First Order Logic and KR

Semantics of First-Order Logic

Ira A. Fulton Schools of Engineering
Arizona State University

Objectives

Objective

Explain the semantics of First-Order Logic

Interpretation: Example

- **For signature $\sigma = \{a, succ, P, Q\}$ where**
  - a is an object constant,
  - succ is a unary function constant,
  - P is a unary predicate constant,
  - Q is a binary predicate constant

$$|I| = \mathbf{N},$$

$$a^I = 10, \qquad succ^I(n) = n + 1$$

$$P^I(n) = \begin{cases} \mathsf{t}, & \text{if } n \text{ is prime,} \\ \mathsf{f}, & \text{otherwise,} \end{cases}$$

$$Q^I(m,n) = \begin{cases} \mathsf{t}, & \text{if } m < n, \\ \mathsf{f}, & \text{otherwise.} \end{cases}$$

Interpretation: Definition

An interpretation (or structure) I of a signature σ consists of

- a non-empty set |I|, called the universe (or domain) of I
- for every object constant c of σ, an element c^I of |I|
- for every function constant f of σ of arity n > 0, a function f^I from $|I|^n$ to |I|
- for every propositional constant R of σ, an element R^I of $\{\mathbf{t}, \mathbf{f}\}$
- for every predicate constant R of σ of arity n > 0, a function R^I from $|I|^n$ to $\{\mathbf{t}, \mathbf{f}\}$

Semantics: Terms

- For ~~term~~ object constant a, a^I is part of interpretation I
- For terms $\mathbf{f}(\mathbf{t}_1, \ldots, \mathbf{t}_n)$,
  - $\mathrm{f}(\mathrm{t}_1, \ldots, \mathrm{t}_n)^{\mathrm{I}} = \mathrm{f}^{\mathrm{I}}(\mathrm{t}_1^{\mathrm{I}}, \ldots, \mathrm{t}_n^{\mathrm{I}})$ for all function constants f of arity n > 0

Example:

- $a^I = 10$ (part of interpretation)
- $succ(a)^I = succ^I(a^I) = succ^I(10) = 11$

Semantics (Formulas): Buggy

We define the truth value F^I of F under interpretation I as:

- $R(t_1, \ldots, t_n)^I = R^I(t_1^I, \ldots, t_n^I)$,
- $\perp^I = \boldsymbol{f}$, $\top^I = \boldsymbol{t}$,
- $(\neg F)^I = \neg(F^I)$,
- $(F \odot G)^I = \odot(F^I, G^I)$ for every binary connective $\odot$
- $\forall w\, F(w)^I = \boldsymbol{t}$ iff, for all object constants c, $F(c)^I = \boldsymbol{t}$
- $\exists w\, F(w)^I = \boldsymbol{t}$ iff, for some object constants c, $F(c)^I = \boldsymbol{t}$

Q: What does this mean?

- $\exists x\, (Q(x, a))^I = t$
  iff for some object constant c,
  $Q(c, a)^I = t$
  iff $Q(a, a)^I = t$

$Q(0, a)^I = t$

$|I| = \mathbf{N}$,

$a^I = 10$,

$$P^I(n) = \begin{cases} \text{t}, & \text{if } n \text{ is prime,} \\ \text{f}, & \text{otherwise,} \end{cases}$$

$$Q^I(m, n) = \begin{cases} \text{t}, & \text{if } m < n, \\ \text{f}, & \text{otherwise.} \end{cases}$$

Semantics: Extended Signature

- **We consider an extended signature σ^I just to define the semantics correctly**
  - σ^I is not available to the knowledge engineer
- **Consider an interpretation I of a signature σ. For any element ξ of its universe |I|, select a new symbol ξ^*, called the name of ξ**
- **By σ^I we denote the signature obtained from σ by adding all names ξ^* as object constants**
- **The interpretation I can be extended to the new signature σ^I by defining $(\xi^*)^I = \xi$ for all ξ ∈ |I|**

