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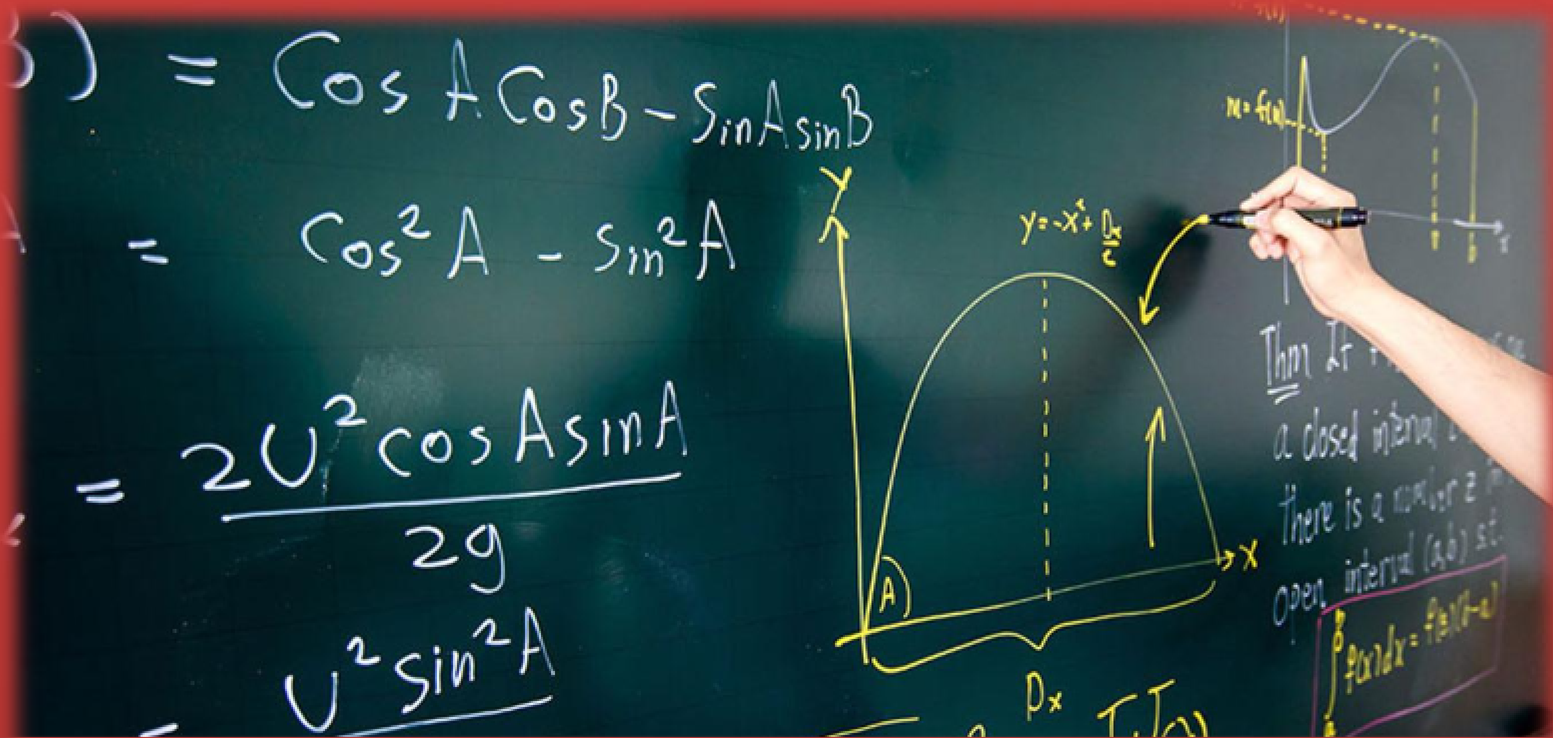

Algebra – 01

Theory of Equations

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1. Introduction

This unit is about how to solve quadratic equations. A quadratic equation is one which must contain a term involving x^2, e.g. $3x^2$, $-5x^2$ or just x^2 on its own. It may also contain terms involving x, e.g. 5x or −7x, or 0.5x. It can also have constant terms - these are just numbers: $6, -7, \frac{1}{2}$

It cannot have terms involving higher powers of x, *like* x^3. It cannot have terms like $\frac{1}{x}x$ in it.

In general a quadratic equation will take the form

$$ax^2 + bx + c = 0$$

a can be any number excluding zero. b and c can be any numbers including zero. If b or c is zero then these terms will not appear.

Key Point

A quadratic equation takes the form

$$\boldsymbol{ax^2 + bx + c = 0}$$

Where a, b and c are numbers. The number a cannot be zero

In this unit we will look at how to solve quadratic equations using for methods:

- Solution by factorisation
- Solution by completing the square
- Solution using a formula
- Solution using graphs

Factorisation and use of the formula are particularly important.

2. Solving quadratic equations by factorisation

In this section we will assume that you already know how to factorise a quadratic expression. If this is not the case you can study other material in this series where factorization is explained.

Example:

Suppose we wish to solve $3x^2 = 27$.

We begin by writing this in the standard form of a quadratic equation by subtracting 27 from each side to give $3x^2 - 27 = 0$.

We now look for common factors. By observation there is a common factor of 3 in both terms. This factor is extracted and written outside a pair of brackets. The contents of the brackets are adjusted accordingly.

$$3x^2 - 27 = 3(x^2 - 9) = 0$$

Notice here the difference of two squares which can be factorized as:

$$3(x^2 - 9) = 3(x - 3)(x + 3) = 0$$

If two quantities are multiplied together and the result is zero then either or both of the quantities must be zero. So either

$$x - 3 = 0 \text{ or } x + 3 = 0$$

So that,

$$x = 3 \text{ or } x = -3$$

These are the two solutions of the equation.

Example:

Suppose we wish to solve $5x^2 + 3x = 0$

We look to see if we can spot any common factors. There is a common factor of x in both terms. This extracted and written in front of pair of brackets:

$$x(5x + 3) = 0$$

Then either $x = 0$ *or* $5x + 3 = 0$ from which $x = \frac{(-3)}{5}$. These are the two solutions.

Example:

Suppose we wish to solve $x^2 - 5x + 6 = 0$

We factorise the quadratic by looking for two numbers which multiply together to give 6, and add to give −5. Now

$$x^2 - 5x + 6 = 0$$
$$x^2 - 3x - 2x + 6 = 0$$
$$x(x-3) - 2(x-3) = 0$$
$$(x-3)(x-2) = 0$$

From which,

$$x - 3 = 0 \; or \; x - 2 = 0$$

So that,

$$x = 3 \; or \; x = 2$$

These are the two solutons.

