Matrix Analysis and Applications

Chapter 4: Positive Semidefinite Matrices

Instructor: Kai Lu

(http://seit.sysu.edu.cn/teacher/1801)

School of Electronics and Information Technology
Sun Yat-sen University

November 22, 2020

Table of Contents

Table of Contents

Quadratic Form

Quadratic Form: given $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, the function

$$\boldsymbol{x}^H \boldsymbol{A}\boldsymbol{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^* x_j a_{ij}, \quad \boldsymbol{x} \in \mathbb{C}^n \tag{1}$$

is called a **quadratic form**.

- for complex-valued Hermitian $\boldsymbol{A}$, $\boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x}$ is real-valued for any $\boldsymbol{x} \in \mathbb{C}^n$.
- for the real-valued case, (1) is often replaced by $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}, \boldsymbol{x} \in \mathbb{R}^n$.

Notation:

$\mathcal{H}^n \subseteq \mathbb{C}^{n\times n}$, the set of all $n \times n$ complex Hermitian matrices.
$\mathcal{S}^n \subseteq \mathbb{R}^{n\times n}$, the set of all $n \times n$ real symmetric matrices.

Positive Semidefinite Matrices

1) A matrix $\boldsymbol{A} \in \mathcal{H}^n$ (resp. $\boldsymbol{A} \in \mathcal{S}^n$) is said to be **positive semidefinite** (PSD) if

$$\boldsymbol{x}^H \boldsymbol{A} \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \in \mathbb{C}^n \tag{2}$$

$$(\text{resp. } \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \in \mathbb{R}^n) \tag{3}$$

2) A matrix $\boldsymbol{A} \in \mathcal{H}^n$ (resp. $\boldsymbol{A} \in \mathcal{S}^n$) is said to be **positive definite** (PD) if

$$\boldsymbol{x}^H \boldsymbol{A} \boldsymbol{x} > 0 \text{ for all } \boldsymbol{x} \in \mathbb{C}^n, \boldsymbol{x} \neq \boldsymbol{0} \tag{4}$$

$$(\text{resp. } \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} > 0 \text{ for all } \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \neq \boldsymbol{0}) \tag{5}$$

3) A matrix $\boldsymbol{A} \in \mathcal{H}^n$ or $\boldsymbol{A} \in \mathcal{S}^n$ is said to be **indefinite** if it is not PSD.

Positive Semidefinite Matrices (cont'd)

Example 1 (Correlation Matrix)

Let $\{\boldsymbol{y}_n\}$, $\boldsymbol{y}_n \in \mathbb{C}^n$, be a WSS process. The correlation matrix $\boldsymbol{R}_{\boldsymbol{y}} = \mathbb{E}\{\boldsymbol{y}_n\, \boldsymbol{y}_n^H\}$ is PSD.

Proof:

$$\boldsymbol{x}^H \boldsymbol{R}_{\boldsymbol{y}} \boldsymbol{x} = \mathbb{E}\left\{\boldsymbol{x}^H \boldsymbol{y}_n \boldsymbol{y}_n^H \boldsymbol{x}\right\} = \mathbb{E}\left\{|\boldsymbol{x}^H \boldsymbol{y}_n|^2\right\} \geq 0, \text{ for any } \boldsymbol{x}.$$

The sample correlation matrix $\hat{\boldsymbol{R}}_{\boldsymbol{y}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{y}_n \boldsymbol{y}_n^H$ is also PSD.

Positive Semidefinite Matrices (cont'd)

Example 1 (Correlation Matrix)

Let $\{\boldsymbol{y}_n\}$, $\boldsymbol{y}_n \in \mathbb{C}^n$, be a WSS process. The correlation matrix $\boldsymbol{R}_{\boldsymbol{y}} = \mathbb{E}\{\boldsymbol{y}_n\, \boldsymbol{y}_n^H\}$ is PSD.

Proof:

$$\boldsymbol{x}^H \boldsymbol{R}_{\boldsymbol{y}} \boldsymbol{x} = \mathbb{E}\left\{\boldsymbol{x}^H \boldsymbol{y}_n \boldsymbol{y}_n^H \boldsymbol{x}\right\} = \mathbb{E}\left\{|\boldsymbol{x}^H \boldsymbol{y}_n|^2\right\} \geq 0, \text{ for any } \boldsymbol{x}.$$

The sample correlation matrix $\hat{\boldsymbol{R}}_{\boldsymbol{y}} = \frac{1}{N} \sum_{n=1}^{N} \boldsymbol{y}_n \boldsymbol{y}_n^H$ is also PSD.

