Percentile Classes

Binomial Theorem

Binomial theorem

An algebraic expression consisting of two terms with $+ve$ or $-ve$ sign between them is called a binomial expression.

For example: $(a+b),(2x-3y),\left(\frac{p}{x^2}-\frac{q}{x^4}\right),\left(\frac{1}{x}+\frac{4}{y^3}\right)$ etc.

Binomial theorem for positive integral index

The rule by which any power of binomial can be expanded is called the binomial theorem.
If n is a positive integer and $x,y,\in C$ then

$$(x+y)^n = {}^nC_0x^{n-0}y^0 + {}^nC_1x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 + \ldots\ldots\ldots\ldots$$
$$+{}^nC_rx^{n-r}y^r + \ldots\ldots + {}^nC_{n-1}xy^{n-1} + {}^nC_nx^0y^n$$

$$i.e.,(x+y)^n = \sum_{r=0}^{n} {}^nC_r.x^{n-r}.y^r$$

Here ${}^nC_0.{}^nC_1.{}^nC_2\ldots\ldots{}^nC_n$ are called binomial coefficient

and ${}^nC_r = \frac{n!}{r!(n-r)!}$ *for* $0 \le r \le n, {}^nC_r$ is also written as $\left(\frac{n}{r}\right)$.

Characteristics of the expansion (x + y)^n

Observing to the expansion of $(x+y)^n\ n \in N,$ we find that

(i) The total number of terms in the expansion $=(n+1)$ i.e. one more than the index n.

(ii) In every successive term of the expansion the power of x decreases by 1 and the power of second term increases by 1. Thus is equal to n (index).
(iii) The binomial coefficients of the terms which are at equidistant from the beginning and from the end are always equal $i.e., = {}^nC_r = {}^nC_{n-r}$

Thus $i.e., {}^nC_0 = {}^nC_n, {}^nC_1 = {}^nC_{n-1}, {}^nC_2 = {}^nC_{n-2}$ *etc.*

(iv) $i.e., {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$.

General terms

In the expansion of $(x+y)^n\ (r+1)^{th}$ term is called the general term which can be represented by T_{r+1}

$$T_{r+1} = {}^nC_rx^{n-r}a^r = {}^nC_r\ (first\ term)^{n-r}\ (second\ term)^r.$$

Independent term or Constant term

Independent term or constant term of a binomial expansion is the term in which exponent of the variable is zero.

Condition: $(n-r)[Power\ of\ x] = +r[Power\ of\ y] = 0,$ in the expression of $(x+y)^n$.

Number of terms in the expansion of $(a+b+c)^n$ *and* $(a+b+c+d)^n$

$(a+b+c)^n$ can be expended as : $(a+b+c)^n = \{(a+b)+c\}^n$

$$(a+b)^n + {}^nc_1(a+b)^{n-1}(c)^1 + {}^nc_2(a+b)^{n-2}(c)^2 + \ldots\ldots{}^nc_nc^n$$

$= (n+1)\ term + n\ term + (n-1) term + ... + 1\ term$

$\therefore$ Total number of terms =

$$(n+1)+(n)+(n-1)+.......+1=\frac{(n+1)(n+2)}{2}.$$

Similarly, number of terms in the expansion of

$(a+b+c+d)^n = \frac{(n+1)(n+2)(n+3)}{6}.$

Middle term

The middle term depends upon the value of n.

(1) When n is even, then total number of terms in The expansion of $(x+y)^n$ *is* $n+1$ (odd). So there is only one middle term *i.e.,* $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term. $T_{\left[\frac{n}{2}+1\right]} = {}^nC_{n/2}x^{n/2}y^{n/2}$

(2) **When n is odd,** then total number of terms in expansion of $(x+y)^n$ *is* $n+1$ (even). So, there are two middle terms *i.e.,* $\left(\frac{n+1}{2}\right)^{th}$ *and* $\left(\frac{n+3}{2}\right)^{th}$ are two middle terms.

$T_{\left[\frac{n+1}{2}\right]} = {}^nC_{\frac{n-1}{2}}x^{\frac{n+1}{2}}y^{\frac{n-1}{2}}$ and $T_{\left[\frac{n+3}{2}\right]} = {}^nC_{\frac{n+1}{2}}x^{\frac{n-1}{2}}y^{\frac{n+1}{2}}$

- When there are two middle terms in the expansion then their binomial coefficients are equal.
- Binomial coefficient of middle term is the greatest binomial coefficient.

