Exercise 1.3.23

(a) $W_1 + W_2 = \{x + y \mid x \in W_1, y \in W_2\}$. To show that $W_1 + W_2$ is a subspace of V, we need to show that it is closed under addition and scalar multiplication, and that it contains the zero vector. For any $u \in W_1 + W_2$ and $v \in W_1 + W_2$, we can write $u = x_1 + y_1$ and $v = x_2 + y_2$ where $x_1, x_2 \in W_1$ and $y_1, y_2 \in W_2$. Then $u + v = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2)$, which is in $W_1 + W_2$. For any scalar c, we have $cu = c(x + y) = cx + cy$, which is in $W_1 + W_2$. Finally, the zero vector is in $W_1 + W_2$ because it can be written as the sum of the zero vectors from each subspace. To show that W_1 is in $W_1 + W_2$, we can write any vector $w \in W_1$ as $w + 0$, where 0 is the zero vector in W_2. Then, this $w + 0$ is in $W_1 + W_2$. Similarly, any vector in W_2 can be written as $0 + w$, where w is a vector in W_2. Then, this $0 + w$ is in $W_1 + W_2$. Therefore, both W_1 and W_2 are contained in $W_1 + W_2$.

(b) We can assume V contains W_1 and W_2 as subspaces. Then, we can write any vector $v \in W_1 + W_2$ as $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. Since W_1 and W_2 are subspaces of V, we have that w_1 and w_2 are also in V. Therefore, the sum of any two vectors in $W_1 + W_2$ is also in V, which means that $W_1 + W_2$ is contained in V.

Exercise 1.3.25

For $P(F) = W_1 \oplus W_2$, we need to show that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = P(F)$. First, we will show that $W_1 \cap W_2 = \{0\}$. W_1 is the set of odd degree polynomials, and W_2 is the set of even degree polynomials. The only polynomial that is both odd and even is the zero polynomial, which means that $W_1 \cap W_2 = \{0\}$. Next, we will show that $W_1 + W_2 = P(F)$. Any polynomial in $P(F)$ can be written as the sum of an odd degree polynomial and an even degree polynomial. For example, if we have a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, we can write it as $p(x) = (a_n x^n + a_{n-2} x^{n-2} + ...) + (a_{n-1} x^{n-1} + a_{n-3} x^{n-3} + ...)$, where the first part is an odd degree polynomial and the second part is an even degree polynomial. Therefore, any polynomial in $P(F)$ can be expressed as the sum of an odd degree polynomial and an even degree polynomial, which means that $W_1 + W_2 = P(F)$. Thus, we have shown that $P(F) = W_1 \oplus W_2$.

Exercise 1.3.28

The set W_1 of all skew-symmetric $n \times n$ matrices with entries from F is a subspace of $M_{n\times n}(F)$. To show that W_1 is a subspace, we need to show that it is closed under addition and scalar multiplication, and that it contains the zero vector. For any $A, B \in W_1$, we have $A^T = -A$ and $B^T = -B$. Then, $(A + B)^T = A^T + B^T = -A - B = -(A + B)$, which means that $A + B$ is also in W_1. For any scalar c, we have $(cA)^T = cA^T = c(-A) = -(cA)$, which means that cA is also in W_1. Finally, the zero matrix is in W_1 because it is skew-symmetric. Therefore, W_1 is a subspace of $M_{n\times n}(F)$. Now we let W_2 be the subspace of $M_{n\times n}(F)$ consisting of all symmetric matrices. We will show that $M_{n\times n}(F) = W_1 \oplus W_2$. First, we will show that $W_1 \cap W_2 = \{0\}$. The only matrix that is both skew-symmetric and symmetric is the zero matrix, which means that $W_1 \cap W_2 = \{0\}$. Next, we will show that $W_1 + W_2 = M_{n\times n}(F)$. Any matrix A in $M_{n\times n}(F)$ can be written as the sum of a skew-symmetric matrix and a symmetric matrix. For example, we can write A as $A = \frac{1}{2}(A - A^T) + \frac{1}{2}(A + A^T)$, where $\frac{1}{2}(A - A^T)$ is a skew-symmetric matrix and $\frac{1}{2}(A + A^T)$ is a symmetric matrix. Therefore, any matrix in $M_{n\times n}(F)$ can be expressed as the sum of a skew-symmetric matrix and a symmetric matrix, which means that $W_1 + W_2 = M_{n\times n}(F)$. Thus, we have shown that $M_{n\times n}(F) = W_1 \oplus W_2$.

