Fixed Points of a Complex Henon Map

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Outline

Complex Dynamics

Fatou Sets

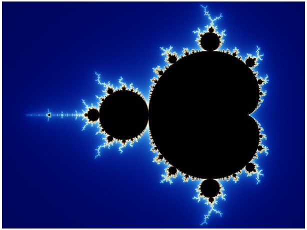
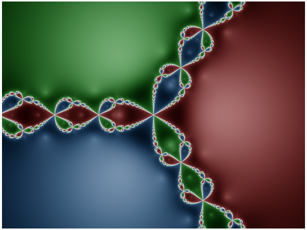
Henon Maps

Research

Complex Dynamics

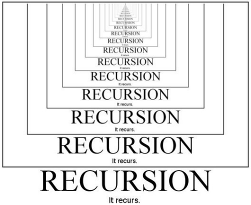
An Introduction and Some Examples

Fractals

- Mandelbrot Set
  - 
- Julia Set for $z^3 - 1$
  - 

What is Complex Dynamics?

- Understanding the behavior of the sequence of functions formed by repeatedly composing a complex function with itself



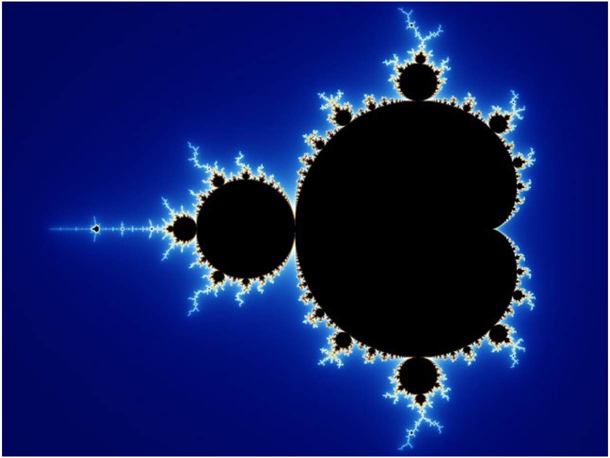
For example: $f(z) = z^2$

$$f_0 = f(1) = 1$$
$$f_1 = f(f_0) = 1$$
$$f_2 = f(f_1) = 1...$$

$$f_0 = f(2) = 4$$
$$f_1 = f(f_0) = 16$$
$$f_2 = f(f_1) = 256...$$

f_0, f_1, f_2: *sequence of iterates*
1: *fixed point*

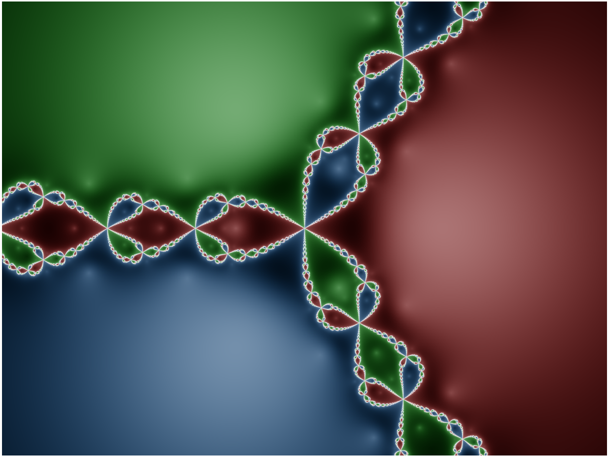
Fatou Sets



Fatou Sets

Defining Fatou Sets with an Example

Fatou Sets



Fatou Sets

Definition: Fatou Set $\mathcal{F}$

Let $q \in \widehat{\mathbb{C}}$ and $\{f^n\}$ be the sequence of iterates. Then, $q \in \mathcal{F}$ if, for every subsequence $f^{n_j}(q)$ of the sequence $f^n(q)$, there exists a further subsequence $f^{n_{j_k}}(q)$ that either diverges to infinity or converges finitely and this convergence is uniform.

Fatou Set Example

Theorem

For the function $f(z) = z^2$, the Fatou Set is the set of all complex numbers excluding those with norm 1.

Let us denote the Fatou Set of f to be $\mathcal{F}$. Let $q \in \mathbb{C}$ be arbitrary, such that $|q| \neq 1$. We will consider 2 scenarios:

