a priori cluster sizes follow $n_k \stackrel{iid}{\sim} \text{TNB}(a,p)$, whose PMF, for any $a > 1 - 2p^{-1}$ (hence any a > -1), always has the mode at $n_k = 1$ and monotonically decreases. In a sample of finite size, for both the gNBP and reparameterized gNBP, as a increases towards one, Figures 3 (a) and (c) show that the PMF of $n_k \sim \text{TNB}(a,p)$ gets a more peaked mode towards left. Figure 3 (b) shows that the PMF of $n_k \sim \text{TNB}(a,p)$ in the gNBP gets heavier tails (decays more slowly) as a increases towards one, whereas Figure 3 (d) shows that the tails decay rapidly at similar rates for different a in the reparameterized gNBP.

Using the PMF of $n_k \sim \text{TNB}(a, p)$ in (14), the probability for $n_k = j$, $j = 1, 2, \cdots$, is $P(n_k = j | a, p) = \frac{\Gamma(j-a)}{j!\Gamma(1-a)}p^{j-1}\frac{ap}{1-(1-p)^a}$, where the term $\frac{ap}{1-(1-p)^a}$ monotonically decreases towards $\max\{a, 0\}$ as p increases towards one. Thus as $\mathbb{E}[m] \to \infty$, we have $p \to 1$ and hence

$$P(n_k = j | a, p) \sim \frac{a\Gamma(j-a)}{j!\Gamma(1-a)} p^{j-1}, \text{ if } 0 < a < 1,$$
 (32)

which are similar to the properties of the normalized generalized gamma process analyzed in Lijoi et al. (2007). The discount parameter determines the lower bounds of the *a priori* ratio of unit-size clusters (clusters with only one sample) as

$$P(n_k = 1 | a, p) \ge \max\{a, 0\}. \tag{33}$$

E.g., if a=0.90, then in the prior at least 90% of the instantiated clusters would be unit size.

Similar to the analysis in Section 4.3.1, we summarize in Table 1 the asymptotic behaviors of the average size of clusters $\mathbb{E}[n_k]$ for both the gNBP and reparameterized gNBP. It is clear that, asymptotically, as the discount parameter a increases, the average size of clusters increases at a higher rate for the gNBP while at a lower rate for the reparameterized gNBP.

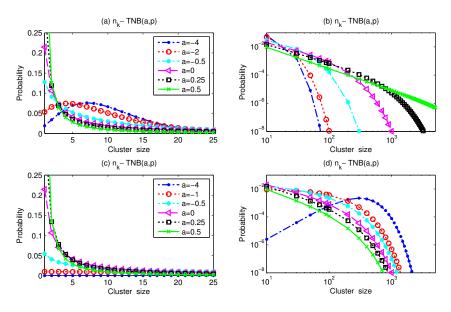


Figure 3: The a priori PMFs of the cluster sizes $n_k \sim \text{TNB}(a,p)$ for a=-4,-2,0,0.25 and 0.5. The first and second rows show the results of the gNBP and reparameterized gNBP, respectively. For the gNBP, the probability parameter p is calculated under the assumption that $\mathbb{E}[m] = \gamma_0 \left(\frac{p}{1-p}\right)^{1-a} = 100$, whereas for the reparameterized gNBP, p is calculated based on $\mathbb{E}[m] = h_0 \frac{p}{1-p} = 100$. We set $\gamma_0 = 1$ for the gNBP and $h_0 = 1$ for the reparameterized gNBP.

4.3.3 Controlling the Asymptotic Behaviors

In summary, for the gNBP, increasing a towards one would encourage a smaller number of larger clusters together with a higher ratio of unit-size clusters to fit the data; whereas decreasing a increases towards $-\infty$ would encourage a larger number of similar-size smaller clusters. For the reparameterized gNBP, increasing a towards one would encourage a larger number of smaller clusters with a higher ratio of unit-size clusters; whereas decreasing a towards $-\infty$ would encourage a smaller number of similar-size larger clusters. When a=0, both the gNBP and reparameterized gNBP become the NB process in Zhou and Carin (2013), which is closely related to the CRP in that they have the same EPPF and prediction rule. The NB process has the advantages over the CRP that given the probability parameter p, which has an analytic beta conditional posterior, the conditional posterior of the mass parameter γ_0 follows an analytic gamma distribution and the priori cluster sizes are known to follow the priori cluster sizes are known to follow the Log(p) distribution. In the CRP, the concentration parameter γ_0 is usually sampled with a data augmentation approach of Escobar and West (1995).

