Introduction to KRR

Notions in Propositional Logic

Ira A. Fulton Schools of Engineering
Arizona State University

Objectives

Objective

Explain the concepts of satisfiability, tautology, equivalence, entailment in propositional logic

Objective

Explain how these concepts are related to each other

Satisfiability, Tautology, Equivalence, Entailment

Satisfiability

A propositional formula F is **satisfiable** if some interpretation satisfies F

Q: Which one is satisfiable? Choose all

1. $(p \rightarrow (q \rightarrow p))$
2. $(p \rightarrow (p \rightarrow q))$
3. $(p \rightarrow q) \wedge (p \wedge \neg q)$

p	q	3
f	f	f
f	t	f
t	f	f
t	t	f

A set of propositional formulas is **satisfiable** if some interpretation satisfies all formulas in the set.

$\{ p \vee q, \neg p \vee \neg q \}$

p	q
t	f

Tautology

A propositional formula F is a **tautology** if every interpretation satisfies F

Q: Which one is a tautology? Choose all

1. $(p \rightarrow q) \rightarrow (\neg p \lor q)$

2. $(p \rightarrow (q \rightarrow p))$

3. $(p \rightarrow (p \rightarrow q))$

t t f

T

Equivalence

F is **equivalent** to G (symbolically, $F \Leftrightarrow G$) if, for every interpretation I, $F^I = \cancel{G^I}\ G^I$

- In other words, $F \leftrightarrow G$ is a tautology

Q: Which formulas are equivalent to each other?

1. $(p \to (q \to p))$ and $(p \lor \neg p)$
2. ~~2.~~ $(p \to (p \to q))$ and p
   f t
3. ~~3.~~ $(p \to q)$ and $(q \to p)$
   f t
4. ~~4.~~ $(p \to q) \to (p \land \neg q)$ and $\perp$
   f t f

p	q
t	f

More examples

- $p \to q \Leftrightarrow \neg q \to \neg p$
- $p \to q \Leftrightarrow \neg p \lor q$
- $\neg(p \to q) \Leftrightarrow (p \land \neg q)$
- $(p \land q) \to r \Leftrightarrow p \to (q \to r)$

Some Useful Equivalence

$$F \to G \Leftrightarrow \neg F \vee G$$

$$F \leftrightarrow G \Leftrightarrow (F \to G) \wedge (G \to F)$$

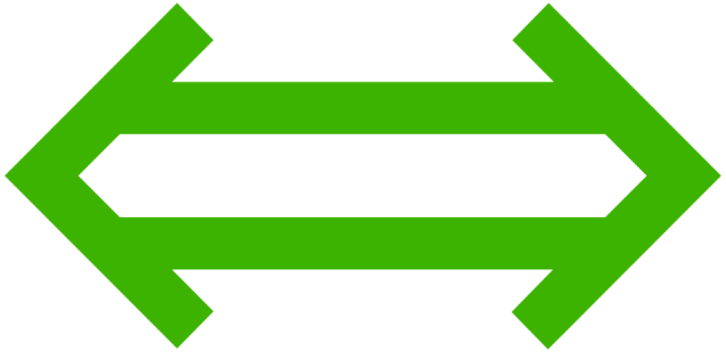
$$\neg\neg F \Leftrightarrow F$$

$$\neg(F \wedge G) \Leftrightarrow \neg F \vee \neg G$$

$$\neg(F \vee G) \Leftrightarrow \neg F \wedge \neg G$$

$$(F \vee (G \wedge H)) \Leftrightarrow (F \vee G) \wedge (F \vee H)$$

$$(F \wedge (G \vee H)) \Leftrightarrow (F \wedge G) \vee (F \wedge H)$$



Entailment

- A set Γ of formulas **entails** a formula F (symbolically, $\Gamma \vDash F$) if, every interpretation that satisfies all formulas in Γ satisfies F also.
  - c.f. Entailment uses the same symbol as satisfaction, the difference being what appears on the left of $\vDash$.

Q: True or false?

- $\{A, B\} \vDash A \wedge B$ O
- $\{A, A \to B\} \vDash B$ O
- $\{A\} \vDash A \vee B$ O
- $\{A\} \vDash A \wedge B$ X
- $\emptyset \vDash B$ X

A	B
t	f

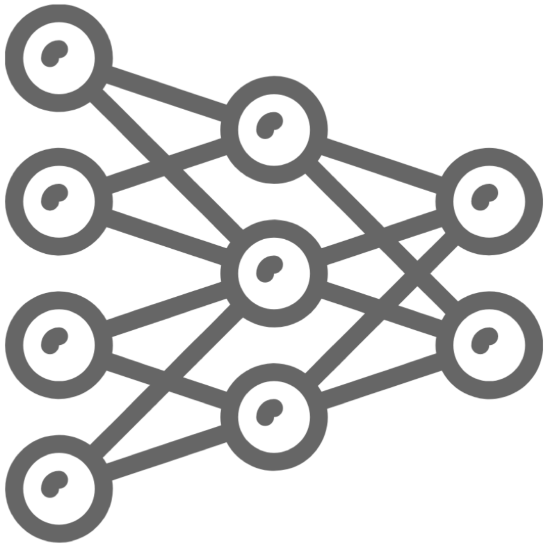
- $\{\perp\} \vDash B$ O

The formulas entailed by Γ are also called the **logical consequences** of Γ.