Exercise: let $\boldsymbol{C}_{\boldsymbol{y}} = \mathbb{E}\left\{(\boldsymbol{y}_n - \mathbb{E}\{\boldsymbol{y}_n\})(\boldsymbol{y}_n - \mathbb{E}\{\boldsymbol{y}_n\})^H\right\}$ denote the covariance matrix of $\boldsymbol{y}_n$. Verify that $\boldsymbol{C}_{\boldsymbol{y}}$ is PSD.

Hessian Matrix of a Function

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, and $\mathrm{dom} f$ be the domain of the function. Assume twice differentiable f.

1) **Gradient**:

$$\Delta f(\boldsymbol{x}) \triangleq \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}. \tag{6}$$

2) **Hessian**: $\Delta^2 f(\boldsymbol{x}) \in \mathcal{S}^n$ is a matrix whose $(i,j)^{\mathrm{th}}$ elements are

$$\left[\Delta^2 f(\boldsymbol{x})\right]_{ij} \triangleq \frac{\partial^2 f}{\partial x_i \, \partial x_j}. \tag{7}$$

Hessian Matrix of a Function (cont'd)

Fact: a twice differentiable function f is convex if (also only if)

$$\Delta^2 f(\boldsymbol{x}) \;\; \text{is PSD for all} \;\; \boldsymbol{x} \in \text{dom} f.$$

- The class of convex functions is important to many areas, esp., optimization.

Hessian Matrix of a Function (cont'd)

Fact: a twice differentiable function f is convex if (also only if)

$$\Delta^2 f(\boldsymbol{x}) \text{ is PSD for all } \boldsymbol{x} \in \mathrm{dom} f.$$

- The class of convex functions is important to many areas, esp., optimization.

Example 2 (Quadratic Function)

The quadratic function

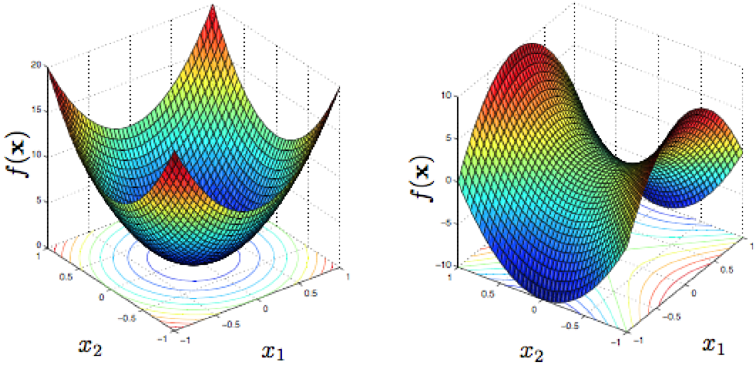
$$f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}^T \boldsymbol{x} + c, \tag{8}$$

where $\boldsymbol{A} \in \mathcal{S}^n, \boldsymbol{b} \in \mathbb{R}^n, c \in \mathbb{R}$. Its gradient and Hessian are

$$\Delta f(\boldsymbol{x}) = 2\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{b}, \quad \Delta^2 f(\boldsymbol{x}) = 2\boldsymbol{A}. \tag{9}$$

- The quadratic function is convex if $\boldsymbol{A}$ is PSD.

Illustration of Quadratic Functions



(a) PSD A. (b) indefinite A.

Table of Contents

Principal Submatrices

A **principal submatrix** of $\boldsymbol{A} \in \mathbb{C}^{n \times n}$, denoted by $\boldsymbol{A}_{\mathcal{I}}$, where $\mathcal{I} = \{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}, m < n$, is a submatrix obtained by keeping only the rows and columns indicated by $\mathcal{I}$.

Property 1

If $\boldsymbol{A}$ is PSD (resp. PD), then any $\boldsymbol{A}_{\mathcal{I}}$ is PSD (resp. PD).

Principal Submatrices

A **principal submatrix** of $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, denoted by $\boldsymbol{A}_{\mathcal{I}}$, where $\mathcal{I} = \{i_1, \cdots, i_m\} \subseteq \{1, \cdots, n\}, m < n$, is a submatrix obtained by keeping only the rows and columns indicated by $\mathcal{I}$.

