Barrier Functions in Control

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Introduction and Motivation – How do we define SAFE?

- Formations of robots/UAVs collectively accomplish tasks – load transportation, surveying, disaster relief, agricultural operations etc.
- Require SAFE operation – how do we define SAFE?
  1. Mutual collisions
  2. Geometric constraints
  3. Obstacle avoidance

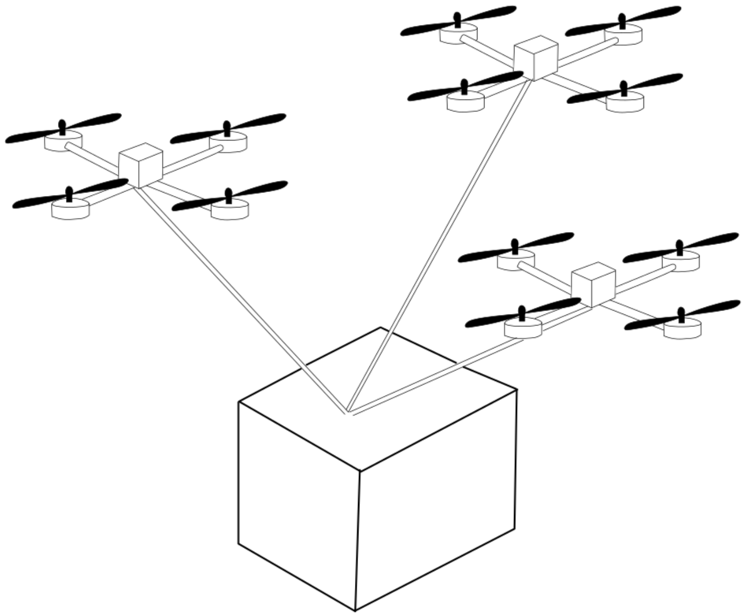
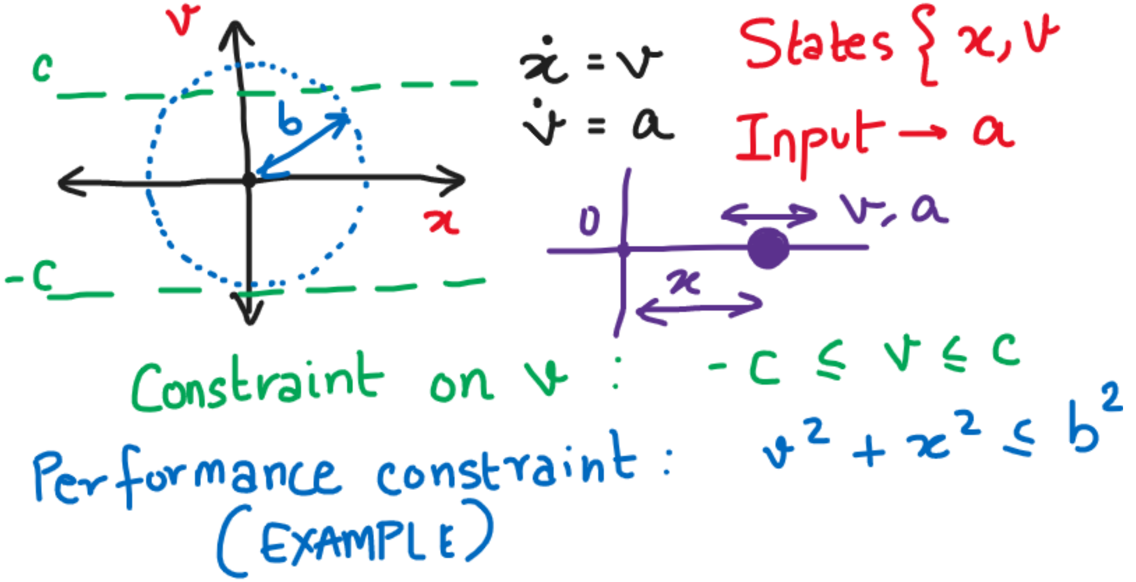


Fig. A three-quadrotor team carrying payload

System Operation and Performance – How do we characterize SAFETY?

- Where do you begin? – requirements, system model (linear or nonlinear)
- Remember system states? – constrained states
- What about system performance? – how do we define performance?