Exercise 1.4.14

To show that $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$, we need to show that both sides are subsets of each other. First, we will show that $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$. Any vector in $\text{span}(S_1 \cup S_2)$ can be written as a linear combination of vectors from S_1 and S_2. We can separate this linear combination into two parts: one part that is a linear combination of vectors from S_1 and another part that is a linear combination of vectors from S_2. Therefore, any vector in $\text{span}(S_1 \cup S_2)$ can be expressed as the sum of a vector from $\text{span}(S_1)$ and a vector from $\text{span}(S_2)$, which means that it is in $\text{span}(S_1) + \text{span}(S_2)$. To show the other inclusion, we will show that $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$. Any vector in $\text{span}(S_1) + \text{span}(S_2)$ can be written as the sum of a vector from $\text{span}(S_1)$ and a vector from $\text{span}(S_2)$. Since each vector in $\text{span}(S_1)$ is a linear combination of vectors from S_1, and each vector in $\text{span}(S_2)$ is a linear combination of vectors from S_2, the sum of these vectors is a linear combination of vectors from $S_1 \cup S_2$. Therefore, any vector in $\text{span}(S_1)+\text{span}(S_2)$ is in $\text{span}(S_1 \cup S_2)$. Thus, we have shown that $\text{span}(S_1 \cup S_2) = \text{span}(S_1)+\text{span}(S_2)$.

Exercise 1.5.17

The columns of a square upper triangular matrix with nonzero diagonal entries are linearly independent. Let A be an $n \times n$ matrix with such properties. The i-th column of A has a nonzero entry in the i-th row and zeros below it. This means that the i-th column cannot be written as a linear combination of the previous columns, which have zeros in the i-th row. Therefore, each column of A is linearly independent from the others, and thus the columns of A are linearly independent.

Exercise 1.6.15

A basis for W is the set of matrices $B = D \cup U$. D is the set of matrices with zero entries everywhere except for the i-th and $i+1$-th diagonal entry, which are two scalars of opposite signs, in which i ranges from 1 to $n-1$. U is the set of matrices with zero entries everywhere except for any entry not on the diagonal, where it is any scalar. In this way, D allows us to construct matrices with a trace of 0, and U allows us to construct matrices entries not on the diagonal. To find the dimension of W, we need to count the number of matrices in the basis. The set D has $n-1$ matrices, and the set U has $n^2 - n$ matrices (since there are n entries on the diagonal in an $n \times n$ matrix). Therefore, the total number of matrices in the basis is $(n-1) + n^2 - n = n^2 - 1$. Thus, the dimension of W is $n^2 - 1$.

Exercise 1.6.17

A basis for W is the set of matrices with zero entries everywhere except for the (i, j)-th entry and the (j, i)-th entry, which are two scalars of opposite signs, where i and j range from 1 to n and $i \neq j$. In this way, we can construct any skew-symmetric matrix by choosing appropriate scalars for the entries. To find the dimension of W, we need to count the number of matrices in the basis. For each pair of indices (i, j) with $i < j$, we have one matrix in the basis. There are $\binom{n}{2}$ such pairs, which means that there are $\binom{n}{2}$ matrices in the basis. Therefore, the dimension of W is $\binom{n}{2} = \frac{n(n-1)}{2}$.

Exercise 1.6.25

If Z is composed of all possible linear combinations of the vectors V and W, then a basis of Z would be comprised of the vectors in V and W that are linearly independent. For V, a basis would be the set of vectors $\{v_1, v_2, \ldots, v_m\}$, which has m vectors. For W, a basis would be the set of vectors $\{w_1, w_2, \ldots, w_n\}$, which has n vectors. To find a basis for Z, we can use $(v_i, 0)$ and $(0, w_j)$ for each i and j, where $v_i \in V$ and $w_j \in W$. This gives us a total of $m + n$ vectors in the basis. Therefore, the dimension of Z is $m + n$.

Exercise 1.6.27

The dimension of the subspace $W_1 \cap P_n(F)$ is the number of vectors in the basis of $W_1 \cap P_n(F)$. A basis for $W_1 \cap P_n(F)$ can be found by taking the basis of W_1 and restricting it to polynomials of degree at most n. Since W_1 is the subspace of all polynomials only odd degree terms, a basis for W_1 is $\{x, x^3, \ldots\}$. When we restrict this basis to polynomials of degree at most n, we get the basis $\{x, x^3, \ldots, x^n\}$ for odd n and $\{x, x^3, \ldots, x^{n-1}\}$ for even n. Therefore, the dimension of $W_1 \cap P_n(F)$ is $\lceil \frac{n}{2} \rceil$.