Property 1

If $\boldsymbol{A}$ is PSD (resp. PD), then any $\boldsymbol{A}_{\mathcal{I}}$ is PSD (resp. PD).

Some immediate results:

- Partition $\boldsymbol{A}$ as
$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix},$$
if $\boldsymbol{A}$ is PSD, then both $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are also PSD.
- If $\boldsymbol{A}$ is PSD, then $a_{ii} \geq 0$ for all i. If $\boldsymbol{A}$ is PD, then $a_{ii} > 0$ for all i.

Eigenvalues

Property 2

Let $\boldsymbol{A} \in \mathcal{H}^n$ (or $\boldsymbol{A} \in \mathcal{S}^n$) and let $\lambda_1, \cdots, \lambda_n$ denote the eigenvalues of $\boldsymbol{A}$. Then, $\boldsymbol{A}$ is PSD (resp. PD) if and only if

$$\lambda_i \geq 0 \ (\textit{resp. } \lambda_i > 0 \textit{ for all } i). \tag{10}$$

Some immediate results:

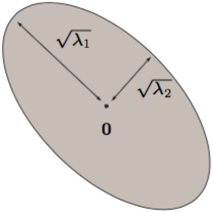
- If $\boldsymbol{A}$ is PSD (resp. PD), then $\det(A) \geq 0$ (resp. $\det(A) > 0$).
- If $\boldsymbol{A}$ is PSD (resp. PD), then tr$(A) \geq 0$ (resp. tr$(A) > 0$).
- If $\boldsymbol{A}$ is PD, then $\boldsymbol{A}$ is nonsingular (and invertible).

Example: Ellipsoid

Ellipsoid:

$$\varepsilon = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,|\, \boldsymbol{x}^T \boldsymbol{P}^{-1} \boldsymbol{x} \leq 1 \right\}, \tag{11}$$

where $\boldsymbol{P} \in \mathcal{S}^n$ is PD.



Let $\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$ be the eigendecomposition.

- $\boldsymbol{Q}$ determines the directions of the semi-axes;
- $\lambda_1, \cdots, \lambda_n$ determine the lengths of the semi-axes.

Example: Multivariate Gaussian Distribution

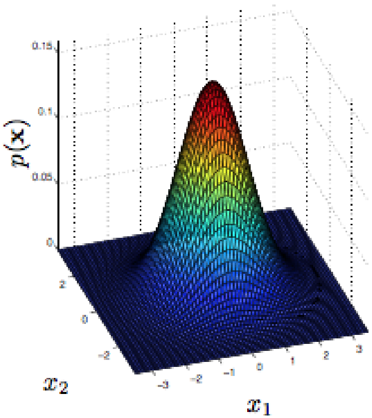
PDF for a Gaussian-distributed vector $\boldsymbol{x} \in \mathbb{R}^n$:

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} (\det(\boldsymbol{\Sigma}))^{\frac{1}{2}}} \exp\left(-\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right), \tag{12}$$

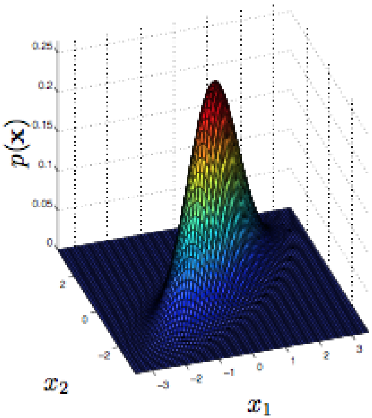
where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the mean and covariance of $\boldsymbol{x}$, resp.

- $\boldsymbol{\Sigma}$ is PD.
- $\boldsymbol{\Sigma}$ determines how $\boldsymbol{x}$ is spread, by the same way as in ellipsoid.

Example: Multivariate Gaussian Distribution



(a) $\mu = 0$, $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.



(b) $\mu = 0$, $\Sigma = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$.

Square-Root Factorization

Theorem 3 (Square-Root Factorization)

A matrix $\boldsymbol{A} \in \mathcal{H}^n$ can be factorized as

$$\boldsymbol{A} = \boldsymbol{B}^H \boldsymbol{B}, \tag{13}$$

for some $\boldsymbol{B}$ of appropriate dimension if and only if $\boldsymbol{A}$ is PSD.

- Eq. (13) is called a **square-root factorization** of $\boldsymbol{A}$, and a $\boldsymbol{B}$ satisfying (13) is called a **square root** of $\boldsymbol{A}$.

