Matrix Analysis and Applications

Chapter 3: Applications of Eigendecomposition

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November 22, 2020

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Preliminaries

Random processes: consider a scalar random process $\{x_n\}$, where $n \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of integers). Let

$$\mu_x(n) \triangleq \mathbb{E}\{x_n\}, \tag{1}$$

$$r_x(n,l) \triangleq \mathbb{E}\{x_n x_l^*\}, \tag{2}$$

$$c_x(n,l) \triangleq \mathbb{E}\left\{(x_n - \mathbb{E}\{x_n\})(x_l - \mathbb{E}\{x_l\})^*\right\}, \tag{3}$$

denote the mean, auto-correlation function and auto-covariance function of x_n, respectively, where $\mathbb{E}\{\cdot\}$ denotes the expectation.

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Examples:

- **Discrete-time signal, time series**: x_n is a signal sample or observation at time indicated by n. The functions $r_x(n,l)$ and $c_x(n,l)$ measure how two signal samples at time n, l are related.
- **Image**: x_n stores the value of a pixel, with the pixel's spatial location indicated by n. The functions $r_x(n,l)$ and $c_x(n,l)$ measure the spatial correlations.
- **Data analysis**: x_n is a data point (e.g., score of student n, preference of user n).

Preliminaries (cont'd)

Wide-sense stationary processes: a random process $\{x_n\}$ is called wide-sense stationary (WSS) if, for any n and l, we have

$$\mu_x(n) = \mu_x(l), \tag{4}$$

$$c_x(n, l) = c_x(n + i, l + i). \tag{5}$$

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- A WSS process is a random process whose first-order and second-order statistics do not change with n (e.g., time in discrete-time signals).
- A zero-mean random process $\{x_n\}$, i.e., $\mu_x(n) = 0$ for all n, is WSS if

$$r_x(n, l) = r_x(n + i, l + i) \quad \text{for any } i.$$

- The same concept applies to a vector random process $\{\boldsymbol{x}_n\}$.

Preliminaries (cont'd)

Let $\{\boldsymbol{x}_j\}$, $\boldsymbol{x}_j \in \mathbb{C}^m$, be a zero-mean complex-valued WSS process.

Correlation matrix:

$$\boldsymbol{R}_{\boldsymbol{x}} \triangleq \mathbb{E}\{\boldsymbol{x}_j \boldsymbol{x}_j^H\} = \begin{bmatrix} \mathbb{E}\{|x_{1j}|^2\} & \mathbb{E}\{x_{1j}x_{2j}^*\} & \cdots & \mathbb{E}\{x_{1j}x_{mj}^*\} \\ \mathbb{E}\{x_{2i}x_{1j}^*\} & \mathbb{E}\{|x_{2j}|^2\} & \cdots & \mathbb{E}\{x_{2j}x_{mj}^*\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}\{x_{mj}x_{1j}^*\} & \cdots & \cdots & \mathbb{E}\{|x_{mj}|^2\} \end{bmatrix}. \quad (6)$$

- Properties:

- $\boldsymbol{R}_{\boldsymbol{x}}$ is Hermitian (and symmetric for the real-valued case).
- $\boldsymbol{R}_{\boldsymbol{x}}$ is positive semidefinite (will be discussed in later lectures).

Representation by Orthonormal Expansion

For a WSS process $\{\boldsymbol{x}_j\}$, $\boldsymbol{x}_j \in \mathbb{C}^m$.

Aim: given a basis $\{\boldsymbol{q}_1, \cdots, \boldsymbol{q}_m\} \subseteq \mathbb{C}^m$, represent each $\boldsymbol{x}_j$ by

$$\boldsymbol{x}_n = \sum_{i=1}^{m} a_{ij}\boldsymbol{q}_i = \boldsymbol{Q}\boldsymbol{a}_j, \quad \text{for some coefficients } a_{ij},\ i = 1, \cdots, m, \tag{7}$$

where $\boldsymbol{a}_j = [a_{1j}, \cdots, a_{mj}]^T$ and $\boldsymbol{Q} = [\boldsymbol{q}_1, \cdots, \boldsymbol{q}_m]$.