To determine a particular term in the expansion

In the expansion of $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$, if x^m occurs in T_{r+1}, then r is given by $n\alpha - r(\alpha+\beta) = m \Rightarrow r = \frac{n\alpha - m}{\alpha+\beta}$

Thus in above expansion if constant term which is independent of x, occurs in T_{r+1} then r is determined by

$n\alpha - r(\alpha+\beta) = 0 \Rightarrow r = \frac{n\alpha}{\alpha+\beta}$

Greatest term and Greatest coefficient

(1) Greatest term: If T_r *and* T_{r+1} be the r^{th} *and* $(r+1)^{th}$ terms in the expansion of $(1+x)^n$, then

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^r}{{}^nC_{r-1}x^{r-1}} = \frac{n-r+1}{r}x$$

Let numerically, T_{r+1} be the greatest term in the above expression, Then $T_{r+1} \ge T_r$ *or* $\frac{T_{r+1}}{T_r} \ge 1$

$\therefore \frac{n-r+1}{r}|x| \ge 1$ *or* $r \le \frac{(n+1)}{(1+|x|)}|x|$ ....(i)

Now substituting values of n and x in (i), we get $r \le m+f$ or $r \le m,$ where m is a positive integer and f is a fraction such that $0 < f < 1$.

When n is even T_{m+1} is the greatest term, when n is odd T_m and T_{m+1} are the greatest terms and both are equal.

Short cut method : To find the greatest term (numerically) in the expansion of $(1+x)^n$.

(i) Calculate $m = \left|\frac{x(n+1)}{x+1}\right|$

(ii) If m is integer, then T_m and T_{m+1} are equal and both are greatest term.

(iii) If m is not integer, then $T_{[m]+1}$ is the greatest term, where $[.]$ denotes the greatest integral part.

(2) Greatest coefficient

(i) If n is even, then greatest coefficient is ${}^nC_{n/2}$.

(ii) If n is odd, then greatest coefficient are ${}^nC_{\frac{n+1}{2}}$ and ${}^nC_{\frac{n+3}{2}}$.

Some important points

(1) Pascal's Triangle

$$\begin{array}{ccccccccccc|c}
 & & & & & 1 & & & & & & (x+y)^0 \\
 & & & & 1 & & 1 & & & & & (x+y)^1 \\
 & & & 1 & & 2 & & 1 & & & & (x+y)^2 \\
 & & 1 & & 3 & & 3 & & 1 & & & (x+y)^3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & (x+y)^4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & (x+y)^5
\end{array}$$

Pascal's triangle gives the direct binomial coefficients.

(2) **Method for finding terms free from radicals or rational terms in the expansion of** $\left(a^{1/p}+b^{1/q}\right)^N \forall a,b \in$ **prime**

numbers: Find the general term $T_{r+1}={}^NC_r\left(a^{1/p}\right)^{N-r}\left(b^{1/q}\right)^r={}^NC_r a^{\frac{N-r}{p}}.b^{\frac{r}{q}}$

Putting the values of $0 \le r \le N$, when indices of a and b are integers.

- Number of irrational terms = Total terms - Number rational terms.

Applications of binomial theorem

(a) With the help of binomial theorem we can find out the value of sq. root, cube root and 4lh root etc. of the given number up to any decimal places.

(b) To find the sum of infinite series. We can compare the given infinite series with the expansion of $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+......$ and by finding the value of x and n and putting in $(1+x)^n$ the sum of the series is determined.

T Tips & Tricks

❖ The number of term in the expansion of $(x+y)^n$ are $(n+1)$.

❖ In any term of expansion of $(x+y)^n$ the sum of the exponents of x and y is always constant = n.

❖ The binomial coefficients ${}^nC_0, {}^nC_1, {}^nC_2,.........$ equidistant from beginning and end are equal *i.e.*, ${}^nC_r={}^nC_{n-r}$.

❖ $(x+y)^n$ = Sum of odd terms + Sum of even terms.

- In the expansion of $(x+y)^n, n \in N \frac{T_{r+1}}{T_r} = \left(\frac{n-r+1}{r}\right)\frac{y}{k}$.
- If the coefficients of p^{th}, q^{th} terms in the expansion of $(1-x)^n$ are equal, then $p+q=n+2$.
- The coefficient of x^{n-1} in the expansion of $(x-1)(x-2).....(x-n) = -\frac{n(n+1)}{2}$
- The coefficient of x^{n-1} in the expansion of $(x+1)(x+2).....(x+n) = \frac{n(n+1)}{2}$
- For finding the greatest term in the expansion of $(x+y)^n$. we rewrite the expansion in this form

  $(x+y)^n. = x^n\left[1+\frac{y}{x}\right]^n$.

