First Order Logic and KR

Syntax of First Order Logic

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Objectives

Objective

Explain the Syntax of First-Order Logic

Signature

Signature consists of two kinds of symbols:

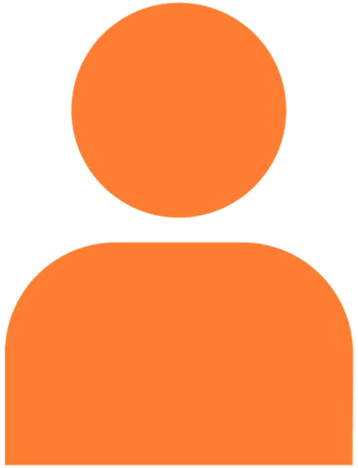
- function constants (with arity n): +/2, a/0, father/1
  - function constants of arity 0 are called object constants
- predicate constants (with arity n): even/1, >/2
  - predicate constants of arity 0 are called propositional constants

Other Symbols Not in Signature

- **These symbols can be used in addition to the symbols in the signature**
  - (object) variables: $x, y, z, x_1, y_1, z_1, x_2, y_2, z_2, \ldots$
  - the propositional connectives: $\bot$, $\top$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
  - the universal quantifier $\forall$ and the existential quantifier $\exists$
  - the parentheses and the comma

Terms

- A **term** is meant to denote an individual. It is defined recursively:
  - an object constant is a term
    - john, andy, mary, …
  - an object variable is a term
    - x, y, z
  - for every function constant f of arity n (n>0), if t_1, …, t_n are terms then so is $f(t_1, \ldots, t_n)$.
    - father(john), +(a, x)
    - father(father(father(john))), +(a, +(x, john))



Syntax of First-Order Formulas

An **atomic formula** is meant to denote a base fact that is either true or false

They work like atoms in propositional logic (smallest unit that can be assigned true or false), but has a more complicated internal structure

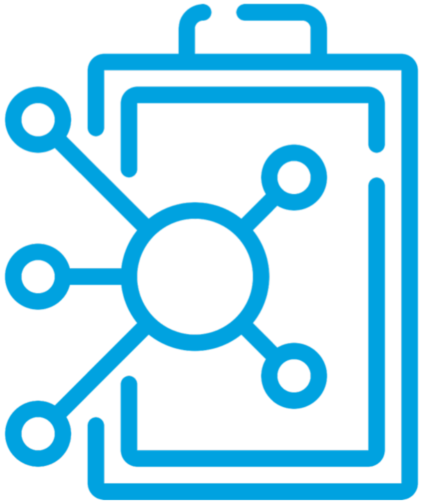
Atomic formulas are either

- Propositional constants R, or
  - TrainLate, TaxiLate
- $R(t_1, \dots, t_n)$ where R is a predicate constant and t_i are terms, or
  - even(2), prime(3), > (3, 2), 3 > 2
- $t_1 = t_2$
  - father(john) = james, 1+2=3

Syntax of First-Order Formulas, cont'd

A (first-order) formula of signature σ is defined recursively:

- every atomic formula of σ is a formula
- both 0-place connectives $(\top, \bot)$ are formulas
- if F is a formula, then $\neg F$ is a formula
- if F, G are formulas, then (F $\odot$ G) is a formulas, where $\odot$ is any binary connective
- if F is a formula then $\forall x$ F and $\exists x$ F are formulas



Examples

Let $\sigma = \{a, P, Q\}$, where a is an object constant, P is a unary and Q is a binary predicate constant

Q: Are these formulas?

1. a
2. P(a)
3. Q(a)
4. $\forall x$ P(a)
5. $\neg P(a) \vee \exists x (P(x) \wedge Q(x, y))$

Bound and Free Variables: Free

An occurrence of a variable v in a formula F is bound if it belongs to a subformula of F that has the form Qv G; otherwise it is free.

- Informally speaking, the occurrence is bound if, in the parse tree, one of its ancestors is Qv.

$$\exists y\ P(x, y) \wedge \neg \exists x\ P(x, y)$$

1 2 3 4 5 6

Q: Which occurrences are free?

v is a free variable of F if v has a free occurrence in F.

Q: Which variables are free?

Example

Formula: $\left(\forall x \left(P(x) \land Q(x)\right)\right) \rightarrow (\neg P(x) \lor Q(y))$

1 2 3 4 5

Q: **What are the free occurrences of a variable?**

Q: **What are the free variables of the formula?**

Bound and Free Variables: Bound

- Bound variables can be renamed without changing meaning:
  - $\forall x\, P(x)$ means the same as $\forall y\, P(y)$.
- A **sentence** is a formula without free variables.
- The **universal closure** of F is the $\forall v_1, \ldots, v_n$ F where $v_1, \ldots, v_n$ are free variables of F.

Assume that the signature consists of the object constant *Me*, the unary predicate constant *Male*, and the binary predicate constant *Parent*, and nothing else. Express each of the given English sentences in first-order logic.

1. I have no daughters
2. I have a granddaughter
3. I have a brother.

EXERCISE

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- **Let the underlying signature be**

$$\{a, P, Q\} \qquad (1)$$

where *a* is an object constant, *P* is a unary predicate constant, and *Q* is a binary predicate constant.

- **We will think of object variables as ranging over the set N of nonnegative integers, and interpret the signature as follows:**
  - A represents the number 10,
  - P(x) represents the condition “x is a prime number,”
  - Q(x, y) represents the condition “x is less than y.”

- **As an example, the sentence All prime numbers are greater than x can be represented by the formula**

$$\forall y\big(P(y) \rightarrow Q(x, y)\big). \qquad (2)$$

EXERCISE

In the follow two problems, represent the given English sentences by predicate formulas.

Problem 1

a) There is a prime number that is less than 10.
b) x equals 0.
c) x equals 9.

Problem 2

There are infinitely many prime numbers

Wrap-Up