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Orthonormal expansion: perform (7) via an orthonormal basis, or unitary $\boldsymbol{Q}$.

- Examples: discrete Fourier transform (DFT), discrete cosine transform (DCT), Haar transform.
- For orthonormal expansion, each $\boldsymbol{a}_j$ can be conveniently obtained via

$$\boldsymbol{a}_j = \boldsymbol{Q}^H\boldsymbol{x}_j. \tag{8}$$

Example: Discrete Fourier Transform (DFT)

The DFT basis matrix is $\boldsymbol{Q} = [\boldsymbol{q}_0, \boldsymbol{q}_1, \cdots, \boldsymbol{q}_{m-1}]$, where

$$\boldsymbol{q}_i = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 \\ e^{\frac{j2\pi i}{m}} \\ e^{\frac{j4\pi i}{m}} \\ \vdots \\ e^{\frac{j2\pi(m-1)i}{m}} \end{bmatrix}, \quad i = 0, 1, \cdots, m-1 \tag{9}$$

and $j = \sqrt{-1}$.

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Exercise: verify that $\|\boldsymbol{q}_i\|_2^2 = 1$, and $\boldsymbol{q}_i^H \boldsymbol{q}_k = 0$ for all $i \neq k$. Also, if we write $\boldsymbol{Q}^H = [\boldsymbol{u}_0, \cdots, \boldsymbol{u}_{m-1}]$, verify that

$$\boldsymbol{u}_i = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 \\ e^{-\frac{j2\pi i}{m}} \\ e^{-\frac{j4\pi i}{m}} \\ \vdots \\ e^{-\frac{j2\pi(m-1)i}{m}} \end{bmatrix}. \tag{10}$$

Karhunen-Loéve Expansion

Assume a zero-mean WSS $\{\boldsymbol{x}_n\}$. From $\boldsymbol{a}_n = \boldsymbol{Q}^H \boldsymbol{x}_n$, one gets

$$\begin{aligned}
\boldsymbol{R}_{\boldsymbol{a}} &\triangleq \mathbb{E}\{\boldsymbol{a}_n \boldsymbol{a}_n^H\} && (11)\\
&= \mathbb{E}\{\boldsymbol{Q}^H \boldsymbol{x}_n \boldsymbol{x}_n^H \boldsymbol{Q}\} && (12)\\
&= \boldsymbol{Q}^H \mathbb{E}\{\boldsymbol{x}_n \boldsymbol{x}_n^H\} \boldsymbol{Q} && (13)\\
&= \boldsymbol{Q}^H \boldsymbol{R}_{\boldsymbol{x}} \boldsymbol{Q}. && (14)
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Problem: find a unitary basis matrix $\boldsymbol{Q}$ such that $\boldsymbol{R}_{\boldsymbol{a}}$ is diagonal.

- Having a diagonal $\boldsymbol{R}_{\boldsymbol{a}}$ means that the elements of $\boldsymbol{a}_n$ are mutually uncorrelated, which is good for applications such as compression.

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Karhunen-Loéve (KL) expansion: set $\boldsymbol{Q} = \boldsymbol{V}$, where $\boldsymbol{V}$ is obtained from the eigendecomposition $\boldsymbol{R}_{\boldsymbol{x}} = \boldsymbol{V} \boldsymbol{\Lambda} \boldsymbol{V}^H$.

- Under KL expansion, $\boldsymbol{R}_{\boldsymbol{a}} = \boldsymbol{\Lambda}$ and $\boldsymbol{R}_{\boldsymbol{a}}$ is diagonal.