Example:
Suppose we wish to solve the equation $2x^2 - 3x - 2 = 0$
To factorise this we seek two numbers which multiply to give -4 (*the coefficient of* x^2 *multiplied by the constant term*) and which add together to give 3.

$$4 \times -1 = -4 \qquad 4 + -1 = 3$$

So the two numbers are 4 *and* -1. We use these two numbers to write $3x$ *as* $4x - x$ and then factorise as follows:

$$2x^2 + 3x - 2 = 0$$
$$2x^2 + 4x - x - 2 = 0$$
$$2x(x+2) - (x+2) = 0$$
$$(x+2)(2x-1) = 0$$

From which

$$x + 2 = 0 \; or \; 2x - 1 = 0$$

So that,

$$x = -2 \; or \; x = \frac{1}{2}$$

These are the two solutions.

Example:
Suppose we wish to solve $4x^2 + 9 = 12x$.
First of all we write this in the standard form;

$$4x^2 - 12x + 9 = 0$$

We should look to see if there is a common factor – but there is not. To factorise we seek two numbers which multiply to give 36 (the coefficient of x^2 multiplied by the constant term) and add to give -12. Now, by inspection.

$$-6 \times -6 = 36 \qquad -6 \pm 6 = -12$$

So the two numbers are -6 *and* -6. We use these two number to write $-12x$ *as* $-6x - 6x$ and proceed to factorise as follows:

$$4x^2 - 12x + 9 = 0$$
$$4x^2 - 6x - 6x + 9 = 0$$
$$2x(2x-3) - 3(2x-3) = 0$$
$$(2x-3)(2x-3) = 0$$

From which,

$$2x - 3 = 0 \; or \; 2x - 3 = 0$$

So that,

$$x = \frac{3}{2} \; or \; x = \frac{3}{2}$$

These are the two solutions, but we have obtained the same answer twice. So we can have quadratic equations for which the solution is repeated.

Example:
Suppose we wish to solve $x^2 - 3x - 2 = 0$
We are looking for two numbers which multiply to give -2 and add together to give -3. Never mind how hard you try you will not find any such two number. So this equation will not factorise. We need another approach. This is the topic of the next section.

3. Solving quadratic equations by completing the square.

Example:

Suppose we wish to solve $x^2 - 3x - 2 = 0$

In order to complete the square we look at the first two terms, and try to write them in the form $(\quad)^2$. Clearly we need an x in the brackets:

$(x+?)^2$ because when the term in brackets is squared this will give the term x^2.

We also need the number $-\frac{3}{2}$, which is half of the coefficient of x in the quadratic equation, $(x-\frac{3}{2})^2$ because when the term in brackets is square this will give the term $-3x$.

However, removing the brackets from $\left(x-\frac{3}{2}\right)^2$ we see there is also a term $(-\frac{3}{2})^2$ which we do not want, and so we subtract this again. So the quadratic equation can be written.

$$x^2 - 3x - 2 = \left(x-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2 - 2 = 0$$

Simplifying,

$$\left(x-\frac{3}{2}\right)^2 - \frac{9}{4} - 2 = 0$$

$$\left(x-\frac{3}{2}\right)^2 - \frac{17}{4} = 0$$

$$\left(x-\frac{3}{2}\right)^2 = \frac{17}{4}$$

$$x - \frac{3}{2} = \frac{\sqrt{17}}{2} \text{ or } -\frac{\sqrt{17}}{2}$$

$$x = \frac{3}{2} + \frac{\sqrt{17}}{2} \text{ or } x = \frac{3}{2} - \frac{\sqrt{17}}{2}$$

We can write these solutions as,

$$x = \frac{3+\sqrt{17}}{2} \text{ or } \frac{3-\sqrt{17}}{2}$$

Again we have two answers. These are exact answers. Approximate values can be obtained using a calculator.

Exercise 2:

(a) Show that $x^2 + 2x = (x+1)^2 - 1$.
Hence, use completing the square to solve $x^2 + 2x - 3 = 0$.

(b) Show that $x^2 - 6x = (x-3)^2 - 9$.
Hence use completing the square to solve $x^2 - 6x = 5$.

(c) Use completing the square to solve $x^2 - 5x + 1 = 0$.

(d) Use completing the square to solve $x^2 + 8x + 4 = 0$.

4. Solving quadratic equations using a formula:

Consider the general quadratic equation $ax^2 + bx + x = 0$

There is a formula for solving this: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. It is so important that you should learn it.

Key Point

Formula for solving $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We will illustrate the use of this formula in the following example.

Example:

Suppose we wish to solve $x^2 - 3x - 2 = 0$

Comparing this with the general form $ax^2 + bx + c = 0$ *we see that* $a = 1, b = -3$ *and* $c = -2$.

These values are substituted into the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$$
$$= \frac{3 \pm \sqrt{9+8}}{2}$$
$$= \frac{3 \pm \sqrt{17}}{2}$$

These solutions are exact.

Example:

Suppose we wish to solve $3x^2 = 5x - 1$.

First we write this in the standard form as $3x^2 - 5x + 1 = 0$ in order to identify the values of $a, b\ and\ c$.

We see that $a = 3, b = -5\ and\ c = 1$. These values are substituted into the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times 1}}{2 \times 3}$$
$$= \frac{5 \pm \sqrt{25 - 12}}{6}$$
$$= \frac{5 \pm \sqrt{13}}{6}$$

Again there are two exact solutions. Approximate values could be obtained using a calculator.

Exercise 3:

Use the quadratic formula to solve the following quadratic equations.

(a) $x^2 - 3x + 2 = 0$
(b) $4x^2 - 11x + 6 = 0$
(c) $x^2 - 5x - 2 = 0$
(d) $3x^2 + 12x + 2 = 0$
(e) $2x^2 = 3x + 1$
(f) $x^2 + 3 = 2x$
(g) $x^2 + 4x = 10$
(h) $25x^2 = 40x - 16$

5. Solving quadratic equations by using graphs

In this section we will see how graphs can be used to solve quadratic equations. If the coefficient of x^2 in the quadratic expression $ax^2 + bx + c$ is positive then a graph of $y = ax^2 + bx + c$ will take the form shown in Figure 1(a). If the coefficient of x^2 is negative the graph will take the form shown in Figure 1(b).