Algorithm for Entailment Checking

- To check a set Γ of formulas **entails** a formula F

- For each interpretation I,
  - For each formula G in $\mathbf{\Gamma}$, check if I satisfies G:
    - If no, continue to next interpretation
    - If yes: check if I satisfies F
      - If yes: continue to next interpretation
      - If no: exit and report "Not Entailed"
  - (When all checking all interpretations are done)
    - Report "Entailed"



Entailment: Example

$TL \wedge \neg T \to JL, \quad TL, \quad \neg JL \quad \not\models T$

TL	T	JL
f	f	f
f	f	t
f	t	f
f	t	t
t	f	f
t	f	t
t	t	f
t	t	t

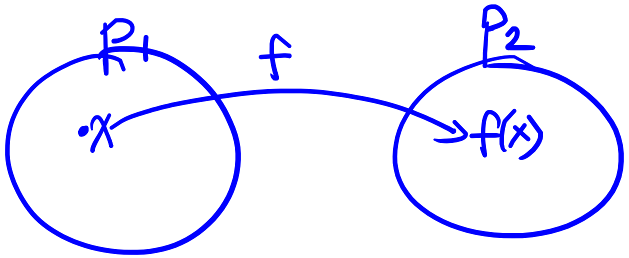
x

Reductions between Problems

Starting Points

- **Intuitively, these problems are strongly related**

- **A reduction from problem P_1 to P_2 is a function f such that**
  - For each input x to P_1, the answer of P_1 for input x coincides with the answer of P_2 for input $f(x)$,
  - Given x, the input $f(x)$ can be efficiently computed.



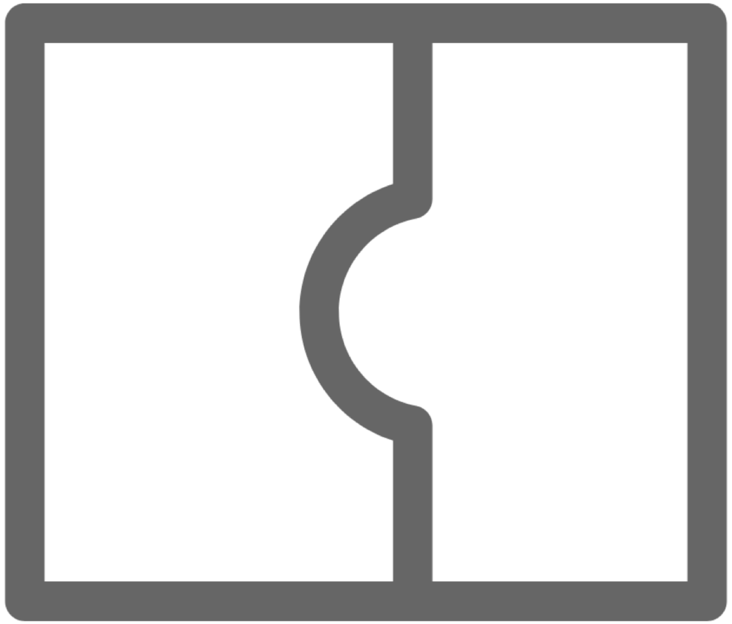
- **These (and many other) problems can be reduced to (un)satisfiability**

- **Satisfiability solvers are a useful tool for KR**

How are Tautology and Satisfiability related?

- F is a tautology iff $\neg F$ is unsatisfiable

- **Example:** $p \vee \neg p$ is a tautology iff $\neg(p \vee \neg p)$ is unsatisfiable



How are Tautology and Tautology related?

- F is equivalent to G iff $F \leftrightarrow G$ is a tautology

- **Example:** $p \to q$ is equivalent to $\neg p \vee q$ iff $(p \to q) \leftrightarrow (\neg p \vee q)$ is a tautology

How are Equivalence and Entailment related?

- F is equivalent to G iff
  - F entails G and
  - G entails F
- Example: $p \rightarrow q$ is equivalent to $\neg p \vee q$ iff
  - $p \rightarrow q$ entails $\neg p \vee q$ and
  - $\neg p \vee q$ entails $p \rightarrow q$

How are Entailment and Tautology related?

- $\{F_1, \ldots, F_n\} \vDash G$ **iff** $(F_1 \land \cdots \land F_n) \rightarrow G$ **is a tautology**
  - $\{p \lor q, \neg p \lor q\} \vDash q$ iff $(p \lor q) \land (\neg p \lor q) \rightarrow q$ is a tautology

- $\emptyset \vDash G$ **iff** G **is a tautology**

How are Entailment and Satisfiability related?

- $F \models G$ iff $F \land \neg G$ is unsatisfiable
  - $\{p \lor q, \neg p \lor q\} \models q$ iff $\{p \lor q, \neg p \lor q, \neg q\}$ is unsatisfiable

Wrap-Up