More About KL Expansion

Since $\boldsymbol{R_a} = \boldsymbol{\Lambda}$, we have

$$\boldsymbol{R_a} = \begin{bmatrix} \mathbb{E}\{|a_{1n}|^2\} & & & \\ & \mathbb{E}\{|a_{2n}|^2\} & & \\ & & \ddots & \\ & & & \mathbb{E}\{|a_{mn}|^2\} \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}, \quad (15)$$

or simply

$$\mathbb{E}\{|a_{in}|^2\} = \lambda_i, \; i = 1, \cdots, m. \quad (16)$$

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or simply

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- The eigenvalues are also the average energies of the KL coefficients.
- There are many situations where the energy in the first few KL coefficients a_{in} dominates that in the remaining ones – which suggests the possibility of approximating $\{\boldsymbol{x}_n\}$ using fewer number of KL coefficients.

Application Example: Coding and Compression

Problem: encode $\{\boldsymbol{a}_n\}$ with a (much) smaller data size than the original $\{\boldsymbol{x}_n\}$; lossy coding is allowed.

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Transform coding (for generic bases): given a unitary basis matrix $\boldsymbol{Q}$,

1) Transform each $\boldsymbol{x}_n$ to $\boldsymbol{a}_n = \boldsymbol{Q}^H \boldsymbol{x}_n$;
2) Select a subset of coefficients in $\boldsymbol{a}_n$ to encode, e.g., by selecting a number of r coefficients whose magnitudes are the largest;
   - The selected coefficients may also be quantized.
3) (Decoding) Reconstruct $\boldsymbol{x}_n$ approximately by $\hat{\boldsymbol{x}}_n = \sum_{i \in \mathcal{I}_n} \hat{a}_{in} \boldsymbol{q}_i$, where $\{\hat{a}_{in}\}_{n \in \mathcal{I}_n}$ and $\mathcal{I}_n \subseteq \{1, \cdots, m\}$ are the selected encoded coefficients and its index set, respectively.

KL transform (KLT): choose $\boldsymbol{Q} = \boldsymbol{V}$.

Remarks

1) To perform KL expansion, information of $\boldsymbol{R}_{\boldsymbol{x}}$ is required. One way is to estimate $\boldsymbol{R}_{\boldsymbol{x}}$ from data: given a number of N samples $\{\boldsymbol{x}_n\}_{n=1}^{N}$, compute the sampled correlation matrix

$$\hat{\boldsymbol{R}}_{\boldsymbol{x}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{x}_n \boldsymbol{x}_n^H. \tag{17}$$

2) Consider the context of coding
   - While the decoder may not require information of $\boldsymbol{R}_{\boldsymbol{x}}$, it still requires information of the eigen-basis vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$.
   - We can encode $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$ too, but that may incur overhead in data size.
   - On the other hand, one may apply fixed models for $\boldsymbol{R}_{\boldsymbol{x}}$. It has been found that for a certain class of correlation models, DCT is, or well approximates, KLT.

3) KLT is almost the same as **principal component analysis (PCA)**, which is extensively used for data analysis and dimensionality reduction.

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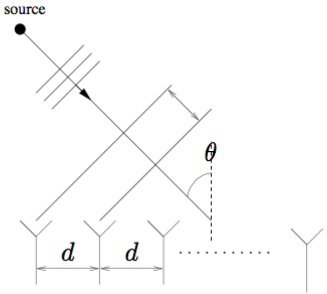
Sensor Array Processing

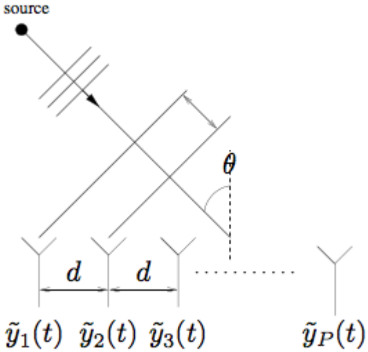
Sensor array processing deals with signal processing problems arising from the use of an array of sensors.

Applications: radar, sonar, wireless communications, seismology, radio astronomy, audio and speech,...



A key problem in this context is **direction-of-arrival (DoA) estimation** of sources.