4.4 Size-Dependent Exchangeable Partition Probability Functions

With $z_{1:j} := (z_1, \dots, z_j)$ and $l_{(j)}$ representing the number of clusters in $z_{1:j}$, we have $\gamma_0 \int_0^\infty r e^{-r} \rho(dr) + \sum_{k=1}^{l_{(j)}} \frac{\int_0^\infty r^{n_k+1} e^{-r} \rho(dr)}{\int_0^\infty r^{n_k} e^{-r} \rho(dr)} = p(\gamma_0 p^{-a} + j - a l_{(j)})$ for the gNBP. Using Corollary 5, we have

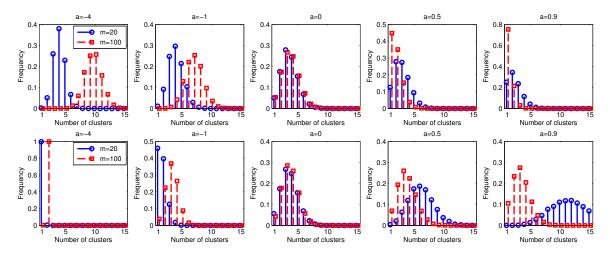


Figure 4: Comparison of the *a priori* distributions of the number of clusters in $z_{1:20}$ between a sample of size m = 20 and a sample of size m = 100. The first and second rows show the results for the gNBP, simulated with $\gamma_0 = 1$ and p = 0.9, and reparameterized gNBP, simulated with $h_0 = 1$ and p = 0.9, respectively. Columns from left to right are the results for a = -4, -1, 0, 0.5 and 0.9, respectively.

$$f(z_{1:m-1}|m,\gamma_0,a,p) = f(z_{1:m-1}|m-1,\gamma_0,a,p) \frac{\sum_{l=0}^{m-1} \gamma_0^l p^{-al} S_a(m-1,l)}{\sum_{l=0}^m \gamma_0^l p^{-al} S_a(m,l)} \left(\gamma_0 p^{-a} + (m-1) - a l_{(m-1)}\right), (34)$$

and we may also marginalize out z_{m-1} as $f(z_{1:m-2}|m,\gamma_0,\rho)=f(z_{1:m-2}|m-2,\gamma_0,\rho)\frac{\sum_{l=0}^{m-2}\gamma_0^lp^{-al}S_a(m-2,l)}{\sum_{l=0}^m\gamma_0^lp^{-al}S_a(m,l)}$ $\left[(\gamma_0p^{-a}+(m-1)-al_{(m-2)})(m-2-al_{(m-2)})+(\gamma_0p^{-a}+(m-1)-a(l_{(m-2)}+1))\gamma_0p^{-a}\right]$. Similar analysis can be further carried out to marginalize out z_{m-3},z_{m-4},\cdots . It is easy to verify that $f(z_{1:j}|m,\gamma_0,\rho)\equiv f(z_{1:j}|j,\gamma_0,\rho)$ for $m\geq j$ when a=0. When $a\neq 0$, $f(z_{1:j}|m,\gamma_0,\rho)$ is usually not equal to $f(z_{1:j}|j,\gamma_0,\rho)$ for m>j. E.g., since

$$f(z_{1:2}|3,\gamma_0,\rho) = f(z_{1:2}|2,\gamma_0,\rho) \frac{(1-a+\gamma_0p^{-a})(\gamma_0p^{-a}+2-l_{(2)}a)}{(1-a)(2-a)+(3-3a)\gamma_0p^{-a}+\gamma_0^2p^{-2a}},$$
(35)

if $a \neq 0$, then $f(z_{1:2}|3, \gamma_0, \rho) \neq f(z_{1:2}|2, \gamma_0, \rho)$ regardless of whether $l_{(2)} = 1$ or $l_{(2)} = 2$. The reparameterized gNBP can be similarly analyzed by replacing γ_0 with $h_0(\frac{p}{1-p})^a$.

In a cluster structure, the sample size m usually plays an important role on how a subset of size j is clustered. As shown in the first row of Figure 4, we compare the simulated a priori cluster-number distributions $f(l_{(20)}|m=20,\gamma_0,a,p)$ and $f(l_{(20)}|m=100,\gamma_0,a,p)$ for the gNBP, where $\gamma_0 = 1$ and p = 0.9; columns from left to right are the results for a = -4, -1, 0, 0.5 or 0.9, respectively. It is clear that except for a = 0, for which $P(\Pi_j|m) \equiv P(\Pi_m)$ for $m \geq j$, $f(l_{(20)}|m=20,\gamma_0,a,p)$ is obviously different from $f(l_{(20)}|m=100,\gamma_0,a,p)$. Similarly, in the second row of Figure 4 we compare the cluster-size distributions $f(l_{(20)}|m=20,h_0,a,p)$ and $f(l_{(20)}|m=100,h_0,a,p)$ for the generalized gNBP, where $h_0 = 1$ and p = 0.9. Except for a = 0, these two cluster-size distributions are also obviously different. Note that for a consistent EPPF, we can simulation an exchangeable random partition by sequentially assigning the i + 1

element to Π_i using the prediction rule until the *m*th element is assigned (Pitman, 2006). For an inconsistent EPPF, as discussed in Section 3.2, we construct a Gibbs sampler based on the prediction rule to simulate exchangeable random partitions; for both m = 20 and m = 100, we consider 15000 Gibbs sampling iterations and record the number of clusters in $z_{1:20}$ of the last 10,000 iterations, which are used to estimate the *a priori* distributions of the number of clusters. The sequential construction of inconsistent exchangeable random partitions for the generalized Chinese restaurant process is currently under investigation.

5 MCMC Inference

We present Markov chain Monte Carlo (MCMC) inference for count-mixture modeling. Using the prediction rule in Theorem 4, a generalized Pólya urn sampling scheme is developed by generalizing the sampling algorithms developed for Dirichlet process mixture models (Escobar and West, 1995, MacEachern and Müller, 1998, Neal, 2000, Green, 2009). The parameters of the completely random measure G can be inferred based on the ECPF in Theorem 3. We consider both the gNBP count-mixture model and its reparameterized version. We develop MCMC inference for both the gNBP based on the ECPF in (17) and the prediction rule in (19), and the reparameterized gNBP based on the ECPF in (24) and prediction rule in (27).