$$|q| > 1$$
$$0 \leq |q| < 1$$

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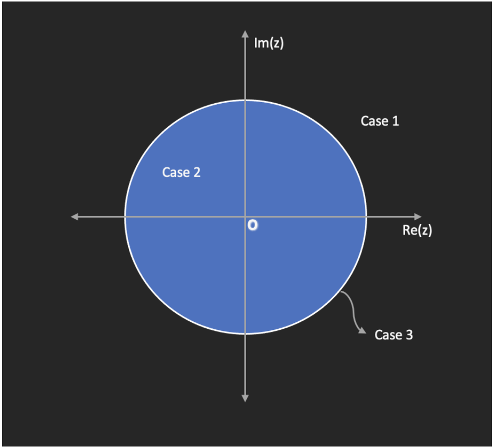
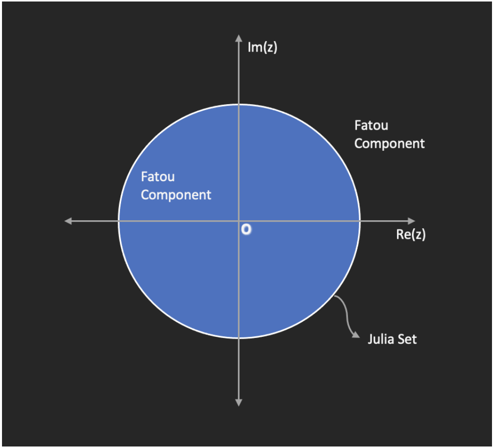
Graphical Representation of Cases



Graphical Representation of Fatou and Julia Sets of z^2



Henon Maps

A Brief Introduction

Henon Maps

A complex Henon map is a holomorphic polynomial diffeomorphism of $\mathbb{C}^2$ (i.e. an invertible mapping of $\mathbb{C}^2$ to itself) of the form:

$$f(x, y) = (y, p(y) - \delta x)$$

where $p(y)$ is a holomorphic polynomial with a degree greater than 2 and $\delta \in \widehat{\mathbb{C}}$ is a non-zero constant.

- A conservative Henon map has $|\delta| = 1$.
- Graphically speaking, conservative Henon maps are volume-preserving.

Question:

Does every bounded Fatou component of a conservative Hénon map have a fixed point?

Bedford's Theorem

Theorem

Let f be conservative, and let Ω be a bounded Fatou component with $f(\Omega) = \Omega$. Let $\mathcal{O} \in \Omega$, and let $\mathrm{A} := D_{\mathcal{O}}f$. If $\mathcal{O}$ is a fixed point for f, then we may diagonalize $\mathrm{A} \sim \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $|\lambda| = |\mu| = 1$.

If:

- $\mathcal{O} \in \Omega$ is a fixed point

Then:

- Derivative of f at $\mathcal{O}$, $D_{\mathcal{O}}f$ is *diagonalizable*

We will use the contrapositive:

If:

- Derivative of f at $\mathcal{O}$, $D_{\mathcal{O}}f$ is **NOT** *diagonalizable*

Then:

- $\mathcal{O}$ is not a fixed point

Diagonalization

Setting up the problem

- Function: $f(x, y) = (y, p(y) - \delta x)$
- Derivative: $D_{\mathcal{O}} f = \begin{bmatrix} 0 & 1 \\ -\delta & p'(y_{\mathcal{O}}) \end{bmatrix}$
- Let $M = D_{\mathcal{O}} f$ have eigenvalues λ
- For all $\mathcal{O}$ in Ω
  - Implies: infinitely many $y_{\mathcal{O}}$ since Ω is an open set
- For a matrix M to be non-diagonalizable: M must have *only* 1 eigenvalue
  - Can be proven for any M of the form $\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$

Diagonalization

Finding the eigenvalues λ of the derivative D of f at a fixed point

Characteristic polynomial of $D_{\mathcal{O}}f$: $\det(D_{\mathcal{O}}f - \lambda I) = 0$

$$
\begin{aligned}
D_{\mathcal{O}}f - \lambda I &= \begin{bmatrix} 0 & 1 \\ -\delta & p'(y_{\mathcal{O}}) \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\
&= \begin{bmatrix} -\lambda & 1 \\ -\delta & p'(y_{\mathcal{O}}) - \lambda \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\det(D_{\mathcal{O}}f - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ -\delta & p'(y_{\mathcal{O}}) - \lambda \end{vmatrix} \\
&= -\lambda p'(y_{\mathcal{O}}) + \lambda^2 + \delta
\end{aligned}
$$

Solving for λ as a root of the charecteristic equation:

$$
\lambda = \frac{p'(y_{\mathcal{O}}) \pm \sqrt{p'(y_{\mathcal{O}})^2 - 4\delta}}{2}
$$

Condition for matrix to be non-diagonalizable

This gives us the following condition for non-diagonalizable as follows:

$$p'(y_{\mathcal{O}})^2 = 4\delta$$

Why does this not work?

$$p'(y_{\mathcal{O}})^2 = 4\delta$$

- **Our goal**: Show that matrix M is non-diagonalizable
- Above equation needs to have infinitely many solutions
- $p'(y_{\mathcal{O}})^2$ has degree $2(n-1)$
  - n is the degree of $p(y_{\mathcal{O}})$
- Above equation has at most $2(n-1)$ unique solutions
  - Fundamental Theorem of Algebra

Bedford's Theorem **cannot** *be used to determine if there exists at least 1 fixed point in every Fatou component of a complex Henon Map*

Thank you!