Successive Shortest Paths and the Assignment Problem

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CSE 551

An Example

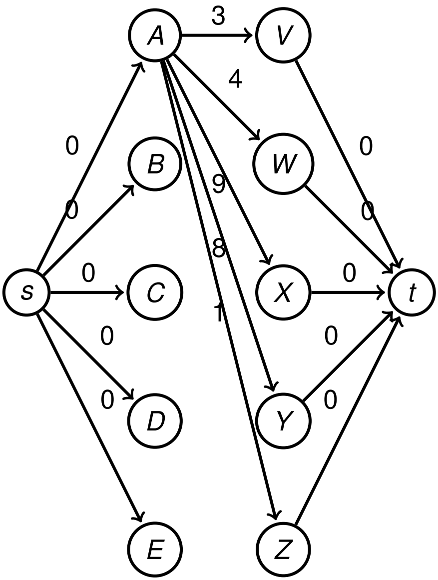
- An instance of the assignment problem

	V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
E	1	7	5	11	9

- Form a bipartite graph to represent residual capacities for the all-0 flow.

An Example

Initial Graph - many edges not shown



Successive Shortest Paths

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- Start with the empty matching, and repeatedly find a shortest path and alternate along the path to increase the number of edges in the matching by 1. To find shortest paths, we have two options.
  - Use the original costs and form the residual graph. Then use Bellman-Ford to find the shortest paths (there may be negative cost edges, but no negative cycles).
  - Maintain prices and use the reduced costs in the residual graph. Then use Dijkstra's algorithm to find shortest paths (all reduced costs are nonnegative).
- Why might you prefer one over the other?

Successive Shortest Paths

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- Dijkstra is faster, despite having to maintain prices and compute reduced costs. Moreover, the final prices give a certificate that the matching found has minimum cost.
- Bellman-Ford provides a minimum cost matching at each stage, so that you get min cost matchings on 1, 2, 3, ... edges as the method runs.
- *Using reduced costs, the intermediate matchings need not be min cost for the original costs. But at each stage the sum of prices of the vertices on the RHS minus the sum of prices on the LHS yields a lower bound on the cost of a min cost perfect matching.*

An Example

- It is easier to work on the array – the picture gets cluttered!

	V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
E	1	7	5	11	9

- What is a shortest s, t-path in the residual graph for the empty matching? Let's run Bellman-Ford.

#iter	s	A	B	C	D	E	V	W	X	Y	Z	t
0	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	0	0	0	0	0	0	∞	∞	∞	∞	∞	∞
2	0	0	0	0	0	0	1	7	5	11	9	∞
3	0	0	0	0	0	0	1	7	5	11	9	1

An Example

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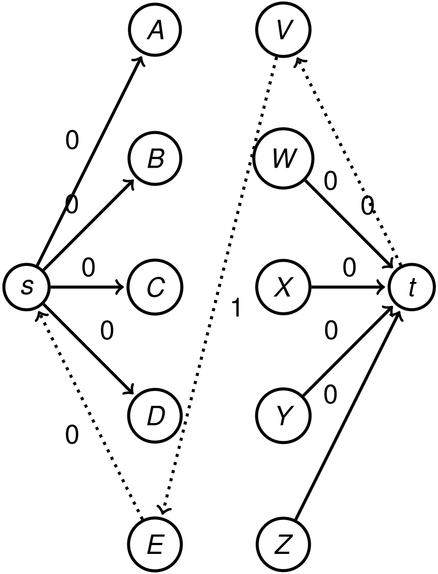
	V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
E	1	7	5	11	9

- What is a shortest s, t-path in the residual graph for the empty matching? Let's run Bellman-Ford.
- $s \to E \to V \to t$ has length 1.
- Matching with one edge EV has cost 1.

An Example

Residual Graph



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Matching with two edges

	*V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
*E	-1	7	5	11	9

- What is a shortest s, t-path in the residual graph for this matching? Let's run Bellman-Ford.

#iter	s	A	B	C	D	E	V	W	X	Y	Z	t
0	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	0	0	0	0	0	∞	∞	∞	∞	∞	∞	∞
2	0	0	0	0	0	∞	3	8	7	15	10	∞
3	0	0	0	0	0	∞	3	8	7	15	10	7
4	0	0	0	0	0	2	3	8	7	15	10	7

Matching with two edges

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	*V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
*E	-1	7	5	11	9

- What is a shortest s,t-path in the residual graph for this matching?
- $s \to A \to V \to E \to X \to t$ has length $3 - 1 + 5 = 7$.
- Matching with two edges AV, EX has cost $1 + 7 = 8$.