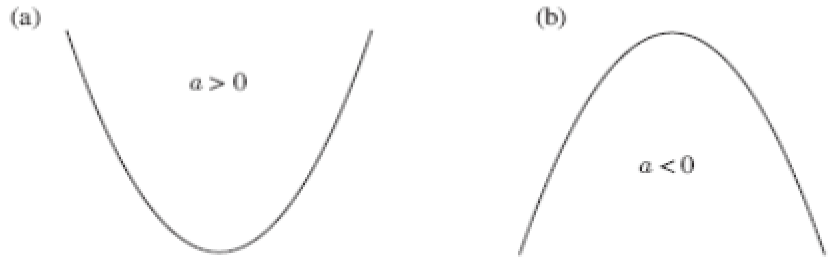

Figure 1. Graphs of $y = ax^2 + bx + c$ have these general shapes

We will now add x and y axes. Figure 2 shows what can happen when we plot a graph of $y = ax^2 + bx + c$ for the case in which a is positive.

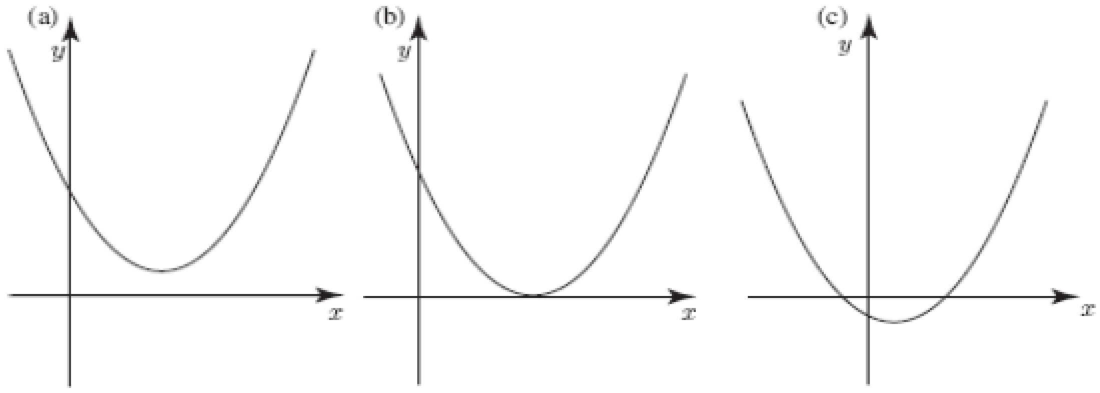

Figure 2. Graphs of $y = ax^2 + bx + c$ when a is positive

The horizontal line, the x axis, corresponds to points on the graph where $y = 0$. So points where the graph touches or crosses this axis correspond to solutions of $ax^2 + bx + c = 0$.

In Figure 2, the graph in (a) never cuts or touches the horizontal axis and so this corresponds to a quadratic equation $ax^2 + bx + c = 0$ having no real roots.

The graph in (b) just touches the horizontal axis corresponding to the case in which the quadratic equation has two equal roots, also called 'repeated roots'.

The graph in (c) cuts the horizontal axis twice, corresponding to the case in which the quadratic equation has two different roots.

What we have done in Figure 2 for the the case in which a is positive we can do for the case in which a is negative. This case is shown in Figure 3.

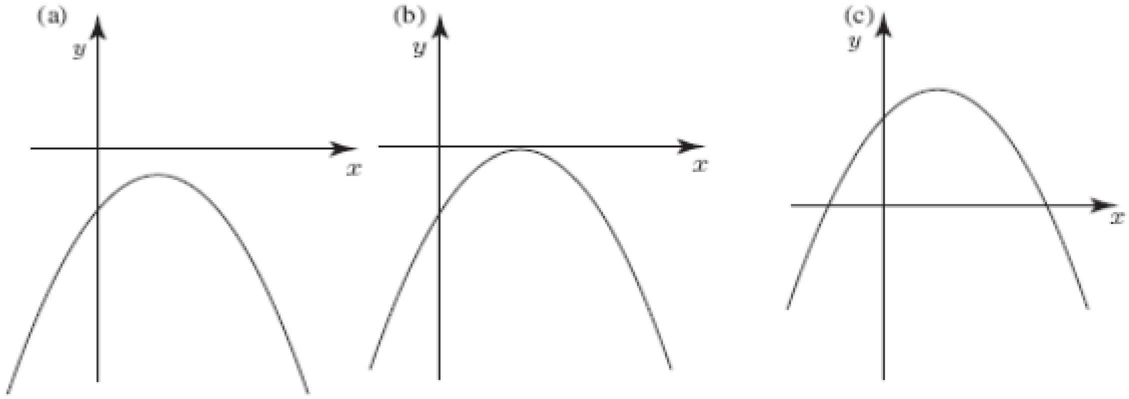

Figure 3. Graphs of $y = ax^2 + bx + c$ when a is negative

Referring to Figure 3: in case (a) there are no real roots. In case (b) there will be repeated roots. Case (c) corresponds to there being two real roots.

Example

Suppose we wish to solve $x^2 - 3x - 2 = 0$.

We consider $y = x^2 - 3x - 2$ and produce a table of values so that we can plot a graph.

x	-2	-1	0	1	2	3	4	5
x^2	4	1	0	1	4	9	16	25
$-3x$	6	3	0	-3	-6	-9	-12	-15
-2	-2	-2	-2	-2	-2	-2	-2	-2
$x^2 - 3x - 2$	8	2	-2	-4	-4	-2	2	8

From this table of values a graph can be plotted, or sketched as shown in Figure 4. From the graph we observe that solutions of the equation $x^2 - 3x - 2 = 0$ lie between -1 and 0, and between 3 and 4.

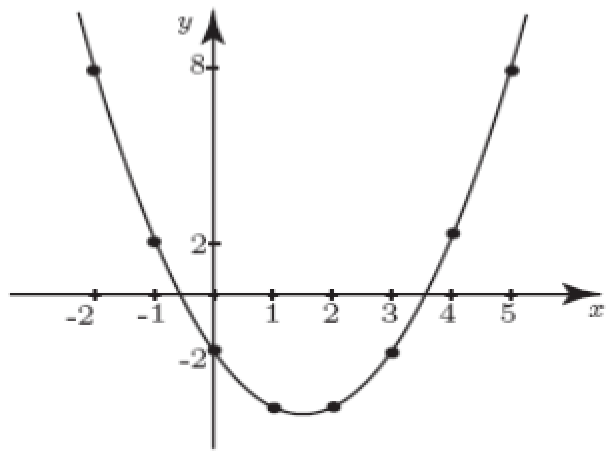

Figure 4. Graph of $y = x^2 - 3x - 2$

Example

We can use the same graph to solve other equations. For example to solve $x^2 - 3x - 2 = 6$ we can simply locate points where the graph crosses the line $y = 6$ as shown in Figure 5.

