Theory of Answer Set Programming

Negation as Failure: Theory

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Objectives

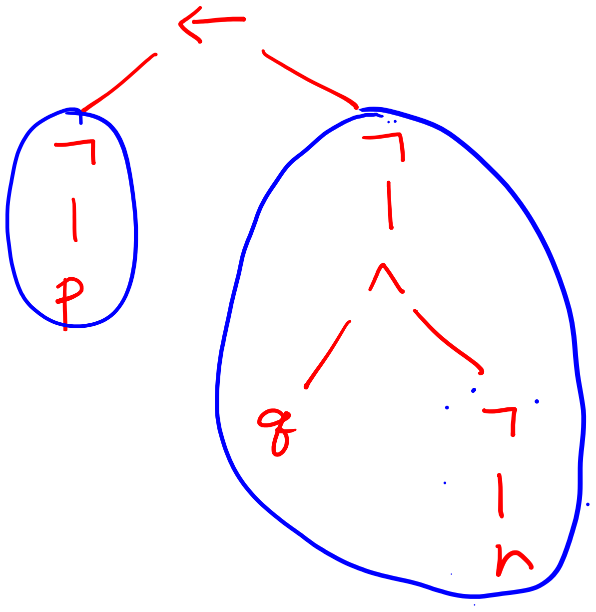
Objective

Compute stable models of programs with negation by hand

Critical Part

- A **critical part** of a propositional rule is a subformula of its head or body that begins with negation but is not part of another subformula that begins with negation.

- **Example: Find the critical parts of the propositional rule**
  - $r \leftarrow p \wedge \underline{\neg s}$
  - $\underline{\neg p} \leftarrow \underline{\neg(q \wedge \neg r)}$
  - $p \leftarrow \underline{\neg\neg p}$
  - $p \vee \underline{\neg p}$



Stable Models of Programs with Negation

The **reduct** Π^X of Π relative to an interpretation X is the **positive** propositional program obtained from Π by replacing each critical part $\neg H$ of each of its rules

- by $\top$ if X satisfies $\neg H$;
- by $\bot$ otherwise

X is a **stable model** of Π If X is a minimal model of the reduct Π^X

Example:

Γ

$$p,$$
$$q,$$
$$r \leftarrow p \wedge \neg s,$$
$$s \leftarrow q.$$

$\Gamma^{\{p,q,s\}}$

SM

p
q
$r \leftarrow p \wedge \bot$
$s \leftarrow q$

$\{p, q, s\}$

$\Gamma^{\{p,q\}}$

Not SM

p
q
$r \leftarrow p \wedge \top$
$s \leftarrow q$

$\{p, q, r, s\}$

$\Gamma^{\{p,q,r\}}$

Not SM

p
q
$r \leftarrow p \wedge \top$
$s \leftarrow q$

$\{p, q, r, s\}$

Steps to Find Stable Models (Succinct)

Given a propositional program Π

1. **Guess an interpretation X**
2. **Find the reduct of Π relative to X (i.e., Π^X)**
3. **Check if X is a minimal model of Π^X (note that Π^X is a positive program; has no negation)**
   a. If yes, conclude X is a stable model of Π
   b. If no, conclude X is **not** a stable model of Π

Steps to Find Stable Models (Verbose)

Given a propositional program Π

1. **Guess an interpretation X**
2. **Find the reduct of Π relative to X (i.e., Π^X)**
3. **Check if X satisfies Π^X (Alternatively, check if X satisfies Π)**
   a. If yes, continue
   b. If no, conclude X is **not** a stable model of Π
4. **Check if no other interpretation that is smaller than X satisfies Π^X. I.e., for each interpretation Y that is smaller than X,**
   a. If Y satisfies Π^X, conclude X is **not** a stable model of Π
   b. Else continue
5. **Conclude X is a stable model of Π**

NOTES:

- Every stable model is a model.
- The red part can't be replaced with Π .

Example (a)

Find all stable models of

$\Pi \quad p \leftarrow \neg q$

$\{p\}$

Inter(X)	reduct Π^X	SM
$\emptyset$	$\Pi^{\emptyset} = p \leftarrow \top$	X
$\{p\}$	$\Pi^{\{p\}} = p \leftarrow \top$	O
$\{q\}$	$\Pi^{\{q\}} = p \leftarrow \bot$	X
$\{p, q\}$	$\Pi^{\{p,q\}} = p \leftarrow \bot$	X

Example (b)

Find all stable models of

$$p \vee q,$$

$$r \leftarrow \neg p.$$

Inter.(X)	Π^X	SM
$\emptyset$	$p \vee q$ $r \leftarrow \top$	×
$\{p\}$, $\emptyset$ $\models$	$p \vee q$ $r \leftarrow \bot$	○
$\{q\}$		
$\{r\}$		
$\{p, q\}$		
$\emptyset$ $\{q\}$ $\{r\}$ $\{q, r\}$	$p \vee q$ $r \leftarrow \top$	○
$\{p, r\}$		
$\{p, q, r\}$		

Example (c)

Find all stable models of each of the following one-rule programs:

- $p \leftarrow \neg p$
- $p \leftarrow \neg\neg p$
- $p \lor \neg p$

X	Π^X	SM
$\emptyset$	$p \leftarrow \top$	✗
$\{p\}$	$p \leftarrow \bot$	✗
$\emptyset$	$p \leftarrow \bot \equiv \top$	O
$\{p\}$	$p \leftarrow \top \equiv p$	O
$\emptyset$	$p \lor \top \equiv \top$	O
$\{p\}$	$p \lor \bot \equiv p$	O

Inclusive vs. Exclusive Or

a) **Find the stable models of**

$$p \lor q$$

$\{p\}$, $\{q\}$

b) **Find the stable models of**

$$p \lor q$$
$$p \leftarrow q$$

$\{p\}$

c) **Find the stable models of**

$$p \lor q$$
$$p \leftarrow q$$
$$q \leftarrow p$$

$\{p, q\}$

Models vs. Stable Models

- Equivalent propositional programs (i.e., having same models) may have different stable models.

- **Example:**

$$p \leftarrow \neg q\ ,\quad q \leftarrow \neg p\ ,\quad p \vee q$$

models $\{p\}$, $\{q\}$, $\{p, q\}$

S.M $\{p\}$ | $\{q\}$ | $\{p\}$. $\{q\}$

$$p \vee \neg p \text{ and } q \vee \neg q$$

SM $\emptyset$, $\{p\}$ | $\emptyset$; $\{q\}$

Minimal Models vs. Stable Models

Are stable models the same as minimal models?

Recall the definition:

X is a stable model of Π if X is a minimal model of $\cancel{\boxed{\Pi^X}}$
Π

Claim: For any program Π,

X is a stable model of Π if X is a minimal model of Π
p

True or false?

$p \lor \neg p$

m.m $\cancel{\emptyset}$

s.m $\emptyset$, $\{p\}$

Recall: Steps to Find Stable Models (Verbose)

Given a propositional program Π

1. **Guess an interpretation X**
2. **Find the reduct of Π relative to X (i.e., Π^X)**
3. **Check if X satisfies Π^X (Alternatively, check if X satisfies Π)**
   a. If yes, continue
   b. If no, conclude X is **not** a stable model of Π
4. **Check if no other interpretation that is smaller than X satisfies Π^X. I.e., for each interpretation Y that is smaller than X,**
   a. If Y satisfies Π^X, conclude X is **not** a stable model of Π
   b. Else continue
5. **Conclude X is a stable model of Π**

NOTES:

- Every stable model is a model.
- The red part can't be replaced with Π .

Lecture Wrap-Up