Exercise 1.4.4

We will show that $\sup T = b$ by contradiction. By assuming $\sup T \neq b$, we know that either $\sup T < b$ or $\sup T > b$. In the first case, we can assume $\sup T < b$. Let $c = \sup T$. Then, by the Density of the Rationals, there must exist a q in the rationals such that $c < q < b$. Then, q is in T because it is in $\mathbb{Q}$ and $q < b$. Since $c < b$, c cannot be the least upper bound of T, as there is an element in T, q, that is greater than it. In the second case, we can assume $\sup T > b$. Again, let $c = \sup T$. Then, by the Density of the Rationals, there must exist a q in the rationals such that $b < q < c$. Now, we know that q is an upper bound for T that is less than c. Therefore, c is not the least upper bound for T. Now, since we arrived at contradictions for both $\sup T < b$ and $\sup T > b$, it must be true that $\sup T = b$.

Exercise 1.4.5

By Density of the Rationals, there must exist a q such that $a - \sqrt{2} < q < b - \sqrt{2}$. Adding $\sqrt{2}$ to the whole inequality, we have $a < q + \sqrt{2} < b$. $q + \sqrt{2} \in \mathbb{I}$ because $a + t \in \mathbb{I}$ if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, which is satisfied by q and $\sqrt{2}$. So, $q + \sqrt{2}$ satisfies a $t \in \mathbb{I}$ for $a < t < b$.

Exercise 1.4.8

(a) Let $A = \{1 - \frac{1}{2^n} | n \in \mathbb{N}\}$ and $B = \{1 - \frac{1}{3^n} | n \in \mathbb{N}\}$. $A \cap B = \emptyset$, as all rationals in each set are in lowest form and A has only powers of 2 in the denominator, while B has only powers of 3 in the denominator. $\sup A = \sup B = 1$ and $1 \notin A \cup B$, so A and B satisfy the conditions.

(b) Let $J_1 = (-1, 1), J_2 = (-\frac{1}{2}, \frac{1}{2}), \ldots, J_n = (-\frac{1}{n}, \frac{1}{n})$. Then, $\bigcap_{n=1}^{\infty} J_n = \{0\}$, as for any real number $y > 0$, $\exists n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$, by the Archimedean Property, and J_n does not contain y. Since the intersection contains a finite amount of elements, the sets J_n satisfies the given conditions.

(c) Let $L_1 = [1, \infty), L_2 = [2, \infty), \ldots, L_n = [n, \infty)$. Then, $\bigcap_{n=1}^{\infty} L_n = \emptyset$, as for any number $x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ satisfying $n > x$, by the Archimedean Property, and L_n does not contain x. Since the intersection is empty, the sets L_n satisfies the given conditions.

(d) This condition is not possible, which can be proven by contradiction. Let $I_n = [a_n, b_n]$, with $a_n \leq b_n$, assuming all the finite intersections are nonempty and the infinite intersection is empty. Because the finite intersections are nonempty, we can define $A_N = \bigcap_{n=1}^{N} I_n$, which is closed and bounded because each I_n is closed and bounded. We know that A_N is nonempty because it is a finite intersection. Each $A_{N+1} = A_N \cap I_{n+1} \subseteq A_N$, so $(A_N)_{N \geq 1}$ is a nested sequence of nonempty closed bounded intervals. By applying the Nested Interval Property, we know that $\bigcap_{N=1}^{\infty} A_N \neq \emptyset$. But, $\bigcap_{N=1}^{\infty} A_N = \bigcap_{n=1}^{\infty} I_n$, so $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. This contradicts the assumption that the infinite intersection is empty, so a sequence of closed bounded intervals cannot satisfy the given conditions.

Exercise 1.5.9

(a) $x^2 - 2 = 0$ is a polynomial which has two real roots, one of which is $\sqrt{2}$, showing that $\sqrt{2}$ is algebraic.

$x^3 - 2 = 0$ is a polynomial which has one real root, which is $\sqrt[3]{2}$, showing that $\sqrt[3]{2}$ is algebraic.

We want $x = \sqrt{2} + \sqrt{3}$. Squaring both sides, we get $x^2 = 5 + 2\sqrt{6}$. Subtracting 5 from both sides and squaring both sides again results in $x^4 - 10x^2 + 25 = 24$. Rearranging, we have the polynomial $x^4 - 10x^2 + 1 = 0$ that has a real root of $\sqrt{2} + \sqrt{3}$, showing that it is algebraic.