Barrier Functions –
How do we enforce SAFETY?

- Controller designed to achieve a primary objective of the system – drive system to desired equilibrium
- How do we make the system operate within bounds? – alter the control based on how close the system states/performance are to the bounds!
- Control is thus altered as a function of distance/margin (any metric) from the bound

u - unaltered control
u_1 - altered control 1 = $f_1(d)$
u_2 - altered control 2 = $f_2(d)$

Barrier Functions – What options do we have?

S. no.	Barrier Lyapunov Functions[1] (BLFs)	Control Barrier Functions[2] (CBFs)
1	Supported by gradient-based control architecture	Supported by optimization-based control architecture (Quadratic Program, affine constraints)
2	Constraints are inherently included in the calculated control	Constraints override the primary control when system is operating close to the safe bounds
3	Assumes that system is initially SAFE, ensures the system is SAFE for all future time	Assumes that system is initially SAFE, ensures the system is SAFE for all future time
4	Robustness to noise, disturbances is questionable (if the system breaches the bounds, return to the SAFE region is not included in the design)	Robustness to noise, disturbances exists (system can be designed to return to the SAFE region if there is an excursion past the bounds)

1. K. P. Tee, S. S. Ge, and E. H. Tay, "Barrier Lyapunov Functions for the control of output-constrained nonlinear systems," Automatica, vol. 45, pp. 918-927, 2009
2. A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in 18th European Control Conference (ECC), 2019, pp. 3420–3431.

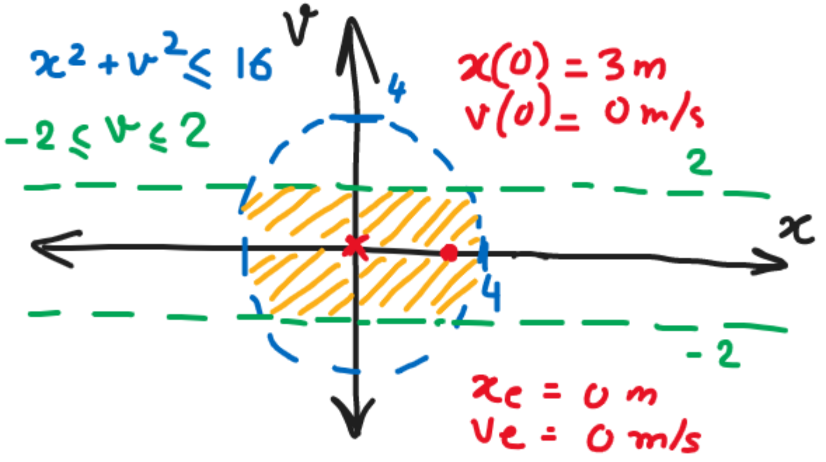
Example Problem

Consider the system on Slide 3 :

$$\dot{x} = v$$
$$\dot{v} = a = u$$

Further, consider the following constraints $(b = 4, c = 2\ m/s)$:

$$-c \leq v \leq c$$
$$x^2 + v^2 \leq b^2, b > c$$



The system has initial conditions

1. $x_1(0) = 3.8\ m, v_1(0) = 0.4\ m/s$,
2. $x_2(0) = -3.5\ m, v_2(0) = -1.8\ m/s$

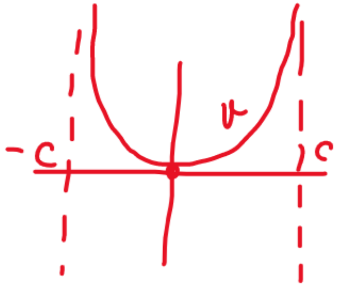
and must be driven to the equilibrium $x_e = 0\ m, v_e = 0\ m/s$

Barrier Lyapunov Function

$V = 0$ at eqb. $(x=0, v=0)$

$$V = \frac{K_d\, v^2}{(c^2 - v^2)} + \frac{K_p\,(x^2+v^2)}{\{b^2 - (x^2+v^2)\}^2}$$

