

Exercise 1.4.4

We will show that $\sup T = b$ by contradiction. By assuming $\sup T \neq b$, we know that either $\sup T < b$ or $\sup T > b$. In the first case, we can assume $\sup T < b$. Let $c = \sup T$. Then, by the Density of the Rationals, there must exist a q in the rationals such that $c < q < b$. Then, q is in T because it is in \mathbb{Q} and $q < b$. Since $c < b$, c cannot be the least upper bound of T , as there is an element in T , q , that is greater than it. In the second case, we can assume $\sup T > b$. Again, let $c = \sup T$. Then, by the Density of the Rationals, there must exist a q in the rationals such that $b < q < c$. Now, we know that q is an upper bound for T that is less than c . Therefore, c is not the least upper bound for T . Now, since we arrived at contradictions for both $\sup T < b$ and $\sup T > b$, it must be true that $\sup T = b$.

Exercise 1.4.5

By Density of the Rationals, there must exist a q such that $a - \sqrt{2} < q < b - \sqrt{2}$. Adding $\sqrt{2}$ to the whole inequality, we have $a < q + \sqrt{2} < b$. $q + \sqrt{2} \in \mathbb{I}$ because $a + t \in \mathbb{I}$ if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, which is satisfied by q and $\sqrt{2}$. So, $q + \sqrt{2}$ satisfies a $t \in \mathbb{I}$ for $a < t < b$.

Exercise 1.4.8

- (a) Let $A = \{1 - \frac{1}{2^n} | n \in \mathbb{N}\}$ and $B = \{1 - \frac{1}{3^n} | n \in \mathbb{N}\}$. $A \cap B = \emptyset$, as all rationals in each set are in lowest form and A has only powers of 2 in the denominator, while B has only powers of 3 in the denominator. $\sup A = \sup B = 1$ and $1 \notin A \cup B$, so A and B satisfy the conditions.
- (b) Let $J_1 = (-1, 1), J_2 = (-\frac{1}{2}, \frac{1}{2}), \dots, J_n = (-\frac{1}{n}, \frac{1}{n})$. Then, $\bigcap_{n=1}^{\infty} J_n = \{0\}$, as for any real number $y > 0$, $\exists n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$, by the Archimedean Property, and J_n does not contain y . Since the intersection contains a finite amount of elements, the sets J_n satisfies the given conditions.
- (c) Let $L_1 = [1, \infty), L_2 = [2, \infty), \dots, L_n = [n, \infty)$. Then, $\bigcap_{n=1}^{\infty} L_n = \emptyset$, as for any number $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ satisfying $n > x$, by the Archimedean Property, and L_n does not contain x . Since the intersection is empty, the sets L_n satisfies the given conditions.
- (d) This condition is not possible, which can be proven by contradiction. Let $I_n = [a_n, b_n]$, with $a_n \leq b_n$, assuming all the finite intersections are nonempty and the infinite intersection is empty. Because the finite intersections are nonempty, we can define $A_N = \bigcap_{n=1}^N I_n$, which is closed and bounded because each I_n is closed and bounded. We know that A_N is nonempty because it is a finite intersection. Each $A_{N+1} = A_N \cap I_{N+1} \subseteq A_N$, so $(A_N)_{N \geq 1}$ is a nested sequence of nonempty closed bounded intervals. By applying the Nested Interval Property, we know that $\bigcap_{N=1}^{\infty} A_N \neq \emptyset$. But, $\bigcap_{N=1}^{\infty} A_N = \bigcap_{n=1}^{\infty} I_n$, so $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradicts the assumption that the infinite intersection is empty, so a sequence of closed bounded intervals cannot satisfy the given conditions.

Exercise 1.5.9

- (a) $x^2 - 2 = 0$ is a polynomial which has two real roots, one of which is $\sqrt{2}$, showing that $\sqrt{2}$ is algebraic.

$x^3 - 2 = 0$ is a polynomial which has one real root, which is $\sqrt[3]{2}$, showing that $\sqrt[3]{2}$ is algebraic.

We want $x = \sqrt{2} + \sqrt{3}$. Squaring both sides, we get $x^2 = 5 + 2\sqrt{6}$. Subtracting 5 from both sides and squaring both sides again results in $x^4 - 10x^2 + 25 = 24$. Rearranging, we have the polynomial $x^4 - 10x^2 + 1 = 0$ that has a real root of $\sqrt{2} + \sqrt{3}$, showing that it is algebraic.