(b) By fixing $n \in \mathbb{N}$, we let A_n denote the set of algebraic numbers obtained as roots of polynomials with integer coefficients of degree n. First, this means that the number of polynomials of integer coefficients of degree n, P_n, must be countable. Each coefficient a_i must be in $\mathbb{Z}$, and the coefficient of the x^n must be in $\mathbb{Z} \setminus \{0\}$. The cardinality of the Cartesian product of these sets represents the cardinality of P_n. Since $|\mathbb{Z}| = |\mathbb{Z} \setminus \{0\}| = |\mathbb{N}|$, $|P_n|$ is equal to the cardinality of the Cartesian product of n sets of $\mathbb{N}$. The finite product of countably infinite sets is also countable, which can be shown by induction. For the base case, we know $\mathbb{N}$ is countable. Using the diagonal method from in class, we know that the product of two countable sets, such as $\mathbb{N} \times \mathbb{N}$, is countable. Assuming $\prod_{i=1}^{m} \mathbb{N}$ is countable, we need to prove $\prod_{i=1}^{m+1} \mathbb{N}$ is countable. $\prod_{i=1}^{m+1} \mathbb{N} = (\prod_{i=1}^{m} \mathbb{N}) \times \mathbb{N}$, which is the product of two countable sets. So, $\prod_{i=1}^{m+1} \mathbb{N}$ is countable. Thus, any finite product of countable sets is countable, and the cardinality of the Cartesian product of n sets of $\mathbb{N}$ is countable. For each $p \in P_n$, the number of roots of p is at most n. Let $R(p)$ be the set of roots of p, and $|R(p)|$ is always finite. So, $A_n = \bigcup_{p \in P_n} R(p)$ and since the countable union of finite sets is countable, A_n is countable.

(c) Since we proved that for a fixed $n \in \mathbb{N}$, A_n is countable, we can also prove that the set of all algebraic numbers, A, is countable. Since a countable union of countable sets is countable, A is countable, as $A = \bigcup_{n=1}^{\infty} A_n$. Using this, we can conclude that the transcendentals are uncountable with a proof by contradiction. The transcendentals are equal to $\mathbb{R} \setminus A$. If $\mathbb{R} \setminus A$ were countable, then $\mathbb{R} = A \cup (\mathbb{R} \setminus A)$ would be countable. Since $\mathbb{R}$ is uncountable, $\mathbb{R} \setminus A$ must be uncountable. Therefore, there are uncountably many transcendentals.

<u>Exercise 1.5.11</u>

(a) We can define $h : X \to Y$ with

$$h(x) = \begin{cases} f(x), & x \in A \\ g^{-1}(x), & x \in A' \end{cases}$$

If f can be shown to be bijective from A to B, and g^{-1} can be shown to exist by g being bijective from B' to A', then their definitions will agree on disjoint subsets of X. Then, h will be a bijection, so $X \sim Y$.

(b) In the case that $A_1 = \emptyset$, then g is surjective, as for every element $x \in X$, $\exists y \in Y$ such that $g(y) = x$. Then, g is bijective, so $X \sim Y$ immediately. Assuming $A \neq \emptyset$, we need to prove that A_i and A_j are disjoint for all $i, j \in \mathbb{N}$. Suppose x is an element of $A_i \cap A_j$, with $i < j$. Then, as $x \in A_i$, there must exist an element $y \in A_1$ such that $i - 1$ applications of g and f maps it to x. Since $x \in A_j$, there must also exist an element $z \in A_1$ such that $j - 1$ applications of g and f maps it to x. So, we have $(g \circ f)^{i-1}(y) = x$ and $(g \circ f)^{j-1}(z) = x$. Then, $(g \circ f)^{i-1}(y) = (g \circ f)^{j-1}(z)$. By the injectivity of g and f and the fact that $i < j$, we can cancel the outermost compositions of $(g \circ f)$ $i - 1$ times. Then, we have $y = (g \circ f)^{j-i}(z)$. Since $j > i$, g is applied at least once to z, so $(g \circ f)^{j-i}(z) \in g(Y)$. However, by definition, $y \in A_1$, and $A_1 \cap g(Y) = \emptyset$. So, by contradiction, $A_i \cap A_j = \emptyset$ and A_i and A_j are disjoint for all $i, j \in \mathbb{N}$. Since the subsets of X are pairwise disjoint and f is injective, each subset $f(A_n)$ for $n \in \mathbb{N}$ of Y is also pairwise disjoint.

(c) To show that f maps A onto B, we need to show that $f(A) \subset B$ and $B \subset f(A)$. Take any $x \in A$. By definition of A, $x \in A_n$ for some n. By applying f, we have $f(x) \in f(A_n) \subset B$, so $f(A) \subset B$. Now, take any $y \in B$. By definition of B, $y \in f(A_n)$ for some n. Then, there must exist an $x \in A_n$ such that $y = f(x)$. Since $A_n \subset A$, $x \in A$. Therefore, $y \in f(A)$ and $B \subset f(A)$. So, f maps A onto B, and since f is injective, f has been proven to be bijective.

(d) To show that g maps B' onto A', we need to show that $g(B') \subset A'$ and $A' \subset g(B')$. Take any $y \in B'$. If $g(y) \in A$, then $g(y) \in A_{n+1} = g(f(A_n))$ for some n. By injectivity of g, we have $y \in f(A_n) \subset B$. This is a contradiction, as $y \in Y \setminus B$. So, $g(y) \notin A$, and $g(y) \in A'$. Therefore, $g(B') \subset A'$. Now, take any $x \in A'$. Then, $x \notin A_n$ for any $n \in \mathbb{N}$. So, $x \notin A_1$, and $x \in g(Y)$. Let $y \in g(Y)$ such that $g(y) = x$. If $y \in B$, then $x \in A$, which is a contradiction, as $x \in A'$. So, $y \in B'$, and since $x = g(y)$, $A' \subset g(B')$. So, g maps B' onto A', and since g is injective, g has been proven to be bijective.