Sample z_i . The data samples are assumed independent conditioning on G and hence exchangeable. Using the prediction rule of the gNBP in (19), since $P(z_i = k | \mathbf{z}^{-i}, m, x_i, \mathcal{D}_m^{-i}) \propto \kappa(x_i | \omega_k) P(z_i = k | \mathbf{z}^{-i}, m)$, where $\mathcal{D}_m^{-i} = \{\omega_k\}_{k:n_k^{-i}>0}$ and $n_k^{-i} = \sum_{j\neq i} \delta(z_j = k)$, we have

$$P(z_{i} = k | \boldsymbol{z}^{-i}, m, x_{i}, \mathcal{D}_{m}^{-i}, \gamma_{0}, a, p) \propto \begin{cases} (n_{k}^{-i} - a) \kappa(x_{i} | \omega_{k}), & \text{for } \omega_{k} \in \mathcal{D}_{m}^{-i}; \\ \gamma_{0} p^{-a} \int_{\Omega} \kappa(x_{i} | \omega) g_{0}(\omega) d\omega, & \text{if } \omega_{k} \in \Omega \backslash \mathcal{D}_{m}^{-i}; \end{cases}$$
(36)

where $\mathbf{x}^{-i} = \mathbf{x} \setminus x_i$ and $P(\omega_k | \mathbf{x}^{-i}, \mathbf{z}^{-i}) \propto g_0(\omega) \prod_{j:(z_j = k, j \neq i)} \kappa(x_j | \omega_k)$. As in (Neal, 2000), we further marginalize out $\omega_k \in \mathcal{D}_m^{-i}$ in (36), leading to

$$P(z_{i} = k | \boldsymbol{z}^{-i}, m, \boldsymbol{x}, \gamma_{0}, a, p) \propto \begin{cases} (n_{k}^{-i} - a) \int_{\Omega} \kappa(x_{i} | \omega_{k}) P(\omega_{k} | \boldsymbol{x}^{-i}, \boldsymbol{z}^{-i}) d\omega_{k}, & \text{for } \omega_{k} \in \mathcal{D}_{m}^{-i}; \\ \gamma_{0} p^{-a} \int_{\Omega} \kappa(x_{i} | \omega) g_{0}(\omega) d\omega, & \text{if } \omega_{k} \in \Omega \setminus \mathcal{D}_{m}^{-i}; \end{cases}$$
(37)

If $g_0(\omega)$ is conjugate to $\kappa(x|\omega)$ in this paper, the conditional posteriors $P(\omega_k|\mathbf{x}^{-i},\mathbf{z}^{-i})$ and the integrals in (36) and (37) can all be analytically calculated. For the reparameterized gNBP, we replace $\gamma_0 p^{-a}$ with $h_0(1-p)^{-a}$ in both (36) and (37).

Sample ω_k . The conditional posterior of an atom ω_k can be expressed as

$$P(\omega_k|-) \propto g_0(\omega) \prod_{i:z_i=k} \kappa(x_i|\omega_k).$$
 (38)

Sample γ_0 or h_0 . For the gNBP, the ECPF in (17) defines a likelihood for γ_0 , a and p. With a gamma prior Gamma $(e_0, 1/f_0)$ placed on γ_0 , we have

$$(\gamma_0|-) \sim \text{Gamma}\left(e_0 + l, \frac{1}{f_0 + \frac{1 - (1-p)^a}{a r^a}}\right).$$
 (39)

As $a \to 0$, we have $\lim_{a\to 0} (\gamma_0|-) \sim \operatorname{Gamma}\left(e_0+l,\frac{1}{f_0-\ln(1-p)}\right)$, the same as that of the NB process count-mixture model (Zhou and Carin, 2013). Similarly, for the reparameterized gNBP, with the ECPF in (24) and a gamma prior $\operatorname{Gamma}(e_0,1/f_0)$ placed on h_0 , we have $(h_0|-) \sim \operatorname{Gamma}\left(e_0+l,\frac{1}{f_0+\frac{1-(1-p)^a}{a(1-p)^a}}\right)$ and $\lim_{a\to 0} (h_0|-) \sim \operatorname{Gamma}\left(e_0+l,\frac{1}{f_0-\ln(1-p)}\right)$.

Sample a. Since a < 1, we have $\tilde{a} = \frac{1}{1+(1-a)} \in (0,1)$. With a uniform prior placed on \tilde{a} in

Sample a. Since a < 1, we have $\tilde{a} = \frac{1}{1+(1-a)} \in (0,1)$. With a uniform prior placed on \tilde{a} in (0,1) and the likelihood of gNBP in (17), we use the griddy-Gibbs sampler (Ritter and Tanner, 1992) to sample a from a discrete distribution

$$(a|-) \propto f(\boldsymbol{z}, m|\gamma_0, a, p) \tag{40}$$

over a grid of points $\frac{1}{1+(1-a)} = 0.0001, 0.0002, \dots, 0.9999$. Similarly, with (24) for the reparameterized gNBP, we sample a from a grid of points with $(a|-) \propto f(\boldsymbol{z}, m|h_0, a, p)$.

