Krishna's

Optimization TECHNIQUES

As per U.P. UNIFIED Syllabus (*w.e.f.* 2012–2013)

(***for B.C.A. IVth Sem. Students of all Colleges Affiliated to Universities in Uttar Pradesh***)

By

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Jai Shri Radhey Shyam

Dedicated

to

Lord

Krishna

Author & Publishers

ABOUT THE BOOK

- The book has been written in a simple, logical and systematic way.
- The basic concept of each topic is carefully and clearly explained.
- Each chapter includes a problem set containing routine, intermediate and challenging exercises.
- Solutions to all problem sets are given at the end of each chapter for learners to practice.
- The goal of the book is to present a wide variety of applications of **Optimization Techniques** in a flexible and accessible format.
- The book has been written to meet the requirements of students pursuing **B.C.A.** in colleges affiliated to **U.P. UNIFIED.**
- The book also contains illustrative examples and theorems along with their proof.

— J.P. Chauhan

PREFACE

It gives me pleasure to place the book **'Optimization Techniques' (All Universities in U.P.)** which has been completely revised and made upto date in the light of suggestions received from the students and the learned teachers of various universities.

The following are the main features of third edition:

- More than 150 new problems and examples have been added.
- Includes recent elegant and elementary proof.
- All chapters are arranged according to the latest **U.P. UNIFIED Syllabus** implemented from 2012 – 2013, keeping in view the requirement of the students of B.C.A. IVth semester of all Universities in Uttar Pradesh (U.P.).
- Errors and omissions have been corrected.
- Up-to-date yearwise references from various universities examination papers have been given throughout the book.
- Each step is made clear by giving reasons with clarification.

We hope that the book in its present form will prove beneficial to the students. We shall also forward to receive your suggestions and comments about the book so that it could be further improved.

— J.P. Chauhan

ACKNOWLEDGEMENT

This book has been written in simple and lucid style for better understanding and appreciation of study. It has been compiled systematically to enable sure one to evaluate detail of the book.

Needless to say, I am deeply indebted to the publisher **Shri Satyendra Rastogi** (***Managing Director***) and **Shri Sugam Rastogi** (***Executive Director***) for providing continuous support and enthusiasm.

This acknowledgement would fall short of completion if I fail to mention the **entire team of Krishna Prakashan.** I would like to thank the contributors for their creativity and especially their excellent work.

Finally, we shall be very grateful to all instructors and students who send us their helpful suggestions.

Meerut (U.P.)

—J.P. Chauhan

SYLLABUS

Optimization Techniques

U.P. UNIFIED (*w.e.f.* 2012-2013)

B.C.A. – S 209 **B.C.A. IVth Semester**

UNIT-1

Linear Programming: Central problem of linear programming various definitions included statements of basic theorem and also their properties, simplex methods, primal and dual simplex method, transport problem, tic-tac problem, and its solution. Assignment problem and its solution. Graphical method formulation, Linear programming problem.

UNIT-2

Queuing Theory: Characteristics of queuing system, Classification of queuing model single channel queuing theory, Generalization of steady state M/M/1 queuing models (Model-I, Model-II).

UNIT-3

Replacement Theory: Replacement of item that deteriorates replacement of items that fail. Group replacement and individual replacement.

UNIT-4

Inventory Theory: Cost involved in inventory problem-single item deterministic model economics long size model without shortage and with shorter having production rate infinite and finite.

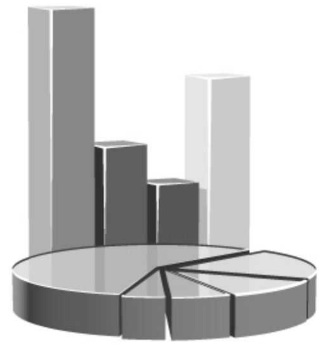
UNIT-5

Job Sequencing: Introduction, solution of sequencing problem johnson s algorithm for n jobs through 2 machines.

Brief Contents

DETAILED CONTENTS

CHAPTER 1



Basic Concepts of Optimization

1.1 Historical Background of Operations Research

No science has ever been born on a specific day. Operations Research is no exception. Its roots are as old as science and society. It was in 1917 when **A.K. Erlang**, a Danish mathematician, published his work on the problem of congestion of telephone traffic. The difficulty was that during busy periods, telephone operators were unable to handle the calls the moment they were made, resulting in delayed calls. A few years after the British Post Office as the basis for calculating circuit facilities. During the 1930, **H.C. Levinson** applied scientific analysis to the problems of merchandising. His work included scientific study of customers 'buying habits, response to advertising and relation of environment to the type of article sold.

Operations Research come to the fore and become established as a subject in its own right during world war II. It was necessary at that time to deploy resources in the most economical and efficient ways.

Past-World War Development

Following the end of world war II, the success of military teams attracted the attention of industrial managers of U.K., who were seeking solutions to their complex executive type problems.

In recent years, operation research had an increasingly great impact on the management of organizations.

Many industries including air craft and missile, automobile, communication, computer, electronics, mining paper, petroleum and transportation have made wide spread use of Operations Research in determining their tactical and strategical decisions more scientifically.

1.2 What is Operations Research?

[B.C.A. (Agra) 2004, 2006, 2009; B.B.A. (Meerut) 2006, 2008, 2008 (Dec)]

Operations Research (O.R.) has been variously described as the "**Science of Use**", "quantitative commonsense", "Scientific approach to decision making problems". But only a few are commonly used and widely accepted, namely.

"O.R. is the art of giving bad answers to problems which otherwise have worse answers".

– T.L. Saaty

"O.R. is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem".

– C.W. Churchman, R.L. Ackoff

"O.R. is a scientific approach to problems solving executive management".

– H.M. Wagner

"O.R. is a scientific method of providing executive departments with quantitative basis for decisions regarding the operations under their control."

– Morse and Kimball.

1.3 The Essential Characteristics of Operations Research

[B.C.A. (Kanpur) 2007; B.B.A. (Meerut) 2003]

The significant features of O.R. are given below:

1. **Decision Making:** A major premise of O.R. is that decision-making, irrespective of the situation involved, can be considered as a general systematic process.
2. **Scientific Approach:** O.R. employs scientific methods for the purpose of solving problems. It is a formalised process of reasoning.
3. **Objective:** O.R. attempts to locate the best or optimal solution to the problem under consideration.

4. **Inter-disciplinary Team Approach:** O.R. is inter-disciplinary in nature and requires a team approach to a solution of the problem. Managerial problems have economic, physical, biological and engineering aspects. This requires a blend of people with expertise in the areas of mathematics, statistics, engineering, economics, management, computer science and so on.
5. **Digital Computer:** Use of digital computer has become an integral part of O.R., approach to decision-making. The computer may be required due to complexity of the model, volume of data required and the computations to be made.

1.4 Modelling in Operations Research

[B.C.A. (Meerut) 2009]

By building a model, the uncertainties and complexities of a decision-making problem can be changed to a logical structure that is amenable to formal analysis. Such as model specifies the decision alternatives and their anticipated consequences for all possible events that may occur, indicates the relevant data for analysing the alternatives and leads to meaningful and informative managerial conclusions. In short we may say, modelling is a means of providing a clear structural framework to the problem for purposes of understanding and dealing with reality. Following are the main characteristics of good modelling.

1. A good model should be capable of taking into account new formulations without having any significant change in its frame.
2. Assumptions made in the model should be as small as possible.
3. The number of variables must be less.
4. The good model should be simple and coherent.
5. It should not take much time in its construction for any problem.

1.4.1 Advantages and Limitations of a Model

[B.C.A. (Meerut) 2010]

1.4.1.1 Advantages of a Model

1. Through a model, the position under consideration becomes controllable.
2. It help in finding evenues for few research and improvements in a system.
3. It indicates the limitations and scope of an activity.
4. It provide some logical and systematic approach to the problem.

1.4.1.2 Limitations of a Model

1. Models are only an attempt in understanding operations and should never be considered as absolute in any sense.
2. Validity of any model with regard to corresponding operation can only be verified by carrying the experiment and relevant data characteristics.

1.4.2 Classification Schemes of Models

Although the classification of models is a subjective problem they may be distinguished as follows:

1. **Models by Degree of Abstraction:** These models based on past data or information of problem under consideration and can be classified into
   (i) Language models
   (ii) Case studies
2. **Models by Function:** These models consist of
   (i) Descriptive models
   (ii) Predictive models
   (iii) Normative models

✦ **Descriptive models** describe, explain and predict facts and relationships among the various activities of the problem. These are used to describe mathematically some particular aspects of the system being modelled.

✦ **Predictive models** indicate 'if this occurs, then what will follow'. They relate dependent and independent variables and permit trying out, 'what if' questions. We can say that these models are used to predict the out comes due to a given set of alternatives for the problem. These models do not have an objective function as a part of the model to evaluate decision alternatives.

✦ **Normative or optimization models** are prescriptive in nature and develop objective decision rules or criteria of optimum solutions.

3. **Models by Structure:** These modes are represented by
   (i) Iconic models
   (ii) Analogue models
   (iii) Symbolic models

✦ **Iconic or physical models** are pictorial representation of real systems and have the appearance of the real thing, example of such models are:

City maps, houses blue prints, globe and so on.

An iconic model is said to be scaled-down or scaled up' according as the dimensions of the model are smaller or greater than those of the real item. Iconic models are easy to observe, build and describe, but are difficult to manipulate and not very useful for the purpose of prediction. Hence we can say these models represent a static event.

✦ **Analogue models** are more abstract than the iconic ones for there is no look alike correspondence between these models and real life items. They are built by utilizing one set of properties to represent another set of properties.

✦ **Mathematical or symbolic models** are more abstract in nature. They employ a set of mathematical symbol to represent the components of the real system. These models are most general and precise. However it is not possible to depict a real system in mathematical formulation. Sometime it is easier to used to mathematical symbol for describing the relationship of the component.

4. **Model by Nature of the Environment:** These model can be classified into
   (i) Deterministic models, and
   (ii) Probabilistic models.

   In deterministic models, all the parameters and functional relationships are assumed to be known with certainly when the decision is to be made. Linear programming and break even models are the examples of deterministic models.

1.5 Limitations of Operations Research

[B.C.A. (Agra) 2006; B.B.A. (Kanpur) 2007]

1. Mathematical models which are essences of O.R. do not take into account quantitative factors or emotional factors which are quite real. All influencing factors which cannot be quantified find no place in mathematical models.
2. Mathematical models are applicable to only specific categories of problems.
3. Being the new field there is resistance from the employees to the new proposals.
4. Management, who has to implement the advised proposals, may itself offer a lot of resistance due to conventional thinking.
5. Young enthusiasts, overtaken by its advantages and exactness generally forget that O.R. is meant for men and not that men are meant for it.

1.6 Management Applications of Operations Research

[B.C.A. (Rohilkhand) 2010; B.B.A. (Meerut) 2003, 2004, 2006, 2008]

Some of the areas of management decision making, where the 'tools' and 'techniques' of O.R. are applied, can be outlined as follows:

1. **Finance, Budgeting and Investments**
   (i) Cash-flow analysis, long range capital requirements, dividend policies, investment portfolios.
   (ii) Credit policies, credit risks and delinquent account procedures.
   (iii) Claim and complaint procedure.
2. **Purchasing, Procurement and Exploration**
   (i) Rules for buying, supplies and stable or varying prices.
   (ii) Determination of quantities and timing of purchases.
   (iii) Bidding policies.
   (iv) Strategies for exploration and exploitation of raw material sources.
   (v) Replacement policies.

3. Production Management

(i) Physical Distribution

(a) Location and size of warehouses distribution centres and retail outlets.

(b) Distribution policy.

(ii) Facilities Planning

(a) Numbers and location of factories, warehouses, hospitals etc.

(b) Loading and unloading facilities railroads and tracks determining the transport schedule.

(iii) Manufacturing

(a) Production scheduling and sequencing.

(b) Stabilization of production and employment training, layoffs and optimum product mix.

(iv) Maintenance and Project Scheduling

(a) Maintenance policies and preventive maintenance.

(b) Maintenance crew siges.

(c) Project scheduling and allocation of resources.

4. Marketing

(i) Product selection, timing, competitive actions.

(ii) Number of salesman, frequency of calling on accounts percent of time spent on prospects.

(iii) Advertising media with respect to cost and time.

5. Personnel Management

(i) Selection of suitable personnel on minimum salary.

(ii) Mixes of age and skills.

(iii) Recruitment policies and assignment of jobs.

6. Research and Development

(i) Determination of the areas of concentration of research and development.

(ii) Project Selection

(iii) Determination of time cost trade-off and control of development projects.

(iv) Reliability and alternative design.

From all above areas of application we may conclude that O.R. has replaced management by personality.

1.7 Techniques Used in Operations Research

[B.C.A. (Rohilkhand) 2008; B.B.A. (Meerut) 2008]

The following techniques used in Operations Research.

1. **Linear Programming:** It is used in the solution of problems concerned with assignment of personal, blending of materials, distribution and transportation and investment properties.

2. **Dynamic Programming:** It is used in such area as planning advertising expenditures distributing sales effort and production scheduling etc.

3. **Queuing Theory:** It is used in solving problem concern with traffic, servicing machines subject to break down, air traffic scheduling, production scheduling, hospital operations, determining optimum number of replacement for a group of machines.

4. **Inventory Theory:** In determing when and how much a production or purchase.

5. **Game Theory:** The primary objective of game theory is to develop rational criteria for selecting a strategy.

6. **Simulations:** The technique of simulation is an important tools of the designer in stimulating airplane flight in a wind tunnel, simulating lines of communication with an organisation chart. With the advent of the high speed digital computer with which to conduct simulated experiments, this technique has become experimental arm of researches.

1.8 The Importance of Operations Research in Decision Theory

[B.B.A. (Meerut) 2008]

Operations Research uses the method of science to understand and explain the phenomena of operating systems. It devises the theories (models) to explain these phenomena. Uses these theories to describe what takes place under altered conditions, and checks these predictions against new observations. Thus, operations research may be regarded as a tool employed to increase the effectiveness of managerial decisions as an objective supplement to the subjective feelings of the decision maker. O.R. may suggest alternative courses of action when a problem is analysed and solution is attempted. However, the study of complex problems by O.R. techniques becomes useful only when a choice between two or more courses of action is possible. O.R. may be regarded as a tool that enables the decision-maker to be objective in choosing an alternative from among many that he can conceive of. Following are the salient advantages of an operations research study approach in decision-making:

1. **Better Decisions:** O.R. models frequently yield actions that do improve on intuitive decision making. A situation may be so complex that the human mind can never hope to assimilate all the significant factors without the aid of O.R. guided computer analysis.
2. **Better Coordination:** Sometimes, operations research has been instrumental in bringing order out of chaos. For instance, an O.R.-oriented planning model becomes a vehicle for coordinating marketing decisions within the limitations imposed on manufacturing capabilities.
3. **Better Control:** The management of large organizations recognize that it is extremely costly to provide continuous executive to devote his attention to more pressing matters. The most frequently adopted application in this category deal with production scheduling and inventory replenishment.
4. **Better Systems:** Often an O.R. study is initiated to analyse a particular decision-problem, such as whether to open a new warehouse. Afterwards, the approach is further developed into a system to be employed repeatedly. Thus, the cost of undertaking the first application may produce benefits.

Problem Set

1. Define O.R. and discuss its scope.
[B.C.A. (Agra) 2010; B.B.A. (Meerut) 2006, 2008]
2. What is operations research? Give main characteristics of O.R. also discuss the importance of O.R. in a decision making.
[B.C.A. (Kanpur) 2009; B.B.A. (Meerut) 2008]
3. What is O.R.? Describe briefly its applications.
[M.B.A. (Garhwal) 2008]
4. Discuss the significance and scope of operations research in modern management.
[B.B.A. (Meerut) 2002; B.B.A. (Delhi) 2007]
5. Discuss scientific method in O.R. **[B.B.A. (Kurukshetra) 2007, 2008]**

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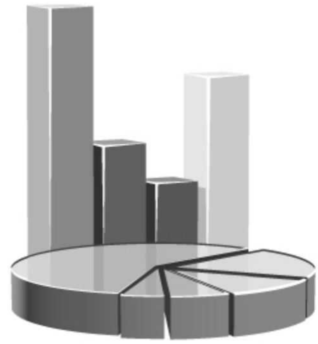
CHAPTER

2



Simplex Methods

2.1 Linear Programming Problems (Simplex Method)

2.1.1 Standard Form of a Linear Programming Problem

A linear programming problem is said to be in standard form if,

1. all the constraints are in the form of equations except for non-negativityrestrictions which remain as inequalities (≥ 0);
2. the right hand side constant (element) of each constraint equation is non-negative;
3. all the decision variables have non-negative values, and
4. the objective function is in maximization form (or minimization form).

Remark: To write a L.P.P. in standard form is often **called Reformulation of L.P.P.**

2.1.2 Conversion of a L.P.P. into Standard Form

[B.C.A. (Meerut) 2006, 2008, 2012]

Step 1: Change the linear constraints of inequality type into equations:

This is done with the help of **slack** or **surplus** variables. These non-negative variables are added to (or subtracted from) the left hand side of each such constraint. These new variables are added if the constraint is ($\leq$) and are called **slack variables.** On the other hand, these new variables are subtracted if the constraint is ($\geq$) and are called **negative slack variables** or **surplus variables.**

Thus, the non-negative variables which are introduced to represent the slack between the left hand side and right hand side of each inequality (constraints) are known as **slack variables.**

For example:

1. The linear constraint $3x_1 + 4x_2 \le 20$ is converted to an equation with **slack variables** S_1 {or x_3} as

$$3x_1 + 4x_2 + S_1 = 20, \text{ where } S_1 \ge 0$$
$$\{\text{or } 3x_1 + 4x_2 + x_3 = 20, \text{ where } x_3 \ge 0\};$$

2. The linear constraint $3x_1 + 7x_2 \ge 28$ is converted to an equation with **surplus variable** S_2 {or x_4}

$$3x_1 + 7x_2 - S_2 = 28, \text{ where } S_2 \ge 0$$
$$\{\text{or } 3x_1 + 7x_2 - x_4 = 28, \text{ where } x_4 \ge 0\};$$

In the objective function the coefficients of slack and surplus variables are shown as zero.

The objective function $Z = 2x_1 + 3x_2$ is written as

$$Z = 2x_1 + 3x_2 + 0 \,.\, S_1 + 0 \,.\, S_2$$
$$\{\text{or } Z = 2x_1 + 3x_2 + 0 \,.\, x_3 + 0 \,.\, x_4\}$$

as their contribution is nil.

Hence, the cost of slack or surplus variable is taking zero in objective function.

Step 2: Make the right hand element of each constraint non-negative.

This is done by multiplying both sides of the resulting constraint by (–1).