The dimension of the subspace $W_2 \cap P_n(F)$ is the number of vectors in the basis of $W_2 \cap P_n(F)$. A basis for $W_2 \cap P_n(F)$ can be found by taking the basis of W_2 and restricting it to polynomials of degree at most n.

Since W_2 is the subspace of all polynomials only even degree terms, a basis for W_2 is $\{1, x^2, x^4, \ldots\}$. When we restrict this basis to polynomials of degree at most n, we get the basis $\{1, x^2, x^4, \ldots, x^n\}$ for even n and $\{x^2, x^4, \ldots, x^{n-1}\}$ for odd n. Therefore, the dimension of $W_2 \cap P_n(F)$ is $\lfloor \frac{n}{2} \rfloor + 1$.

Exercise 1.6.30

W_1 is a subspace if it is closed under addition and scalar multiplication, and contains the zero vector. For any $u, v \in W_1$, we have $u = \begin{bmatrix} a_1, b_1 \\ c_1, a_1 \end{bmatrix}$ and $v = \begin{bmatrix} a_2, b_2 \\ c_2, a_2 \end{bmatrix}$. So, their sum is $u + v = \begin{bmatrix} a_1 + a_2, b_1 + b_2 \\ c_1 + c_2, a_1 + a_2 \end{bmatrix}$, which is in W_1. For any scalar k, we have $ku = \begin{bmatrix} ka_1, kb_1 \\ kc_1, ka_1 \end{bmatrix}$, which is also in W_1. Finally, the zero vector $\begin{bmatrix} 0, 0 \\ 0, 0 \end{bmatrix}$ is in W_1. Therefore, W_1 is a subspace. Now we will show that W_2 is a subspace. For any $u, v \in W_2$, we have $u = \begin{bmatrix} 0, a_1 \\ -a_1, b_1 \end{bmatrix}$ and $v = \begin{bmatrix} 0, a_2 \\ -a_2, b_2 \end{bmatrix}$. So, their sum is $u + v = \begin{bmatrix} 0, a_1 + a_2 \\ -(a_1 + a_2), b_1 + b_2 \end{bmatrix}$, which is in W_2. For any scalar k, we have $ku = \begin{bmatrix} 0, ka_1 \\ -ka_1, kb_1 \end{bmatrix}$, which is also in W_2. Finally, the zero vector $\begin{bmatrix} 0, 0 \\ 0, 0 \end{bmatrix}$ is in W_2. Therefore, W_2 is a subspace. The dimension of W_1 is 3, since a basis for W_1 is $\left\{ \begin{bmatrix} 1, 0 \\ 0, 1 \end{bmatrix}, \begin{bmatrix} 0, 1 \\ 0, 0 \end{bmatrix}, \begin{bmatrix} 0, 0 \\ 1, 0 \end{bmatrix} \right\}$. The dimension of W_2 is 2, since a basis for W_2 is $\left\{ \begin{bmatrix} 0, 1 \\ -1, 0 \end{bmatrix}, \begin{bmatrix} 0, 0 \\ 0, 1 \end{bmatrix} \right\}$. The dimension of $W_1 + W_2$ is 4, since a basis for $W_1 + W_2$ is $\left\{ \begin{bmatrix} 1, 0 \\ 0, 0 \end{bmatrix}, \begin{bmatrix} 0, 1 \\ 0, 0 \end{bmatrix}, \begin{bmatrix} 0, 0 \\ 1, 0 \end{bmatrix}, \begin{bmatrix} 0, 0 \\ 0, 1 \end{bmatrix} \right\}$. The top left and bottom right entries are W_1 and W_2 allow for the top left and bottom right entries of $W_1 + W_2$ to be any scalar, and the top right and bottom left entries of W_1 allow for the top right and bottom left entries of $W_1 + W_2$ to be any scalar. Therefore, the dimension of $W_1 + W_2$ is 4. The dimension of $W_1 \cap W_2$ is 1, since a basis for $W_1 \cap W_2$ is $\left\{ \begin{bmatrix} 0, 1 \\ -1, 0 \end{bmatrix} \right\}$. The top left entry of W_2 forces the top left entry of $W_1 \cap W_2$ to be 0, and the bottom right entry of W_1 being the same as the top left entry of W_1 forces the bottom right entry of $W_1 \cap W_2$ to be 0. The top right and bottom left entries of W_1 force the top right and bottom left entries of $W_1 \cap W_2$ to be opposite scalars. Therefore, the dimension of $W_1 \cap W_2$ is 1.