- (b) By fixing $n \in \mathbb{N}$, we let A_n denote the set of algebraic numbers obtained as roots of polynomials with integer coefficients of degree n . First, this means that the number of polynomials of integer coefficients of degree n , P_n , must be countable. Each coefficient a_i must be in \mathbb{Z} , and the coefficient of the x^n must be in $\mathbb{Z} \setminus \{0\}$. The cardinality of the Cartesian product of these sets represents the cardinality

of P_n . Since $|\mathbb{Z}| = |\mathbb{Z} \setminus \{0\}| = |\mathbb{N}|$, $|P_n|$ is equal to the cardinality of the Cartesian product of n sets of \mathbb{N} . The finite product of countably infinite sets is also countable, which can be shown by induction. For the base case, we know \mathbb{N} is countable. Using the diagonal method from in class, we know that the product of two countable sets, such as $\mathbb{N} \times \mathbb{N}$, is countable. Assuming $\prod_{i=1}^m \mathbb{N}$ is countable, we need to prove $\prod_{i=1}^{m+1} \mathbb{N}$ is countable. $\prod_{i=1}^{m+1} \mathbb{N} = (\prod_{i=1}^m \mathbb{N}) \times \mathbb{N}$, which is the product of two countable sets. So, $\prod_{i=1}^{m+1} \mathbb{N}$ is countable. Thus, any finite product of countable sets is countable, and the cardinality of the Cartesian product of n sets of \mathbb{N} is countable. For each $p \in P_n$, the number of roots of p is at most n . Let $R(p)$ be the set of roots of p , and $|R(p)|$ is always finite. So, $A_n = \bigcup_{p \in P_n} R(p)$ and since the countable union of finite sets is countable, A_n is countable.

- (c) Since we proved that for a fixed $n \in \mathbb{N}$, A_n is countable, we can also prove that the set of all algebraic numbers, A , is countable. Since a countable union of countable sets is countable, A is countable, as $A = \bigcup_{n=1}^{\infty} A_n$. Using this, we can conclude that the transcendentals are uncountable with a proof by contradiction. The transcendentals are equal to $\mathbb{R} \setminus A$. If $\mathbb{R} \setminus A$ were countable, then $\mathbb{R} = A \cup (\mathbb{R} \setminus A)$ would be countable. Since \mathbb{R} is uncountable, $\mathbb{R} \setminus A$ must be uncountable. Therefore, there are uncountably many transcendentals.

Exercise 1.5.11

- (a) We can define $h : X \rightarrow Y$ with

$$h(x) = \begin{cases} f(x), & x \in A \\ g^{-1}(x), & x \in A' \end{cases}$$

If f can be shown to be bijective from A to B , and g^{-1} can be shown to exist by g being bijective from B' to A' , then their definitions will agree on disjoint subsets of X . Then, h will be a bijection, so $X \sim Y$.

- (b) In the case that $A_1 = \emptyset$, then g is surjective, as for every element $x \in X$, $\exists y \in Y$ such that $g(y) = x$. Then, g is bijective, so $X \sim Y$ immediately. Assuming $A \neq \emptyset$, we need to prove that A_i and A_j are disjoint for all $i, j \in \mathbb{N}$. Suppose x is an element of $A_i \cap A_j$, with $i < j$. Then, as $x \in A_i$, there must exist an element $y \in A_1$ such that $i - 1$ applications of g and f maps it to x . Since $x \in A_j$, there must also exist an element $z \in A_1$ such that $j - 1$ applications of g and f maps it to x . So, we have $(g \circ f)^{i-1}(y) = x$ and $(g \circ f)^{j-1}(z) = x$. Then, $(g \circ f)^{i-1}(y) = (g \circ f)^{j-1}(z)$. By the injectivity of g and f and the fact that $i < j$, we can cancel the outermost compositions of $(g \circ f)$ $i - 1$ times. Then, we have $y = (g \circ f)^{j-i}(z)$. Since $j > i$, g is applied at least once to z , so $(g \circ f)^{j-i}(z) \in g(Y)$. However, by definition, $y \in A_1$, and $A_1 \cap g(Y) = \emptyset$. So, by contradiction, $A_i \cap A_j = \emptyset$ and A_i and A_j are disjoint for all $i, j \in \mathbb{N}$. Since the subsets of X are pairwise disjoint and f is injective, each subset $f(A_n)$ for $n \in \mathbb{N}$ of Y is also pairwise disjoint.
- (c) To show that f maps A onto B , we need to show that $f(A) \subset B$ and $B \subset f(A)$. Take any $x \in A$. By definition of A , $x \in A_n$ for some n . By applying f , we have $f(x) \in f(A_n) \subset B$, so $f(A) \subset B$. Now, take any $y \in B$. By definition of B , $y \in f(A_n)$ for some n . Then, there must exist an $x \in A_n$ such that $y = f(x)$. Since $A_n \subset A$, $x \in A$. Therefore, $y \in f(A)$ and $B \subset f(A)$. So, f maps A onto B , and since f is injective, f has been proven to be bijective.
- (d) To show that g maps B' onto A' , we need to show that $g(B') \subset A'$ and $A' \subset g(B')$. Take any $y \in B'$. If $g(y) \in A$, then $g(y) \in A_{n+1} = g(f(A_n))$ for some n . By injectivity of g , we have $y \in f(A_n) \subset B$. This is a contradiction, as $y \in Y \setminus B$. So, $g(y) \notin A$, and $g(y) \in A'$. Therefore, $g(B') \subset A'$. Now, take any $x \in A'$. Then, $x \notin A_n$ for any $n \in \mathbb{N}$. So, $x \notin A_1$, and $x \in g(Y)$. Let $y \in g(Y)$ such that $g(y) = x$. If $y \in B$, then $x \in A$, which is a contradiction, as $x \in A'$. So, $y \in B'$, and since $x = g(y)$, $A' \subset g(B')$. So, g maps B' onto A' , and since g is injective, g has been proven to be bijective.