Sample p. When $a \to 0$, the likelihood of the gNBP in (17) becomes proportional to $p^m(1-p)^{\gamma_0}$, with a beta prior $\text{Beta}(a_0,b_0)$ on p, we have $\lim_{a\to 0}(p|-) \sim \text{Beta}(a_0+m,b_0+\gamma_0)$, the same as that of the NB process count-mixture model (Zhou and Carin, 2013). When $a \neq 0$, with a uniform prior placed on p in (0,1) and the likelihood in (17), we use the griddy-Gibbs sampler to sample p from a discrete distribution

$$(p|-) \propto e^{-\frac{\gamma_0(1-(1-p)^a)}{ap^a}} p^{m-al}$$
 (41)

over a grid of points $p = 0.0001, 0.0002, \dots, 0.9999$. Similarly, for the reparameterized gNBP, for a = 0, we have $(p|-) \sim \text{Beta}(a_0 + m, b_0 + h_0)$, and for $a \neq 0$, we sample p from a grid of points using $(p|-) \propto e^{-\frac{h_0(1-(1-p)^a)}{a(1-p)^a}} p^m (1-p)^{-al}$.

Note that for the normalized generalized gamma process mixture model (Lijoi et al., 2007, James et al., 2009), since its likelihood with the auxiliary variable $\beta = \frac{p}{1-p}$ is related to the ECPF of the reparameterized gNBP as in (29), its inference closely follows that of the reparameterized gNBP, except that a < 0 is not allowed and p is an auxiliary variable that cannot be fixed and shall be sampled with $(p|-) \propto e^{-\frac{h_0(1-(1-p)^a)}{a(1-p)^a}} p^{m-1}(1-p)^{-al+1}$ for 0 < a < 1 and with $(p|-) \propto p^{m-1}(1-p)^{1+h_0}$ for a=0. We find that this difference in inference between the normalized generalized gamma process and reparameterized gNBP is not significant enough to result in major performance differences in our experiments for $0 \le a < 1$.

6 Example Results

To cluster m P dimensional data vectors $\{x_i\}_{1,m}$, we construct a generalized negative binomial process (gNBP) Gaussian count-mixture model as

$$\mathbf{x}_{i} \sim \mathcal{N}(\boldsymbol{\mu}_{z_{i}}, \varphi^{-1}\mathbf{I}_{P}), \ \boldsymbol{\mu}_{k} \sim \mathcal{N}(\boldsymbol{\mu}_{0}, \varphi_{0}^{-1}\mathbf{I}_{P}), \ \varphi \sim \operatorname{Gamma}(c_{0}, 1/d_{0}),$$

$$z_{i} \sim \sum_{k=1}^{K} \frac{r_{k}}{G(\Omega)} \delta_{k}, \ m \sim \operatorname{Pois}(G(\Omega)), \ G \sim \operatorname{gGaP}(G_{0}, a, (1-p)/p),$$

$$(42)$$

where $K = \infty$ if $0 \le a < 1$ and $K \sim \text{Pois}(\frac{\gamma_0(1-p)^a}{-ap^a})$ if a < 0. The reparameterized version of the model is constructed by replacing $m \sim \text{Pois}(G(\Omega))$, $G \sim \text{gGaP}(G_0, a, (1-p)/p)$ with

$$m \sim \text{Pois}\left(G(\Omega)p/(1-p)\right), \ G \sim \text{gGaP}(H_0, a, 1).$$
 (43)

Similar to the settings in Ishwaran and Zarepour (2000), Griffin (2010), we consider noninformative priors by letting $\mu_0 \sim \mathcal{N}(0, 1000\mathbf{I}_P)$, $\varphi_0 \sim \text{Gamma}(0.001, 1/0.001)$, and $c_0 = d_0 = 0.001$. We set the gamma hyper-parameters for both γ_0 and h_0 as $e_0 = f_0 = 1$. We place a noninformative beta prior Beta(0.01, 0.01) on p when a = 0, or a uniform prior on p in (0, 1) when $a \neq 0$. Since a < 1, we place a uniform prior on $\tilde{a} = \frac{1}{1+(1-a)}$ in (0, 1).

For the gNBP, using (37), we have

$$P(z_i = k | \boldsymbol{z}^{-i}, m, \boldsymbol{x}, \gamma_0, a, p) \propto \begin{cases} (n_k^{-i} - a) \mathcal{N}\left(\boldsymbol{x}_i; \boldsymbol{\mu}_k^{-i}, \varphi^{-1} \mathbf{I}_P + (\varphi_0 + n_k \varphi)^{-1} \mathbf{I}_P\right), & \text{for } \boldsymbol{\mu}_k \in \mathcal{D}_m^{-i}; \\ \gamma_0 p^{-a} \mathcal{N}\left(\boldsymbol{x}_i; \boldsymbol{\mu}_0, \varphi_0^{-1} \mathbf{I}_P + \varphi^{-1} \mathbf{I}_P\right), & \text{if } \boldsymbol{\mu}_k \in \Omega \setminus \mathcal{D}_m^{-i}; \end{cases}$$