Square-Root Factorization

Theorem 3 (Square-Root Factorization)

A matrix $\boldsymbol{A} \in \mathcal{H}^n$ can be factorized as

$$\boldsymbol{A} = \boldsymbol{B}^H \boldsymbol{B}, \tag{13}$$

for some $\boldsymbol{B}$ of appropriate dimension if and only if $\boldsymbol{A}$ is PSD.

- Eq. (13) is called a **square-root factorization** of $\boldsymbol{A}$, and a $\boldsymbol{B}$ satisfying (13) is called a **square root** of $\boldsymbol{A}$.

Exercise 1: For the square-root factorization in (13), does $\boldsymbol{B}$ have to be square?

Exercise 2: Suppose that $\boldsymbol{A}$ is PSD, and that the matrix dimension of its square root $\boldsymbol{B}$ is fixed. Is $\boldsymbol{B}$ unique? Can we find a PSD $\boldsymbol{B}$?

Transformation

Theorem 4

Let $\boldsymbol{A} \in \mathcal{H}^n$, $\boldsymbol{B} \in \mathbb{C}^{n\times m}$ *and*

$$\boldsymbol{C} \triangleq \boldsymbol{B}^H \boldsymbol{A}\boldsymbol{B}. \tag{14}$$

The following properties hold:

- *If* $\boldsymbol{A}$ *is PSD, then* $\boldsymbol{C}$ *is PSD.*
- *If* $\boldsymbol{A}$ *is PD, then* $\boldsymbol{C}$ *is PD if and only if rank(*$\boldsymbol{B}$*)* $= m$.
- *If* $\boldsymbol{B}$ *is square and nonsingular, then* $\boldsymbol{C}$ *is PD (resp. PSD) if and only if* $\boldsymbol{A}$ *is PD (resp. PSD).*

Table of Contents

Matrix Inequality

Notation: the symbols '$\succeq$' and '$\succ$' mean that

$$\begin{aligned} \boldsymbol{A} \succeq \boldsymbol{B} \quad &\Longleftrightarrow \quad \boldsymbol{A} - \boldsymbol{B} \text{ is PSD.} \\ \boldsymbol{A} \succ \boldsymbol{B} \quad &\Longleftrightarrow \quad \boldsymbol{A} - \boldsymbol{B} \text{ is PD.} \end{aligned}$$

[1] Dennis S. Bernstein, *Scalar, Vector, and Matrix Mathematics: Theory, Facts, and Formulas (Revised and Expanded Edition)*, Princeton University Press, 2018.

Matrix Inequality

Notation: the symbols '$\succeq$' and '$\succ$' mean that

$$\begin{aligned} \boldsymbol{A} \succeq \boldsymbol{B} \quad &\Longleftrightarrow \quad \boldsymbol{A} - \boldsymbol{B} \text{ is PSD.} \\ \boldsymbol{A} \succ \boldsymbol{B} \quad &\Longleftrightarrow \quad \boldsymbol{A} - \boldsymbol{B} \text{ is PD.} \end{aligned}$$

Basic results (assuming $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{H}^n$):

- If $\boldsymbol{A} \succeq \mathbf{0}, \alpha \geq 0$ (resp. $\boldsymbol{A} \succ \mathbf{0}, \alpha > 0$), then $\alpha\boldsymbol{A} \succeq \mathbf{0}$ (resp. $\alpha\boldsymbol{A} \succ \mathbf{0}$).
- If $\boldsymbol{A} \succeq \mathbf{0}, \boldsymbol{B} \succeq \mathbf{0}$ (resp. $\boldsymbol{A} \succ \mathbf{0}$), then $\boldsymbol{A} + \boldsymbol{B} \succeq \mathbf{0}$ (resp. $\boldsymbol{A} + \boldsymbol{B} \succ \mathbf{0}$).
- If $\boldsymbol{A} \succeq \boldsymbol{B}, \boldsymbol{B} \succeq \boldsymbol{C}$ (resp. $\boldsymbol{B} \succ \boldsymbol{C}$), then $\boldsymbol{A} \succeq \boldsymbol{C}$ (resp. $\boldsymbol{A} \succ \boldsymbol{C}$).

[1] Dennis S. Bernstein, *Scalar, Vector, and Matrix Mathematics: Theory, Facts, and Formulas (Revised and Expanded Edition)*, Princeton University Press, 2018.