  Greatest term in $(x+y)^n. = x^n$. Greatest term in $\left(1+\frac{y}{x}\right)^n$.
- If n is odd, then $(x+y)^n + (x-y)^n$ *and* $(x+y)^n - (x-y)^n$, both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$.
- If n is even, then $(x+y)^n + (x-y)^n$ has $\left(\frac{n}{2}+1\right)$. terms and $(x+y)^n - (x-y)^n$ has $\frac{n}{2}$ terms.
- There are infinite number of terms in the expansion of $(1+x)^n$, when n is a negative integer or a fraction.
- The number of terms in the expansion of $(x_1+x_2+......+x_r)^n = {}^{n+r-1}C_{r-1}$.
- If the coefficient of the $r^{th}, (r+1)^{th}$ *and* $(r+2)^{th}$ terms in the expansion of $(1+x)^n$ are in H.P., then $n+(n-2r)^2 = 0$.
- If coefficient of $r^{th}, (r+1)^{th}$ *and* $(r+2)^{th}$ terms in the expansion of $(1+x)^n$ are in A.P., then $n^2 + n(4r+1) + 4r^2 - 2 = 0$.

Exercise 01
Binomial

1. $\frac{1}{25} + \frac{1}{5.8} + \frac{1}{8.11} + ...\frac{1}{(3n-1)(3n+2)} =$

   (a) $\frac{n}{6n-4}$ (b) $\frac{n}{6n+3}$
   (c) $\frac{n}{6n+4}$ (d) $\frac{n+1}{6n+4}$

2. The total number of terms in the expansion of $(x+a)^{100} + (x-a)^{100}$ after simplification will be

   (a) 202 (b) 51 (c) 50 (d) None of these

3. The value of $\left(\sqrt{5}+1\right)^5 - \left(\sqrt{5}-1\right)^5$ is

   (a) 252 (b) 352 (c) 452 (d) 532

4. The larger of $99^{50} + 100^{50}$ and 101^{50} is

   (a) $99^{50} + 100^{50}$ (b) Both are equal

(c) 101^{50} (d) None of these

5. The greatest integer which divides the number $101^{50}-1$ is
(a) 100 (b) 1000 (c) 10000 (d) 100000

6. The last digit in 7^{300} is
(a) 7 (b) 9 (c) 1 (d) 3

General term, Coefficient of any power of x, Independent term, Middle term and Greatest term & Greatest coefficient

7. Coefficient of x^5 in $\left(1+2x+3x^2+....\right)^{3/2}$ is
(a) 19 (b) 20 (c) 21 (d) 22

8. Coefficient of x in the expansion of $\left(x^2+\frac{a}{x}\right)^5$ is.
(a) $9a^2$ (b) $10a^3$ (c) $10a^2$ (d) $10a$

9. The ninth term of the expansion $\left(3x-\frac{1}{2x}\right)^8$ is
(a) $\frac{1}{512x^9}$ (b) $\frac{-1}{512x^9}$ (c) $\frac{-1}{256x^8}$ (d) $\frac{1}{256x^8}$

10. If p and q be positive, then the coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ will be
(a) Equal (b) Equal in magnitude but opposite in sign
(c) Reciprocal to each other (d) None of these

11. In the expansion of $\left(x-\frac{1}{x}\right)^6$, the constant term is
(a) –20 (b) –82 (c) –81 (d) 0

12. In the coefficient of x^5 in the expansion of $\left(x^2-x-2\right)^5$ is
(a) –83 (b) –82 (c) –81 (d) 0

13. In the expansion of $\left(1+x+x^3+x^4\right)^{10}$ the coefficient of x^4 is.
(a) ${}^{40}C_4$ (b) ${}^{10}C_4$ (c) 210 (d) 310

14. The term independent of x in the expansion $\left(x^2-\frac{1}{3x}\right)^9$ is
(a) $\frac{28}{81}$ (b) $\frac{28}{243}$ (c) $-\frac{28}{243}$ (d) $-\frac{28}{81}$

15. The largest term in the expansion of $(4+2x)^{49}$ where $x=1/3$ is
(a) 3^{rd} (b) 5^{th} (c) 8^{th} (d) None of these