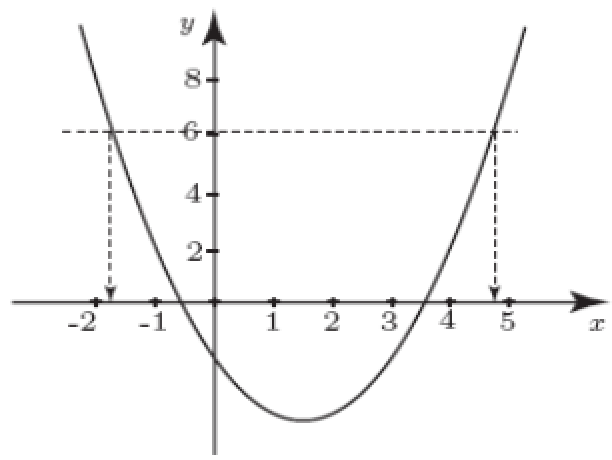

Figure 5. Using the graph of $y = x^2 - 3x - 2$ to solve $x^2 - 3x - 2 = 6$

Example

We can use the same graph to solve $x^2 - 3x - 5 = 0$ by rewriting the equation as $x^2 - 3x - 2 - 3 = 0$ and then as $x^2 - 3x - 2 = 3$. We can then locate points where the graph crosses the line $y = 3$ in order to solve the equation.

Exercise 4

By plotting the graph $y = x^2 - 5x + 2$, solve the equation $x^2 - 5x + 2 = 0$, giving your answers to 1 decimal place.

Use your graph to solve the equations $x^2 - 5x + 2 = 4$, $x^2 - 5x - 1 = 0$, $x^2 - 5x + 2 = 2x$.

Answers

Exercise 1

a) 1, 2 b) $2, -2$ c) 5, -1 d) 0, 5 e) $-4, -15$ f) -2, $\frac{3}{2}$ g) 2, $-\frac{3}{2}$ h) 2, $\frac{3}{4}$

Exercise 2

a) $1, -3$ b) $3 \pm \sqrt{14}$ c) $\dfrac{5 \pm \sqrt{21}}{2}$ d) $-4 \pm \sqrt{12}$

Exercise 3

a) 1, 2 b) 2, $\frac{3}{4}$ c) $\dfrac{5 + \sqrt{33}}{2}$ d) $\dfrac{-12 \pm \sqrt{120}}{6}$ e) $\dfrac{3 \pm \sqrt{17}}{4}$ f) No real roots
g) $-2 \pm \sqrt{14}$ h) $\frac{4}{5}$ repeated.

Exercise 4

a) 4.6, 0.4 b) 5.4, -0.4 c) 5.2, -0.2 d) 6.7, 0.3

LANGUAGE OF ALGEBRA

Statement	Meaning
$a = b$	*a is equal to b*
$a \neq b$	*a is not equal to b*
$a < b$	*a is less than b*
$a > b$	*a is greater than b*
$a \leq b$	*a is less than or equal to b*
$a \geq b$	*a is greater than or equal to b*
$a \nless b$	*a is not less than*
$a \ngtr b$	*a is not greater than b.*

QUADRATIC INEQUATIONS

The following properties of inequations will be useful in solving the problem on inequations:

1. Square of any real quantity ≥ 0
2. If a, b, c are three real numbers such that $a \geq b$ then, $a \pm c \geq b \pm c$
3. (i) If a, b, c are three real numbers such that $a \geq b$ and $c > 0$, then $ac \geq bc$.
   (ii) If $a \leq b$ and $c > 0$, then $ac \leq bc$
4. (i) If a, b, c are three real numbers such that $a \geq b$ and $c < 0$, then $ac \leq bc$
   (ii) If $a \leq b$ and $c < 0$, then $ac \geq bc$
5. (i) If $ab > 0$, then a and b are either both positive or both negative.
   (ii) If $ab < 0$, then a and b are of opposite signs, i.e., either $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

MAXIMUM OR MINIMUM VALUE OF A QUADRATIC EQUATION

At $x = -b/2a$ we get the maximum or minimum value of the quadratic expression.

(i) When $a > 0$ (in the equation $ax^2 + bx + c$) the expression gives minimum value $y = \frac{4ac-b^2}{4a}$

(ii) When $a < 0$ (in the equation $ax^2 + bx + c$) the expression gives maximum value $y = \frac{4ac-b^2}{4a}$

Tip – 1

General Quadratic equation will be in the form of $ax^2 + bx + x = 0$

The value of 'x' satisfying the equation are called roots of the equation.

- The value of roots. α and $\beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Sum of the roots $= \alpha + \beta = \frac{-b}{a}$
- Product of the roots $= \alpha \times \beta = \frac{c}{a}$
- If c and a are equal then the roots are reciprocal to each other.
- If $b = 0$, then the roots are equal and are opposite in sign.

Tip – 2

Let D denote the discriminant, $D = b^2 - 4ac$. Depending on the sign and value of D, nature of the roots would be as follows:

- $D < 0$ and $|D|$ is not a perfect square:
  Roots will be in the form of $\alpha + i\beta$ and $\alpha - i\beta$ where p and q are the real and imaginary parts of the complex roots. α is rational and β is irrational.
- $D < 0$ and $|D|$ is a perfect square:
- $D = 0$
  Roots are real and equal. $X = \frac{-b}{2a}$

Tip – 3

- $D > 0$ and D is not a perfect square:
  Roots are conjugate surds
- $D > 0$ and D is a perfect square:
  Roots are real, rational and unequal.

Tip – 4

Signs of the roots: Let P be product of roots and S be their sum.

- $P > 0, S > 0$: Both roots are positive
- $P > 0, S < 0$: Both roots are negative
- $P < 0, S > 0$: Numerical smaller root is negative and the other root is positive
- $P < 0, S < 0$: Numerical larger root is negative and the other root is positive.

Tip – 5

- Minimum and maximum values of $ax^2 + bx + c = 0$
- If $a > 0$: minimum value $= \frac{4ac - b^2}{4a}$ and occurs at $x = \frac{-b}{2a}$
- If $a < 0$: maximum value $= \frac{4ac - b^2}{4a}$ and occurs at $x = \frac{-b}{2a}$

Tip – 6

If $A_nX^n + A_{n-1}X^{n-1} + \cdots A_1X + A_0 = 0$ then

- Sum of the roots $= \frac{-A_{n-1}}{A_n}$
- Sum of roots taken two at a time $= A_{n-2}/A_n$
- Sum of roots taken three at a time $= \frac{-A_{n-3}}{A_n}$ and so on
- Product of the roots $= \frac{[(-1)^n A_0]}{A_n}$

Tip – 7

Finding a quadratic equation:

- If roots are given: $(x - a)(x - b) = 0 \Rightarrow x^2 - (a + b)x + ab = 0$
- If sum s and product p of roots are given: $x^2 - sx + p = 0$
- If roots are reciprocals of roots of equation $ax^2 + bx + c = 0$, then equation is $cx^2 + bx + a = 0$