Matching with three edges

	*V	W	*X	Y	Z
*A	-3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
*E	1	7	-5	11	9

- What is a shortest s,t-path in the residual graph for this matching? Let's run Bellman-Ford.

#iter	s	A	B	C	D	E	V	W	X	Y	Z	t
0	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	0	0	0	∞	∞	∞	∞	∞	∞	∞
2	0	∞	0	0	0	∞	4	10	7	16	10	∞
3	0	1	0	0	0	2	4	10	7	16	10	10
4	0	1	0	0	0	2	4	9	7	13	10	10
5	0	1	0	0	0	2	3	9	7	13	10	9
6	0	0	0	0	0	2	3	9	7	13	10	9
7	0	0	0	0	0	2	3	8	7	13	10	9
8	0	0	0	0	0	2	3	8	7	13	10	8

Matching with three edges

	*V	W	*X	Y	Z
*A	-3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
*E	1	7	-5	11	9

- What is a shortest s, t-path in the residual graph for this matching?
- $s \rightarrow B \rightarrow X \rightarrow E \rightarrow V \rightarrow A \rightarrow W \rightarrow t$ has length 8.
- Matching with three edges AW, BX, EV has cost $8 + 8 = 16$.

Matching with four edges

	*V	*W	*X	Y	Z
*A	3	-8	9	15	10
*B	4	10	-7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
*E	-1	7	5	11	9

- What is a shortest s, t-path in the residual graph for this matching? Let's run Bellman-Ford.

#iter	s	A	B	C	D	E	V	W	X	Y	Z	t
0	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	∞	0	0	∞	∞	∞	∞	∞	∞	∞
2	0	∞	∞	0	0	∞	8	13	11	19	10	∞
3	0	5	4	0	0	7	8	13	11	19	10	10

Matching with four edges

	*V	*W	*X	Y	Z
*A	3	-8	9	15	10
*B	-4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
*E	1	7	-5	11	9

- What is a shortest s, t-path in the residual graph for this matching?
- $s \to C \to Z \to t$ has length 10.
- Matching with four edges AW, BX, CZ, EV has cost $16 + 10 = 26$.

Matching with five edges

	*V	*W	*X	Y	*Z
*A	3	-8	9	15	10
*B	4	10	-7	16	14
*C	9	13	11	19	-10
D	8	13	12	20	13
*E	-1	7	5	11	9

- What is a shortest s, t-path in the residual graph for this matching? Let's run Bellman-Ford!

#iter	s	A	B	C	D	E	V	W	X	Y	Z	t
0	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	∞	∞	0	∞	∞	∞	∞	∞	∞	∞
2	0	∞	∞	∞	0	∞	8	13	12	20	13	∞
3	0	5	5	3	0	7	8	13	12	20	13	20
4	0	5	5	3	0	7	8	13	12	18	13	20
5	0	5	5	3	0	7	8	13	12	18	13	18

Matching with five edges

	*V	*W	*X	Y	*Z
*A	3	-8	9	15	10
*B	4	10	-7	16	14
*C	9	13	11	19	-10
D	8	13	12	20	13
*E	-1	7	5	11	9

- What is a shortest s, t-path in the residual graph for this matching?
- $s \rightarrow D \rightarrow V \rightarrow E \rightarrow Y \rightarrow t$ has length 18.
- Matching with five edges AW, BX, CZ, DV, EY has cost $26 + 18 = 44$.

Matching with five edges

	V	W	X	Y	Z
A	3	$\boxed{8}$	9	15	10
B	4	10	$\boxed{7}$	16	14
C	9	13	11	19	$\boxed{10}$
D	$\boxed{8}$	13	12	20	13
E	1	7	5	$\boxed{11}$	9

- cost 44.

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Matching with five edges

Prices and Reduced Costs

	V	W	X	Y	Z
A	3	$\boxed{8}$	9	15	10
B	4	10	$\boxed{7}$	16	14
C	9	13	11	19	$\boxed{10}$
D	$\boxed{8}$	13	12	20	13
E	1	7	5	$\boxed{11}$	9

	V 8	W 13	X 11	Y 18	Z 13
A 5	0	$\boxed{0}$	3	2	2
B 4	0	1	$\boxed{0}$	2	5
C 3	4	0	3	4	$\boxed{0}$
D 0	$\boxed{0}$	0	1	2	0
E 7	0	1	1	$\boxed{0}$	3

- Shows that the solution is indeed minimum cost!