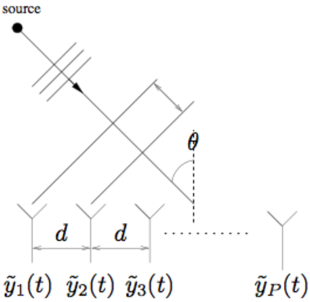
DoA Estimation: Preliminaries



Settings:

- Uniform linear array (ULA) with inter-sensor distance d and with P sensors,
- Far-field wave propagation so that the arriving source waves are planar.

DoA Estimation: Preliminaries (cont'd)



For one radiating source, the output of sensor p can be modeled as

$$\tilde{y}_p(t) = x\left(t - (p-1)\frac{d\sin\theta}{c}\right), \tag{18}$$

where $x(t)$ is the source signal impinging on sensor 1, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ denotes the source's DoA, and $c = 3 \times 10^8$ m/s refers to the wave propagation velocity.

DoA Estimation: Preliminaries (cont'd)

Assumption: narrowband carrier-modulated source waveform

$$x(t) = e^{j\omega_c t} s(t), \tag{19}$$

where ω_c is the center frequency, and $s(t)$ is narrowband in the sense that it satisfies

$$s(t-\tau) \simeq s(t), \quad \text{for any } \tau \in [-T_0, T_0] \tag{20}$$

and for some time duration T_0.

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and for some time duration T_0.

Let $y_p(t) = e^{-j\omega_c t}\tilde{y}_p(t)$ be a demodulated signal for sensor p. Assuming $T_0 > Pd/c$, we get

$$\begin{aligned} y_p(t) &= e^{-j\omega_c t} x\left(t - (p-1)\frac{d\sin\theta}{c}\right) && (21) \\ &= e^{-j(p-1)\omega_c \frac{d\sin\theta}{c}} s\left(t - (p-1)\frac{d\sin\theta}{c}\right) && (22) \\ &\simeq e^{-j(p-1)\omega_c \frac{d\sin\theta}{c}} s(t). && (23) \end{aligned}$$

DoA Estimation: Preliminaries (cont'd)

Let $\boldsymbol{y}(t) = [y_1(t), \cdots, y_P(t)]^T$, we know

$$\boldsymbol{y}(t) = \boldsymbol{a}(\theta)s(t), \tag{24}$$

where

$$\boldsymbol{a}(\theta) \triangleq \left[1, e^{-j\phi(\theta)}, e^{-j2\phi(\theta)}, \cdots, e^{-j(P-1)\phi(\theta)}\right]^T, \tag{25}$$

is called the **steering vector**, in which

$$\phi(\theta) \triangleq \omega_c \frac{d\sin\theta}{c} = 2\pi \frac{d\sin\theta}{\lambda}, \tag{26}$$

and λ is the wavelength corresponding to the center frequency ω_c.

- To avoid **spatial aliasing**, i.e., $\boldsymbol{a}(\theta_1) = \boldsymbol{a}(\theta_2)$ for some $\theta_1 \neq \theta_2$, $\theta_1, \theta_2 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we need

$$d < \frac{\lambda}{2}. \tag{27}$$

DoA Estimation: Problem Setup and Model

Scenario: K radiating sources, noisy observation.

Model: consider discrete-time sampling $\boldsymbol{y}[n] = \boldsymbol{y}(nT_s)$, where T_s is the sampling period. At time stamp $\boldsymbol{n}$, the spatial signal vector $\boldsymbol{y}[n]$ is modeled as

$$\boldsymbol{y}[n] = \sum_{k=1}^{K} \boldsymbol{a}(\theta_k) s_k[n] + \boldsymbol{w}[n] \tag{28}$$

$$= \boldsymbol{A}\boldsymbol{s}[n] + \boldsymbol{w}[n],\ n = 1, 2, \cdots, \tag{29}$$