where
$$\boldsymbol{\mu}_k^{-i} := \frac{\varphi_0 \boldsymbol{\mu}_0 + \varphi \sum_{j:(z_j = k, j \neq i)} \boldsymbol{x}_j}{\varphi_0 + n_k^{-i} \varphi}$$
, $(\varphi|-) \sim \operatorname{Gamma}\left(c_0 + \frac{mP}{2}, \frac{1}{d_0 + \sum_k \sum_{i:z_i = k} \frac{\|\boldsymbol{x}_i - \boldsymbol{\mu}_k\|_2^2}{2}}\right)$, $(\boldsymbol{\mu}_0|-) \sim \mathcal{N}\left(\frac{\varphi_0 \sum_k \boldsymbol{\mu}_k}{10^{-3} + l\varphi_0}, \frac{1}{10^{-3} + l\varphi_0}\mathbf{I}_P\right)$ and $(\varphi_0|-) \sim \operatorname{Gamma}\left(10^{-3} + \frac{lP}{2}, \frac{1}{10^{-3} + \sum_k \frac{\|\boldsymbol{\mu}_k - \boldsymbol{\mu}_0\|_2^2}{2}}\right)$. The mass parameter γ_0 , probability parameter p and discount parameter q are sampled as in Section 5. For the reparameterized gNBP Gaussian count-mixture model, the update equations can be similarly derived.

We consider the Galaxy dataset (Roeder, 1990), which consists of the relative velocities of m=82 galaxies. We consider 15,000 MCMC iterations, with the last 10,000 samples collected. The m data points appear in a random order in each MCMC iteration. We let both p and γ_0 be inferred from the data. We consider either fixing a=-4,-2,-1,-0.50,0,0.25,0.50,0.90 or 0.99, or letting a be inferred from the data. We record in each MCMC iteration the number of clusters, the ratio of unit-size clusters and the cluster-size distribution.

Figure 5 (a) shows that the posterior mean of the number of clusters tends to decrease as the discount parameter a increases towards one for the gNBP, but rapidly increases as a increases for

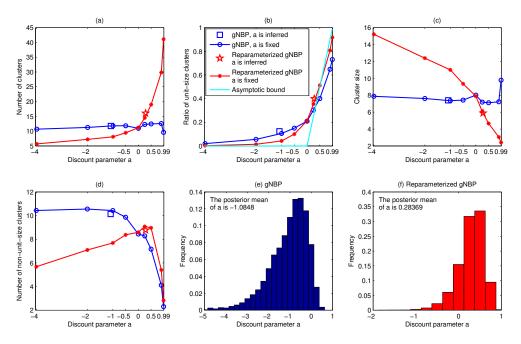


Figure 5: The posterior means of (a) the number of clusters, (b) the ratio of unit-size clusters, (c) the average size of clusters and (d) the number of non-unit-size clusters, as a function of the discount parameter a, which is either fixed at a = -4, -2, -0.50, 0, 0.25, 0.50, 0.90, 0.99 or inferred from the data, for both the generalized negative binomial process (gNBP) count-mixture model and its reparameterized version. The normalized histograms of the inferred discount parameter a for the gNBP and its reparameterized version are shown in (e) and (f), respectively. The prior asymptotic lower bound $\max\{a,0\}$ for the ratio of unit-size clusters is shown in (b). The horizontal and vertical positions of the \square sign for the gNBP and \bigstar sign for the reparameterized gNBP are the posterior means of the a inferred from the data and the corresponding y-axis values, respectively.

the reparameterized gNBP. Figure 5 (b) shows that the posterior mean of the ratio of unit-size clusters, as lower bounded by $\max\{a,0\}$ in the prior, generally increases for both the gNBP and reparameterized gNBP as a increases. Figure 5 (c) shows that the posterior mean of the average size of clusters tends to increases as a increases towards one for the gNBP, but rapidly decreases as a increases for the reparameterized gNBP. Figure 5 (d) shows that the posterior mean of the non-unit-size clusters rapidly decrease as a increases towards one for both the gNBP and reparameterized gNBP. The normalized histograms of the inferred discount parameter a are shown in Figures 5 (e) and (f) for the gNBP and reparameterized gNBP, respectively. Note that for the gNBP, the total number of clusters is reluctant to decrease as a increases towards one, this is because the cluster sizes follow the TNB(a, p) distribution in the prior, which favors a single non-unit-size cluster to be accompanied with at least a/(1-a) unit-size clusters; the total number of clusters is also reluctant to increase as the negative a decreases towards $-\infty$, this is because a negative a with a large absolute value would favor the average size of clusters to be large. Although the gNBP and reparameterized gNBP exhibit distinct behaviors on both the number and sizes of clusters, Figures 5 (b) and (d) show that they share similar trends on both the ratio and number of non-unit-size clusters as a function of the discount parameter a.