For example: Consider the constraint $2x_1 - 3x_2 + 4x_3 = -17$

This constraint is changed (in standard form) to

$$-(2x_1 - 3x_2 + 4x_3) = (-1)(-17)$$
$$\Rightarrow \quad -2x_1 + 3x_2 - 4x_3 = 17$$

Step 3: Make the unrestricted variables as non-negative :

This is done by replacing the unrestricted variable by a difference of two non-negative variables. For example, if x_2 is unrestricted in sign, it can be replaced by

$$(x_2' - x_2''), \text{ where } x_2' \ge 0, \; x_2'' \ge 0$$

or by $(x_3 - x_4)$, where $x_3 \ge 0, \quad x_4 \ge 0$

or by $(y_1 - y_2)$, where $y_1 \ge 0, \quad y_2 \ge 0$

Step 4: Convert the objective function in maximization form: [Although it is not necessary]. This is done by changing the sign of the objective function.

For example: Minimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ is equivalent to the expression

$$\text{Maximize } (-Z) = -c_1x_2 - c_2x_2 - \ldots - c_nx_n$$

After going through the steps 1 to 4 a given L.P.P. in standard form as:

Optimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0 \,.\, S_1 + \ldots + 0 \,.\, S_m$

Subject to

$$\begin{aligned}
& a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \pm S_1 = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \pm S_2 = b_2 \\
& \quad \vdots \quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\
& a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \pm S_i = b_i \\
& \quad \vdots \quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\
& a_{m1}\,x_1 + a_{m2}x_2 + \ldots + a_{mn}\,x_n \pm S_m = b_m
\end{aligned}$$

and $x_1, x_2, x_3, \ldots, x_n, S_1, \ldots, S_m \geq 0$

where $b_1, b_2, \ldots, b_m \geq 0$

Since standard form involves only equations, we can write it in matrix form as follows:

$$\text{Optimize } Z = CX$$

Such that $AX = b$

and $X \geq 0$

Solved Examples

Example 1: *Express the following linear programming problem in standard form:*

Maximize $Z = 3x_1 + 4x_2 + 7x_3$

subject to

$$\begin{aligned}
2x_1 + 3x_2 - 2x_3 &\leq 30 \\
4x_1 - 2x_2 + x_3 &\leq 22 \\
x_1 - 5x_2 - 6x_3 &\geq 4
\end{aligned}$$

$x_1 \geq 0,\ x_2, x_3$ *unrestricted.*

[B.C.A. (Kanpur) 2004, 2007; B.C.A. (Agra) 2003]

Solution: Since x_2, x_3 are unrestricted, we express x_1, x_2 and x_3 as

$$\begin{aligned}
x_1 &= y_1 & y_1 &\geq 0 \\
x_2 &= y_2 - y_3, & y_2, y_3 &\geq 0 \\
x_3 &= y_4 - y_5, & y_4, y_5 &\geq 0
\end{aligned}$$

Thus, the given L.P.P. is

Maximize $Z = 3y_1 + 4(y_2 - y_3) + 7(y_4 - y_5)$

such that $2y_1 + 3(y_2 - y_3) - 2(y_4 - y_5) \le 30$

$$4y_1 - 2(y_2 - y_3) + (y_4 - y_5) \le 22$$

$$y_1 - 5(y_2 - y_3) - 6(y_4 - y_5) \ge 4$$

$$y_1, y_2, y_3, y_4, y_5 \ge 0$$

or Maximize $Z = 3y_1 + 4y_2 - 4y_3 + 7y_4 - 7y_5$

such that $2y_1 + 3y_2 - 3y_3 - 2y_4 + 2y_5 \le 30$

$$4y_1 - 2y_2 + 2y_3 + y_4 - y_5 \le 22$$

$$y_1 - 5y_2 + 5y_3 - 6y_4 + 6y_5 \ge 4$$

Introducing slack variables S_1, S_2 and surplus variables S_3, the required standard form is

Maximize $Z = 3y_1 + 4y_2 - 4y_3 + 7y_4 - 7y_5$

$$2y_1 + 3y_2 - 3y_3 - 2y_4 + 2y_5 + S_1 = 30$$

$$4y_1 - 2y_2 + 2y_3 + y_4 - y_5 + S_2 = 22$$

$$y_1 - 5y_2 + 5y_3 - 6y_4 + 6y_5 - S_3 = 4$$

and $y_1, y_2, y_3, y_4, y_5 \ge 0$

Example 2: *Reformulate the following L.P.P. into standard form:*

Minimize $Z = 2x_1 + 5x_2 + 4x_3$

subject to the constraints $-2x_1 + 4x_2 \le 4$

$$x_1 + 2x_2 + x_3 \ge 5$$

$$2x_1 + 3x_3 \le 3$$

$x_1, x_2 \ge 0$ *and* x_3 *unrestricted is sign.*

[B.C.A. (Lucknow) 2009; B.C.A. (Agra) 2005, 2009]

Solution: Since x_3 is unrestricted we can express x_3 as $x_3 = x_3' - x_3''$.

Then the given L.P.P. is

Minimize, $Z = 2x_1 + 5x_2 + 4x_3' - 4x_3''$

such that $-2x_1 + 4x_2 \le 4$

$$x_1 + 2x_2 + x_3' - x_3'' \ge 5$$

$$2x_1 + 3x_3' - 3x_3'' \le 3$$

$$x_1, x_2, x_3', x_3'' \ge 0$$

Introducing the slack S_1, S_3 and surplus variable S_2, we have the required standard form.

Minimize $Z = 2x_1 + 5x_2 + 4x_3' - 4x_3'' + 0 . S_1 - 0 . S_2 + 0 . S_3$

subject to the constraints

$$-2x_1 + 4x_2 + S_1 = 4$$

$$x_1 + 2x_2 + x_3' - x_3'' - S_2 = 5$$

$$2x_1 + 3x_3' - 3x_3'' + S_3 = 3$$

With non-negative conditions:

$$x_1 \geq 0, x_2 \geq 0, x_3' \geq 0, x_3'' \geq 0, S_1 \geq 0, S_2 \geq 0, S_3 \geq 0.$$

However, if the objective function is converted into maximization, then we have

Maximize $(-Z) = Z^* = -2x_1 - 5x_2 - 4x_3' + 4x_3'' - 0 . S_1 + 0 . S_2 - 0 . S_3$

2.1.3 Some Important Definitions Regarding the Solution of L.P.P. Using Standard Form/Canonical Form

1. The solution of an L.P.P. obtained from a canonical system by setting the **non-basic variables to zero** and solving for the basic variables is called a **Basic Solution.**

   For a general linear programming problem with n decision variables and m constraints, a basic solution may be found by setting $(n - m)$ of the decision variables equal to zero and solving for the remaining m variables. If a unique solution exists it is a basic solution.

2. A basic solution is said to be **basic feasible solution** (BFS) if all basic variables are non-negative.

3. A basic feasible solution is said to be **non-degenerate basic feasible solution** (N-*d* BFS) if the value of all basic variables are positive.

4. A basic feasible solution is said to be **degenerate basic feasible solution** (DBFS) if the values of one or more basic variables are zero.

5. A solution of L.P.P. is called a **Feasible solution** if it satisfies all the constraints and the non-negativity restrictions.

6. A basic feasible solutions is said to be optimum if it **optimizes** (maximizes or minimizes) the objective function.

7. If the value of the objective function Z can be increased or decreased indefinitely, then such solutions are called **unbounded solutions.**

2.1.4 Canonical Form of a General L.P.P.

A general L.P.P. can always be put in following form called the canonical form:

$$\text{Maximize } Z = \sum_{j=1}^{n} c_j x_j$$

subject to $$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \ldots, m$$

and $$x_j \ge 0, \quad \text{for } j = 1, 2, \ldots, n$$

The characteristics of canonical form are :

1. All decision variables are non-negative.
2. All constraints are of the ($\le$) type.

   An inequality in directions ($\ge$) can be changed to an inequality in the direction ($\le$) by multiplying with sides of the inequality by -1.
3. Objective function is of maximization type.

   The minimization of objective function Z is equivalent to the maximization of $(-Z)$.

Problem Set

1. What do you mean by a Linear Programming problem? Explain the procedure to convert a L.P.P. into standard form. **[B.C.A. (Purvanchal) 2005, 2007]**

Express the following L.P.P. in standard form:

2. Minimize $Z = 2x_1 + 3x_2$

subject to
$$2x_1 - 3x_2 \le 5$$
$$-x_1 + 3x_2 \ge -21$$
$$x_1 \ge 0, x_2 \text{ is unrestricted.}$$

3. Maximize $Z = 2x_1 + x_2 - 5x_3 + 3x_4$

subject to
$$3x_1 + 2x_2 \le 15$$
$$4x_1 + 5x_2 \ge 20$$
$$x_1 + x_2 - x_3 + 2x_4 = 10$$
$$2 \le 2x_1 + 4x_2 - x_3 \le 30$$
$$x_i \ge 0, \ i = 1, 2, 3, 4.$$

[B.C.A. (Meerut) 2007]

Answers

2.	Minimize $Z = 2x_1 + 3(x_3 - x_4) + 0 . S_1 + 0 . S_2$ subject to $2x_1 - 3(x_3 - x_4) + S_1 = 5$ $x_1 - 3(x_3 - x_4) + S_2 = 21$ (using $x_1 - 3x_2 \le 21$) $x_1, x_3, x_4, S_1, S_2 \ge 0$
3.	Maximize $Z = 2x_1 + x_2 - 5x_3 + 3x_4 + 0 . S_1 + 0 . S_2 + 0 . S_3 + 0 . S_4$ subject to $3x_1 + 2x_2 + S_1 = 15$ $4x_1 + 5x_2 - S_2 = 20$ $x_1 + x_2 - x_3 + 2x_4 = 10$ $2x_1 + 4x_2 - x_3 + S_3 = 30$ $2x_1 + 4x_2 - x_3 - S_4 = 2$ $x_j \ge 0,\ S_j \ge 0,\ j = 1, 2, 3, 4$

2.2 Procedure of the Simplex Method

[B.C.A. (Meerut) 2006]

The steps taken in the simplex method are as follows:

Step 1: Formulate the L.P.P. if given otherwise.

Step 2: Convert the L.P.P. in maximization problem if it is a minimization problem.

Step 3: Express the L.P.P. in standard form and then in canonical form. We use slack or surplus variables to convert the inequality to an equality.

Step 4: Start with an initial basic feasible solution by constructing a simplex table often called **Simplex tableau.**

When slack or surplus variables cannot provide the initial basic feasible solution, we use **artificial variables.**

Step 5: Check weather the initial basic feasible solution is optimum or not. If optimum stop, otherwise go to step 6.

Step 6: Decide the non-basic variable to enter the basic and the basic variable to be replaced by this non-basic variable using **minimum ration rule.**

Step 7: Obtain the current basic feasible solution. An adjacent basic feasible solution differ from the present basic feasible solution only in exactly one basic variable.

Test it for optimality. If optimum stop, otherwise go to step 8.

Step 8: Continue to find basic feasible solutions and test for optimality. When a particular basic feasible solution is found to be optimal the simplex method terminates.

2.2.1 Brief Explanation of Simplex Tableau 1 and the Various Factors Connected with its Construction

The form
$$a_{11}x_1 + a_{12}x_2 + 1 . x_3 + 0 . x_4 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + 0 . x_3 + 1 . x_4 = b_2$$

is often known as tableau form; its matrix contains two column form identity matrix. This is written prior to setting up the initial simplex tableau or simplex tableau 1.

Simplex Tableau 1

C_B	C_j / Basis	c_1 x_1	c_2 x_2	0 x_3	0 x_4	x_B	Ratio
. .	. . . .	a_{11} a_{21} Body Matrix X_1	a_{12} a_{22} X_2	1 0 Identity Matrix X_3	0 1 $X_4 \downarrow$	b_1 b_2	minimum (key row) . . .
	Z_j	Z_1	Z_2	Z_3	Z_4		
	$\Delta_j =$ $C_j - Z_j$	Δ_1	Δ_2 maximum (key column)	Δ_3	Δ_4		$Z = \ldots$

Remark: After the initial simplex tableau or (including this) the elements of coefficient matrix A are often denoted by y_{ij} which are identical to a_{ij} in the initial simplex tableau and obtained by transformation in the consecutive simplex tableau. The elements of the column X_B are often denoted as X_{B_i}.

1. **C_j Row:** The C_j row at the top of the simplex tableau is known as the objective row.

   It represents the coefficients of the decision variables, slack variables, surplus variables in the objective function. The row just below it represents the respective variables of the objective function.

2. **C_B Column:** The C_B column to the left of the simplex tableau is known as the objective column. It represents the contribution per unit of the feasible variables, *i.e.,* coefficients of those variables in the objective function which form the identity matrix.

3. **Basis Column:** The basis column represents the variables which are feasible for solution (*i.e.* basic variables whose coefficients are written in the column C_B. The variables in this column are so arranged that the row headed by a variable and the column headed by the same variables cross in the identity, *i.e.* where the 1 occurs.

4. **Body Matrix:** It is made up of the coefficients of the variable (which are not in the identity matrix) on LHS of the constraints. The elements of body matrix can be positive, negative and 0 as well.

5. **Identity Matrix :** It represents the matrix formed by the coefficients of feasible variables in LHS of the constraints which has 1 at the diagonal and 0 at other places.

6. **X_B Column:** This column is also known as **quantity column** which gives the values of the corresponding variables in basis column.

7. **Vector Row:** This row represents the vector corresponding to the variables in the Body Matrix and Identity Matrix.

8. **Index Row or Net Evaluation Row or Δ_j Row:** This row represents the relative profit coefficient Δ_j , Δ_j is obtained by a simplex formula :

$$\Delta_j = C_j - Z_j$$

$$= C_j - C_B' X_j', \text{ where } C_B' = \text{transpose of } C_B$$

$$= C_j - (\text{inner product of } C_B \text{ and } X_j \text{ in the canonical system}).$$

Note that Δ_j for basic variables are zero. Optimality is tested on the basic of Δ_j.

(i) If none of the columns in the index row shows a positive value (*i.e.*, the values of Δ_j are either 0 or negative) then the solution is optimal.

If none of Δ_j is positive, but any are zero, then other optimal solution exists with the same value of Z.

If all of Δ_j are negative, the solution under consideration is unique optimal solution.

(ii) If $\Delta_j > 0$ for some j, the solution under consideration is not optimal, so we proceed to next interation.

(iii) If corresponding to maximum positive Δ_j all elements in the column X_j are negative or zero, the solution under consideration will be **unbounded.**

9. **Key Column:** The column showing maximum positive Δ_j is indicated as the key column (or pivot column). The vector X_j corresponding to this column enters into the subsequent simplex tableau. This is marked with an upward arrow (↑).

10. **Index Column:** This column (the last column) is also known as ratio column. The various ratios in this column are found out by dividing each quantity in quantity column by their corresponding value in the key column.

11. **Key Row:** The row with the smallest (non-negative) ratio shown in the index column is indicated as the key row (or pivot row). The vector corresponding to the variable for which this ratio is minimum becomes out going vector and marked with a downward arrow (↓).

12. **Key Factor (or Key Element):** The number that lies at the intersection of the key column and the key row of a simplex tableau is known as key factor (or pivot factor). This number is reduced to unity to obtain the row values of the main row relating to entering vector of the next simplex tableau. The row of a subsequent tableau that replaces the key row of its immediately proceeding table is known as the main row.

Example 3: *A firm produces two types of products A_1 and A_2 through two machines P_1 and P_2. A_1 needs 2 hours of P_1 and 2 hours of P_2. A_2 needs 3 hours of P_1 and 1 hour of P_2. Machine P_1 can run at least for 24 hours per day. Machine P_2 can run at least 16 hours per day. If profit from A_1 and A_2 are ₹ 4 and ₹ 5 per unit respectively, ascertain by simplex method how many units of A_1 and A_2 should be produced to maximize the profit?* **[B.C.A. (Bhopal) 2006, 2011]**

Solution: **Step 1: (Formulation of L.P.P.):**

Let x_1 = number of units of Product A_1

x_2 = number of units of Product A_2

Then we have the L.P.P.:

$$\text{Maximize } Z = 4x_1 + 5x_2$$

subject to
$$2x_1 + 3x_2 \le 24$$
$$2x_1 + x_2 \le 16$$

and
$$x_1, x_2 \ge 0$$

Step 2: (Conversion of L.P.P. into Standard Form): Convert the constraint inequalities into equations with the introduction of slack variables S_1 and S_2 as follows:

Constraints
$$\begin{cases} 2x_1 + 3x_2 + S_1 = 24 \\ 2x_1 + x_2 + S_2 = 16 \end{cases}$$

$$\begin{cases} 2x_1 + 3x_2 + S_1 + 0\,.\,S_2 = 24 \\ 2x_1 + x_2 + 0\,.\,S_1 + S_2 = 16 \end{cases}$$

Consequently, the objective function becomes

$$Z = 4x_1 + 5x_2 + 0 \,.\, S_1 + 0 \,.\, S_2$$

Step 3: (Construction of Simplex Tableau 1): The presentation in this form is often known as tableau form, its matrix contains two columns to form identity matrix.

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Simplex Tableau 1

C_B	C_j / Basis	4 x_1	5 x_2	0 S_1	0 S_2	x_B	Ratio
0	S_1	2	$\boxed{3}$	1	0	24	$\frac{24}{3} = 8$ (min) key row
0	S_2	2	1	0	1	16	$\frac{16}{1} = 16$
		X_1	$X_2 \uparrow$	$S_1 \downarrow$	S_2		
	Z_j	0	0	0	0		
	Δ_j $= C_j - Z_j$	4	5 (maximum key column)	0	0		$Z = 0$

Initial basic feasible solution is $x_1 = 0, x_2 = 0, S_1 = 24, S_2 = 16$

$$Z = C_B{}' X_B = [0 \;\; 0] \begin{bmatrix} 24 \\ 16 \end{bmatrix} = 0 \times 24 + 0 \times 16 = 0$$

Step 4: (Test for Optimal Solution):

$$Z_1 = C_B{}' X_1 = [0 \;\; 0] \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0$$

$$Z_2 = C_B{}' X_2 = [0 \;\; 0] \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0$$

$$Z_3 = C_B{}' S_1 = [0 \;\; 0] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$Z_4 = C_B' S_2 = [0 \quad 0]\begin{bmatrix}0\\1\end{bmatrix} = 0$$

$$\Delta_j = C_j - Z_j = C_j - C_B' X_j$$
$$= C_j - [0 \quad 0] X_j$$
$$= C_j \text{ for all } j$$

Since there are some positive values in the Δ_j row, the initial basic feasible solution is not optimal.

Step 5: (To Find Incoming and Outgoing Vector): Δ_j is maximum for x_2. Hence, X_2 is the entering vector.

The ratio $\dfrac{X_B}{\text{Corresponding element in } X_2}$ is minimum for S_1. Hence, S_1 is the outgoing vector.

Step 6: (Construction of Tableau 2): The element 3 in X_2 is the pivot number or key element; hence it is to be reduced to 1 and the other element 1 in X_2 is to be reduced to zero.