- If root are k more than roots of $ax^2+bx+c=0$ then equation is a $a(x-k)^2+b(x-k)+c=0$
- If roots are k time roots of $ax^2+bx+c=0$ then equation is a $a(x/k)^2+b\left(\frac{x}{k}\right)+c=0$
- A quadratic equation whose roots are the negative of the roots of the equation $ax^2+bx+c=0$ $(i.e., the\ roots\ are -\alpha\ and -\beta)$
  The required equation is $a(-x)^2+b(-x)+c=0$
  $\Rightarrow\ ax^2-bx+c=0$
- A quadratic equation whose roots are the squares of the roots of the equation $ax^2+bx+c=0$. (i.e., the roots are $a^2\ and\ \beta^2$)
  The required equation is $a(\sqrt{x})^2+b(\sqrt{x})+c=0$
  $\Rightarrow ax+b\sqrt{x}+c=0$
- A quadratic equation whose roots are the cubes of the roots of the equation $ax^2+bx+c=0$. (i.e., the roots are $a^3\ and\ \beta^3$)
  The required equation is $a(x^{1/3})^2+b(x^{1/3})+c=0$
  $\Rightarrow\ \ ax^{2/3}+bx^{1/3}+c=0$

Tip – 8

- Descartes Rules: A polynomial equation with n sign changes can have a maximum of n positive roots. To find the maximum possible number of negative roots, find the number of positive roots of $f(-x)$.
- An equation where highest power is odd must have at least one real root.

Important Formulae

1. $(a+b)^2=\ a^2+b^2+2ab$
2. $(a-b)^2=a^2+b^2-2ab$
3. $(a+b)^3=(a-b)^2+4ab$
4. $(a-b)^2=(a+b)^2-4ab$
5. $(a^2+b^2)=\frac{1}{2}[(a+b)^2+(a-b)^2]$
6. $a^2-b^2=(a+b)(a-b)$
7. $(a+b)^3=a^3+b^3+3ab(a+b)$
8. $(a-b)^3=a^3-b^3-3ab(a-b)$
9. $a^3+b^3=(a+b)(a^2+b^2-ab)$
10. $a^3-b^3=(a-b)(a^2+b^2+ab)$
11. $(a+b+c)^2=(a^2+b^2+c^2)+2(ab+bc+ac)$
12. $a^3+b^3+c^3-3abc$
    $=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$
    $If\ a+b+c=0, then\ a^3+b^3+c^3=3abc$
13. $a^4+b^4+a^2b^2=(a^2+b^2+ab)(a^2+b^2-ab)$
14. $a^4-b^4=(a-b)(a+b)(a^2+b^2)$
15. $a^8-b^8=(a-b)(a+b)(a^2+b^2)(a^4+b^4)$

Exercise 01 (Solve the Questions)

1. Which of the following are quadratic equations?
   (a) $3x^2 - 2 = 0$
   (b) $3x^2 - 2x + 4 = \frac{1}{4}(9x^2 - x + 3)$
   (c) $x^2 - 6x = x^2 - 3$
   (d) $x + \frac{1}{x} = 4$

2. Determine whether the given values are solutions (i.e., roots) of the equation or not.
   $3x^2 + 5x + 2 = 0, x = 3, x = -1$

3. Solve the following equation by factorization.
   $6x^2 - x - 2 = 0$

4. Find the value of K for which $x = 2$ is a root of the equation $3x^2 - 2Kx + 5 = 0$.

5. Solve the following equation by using the formula:
   $8x - 2x^2 = 5$

6. Solve for $y = 9y^4 - 29y^2 + 20 = 0$

7. Solve for $x: x^6 - 26x^3 - 27 = 0$.

8. Solve: $2x - \frac{3}{x} = 5$.

9. Solve $\sqrt{2x+9} + x = 13$.

10. Solve $\sqrt{2x+9} - \sqrt{x-4} = 3$

11. Solve for $x: 2\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) + 14 = 0$

12. Solve $6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x - \frac{1}{x}\right) + 12 = 0$

13. $x: \sqrt{x^2 + x - 6} - x + 2 = \sqrt{x^2 - 7x + 10},\ x \in R$

14. Solve for $x: 3^{x+2} + 3^{-x} = 10$

15. Solve for $x = (x+1)(x+2)(x+3)(x+4) = 24$

16. Find the sum and product of the roots of the quadratic equation:
    $3x^2 + 2x - 1 = 0$

17. Find the quadratic equation whose roots are, $-3, 5$

18. If α, β are the roots of the equation $2x^2 - 3x + 2 = 0$, form the equation whose roots are α^2, β^2.

19. Determine k for which the roots of the equation $9x^2 + 2kx + 4 = 0$ *are equal*.

20. Find the set of values of p for which the quadratic equation has real linear factors.
    $$9x^2 - px + 4$$

21. The length of a rectangle is 7 cm more than its width. The area is 60 cm^2. Find the dimensions of the rectanlgle.

22. For which value of k will the equations $x^2 - kx - 21 = 0$ and $x^2 - 3kx + 35 = 0$ have one common root.

23. If x be real, find the maximum value of $7 + 10x - 5x^2$.

24. If x be real, find the maximum value of $\frac{(x+2)}{(2x^2+3x+6)}$

25. Solve $x^2 - 5x + 4 > 0$.

26. Solve $-x^2 + 6x - 8 > 0$

27. Solve the inequality $\frac{x-2}{x+2} > \frac{2x-3}{4x+1}$

28. Determine the range of values of x for which $\frac{x^2-2x+5}{3x^2-2x-5} > \frac{1}{2}$.

29. $2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 1 = 0$:

30. $\sqrt{2x^2 - 2x + 1} - 2x + 3 = 0$: find roots

Exercise 02 (MCQs)

1. Find the maximum value of the expression $(x^2 + 8x + 20)$.
(a) 4 (b) 2 (c) 29 (d) None of these

2. If the product of roots of the equation $x^3 - 3(2a + 4)x + a^2 + 18a + 81 = 0$ is unity, then a can take the values as
(a) 3, - 6 (b) 10, - 8 (c) - 10, - 8 (d) - 10, - 6

3. If the toots of equation $x^2 + bx + c = 0$ differ by 2, then which of the following is true?
(a) $a^2c^2 = 4(1 + c)$ (b) $4b + c = 1$ (c) $c^2 = 4 + b$ (d) $b^2 = 4\,(c + 1)$

4. Find the value of the expression $\left(\sqrt{x + (\sqrt{x + (\sqrt{x \ldots +}))}}\,)\right)$
(a) $\frac{1}{2}\left[2\sqrt{(2x-1)}\right] + 1$ (b) $\frac{1}{2}\left[\sqrt{(4x+1}\right] + 1$
(c) $\frac{1}{2}\left[2\sqrt{(2x-1}\right] - 1$ (d) $\frac{1}{2}\left[\sqrt{(4x-1}\right] - 1$