16. In the expansion of $\left(2x^2-\frac{1}{x}\right)^{12}$, the term independent of x is

(a) 10^{th} (b) 9^{th} (c) 8^{th} (d) 7^{th}

17. If the middle term in the expansion of $\left(x^2+\frac{1}{x}\right)^n$ is $924x^6$ then $n=$

(a) 10 (b) 12 (c) 14 (d) None of these

18. The middle term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$ is

(a) ${}^{10}C_4\frac{1}{x}$ (b) ${}^{10}C_5$ (c) ${}^{10}C_5x$ (d) ${}^{10}C_7x^4$

19. The term independent of x in the expansion of $\left(x^2-\frac{3\sqrt{3}}{x^3}\right)$ is

(a) 153090 (b) 150000 (c) 150090 (d) 153180

Exercise 01 (Solutions)

1.Sol **(c)** $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.....\frac{1}{(3n-1)(3n+2)}$

$$=\frac{1}{3}\left[\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{11}\right)+\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)\right]$$

$$=\frac{1}{3}\left[\frac{1}{2}-\frac{1}{3n+2}\right]=\frac{1}{3}\left[\frac{3n+2-2}{2(3n+2)}\right]=\frac{(3n)}{6(3n+2)}=\frac{n}{6n+4}$$

2.**Sol** **(b)** We know

$\frac{1}{2}\left\{(1+a)^n+(1-a)^n\right\}={}^nC_2a^2+{}^nC_4a^4+.....$

Therefore, number of terms in expansion of

$\left\{(x+a)^{100}+(x=a)^{100}\right\}$ is 51

3.Sol **(b)** $\left(\sqrt{5}+1\right)^5-\left(\sqrt{5}-1\right)^5$

$$=2\left\{{}^5C_1\left(\sqrt{5}\right)^4+{}^5C_3\left(\sqrt{5}\right)^2+{}^5C_5.1\right\}=352$$

4.**Sol** **(c)** We have

$101^{50}=(100+1)^{50}=100^{50}+50.100^{49}+\frac{50.49}{2.1}100^{48}+....$ .... (i)

and $99^{50}=(100-1)^{50}=100^{50}-50.100^{49}+\frac{50.49}{2.1}100^{48}.....$ .... (ii)

Subtracting (ii) from (i), we get

$$101^{50}-99^{50}=100^{50}+2\frac{50.49.48}{1.2.3}100^{47}>100^{50}$$

Hence, $101^{50} > 100^{50} + 99^{50}$

5.Sol **(c)** $(1+100)^{100} = 1 + 100.100 + \frac{100.99}{1.2}.(100)^2 + \frac{100.99.98}{1.2.3}(100)^3 + ...$

$$(100)^{100} - 1 = 100.100\left[1 + \frac{100.99}{1.2} + \frac{100.99.98}{1.2.3}.100 + ...\right]$$

From above it is clear that,

$(100)^{100} - 1$ is divisible by $(100)^2 = 10000$

6. **Sol** **(c)** We have $7^2 = 49 = 50 - 1$

Now, $7^{300} = \left(7^2\right)^{150} = (50-1)^{150}$

$$= {}^{150}C_0(50)^{150}(-1)^0 + {}^{150}C_1(50)^{149}(-1)^1 + ... + {}^{150}C_{150}(50)^0(-1)^{150}$$

Thus the last digits of 7^{300} are ${}^{150}C_{150}.1.1$ i.e. 1

7.Sol **(c)** $\left(1+2x+3x^2+4x^2+....\right)^{3/2}$

$$\because \{1+x\}^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

$$\because \left\{(1-x)^{-2}\right\}^{\frac{3}{2}} = (1-x)^{-3}$$

$$= 1 + \frac{3x}{1!} + \frac{3.4}{2!}x^2 + \frac{3.4.5.}{3!}x^3 + \frac{3.4.5.6}{4!}x^4 + \frac{3.4.5.6.7}{5!}x^5 + ...$$

$$\therefore x^5 = \frac{3.4.5.6.7}{5!} = 21$$

8.**Sol** **(b)** In the expansion of $\left(x^2 + \frac{a}{x}\right)^5$ the general terms is

$$T_{r+1} = {}^5C_r\left(x^2\right)^{5-r}\left(\frac{a}{x}\right)^r = {}^5C_r a^r x^{10-3r}$$

Here, exponent x is $x = 10a^3.x$

Hence coefficient of x is $10a^3$

9.Sol **(d)** $\left(3x\frac{1}{2x}\right)^8$

When we expand the given binomial, we get 9 terms,

9^{th} term is the last term.