Constraints



$$\dot V = \nabla V \cdot f(X), \quad \begin{bmatrix}\dot x\\ \dot v\end{bmatrix} = f(X) = \begin{bmatrix} v \\ u\end{bmatrix}$$

$$= (\nabla V)_1 v + (\nabla V)_2 u \le 0$$

$$u = -\frac{(v+\beta)}{\alpha} \qquad \alpha = \left[\frac{2K_d c^2}{(c^2 - v^2)^2} + \frac{2K_p b^2}{\{b^2 - (x^2+v^2)\}^2}\right]$$

$$\beta = \frac{2K_p b^2 x}{\{b^2 - (x^2+v^2)\}^2}$$

Control Barrier Function

$u_r = -K_p x - K_d v$ } EXAMPLE

$$u^* = \underset{u}{\operatorname{argmin}} \|u - u_r\|$$

s.t. $-c \le v \le c$

$v^2 + x^2 \le b^2$

} How to convert these to constraints on u?

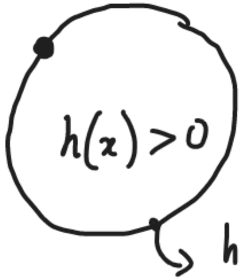
$$h_1(x) = c^2 - v^2 \ge 0$$

$$h_2(x) = b^2 - (x^2+v^2) \ge 0$$

$$\left.\begin{aligned}\dot h_1(X) &\ge -h_1(X)\\ \dot h_2(X) &\ge -h_2(X)\end{aligned}\right\} \rightarrow \begin{aligned} -2vu &\ge v^2 - c^2 \\ -2xv - 2vu &\ge (x^2+v^2) - b^2\end{aligned}$$