Example: for the signature $\boldsymbol{\sigma} = \{\mathbf{a}, \mathbf{P}, \mathbf{Q}\}$ **and interpretation** I such that

$$|I| = \mathbf{N},$$

$$a^I = 10,$$

$$P^I(n) = \begin{cases} \mathsf{t}, & \text{if } n \text{ is prime,} \\ \mathsf{f}, & \text{otherwise,} \end{cases}$$

$$Q^I(m,n) = \begin{cases} \mathsf{t}, & \text{if } m < n, \\ \mathsf{f}, & \text{otherwise.} \end{cases}$$

Q: What is $\boldsymbol{\sigma}^I$?

$\sigma^I = \{a, P, Q, 0^*, 1^*, 2^*, \ldots\}$

Semantics (Formulas): Corrected

- $R(t_1, \ldots, t_n)^I = R^I(t_1^I, \ldots, t_n^I)$
- $\perp^I$ = f, $\top^I$ = t
- $(\neg F)^I = \neg(F^I)$
- $(F \odot G)^I = \odot(F^I, G^I)$ for every binary connective $\odot$
- $\forall w F(w)^I$ = t if $F(\xi^*)^I$ = t for all $\xi \in |I|$
- $\exists w F(w)^I$ = t if $F(\xi^*)^I$ = t for some $\xi \in |I|$

Q: $\exists x \left(Q(x, a)\right)^I = ?$ t

iff $Q(\xi^*, a)^I = t$ for some $\xi \in N$

iff true ($\because$ take $\xi = 0$)

$$|I| = \mathbf{N},$$

$$a^I = 10,$$

$$P^I(n) = \begin{cases} \mathsf{t}, & \text{if } n \text{ is prime,} \\ \mathsf{f}, & \text{otherwise,} \end{cases}$$

$$Q^I(m, n) = \begin{cases} \mathsf{t}, & \text{if } m < n, \\ \mathsf{f}, & \text{otherwise.} \end{cases}$$

Satisfaction, Logical Validity (Tautology)

- The notions are simply carried over from propositional logic

- We say that an interpretation I **satisfies** a **sentence** F, or is a **model** of F, and write $\mathrm{I} \vDash \mathrm{F}$, if $\mathrm{F}^{\mathrm{I}} = \mathrm{t}$
  - A sentence is a formula with no free variables

- A sentence F is **logically valid** if every interpretation satisfies F (c.f. tautology in propositional logic)

Satisfaction, Logical Validity (Tautology), Equivalence

Two sentences, or sets of sentences, are **equivalent** to each other if they are satisfied by the same interpretations

A formula with free variables is said to be **logically valid** if its universal closure is logically valid

Formulas F and G that may contain free variables are equivalent to each other if $F \leftrightarrow G$ is logically valid

Entailment

A set Γ of sentences is **satisfiable** if there exists an interpretation satisfying all sentences in Γ

A set Γ of sentences **entails** a formula F (symbolically, $\Gamma \vDash F$) if every interpretation satisfying Γ satisfies the universal closure of F

Q: True or false?

- $\{\exists x\, P(x),\ \exists x\, Q(x)\} \vDash \exists x\big(P(x) \wedge Q(x)\big)$ false
- $\{\forall x\, P(x),\ \forall x\, Q(x)\} \vDash \forall x\big(P(x) \vee Q(x)\big)$ true

Undecidability of FOL

- **The Validity Problem of FOL: given any sentence F, is F logically valid?**
- **Theorem: The Validity Problem is undecidable**
- **Corollary: The satisfiability problem of first-order logic (i.e., given a sentence F, is F satisfiable?) is undecidable:**
  - Proof: By reducing the validity problem to it
  - F is logically valid iff $\neg F$ is unsatisfiable
- **We need to restrict FOL in a meaningful way!**

Wrap-Up