The elements of the key row will become

	x_1	x_2	S_1	S_2	X_B
	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{1}{3}$	$\frac{0}{3}$	$\frac{24}{3}$
i.e. R_1	$\frac{2}{3}$	1	$\frac{1}{3}$	0	8

The element of the other row will be

	x_1	x_2	S_1	S_2	X_B
$R_2 = R_2 - R_1$	$2 - \frac{2}{3}$	$1 - 1$	$0 - \frac{1}{3}$	$1 - 0$	$16 - 8$
i.e.	$\frac{4}{3}$	0	$-\frac{1}{3}$	1	8

Simplex Tableau 2

C_B	C_j / Basis	4 x_1	5 x_2	0 S_1	0 S_2	X_B	Ratio
5	x_2	$\frac{2}{3}$	1	$\frac{1}{3}$	0	8	$8 \div \frac{2}{3} = 12$
0	S_2	$\boxed{\frac{4}{3}}$	0	$-\frac{1}{3}$	1	8	$8 \div \frac{4}{3} = 6$ (min) key row
		$X_1 \uparrow$	X_2	S_1	$S_2 \downarrow$		
	Z_j	$\frac{10}{3}$	5	$\frac{5}{3}$	0		
	$\Delta_j = C_j - Z_j$	$\frac{2}{3}$ (maximum key column)	0	$-\frac{5}{3}$	0		$Z = 40$

The current basic feasible solution is $x_1 = 0, x_2 = 8, S_1 = 0, S_2 = 8$

$$Z = C_B' \, X_B = [5 \quad 0]\begin{bmatrix} 8 \\ 8 \end{bmatrix} = 40$$

Step 7: (Test for Optimal Solution):

$$\Delta_j = C_j - Z_j = C_j - C_B' \, X_j$$

$$\Delta_1 = C_1 - C_B' \, X_1 = 4 - [5 \quad 0]\begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \end{bmatrix} = 4 - \frac{10}{3} = \frac{2}{3}$$

$$\Delta_3 = C_3 - C_B' \, X_3 = 0 - [5 \quad 0]\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = 0 - \frac{5}{3} = -\frac{5}{3}$$

Since Δ_1 is positive, the current feasible solution is not optimal.

Step 8: (To Find Entering and Outgoing Vectors): Since Δ_1 is maximum, the entering vector is X_1.

Since the ratio is minimum for S_2, the outgoing vector is S_2.

Step 9: (Construction of Tableau 3): The element $\frac{4}{3}$ in vector X_1 is the pivot number. Hence, it is to be reduced to zero.

The elements of the key row will become

	x_1	x_2	S_1	S_2	X_B
R_2	$\frac{4}{3} \div \frac{4}{3}$	$0 \div \frac{4}{3}$	$-\frac{1}{3} \div \frac{4}{3}$	$1 \div \frac{4}{3}$	$8 \div \frac{4}{3}$
R_2 *i.e.*	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	6

The elements of the other row will be

	x_1	x_2	S_1	S_2	X_B
$R_1 = R_1 - \frac{2}{3} R_2,$	$\frac{2}{3} - 1 \times \frac{2}{3}$	$1 - 0 \times \frac{2}{3}$	$\frac{1}{3} - \left(-\frac{1}{4} \times \frac{2}{3}\right)$	$0 - \frac{3}{4} \times \frac{2}{3}$	$8 - 6 \times \frac{2}{3}$
i.e.	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	4

Simplex Tableau 3

C_B	C_j / Basis	4 x_1	5 x_2	0 S_1	0 S_2	X_B	Ratio
5	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	4	No need
4	x_1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	6	
		X_1	X_2	S_1	S_2		
	Z_j	4	5	$\frac{3}{2}$	$\frac{1}{2}$		
$S_1 = 0$, $S_2 = 0$	$\Delta_j = C_j - Z_j$	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$		$Z = 44$

The current basic feasible solution is $x_1 = 6, x_2 = 4, S_1 = 0, S_2 = 0$

$$Z = C_B' X_B = [5 \quad 4]\begin{bmatrix}4\\6\end{bmatrix} = 20 + 24 = 44$$

Step 10: (Test for Optical Solution):

$$\Delta_j = C_j - Z_j = C_j - C_B' X_j$$

$$\Delta_3 = 0 - [5 \quad 4]\begin{bmatrix} \frac{1}{2} \\ \frac{-1}{4} \end{bmatrix} = 0 - \left(\frac{5}{2} - 1\right) = -\frac{3}{2}$$

$$\Delta_4 = 0 - [5 \quad 4]\begin{bmatrix} \frac{-1}{2} \\ \frac{3}{4} \end{bmatrix} = 0 - \left[-\frac{5}{2} + 3\right] = -\frac{1}{2}$$

Since all $\Delta_j \leq 0$, hence the current basic feasible solution is optimal with $x_1 = 6$, $x_2 = 4$ and $Z_{\max} =$ ₹ 44.

Example 4: *Solve the following linear programming problem by simplex method*

$$\text{Maximize } Z = 3x_1 + 2x_2$$

$$\text{subject to } \quad x_1 + x_2 \leq 4$$

$$x_1 - x_2 \leq 2$$

and $\quad x_1, x_2 \geq 0.$ **[B.C.A. (Meerut) 2011; B.C.A. (Agra) 2002, 2005]**

Solution: **Step 1: (Write L.P.P. in Standard Form):** For this all the right hand side constants should be positive. The inequality constraints are converted to equations by introducing the non-negative slack or surplus variables. The coefficients of slack and surplus variables are taken zero in the objective function. The given L.P.P. in standard form is

$$\text{Maximize } Z = 3x_1 + 2x_2 + 0\,.\,x_3 + 0\,.\,x_4$$

$$\text{subject to } \quad x_1 + x_2 + x_3 = 4 \qquad \ldots(1)$$

$$x_1 - x_2 + x_4 = 2 \qquad \ldots(2)$$

and $\quad x_1, x_2, x_3, x_4 \geq 0$

The tableau form of the L.P.P. is

$$x_1 + x_2 + x_3 + 0\,.\,x_4 = 4$$

$$x_1 - x_2 + 0\,.\,x_3 + x_4 = 2$$

which forms identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 2: The system of equations (1) and (2) is in canonical form, x_3 is basic variable in equation (1) and x_4 is basic variable in equation (2).

Step 3: Let C_j = coefficient of x_j in the objective function.

C_B = column vector of the coefficients of basic variables in the objective function.

X_j = column vector of the coefficients of x_j in the linear constraints.

X_B = Basic feasible solution.

For initial basic feasible solution, $C_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

and X_B = vector of the right hand side constants of the linear constraints = B.

Using the method of detached coefficients the Simplex Tableau 1 is as follows:

Simplex Tableau 1

C_B	Basis \ C_j	3 x_1	2 x_2	0 x_3	0 x_4	X_B	Ratio (to be computed in step 5)
0	x_3	1	1	1	0	4	$\frac{4}{1} = 4$
0	x_4	$\boxed{1}$	−1	0	1	2	$\frac{2}{1} = 2$ (minimum key row)
		$\uparrow X_1$	X_2	X_3	$\downarrow X_4$		
	Z_j	0	0	0	0		
$\Delta_j = C_j - Z_j$ To be computed in step 4		3 (maximum key row)	2	0	0		$Z = C_B' X_B = 0$

From the above table the initial basic feasible solution is

$$x_1 = 0, x_2 = 0, x_3 = 4, x_4 = 2$$

The value of objective function is given by

$$Z = 3 \times 0 + 2 \times 0 + 0 \times 4 + 0 \times 2 = 0$$

or Z = The inner product of the vectors C_B and X_B

$$= C_B' X_B = [0 \quad 0] \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 0 \times 4 + 0 \times 2 = 0$$

Step 4: (Test for Optimal Solution): To check if the initial basic feasible solution is optimal, calculate the relative profit coefficient Δ_j by a simple formula known as the **inner product rule.**

Relative profit coefficient of the variable x_j, denoted by Δ_j is given by

$$C_j - Z_j = C_j - \text{(inner product of } C_B \text{ and } X_j \text{ in the canonical system)}$$
$$= C_j - C_B' X_j$$

The relative profit coefficient Δ_j corresponding to the column of basic variables is always zero. So we calculate for non-basic variables. Thus

$$\Delta_1 = C_i - C_B' X_1 = 3 - [0, 0]\begin{bmatrix}1\\1\end{bmatrix} = 3 - 0 = 3$$

$$\Delta_2 = C_2 - C_B' X_2 = 2 - [0, 0]\begin{bmatrix}1\\-1\end{bmatrix} = 2,$$

similarly $\quad \Delta_3 = 0, \ \Delta_4 = 0.$

The initial B.F. solution is optimal (in case of maximization problem) if the relative profit coefficients of its non-basic variables are all negative or zero.

Since, there are some positive values in the Δ_j row, the initial basic feasible solution is not optimal.

Step 5: (Determine the Incoming Vector and Outgoing Vector): The non-basic variable x_1 gives the greatest value of Δ_j hence it will be chosen as the new basic variable to enter in the basis (*i.e.* vector X_1 is incoming vector).

In order to decide with basic variable is going to be replaced (*i.e.* which X_j. In mathematical form, we calculate the ratio

$$\left\{\frac{\text{Element of } X_B}{\text{Corresponding element of the Incoming Vector } X_j}\right\}$$

For each constraint row (in the last column of tableau) as follows :

Row Number	Basic Variable	Ratio (over X_1)
1	x_3	$\frac{4}{1} = 4$
2	x_4	$\frac{2}{1} = 2$ (minimum)

Since the minimum ratio is for x_4, so x_4 is the outgoing vector (*i.e.* second row is pivot row).

The element 1 denoted as [1] is the key number or pivot number.

Step 6: (Construction of Tableau 2): To have the vector X_1 in the basis, apply the operation $R_1 \to R_1 - R_2$ in tableau1 to make the vector X_1 as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to construct the tableau 2.

The key row in Tableau 1 becomes.

x_1	x_2	x_3	x_4	X_B
1	−1	0	1	2

in tableau 2.

The other row in tableau 2 is

	x_1	x_2	x_3	x_4	X_B
	1 − 1	1 − (− 1)	1 − 0	0 − 1	4 − 2
i.e.	0	2	1	− 1	2

Now, the system of equations in row 1 and row 2 is in canonical form, where x_3 is basic variable in row 1 and x_1 is basic variable in row 2.

Simplex Tableau 2

C_B	C_J / Basis	3 x_1	2 x_2	0 x_3	0 x_4	X_B	Ratio (to be computed in step 8)
0	x_3	0	$\boxed{2}$	1	− 1	2	$\frac{2}{2} = 1$ (min. key row)
3	x_1	1	− 1	0	1	2	$\frac{2}{-1}$ neglect
		X_1	$X_2 \uparrow$	$X_3 \downarrow$	X_4		
	Z_j	3	− 3	0	3		
Δ_j To be computed in step 7		0	5 (maximum key column)	0	− 3		$Z = C_B' X_B = 6$

Here, the current basic feasible solution is

$$x_2 = 0, x_4 = 0 \text{ and } \begin{cases} 0 \,.\, x_1 + 1 \,.\, x_3 = 2 \Rightarrow x_3 = 2 \\ 1 \,.\, x_1 + 0 \,.\, x_3 = 2 \Rightarrow x_1 = 2 \end{cases}$$

(non-basic variable)

The value of the objective function is given by the inner product of the vectors C_B and X_B:

$$C_B' \, X_B = [0 \quad 3] \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0 + 6 = 6$$

$$[\text{or } Z = 3 \times 2 + 2 \times 0 + 0 \times 2 + 0 \times 0 = 6]$$

Step 7: (Test for Optimal Solution):

$$\Delta_1 = 0$$

$$\Delta_2 = C_2 - Z_2 = 2 - [0 \quad 3] \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 + 3 = 5$$

$$\Delta_3 = 0$$

$$\Delta_4 = C_4 - Z_4 = 0 - [0 \quad 3] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0 - 3 = -3$$

Since there is one positive value of Δ_j the current basic feasible solution is not optimal.

Step 8: (Determine Incoming Vector and Outgoing Vector): The non-basic variable x_2 gives the greatest value of Δ_j hence it will be chosen as the new basic variable to enter the basis (*i.e.* X_2 is the incoming vector).

In order to decide which basic variables is going to be replaced (*i.e.* which $X_{j,}\ j = 1, 4$ is outgoing vector) we use minimum ratio rule. In mathematical form we calculate the ratio:

$$\left\{ \frac{\text{Element of } X_B}{\text{Corresponding element of the Incoming Vector } X_j} \right\}$$

For each constraint row as follows:

Row Number	**Basic Variable**	**Ratio (over X_2)**
1	x_3	$\frac{2}{2} = 1$ (minimum)
2	x_1	$\frac{2}{-1} = -2$ (negative value is never considered)

The ratio is minimum for x_3 so the vector X_3 is the outgoing vector. The element 2 denoted as $\boxed{2}$ is the key number of pivot number.

Step 9: (Construction of Tableau 3): To have the vector X_2 in the basis, divide first row by 2 in tableau 2 and apply the operation $R_2 = R_2 + R_1$ to make the vector $X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The key row in tableau 2 becomes

	x_1	x_2	x_3	x_4	X_B
	$0 \div 2$	$2 \div 2$	$1 \div 2$	$-1 \div 2$	$2 \div 2$
i.e.	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1

in tableau 3.

The other row in tableau 3 will be

	x_1	x_2	x_3	x_4	X_B
	$1+0$	$-1+1$	$0+\frac{1}{2}$	$1-\frac{1}{2}$	$2+1$
i.e.	1	0	1/2	1/2	3

Simplex Tableau 3

C_B	C_j / Basis	3 x_1	2 x_2	0 x_3	0 x_4	X_B	Ratio
2	x_2	0	1	1/2	–1/2	1	—
3	x_1	1	0	1/2	1/2	3	—
		X_1	X_2	X_3	X_4		
	Z_j	3	2	5/2	1/2		
$\Delta_j = C_j - Z_j$ (to be computed in step 10)		0	0	–5/2	–1/2	$Z = C_B' X_B = 11$	

Form this table the current basic feasible solution is $x_3 = 0 = x_4$ and $0 \,.\, x_1 + 1 \,.\, x_2 = 1 \Rightarrow x_2 = 1$.

(non-basic variable) $1 \,.\, x_1 + 0 \,.\, x_3 = 3 \Rightarrow x_1 = 3$

The values of objective function is given by the inner product of the vectors C_B and X_B.

$$C_B' X_B = [2 \quad 3]\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2 + 9 = 11$$

$$[\text{or} \quad Z = 3 \times 3 + 2 \times 1 + 0 \times 0 \times 0 + 0 = 9 + 2 = 11]$$

Step 10: (Test for Optimal Solution):

$$\Delta_1 = 0$$

$$\Delta_2 = 0$$

$$\Delta_3 = C_3 - Z_3 = 0 - [2 \quad 3]\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 0 - \frac{5}{2} = -\frac{5}{2}$$

$$\Delta_4 = C_4 - Z_4 = 0 - [2 \quad 3]\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = 0 - \frac{1}{2} = -\frac{1}{2}$$

Since there is no positive value in the Δ_j row, the current basic feasible solution is optimal.

Thus, the optimal solution is $x_1 = 3, \; x_2 = 1$

Remark 1: The computation in simplex method may be shown in a single table as given below (without giving an expression in words):

C_B	Basis \ C_j	3 x_1	2 x_2	0 x_3	0 x_4	X_B	Ratio
0	x_3	1	1	1	0	4	$\frac{4}{1} = 4$
0	x_4	$\boxed{1}$	−1	0	1	2	$\frac{2}{1} = 2$ (min.)
		$X_1 \uparrow$	X_2	X_3	$\downarrow X_4$		
$\Delta_j = C_j - Z_j$ $= C_j - C_B' X_j$		3 (max.)	2	0	0	$x_1 = 0$ $x_2 = 0$	$Z = C_B' X_B = 0$
0	x_3	0	$\boxed{2}$	1	−1	2	$\frac{2}{2} = 1$ (min)
3	x_1	1	−1	1	1	2	$\frac{2}{-1} = -1$ (neglected)
Δ_j		0	$5\uparrow$ (max.)	$0\downarrow$	−3	$x_3 = 0$ $x_4 = 0$	$Z = C_B' X_B = 6$
2	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1	No need
3	x_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	3	
Δ_j		0	0	$-\frac{5}{2}$	$-\frac{1}{2}$	$x_3 = 0$ $x_4 = 0$	$Z = C_B' X_B = 11$

Thus, optimum solution: $x_1 = 3, x_2 = 1$, with maximum $Z = 11$.

Remark 2: Some authors use $\Delta_j = Z_j - C_j = C_B' X_j - C_j$

To check whether the basic feasible solution is optimal (maximum), we proceed as follows:

1. If all $\Delta_j \geq 0$, the basic feasible solution is **optimal.**

2. If at least one Δ_j is negative, the basic feasible solution is **not optimal** then proceed the next step.

3. If corresponding to most negative Δ_j, all elements of the column X_j are negative or zero (≤ 0), then basic feasible solution will be **unbounded.**

4. To determine incoming vector use the minimum Δ_j.

5. To find outgoing vector use the minimum ratio.

Remark 3: However, some author use

$$\Delta_j = C_j - Z_j = C_j - C_B' X_j$$

So, the students are advised to be careful while going through various notations.

Example 5: *Maximize* $Z = 3x_1 + 2x_2$

subject to $$2x_1 + x_2 \leq 5$$

$$x_2 + x_2 \leq 3$$

and $$x_1, x_2 \geq 0.$$

[B.C.A. (Rohilkhand) 2006, 2010, 2012]

Solution: Introducing x_3, x_4 as slack variables, we have

$$\text{Maximize } Z = 2x_1 + 2x_2 + 0 \,.\, x_3 + 0 \,.\, x_4$$

subject to $$2x_1 + x_2 + x_3 + 0 \,.\, x_4 = 5$$

$$x_1 + x_2 + 0 . x_3 + x_4 = 3$$

and $$x_1, x_2, x_3, x_4 \geq 0.$$

With usual notation, the computation in a single table without any explanation is as follows:

C_B	Basis \ C_j	3 x_1	2 x_2	0 x_3	0 x_4	X_B	Ratio
0	x_3	$\boxed{2}$	1	1	0	5	$\frac{5}{2}$ (min)
0	x_4	1	1	0	1	3	3
		$\uparrow X_1$	X_2	$X_3 \downarrow$	X_4		
	Z_j	0	0	0	0		
$\Delta_j = Z_j - C_j$		-3 (most negative)	-2	0	0		$Z = C_B' X_B = 0$
3	x_1	1	1/2	1/2	0	5/2	$5/2 \div \frac{1}{2} = 5$
0	x_4	0	$\boxed{1/2}$	$-1/2$	1	1/2	$\frac{1}{2} \div \frac{1}{2} = 1$ (Min.)
		X_1	$\uparrow X_2$	X_3	$\downarrow X_4$		
	Z_j	3	3/2	3/2	0		
$\Delta_j = Z_j - C_j$		0	$-1/2$ (most negative)	3/2	0		$Z = C_B' X_B = \frac{15}{2}$
3	x_1	1	0	1	-1	2	No need
2	x_2	0	1	-1	2	1	
		X_1	X_2	X_3	X_4		
	Z_j	3	2	1	1		
$D_j = Z_j - C_j$		0	0	1	1		$Z = C_B' X_B = 8$

the required optimal solution is $x_1 = 2 \;\; x_2 = 1, Z_{\max} = 8$.