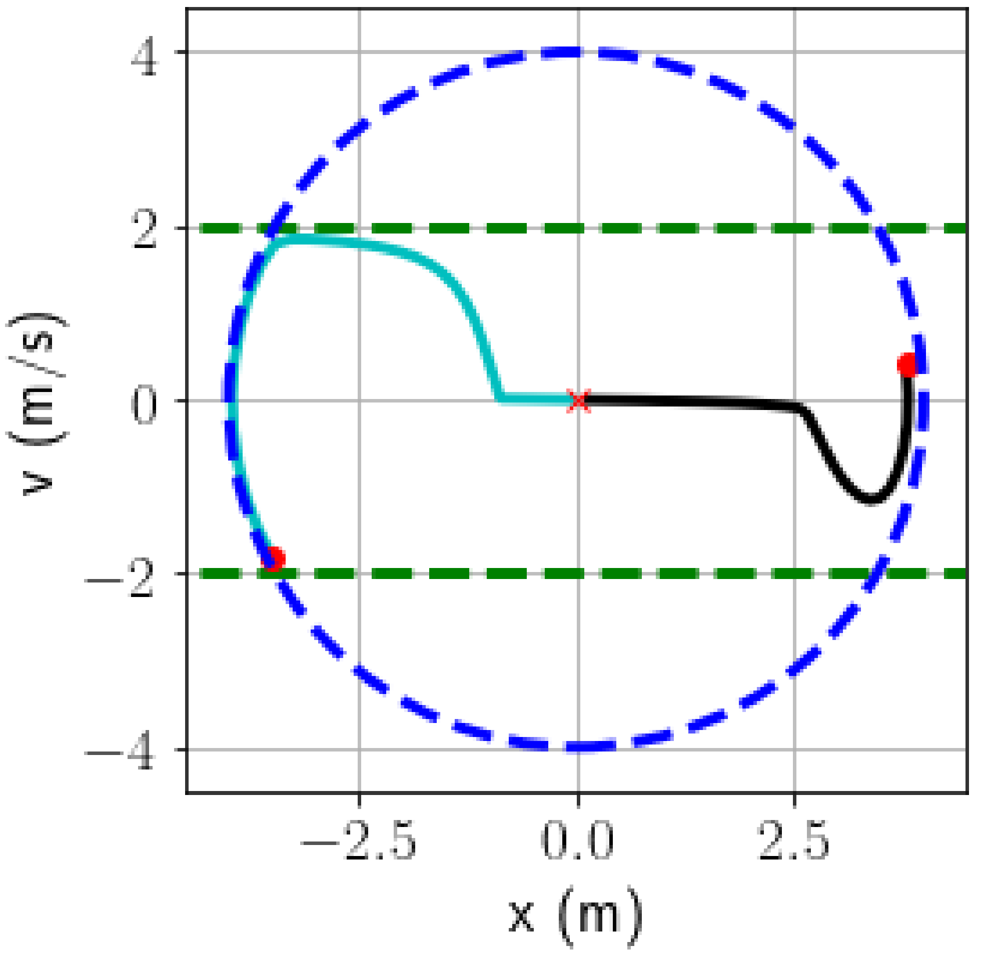
$\dot h(x) = 0$ at boundary

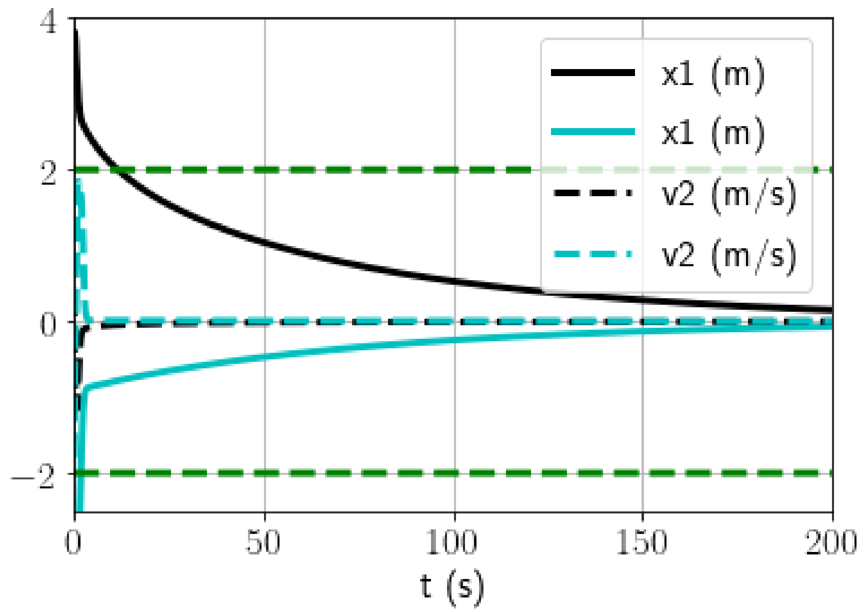
$\dot h(x) \le 0$ allowed inside safe boundary