5. If m, n are the roots of the equation $px^2 + qx + r = 0$ and km, kn are the roots of the equation $ax^2 + bx + c = 0, then\ k =$
(a) $\sqrt{\frac{cp}{ar}}$ (b) $\sqrt{\frac{cr}{ap}}$ (c) $\sqrt{\frac{ap}{cr}}$ (d) None of these

6. In the Maths Olympiad of 2020 at Animal Planets, two representatives from the donkey's side, while solving a quadratic equation, committed the following mistakes:
(i) One of them made a mistake in the constant term and got the roots as 5 and 9.
(ii) Another one committed an error in the coefficient of x and he got the roots as 12 and 4.
But in the meantime, they realized that they are wrong and they managed to get it right jointly. Which of the following could be the quadratic equation?
(a) $x^2 + 4x + 14 = 0$ (b) $2x^2 + 7x - 24 = 0$
(c) $x^2 - 14x + 48 = 0$ (d) $3x^2 - 17x + 52 = 0$

7. If $x^2 - 3x + 2$ is a factory of $x^4 - ax^2 + b = 0$ then the values of $a\ and\ b$ are
(a) $-5, -4$ (b) 5, 4 (c) -5, 4 (d) 5, -4

8. The value of p satisfying $log_3(p^2+4p+12)=2$ are
(a) 1, - 3 (b) - 1, - 3 (c) -4, 2 (d) -4, -2

9. If one root of the equation $(I-m)x^2+Ix+1=0$ is double of the other and is real, find the grates value of m.
(a) $\frac{9}{8}$ (b) $\frac{8}{7}$ (c) $\frac{8}{6}$ (d) $\frac{7}{5}$

10. The set of values of p for which the roots of the equation $3x^2+2x+p(p-1)=0$ are of opposite sign is
(a) $(-\infty,0)$ (b) $(0,1)$ (c) $(1,\infty)$ (d) $(0,\infty)$

11. If α,β are the roots of the quadratic equation $x^2+mx+1=0$ and γ,δ are the roots of the equation $x^2+nx+1=0$, then the value of $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$ is equal to
(a) n^2-m^2. (b) m^2-n^2. (c) $2\,m^2-n^2$. (d) None of these

12. Find the roots of the quadratic equation $bx^2-2ax+a=0$
(a) $\frac{\sqrt{b}}{\sqrt{b\pm\sqrt{a-b}}}$ (b) $\frac{\sqrt{a}}{\sqrt{b\pm\sqrt{a-b}}}$ (c) $\frac{a}{\sqrt{a\pm\sqrt{a-b}}}$ (d) $\frac{\sqrt{a}}{\sqrt{a\pm\sqrt{a+b}}}$

13. What is the value of k when the equation $x^2-(k+6)x+2(2k-1)=0$ has sum of the roots equal to half of their product?
(a) -2 (b) 7 (c) 9 (d) 12

14. What is the value of P when the sum of the squares of roots of the equation $x^2-(p-2)x-p-1=0$? Assume the least value.
(a) -2 (b) 1 (c) 9 (d) 12

15. If the equation $x^2+2(p+1)x+9p-5=0$ has only positive roots, then which of the following is true?
(a) $p\geq 6$ (b) $p\leq 0$ (c) $p\leq -6$ (d) None of these

16. The quadratic equations $x^2-ax-21=0$ and $x^2-3ax+35=0$, where $a>0$ have a common root. What is the value of a?
(a) 0 (b) 1 (c) 4 (d) 6

17. What is the sum of the roots of quadratic equation $3x^2+2x-1=0$?
(a) 1/3 (b) 2/3 (c) 3 (d) -2/3

18. What is the product of the roots of quadratic equation $3x^2+2x-1=0$?
(a) -1/3 (b) 2/3 (c) 3 (d) -2/3

19. What is the sum of the roots of cubic equation $x^3+2x^2+x-1=0$?
(a) -2 (b) 2/3 (c) -1 (d) 2

20. Five burgers, six pizzas, and seven cold drinks cost Rs. 178, and six burgers, four pizzas, and two cold drinks cost Rs. 124. What is the cost of 3 (pizzas + burgers + cold drinks)?
(a) Rs. 60 (b) Rs. 62.5 (c) Rs. 90 (d) Cannot be determined

21. If $x<0$, then what is the maximum value of $\frac{9}{x}+\frac{x}{9}$?
(a) 2 (b) -2 (c) $+\infty$ (d) None of these

22. If the coefficient of x^2 and the constant term of a quadratic equation are interchanged, then which of the following will not get changed?
(a) Sum of the roots (b) Product of the roots
(c) Roots of the equation (d) None of these

23. How many real roots will be there of the equation $x^4+x^2=0$?
(a) 0 (b) 2 (c) 4 (d) None of these

24. Given that $as^4 + bs^3 + cs^2 + e = 0$ is bi-quadratic equation in s and $a \neq 0$, what is the value of $(1 - s_1)(1 - s_2)(1 - s_4)$?
(a) 1
(b) $(a + b + c + d + e)/a$
(c) 0
(d) $a + b + c + d + e$

25. $x^2 - 5x + 6 = 0$ *and* $x^2 + m\,x + 3 = 0$ have a root in common. What is the value of *m*?
(a) -4 (b) -7/2 (c) Both 1 and 2 (d) None of these

Exercise 03 (MCQs)

1. If $x^2 + 3x - 10$ is a factor of $3x^4 + 2x^3 - ax^2 + bx - a + b - 4$, then the closest approximate values of a *and* b are
(a) 25, 43 (b) 52, 43 (c) 52, 67 (d) None of these

2. If x is real, the smallest value of the expression $3x^2 - 4x + 7$ is:
(a) 2/3 (b) 3/4 (c) 7/9 (d) None of these

3. If $0 < p < 1$ then the roots of the equation $(1 - p)x^2 + 4x + p = 0$ *are* _____?
(a) Real and of opposite sign.
(b) Real and both negative
(c) Imaginary
(d) Real and both positive

4. The number of possible real solution(s) of y in the equation $y^2 - 2y\cos x + 1 = 0$ *is* _____?
(a) 0 (b) 1 (c) 2 (d) 3

5. A polynomial $ax^3 + bx^2 + cx + d$ intersects the $x - axis\ at\ 1\ and - 1, and\ y - axis\ at\ 2$. The value of b is:
(a) -2 (b) 0 (c) 1 (d) 2

6. If the equation $ax^2 + bx + a = 0 (a > 0)$ has real and positive roots then which of the following is always true?
(a) $b < 2a$ (b) $b < 0$ (c) $b \leq -2a$ (d) All the options are true.