Matrix Inequality

Notation: the symbols '$\succeq$' and '$\succ$' mean that

$$\begin{aligned} \boldsymbol{A} \succeq \boldsymbol{B} \quad &\Longleftrightarrow \quad \boldsymbol{A} - \boldsymbol{B} \text{ is PSD.} \\ \boldsymbol{A} \succ \boldsymbol{B} \quad &\Longleftrightarrow \quad \boldsymbol{A} - \boldsymbol{B} \text{ is PD.} \end{aligned}$$

Basic results (assuming $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{H}^n$):

- If $\boldsymbol{A} \succeq \mathbf{0}, \alpha \geq 0$ (resp. $\boldsymbol{A} \succ \mathbf{0}, \alpha > 0$), then $\alpha\boldsymbol{A} \succeq \mathbf{0}$ (resp. $\alpha\boldsymbol{A} \succ \mathbf{0}$).
- If $\boldsymbol{A} \succeq \mathbf{0}, \boldsymbol{B} \succeq \mathbf{0}$ (resp. $\boldsymbol{A} \succ \mathbf{0}$), then $\boldsymbol{A} + \boldsymbol{B} \succeq \mathbf{0}$ (resp. $\boldsymbol{A} + \boldsymbol{B} \succ \mathbf{0}$).
- If $\boldsymbol{A} \succeq \boldsymbol{B}, \boldsymbol{B} \succeq \boldsymbol{C}$ (resp. $\boldsymbol{B} \succ \boldsymbol{C}$), then $\boldsymbol{A} \succeq \boldsymbol{C}$ (resp. $\boldsymbol{A} \succ \boldsymbol{C}$).

Exercise: Suppose $\boldsymbol{A} \not\succeq \boldsymbol{B}$, which means that $\boldsymbol{A} - \boldsymbol{B}$ is not PSD. Does this imply $\boldsymbol{B} \succeq \boldsymbol{A}$?

Note: There are more results arising from the PSD matrix inequality.[1]

[1] Dennis S. Bernstein, *Scalar, Vector, and Matrix Mathematics: Theory, Facts, and Formulas (Revised and Expanded Edition)*, Princeton University Press, 2018.

Some Matrix Inequality Results

Denote $\lambda_1(\boldsymbol{A}), \cdots, \lambda_n(\boldsymbol{A})$ as the eigenvalues of $\boldsymbol{A} \in \mathcal{H}^n$.

Some results (assuming $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{H}^n$)[2]:

- $\boldsymbol{A} \succeq \boldsymbol{I}$ (resp. $\boldsymbol{A} \succ \boldsymbol{I}$) $\Longleftrightarrow \lambda_k(\boldsymbol{A}) \geq 1$ (resp. $\lambda_k(\boldsymbol{A}) > 1$) for all $k = 1, \cdots, n$.
  $\boldsymbol{I} \succeq \boldsymbol{A}$ (resp. $\boldsymbol{I} \succ \boldsymbol{A}$) $\Longleftrightarrow \lambda_k(\boldsymbol{A}) \leq 1$ (resp. $\lambda_k(\boldsymbol{A}) < 1$) for all $k = 1, \cdots, n$.
- If $\boldsymbol{A} \succeq \boldsymbol{B}$, then $\lambda_k(\boldsymbol{A}) \geq \lambda_k(\boldsymbol{B})$, $k = 1, \cdots, n$.
- Suppose $\boldsymbol{A}, \boldsymbol{B} \succ \boldsymbol{0}$, then $\boldsymbol{A} \succeq \boldsymbol{B} \Longleftrightarrow \boldsymbol{B}^{-1} \succeq \boldsymbol{A}^{-1}$.

[2] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd Ed., Cambridge University Press, 2013.

Some Matrix Inequality Results

Denote $\lambda_1(\boldsymbol{A}), \cdots, \lambda_n(\boldsymbol{A})$ as the eigenvalues of $\boldsymbol{A} \in \mathcal{H}^n$.