Example 6: *Solve the following problem by simplex method:*

$$Max\ (Z) = 3x_1 + 5x_2 + 4x_3$$

subject to $2x_1 + 3x_2 \le 8$

$$0x_1 + 2x_2 + 5x_3 \le 10$$

$$3x_1 + 2x_2 + 4x_3 \le 15$$

and $x_1, x_2, x_3 \ge 0.$ **[B.C.A. (Meerut) 2004, 2009, 2012]**

Solution: L.P.P. in standard form.

$$\text{Max } (Z) = 3x_1 + 5x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6$$

subject to

$$2x_1 + 3x_2 + 0x_3 + x_4 + 0x_5 + 0x_6 = 8$$

$$0x_1 + 2x_2 + 5x_3 + 0x_4 + x_5 + 0x_6 = 10$$

$$3x_1 + 2x_2 + 4x_3 + 0x_4 + 0x_5 + x_6 = 15$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

Simplex Tableau 1

C_B	Basis \ C_j	3	5	4	0	0	0	X_B	Ratio
		x_1	x_2	x_3	x_4	x_5	x_6		
0	x_4	2	$\boxed{3}$	0	1	0	0	8	$\frac{8}{3} = 2.6$ (min.)
0	x_5	0	2	5	0	1	0	10	$\frac{10}{2} = 5$
0	x_6	3	2	4	0	0	1	15	$\frac{15}{2} = 7.5$
		X_1	$X_2 \uparrow$	X_3	$X_4 \downarrow$	X_5	X_6	$x_1 = 0$	
$\Delta_j = Z_j - C_j$		-3	-5 (min.)	-4	0	0	0	$x_2 = 0$ $x_3 = 0$	$Z = C_B' X_B = 0$

$$\Delta_1 = C_B' X_1 - C_1 = 0 - 3 = -3$$

$$\Delta_2 = C_B' X_2 - C_2 = 0 - 5 = -5$$

$$\Delta_3 = C_B' X_3 - C_3 = 0 - 4 = -4$$

Since all the Δ_j are not ≥ 0, the initial basic feasible solution ($x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 8, x_5 = 10, x_6 = 15$) is not optimal.

Since Δ_2 is minimum, hence X_2 is the entering or incoming vector.

Since the ratio $\frac{X_B}{X_2}$ is minimum for x_4, the departing or outgoing vector is X_4.

To convert the vector X_2 as $\begin{bmatrix}1\\0\\0\end{bmatrix}$, we apply operations

$R_1 \to \frac{1}{3}R_1, R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$ successively and have.

Simplex Tableau 2

C_B	C_j / Basis	3 x_1	5 x_2	4 x_3	0 x_4	0 x_5	0 x_6	X_B	Ratio
5	x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	8/3	$\frac{8}{3}\times\frac{1}{0}=\infty$
0	x_5	$-\frac{4}{3}$	0	$\boxed{5}$	$-\frac{2}{3}$	1	0	14/3	$\frac{14}{3}\times\frac{1}{5}=\frac{14}{15}$ (min.)
0	x_6	$\frac{5}{3}$	0	4	$-\frac{2}{3}$	0	1	29/3	$\frac{29}{3}\times\frac{1}{4}=\frac{29}{12}$
		X_1	X_2	$X_3\uparrow$	X_4	$X_5\downarrow$	X_6	$x_1=0$	
$\Delta_j = Z_j - C_j$		1/3	0	-4	5/3	0	0	$x_2=0$ $x_3=0$	$Z = C_B'X_B = \frac{40}{3}$

$$\Delta_j = C_B'X_1 - C_1 = \frac{10}{3} - 3 = \frac{1}{3}$$

$$\Delta_3 = C_B'X_3 - C_3 = 0 - 4 = -4$$

$$\Delta_4 = C_B'X_4 - C_4 = \frac{5}{3} - 0 = \frac{5}{3}$$

Since Δ_3 is negative, the current basic feasible solution $x_1 = 0, x_2 = \frac{8}{3}, x_3 = 0, x_4 = 0,$ $x_5 = \frac{14}{3}\ x_6 = \frac{29}{3}$ is not optimal.

Since Δ_3 is the minimum net evaluation, hence X_3 is the entering or incoming vector.

Since the ratio $\frac{x_B}{X_3}$ is minimum for x_5, the departing or outgoing vector is x_5.

To convert the vector X_3 as $\begin{bmatrix}0\\1\\0\end{bmatrix}$, we apply operations

$R_2 \to \frac{1}{5}R_2 R_1 \to R_1,\ R_3 \to R_3 - 4R_2$ successively and have.

Simplex Tableau 3

C_B	C_j / Basis	3 x_1	5 x_2	4 x_3	0 x_4	0 x_5	0 x_6	X_B	Ratio
5	x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	8/3	$\frac{8}{3}\times\frac{3}{2}=4$
4	x_3	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	$\frac{1}{5}$	0	14/15	negative
0	x_6	$\boxed{\frac{41}{15}}$	0	0	$-\frac{2}{15}$	$-\frac{4}{5}$	1	89/15	$\frac{89}{15}\times\frac{15}{41}=\frac{89}{41}$
		$X_1\uparrow$	X_2	X_3	X_4	X_5	$X_6\downarrow$	$x_1=0$	(min.)
$\Delta_j = Z_j - C_j$		−11/15 (min.)	0	0	17/15	4/5	0	$x_2=0$ $x_3=0$	$Z = C_B' X_B = \frac{256}{15}$

$$\Delta_1 = C_B' X_1 - C_1 = \left(\frac{10}{3} - \frac{16}{15}\right) - 3 = \frac{24}{15} - 3 = -\frac{11}{15}$$

$$\Delta_4 = C_B' X_4 - C_4 = \left(\frac{5}{3} - \frac{8}{15}\right) - 0 = \frac{17}{15} - 0 = \frac{17}{15}$$

$$\Delta_5 = C_B' X_5 - C_5 = \frac{4}{5} - 0 = \frac{4}{5}$$

Since Δ_1 is negative, the current basic feasible solution

$$\left[x_1 = 0, x_4 = 0, x_5 = 0, x_2 = \frac{8}{3}, x_3 = \frac{14}{15}, x_6 = \frac{89}{15}\right]$$

is not optimal since Δ_1 is the minimum net evaluation, hence X_1 is the entering or incoming vector.

Since the ratio $\frac{X_B}{X_1}$ is minimum for X_6, the departing or outgoing vector is X_6.

To convert the vector X_1 as $\begin{bmatrix}0\\0\\1\end{bmatrix}$ we apply operations

$$R_3 \to \frac{15}{41} R_1 \to R_1 - \frac{2}{3} R_3, R_2 \to R_2 + \frac{4}{15} R_3$$

successively and have.

Simplex Tableau 4

C_B	C_j / Basis	3 x_1	5 x_2	4 x_3	0 x_4	0 x_5	0 x_6	X_B	Ratio
5	x_2	0	1	0	$\frac{15}{41}$	$\frac{8}{41}$	$-\frac{10}{41}$	$\frac{50}{41}$	No need
4	x_3	0	0	1	$-\frac{6}{41}$	$\frac{5}{41}$	$\frac{4}{41}$	$\frac{62}{41}$	
3	x_1	1	0	0	$-\frac{2}{41}$	$-\frac{12}{41}$	$\frac{15}{41}$	$\frac{89}{41}$	
$\Delta_j = Z_j - C_j$		X_1	X_2	X_3	X_4	X_5	X_6	$x_4 = 0$	$Z = C_B' X_B$
		0	0	0	$\frac{45}{41}$	$\frac{24}{41}$	$\frac{11}{41}$	$x_5 = 0$	$= \frac{765}{41}$
								$x_6 = 0$	

$$\Delta_4 = C_B' X_4 - C_4 = [543]\begin{bmatrix} \frac{15}{41} \\ -\frac{6}{41} \\ \frac{-2}{41} \end{bmatrix} - 0 = \frac{75}{41} - \frac{24}{41} - \frac{60}{41} = \frac{45}{41}$$

$$\Delta_5 = C_B' X_5 - C_5 = [543]\begin{bmatrix} \frac{8}{41} \\ \frac{5}{41} \\ -\frac{12}{41} \end{bmatrix} - 0 = \frac{40}{41} + \frac{20}{41} - \frac{30}{41} = \frac{24}{41}$$

$$\Delta_6 = C_B' X_6 - C_6 = [543]\begin{bmatrix} -\frac{10}{41} \\ \frac{4}{41} \\ \frac{15}{41} \end{bmatrix} - 0 = -\frac{50}{41} + \frac{16}{41} + \frac{45}{41} = \frac{11}{41}$$

Since all Δ_j are ≥ 0, the current basic feasible solution

$$\left[x_1 = \frac{89}{41}, x_2 = \frac{50}{41}, x_3 = \frac{62}{41}, x_4 = 0, x_5 = 0, x_6 = 0\right]$$

is optimal with maximum $Z = C_B' X_B$

$$= [5 \;\; 4 \;\; 3]\begin{bmatrix} \frac{50}{41} \\ \frac{62}{41} \\ \frac{84}{41} \end{bmatrix} = \frac{250}{41} + \frac{248}{41} + \frac{267}{41} = \frac{765}{41}$$

Example 7: *Solve the following linear programming problem by simplex method.*

Minimize $Z = x_1 - 3x_2 + 2x_3$

subject to $3x_1 - x_2 + 3x_3 \le 7$

$-2x_1 + 4x_2 \le 12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

and $x_1, x_2, x_3 \ge 0.$ **[B.C.A. (Meerut) 2002, 2009]**

Solution: Convert the objective function into maximization form:

$$\text{Maximize } Z^* = -x_1 + 3x_2 - 2x_3, \text{ where } Z^* = -Z$$

The L.P.P. in standard form (adding slack variables x_4, x_5 and x_6).

$$\text{Maximize } Z^* = -x_1 + 3x_2 - 2x_3 + 0 \,.\, x_4 + 0 \,.\, x_5 + 0 \,.\, x_6$$

subject to

$$3x_1 - x_2 + 3x_3 + x_4 + 0 \,.\, x_5 + 0 \,.\, x_6 = 7$$

$$-2x_1 + 4x_2 + 0 \,.\, x_3 + 0 \,.\, x_4 + x_5 + 0 \,.\, x_6 = 12$$

$$-4x_1 + 3x_2 + 8x_3 + 0 \,.\, x_4 + 0 \,.\, x_5 + x_6 = 10$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

with usual notation, we have

Simplex Tableau 1

C_B	C_j / Basis	**– 1**	**3**	**– 2**	**0**	**0**	**0**	X_B	Ratio
		x_1	x_2	x_3	x_4	x_5	x_6		
0	x_4	3	– 1	3	1	0	0	7	$\frac{7}{-1} = -7$ (neg.)
0	x_5	– 2	$\boxed{4}$	0	0	1	0	12	$\frac{12}{4} = 3$ (min.)
0	x_6	– 4	3	8	0	0	1	10	$\frac{10}{3} = 3.3$
$\Delta_j = Z_j - C_j$		X_1	$\uparrow X_2$	X_3	X_4	$\downarrow X_5$	X_6	$x_1 = 0$	
		– 1	3 (max.)	– 2	0	0	0	$x_2 = 0$	$Z_* = 0, Z = 0$
								$x_3 = 0$	

Since there are some positive values of Δ_j the initial basic feasible solution $(x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 7, x_5 = 12, x_6 = 10)$ is not optimal.

Since C_2 is maximum so, X_2 is the incoming vector

Since ratio is minimum in second row so X_5 is the outgoing vector.

To make the vector X_2 as $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ apply the operations $R_2 \to \frac{1}{4} R_2,\ R_1 \to R_1 + R_2,$ $R_3 \to R_3 - 3R_2$ successively, then we have

Simplex Tableau 2

C_B	Basis \ C_j	–1	3	–2	0	0	0	X_B	Ratio
		x_1	x_2	x_3	x_4	x_5	x_6		
0	x_4	$\boxed{5/2}$	0	3	1	1/4	0	10	$10 \div \frac{5}{2} = 4$ (minimum)
3	x_2	–1/2	1	0	0	1/4	0	3	$3 \div \left(-\frac{1}{2}\right) = -6$ (negative)
0	x_6	–5/2	0	8	0	–3/4	1	1	$1 \div \left(-\frac{5}{2}\right) = -\frac{5}{2}$ (negative)
$\Delta_j = C_j - Z_j$		$X_1 \uparrow$	X_2	X_3	$\downarrow X_4$	X_5	X_6	$x_1 = 0$	
		1/2 (max.)	0	–2	0	–3/4	0	$x_3 = 0$ $x_5 = 0$	$Z^* = 9, Z = -9$

Since $\Delta_1 = \frac{1}{2}$, the current basic feasible solution

$$[x_1 = 0, x_3 = 0, x_5 = 0, x_4 = 10, x_2 = 3, x_6 = 1] \text{ is not optimum.}$$

Since Δ_1 is maximum, so X_1 is the incoming vector, since ratio is minimum in first row so X_4 is the outgoing vector.

To make the vector X_1 as $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ apply the operations

$R_1 \to \frac{2}{5} R_1, R_2 \to R_2 + \frac{1}{2} R_1, R_3 \to R_3 + \frac{5}{2} R_1$ successively; then we have

Simplex Tableau 3

C_B	Basis \ C_j	−1 x_1	3 x_2	−2 x_3	0 x_4	0 x_5	0 x_6	X_B	Ratio
−1	x_1	1	0	6/5	2/5	1/10	0	4	No Need
3	x_2	0	1	3/5	1/5	3/10	0	5	
0	x_6	0	0	11	1	−1/2	1	11	
		X_1	X_2	X_3	X_4	X_5	X_6	$x_3=0$	
$\Delta_j = C_j - Z_j$		0	0	$-\frac{13}{5}$	$-\frac{1}{5}$	$-\frac{8}{10}$	0	$x_4=0$, $x_5=0$	$Z^*=11$, $Z=-11$

$$\Delta_3 = -2 - [-1 \quad 3 \quad 0]\begin{bmatrix} 6/5 \\ 3/5 \\ 11 \end{bmatrix} = -2 - \left(-\frac{6}{5} + \frac{9}{5} + 0\right) = -2 - \left(\frac{3}{5}\right) = -\frac{13}{5}$$

$$\Delta_4 = 0 - [-1 \quad 3 \quad 0]\begin{bmatrix} 2/5 \\ 1/5 \\ 1 \end{bmatrix} = 0 - \left(-\frac{2}{5} + \frac{3}{5} + 0\right) = -\frac{1}{5}$$

$$\Delta_5 = 0 - [-1 \quad 3 \quad 0]\begin{bmatrix} 1/10 \\ 3/10 \\ -1/2 \end{bmatrix} = 0 - \left(-\frac{1}{10} + \frac{9}{10} + 0\right) = -\frac{8}{10}$$

Since all the Δ_j are ≤ 0, hence the current basic feasible solution $(x_1 = 4, x_2 = 5, x_6 = 11, x_3 = 0, x_5 = 0, x_4 = 0)$ is optimal. Thus, $x_1 = 4, x_2 = 5$ and $x_3 = 0$ with minimum $Z = -11$.

Problem Set

Solve the following problems using simplex method:

1. Maximize $Z = 5x_1 + 3x_2$

subject to $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

and $x_1, x_2 \ge 0$ **[B.C.A. (Rohilkhand) 2003]**

2. Maximize $Z = 3x_1 + 2x_2 + 5x_3 + 1000$

subject to $x_1 + 2x_2 + x_3 \le 430$

$3x_1 + 2x_2 \le 460$

$x_1 + 4x_2 \le 420$

$x_1, x_2, x_3 \ge 0$

3. Maximize $Z = 3x_1 + 5x_2 + 4x_3$

subject to $2x_1 - x_2 \le 8$

$2x_2 + 5x_3 \le 10$

$3x_1 + 2x_2 + 4x_3 \le 15$

$x_1 \ge 2, x_2 \ge 4, x_3 \ge 0$ **[B.C.A. (Lucknow) 2010]**

4. Maximize $Z = 2x_1 + 5x_2 + 7x_3$

subject to $3x_1 + 2x_2 + 4x_3 \le 100$

$x_1 + 4x_2 + 2x_3 \le 100$

$x_1 + x_2 + 3x_3 \le 100$

$x_1, x_2, x_3 \ge 0$

5. Maximize $Z = 4x_1 + x_2 + 3x_3 + 5x_4$

subject to $4x_1 - 6x_2 - 5x_3 + 4x_4 \ge -20$

$3x_1 - 2x_2 + 4x_3 + x_4 \le 10$

$8x_1 - 3x_2 + 3x_3 + 2x_4 \le 20$

and $x_1, x_2, x_3, x_4 \ge 0$

6. Minimize $Z = x_2 - 3x_3 + 2x_5$

subject to
$$3x_2 - x_3 + 2x_5 \le 7$$
$$-2x_2 + 4x_3 \le 12$$
$$-4x_2 + 3x_3 + 8x_5 \le 10$$
and $x_2, x_3, x_5 \ge 0$ **[B.C.A. (Kanpur) 2009]**

7. Maximize $Z = 2x + 5y$

subject to
$$x + y \le 600$$
$$0 \le x \le 400$$
$$0 \le y \le 30$$
[B.C.A. (Purvanchal) 2009; B.C.A. (Kanpur) 2008]

8. Min $(Z) = x_1 - 3x_2 + 2x_3$

subject to
$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 4x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

9. Min $(Z) = 2x_1 + 4x_2 + x_3 + x_4$

subject to
$$x_1 + 3x_2 + x_4 \le 4$$
$$2x_1 + x_2 \le 3$$
$$x_2 + 4x_3 + x_4 \le 3$$
$$x_1, x_2, x_3, x_4 \ge 0$$
[B.C.A. (Aligarh) 2008, 2012]

10. Max $(Z) = 5x_1 + 3x_2$

subject to
$$3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$x_1, x_2 \ge 0$$
[B.C.A. (Avadh) 2011, B.C.A. (Kanpur) 2007]

11. Max $(Z) = 3x_1 + 5x_2$

subject to
$$3x_1 + 2x_2 \le 18$$
$$x_1 \le 4$$
$$x_2 \le 6$$
$$x_1, x_2 \ge 0$$
[B.C.A. (Kashi) 2004, 2012]

12. Max $(Z) = 4x_1 + 10x_2$

subject to
$$2x_1 + x_2 \le 50$$
$$2x_1 + 5x_2 \le 100$$
$$2x_1 + 3x_2 \le 90$$
$$x_1, x_2 \ge 0$$
[B.C.A. (Agra) 2003, 2007]

13. Minimize $(Z) = x_1 + x_2 + 3x_3$

subject to
$$3x_1 + 2x_2 + x_3 \le 3$$
$$2x_1 + x_2 + 2x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

14. Write short notes on the simplex method. **[B.C.A. (Meerut) 2007, 2011]**

15. Give the brief explanation of simplex method. **[B.C.A. (Bhopal) 2008, 2010, 2012]**

16. How do the graphical and simplex methods of solving a L.P.P.? **[B.C.A. (Meerut) 2008, 2009, 2011]**

17. Define the following :

(i) Optimal solution

(ii) Surplus variable

(iii) Basic feasible solution **[B.C.A. (Meerut) 2008, 2011]**

(iv) Slack and surplus variable **[B.C.A. (Meerut) 2004, 2006]**

18. Write notes on the following terms as used in a L.P.P.:

(i) Objective function

(ii) Surplus variables

(iii) Basic feasible solution **[B.C.A. (Meerut) 2006, 2010]**

19. Give an account of linear programming problem with its important components. What does the non-negativity restriction mean?