where

- $\boldsymbol{s}_k[n]$ is the k^{th} source signal strength.
- θ_k is the DoA of the k^{th} source.
- $\boldsymbol{w}[n]$ is zero-mean spatially white noise, i.e., $\mathbb{E}\{\boldsymbol{w}[n]\boldsymbol{w}^H[n]\} = \sigma_{\boldsymbol{w}}^2 \boldsymbol{I}$.
- $\boldsymbol{A} = [\boldsymbol{a}(\theta_1), \cdots, \boldsymbol{a}(\theta_K)]$.
- $\boldsymbol{s}[n] = [s_1[n], \cdots, s_K[n]]^T$.

DoA Estimation: Problem Setup and Model (cont'd)

Assumptions:

- every $s_k[n]$ is zero-mean WSS.
- $\boldsymbol{s}[n]$ and $\boldsymbol{w}[n]$ are uncorrelated.

The correlation matrix of $\boldsymbol{y}[n]$ is given by

$$\begin{aligned}
\boldsymbol{R_y} &\triangleq \mathbb{E}\left\{\boldsymbol{y}[n]\boldsymbol{y}^H[n]\right\} && (30)\\
&= \mathbb{E}\left\{\boldsymbol{A}\boldsymbol{s}[n]\boldsymbol{s}^H[n]\boldsymbol{A}^H\right\} + \mathbb{E}\left\{\boldsymbol{w}[n]\boldsymbol{w}^H[n]\right\} && (31)\\
&= \boldsymbol{A}\boldsymbol{R_s}\boldsymbol{A}^H + \sigma_w^2\boldsymbol{I}, && (32)
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where $\boldsymbol{R}_{\boldsymbol{s}} = \mathbb{E}\left\{\boldsymbol{s}[n]\boldsymbol{s}^H[n]\right\}$.

Problem: estimate $\theta_1, \cdots, \theta_K$ from $\boldsymbol{R}_{\boldsymbol{y}}$, assuming the model

$$\boldsymbol{R}_{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{R}_{\boldsymbol{s}}\boldsymbol{A}^H + \sigma_w^2\boldsymbol{I}. \tag{33}$$

DoA Estimation: Subspace Approach

Idea: use eigendecomposition of $\boldsymbol{R_y}$ and subspace properties to identify $\theta_1, \cdots, \theta_K$.

Assumptions:

- $P > K$ (more sensors than sources).
- $\theta_1, \cdots, \theta_K$ are distinct.
- $\boldsymbol{R_s}$ has full rank (i.e., the sources may be correlated, but not coherent).

Property 1

The array response matrix $\boldsymbol{A}$ has full rank.

Vandemonde Matrix

A matrix $\boldsymbol{A}$ is said to be **Vandemonde** if it takes the form

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ z_1^2 & z_2^2 & \cdots & z_n^2 \\ \vdots & \ldots & \ddots & \vdots \\ z_1^{m-1} & z_2^{m-1} & \cdots & z_n^{m-1} \end{bmatrix} \in \mathbb{C}^{m\times n}, \tag{34}$$

where $z_1, z_2, \cdots, z_n \in \mathbb{C}$.

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where $z_1, z_2, \cdots, z_n \in \mathbb{C}$.

Property 2

A Vandemonde matrix with $z_i \neq z_k$ for all $i \neq k$ has full rank.

- The array response matrix $\boldsymbol{A}$ in DoA estimation is a special case of the Vandemonde matrix where $z_i = e^{-j\phi(\theta_i)}$.
- Property 1 directly follows from Property 2.

DoA Estimation: Subspace Approach

Denote the eigendecomposition of the signal correlation matrix $\boldsymbol{A}\boldsymbol{R_s}\boldsymbol{A}^H$ by

$$\boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^H. \tag{35}$$

Assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_P$.