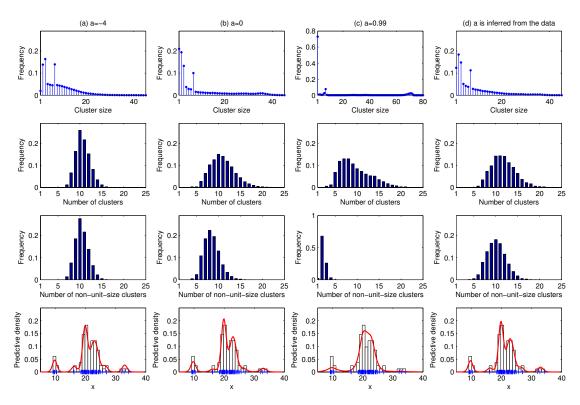


Figure 6: For the generalized negative binomial process Gaussian count-mixture model, rows from top to bottom show the posterior distributions of the sizes of clusters, number of clusters, number of non-unit-size clusters and predictive data densities, respectively. Columns from left to right show the results with a = -4, 0 and 0.9 and with a inferred from the data, respectively.

To better visualize the distinctions between the gNBP and reparameterized gNBP, we compare their posterior cluster-number and cluster-size distributions and predictive densities in Figures 6 and 7. The first row of Figures 6 shows that for the gNBP, the increase of a makes the posterior cluster-size distribution not only has a larger probability at $n_k = 1$ but also has heavier tails, encouraging large-size clusters. Whereas the first row of Figures 7 shows that for the reparameterized gNBP, the increase of a makes the posterior cluster-size distribution has a larger probability at $n_k = 1$ and lower probabilities on the tails, discouraging large-size clusters. It is interesting to notice that large posterior probabilities are often assigned to the cluster sizes of $n_k = 2$, $n_k = 3$ and $n_k = 7$. The reason is that there are seven data points around x = 10, two around x = 16, two around x = 27 and three around x = 33 that are clearly separately from the other ones and hence usually clustered together.

The second row of Figure 6 shows that for the gNBP, the increase of a drives the high density region of the posterior cluster-number PMF towards left, encouraging fewer clusters, whereas the second row of Figure 7 shows that for the reparameterized gNBP, the increase of a drives the high density region of the posterior cluster-number PMF towards right, encouraging more clusters. The thirds rows of Figures 6 and 7 show that a discount parameter a close to one would drive down the number of non-unit-size clusters for both the gNBP and reparameterized

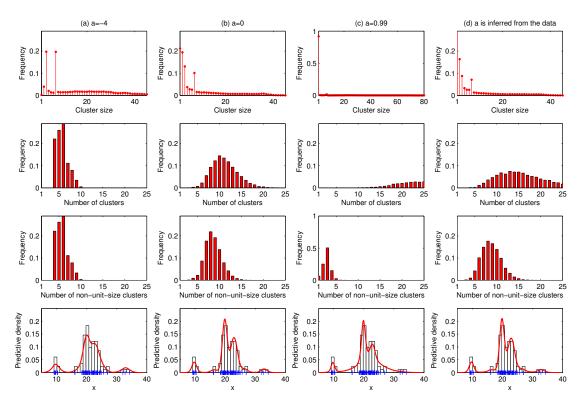


Figure 7: For the reparameterized generalized negative binomial process Gaussian count-mixture model, rows from top to bottom show the posterior distributions of the sizes of clusters, number of clusters, number of non-unit-size clusters and predictive data densities, respectively. Columns from left to right show the results with a = -4, 0 and 0.9 and with a inferred from the data, respectively.

gNBP. The estimated predictive data densities are shown in the last rows of Figures 6 and 7.

Figures 6 and 7 clearly show that different discount parameters lead to distinct model behaviors in terms of the distributions of both the number and sizes of clusters. For both the gNBP and reparameterized gNBP, by allowing the learning of a, the inferred posteriors have a large support over the values of a in $(-\infty, 1)$, covering a wide array of models with distinct cluster structures.

To show that in the posterior, the EPPFs of both the gNBP $f(z_{1:20}|m, \gamma_0, a, p)$ and reparameterized gNBP $f(z_{1:20}|m, h_0, a, p)$ are dependent on the sample size m, we test both a sample consisting of the m=20 galaxies with the lowest relative velocities and a sample consisting of all the m=82 galaxies. The parameters are set as p=0.9, $\gamma_0=1$, $h_0=1$ and a=-4, -1, 0, 0.5 or 0.9. It is clear that for both the gNBP and reparameterized gNBP, when $a \neq 0$, with the same parameters γ_0 , h_0 , a and p, the posterior distribution of the number of clusters in $z_{1:20}$ for the sample of $x_{1:20}$ is clearly different from that in $z_{1:20}$ for the sample of $x_{1:82}$. Allowing the partition probability function $P(\Pi_j|m)$ to be dependent on the sample size m is a unique feature of the cluster structure, which is not permitted in partition structures whose EPPFs are subject to the addition rule.

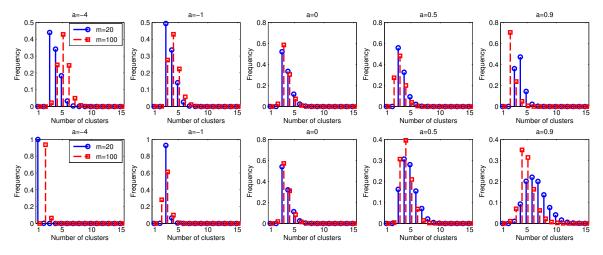


Figure 8: Comparison of the posterior distributions of the number of clusters in $z_{1:20}$ between a sample consisting of the m=20 galaxies with the lowest velocities and a sample consisting of all the m=82 galaxies, from the Galaxy dataset. The first and second rows show the results for the gNBP and reparameterized gNBP respectively. Columns from left to right are the results for a=-4, -1, 0, 0.5 and 0.9, respectively. The parameters are set as $\gamma_0=1$, $h_0=1$ and p=0.9.