7. If the sum of the roots of the quadratic equation $px^2 + qx + r = 0$ is equal to the sum of the square of their reciprocals, marks all the correct statements.
(a) *r/p, p/q and q/r are in A. P.*
(b) *p/r, q/p and r/q are in G. P.*
(c) *p/r, q/p and r/q are in H. P.*
(d) Option *(a) and (c) both.*

8. Find the number of integral roots of equation $x^3 - x^2 + 2x - 17 = 0$.
(a) 1 (b) 2 (c) 3 (d) None of these

9. The number of real solutions of $x - \frac{1}{x^2 - 4} = 2 - \frac{1}{x^2 - 4}$
(a) 0 (b) 1 (c) 2 (d) Infinite

10. If one root of the equation $(k^2 + 1)x^2 + 13x + 4k = 0$ is reciprocal of the other, then *k* has the value:
(a) $-2 + \sqrt{3}$ (b) $2 - \sqrt{3}$ (c) 1 (d) None of these

11. If α, β are roots of the equation $(x - a)(x - b) = c, c \neq 0$, then find the roots of the equation $(x - \alpha)(x - \beta) + c = 0$.
(a) $a,\ c$ (b) b, c (c) a, b (d) $a + c, b + c$

12. If a, b are roots of $x^2 + px + q = 0$ *and* g, d are the roots of $x^2 + px - r = 0$, then $(a - g)(a - d)$ is equal to:
(a) $q + r$ (b) $q - r$ (c) $-(q + r)$ (d) $-(p + q + r)$

13. If α, β, γ be the roots of the equation $x(1 + x^2) + x^2(6 + x) + 2 = 0$, then find the value of $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$.
(a) -3 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) None of these

14. If $(x - 1)^3$ is a factor of $x^4 + ax^3 + bx^2 + cx - 1$, then find the other factor.
(a) $x - 3$ (b) $x + 1$ (c) $x + 2$ (d) None of these

15. If a and b are the roots of equation $2x^2 - 3x - 2 = 0$, find the value of $(a^3 + b^3)$ without finding the roots of the equation.
(a) $\frac{8}{3}$ (b) $\frac{64}{3}$ (c) $\frac{63}{8}$ (d) 16

16. $\sqrt{x^2 - 9x + 20} - \sqrt{x^2 - 12x + 32} = \sqrt{2x^2 - 25x + 68}$:
(a) 4, 9 (b) 3, 9 (c) 2, 4 (d) 2, 6

17. Determine k such that the quadratic equation $x^2 + 7(3 + 2k) - 2x(1 + 3k) = 0$ has equal roots:
(a) 2, 7 (b) 7, 5 (c) $2, -\frac{10}{9}$ (d) None of these

18. Discriminant of the equation $-3x^2 + 2x - 8 = 0$ is:
(a) -92 (b) -29 (c) 39 (d) 49

19. The nature of the roots of the equation $x^2 - 5x + 7 = 0$ is
(a) no real roots (b) 1 real root (c) can't be determined (d) none of these

20. Find the value of k so that the sum of the roots of the quadratic equation is equal to the product of the roots:
$$(k + 1)^2 + 2kx + 4 = 0$$
(a) -2 (b) -4 (c) 6 (d) 8

21. The value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots \infty}}}}$ is :
(a) 2 (b) 3 (c) 4 (d) 5

22. If $x \in R$, and $K = \frac{(x^2-x+1)}{(x^2+x+1)}$, then:
(a) $x \leq 0$ (b) $\frac{1}{3} \leq k \leq 3$ (c) $k \geq 5$ (d) none of these

23. If $x^2 - ax - 21 = 0$ and $x^2 - 3ax + 35 = 0; a > 0$ have a common root, then a is equal to:
(a) 1 (b) 2 (c) 4 (d) 5

24. The number of real solution of the equation $2^{3x^2-7x+4} = 1$ is:
(a) 0 (b) 4 (c) 2 (d) infinitely many

25. The number of real roots of the equation $(x - 1)^2 + (x - 2)^2 + (x - 3)^2 = 0$:
(a) 0 (b) 2 (c) 3 (d) 6

Exercise 04 (MCQs)

1. If the roots of the equation $3ax^2 + 2bx + c = 0$ are in the ratio of 2 : 3, then:
(a) $8ac = 25b$ (b) $8ac = 9b^2$ (c) $8b^2 = 9ac$ (d) $8b^2 = 25ac$

2. Ramesh bought certain number of chairs for Rs. 2400. If the price of each chair is reduced by Rs. 20, then 10 more chairs can be purchased for the same amount. Find the number of chairs he purchased initially.
(a) 20 (b) 30 (c) 40 (d) 50

3. $If\ x^2 - kx + 5 = 0$ has 3 as a root, then find the value of K.
(a) 17/5 (b) 13/2 (c) 14/3 (d) 16/5

4. If $x^4 + \frac{1}{x^4} = 47$, then find the value of $x^3 + \frac{1}{x^3}$.
(a) 9 (b) 18 (c) 12 (d) 27

5. If one of the roots of the cubic expression $x^3 - ax^2 + 11x - 6\ is\ 3$, what are the other roots?
(a) 6 and 2 (b) -1 and 2 (c) 1 and 2 (d) -2 and -1

6. If $m\ and\ n$ are natural numbers, then:
(a) $m^{100} + n^{100} > (m+n)^{100}$ (b) $m^{100} + n^{100} \geq (m+n)^{100}$
(c) $m^{100} + n^{100} < (m+n)^{100}$ (d) $m^{100} + n^{100} \leq (m+n)^{100}$

7. How many real roots will be there of the equation $= x^2 + 5|x| + 6 = 0$?
(a) 0 (b) 2 (c) 4 (d) None of these

Direction: $x_1, x_2, and\ x_3$ are the roots of two distinct quadratic equations (with coefficient of x^2 being 1 in both the equations) in which x_1 is a common root.

8. If $x_1, x_2, and\ x_3$ are in AP and the coefficient of x in the equation with $x_1\ and\ x_2$ as its root is -4, then find one of the root.
(a) 1 (b) 3 (c) 5 (d) Cannot be determined

Direction: One of the +ve roots of a quadratic equation $ax^2 + m\,x + c = 0$, where $a = 1$, is twice that of the other. The sum of the coefficient of x and constant term is 2.