Property 3

The eigenvalues $\lambda_1, \cdots, \lambda_P$ *of* $\boldsymbol{A}\boldsymbol{R_s}\boldsymbol{A}^H$ *satisfy*

$$\lambda_i > 0,\ i = 1, \cdots, K \textit{ while } \lambda_{K+1} = \cdots = \lambda_P = 0. \tag{36}$$

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Property 4

The eigendecomposition of $\boldsymbol{R_y}$ is

$$\boldsymbol{R_y} = \boldsymbol{V}(\boldsymbol{\Lambda} + \sigma_w^2\boldsymbol{I})\boldsymbol{V^H}, \tag{37}$$

where $\boldsymbol{V}$ and $\boldsymbol{\Lambda}$ denote the eigenvector and eigenvalue matrices of $\boldsymbol{A}\boldsymbol{R_s}\boldsymbol{A}^H$, resp.

- Property 4 means that the eigenvector matrix $\boldsymbol{V}$ of the signal correlation matrix can be obtained from the eigendecomposition of $\boldsymbol{R_y}$.

DoA Estimation: Subspace Approach (cont'd)

Theorem 1

Partition $\boldsymbol{V} = [\boldsymbol{V}_1, \boldsymbol{V}_2]$, *where*

$$\boldsymbol{V}_1 = [\boldsymbol{v}_1, \cdots, \boldsymbol{v}_K], \quad \boldsymbol{V}_2 = [\boldsymbol{v}_{K+1}, \cdots, \boldsymbol{v}_P]. \tag{38}$$

The equation

$$\boldsymbol{V}_2^H \boldsymbol{a}(\theta) = \boldsymbol{0}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{39}$$

is satisfied if and only if $\theta = \theta_i$ *for any* $i = 1, \cdots, K$.

- The subspace approach amounts to using $\boldsymbol{V}_2^H \boldsymbol{a}(\theta) = \boldsymbol{0}$ (or its approximation) to identify the source DoAs.
- There are many ways to do so, leading to many algorithms.
- A classical and most popular one is multiple signal classification (MUSIC).

DoA Estimation: The MUSIC Algorithm

MUSIC: Multiple Signal Classification[1]

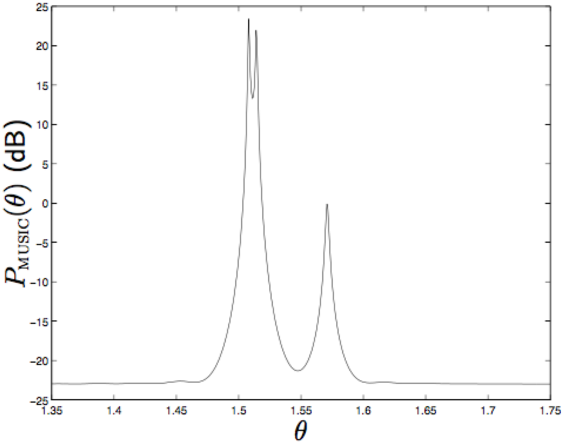
- **Step1.** compute the sample correlation matrix $\hat{\boldsymbol{R}}_{\boldsymbol{y}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{y}_n[n]\, \boldsymbol{y}_n^H[n]$.
- **Step2.** compute the eigenvector matrix of $\hat{\boldsymbol{R}}_{\boldsymbol{y}}$, denoted by $\hat{\boldsymbol{V}}$.
- **Step3.** evaluate the 'pseudo-spectrum'

$$P_{\mathsf{MUSIC}}(\theta) = \frac{1}{\|\hat{\boldsymbol{V}}_2^H \boldsymbol{a}(\theta)\|_2^2}, \quad \text{for } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{40}$$

where $\hat{\boldsymbol{V}} = [\hat{\boldsymbol{v}}_{K+1}, \cdots, \hat{\boldsymbol{v}}_P]$, and determine the DoAs by finding the peaks of $P_{\mathsf{MUSIC}}(\theta)$.

[1] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Transactions on Antenna Propagation*, vol. AP-34, pp. 276–290, March 1986.

