Q1.

1. When we divided the time duration T into discrete time intervals of length h each, we then have $n+1$ time instances, with x_i, v_i, a_i, K_i, p_i for each of them. We know that $x_i, a_i, K_i, p_i, p_{d,i}$ are all related to v_i, hence we can set v_i (velocity) and $p_{b,i}$ (braking power) as our decision variables.

The optimisation problem then becomes the following:

$$\min \; h \sum_{r=0}^{n-1} p_r$$

$$\text{s.t.} \quad x_0 = 0$$

$$x_n \geq x_f$$

$$a_r \leq A \qquad \forall r = \{0,1,2 \ldots n-1\}$$

$$0 \leq v_r \leq V \qquad \forall r = \{0,1,2 \ldots n-1\}$$

$$v_n \leq v_f$$

subject to

$$x_{r+1} = x_r + h\frac{(v_r + v_{r+1})}{2} \quad \forall r = \{0,1, \ldots n-1\}$$

$$a_r = \frac{v_{r+1} - v_r}{h} \qquad \forall r = \{0,1 \ldots n-1\}$$

$$p_{d,r} = c v_r^3 \qquad \forall r = \{0,1,2 \ldots n-1\}$$

$$p_r = p_{b,r} + p_{d,r} + \frac{(K_{r+1} - K_r)}{h} \quad \forall r = \{0,1 \ldots n-1\}$$

$$K_r = \frac{v_r^2}{2} \qquad \forall r = \{0,1,2 \ldots n\}$$

2. Done in code

3. The constraints with equality are not convex, i.e. $p_{d,r} = c v_r^3$ and $K_r = v_r^2/2$. If we rephrase them as: $p_{d,r} \geq c v_r^3$ and $K_r \geq v_r^2/2$, which are convex on $v_r \geq 0$.

$\Rightarrow$ At optimal case, inequalities will be tight.

Q2.

Since we know that f_k is non-decreasing, we have

$\int_0^{i_k} f(u_k)\,du_k$ is convex in i_k.

Our optimisation problem is:

minimize $\phi(i)$

s.t. $Ai = 0$ with $i \in \mathbb{R}^m$

Consider a dual variable associated as p:

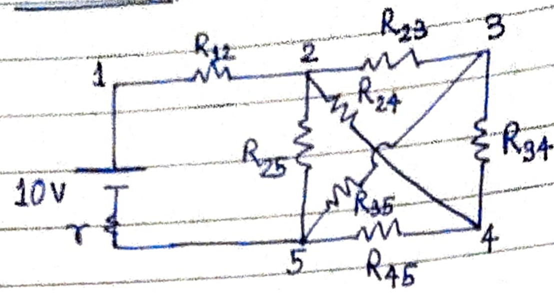
$Ai = 0$ and $\nabla \phi(i) + A^T p = 0$

Let $v = \nabla \phi(i)$ and $e = -p$

$\Rightarrow$ Optimality conditions (dual):

$Ai = 0 \quad A^T e = v \quad v_k = f_k(i_k) \quad k = 1, 2 \ldots m$

3. Circuit 'a' :



Given that, $R_{12} = R_{34} = R_{35} = R_{24} = R_{25} = 10\,\Omega$

and $R_{23} = R_{45} = 100\,\Omega$

and $r = 2\,\Omega$, $V = 10V$

$\Rightarrow i \in \mathbb{R}^8$ and $A \in \mathbb{R}^{5\times 8}$

We obtain:

$$A = \begin{bmatrix} +1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 \end{bmatrix}$$

Some results (useful in circuit 'b' too):

- For $v_k = i_k R_k$ (resistor) $\Rightarrow \int_0^{i_k} z_k R_k \, dz_k = \frac{i_k^2 R_k}{2}$

- For $v_k = V_k + i_k r_k \Rightarrow \int_0^{i_k} (V_k + z_k r_k)\, dz_k = V_k i_k + \frac{i_k^2 r_k}{2}$

- For $v_k = V_T \log\left(1 + \frac{i_k}{I_s}\right) \Rightarrow \int_0^{i_k} V_T \log\left(1 + \frac{z_k}{I_s}\right) dz_k$

$$= V_T I_s \left(\left(1 + \frac{i_k}{I_s}\right) \log\left(1 + \frac{i_k}{I_s}\right) - \frac{i_k}{I_s} \right)$$

For circuit 'a':

$$\phi(i) = -V_8 i_8 + i_8^2 r/2 + \frac{i_1^2 R_1}{2} + \frac{i_2^2 R_2}{2} + \frac{i_3^2 R_3}{2} + \frac{i_4^2 R_4}{2} + \frac{i_5^2 R_5}{2} + \frac{i_6^2 R_6}{2} + \frac{i_7^2 R_7}{2}$$

The primal and dual for the circuit 'a' are:

Primal: $\min \phi(i)$

$\text{s.t. } Ai = 0$

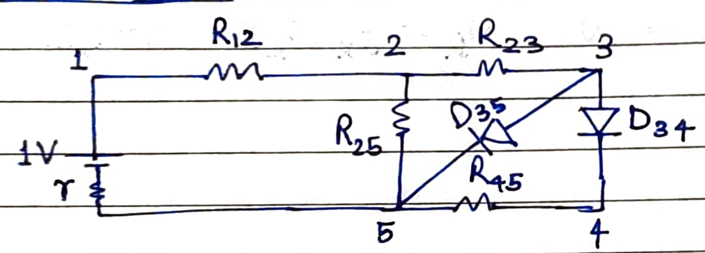
Dual: $\max_{b} L(i, b)$

$\text{s.t. } Ai = 0 \qquad A^T e = v \quad (b = -e)$

$\Rightarrow A^T b = -v$

and $V_K = \begin{cases} i_K R_K & \text{for } K = 1, 2 \ldots 7 \\ V_K + i_K r_K & \text{for } K = 8 \end{cases}$

Circuit 'b':



Given: $R_{12} = R_{23} = R_{25} = 1000\,\Omega$

and $R_{45} = 100\,\Omega$

and for diodes D_{34}, D_{35}: $V_T = 30 \times 10^{-3}\,V$

$I_s = 0.5\,mA$

$\Rightarrow i \in \mathbb{R}^7$ and $A \in \mathbb{R}^{5 \times 7}$

We obtain:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{bmatrix}$$

For circuit 'b':

$$\phi(i) = -V_7 i_7 + \frac{i_7^2 r}{2} + V_T I_s\left(\left(1+\frac{i_3}{I_s}\right)\log\left(1+\frac{i_3}{I_s}\right) - \frac{i_3}{I_s}\right)$$
$$+ V_T I_s\left(\left(1+\frac{i_5}{I_s}\right)\log\left(1+\frac{i_5}{I_s}\right) - \frac{i_5}{I_s}\right)$$
$$+ \sum_{k} \frac{i_k^2 R_k}{2}$$

$\downarrow$

$k = 1, 2, 4, 6$

The primal and dual for circuit 'b' are:

Primal: $\min \phi(i)$

s.t. $Ai = 0$

Dual: $\max_{b} L(i, b)$

s.t. $Ai = 0 \quad -v = A^T b$

$$V_k = \begin{cases} i_k R_k & \text{for } k = 1,2,4,6 \leftarrow \text{Resistor} \\ V_k + i_k R_k & \text{for } k = 7 \leftarrow \text{Battery} \\ V_T \log\left(1 + \frac{i_k}{I_s}\right) & \text{for } k = 3,5 \leftarrow \text{Diode} \end{cases}$$