Using reduced costs

	V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
E	1	7	5	11	9

- Initial prices and reduced costs

		V	W	X	Y	Z
		1	7	5	11	9
A	0	2	1	4	4	1
B	0	3	3	3	5	5
C	0	8	6	6	8	1
D	0	7	6	7	9	4
E	0	0	0	0	0	0

Using reduced costs

First iteration

- Initial prices and reduced costs

		V	W	X	Y	Z
		1	7	5	11	9
A	0	2	1	4	4	1
B	0	3	3	3	5	5
C	0	8	6	6	8	1
D	0	7	6	7	9	4
E	0	0	0	0	0	0

- What is a shortest s, t-path in the residual reduced graph for this matching? Let's run Dijkstra!

node	s	A	B	C	D	E	V	W	X	Y	Z	t
dist	0	0	0	0	0	0	0	0	0	0	0	0
pred	-	s	s	s	s	s	E	E	E	E	E	Z

- A shortest s, t-path is $s \to E \to Z \to t$, so matching is EZ.

Using reduced costs

Second iteration

- Updated prices and reduced costs

	V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
E	1	7	5	11	9

		V	W	X	Y	*Z
		1	7	5	11	9
A	0	2	1	4	4	1
B	0	3	3	3	5	5
C	0	8	6	6	8	1
D	0	7	6	7	9	4
*E	0	0	0	0	0	-0

- What is a shortest s, t-path in the residual reduced graph for this matching? Let's run Dijkstra!

node	s	A	B	C	D	E	V	W	X	Y	Z	t
dist	0	0	0	0	0	1	1	1	1	1	1	1
pred	-	s	s	s	s	Z	E	A	E	E	A	W

- A shortest s, t-path is $s \to A \to W \to t$, so matching is AW, EZ.

Using reduced costs

Third iteration

- Updated prices and reduced costs

	V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
E	1	7	5	11	9

		V	*W	X	Y	*Z
		2	8	6	12	10
*A	0	1	-0	3	3	0
B	0	2	2	1	4	4
C	0	7	6	6	7	0
D	0	6	5	6	8	3
*E	1	0	0	0	0	-0

- What is a shortest s, t-path in the residual reduced graph for this matching? Let's run Dijkstra!

node	s	A	B	C	D	E	V	W	X	Y	Z	t
dist	0	0	0	0	0	0	0	0	0	0	0	0
pred	-	W	s	s	s	Z	E	E	E	E	C	X

- A shortest s, t-path is $s \to C \to Z \to E \to X \to t$, so matching is AW, CZ, EX.

Using reduced costs

Fourth iteration

- Updated prices and reduced costs

	V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
E	1	7	5	11	9

		V	*W	*X	Y	*Z
		2	8	6	12	10
*A	0	1	-0	3	3	0
B	0	2	2	1	4	4
*C	0	7	6	6	7	-0
D	0	6	5	6	8	3
*E	1	0	0	-0	0	0

- What is a shortest s,t-path in the residual reduced graph for this matching? Let's run Dijkstra!

node	s	A	B	C	D	E	V	W	X	Y	Z	t
dist	0	1	0	1	0	1	1	1	1	1	1	1
pred	-	W	s	Z	s	X	E	E	B	E	E	Y

- A shortest s,t-path is $s \to B \to X \to E \to Y \to t$, so matching is AW, BX, CZ, EY.

Using reduced costs

Fifth iteration

- Updated prices and reduced costs

	V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
E	1	7	5	11	9

		V	*W	*X	*Y	*Z
		3	9	7	13	11
*A	1	1	-0	3	3	0
*B	0	1	1	-0	3	3
*C	1	7	6	6	7	-0
D	0	5	4	5	7	2
*E	2	0	0	0	-0	0

- What is a shortest s, t-path in the residual reduced graph for this matching? Let's run Dijkstra!

node	s	A	B	C	D	E	V	W	X	Y	Z	t
dist	0	4	5	2	0	7	5	4	5	7	2	5
pred	-	W	X	Z	s	Y	D	D	D	D	D	V

- A shortest s, t-path is $s \to D \to V \to t$, so matching is AW, BX, CZ, DV, EY.

Using reduced costs

Check

- Updated prices and reduced costs

	V	W	X	Y	Z
A	3	8	9	15	10
B	4	10	7	16	14
C	9	13	11	19	10
D	8	13	12	20	13
E	1	7	5	11	9

		*V	*W	*X	*Y	*Z
		8	13	12	20	13
*A	5	0	-0	2	0	2
*B	5	1	2	-0	1	6
*C	3	4	3	2	2	-0
*D	0	-0	0	0	0	0
*E	9	2	3	2	-0	5