7 Conclusions

Every completely random measure with a finite total random mass can be used to construct a count-mixture model, leading to a cluster structure where the cluster-number and clustersize distributions are determined by the Lévy measure. A cluster structure is distinct from a partition structure that the partition probability function of a subset of the sample is allowed to be dependent on the sample size, and the generated exchangeable random partitions are permitted to be inconsistent in distribution as the sample size varies. The paper presents a generalized negative binomial process count-mixture model, which generates a cluster structure with a Poisson distributed finite number of clusters and truncated negative binomial distributed cluster sizes. The exchangeable cluster probability function defines a fully factorized marginal likelihood for parameter estimation. The exchangeable partition probability function, which violates the addition rule when a < 0 and 0 < a < 1, defines a generalized Chinese restaurant process, whose simple prediction rule is used to develop a generalized Pólya urn sampling scheme. Both theoretic analyses and experimental results confirm that the generalized negative binomial process and its reparameterized version can be controlled to exhibit distinct behaviors on both the number and sizes of clusters. In both the original and reparameterized versions, the selection of $a \in (-\infty, 1)$ is able to effectively reflect one's prior preference on cluster-number and clustersize distributions, and the learning of a sidesteps the requirement of model selection, allowing the posterior to be averaged over a wide array of models with distinct cluster structures. Hierarchical constructions (Zhou et al., 2012b, Broderick et al., 2013, Zhou and Carin, 2012) based on the generalized negative binomial process are worth further investigation.

Acknowledgements

The author is grateful to Stephen G. Walker, Lawrence Carin, Fernando A. Quintana and Peter Müller for their helpful comments and suggestions.

References

- O. O. Aalen. Modelling heterogeneity in survival analysis by the compound Poisson distribution.

 Annals of Applied Probability, 1992.
- D. Aldous. Exchangeability and related topics. In *Ecole d'Ete de Probabilities de Saint-Flour XIII*, pages 1–198. Springer, 1983.
- C. E. Antoniak. Mixtures of Dirichlet processes with applications to Bayesian nonparametric problems. Ann. Statist., 1974.
- S. K. Bar-Lev and P. Enis. Reproducibility and natural exponential families with power variance functions. *Annals of Statistics*, 1986.
- E. Barrios, A. Lijoi, L. E. Nieto-Barajas, and I. Pruenster. Modeling with normalized random measure mixture models. *Carlo Alberto Notebooks, No. 276*, 2012.
- A. Brix. Generalized gamma measures and shot-noise Cox processes. Advances in Applied Probability, 1999.
- T. Broderick, L. Mackey, J. Paisley, and M. I. Jordan. Combinatorial clustering and the beta negative binomial process. arXiv:1111.1802v5, 2013.
- C. A Charalambides. Combinatorial methods in discrete distributions. Wiley, 2005.
- S. Engen. On species frequency models. *Biometrika*, 1974.
- M. D. Escobar and M. West. Bayesian density estimation and inference using mixtures. JASA, 1995.
- W. J. Ewens. The sampling theory of selectively neutral alleles. Theoretical Population Biology, 1972.
- S. Favaro and Y. W. Teh. MCMC for normalized random measure mixture models. to appear in Statistical Science, 2013.
- T. Ferguson. A Bayesian analysis of some nonparametric problems. Annals of Statistics, 1973.
- H. U. Gerber. From the generalized gamma to the generalized negative binomial distribution. *Insurance:* mathematics and economics, 1992.
- A. Gnedin and J. Pitman. Exchangeable Gibbs partitions and Stirling triangles. *Journal of Mathematical Sciences*, 2006.
- A. Gnedin, C. Haulk, and J. Pitman. Characterizations of exchangeable partitions and random discrete distributions by deletion properties. In N.H. Bingham and C.M. Goldie, editors, *Probability and Mathematical Genetics: Papers in Honour of Sir John Kingman*. 2009.
- P. Green. Colouring and breaking sticks: random distributions and heterogeneous clustering. In N.H. Bingham and C.M. Goldie, editors, Probability and Mathematical Genetics: Papers in Honour of Sir John Kingman. 2009.
- J. E. Griffin. Default priors for density estimation with mixture models. Bayesian Analysis, 2010.

