CSE 551 Homework 1 Solutions

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1 Question 1

Prove that the choice of data structure used in the Gale-Shapley algorithm to store "free" men does not affect the stable matching found.

Solution: The choice of data structure only affects the order in which the "free" men propose each day (and possibly the efficiency of the algorithm, but that is irrelevant to this question). The Gale-Shapley algorithm, however, always provides a stable matching that does not depend on the order in which "free" men propose each day. Suppose that using some data structure, some man m is matched with w and using another data structure m is matched to w'. The Gale-Shapely algorithm is male optimal so m must like w and w' equally, but this is impossible. Contradiction.

2 Question 2

What is the worst-case runtime of the Gale-Shapley algorithm? Describe an example of preference lists for each n that requires the largest number of proposals before it terminates.

Solution: If a man were rejected n times, this would mean he was rejected by every woman and no stable matching was found. So every man can be rejected at most $(n-1)$ times. Then consider a scenario where (except for the first round where every man is free) only one man is free each round. Then $n(n-1)$ rounds are needed in addition to one more round where no one is rejected, hence $n(n-1)+1$ is an upper bound.

(Another solution is as follows: Once all women have received a proposal, every woman must be engaged and there can be no free men, so the Gale-Shapley algorithm terminates. Therefore, the last woman to receive a proposal receives only one. The other $n-1$ women can each receive at most n proposals. Hence, there cannot be more than $n(n-1)+1$ proposals in total.)

Here is an instance of the stable marriage problem that for any size the Gale-Shapley algorithm requires $n(n-1)+1$ iterations:

$$
\begin{array}{c|cccccc}
M_1 & W_1 & W_2 & \cdots & W_{n-2} & W_{n-1} & \underline{W_0} \\
M_2 & W_2 & W_3 & \cdots & W_{n-1} & \underline{W_1} & W_0 \\
M_3 & W_3 & W_4 & \cdots & W_1 & \underline{W_2} & W_0 \\
\vdots & & & & & & \\
M_{n-1} & W_{n-1} & W_1 & \cdots & W_{n-3} & \underline{W_{n-2}} & W_0 \\
M_0 & W_1 & W_2 & \cdots & W_{n-2} & \underline{W_{n-1}} & W_0
\end{array}
$$

$$
\begin{array}{c|ccccc}
W_1 & \underline{M_2} & M_3 & \cdots & M_0 & M_1 \\
W_2 & \underline{M_3} & M_4 & \cdots & M_1 & M_2 \\
W_3 & \underline{M_4} & M_5 & \cdots & M_2 & M_3 \\
\vdots & & & & & \\
W_{n-1} & \underline{M_0} & M_1 & \cdots & M_{n-2} & M_{n-1} \\
W_0 & \underline{M_1} & M_2 & \cdots & M_{n-1} & M_0
\end{array}
$$

The only stable matching is the one underlined (it is man-optimal). Because men propose in order we have $(n-1)(n-1)+n = n(n-1)+1$ proposals.

3 Question 3

For every n, give preference lists for which there is more than one stable matching; if, for some value(s) of n, this cannot be done, show that for this value of n there must always be only one stable matching. Also, for some value of n give an example that has at least three stable matchings, or show that there are at most two stable matchings for every n.

Solution: First observe that 2 sable matchings cannot be achieved when there is just one man and one woman. Second observe the following stable marriage instance with two men and two women provides two stable matchings:

X	A	B
Y	B	A

and:

A	X	Y
B	X	Y

Then $\{(X,A),(Y,B)\}$ and $\{(X,B),(Y,A)\}$ are stable matchings.

We can use this construction to ensure more than one stable matching for every $n \geq 2$. Label the men as $X, Y, M_3, M_4, \ldots M_n$, and label the women as $A, B, W_3, W_4, \ldots W_n$. For every man M_i except X, Y it only matters their number one preference is W_i, and that for every woman except A, B it only matters that their number one preference is M_i. For X it only matters that his first two preferences are A, B in that order, and for Y it only matters that his first two preferences are B, A in that order. Similarly for A it only matters that her first two preference are X, Y in that order and that for B her first two preferences are X, Y in that order.

This construction works as everyone (barring X, Y, A, and B) matches with their first preference. Then X, Y, A, and B can be paired two ways as described in the four person scenario.

There are many answers for more than two stable matchings. For instance:

X	B	A	C
Y	C	B	A
Z	A	C	B

and:

A	Y	X	Z
B	Z	Y	X
C	X	Z	Y

Then $\{(Y,A),(Z,B),(X,C)\}$, $\{(X,A),(Y,B),(Z,C)\}$ and $\{(Z,A),(X,B),(Y,C)\}$ are stable matchings.

4 Question 4

Consider a variation of the Gale-Shapley algorithm where there are 2N applicants and N employers that has 2 open positions each (The positions may not be the same). Every applicant has a job preference list and every employer has an applicant preference list. How do you change the algorithm such that we find a stable matching?

Solution: Observe that any job is equivalent to a woman in the stable marriage problem that has the preference list of the company the job is listed for. Thus, let every applicant be a man, and let every job position be a woman whose preference list is the same as the company the position is for. This then becomes an instance of the stable marriage problem and can be solved using the Gale-Shapely algorithm.

5 Question 5

a) Consider another variation of the Gale-Shapley algorithm where women may prefer some men equally. For example: a woman's preference could look like [Adam > Bob = Charlie > Dave ...]. Define "stable matching" precisely for this situation. Does this variation have such a stable matching? If so, how do you modify the algorithm to achieve stable matching (in particular, how can you make an inverse preference list)? If not, why? b) Suppose that all women rank all men as equal (no preference). Must there be a man-optimal stable matching for every way for the men to give their preferences? Explain.

Solution:

a) A matching, in this situation, is not stable if there exists married pairs (M, W) and (M_0, W') such that on W's preference list, $M_0 > M$ and on M_0's preference list, $W > W'$. A stable matching is, therefore, a matching that is not not stable.

   Use the Gale-Shapely algorithm with the provision, that in the case a woman is proposed to by a man m she prefers just as much as her current partner, she should reject m. To show this "new" algorithm (1) terminates and (2) results in a stable matching, the argument is nearly identical to the proof that the Gale-Shapely terminates and results in a stable matching. To show (1), a man will propose to everyone if he must. So every woman must get engaged at some point, and since a female cannot become unengaged at any point, even when a woman prefers two men equally she will always pick one man using the scheme above and remain engaged. Thus, every man and woman must get engaged. To show (2), suppose there is a man M and woman W such that they are not engaged but they prefer each other to their resulting partners. Then M must have proposed to W at some point and W only rejected if she preferred her partner more *or the same* as M, in neither case did she prefer M over the partner she had.

b) Observe that this version of the algorithm may depend on the order in which the men propose. In the case all men prefer woman W the most, the man who marries W is the one to propose to her first. If there are n men, the algorithm could generate any n possible situations where one of the M_i marries W. However, any one of the M_i could marry W and a stable matching could be achieved. Thus, only one man is lucky enough to marry W and the matching is not optimal for every man.