<u>Exercise 1.6.1</u>

We must prove that $\mathbb{R}$ being uncountable $\implies$ $(0,1)$ being uncountable, and $(0,1)$ being uncountable $\implies$ $\mathbb{R}$ being uncountable. Assuming $\mathbb{R}$ is uncountable, we need to create a bijection $f : \mathbb{R} \to (0,1)$. Let $f(x) = \frac{1}{1+e^{-x}}$. This function has an inverse, $f^{-1} : (0,1) \to \mathbb{R}$, which is defined as $f^{-1}(x) = -\ln(\frac{1}{x} - 1)$. Since f has an inverse function, f is bijective, meaning $\mathbb{R}$ and $(0,1)$ have the same cardinality. Since we are assuming $\mathbb{R}$ is uncountable, $(0,1)$ is also uncountable. To prove that $(0,1)$ being uncountable $\implies$ $\mathbb{R}$ being uncountable, we can assume $(0,1)$ is uncountable and use $f^{-1} : (0,1) \to \mathbb{R}$ as our function. Since it has an inverse, specifically f, f^{-1} is a bijection, so $(0,1)$ and $\mathbb{R}$ have the same cardinality. Since we are assuming $(0,1)$ is uncountable, $\mathbb{R}$ is also uncountable. Therefore, $(0,1)$ is uncountable if and only if $\mathbb{R}$ is uncountable.

Exercise 1.6.9

The power set of $\mathbb{N}$ is the collection of subsets of the natural numbers. Let each subset A be represented by a sequence of binary numbers, $(a_1, a_2, a_3, \dots)$, where $a_n = 1$ if $n \in A$ and $a_n = 0$ if $n \notin A$. This correspondence between the subsets of the naturals and a binary sequence is bijective, so $|P(\mathbb{N})| = |\{0,1\}^{\mathbb{N}}|$. Every binary sequence can be mapped to a real number in $[0,1]$. Let x be defined as $\sum_{n=1}^{\infty} \frac{a_n}{2^n}$. However, some real numbers have multiple expansions in this form, such as $\frac{1}{2}$, which can be represented as $(1,0,0,\dots)$, but also as $(0,1,1,\dots)$. However, this is true for only a countable amount of real numbers, so cardinality is not affected. Therefore, $|\{0,1\}^{\mathbb{N}}| = |[0,1]|$. Since the cardinality of $[0,1]$ is equal to the cardinality of $\mathbb{R}$, we have $|P(\mathbb{N})| = |\{0,1\}^{\mathbb{N}}| = |[0,1]| = |\mathbb{R}|$. So, $P(\mathbb{N}) \sim \mathbb{R}$.

Exercise 1.6.10

(a) The set of all functions from $\{0,1\}$ to $\mathbb{N}$ can be represented as a set of ordered pairs. Each function $f : \{0,1\} \to \mathbb{N}$ can be written as an ordered pair $(f(0), f(1))$. The values of $f(0)$ and $f(1)$ vary across the natural numbers, so the set of all of these ordered pairs is equivalent to $\mathbb{N} \times \mathbb{N}$. Since there is a 1-1 correspondence from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, the set of all functions from $\{0,1\}$ to $\mathbb{N}$ is countable.

(b) We can show that the set of all functions from $\mathbb{N}$ to $\{0,1\}$ is uncountable using Cantor's diagonal argument. Assume that the set is countable; then it should be able to be enumerated in correspondence with the natural numbers. Let $f_n : \mathbb{N} \to \{0,1\}$ be the function that corresponds to n, and n varies across the natural numbers. We can create a function $g : \mathbb{N} \to \{0,1\}$ such that $g(n) = 1$ if $f_n(n) = 0$ and $g(n) = 0$ if $f_n(n) = 1$. g differs from every f_n at least at the n-th element, so g is not in the list of functions. This contradicts the assumption that the set of all functions from $\mathbb{N}$ to $\{0,1\}$ is countable, so it must be uncountable.

(c) Let $A = 2n+1 : n \in \mathbb{N}$, the set of all odd natural numbers. Let $B = 2n : n \in \mathbb{N}$, the set of all even natural numbers. We can define a set in such a way that the n-th element of the set is either the n-th element of A or the n-th element of B, and each such set is infinite. Let C be the collection of such all such sets. Then, this set C is a subset of the power set of $\mathbb{N}$. It is an antichain of $P(\mathbb{N})$, as for any two sets in C, there must exist an element that is in one set but not the other, so no set in C is a subset of another set in C. To prove that C is uncountable, we can see that it corresponds to the set of all functions from $\mathbb{N}$ to $\{0,1\}$, which we proved is uncountable in part (b). Each function corresponds to a set in C by mapping n to the n-th element of A if the function maps n to 1, and mapping n to the n-th element of B if the function maps n to 0. Since there is a bijection from the set of all functions from $\mathbb{N}$ to $\{0,1\}$ to C, C is uncountable.