[B.C.A. (Kanpur) 2006; B.C.A. (Meerut) 2003, 2005, 2010]

Answers

1.	$x_1 = \frac{20}{19}, x_2 = \frac{45}{19}, Z_{\max} = \frac{235}{19}$
2.	$x_1 = 0, x_2 = 100, x_3 = 230, Z_{\max} = 1350 + 1000 = 2350$
3.	$x_1 = \frac{89}{41}, x_2 = \frac{50}{41}, x_3 = \frac{62}{41}, \text{Max}(Z) = \frac{765}{41}$
4.	$x_1 = 0, x_2 = \frac{50}{3}, x_3 = \frac{50}{3}, Z_{\max} = 200$
5.	The solution is unbound.
6.	$x_2 = 4, x_3 = 5, x_5 = 0, Z_{\min} = -11$
7.	$x = 300, y = 300, Z_{\max} = 21$
8.	$x_1 = 4, x_2 = 5, x_3 = 0,$ Min $(Z) = -11$
9.	$x_1 = 1, x_2 = 1, x_3 = \frac{1}{2}, x_4 = 0, \text{Max}(Z) = \frac{13}{2}$
10.	$x_1 = \frac{20}{19}, x_2 = \frac{45}{19}, \text{Max}(Z) = \frac{235}{19}$
11.	$x_1 = 2, x_2 = 6, \text{Max}(Z) = 36$
12.	$x_1 = 0, x_2 = 20, \text{Max}(Z) = 200$
	or $x_1 = \frac{75}{4}, x_2 = \frac{25}{2}, \text{Max}(Z) = 200$
13.	$x_1 = x_2 = x_3 = 0, \text{Min}(Z) = 0$

2.3 Linear Programming Problems Big 'M' Method and Two Phase Method

2.3.1 Use of Artificial Variables

The solution in case of constraints of the (= or ≥) type we introduce non-negative variables to the left hand side of all such constraints. The variables are called **artificial variables.** The artificial variables play the same role as that of slack variables in providing an initial basic feasible solution. Use of artificial variables is a systematic way of getting a canonical system with a basic feasible solution when none is available by inspection.

There are two methods to solve such problems:

1. The M-methods or Big M-method or Charne's method of penalty
2. The **two** phase method.

2.3.2 Method of Penalties

When slack variables do not provide the initial basic feasible solution, we use artificial variables. That is in order to complete the identity matrix, we introduce artificial variables along with slack and/or surplus variables.

As the coefficients of slack variables are taken as 0 in the objective function, we assign a very **high penalty cost** (say $-M, M \geq 0$) to the artificial variables in the objective function (that is their coefficients are taken as $-M$ in case of maximization) and we assign a very high penalty cost (say, $M, M \geq 0$) in case of minimization so that they may be driven to zero while reaching optimality. Then L.P.P. is solved as usual by simplex method. The method of solving a linear programming problem in which a high penalty cost has been assigned to the artificial variables is known as the **Charne's Method** of **Penalties** of **Big -M Method.** If the artificial variable introduced in L.P.P. is present in the final tableau at zero level the solution so obtained is optimal. Otherwise the solution is known as **Pseudo-Optimal solution.**

(Big – 'M' Method)

Example 8: *Solve the following L.P.P. by simplex method.*

$$\text{Minimize } Z = 3x_1 + 2.5x_2$$

such that

$$x_1 + 2x_2 \geq 20$$
$$3x_1 + 2x_2 \geq 50$$
$$x_1, x_2 \geq 0.$$

Solution: Standard form

$$\text{Minimize } Z = 3x_1 + 2.5x_2 + 0 . S_1 + 0 . S_2$$

such that

$$x_1 + 2x_2 - 1. S_1 + 0. S_2 = 20$$
$$3x_1 + 2x_2 + 0. S_1 - 1. S_2 = 50$$

and

$$x_1, x_2, S_1 . S_2 \geq 0$$

Here, identity matrix is not complete so we use artificial variables a_1, a_2. The coefficients of a_1 and a_2 are taken as M in the objective function Z. Thus, the L.P.P. is

$$\text{Minimize } Z = 3x_1 + 2.5x_2 + 0. S_1 + 0. S_2 + Ma_1 + Ma_2$$

such that

$$x_1 + 2x_2 - 1. S_1 + 0. S_2 + 1. a_1 + 0. a_2 = 20$$
$$3x_1 + 2x_2 + 0. S_1 - 1. S_2 + 0. a_1 + 1. a_2 = 50$$

and

$$x_1, x_2, S_1, S_2, a_1, a_2 \geq 0$$

Simplex Tableau 1

C_B	Basis \ C_j	3	2.5	0	0	M	M	X_B	Ratio
		x_1	x_2	S_1	S_2	a_1	a_2		
M	a_1	1	$\boxed{2}$	–1	0	1	0	20	$\frac{20}{2}=10$ (min)
M	a_2	3	2	0	–1	0	1	50	$\frac{50}{2}=25$
		X_1	$X_2\uparrow$	S_1	S_2	$A_1\downarrow$	A_2		
	Z_j	$4M$	$4M$	$-M$	$-M$	M	M	$x_1=0$ $x_2=0$	
	$\Delta_j = C_j - Z_j$	$3-4M$	$2.5-4M$ (key column)	M	M	0	0	$S_1=0$ $S_2=0$	$Z=70M$

Conclusion: Since some Δ_j are negative, hence the initial basic feasible solution is not optimal. The highest negative value of Δ_j is $\Delta_2 = 2.5 - 4M$, hence vector x_2 will enter. Since the corresponding ratio is minimum in first row so vector A_1 is outgoing vector. Key element is $\boxed{2}$

Simplex Table 2

	x_1	x_2	S_1	S_2	a_1	a_2	X_B
First row (First row ÷ 2)	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	10

Second row (old second row –2 × new first row)

	x_1	x_2	S_1	S_2	a_1	a_2	X_B
	$3-2\times\frac{1}{2}$	$2-2\times 1$	$0-2\times\left(-\frac{1}{2}\right)$	$-1-2\times 0$	$0-2\times\frac{1}{2}$	$1-2\times 0$	$50-2\times 10$
i.e.	2	0	1	–1	–1	1	30

Simplex (Revised) Tableau 2

C_B	C_j / Basis	3 x_1	2.5 x_2	0 S_1	0 S_2	M a_1	M a_2	X_B	Ratio
2.5	x_2	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	10	$\frac{10}{1/2}=20$
M	a_2	$\boxed{2}$	0	1	-1	-1	1	30	$\frac{30}{2}=15$ (min.)
		$X_1\uparrow$	X_2	S_1	S_2	A_1	$\downarrow A_2$		
	Z_j	$\frac{5}{4}+2M$	2.5	$-\frac{5}{4}+2M$	$-M$	$\frac{5}{4}-M$	M		
	$\Delta_j=C_j-Z_j$	$\frac{7}{4}-2M$ (key column)	0	$\frac{5}{4}-M$	M	$-\frac{5}{4}+M$	0		$Z=25+30M$

Conclusion: Some Δ_j are negative, hence this solution is not optimal. The largest negative value of Δ_1 is $\Delta_j=\frac{7}{4}-2M$, hence vector X_1 will enter. The ratio is minimum for second row, hence, vector A_2 is outgoing vector. The key element is $\boxed{2}$

For Simplex (Revised) Table 3

Main row (Second old row ÷ 2), Second new row

	x_1	x_2	S_1	S_2	a_1	a_2	X_B
	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{30}{2}$
i.e.	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	15

First row (old first row $-\frac{1}{2}$ main first row)

	x_1	x_2	S_1	S_2	a_1	a_2	X_B
	$\frac{1}{2}-\frac{1}{2}$	$1-\frac{1}{2}\times 0$	$-\frac{1}{2}-\frac{1}{2}\times\frac{1}{2}$	$0-\frac{1}{2}\left(-\frac{1}{2}\right)$	$\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}\right)$	$0-\frac{1}{2}\times\frac{1}{2}$	$10-\frac{15}{2}$
i.e.	0	1	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{5}{2}$

C_B	Basis \ C_j	3 x_1	2.5 x_2	0 s_1	0 s_2	M a_1	M a_2	X_B	Ratio
2.5	x_2	0	1	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{5}{2}$	No Need
3	x_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	15	
		X_1	X_2	S_1	S_2	A_1	A_2		
	Z_j	3	2.5	$-\frac{3}{8}$	$-\frac{7}{8}$	$\frac{3}{8}$	$\frac{7}{8}$		$Z=\frac{25}{4}+45$
	$\Delta_j = C_j - Z_j$	0	0	$\frac{3}{8}$	$\frac{7}{8}$	$M-\frac{3}{8}$	$M-\frac{7}{8}$		$=51.25$

Conclusion: Since all Δ_j are positive, hence present solution is optimal.

Where, $$x_1 = 15, x_2 = \frac{5}{2}$$

and $$Z = 3 \times 15 + 2.5 \times \frac{5}{2}$$
$$= 45 + 6.25 = 51.25$$

Example 9: *Use penalty (or Big-M) method due to A. Charnes to solve the following L.P.P. .*

$$\text{Minimize } Z = 4x_1 + 3x_2$$

subject to
$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$

and $$x_1, x_2 \geq 0$$

Solution: Convert the L.P.P. in maximization form. Introduce surplus variables $x_3 \geq 0$ and $x_5 \geq 0$ and slack variable $x_4 \geq 0$ in the constraints inequalities. Then the problem in standard form becomes

$$\text{Maximize } (-Z) = Z^* = -4x_1 - 3x_2 + 0.x_3 - 0.x_4 + 0.x_5$$

subject to
$$2x_1 + x_2 - x_3 = 10$$
$$-3x_1 + 2x_2 + x_4 = 6$$
$$x_1 + x_2 - x_5 = 6$$

clearly we do not have ready initial basic feasible solution. The surplus variables carry negative coefficients (– 1). So, introduce two artificial variables $a_1 \geq 0$ and $a_2 \geq 0$ in first and third equations respectively. Then we have

$$\text{Maximize } Z^* = -4x_1 - 3x_2 + 0.x_3 + 0.x_4 + 0.x_5 - M.a_1 - M.a_2$$

subject to
$$2x_1 + x_2 - x_3 + a_1 = 10$$
$$-3x_1 + 2x_2 + x_4 = 6$$
$$x_1 + x_2 - x_5 + a_2 = 6$$

with usual notations, we have.

Simplex Tableau 1

C_B	Basis \ C_j	-4 x_1	-3 x_2	0 x_3	0 x_4	0 x_5	$-M$ a_1	$-M$ a_2	X_B	Ratio
$-M$	a_1	$\boxed{2}$	1	-1	0	0	1	0	10	$\frac{10}{2}=5$ (min)
0	x_4	-3	2	0	1	0	0	0	6	negative
$-M$	a_2	1	1	0	0	-1	0	1	6	$\frac{6}{1}=6$
$\Delta_j = Z_j - C_j$		$\uparrow X_1$	X_2	X_3	X_4	X_5	$\downarrow A_1$	A_2	$x_1=0$	$Z^* = -16M$
		$\begin{pmatrix}-3M\\+4\end{pmatrix}$ $\downarrow$ most negative	$\begin{pmatrix}-2M\\+3\end{pmatrix}$	M	0	M	0	0	$x_2=0$, $x_3=0$, $x_5=0$	$Z=16M$

$$Z^* = C_B' X_B = \begin{bmatrix} -M\ 0\ M \end{bmatrix}\begin{bmatrix}10\\6\\6\end{bmatrix} = -16\ M$$

Test for Optimality:

$$\Delta_1 = C_B' X_1 - C_1 = [-M\ \ 0\ -M]\begin{bmatrix}2\\-3\\1\end{bmatrix} - (-4)$$

$$= (-2M + 0 - M) + 4 = (-3M + 4)$$

$$\Delta_2 = C_B' X_2 - C_2 = (-M\ 0 - M)\begin{bmatrix}1\\2\\1\end{bmatrix} - (-3)$$

$$= (-M + 0 - M) + 3 = (-2M + 3)$$

$$\Delta_3 = C_B' X_2 - C_3 = [-M\ 0 - M]\begin{bmatrix}-1\\0\\0\end{bmatrix} - 0 = M$$

$$\Delta_4 = 0$$

$$\Delta_5 = C_B' X_5 - C_5 = [-M\ 0 - M]\begin{bmatrix}0\\0\\-1\end{bmatrix} - 0 = M$$

$$\Delta_6 = 0$$

$$\Delta_7 = 0$$

Since some of the net evaluations Δ_j are negative, the initial basic feasible solution $[x_1 = 0, x_2 = 0, x_3 = 0, x_5 = 0, a_1 = 10, x_4 = 6, a_2 = 6]$ is not optimal. The most negative Δ_j is $\Delta_1 = (-3M + 4)$. The corresponding column vector X_1, therefore enters the basis.

Further the ratio is minimum for a_1, therefore A_1 is the outgoing vector, (we drop A_1 forever, because we do not like it to re-enter in any subsequent iteration).

To convert the vector X_1 as $\begin{bmatrix}1\\0\\0\end{bmatrix}$ we apply the transformation $R_1 \to \frac{1}{2}R_1, R_2 \to R_3 + 3R_1,$

$R_3 \to R_3 - R_1$ successively.

Simplex Tableau 2

C_B	C_j / Basis	-4 x_1	-3 x_2	0 x_3	0 x_4	0 x_5	$-M$ x_6	X_B	Ratio = $\frac{x_B}{X_2}$
-4	x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	5	$\frac{5}{1/2} = 10$
0	x_4	0	$\frac{7}{2}$	$-\frac{3}{2}$	1	0	0	21	$\frac{21}{7/2} = \frac{42}{7} = 6$
$-M$	a_2	0	$\boxed{\frac{1}{2}}$	$\frac{1}{2}$	0	-1	1	1	$\frac{1}{1/2} = 2$ (min.)
$\Delta_j = Z_j - C_j$		X_1	$\uparrow X_2$	X_3	X_4	X_5	$\downarrow A_2$	$x_1 = 0$	
		0	$\left[-\frac{M}{2}+1\right]$	$\left[-\frac{M}{2}+1\right]$	0	M	0	$x_2 = 0$	$Z^* = -20 - M$
			most negative					$x_5 = 0$	$Z = 20 + M$

$$Z^* = C_B' X_B = [-4 \quad 0 \quad M]\begin{bmatrix}5\\21\\1\end{bmatrix} = -20 - M$$

Test for Optimality:

$$\Delta_1 = 0$$

$$\Delta_2 = C_B' X_2 - C_2 = [-4 \quad 0 - M]\begin{bmatrix}1/2\\7/2\\1/2\end{bmatrix} - (-3)$$

$$= \left[-\frac{4}{2} + 0 - \frac{M}{2}\right] + 3 = -\frac{M}{2} + 1$$

$$\Delta_3 = C_B' X_3 - C_3 = (-4 \quad 0 - M)\begin{bmatrix} -1/2 \\ -3/2 \\ -1/2 \end{bmatrix} - 0$$

$$= \left[\frac{4}{2} + 0 - \frac{M}{2}\right] = -\frac{M}{2} + 2$$

$$\Delta_4 = 0$$

$$\Delta_5 = C_B' X_5 - C_5 = [-4 \quad 0 - M]\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - 0 = M$$

$$\Delta_6 = 0$$

Since some of the net evaluation Δ_j are negative, the current basic feasible solution $[x_1 = 5, x_2 = 0, x_3 = 0, x_4 = 21, x_5 = 0, a_2 = 1]$ is not optimal.

The most negative Δ_j is $\Delta_2 = -\dfrac{M}{2} + 1$. The corresponding column vector x_2, therefore enter the basis.

Further the ratio minimum for a_2, therefore, A_2 is the outgoing vector we drop A_2 forever.

To convert the vector X_2 as $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we apply the transformation

$$R_3 \to 2R_3,\ R_1 \to R_1 - \frac{1}{2} R_3,\ R_2 \to R_2 - \frac{7}{2} R_3 \text{ successively}$$

Simplex Tableau 3

C_B	C_j / Basis	-4	-3	0	0	0	X_B	Ratio
		x_1	x_2	x_3	x_4	x_5		
–4	x_1	1	0	–1	0	1	4	No need
0	x_4	0	0	–5	1	7	14	
–3	x_2	0	1	1	0	–2	2	
$\Delta_j = Z_j - C_j$		X_1	X_2	X_3	X_4	X_5	$x_3 = 0$	$Z^* = C_B' X_B = -22$
		0	0	1	0	2	$x_5 = 0$	$Z = 22$

Test for Optimality

$$\Delta_1 = 0$$

$$\Delta_2 = 0$$

$$\Delta_3 = C_B' X_3 - C_3 = [-4\ 0\ -3]\begin{bmatrix}-1\\-5\\1\end{bmatrix} - 0 = 4 - 3 = 1$$

$$\Delta_4 = 0$$

$$\Delta_5 = C_B' X_5 - C_5 = [-4\ 0\ -3]\begin{bmatrix}1\\7\\-2\end{bmatrix} - 0 = -4 + 6 = 2$$

Since all net evaluation Δ_j are positive. The current basic feasible solution $(x_1 = 4, x_2 = 2, x_3 = 0, x_4 = 14, x_5 = 0)$ is optimal. Thus, the required solution is $x_1 = 4$, $x_2 = 2$ with minimum $Z = 22$.

Example 10: *Solve the following linear programming problem by 'Big M' method:*

$$\text{Maximize } Z = x_1 + 2x_2 + 3x_3 - x_4$$

subject to

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and $x_1, x_2, x_3, x_4 \geq 0$ **[B.C.A. (Lucknow) 2010]**

Solution: Introducing the artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the L.P.P. in standard form is as follows:

$$\text{Maximize } Z = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1 - Ma_2$$

subject to

$$x_1 + 2x_2 + 3x_3 + 0.x_4 + a_1 = 15$$
$$2x_1 + x_2 + 5x_3 + 0.x_4 + a_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$

and $x_1, x_2, x_3, x_4, a_2 \geq 0$

with usual notations, we have

Simplex Tableau 1

C_B	C_j / Basis	1 x_1	2 x_2	3 x_3	−1 x_4	$-M$ a_1	$-M$ a_2	C_B	Ratio
$-M$	a_1	1	2	3	0	1	0	15	$\frac{15}{3} = 5$
$-M$	a_2	2	1	$\boxed{5}$	0	0	1	20	$\frac{20}{5} = 4$ (min.)
−1	x_4	1	2	1	1	0	0	10	$\frac{10}{1} = 10$
		X_1	X_2	$X_3\uparrow$	X_4	A_1	$A_2\downarrow$	$x_1=0$	
$\Delta_j = Z_j - C_j$		$(-3M-2)$	$(-3M-4)$	$(-8M-4)$ (min)	0	0	0	$x_2=0$ $x_3=0$	$Z = -35$ $M - 10$

$$\Delta_1 = C_B' X_1 - C_1 = [-M \ -M \ -1]\begin{bmatrix}1\\2\\1\end{bmatrix} - 1$$

$$= [-M - 2M - 1] - 1 = -3M - 2$$

$$\Delta_2 = C_B' X_2 - C_2 = [-M \ -M \ -1]\begin{bmatrix}2\\1\\2\end{bmatrix} - 2$$

$$= [-2M - M - 2] - 2 = -3M - 4$$

$$\Delta_3 = C_B' X_3 - C_3 = [-M \ -M \ -1]\begin{bmatrix}3\\5\\1\end{bmatrix} - 3$$

$$= [-3M - 5M - 1] - 3 = -8M - 4$$

$$Z = C_B' X_B = -M \ -M \ -1]\begin{bmatrix}15\\20\\10\end{bmatrix} = -35M - 10$$

Since some net evaluations Δ_j are negative, the initial basic feasible solution $(x_1 = 0, x_2 = 0, x_3 = 0, a_1 = 15, a_2 = 20, x_4 = 10)$ is not optimal.