9. What is the value of m in the equation?
(a) -4 (b) 4 (c) -6 (d) 6

10. The polynomial $(ax^2 + bx + c)(ax^2 - bx - c), ac \neq 0$, has:
(a) Four real zeros (b) At least two real zeros
(c) At most two real zeros (d) No real zeros

11. If $y \neq 0$, then the number of values of the pair (x, y) such that $x + y + \frac{x}{y} = \frac{1}{2}(x+y)\frac{x}{y} = -\frac{1}{2}, is:$
(a) 1 (b) 2 (c) 0 (d) None of these

12. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has.
(a) No solution (b) One solution
(c) Two solutions (d) More than two solutions

13. Find the maximum and the minimum values of the function $\frac{x^2-x+1}{x^2+x+1}$ for real values of x.
(a) 3 and -3 (b) $\frac{1}{3}\ and\ -\frac{1}{3}$ (c) $3\ and\ \frac{1}{3}$ (d) None of these

14. If $2a^{-1/3} + 2a^{1/3} = 5$, then find the value of a.
(a) $8\ or\ \frac{1}{8}$ (b) $2\ or\ \frac{1}{2}$ (c) $3\ or\ \frac{1}{3}$ (d) None of these

15. If $(p^2 + q^2)x^2 - \sqrt{2}(p+q)x + 1 = 0$, what must be the relation between $q\ and\ p$ if the equation has equal and real roots?
(a) $p = q$ (b) $p > q$ (c) $p < q$ (d) None of these

16. If $x^2 + y^2 + z^2 = 1$, then the value of $xy + yz + zx$ lies in the interval of:
(a) $[\frac{1}{2}, 2]$ (b) $[-1, 2]$ (c) $[-\frac{1}{2}, 1]$ (d) $[-1, \frac{1}{2}]$

17. Find all the values of p for which one root of the equation $x^2 - (p+1)x + p^2 + p - 8 = 0$, is greater than 2 and the other root is smaller than 2.
(a) $-2 < p < 3$ (b) $-3 < p < 2$ (c) $-3 < p$ (d) $p < 2$

18. If α, β are the roots of the equation $8x^2 - 3x + 27 = 0$, then the value of $[(\alpha^2/\beta)^{\frac{1}{3}} + (\beta^2/\alpha)^{\frac{1}{3}}]$ is:
(a) 1/3 (b) 1/4 (c) 1/5 (d) 1/6

19. The number of solutions of the equation $3^{x-1} + 5^{x-1} = 34$:

(a) 0 (b) 1 (c) 2 (d) none of these

20. If the roots of $ax^2 + bx + c = 0, a > 0$, be each greater than unity, then:
(a) $a + b + c = 0$ (b) $a + b + c > 0$
(c) $a + b + x < 0$ (c) none of these

21. The value of x satisfying the equation
$|x-1|^{log_3 x^2 - 2log_x 9} = (x-1)^7$:
(a) $\sqrt{3}$ (b) 3^4 (c) 3 (d) $log_4{}^3$

22. If $x + y + z = 0, then\ x^3 + y^3 + z^3\ is\ equal\ to$:
(a) 0 (b) $3\,xyz$ (c) $\frac{xy+yz+zx}{xyz}$ (d) $xyz(xy + yz + zx)$

23. If $x - \frac{1}{x} = 2, then\ the\ value\ of\ x^4 + \frac{1}{x^4}\ is$:
(a) 4 (b) 8 (c) 12 (d) 34

24. If $\left(x + \frac{1}{x}\right) = 3, then\ the\ value\ of\ \left(x^6 + \frac{1}{x^6}\right)\ is$:
(a) 927 (b) 414 (c) 364 (d) 322

25. Let $f(x) = x^2 - 27x + 196. If\ (x) = x, then\ what\ is\ the\ value\ of\ x$?
(a) 28 (b) 14 (c) 7 (d) 4

Exercise 05 (TITA or Short Answers)

1. Find the minimum value of the expression $\left(p + \frac{1}{p}\right); p > 0$.

....

2. Find the number of real roots of the equation $|x|^2 - 2|x| - 3 = 0$.

....

3. A, B and C are real values such that A + B + C = 2, $A^2 + B^2 + C^2 = 6\ and\ A^3 + B^3 + C^3 = 8$, then find the value of $A^4 + B^4 + C^4$?

....

4. The sum of the real roots of equation $x^2 + |x| - 6 = 0$is:

....

5. If the sum of the roots of quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then $\frac{b^2}{ac} + \frac{bc}{a^2}$ is equal to:

....

6. If $x^3 - ax^2 + bx + 10$ is perfectly divisible by $(x + 5) and\ x^4 + x^3 + bx^2 - ax + 42$ is perfectly divisible by $(x - 3)$, find the value of $(a - 3b)$.

....

7. If $\alpha\ and\ \beta$ are the roots of $x^2 + x + 2 = 0$, then $(\alpha^{10} + \beta^{10})/(\alpha^{-10} + \beta^{-10})$ is equal to:

....

8. If $s = 2 + 2^{1/3} + 2^{2/3}$, then what is the value of $s^3 - 6s^2 + 6s$?

....

9. Find the number of solutions of $log_4(x - 1) = log_2(x - 3)$.

....

10. If 2 and 4 are two roots of the expression $x^4 + 2x^3 + mx^2 + nx + 3 = 0 (m \text{ and } n \text{ are constants})$, then what is the value of $2m + 3n$?

....

11. Quadratic equations $(2p-1)z^2 + (2p+1)z + c = 0 \text{ and } (q+1)y^2 + (4q+1)y + 3c = 0$ have the same pair of roots. Given that $c \neq 0$, what is the value of $(p+q)$?
(a) 3 (b) 4 (c) 2 (d) Cannot be determined

....

12. If $x + \frac{1}{x} = 1 \text{ and } p = x^{4000} + \frac{1}{x^{4000}}$ and q be the digit at units place in the number $2^{2^n} + 1, n$ being a natural number greater than 1, then $p + q =$

....

13. Find the value of k such that the equation. $x^2 - (k+6)x + 2(2k-1) = 0$ has sum of the roots equal to half of their product:

....

14. If $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \infty}}}} \text{ then } x \text{ is:}$

....

15. The number of real solutions of the equation $2|x|^2 - 5|x| + 2 = 0$ is:

....

16. If $xy = 2(x+y), x \leq y \text{ and } x, y \in N,$ the number of solutions of the equation:

....

17. If $x = 7 + 4\sqrt{3} \text{ and } xy = 1, \text{then the value of } \frac{1}{x^2} + \frac{1}{y^2} \text{ is:}$

....

18. If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in HP, then find the value of c:

....

19. Find the number pairs for (x, y) from the following equations:

$$log_{100}|x + y| = \frac{1}{2}$$
$$log_{10}y - log_{10}|x| = log_{100}{}^4$$

....

20. The sum of all the real roots of the equation $|x-2|^2 + |x-2| - 2 = 0$ is:

....