DoA Estimation: Numerical Demonstration



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Preliminary: Circulant Matrix

A matrix $\boldsymbol{H}$ is said to be **circulant** if it takes the following structure

$$\boldsymbol{H} = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & \cdots & h_3 & h_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix} \tag{41}$$

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Eigenvectors: let $\boldsymbol{f}_k \triangleq \frac{1}{\sqrt{N}}\left[1, e^{j\frac{2\pi k}{N}}, \cdots, e^{j\frac{2\pi(N-1)k}{N}}\right]^T \in \mathbb{C}^N$. It can be verified that

$$\boldsymbol{H}\boldsymbol{f}_k = d_k \boldsymbol{f}_k, \quad k = 0, \cdots, N-1, \tag{42}$$

where

$$d_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} h_n e^{-j\frac{2\pi nk}{N}}. \tag{43}$$

- $\{d_k\}$ is the normalized DFT of $\{h_n\}$.

Preliminary: Circulant Matrix (cont'd)

Let $\boldsymbol{F} \triangleq [\boldsymbol{f}_0, \cdots, \boldsymbol{f}_{N-1}] \in \mathbb{C}^{N\times N}$.

- $\boldsymbol{F}$ is unitary.
- $\boldsymbol{F}$ is seen to be the inverse DFT (IDFT) matrix, while $\boldsymbol{F}^{-1} = \boldsymbol{F}^H$ the DFT matrix.
- DFT and IDFT can be efficiently computed (complexity order of $N\log(N)$).

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By the unitarity of $\boldsymbol{F}$, a circulant $\boldsymbol{H}$ admits an eigendecomposition

$$\boldsymbol{H} = \boldsymbol{F}\boldsymbol{D}\boldsymbol{F}^H, \tag{44}$$

$$\boldsymbol{D} = \text{diag}(d_0, \cdots, d_{N-1}). \tag{45}$$

- Application: **orthogonal frequency division multiplexing (OFDM)** in wireless communications.

OFDM: Background

Scenario: digital transmission over a linear time-invariant channel.
System Model:

$$y[n] = \sum_{l=0}^{L} h_l s[n-l] + \nu[n], \quad n \in \mathbb{Z}, \tag{46}$$

where

- $s[n]$: the transmitted data symbol stream,
- $y[n]$: the received signal (discrete-time sampled),
- h_l: the channel impulse response, with length $L+1$,
- $\nu[n]$: AWGN noise.

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Problem formulation:
Originally, we wish to detect each $s[n]$ from $y[n]$ – which works if $h_0 \neq 0$, $h_l = 0$ for $l > 1$. However, in general, we have **intersymbol interference (ISI)**

$$y[n] = h_0 s[n] + \underbrace{\sum_{l \neq 0} h_l s[n-l]}_{\text{ISI}} + \nu[n]. \tag{47}$$

OFDM

Aim: precode $\{s[n]\}$ so that it can be received in a zero-ISI manner.

Consider a block transmission, where transmission is parsed into time blocks, and every block has N symbols, $s[0], s[1], \cdots, s[N-1]$, to transmit.

Model for one time block:

$$y[n] = \sum_{l=0}^{L} h_l \tilde{x}[n-l] + \nu[n], \quad n = 0, 1, \cdots, N+L-1, \tag{48}$$

where

$$\left.\begin{aligned} \tilde{x}[0] &= x[N-L] \\ \tilde{x}[1] &= x[N-L+1] \\ &\vdots \\ \tilde{x}[L-1] &= x[N-1] \end{aligned}\right\} \text{cyclic prefix} \qquad \begin{aligned} \tilde{x}[L] &= x[0] \\ \tilde{x}[L+1] &= x[1] \\ &\vdots \\ \tilde{x}[N+L-1] &= x[N-1] \end{aligned}$$

and $x[n]$ is the precoded signal, which will be specified shortly.