Since Δ_3 is most negative, hence vector x_3 is the incoming vector.

Since the ratio is minimum for a_2, so vector a_2 is dropped forever (*i.e.* A_2 is the outgoing vector).

To convert the vector X_3 as $\begin{bmatrix}0\\1\\0\end{bmatrix}$, we apply the transformations.

$$R_2 \to \frac{1}{5}R_2,\ R_1 - 3R_2,\ R_3 \to R_3 - R_1 \text{ respectively.}$$

Simplex Tableau 2

C_B	Basis \ C_j	**1**	**2**	**3**	**–1**	**–M**	X_B	**Ratio**
		x_1	x_2	x_3	x_4	a_2		
$-M$	a_1	$-\frac{1}{5}$	$\boxed{\frac{7}{5}}$	0	0	1	3	$3 \div \frac{7}{5} = \frac{15}{7}$ (min.)
3	x_3	$\frac{2}{5}$	$\frac{1}{5}$	1	0	0	4	$4 \div \frac{1}{5} = 20$
–1	x_4	$\frac{3}{5}$	$\frac{9}{5}$	0	1	0	6	$6 \div \frac{9}{5} = \frac{10}{3}$
$\Delta_j = Z_j - C_j$		X_1	$\uparrow X_2$	X_3	X_4	$\downarrow A_1$	$x_4 = 0$	$Z = -3M + 6$
		$\left(\frac{M}{5} - \frac{2}{5}\right)$	$\left[\frac{-7M}{15} - \frac{16}{5}\right]$	0	0	0	$x_2 = 0$	

$$\Delta_1 = C_B' X_1 - C_1 = [-M \ 3 \ -1]\begin{bmatrix} -1/5 \\ 2/5 \\ 3/5 \end{bmatrix} - 1$$

$$= \left[\frac{M}{5} + \frac{6}{5} - \frac{3}{5}\right] - 1 = \frac{M}{5} - \frac{2}{5}$$

$$\Delta_2 = C_B' X_2 - C_2 = [-M \ 3 \ -1]\begin{bmatrix} 7/5 \\ 1/5 \\ 9/5 \end{bmatrix} - 2$$

$$= \left[-\frac{7M}{5} + \frac{3}{5} - \frac{9}{5}\right] - 2 = -\frac{7M}{5} - \frac{16}{5}$$

Since Δ_2 is negative, the current basic feasible solution ($x_1 = 0, x_2 = 0, a_1 = 3, x_3 = 4, x_4 = 6$) is not optimal. So X_2 is the incoming vector. Since the ratio is minimum for a_1 so vector A_1 is the outgoing vector. To convert the vector X_2 as $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ we apply the transformation $R_1 \to \frac{5}{7} R_1, R_2 \to R_2 - \frac{1}{5} R_1, R_3 \to R_3 - \frac{9}{5} R_1$ successively.

Simplex Tableau 3

C_B	C_j / Basis	1	2	3	–1	X_B	Ratio
		x_1	x_2	x_3	x_4		
2	x_2	$-\frac{1}{7}$	1	0	0	15/7	Negative
3	x_3	$\frac{3}{7}$	0	1	0	25/7	$\frac{25}{7} \div \frac{3}{7} = \frac{25}{3}$
–1	x_4	$\boxed{\frac{6}{7}}$	0	0	1	15/7	$\frac{15}{7} \div \frac{6}{7} = \frac{5}{2}$ (min.)
		$\uparrow X_1$	X_2	X_3	$X_4 \downarrow$	$x_1 = 0$	
$\Delta_j = Z_j - C_j$		$-\frac{6}{7}$ Key Column	0	0	0		

$$\Delta_1 = C_B' X_1 - C_1 = [2 \ \ 3 \ \ -1]\begin{bmatrix} -1/7 \\ 3/7 \\ 6/7 \end{bmatrix} - 1$$

$$= \left[-\frac{2}{7} + \frac{9}{7} - \frac{6}{7}\right] - 1 = \frac{1}{7} - 1 = -\frac{6}{7}$$

Since Δ_1 is negative, the current basic feasible solution

$$\left(x_1 = 0, x_2 = \frac{15}{7}, x_3 = \frac{25}{7}, x_4 = \frac{15}{7}\right)$$

is not optimal. Hence, the vector X_1 is the incoming vector, since the ratio is minimum for X_4 so the vector X_4 is the outgoing vector.

To convert the vector X_1 as $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we apply the transformation

$$R_3 \to \frac{7}{6} R_3,\ R_1 \to R_1 + \frac{1}{7} R_3, R_2 \to R_2 - \frac{3}{7} R_3 \text{ successively.}$$

Simplex Tableau 4

C_B	C_j / Basis	1 x_1	2 x_2	3 x_3	-1 x_4	x_B	Ratio
2	x_2	0	1	0	1/6	5/2	No need
3	x_3	0	0	1	–1/2	5/2	
1	x_1	1	0	0	7/6	5/2	
		X_1	X_2	X_3	X_4	$x_4 = 0$	
$\Delta_j = Z_j - C_j$		0	0	0	1		$Z = C_B' X_B = 15$

$$\Delta_4 = [2 \;\; 3 \;\; 1] \begin{bmatrix} 1/6 \\ -1/2 \\ 7/6 \end{bmatrix} - (-1) = \left(\frac{2}{6} - \frac{3}{2} + \frac{7}{6}\right) + 1 = 1$$

Since all $\Delta_j \geq 0$ the current basic feasible solution is optimal.

Hence, $\quad x_1 = \frac{5}{2}, x_2 = \frac{5}{2}, x_3 = \frac{5}{2}, x_4 = 0$ with maximum $Z = 15$.

Example 11: *Maximize* $Z = 4x_1 + 5x_2 - 3x_3$

subject to
$$x_1 + x_2 + x_3 = 10$$
$$x_1 - x_2 \geq 1$$
$$2x_1 + 3x_2 + x_3 \leq 40$$
and $\quad x_1, x_2, x_3 \geq 0$ **[B.C.A. (Bhopal) 2012]**

Solution: Using $a_1 \geq 0, a_2 \geq 0$ artificial variables and $x_4 \geq 0, x_5 \geq 0$ as slack variables, the given L.P.P. is of the form:

$$\text{Maximize } Z = 4x_1 + 5x_2 - 3x_3 - 0.x_4 - Ma_1 - Ma_2 + 0.x_5$$

subject to
$$x_1 + x_2 + x_3 + a_1 = 10$$
$$x_1 - x_2 - x_4 + a_2 = 1$$
$$2x_1 + 3x_2 + x_3 + x_5 = 40$$
and $\quad x_1, x_2, x_3, x_4, x_5, a_1, a_2 \geq 0$

with usual notations, we have

Simplex Tableau 1

C_B	C_j / Basis	4 x_1	5 x_2	–3 x_3	0 x_4	$-M$ a_1	$-M$ a_2	0 x_5	X_B	Ratio
$-M$	a_1	1	1	1	0	1	0	0	10	10
$-M$	a_2	$\boxed{1}$	–1	0	–1	0	1	0	1	1 (min.)
0	x_5	2	3	1	0	0	0	1	40	20
		$X_1\uparrow$	X_2	X_3	X_4	A_1	$\downarrow A_2$	X_5	$x_1=0$	
$\Delta_j = Z_j - C_j$		$(-2M-4)$ (Most Negative)	–5	$(-M+3)$	M	0	0	0	$x_2=0$ $x_3=0$ $x_4=0$	$Z=-11M$

To convert the vector X_1 as $\begin{bmatrix}0\\1\\0\end{bmatrix}$, we apply the transformation $R_2 \to R_2, R_1 \to R_1 - R_2$, $R_3 \to R_3 - 2R_2$ successively. Drop vector A_2 forever.

Simplex Tableau 2

C_B	C_j / Basis	4 x_1	5 x_2	–3 x_3	0 x_4	$-M$ a_1	0 x_5	X_B	Ratio
$-M$	a_1	0	$\boxed{2}$	1	1	1	0	9	$\frac{9}{2}=4.5$ (min.)
4	x_1	1	–1	0	–1	0	0	1	Negative
0	x_5	0	5	1	2	0	1	38	$\frac{28}{5}=7.6$
		X_1	$X_2\uparrow$	X_3	X_4	$\downarrow A_1$	X_5	$x_2=0$	
$\Delta_j = Z_j - C_j$		0	$(-2M-9)$	$(-M+3)$	$(-M-4)$	0	0	$x_3=0$ $x_4=0$	$Z = -9M+4$

To convert the vector X_2 as $\begin{bmatrix}1\\1\\0\end{bmatrix}$, we apply the transformation $R_1 \to \frac{1}{2}R_1, R_2 \to R_2 + R_1$, $R_3 \to R_3 - 5R_1$ successively. Drop vector A_1 forever.

Simplex Tableau 3

C_B	Basis \ C_j	4	5	–3	0	0	X_B	Ratio
		x_1	x_2	x_3	x_4	x_5		
5	x_2	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{9}{2}$	No need
4	x_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{11}{2}$	
0	x_5	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{31}{2}$	
		X_1	X_2	X_3	X_4	X_5	$x_3 = 0$	$Z = \frac{89}{2}$
$\Delta_j = Z_j - C_j$		0	0	$\frac{15}{2}$	$\frac{1}{2}$	0	$x_4 = 0$	

$$\Delta_3 = \frac{5}{2} + \frac{4}{2} - (-3) = \frac{15}{2}$$

$$\Delta_4 = \frac{5}{2} - \frac{4}{2} - 0 = \frac{1}{2}$$

Since the net evaluation for not basic variables x_3 and x_4 are positive, hence the current basic feasible solution $\left[x_1 = \frac{11}{2}, x_2 = \frac{9}{2}, x_3 = 0, x_4 = 0, x_5 = \frac{31}{2}\right]$ is optimum with maximum $Z = 4 \times \frac{11}{2} + \frac{5 \times 9}{2} - 3 \times 0 = \frac{89}{2}$.

2.3.3 Two Phase Method (Due to Dantzing, Orden and Wolfe)

In the two phase method the L.P.P. is solved in two phase using the artificial variables.

Phase I: In this phase the artificial variables are eliminated and a basic feasible solution of the original linear programming problem is found. For this an artificial objective function is created which is the sum of all the artificial variables. Assign a cost ' – 1' to each artificial variables and cost 0 to all other variables in the new objective function. The artificial objective function is then minimized using the simplex method. If the minimum value of the artificial objective function is zero, then all the artificial variables will be zero and we have a basic feasible solution to the original problem. Then we go to phase II.

In case the minimum value of the artificial objective function is positive then at least one of the artificial variables is positive. This means that the original L.P.P. without the artificial variables is feasible and we terminate.

Phase II: In this phase, the basic feasible solution found at the end of Phase I is optimized (*i.e.* improved) with respect to original objective function. That is the final tableau of Phase I becomes the initial tableau for Phase II after changing the objective function.

On Two Phase Method

Example 12: *Solve the following L.P.P. by using two phase method.*

$$\text{Minimize } Z = x_1 + x_2$$

$$\text{subject to } 2x_1 + x_2 \geq 4$$

$$x_1 + 7x_2 \geq 7$$

and $$x_1, x_2 \geq 0$$

Solution: Convert the problem of minimization to maximization.

$$\text{Maximization } (-Z) = -x_1 - x_2$$

Introduce surplus variables $x_3 \geq 0$, $x_4 \geq 0$ in the constraints:

$$2x_1 + x_2 - x_3 = 4$$

$$x_1 + 7x_2 - x_4 = 7$$

The initial basic feasible solution cannot be obtained (*i.e.* the basic variables do not exist). Introduce artificial variables $a_1 \geq 0, a_2 \geq 0$ then the constraints are:

$$2x_1 + x_2 - x_3 + a_1 + 0.a_2 = 4$$

$$x_1 + 7x_2 - x_4 + 0.a_1 + a_2 = 7$$

where $a_1 < x_3$ and $a_2 < x_4$ otherwise the constraints of the problem will not hold.

Now, the basic variables are a_1 and a_2 which complete the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So the basic initial basic feasible solution is

$$[x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, a_1 = 4, a_2 = 7].$$

Phase I: Assign a cost (-1) to artificial variables a_1 and a_2 and a cost 0 to all other variables, the new objective function is:

$$Z^* = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 - a_1 - a_2$$

Simplex Tableau 1

C_B	C_j / Basis	0 x_1	0 x_2	0 x_3	0 x_4	−1 a_1	−1 a_2	X_B	Ratio
−1	a_1	2	1	−1	0	1	0	4	$\frac{4}{1} = 4$
−1	a_2	1	$\boxed{7}$	0	−1	0	1	7	$\frac{7}{7} = 1$ (min.)
		X_1	$\uparrow X_2$	X_3	X_4	A_1	$A_2 \downarrow$	$x_1 = 0$	
$\Delta_j = Z_j - C_j$		−3	−8 (min.)	1	1	0	0	$x_2 = 0$, $x_3 = 0$, $x_4 = 0$	$Z^* = -11$

Now, the vector X_2 is incoming vector and A_2 is the outgoing vector (drop it forever). To make the vector X_2 as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, apply the transformation $R_2 = \frac{1}{7} R_2$ and $R_1 \to R_1 - R_2$ successively.

Simplex Tableau 2

C_B	Basis \ C_j	0	0	0	0	−1	X_B	Ratio
		x_1	x_2	x_3	x_4	a_1		
−1	a_1	$\boxed{\frac{13}{7}}$	0	−1	$\frac{1}{7}$	1	3	$\frac{21}{13}$ (min.)
0	x_2	$\frac{1}{7}$	1	0	$-\frac{1}{7}$	0	1	7
		$\uparrow X_1$	X_2	X_3	X_4	$\downarrow A_1$	$x_1 = 0$	$Z^* = -3$
$\Delta_j = Z_j - C_j$		$-\frac{13}{7}$ (min.)	0	1	$-\frac{1}{7}$	0	$x_3 = 0$, $x_4 = 0$	

Now, the vector X_1 is incoming vector and A_1 is the outgoing vector (drop it forever). To make the vector X_1 as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ apply the transformation. $R_1 \to \frac{7}{13} R_1$ and $R_2 \to R_2 - \frac{1}{7} R_1$ successively.

Simplex Tableau 3

C_B	Basis \ C_j	0	0	0	0	X_B	Ratio
		x_1	x_2	x_3	x_4		
0	x_1	1	0	$-\frac{7}{13}$	$\frac{1}{13}$	$\frac{21}{13}$	No need
0	x_2	0	1	$\frac{1}{13}$	$-\frac{2}{13}$	$\frac{10}{13}$	
		X_1	X_2	X_3	X_4	$x_3 = 0$	
$\Delta_j = Z_j - C_j$		0	0	0	0	$x_4 = 0$	$Z^* = 0$

Since all $\Delta_j \geq 0$, the current basic feasible solution is optimum.

Phase II: Consider the original objective function, $-Z = x_1 - x_2$ and the constraints from simplex tableau 3, we have:

Simplex Tableau 4

C_B	Basis \ C_j	-1 x_1	-1 x_2	0 x_3	0 x_4	X_B	Ratio
-1	x_1	1	0	$-\frac{7}{13}$	$\frac{1}{13}$	$\frac{21}{13}$	No need
-1	x_2	0	1	$\frac{1}{13}$	$-\frac{2}{13}$	$\frac{10}{13}$	
		X_1	X_2	X_3	X_4	$x_3=0$	
$\Delta_j = Z_j - C_j$		0	0	$\frac{6}{13}$	$\frac{1}{13}$	$x_4=0$	$-Z=-\frac{21}{13}-\frac{10}{13}=-\frac{31}{13}$

Since all $\Delta_j \geq 0$, hence the current basic feasible solution is optimal.

Thus, $x_1 = \frac{21}{13}, x_2 = \frac{10}{13}$ with maximum $(-Z) = -\frac{31}{13}$ of minimum $Z = \frac{31}{13}$.

Example 13: *Solve the following L.P.P. by two phase method.*

$$\text{Maximize } Z = 2x_1 - x_2 + x_3$$

$$\text{subject to } x_1 + x_2 - 3x_3 \leq 8$$

$$4x_1 - x_2 + x_3 \geq 2$$

$$2x_1 + 3x_2 - x_3 \geq 4$$

and $\quad x_1, x_2, x_3 \geq 0$ **[B.C.A. (Rohtak) 2012]**

Solution: Introducing $x_4 \geq 0$ as slack variables, $x_5 \geq 0$ and $x_6 \geq 0$ as surplus variable the given L.P.P. becomes

$$\text{Maximize } Z = 2x_1 - x_2 + x_3 + 0.x_4 + 0.x_5 + 0.x_6$$

$$\text{subject to } x_1 + x_2 - 3x_3 + x_4 + 0.x_5 + 0.x_6 = 8$$

$$4x_1 - x_2 + x_3 + 0.x_4 - x_5 + 0.x_6 = 2$$

$$2x_1 + 3x_2 - x_3 + 0.x_4 + 0.x_5 - x_6 = 4$$

and $\quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Since no identity matrix exists, we introduces artificial variable $a_1 \geq 0$ and $a_2 \geq 0$. Then the constraints becomes

$$x_1 + x_2 - 3x_3 + x_4 + 0.x_5 + 0.x_6 + 0.a_1 + 0.a_2 = 8$$

$$4x_1 - x_2 + x_3 + 0.x_4 - x_5 - 0.x_6 + a_1 + 0.a_2 = 2$$

$$2x_1 + 3x_2 - x_3 + 0.x_4 + 0.x_5 - x_6 + 0.a_1 + a_2 = 4$$

and $\quad x_1, x_2, x_3, x_4, x_5, x_6, a_1, a_2 \geq 0$

Phase I: Assign a cost (-1) to each artificial variable and a cost 0 to all other variables to form a new objective function.