$T_{r+1} = {}^nC_r x^{n-r} a^r$. For $(x+a)^n$

$$T_9 = {}^8C_8(3x)^0\left(\frac{-1}{2x}\right)^8 = \frac{+1}{256x^8}$$

10. **Sol** **(a)** Coefficient of x^p is ${}^{(p+q)}C_p$ and coefficient of x^q is ${}^{(p+q)}C_q$. But ${}^{(p+q)}C_q, \left(\therefore {}^nC_r = {}^nC_{n-r}\right)$

11. Sol **(a)** $6 - r(2) = 0 \Rightarrow r = 3$

Hence, ${}^6C_3 (x)^3 \left(\frac{-1}{x}\right)^3 = -20$

12. Sol **(c)** $\left(x^2 - x - 2\right)^5 = (x-2)^5 (1+x)^5$

$= \left[{}^5C_0 x^5 - {}^5C_1 x^4 \times 2 + ...\right]\left[{}^5C_0 + {}^5C_1 x + ...\right]$

Collecting the coefficient of x^5

$1 - 5.5.2 + 10.10.4 - 10.10.8 + 5.5.16 - 32$

$1 - 50 + 400 - 800 + 400 - 32 = -81.$

13. **Sol** **(d)** $\left(1 + x + x^3 + x^4\right)^{10} = (1+x)^{10}\left(1+x^3\right)^{10}$

$= \left(1 + {}^{10}C_1.x + {}^{10}C_2.x^2 + ...\right)\left(1 + {}^{10}C_1.x^3 + {}^{10}C_2.x^6 + ...\right)$

$\therefore$ Coefficient of $x^4 = {}^{10}C_1 . {}^{10}C_1 + {}^{10}C_4 = 310.$

14. **Sol** **(b)** In $\left(x^2 - \frac{1}{3x}\right)^9$

$T_{r+1} = {}^9C_r \left(x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r = {}^9C_r x^{18-2r} \frac{(-1)^r}{3^r} x^{-r}$

It is independent of x.

$\therefore 18 - 3r = 0 \Rightarrow r = 6$

$\therefore T_7 = {}^9C_6 x^{18-12} \frac{(-1)^6}{3^6} x^{-6} = {}^9C_6 \frac{(-1)^6}{3^6} = \frac{28}{243}$

15. **Sol** **(c)** $(4+2x)^{49}$ where $x = \frac{1}{3}$

Let T_r and T_{r+1} denote r^{th} and $(r+1)^{th}$

term in the expansion of $(4+2x)^{49}$

$T_{r+1} = {}^{49}C_r (4)^{49-r} (2x)^r$

$T_r = {}^{49}C_{r-1} (4)^{50-r} (2x)^{r-1}$

$\frac{T_{r+1}}{T_r} = \frac{{}^{49}C_r}{{}^{49}C_{r-1}} (4)^{49-r-50} (2x)^{r-r+1}$

$= \frac{50-r}{r} = \frac{1}{4}.2x = \frac{50-r}{r}\frac{x}{2}$

When $x = \frac{1}{3}, \frac{T_{r+1}}{T_r} = \frac{50-r}{r}.\frac{1}{6}$

Now, $x = \frac{T_{r+1}}{T_r} \geq 1 \Rightarrow \frac{50-r}{r}.\frac{1}{6} \geq 1 \Rightarrow 50 - r \geq 6r$

$\Rightarrow r \leq \frac{50}{7} = 7.1$

So, $r = 7$

$T_{r+1} = T_{7+1} = T_8$

$\therefore 8^{th}$ term is the greatest term.

16. Sol **(b)** $T_{r+1} = {}^{12}C_1\left(2x^2\right)^{12-r}(-1)^r\left(\frac{1}{x}\right)^r$

For term independent of x.

$24-3r=0 \Rightarrow r=8.$ So, 9^{th} term is independent of x.

17. **Sol** **(b)** Since n is even therefore $\left(\frac{n}{2}+1\right)^{th}$ term is middle term.

hence ${}^{n}C_{n/2}\left(x^2\right)^{n/2}\left(\frac{1}{x}\right)^{n/2}=924x^6$

$\Rightarrow x^{n/2}=x^6 \Rightarrow n=12.$

18. **Sol** **(b)** Middle term of $\left(x+\frac{1}{x}\right)^{10}$ is $T_6 = {}^{10}C_5$.

19. Sol **(a)** $T_{r+1} = {}^{10}C_r\left(x^2\right)^{10-r}\left(\frac{-3\sqrt{3}}{x^3}\right)^r$

For term independent of $x, 20-2r-3r=0 \Rightarrow r=4$

$\therefore T_{4+1} = {}^{10}C_4(-3)^4\left(\sqrt{3}\right)^4 = 153090.$