Some results (assuming $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{H}^n$)[2]:

- $\boldsymbol{A} \succeq \boldsymbol{I}$ (resp. $\boldsymbol{A} \succ \boldsymbol{I}$) $\Longleftrightarrow \lambda_k(\boldsymbol{A}) \geq 1$ (resp. $\lambda_k(\boldsymbol{A}) > 1$) for all $k = 1, \cdots, n$.
  $\boldsymbol{I} \succeq \boldsymbol{A}$ (resp. $\boldsymbol{I} \succ \boldsymbol{A}$) $\Longleftrightarrow \lambda_k(\boldsymbol{A}) \leq 1$ (resp. $\lambda_k(\boldsymbol{A}) < 1$) for all $k = 1, \cdots, n$.
- If $\boldsymbol{A} \succeq \boldsymbol{B}$, then $\lambda_k(\boldsymbol{A}) \geq \lambda_k(\boldsymbol{B})$, $k = 1, \cdots, n$.
- Suppose $\boldsymbol{A}, \boldsymbol{B} \succ \boldsymbol{0}$, then $\boldsymbol{A} \succeq \boldsymbol{B} \Longleftrightarrow \boldsymbol{B}^{-1} \succeq \boldsymbol{A}^{-1}$.

Some results from the above results:

- If $\boldsymbol{A} \succeq \boldsymbol{B}$ and $\boldsymbol{B} \succeq \boldsymbol{0}$, then $\det(\boldsymbol{A}) \geq \det(\boldsymbol{B})$.
- If $\boldsymbol{A} \succeq \boldsymbol{B}$, then $\mathrm{tr}(\boldsymbol{A}) \geq \mathrm{tr}(\boldsymbol{B})$.
- If $\boldsymbol{A}, \boldsymbol{B} \succ \boldsymbol{0}$ and $\boldsymbol{A} \succeq \boldsymbol{B}$, then $\mathrm{tr}(\boldsymbol{A}^{-1}) \leq \mathrm{tr}(\boldsymbol{B}^{-1})$.

[2] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd Ed., Cambridge University Press, 2013.

Schur Complement

Let

$$\boldsymbol{X} \triangleq \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^H & \boldsymbol{C} \end{bmatrix} \in \mathcal{H}^{m+n}, \tag{15}$$

where $\boldsymbol{A} \in \mathcal{H}^m, \boldsymbol{B} \in \mathbb{C}^{m\times n}, \boldsymbol{C} \in \mathcal{H}^n, \boldsymbol{C} \succ \mathbf{0}$.

Define $\boldsymbol{S} \triangleq \boldsymbol{A} - \boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{B}^H$, which is called the **Schur complement** of the block $\boldsymbol{C}$ of the matrix $\boldsymbol{X}$. Then,

$$\boldsymbol{X} \succeq \mathbf{0} \ \ (\text{resp.}\ \ \boldsymbol{X} \succ \mathbf{0}) \Longleftrightarrow \boldsymbol{S} \succeq \mathbf{0} \ \ (\text{resp.}\ \ \boldsymbol{S} \succ \mathbf{0}). \tag{16}$$

Likewise, $\boldsymbol{S}' \triangleq \boldsymbol{C} - \boldsymbol{B}^H\boldsymbol{A}^{-1}\boldsymbol{B}$, called the **Schur complement** of the block $\boldsymbol{A}$ of the matrix $\boldsymbol{X}$.

Schur Complement

Let

$$\boldsymbol{X} \triangleq \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^H & \boldsymbol{C} \end{bmatrix} \in \mathcal{H}^{m+n}, \tag{15}$$

where $\boldsymbol{A} \in \mathcal{H}^m, \boldsymbol{B} \in \mathbb{C}^{m\times n}, \boldsymbol{C} \in \mathcal{H}^n, \boldsymbol{C} \succ \mathbf{0}$.

Define $\boldsymbol{S} \triangleq \boldsymbol{A} - \boldsymbol{B}\boldsymbol{C}^{-1}\boldsymbol{B}^H$, which is called the **Schur complement** of the block $\boldsymbol{C}$ of the matrix $\boldsymbol{X}$. Then,

$$\boldsymbol{X} \succeq \mathbf{0} \ \ (\text{resp.} \ \ \boldsymbol{X} \succ \mathbf{0}) \Longleftrightarrow \boldsymbol{S} \succeq \mathbf{0} \ \ (\text{resp.} \ \ \boldsymbol{S} \succ \mathbf{0}). \tag{16}$$

Likewise, $\boldsymbol{S}' \triangleq \boldsymbol{C} - \boldsymbol{B}^H\boldsymbol{A}^{-1}\boldsymbol{B}$, called the **Schur complement** of the block $\boldsymbol{A}$ of the matrix $\boldsymbol{X}$.

Application: see `https://en.wikipedia.org/wiki/Schur_complement`

Thank you
for your attention!

Kai Lu

E-mail: lukai86@mail.sysu.edu.cn

Web: http://seit.sysu.edu.cn/teacher/1801