Let $\quad Z^* = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 + 0.x_5 + 0.x_6 - a_1 - a_2$

Simplex Tableau 1

C_B	C_j / Basis	0 x_1	0 x_2	0 x_3	0 x_4	0 x_5	0 x_6	−1 a_1	−1 a_2	X_B	Ratio
0	x_4	1	1	−3	1	0	0	0	0	8	8
−1	a_1	$\boxed{4}$	−1	1	0	−1	0	1	0	2	$\frac{1}{2}$ (min.)
−1	a_2	2	3	−1	0	0	−1	0	1	4	2
		$\uparrow X_1$	X_2	X_3	X_4	X_5	X_6	$A_1 \downarrow$	A_2	$x_1 = 0$	
$\Delta_j = Z_j - C_j$		−6 (min.)	−2	0	0	1	1	0	0	$x_2 = 0$ $x_3 = 0$ $x_5 = 0$ $x_6 = 0$	$Z^* = -6$

The incoming vector is X_1 and the outgoing vector is A_1 (drop it forever). To convert the vector X_1 as $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, apply the transformation $R_2 \to \frac{1}{4} R_2, R_1 \to R_1 - R_2, R_3 \to R_3 - 2R_1$ successively.

Simplex Tableau 2

C_B	C_j / Basis	0 x_1	0 x_2	0 x_3	0 x_4	0 x_5	0 x_6	−1 a_2	X_B	Ratio
0	x_4	0	$\frac{5}{4}$	$-\frac{13}{4}$	1	$\frac{1}{4}$	0	0	$\frac{15}{2}$	$\frac{15}{2} + \frac{5}{4} = \frac{3}{2}$
0	x_1	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	$\frac{1}{2}$	Negative
−1	a_2	0	$\boxed{\frac{7}{2}}$	$-\frac{3}{2}$	0	$\frac{1}{2}$	−1	1	3	$3 + \frac{7}{2} = \frac{6}{7}$ (min.)
		X_1	$\uparrow X_2$	X_3	X_4	X_5	X_6	$\downarrow A_2$	$x_2 = 0$	
$\Delta_j = Z_j - C_j$		0	$-\frac{7}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	1	0	$x_3 = 0$	$Z = -3$
			(Key Column)						$x_5 = 0$ $x_6 = 0$	

The incoming vector is X_2 and the outgoing vector is A_2 (drop it forever). To convert the vector X_2 as $\begin{bmatrix}0\\0\\1\end{bmatrix}$ apply the transformation. $R_3 \to \frac{2}{7}R_3$, $R_1 \to R_1 - \frac{5}{4}R_3$ and $R_2 \to R_2 + \frac{1}{4}R_3$ successively.

Simplex Tableau 3

C_B	C_j / Basis	0	0	0	0	0	0	X_B	Ratio
		x_1	x_2	x_3	x_4	x_5	x_6		
0	x_4	0	0	$-\frac{19}{7}$	1	$\frac{1}{14}$	$\frac{5}{14}$	$\frac{45}{7}$	No need
0	x_1	1	0	$\frac{1}{7}$	0	$-\frac{3}{14}$	$-\frac{1}{14}$	$\frac{5}{7}$	
0	x_2	0	1	$-\frac{3}{7}$	0	$\frac{1}{7}$	$-\frac{2}{7}$	$\frac{6}{7}$	
$\Delta_j = Z_j - C_j$		X_1	X_2	X_3	X_4	X_5	X_6	$x_3 = 0$	
		0	0	0	0	0	0	$x_5 = 0$	$Z^* = 0$
								$x_6 = 0$	

Since $\Delta_j \geq 0$ and none of the artificial vector appears in the basis, the current basic feasible solution $\left[x_1 = \frac{5}{7}, x_2 = \frac{6}{7}, x_4 = \frac{45}{7}, x_5 = 0, x_6 = 0\right]$ will serve a BFS for phase II.

Phase II: Consider $Z = 2x_1 - x_2 + x_3 + 0.x_4 + 0.x_5 + 0.x_6$ and the constraints as in simplex tableau 3, we have

Simplex Tableau 4

C_B	C_j / Basis	2	−1	1	0	0	0	X_B	Ratio
		x_1	x_2	x_3	x_4	x_5	x_6		
0	x_4	0	0	$-\frac{19}{7}$	1	$\frac{1}{14}$	$\frac{5}{14}$	$\frac{45}{7}$	90
2	x_1	1	0	$\frac{1}{7}$	0	$-\frac{3}{14}$	$-\frac{1}{14}$	$\frac{5}{7}$	Negative
−1	x_2	0	1	$-\frac{3}{7}$	0	$\boxed{\frac{1}{7}}$	$-\frac{2}{7}$	$\frac{6}{7}$	6 (min.)
$\Delta_j = Z_j - C_j$		X_1	$\downarrow X_2$	X_3	X_4	$\uparrow X_5$	X_6	$x_3 = 0$	
		0	0	$-\frac{2}{7}$	0	$-\frac{4}{7}$	$\frac{1}{7}$	$x_5 = 0$	$Z = \frac{4}{7}$
								$x_6 = 0$	

The incoming vector is X_5 and the outgoing vector is X_2. To convert the vector X_5 as $\begin{bmatrix}0\\0\\1\end{bmatrix}$, apply the transformation $R_3 \to 7R_3, R_1 \to R_1 - \frac{R_2}{14}, R_3 \to R_2 + \frac{3}{14} R_3$ successively.

Simplex Tableau 5

C_B	C_j / Basis	2 x_1	−1 x_2	1 x_3	0 x_4	0 x_5	0 x_6	X_B	Ratio
0	x_4	0	$-\frac{1}{2}$	$-\frac{5}{2}$	1	0	$\frac{1}{2}$	6	Negative
2	x_1	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	2	”
0	x_5	0	7	−3	0	1	−2	6	”
		X_1	X_2	X_3	X_4	X_5	X_6	$x_2=0$	
$\Delta_j = Z_j - C_j$		0	4	−2	0	0	−1	$x_3=0$, $x_6=0$	$Z=4$

Since the most negative value of net evaluation Δ_j is for Δ_3 but the elements of vector X_3 are all negative, the given L.P.P. admits an unbounded solution.

Example 14: *Use two phase method to solve the following L.P.P.:*

$$\text{Maximize } Z = 5x_1 + 3x_2$$

$$\text{subject to } 2x_1 + x_2 \le 1$$

$$x_1 + 4x_2 \ge 6$$

and $$x_1, x_2 \ge 0$$ **[B.C.A. (Kurukshetra) 2010]**

Solution: Introducing slack variables $x_3 \ge 0$ and surplus variable $x_4 \ge 0$ we have the given L.P.P. in standard form.

$$\text{Maximize } Z = 5x_1 + 3x_2 + 0.x_3 + 0.x_4$$

$$\text{subject to } 2x_1 + x_2 + x_3 = 1$$

$$x_1 + 4x_2 - x_4 = 6$$

and $$x_1, x_2, x_3, x_4 \ge 0$$

Here is no basic variable in second constraint, so we introduce an artificial variable $a_1 \ge 0$. Then the constraints are:

$$2x_1 + x_2 + x_3 + 0.x_4 + 0.a_1 = 1$$

$$x_1 + 4x_2 + 0.x_3 - x_4 + a_1 = 6$$

and $$x_1, x_2, x_3, x_4, a_1 \ge 0$$

Here the basic variables are x_3 and a_1, which complete the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

So, the initial basic feasible solution is $x_1 = 0, x_2 = 0, x_4 = 0, x_3 = 1$ and $a_1 = 6$.

Phase I: Assign a cost '−1' to artificial variable a_1 and a cost 0 to all other variables, the new objective function is:

$$Z^* = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 - a_1$$

Simplex Tableau 1

C_B	C_j / Basis	0 x_1	0 x_2	0 x_3	0 x_4	−1 a_1	X_B	Ratio
0	x_3	2	$\boxed{1}$	1	0	0	1	$\frac{1}{1}$ (min.)
−1	a_1	1	4	0	−1	1	6	$\frac{6}{4}$
		X_1	$\uparrow X_2$	$\downarrow X_3$	X_4	A_1	$x_1 = 0$	
$\Delta_j = Z_j - C_j$		−1	−4 (min.)	0	1	0	$x_2 = 0$ $x_4 = 0$	$Z^* = -6$

Now, the vector X_2 is incoming vector and X_3 is the outgoing vector. Apply the transformation $R_1 \to R_1$ and $R \to R_2 - 4R_1$.

Simplex Tableau 2

C_B	C_j / Basis	0 x_1	0 x_2	0 x_3	0 x_4	−1 a_1	X_B	Ratio
0	x_2	2	1	1	0	0	1	No need
−1	a_1	−7	0	−4	−1	1	2	
		X_1	X_2	X_3	X_4	A_1		
$\Delta_j = Z_j - C_j$		7	0	4	1	0		$Z^* = -2$

Since all $\Delta_j \geq 0$ an optimum basic feasible solution to the new L.P.P. is obtained. But maximum Z^* is negative and an artificial vector A_1 is in the basis at a positive level. Therefore, the original L.P.P. does not possess any feasible solution.

Problem Set

Solve the Following L.P.P. Using two Phase Method:

1. $\text{Min }(Z) = x_1 + x_2$

subject to $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$ **[B.C.A. (Meerut) 2004, 2007]**

2. $\text{Max }(Z) = 5x_1 + 8x_2$

subject to $3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0$ **[B.C.A. (Aligarh) 2009]**

3. $\text{Max }(Z) = 3x_1 + 2x_2 + x_3 + 4x_4$

subject to $4x_1 + 5x_2 + x_3 + 5x_4 = 5$

$2x_1 - 3x_2 - 4x_3 + 5x_4 = 7$

$x_1 + 4x_2 + 5x_3 - 4x_4 = 6$

$x_1, x_2, x_3, x_4 \geq 0$ **[B.C.A. (Agra) 2009]**

4. $\text{Maximum }(Z) = 5x_1 - 2x_2 + 3x_3$

subject to $2x_1 + 2x_2 - x_3 \geq 2$

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$

$x_1, x_2, x_3, x_4 \geq 0$ **[B.C.A. (Kashi) 2006, 2008]**

Apply Big-M Method to Solve the Following Problems:

1. $\text{Max }(Z) = -2x_1 - x_2$

subject to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$ **[B.C.A. (Kanpur) 2009]**

2. $\text{Max }(Z) = 4x_1 + 5x_2 - 3x_3$

subject to $x_1 + x_2 + x_3 = 10$

$x_1 - x_2 \geq 1$

$2x_1 + 3x_2 + x_3 \leq 40$

$x_1, x_2, x_3 \geq 0$ **[B.C.A. (Bundelkhand) 2007]**

3. $\text{Min}\ (Z) = 5x_1 + 6x_2$

subject to $2x_1 + 5x_2 \geq 1500$

$3x_1 + x_2 \geq 1200$

$x_1, x_2 \geq 0$ **[B.C.A. (Rohilkhand) 2009]**

4. $\text{Min}\ (Z) = 2y_1 + 3y_2$

subject to $y_1 + y_2 \geq 5$

$y_1 + 2y_2 \geq 6$

$y_1, y_2 \geq 0$ **[B.C.A. (Indore) 2010]**

5. $\text{Min}\ (Z) = x_1 + x_2 + 3x_3$

subject to $3x_1 + 2x_2 + x_3 \leq 3$

$2x_1 + x_2 + 2x_3 \geq 3$

$x_1, x_2, x_3 \geq 0$ **[B.C.A. (Rohilkhand) 2002, 2006]**

Answers

Two Phase Method:

1.	$x_1 = \frac{21}{13},\ x_2 = \frac{10}{13},\ \text{Min}\ (Z) = \frac{31}{13}$
2.	$x_1 = 0, x_2 = 5, \text{Max}\ (Z) = 40$
3.	No feasible solution
4.	$x_1 = \frac{23}{3},\ x_2 = 5,\ x_3 = 0,\ \text{Max}\ (Z) = \frac{85}{3}$

Big-M Method:

1.	$x_1 = 3/5,\ x_2 = 6/5,\ \text{Max}\ (Z) = -12/5$
2.	$x_1 = 11/2,\ x_2 = 9/2,\ \text{Max}\ (Z) = 89/2$
3.	$x_1 = 4500/13,\ x_2 = 2100/13,\ \text{Min}\ (Z) = 2700$
4.	$y_1 = 4,\ y_2 = 1,\ \text{Min}\ (Z) = 11$
5.	$x_1 = \frac{3}{4}, x_2 = 0, x_3 = \frac{3}{4}, \min Z = 3$

2.4 Duality in Linear Programming

[B.C.A. (Lucknow) 2010]

The concept of duality was one of the most important discoveries in the early development of linear programming. It explains that corresponding to every linear programming problem there is another linear programming problem. The original problem is known as **primal** and the other, the related problem is known as the **dual.** The two problems are replicas of each other. If the primal problem is of maximization type, the dual will be of the minimization type. If the optimal solution to one problem is known to us, we can easily find the optimal solution of the other. This fact is important because at times it is easier to solve the dual than the primal.

2.4.1 Symmetric Primal Dual Problems

Consider the following linear programming problem, we may call it the symmetric primal problem.

Primal L.P.P.: Find the variables $x_1, x_2, \dots, x_n$ which maximize

$$Z_x = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m \end{array}$$

and $$x_1, x_2, \dots, x_n \ge 0$$

the signs of the parameters a, b, c are arbitrary. To obtain the dual of the above primal following steps are required:

1. Minimize the objective function in place of maximizing it.
2. Interchange the role of constant terms and the coefficients of the objective function.
3. Find A' where A' denotes the transpose of the coefficient matrix A.
4. Reverse the direction of inequalities.

Thus, the dual problem is:

Find the variables $w_1, w_2, \ldots, w_m$ which minimize

$$Z_w = b_1 w_1 + b_2 w_2 + \ldots + b_m w_m$$

subject to
$$a_{11} w_1 + a_{21} w_2 + \ldots + a_{m1} w_m \geq c_1$$
$$a_{12} w_1 + a_{22} w_2 + \ldots + a_{m2} w_2 \geq c_2$$
$$\ldots \quad \ldots \quad \ldots \qquad \ldots \quad \ldots \quad \ldots \qquad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \qquad \ldots \quad \ldots \quad \ldots \qquad \ldots \quad \ldots \quad \ldots$$
$$a_{1n} w_1 + a_{2n} w_2 + \ldots + a_{mn} w_m \geq c_n$$

and
$$w_1, w_2, \ldots, w_n \geq 0$$

2.4.2 Matrix form of Symmetric Primal Dual Problem

[B.C.A. (Meerut) 2011]

Primal Problem: Find a column vector $x \in R^n$ which maximizes

$$Z_x = cx, \; c \in R^n$$

subject to
$$Ax \leq b, \; b \in R^m$$

$x \geq 0$ and A is an $m \times n$ real matrix.

Dual Problem: Find a column vector $w \in R^m$ which minimizes

$$Z_w = b' w$$

subject to
$$A' w \geq c'$$

$w \geq 0$, A', b', c' are the transposes of A, b and c respectively.

Example 15: *Consider the symmetric primal problem*

$$Max\ Z_x = 5x_1 + 9x_2$$

subject to
$$x_1 \leq 6$$
$$x_1 + x_2 \leq 13$$
$$x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

Solution: The corresponding dual problem is

$$\text{Min } Z_w = 6w_1 + 13w_2 + 8w_3$$

subject to
$$w_1 + w_2 \geq 5$$
$$w_2 + w_3 \geq 9$$
$$w_1, w_2, w_3 \geq 0$$

2.5 Unsymmetric Primal Dual Problems

[B.C.A. (Delhi) 2010, 2012]

Primal Problem: Find a column vector $x \in R^n$ which maximizes.

$$Z_x = cx, \ c \in R^n$$

subject to $\quad Ax = b, \ b \in R^m$

$$x \geq 0 \text{ and } A \text{ is an } m \times n \text{ real matrix.}$$

Dual Problem: Find a column vector $w \in R^m$ which minimizes

$$Z_w = b' w$$

subject to $\quad A' w \geq c'$

In this case the dual variables are unrestricted in sign.

2.5.1 Dual of an L.P.P. with Mixed Restrictions

[B.C.A. (Lucknow) 2011]

Sometimes a given L.P.P. contains a mixture of inequalities (≥, ≤) equations; non-negative variables and unrestricted variables, then to obtain its dual we proceed in the following manner:

1. If a constraint is an equation (has = sign) replace it by two constraints involving the inequalities going in the opposite directions.

   For Example, the equation $x_1 + 2x_2 = 3$ is replaced by

   $$x_1 + 2x_2 \leq 3 \qquad \text{...(1)}$$

   and $$x_1 + 2x_2 \geq 3 \qquad \text{...(2)}$$

2. If the given problem is of maximization, all constraints should have ≤ sign. If some constraints has ≥ sign, multiply both sides by −1 and make the sign ≤.

   Thus, in the above example multiply both sides of (2) by −1 so that it becomes

   $$-x_1 - 2x_2 \leq -3$$

   Similarly, if the problem is of minimization, all constraints should have ≥ sign.

3. If there is some unrestricted variable, replace it by the difference of two variables.

4. Now to find the dual problem.

Standard Primal Form: A linear programming problem is said to be in standard primal form if:

1. For a maximization problem all the constraints have ≤ sign
2. For a minimization problem all the constraints have ≥ sign.

Example 16: *Find the dual of the following L.P.P.*

$$Min\ Z = 10x_1 + 20x_2$$

subject to $3x_1 + x_2 \geq 18$

$$x_1 + 3x_2 \geq 8$$

$$2x_1 - x_2 \leq 6,$$

$$x_1, x_2 \geq 0$$

[B.C.A. (Meerut) 2004]

Solution: The given L.P.P. in the standard primal form is

$$\text{Min } Z = 10x_1 + 20x_2$$

subject to $3x_1 + 2x_2 \geq 18$

$$x_1 + 3x_2 \geq 8$$

$$-2x_1 + x_2 \geq -6$$

and $x_1, x_2 \geq 0$

The matrix form of this problem is

$$\text{Min } Z = 10x_1 + 20x_2 = (10, 20)\,[x_1, x_2] = cx$$

subject to $\begin{bmatrix} 3 & 2 \\ 1 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 18 \\ 8 \\ -6 \end{bmatrix}$

or $AX \geq b, x_1, x_2 \geq 0$

$\therefore$ the dual of this problem is

$$\text{Max. } Z_D = b'\ y = (18, 8, -6)\,(y_1, y_2, y_3)$$

$$= 18y_1 + 8y_2 - 6y_3$$

Subject to $A'\ y \leq c'$ or $\begin{bmatrix} 3 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

or $\begin{bmatrix} 3y_1 + y_2 - 2y_3 \\ 2y_1 + 3y_2 + y_3 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

or $3y_1 + y_2 - 2y_3 \leq 10, 2y_1 + 3y_2 + y_3 \leq 20, y_1, y_2, y_3 \geq 0$

Hence, dual of the given problem is

$$\text{Max } Z_D = 18y_1 + 8y_2 - 6y_3$$

Subject to $3y_1 + y_2 - 2y_3 \leq 10, 2y_1 + 3y_2 + y_3 \leq 20, y_1, y_2, y_3 \geq 0$

Example 17: *Write the dual of the following L.P.P.*

$$\text{Max } Z = 2x_1 + 3x_2 + x_3$$

subject to $4x_1 + 3x_2 + x_3 = 6$

$$x_1 + 2x_2 + 5x_3 = 4$$

$$x_1, x_2, x_3 \geq 0.$$

[B.C.A. (Meerut) 2005, 2010]

Solution: Now to convert the given L.P.P. into standard primal form.