OFDM (cont'd)

It can be verified that

$$\underbrace{\begin{bmatrix} y[L] \\ y[L+1] \\ \vdots \\ \vdots \\ \vdots \\ y[N+L-1] \end{bmatrix}}_{\triangleq \boldsymbol{y} \in \mathbb{C}^N} = \underbrace{\begin{bmatrix} h_0 & \cdots & \cdots & h_L & \cdots & h_1 \\ h_1 & h_0 & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \cdots & h_L \\ h_L & & & \ddots & & \\ & \ddots & & & \ddots & \\ & & h_L & \cdots & h_1 & h_0 \end{bmatrix}}_{\triangleq \boldsymbol{H} \in \mathbb{C}^{N \times N}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ \vdots \\ \vdots \\ x[N-1] \end{bmatrix}}_{\triangleq \boldsymbol{x} \in \mathbb{C}^N} + \underbrace{\begin{bmatrix} \nu[L] \\ \nu[L+1] \\ \vdots \\ \vdots \\ \vdots \\ \nu[N+L-1] \end{bmatrix}}_{\triangleq \boldsymbol{\nu} \in \mathbb{C}^N}, \quad (49)$$

where a linear system $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\nu}$ with a circulant system response $\boldsymbol{H}$ is obtained, in other words, we have

$$\boldsymbol{y} = \boldsymbol{F}\boldsymbol{D}\boldsymbol{F}^H \boldsymbol{x} + \boldsymbol{\nu}. \quad (50)$$

OFDM (cont'd)

1) At the transmitter side (IDFT), let $\boldsymbol{s} = [s[0], \cdots, s[N-1]]^T$.
OFDM: choose $\boldsymbol{x} = \boldsymbol{F}\boldsymbol{s}$ (i.e., IDFT the symbol block).
The system model reduces to

$$\boldsymbol{y} = \boldsymbol{F}\boldsymbol{D}\boldsymbol{F}^H\boldsymbol{x} + \boldsymbol{\nu} = \boldsymbol{F}\boldsymbol{D}\boldsymbol{F}^H\boldsymbol{F}\boldsymbol{s} + \boldsymbol{\nu} = \boldsymbol{F}\boldsymbol{D}\boldsymbol{s} + \boldsymbol{\nu}. \tag{51}$$

2) At the receiver side (DFT), consider a transformation $\boldsymbol{z} = \boldsymbol{F}^H\boldsymbol{y}$ (i.e., DFT $\boldsymbol{y}$, done by the receiver). Since

$$\boldsymbol{z} = \boldsymbol{D}\boldsymbol{s} + \boldsymbol{F}^H\boldsymbol{\nu} \Longleftrightarrow z_k = d_k s_k + [\boldsymbol{F}^H\boldsymbol{\nu}]_k, \quad k = 0, \cdots, N-1, \tag{52}$$

zero ISI is achieved.

- The noise term $\boldsymbol{F}^H\boldsymbol{\nu}$ is only rotated (by the unitarity of $\boldsymbol{F}$), but not magnified.
- A related scheme is single-carrier modulation (SCM), which chooses $\boldsymbol{x} = \boldsymbol{s}$. The receiver can achieve zero ISI by $\boldsymbol{z} = \boldsymbol{F}\boldsymbol{D}^{-1}\boldsymbol{F}^H\boldsymbol{y}$ (can be efficiently computed via DFT/IDFT), assuming an invertible $\boldsymbol{D}$. However, noise may be magnified.

OFDM (cont'd)

Remark 1

1) *To capture the central idea, we have introduced OFDM using a simplified model, namely, no interference between time blocks is assumed, but actually there is.*

   - *The inter-block interference can be easily handled by the so-called guard time removal; find textbooks for details, see e.g., [R1].*[2]

2) *OFDM (also SCM) uses a block length of $N + L$ to transmit N symbols. For good efficiency, N should be much greater than L.*

[2] [R1] J. Li, X. Wu, and R. Laroia, *OFDMA Mobile Broadband Communications: A Systems Approach*, Cambridge University Press, 2013.

Thank you for your attention!

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