First Order Logic and KR

Herbrand Models

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Objectives

Objective

Explain how Herbrand models are defined and reduces the complexity of FOL reasoning

Herbrand Models

- **A Hebrand interpretation is a special case of first-order interpretation**
- **A Herbrand interpretation of signature σ (containing at least one object constant) is an interpretation of σ such that**
  - its universe (Herbrand Universe) is the set of all ground (i.e., variable-free) terms of σ,
  - every ground term is interpreted as itself ($t^I = t$)

$a^I = a$

Herbrand Models, cont'd

- Herbrand interpretations of the signature {P, a}

$|I| = \{a\}$ | $|I| = \{a\}$

$a^I = a$ | $a^I = a$

$P^I(a) = t$ | $P^I(a) = f$

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- Herbrand models of the signature {P, a} that satisfies formula P(a)

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Herbrand Models, cont'd

- Herbrand interpretations of the signature {P, a, b}

$|I| = \{a, b\}$
$a^I = a$
$b^I = b$

①	②	③	④
$P^I(a) = f$	$P^I(a) = f$	$P^I(a) = t$	$P^I(a) = t$
$P^I(b) = f$	$P^I(b) = t$	$P^I(b) = f$	$P^I(b) = t$

- Herbrand models of the signature {P, a, b} that satisfies formula P(a)

③, ④

- An Herbrand interpretation can be identified with the set of ground atoms to which it assigns the value true.

①	②	③	④
$\emptyset$	$\{p(b)\}$	$\{p(a)\}$	$\{p(a), p(b)\}$

Exercise

1. Let F_1 be $P(a) \wedge \exists x \neg p(x)$.

   Find the Herbrand models of F_1 whose signature is {a,P}.

   $|I| = \{a\}$ ① $P^I(a) = t$ ✗ | ② $P^I(a) = f$ ✗

   Find the Herbrand models of F_1 whose signature {a,b,P}

   $|I| = \{a, b\}$ ① $P^I(a) = f$, $P^I(b) = f$ ✗ | ② $P^I(a) = f$, $P^I(b) = t$ ✗ | ③ $P^I(a) = t$, $P^I(b) = f$ O | ④ $P^I(a) = t$, $P^I(b) = t$ ✗

2. Let F_2 be $P(a) \wedge \neg P(b) \wedge \exists x \neg P(x)$.

   Find the Herbrand models of F_2 whose signature is {a,b,P}.

Entailment and Herbrand Models

- **Without functions, entailment restricted to Herbrand models is decidable**
  - Herbrand models are finitely enumerable
- **With functions, this is not the case**
  - Consider {P, f, a} $\{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$
- **When the Herbrand universe is finite, quantified formulas can be identified with propositional formulas**
  - $\forall x\, P(x)$ vs. $P(a) \wedge P(b) \wedge P(c)$
  - $\exists x\, P(x)$ vs. $P(a) \vee P(b) \vee P(c)$

Wrap-Up