Exercise 1.6.1

We must prove that \mathbb{R} being uncountable $\implies (0, 1)$ being uncountable, and $(0, 1)$ being uncountable $\implies \mathbb{R}$ being uncountable. Assuming \mathbb{R} is uncountable, we need to create a bijection $f : \mathbb{R} \rightarrow (0, 1)$. Let $f(x) = \frac{1}{1+e^{-x}}$. This function has an inverse, $f^{-1} : (0, 1) \rightarrow \mathbb{R}$, which is defined as $f^{-1}(x) = -\ln(\frac{1}{x} - 1)$. Since f has an inverse function, f is bijective, meaning \mathbb{R} and $(0, 1)$ have the same cardinality. Since we are assuming \mathbb{R} is uncountable, $(0, 1)$ is also uncountable. To prove that $(0, 1)$ being uncountable $\implies \mathbb{R}$ being uncountable, we can assume $(0, 1)$ is uncountable and use $f^{-1} : (0, 1) \rightarrow \mathbb{R}$ as our function. Since it has an inverse, specifically f , f^{-1} is a bijection, so $(0, 1)$ and \mathbb{R} have the same cardinality. Since we are assuming $(0, 1)$ is uncountable, \mathbb{R} is also uncountable. Therefore, $(0, 1)$ is uncountable if and only if \mathbb{R} is uncountable.

Exercise 1.6.9

The power set of \mathbb{N} is the collection of subsets of the natural numbers. Let each subset A be represented by a sequence of binary numbers, (a_1, a_2, a_3, \dots) , where $a_n = 1$ if $n \in A$ and $a_n = 0$ if $n \notin A$. This correspondence between the subsets of the naturals and a binary sequence is bijective, so $|P(\mathbb{N})| = |\{0, 1\}^{\mathbb{N}}|$. Every binary sequence can be mapped to a real number in $[0, 1]$. Let x be defined as $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$. However, some real numbers have multiple expansions in this form, such as $\frac{1}{2}$, which can be represented as $(1, 0, 0, \dots)$, but also as $(0, 1, 1, \dots)$. However, this is true for only a countable amount of real numbers, so cardinality is not affected. Therefore, $|\{0, 1\}^{\mathbb{N}}| = |[0, 1]|$. Since the cardinality of $[0, 1]$ is equal to the cardinality of \mathbb{R} , we have $|P(\mathbb{N})| = |\{0, 1\}^{\mathbb{N}}| = |[0, 1]| = |\mathbb{R}|$. So, $P(\mathbb{N}) \sim \mathbb{R}$.

Exercise 1.6.10

- (a) The set of all functions from $\{0, 1\}$ to \mathbb{N} can be represented as a set of ordered pairs. Each function $f : \{0, 1\} \rightarrow \mathbb{N}$ can be written as an ordered pair $(f(0), f(1))$. The values of $f(0)$ and $f(1)$ vary across the natural numbers, so the set of all of these ordered pairs is equivalent to $\mathbb{N} \times \mathbb{N}$. Since there is a 1-1 correspondence from $\mathbb{N} \times \mathbb{N}$ to \mathbb{N} , the set of all functions from $\{0, 1\}$ to \mathbb{N} is countable.
- (b) We can show that the set of all functions from \mathbb{N} to $\{0, 1\}$ is uncountable using Cantor's diagonal argument. Assume that the set is countable; then it should be able to be enumerated in correspondence with the natural numbers. Let $f_n : \mathbb{N} \rightarrow \{0, 1\}$ be the function that corresponds to n , and n varies across the natural numbers. We can create a function $g : \mathbb{N} \rightarrow \{0, 1\}$ such that $g(n) = 1$ if $f_n(n) = 0$ and $g(n) = 0$ if $f_n(n) = 1$. g differs from every f_n at least at the n -th element, so g is not in the list of functions. This contradicts the assumption that the set of all functions from \mathbb{N} to $\{0, 1\}$ is countable, so it must be uncountable.
- (c) Let $A = 2n + 1 : n \in \mathbb{N}$, the set of all odd natural numbers. Let $B = 2n : n \in \mathbb{N}$, the set of all even natural numbers. We can define a set in such a way that the n -th element of the set is either the n -th element of A or the n -th element of B , and each such set is infinite. Let C be the collection of such all such sets. Then, this set C is a subset of the power set of \mathbb{N} . It is an antichain of $P(\mathbb{N})$, as for any two sets in C , there must exist an element that is in one set but not the other, so no set in C is a subset of another set in C . To prove that C is uncountable, we can see that it corresponds to the set of all functions from \mathbb{N} to $\{0, 1\}$, which we proved is uncountable in part (b). Each function corresponds to a set in C by mapping n to the n -th element of A if the function maps n to 1, and mapping n to the n -th element of B if the function maps n to 0. Since there is a bijection from the set of all functions from \mathbb{N} to $\{0, 1\}$ to C , C is uncountable.