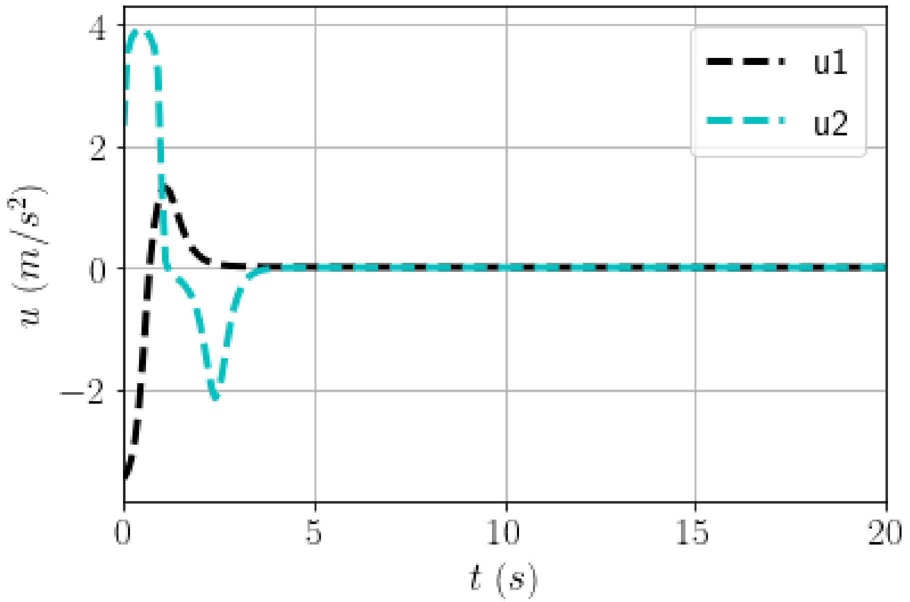


$h(x) = 0$

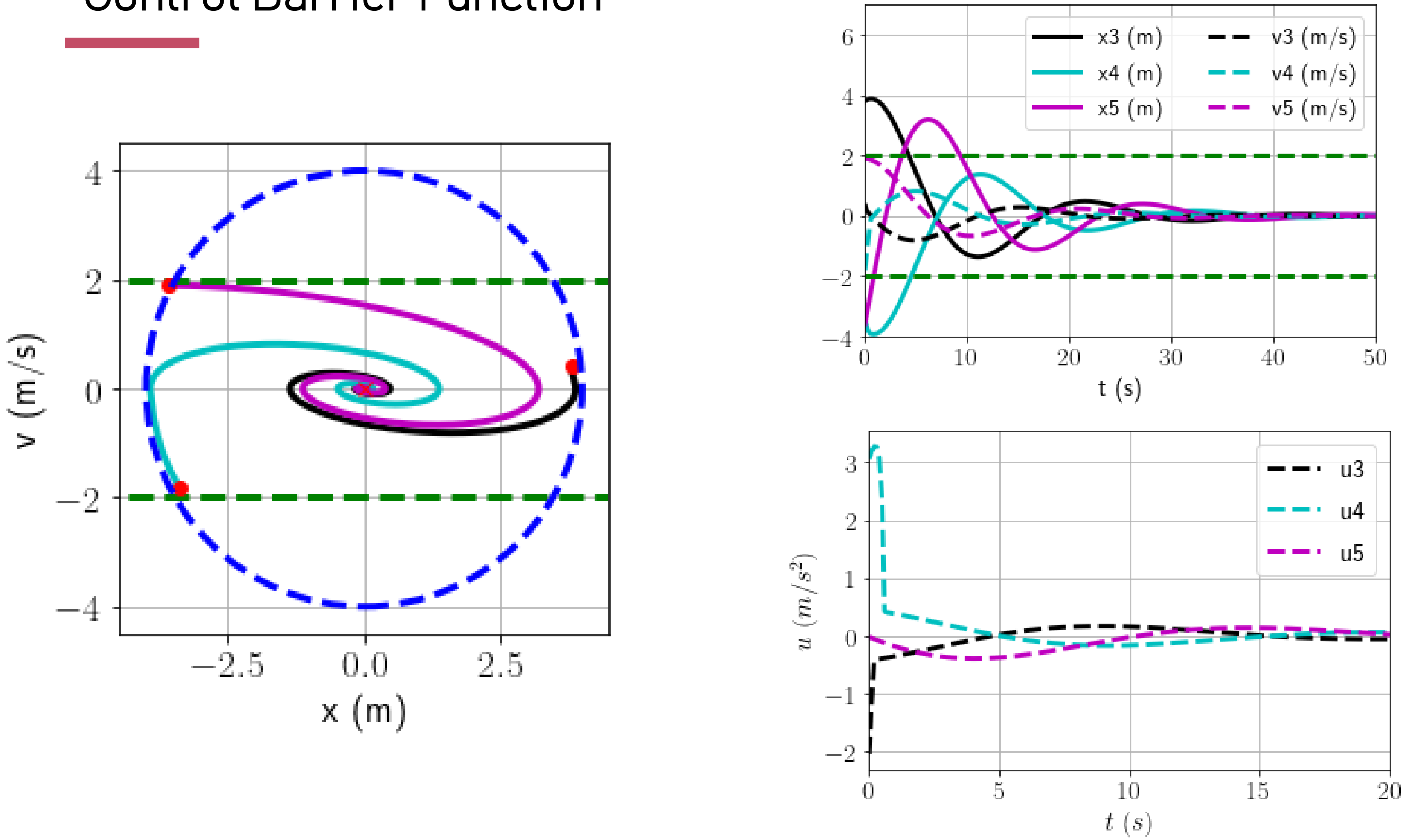
Barrier Lyapunov Function







Control Barrier Function



Conclusions

- Barrier functions can be used to enforce constraints in controlled systems
- The control is altered to keep system within SAFE operating limits
- CBFs and BLFs are two ways in which constraints are enforced
- These can be extended to multi-agent systems with applications in robotics and networked systems

THANK YOU!

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