- J. E. Griffin and S. G. Walker. Posterior simulation of normalized random measure mixtures. *Journal of Computational and Graphical Statistics*, 2011.
- K. M. Helgen, C. M. Pinto, R. Kays, L. E. Helgen, M. T. N. Tsuchiya, A. Quinn, D. E. Wilson, and J. E. Maldonado. Taxonomic revision of the olingos (Bassaricyon), with description of a new species, the Olinguito. ZooKeys, 2013.
- N. L. Hjort. Nonparametric Bayes estimators based on beta processes in models for life history data. Ann. Statist., 1990.
- P. Hougaard. Survival models for heterogeneous populations derived from stable distributions. Biometrika, 1986.
- H. Ishwaran and L. F. James. Gibbs sampling methods for stick-breaking priors. JASA, 2001.
- H. Ishwaran and M. Zarepour. Markov chain Monte Carlo in approximate Dirichlet and beta two-parameter process hierarchical models. *Biometrika*, 2000.
- L. F. James, A. Lijoi, and I. Prünster. Posterior analysis for normalized random measures with independent increments. *Scandinavian Journal of Statistics*, 2009.
- N. L. Johnson, A. W. Kemp, and S. Kotz. *Univariate Discrete Distributions*. John Wiley & Sons, 2005.
- B. Jørgensen. The Theory of Dispersion Models. London: Chapman & Hall, 1997.
- J. F. C. Kingman. Completely random measures. Pacific Journal of Mathematics, 1967.
- J. F. C. Kingman. Random partitions in population genetics. Proceedings of the Royal Society of London. A., 1978a.
- J. F. C. Kingman. The representation of partition structures. *Journal of the London Mathematical Society*, 1978b.
- J. F. C. Kingman. *Poisson Processes*. Oxford University Press, 1993.
- J. Lee, F. A. Quintana, P. Müller, and L. Trippa. Defining predictive probability functions for species sampling models. *Statistical Science*, 2013.
- A. Lijoi and I. Prünster. Models beyond the Dirichlet process. In N. L. Hjort, C. Holmes, P. Müller, and S. G. Walker, editors, *Bayesian nonparametrics*. Cambridge University Press, 2010.
- A. Lijoi, R. H. Mena, and I. Prünster. Hierarchical mixture modeling with normalized inverse-Gaussian priors. *Journal of the American Statistical Association*, 2005.
- A. Lijoi, R. H. Mena, and I. Prünster. Controlling the reinforcement in Bayesian non-parametric mixture models. *Journal of the Royal Statistical Society: Series B*, 2007.
- S. N. MacEachern and P. Müller. Estimating mixture of Dirichlet process models. *Journal of Computational and Graphical Statistics*, 1998.
- P. Müller and R. Mitra. Bayesian nonparametric inference why and how. Bayesian Analysis, 2013.
- P. Müller and F. A. Quintana. Nonparametric Bayesian data analysis. Statistical Science, 2004.
- R. M. Neal. Markov chain sampling methods for Dirichlet process mixture models. *Journal of computational and graphical statistics*, 2000.
- M. Perman, J. Pitman, and M. Yor. Size-biased sampling of poisson point processes and excursions. *Probability Theory and Related Fields*, 1992.
- J. Pitman. Exchangeable and partially exchangeable random partitions. Probability Theory and Related

- Fields, 1995.
- J. Pitman. Some developments of the Blackwell-Macqueen urn scheme. In Statistics, Probability and Game Theory; Papers in honor of David Blackwell, 1996.
- J. Pitman. Poisson-Kingman partitions. Lecture Notes-Monograph Series, pages 1–34, 2003.
- J. Pitman. Combinatorial stochastic processes. Lecture Notes in Mathematics. Springer-Verlag, 2006.
- J. Pitman and M. Yor. The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator. *The Annals of Probability*, 1997.
- M. H. Quenouille. A relation between the logarithmic, Poisson, and negative binomial series. *Biometrics*, 1949.
- E. Regazzini, A. Lijoi, and I. Prünster. Distributional results for means of normalized random measures with independent increments. *Annals of Statistics*, 2003.
- C. Ritter and M. A. Tanner. Facilitating the Gibbs sampler: the Gibbs stopper and the griddy-Gibbs sampler. *Journal of the American Statistical Association*, 1992.
- K. Roeder. Density estimation with confidence sets exemplified by superclusters and voids in the galaxies. JASA, 1990.
- R. Thibaux and M. I. Jordan. Hierarchical beta processes and the Indian buffet process. In AISTATS, 2007.
- M. C. K. Tweedie. An index which distinguishes between some important exponential families. In Statistics: Applications and New Directions: Proc. Indian Statistical Institute Golden Jubilee International Conference, pages 579–604, 1984.
- G. E. Willmot. A remark on the poisson-pascal and some other contagious distributions. *Statistics & probability letters*, 1988.
- R. L. Wolpert and K. Ickstadt. Poisson/gamma random field models for spatial statistics. *Biometrika*, 1998.
- R. L. Wolpert, M. A. Clyde, and C. Tu. Stochastic expansions using continuous dictionaries: Lévy Adaptive Regression Kernels. *Annals of Statistics*, 2011.
- M. Zhou and L. Carin. Augment-and-conquer negative binomial processes. In NIPS, 2012.
- M. Zhou and L. Carin. Negative binomial process count and mixture modeling. arXiv:1209.3442, accepted for publication in IEEE Trans. Pattern Analysis and Machine Intelligence, 2013.
- M. Zhou, H. Yang, G. Sapiro, D. Dunson, and L. Carin. Dependent hierarchical beta process for image interpolation and denoising. In *AISTATS*, 2011.
- M. Zhou, H. Chen, J. Paisley, L. Ren, L. Li, Z. Xing, D. Dunson, G. Sapiro, and L. Carin. Nonparametric Bayesian dictionary learning for analysis of noisy and incomplete images. *IEEE Trans. Image Processing*, 2012a.
- M. Zhou, L. Hannah, D. Dunson, and L. Carin. Beta-negative binomial process and Poisson factor analysis. In AISTATS, 2012b.