Here, both constraints are qualities, so replacing each by two inequalities, we get the constraints

$$4x_1 + 3x_2 + x_3 \leq 6 \quad \text{and} \quad 4x_1 + 3x_2 + x_3 \geq 6$$

$$x_1 + 2x_2 + 5x_3 \leq 4$$

and $x_1 + 2x_2 + 5x_3 \geq 4$

Since, the given problem is of maximization, so all the constraints should have the sign $\leq$.

The standard primal form of the given L.P.P. is

$$\text{Max } Z = 2x_1 + 3x_2 + x_3$$

subject to $4x_1 + 3x_2 + x_3 \leq 6$

$$-4x_1 - 3x_2 - x_3 \leq -6$$

$$x_1 + 2x_2 + 5x_3 \leq 4$$

$$-x_1 - 2x_2 - 5x_3 \leq -4$$

$$x_1, x_2, x_3 \geq 0$$

The matrix form of the above problem is

$$\text{Max } Z = 2x_1 + 3x_2 + x_3 = (2, 3, 1)\,[x_1, x_2, x_3] = c\,.\,x$$

subject to

$$\begin{bmatrix} 4 & 3 & 1 \\ -4 & -3 & -1 \\ 1 & 2 & 5 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 6 \\ -6 \\ 4 \\ -4 \end{bmatrix}$$

or $AX \leq b$

$$x_1, x_2, x_3 \geq 0$$

$\therefore$ the dual of the given primal is

$$\text{Min } Z_D = b'\,.\,y = (6, -6, 4, -4)\,[y_1', y_1'', y_2', y_2'']$$

$$= 6\,(y_1' - y_1'') + 4\,(y_2' - y_2'')$$

Subject to $A'\,y \geq c'$

or $$\begin{bmatrix} 4 & -4 & 1 & -1 \\ 3 & -3 & 2 & -2 \\ 1 & -1 & 5 & -5 \end{bmatrix} \begin{bmatrix} y_1' \\ y_1'' \\ y_2' \\ y_2'' \end{bmatrix} \geq \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

or $$\begin{bmatrix} 4y_1' - 4y_1'' + y_2' - y_2'' \\ 3y_1' - 3y_1'' + 2y_2' - 2y_2'' \\ y_1' - y_1'' + 5y_2' - 5y_2'' \end{bmatrix} \geq \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

or $$4(y_1' - y_1'') + y_2' - y_2'' \geq 2$$
$$3(y_1' - y_1'') + 2(y_2' - y_2'') \geq 3$$
$$(y_1' - y_1'') + 5(y_2' - y_2'') \geq 1$$
$$y_1', y_1'', y_2', y_2'' \geq 0$$

Substituting $y_1 = y_1' - y_1''$, $y_2 = y_2' - y_2''$, the required dual is

$$\text{Min. } Z_D = 6y_1 + 4y_2$$

$$\text{subject to } 4y_1 + y_2 \geq 2, 3y_1 + 2y_2 \geq 3, y_1 + 5y_2 \geq 1$$

where y_1, y_2 are unrestricted in sign.

Example 18: *Write the dual of the following problem:*

$$Min\ Z = x_1 + x_2 + x_3$$

subject to
$$x_1 - 3x_2 + 4x_3 = 5,$$
$$x_1 - 2x_2 \leq 3,$$
$$2x_2 - x_3 \geq 4,$$
$$x_1, x_2 \geq 0, x_3 \textit{ is unrestricted.}$$ **[B.C.A. (Meerut) 2002, 2008]**

Solution: Write the given L.P.P. in the standard primal form, substituting $x_3 = x_3' - x_3''$, $x_3' \geq 0$, $x_3'' \geq 0$. The given problem can be written in the standard primal form as

$$\text{Min } Z = x_1 + x_2 + x_3' - x_3''$$

subject to
$$-x_1 + 3x_2 - 4(x_3' - x_3'') \geq -5$$
$$x_1 - 3x_2 + 4(x_3' - x_3'') \geq 5$$
$$-x_1 + 2x_2 \geq -3$$
$$2x_2 - (x_3' - x_3'') \geq 4$$
$$x_1, x_2, x_3', x_3'' \geq 0$$

The matrix form of the above problem is

$$\text{Min } Z = (1, 1, 1, -1)[x_1, x_2, x_3', x_3''] = c.x$$

subject to

$$\begin{bmatrix} -1 & 3 & -4 & 4 \\ 1 & -3 & 4 & -4 \\ -1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3' \\ x_3'' \end{bmatrix} \geq \begin{bmatrix} -5 \\ 5 \\ -3 \\ 4 \end{bmatrix}$$

or $$AX \geq b, \; x_1, x_2, x_3', x_3'' \geq 0$$

Now, the dual of the given primal is

$$\text{Max } Z_D = (-5, 5, -3, 4) \; [y_1', y_1'', y_2, y_3]$$
$$= -5\,(y_1' - y_1'') - 3\,y_2 + 4\,y_3$$

subject to $$A'\,y \leq c'$$

or $$\begin{bmatrix} -1 & 1 & -1 & 0 \\ 3 & -3 & 2 & 2 \\ -4 & 4 & 0 & -1 \\ 4 & -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1' \\ y_1'' \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

or $$-(y_1' - y_1'') - y_2 \leq 1$$
$$3\,(y_1' - y_1'') + 2\,y_2 + 2\,y_3 \leq 1$$
$$-4\,(y_1' - y_1'') - y_3 \leq 1$$
$$4\,(y_1' - y_1'') + y_3 \leq -1$$
$$y_1', y_1'', y_2, y_3 \geq 0$$

Substituting $y_1 = y_1' - y_1''$, the required dual is

$$\text{Max } Z_D = -5\,y_1 - 3\,y_2 + 4\,y_3$$

subject to $$-y_1 - y_2 \leq 1, 3\,y_1 + 2\,y_2 + 2\,y_3 \leq 1, -4\,y_1 - y_3 \leq 1$$
$$4\,y_1 + y_3 \leq -1$$

or $$-4\,y_1 - y_3 \geq 1$$
$$y_2, y_3 \geq 0 \text{ and } y_1 \text{ is unrestricted.}$$

Hence, the required dual is

$$\text{Max } Z_D = -5\,y_1 - 3\,y_2 + 4\,y_3$$

subject to $$-y_1 - y_2 \leq 1, 3\,y_1 + 2\,y_2 + 2\,y_3 \leq 1, -4\,y_1 - y_3 = 1$$
$$y_2, y_3 \geq 0, \; y_1 \text{ is unrestricted in sign.}$$

Example 19: *Write the dual of the following problem:*

$$Min\ Z = 2x_2 + 5x_3$$

subject to $$x_1 + x_2 \geq 2$$
$$2x_1 + x_2 + 6x_3 \leq 6$$
$$x_1 - x_2 + 3x_3 = 4$$
$$x_1, x_2, x_3 \geq 0$$

[B.C.A. (Bhopal) 2011]

Solution: First we shall convert the given problem to standard primal form:

1. Since the problems of minimization, therefore all the constraints should have the sign $\geq$.

2. Multiplying the second constraint by -1, it becomes

$$-2x_1 - x_2 - 6x_3 \geq -6$$

3. Since, the third constraint is an equality, so replacing it by the following two constraints

$$x_1 - x_2 + 3x_3 \geq 4$$

and $$x_1 - x_2 + 3x_3 \leq 4$$

or $$x_1 - x_2 + 3x_3 \geq 4$$

and $$-x_1 + x_2 - 3x_3 \geq -4$$

$\therefore$ the given problem is standard primal form is

$$\text{Min } Z = 0x_1 + 2x_2 + 5x_3$$

subject to
$$\begin{aligned} x_1 + x_2 &\geq 2 \\ -2x_1 - x_2 - 6x_3 &\geq -6 \\ x_1 - x_2 + 3x_3 &\geq 4 \\ -x_1 + x_2 - 3x_3 &\geq -4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The matrix form of the above problem is

$$\text{Min } Z = (0, 2, 5)\,[x_1, x_2, x_3] = c.x$$

subject to $$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -6 \\ 1 & -1 & 3 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ -6 \\ 4 \\ -4 \end{bmatrix} \quad \text{or} \quad AX \geq b, x_1, x_2, x_3 \geq 0$$

$\therefore$ the dual of the given primal is

$$\begin{aligned} \text{Max } Z_D &= b'\,y = (2, -6, 4, -4)\,[y_1, y_2, y_3', y_3''] \\ &= 2y_1 - 6y_2 + 4\,[y_3' - y_3''] \end{aligned}$$

subject to $$b'\,y \leq c'$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 0 & -6 & 3 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3' \\ y_3'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

or $$\begin{bmatrix} y_1 - 2y_2 + y_3' - y_3'' \\ y_1 - y_2 - y_3' + y_3'' \\ 0.\,y_1 - 6y_2 + 3y_3' - 3y_3'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$y_1, y_2, y_3', y_3'' \geq 0$$

or $$\text{Max } Z_D = 2y_1 - 6y_2 - 4\,(y_3' - y_3'')$$

subject to
$$\begin{aligned} y_1 - 2y_2 + (y_3' - y_3'') &\leq 0 \\ y_1 - y_2 - (y_3' - y_3'') &\leq 2 \\ -6y_2 + 3\,(y_3' - y_3'') &\leq 5 \\ y_1, y_2, y_3', y_3'' &> 0 \end{aligned}$$

Substituting $y_3 = y_3' - y_3''$, the required dual is

$$\text{Max } Z_D = 2y_1 - 6y_2 + 4y_3$$

subject to $y_1 - 2y_2 + y_3 \le 0,\ y_1 - y_2 - y_3 \le 2,\ -6y_2 + 3y_3 \le 5$

$y_1, y_2 \ge 0$ and y_3 is unrestricted in sign.

NOTE:
The variable corresponding to the equality equation (here, third equation) in the constraint will be unrestricted in sign.

Problem Set

Find the Dual of the Following Linear Programming Problems:

1. Max $Z = x_1 - x_2 + 3x_3$

subject to $x_1 + x_2 + x_3 \le 10$

$2x_1 - x_3 \le 2$

$2x_1 - 2x_2 + 3x_3 \le 6$

$x_1, x_2, x_3 \ge 0$

2. Min $Z = 3x_1 + x_2$

subject to $2x_1 + 3x_2 \ge 2$

$x_1 + x_2 \ge 1$

$x_1, x_2 \ge 0$

3. Min $Z = 4x_1 + 6x_2 + 18x_3$

subject to $x_1 + 3x_3 \ge 3$

$x_2 + 2x_3 \ge 5$

$x_1, x_2, x_3 \ge 0$

4. Min $Z = 2x_1 + 3x_2 + 4x_3$

subject to $2x_1 + 3x_2 + 5x_3 \ge 2$

$3x_1 + x_2 + 7x_3 = 3$

$x_1 + 4x_2 + 6x_3 \le 5$

$x_1, x_2 \ge 0$, x_3 is unrestricted

[B.C.A. (I.G.N.O.U.) 2006]

5. Min $Z = x_1 - 3x_2 - 2x_3$

subject to $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 - 4x_2 \ge 12$

$-4x_1 + 3x_2 + 8x_3 = 10$

$x_1, x_2 \ge 0$, x_3 is unrestricted.

6. Max $Z = 3x_1 + x_2 + 4x_3 + x_4 + 9x_5$

subject to $4x_1 - 5x_2 - 9x_3 + x_4 - 2x_5 \le 6$

$2x_1 + 3x_2 + 4x_3 - 5x_4 + x_5 \le 9$

$x_1 + x_2 - 5x_3 - 7x_4 + 11x_5 \le 10,$

$x_1, x_2, x_3, x_4, x_5 \ge 0$

[B.B.A. (Meerut) 2006]

7. How do the graphical and simplex methods of solving as L.P.P. differ?

[B.B.A. (Meerut) 2004]

8. Explain in brief duality in L.P.P. **[B.B.A. (Meerut) 2006]**

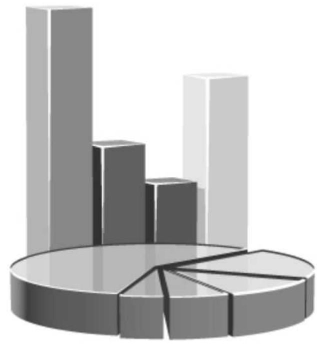
Answers

1.	Min $Z_D = 10y_1 + 2y_2 + 6y_3$ subject to $y_1 + 2y_2 + 2y_3 \geq 1$ $y_1 - 2y_3 \geq -1$ $y_1 - y_2 + 3y_3 \geq 3$ $y_1, y_2, y_3 \geq 0$	**2.**	Max $Z_D = 2y_1 + y_2$ subject to $2y_1 + y_2 \leq 3$ $3y_1 + y_2 \leq 1$ $y_1, y_2 \geq 0$
3.	Max $Z_D = 3y_1 + 5y_2$ subject to $y_1 \leq 4$ $y_2 \leq 6$ $3y_1 + 2y_2 \leq 18$ $y_1, y_2 \geq 0$	**4.**	Max $Z_D = 5y_1 - 3y_2 - 5y_3$ subject to $2y_1 - 3y_2 - y_3 \leq 2$ $3y_1 - y_2 - 4y_3 \leq 3$ $5y_1 - 7y_2 - 6y_3 = 4$ $y_1, y_3 \geq 0$, y_2 is unrestricted.
5.	Max $Z_D = -7y_1 + 12y_2 + 10y_3$ subject to $-3y_1 + 2y_2 - 4y_3 \leq 1$ $-y_1 + 4y_2 - 3y_3 \geq 3$ $2y_1 - 8y_3 = 2$ $y_1, y_2 \geq 0$, y_3 is unrestricted.	**6.**	Min $Z_D = 6y_1 + 9y_2 + 10y_3$ subject to $4y_1 + 2y_2 + y_3 \geq 3$ $-5y_1 + 3y_2 + y_3 \geq 1$ $-9y_1 + 4y_2 - 5y_3 \geq 4$ $y_1 - 5y_2 - 7y_3 \geq 1$ $-2y_1 + y_2 + 11y_3 \geq 9$ $y_1, y_2, y_3 \geq 0$

❍❍❍

CHAPTER

3



Transportation Problem

3.1 Transportation Problem

[B.C.A. (Meerut) 2004, 2007, 2011; B.C.A. (Agra) 2006, 2010; B.C.A. (Rohilkhand) 2011; B.C.A. (Kanpur) 2006; B.B.A. (Meerut) 2002, 2003, 2004, 2007, 2009]

Transportation problem is a special kind of linear programming problem in which goods are transported from a set of **sources** to a set of **destination** subject to the supply and demand of the source and destination, respectively, such that the total cost of transportation is minimized.

Let m = the number of sources

n = the number of destinations

a_i = the supply at the source i

b_j = the demand at the destination j

c_{ij} = the cost of transportation per unit from ith source to jth destination

x_{ij} = the number of units to be transported from the source ith to the jth the destination

3.2 Mathematical Formulation of Transportation Problem

[B.C.A. (Kashi) 2011; B.C.A. (Lucknow) 2008; B.C.A. (Agra) 2002, 2004, 2009; B.B.A. (Meerut) 2002]

Mathematically, the problem may be stated as follows:

$$\text{Minimize } (Z) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

Subject to the constraints:

$$\sum_{j=1}^{n} x_{ij} = a_i \text{ for } i = 1, 2, 3, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{ij} = b_j \text{ for } j = 1, 2, 3, \ldots, n$$

where $x_{ij} \geq 0$ for all i and j

For a feasible solution to exist, it is necessary that total supply equals total requirement *i.e.*

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \text{[Rim Condition]}$$

The above information can be put in the form of a general matrix shown below:

Destination

		w_1	w_2	...	w_j	...	w_n	**Supply**
	F_1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}	a_1
	F_2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}	a_2
Origin	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	F_i	c_{i1}	c_{i2}	...	c_{ij}	...	c_{in}	a_i
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	F_m	c_{m1}	c_{m2}	...	c_{mj}	...	c_{mn}	a_m
Demand		b_1	b_2	...	b_j	...	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

3.2.1 Definition

1. **Feasible Solution:** A set of non-negative values x_{ij} $(i = 1, 2, \ldots m, j = 1, 2, 3, \ldots n)$ that satisfies the constraints is called a **feasible solution** to the transportation problem.

[B.C.A. (Meerut) 2009, 2010]

2. **Basic Feasible Solution:** A feasible solution that contains no more than $m + n - 1$ non-negative allocations is called a **basic feasible solution** to the transportation problem. **[B.C.A. (Meerut) 2009]**

3. **Optimal Solution:** A feasible solution is said to be **optimal** if it minimizes the total transportation cost. **[B.C.A. (Meerut) 2009]**

4. **Non-degenerate Basic Feasible Solution:** A basic feasible solution to $(m \times n)$ transportation problem that contains exactly $m + n - 1$ allocations in independent positions.

5. **Degenerate Basic Feasible Solution:** A basic feasible solution that contains less than $m + n - 1$ non-negative allocations.

6. **Balanced and Unbalanced Transportation Problems:** A transportation problem is said to be balanced if the total supply from all the sources equals the total demand in all the destinations and is called unbalanced otherwise. Thus,

For balance problem $$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

For unbalance problem $$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j.$$

3.2.2 Existence of Feasible Solution

Theorem 1: A necessary and sufficient condition for the existence of feasible solution of a transportation problem is:

$$\boldsymbol{\Sigma a_i = \Sigma b_j \ (i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n).}$$

[B.C.A. (Lucknow) 2010; B.C.A. (I.G.N.O.U.) 2001, 2004, 2012]

Proof: The Condition is Necessary: Let there exist a feasible solution to the transportation problem. Then,

$$\sum_{j=1}^{n} x_{ij} = a_i, \ \ i = 1, 2, \ldots, m$$

and $$\sum_{i=1}^{m} x_{ij} = b_j, \ j = 1, 2, \ldots, n$$

Summing over all i and j respectively, we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i$$

and $$\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j$$

$$\Rightarrow \qquad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The Condition is Sufficient: Let $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = k$ (say).

If $x_{ij} = \lambda_i\, b_j$ for all i and j, where $\lambda_i \neq 0$ is any real number, then

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \lambda_i b_j = \lambda_i \sum_{j=1}^{n} b_j = k\,\lambda_i$$

$$\Rightarrow \qquad \lambda_i = \frac{1}{k}\sum_{j=1}^{n} x_{ij} = \frac{a_i}{k}$$

Thus, $$x_{ij} = \lambda_i b_j = \frac{a_i b_j}{k}, \text{ for all } i \text{ and } j.$$

As $a_i \geq 0, b_j \geq 0$ so $x_{ij} \geq 0$ for all i and j.
Hence, a feasible solution exists.

Theorem 2: Out of $(m + n)$ equations, there are only $m + n - 1$ independent equations in a transportation problem, m and n being the number of origins and destinations and any one equation can be dropped as the redundant equation.

Proof: Consider m row equations and $n - 1$ column equations of the transportation problem as:

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m \qquad \text{...(1)}$$

and $$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n-1 \qquad \text{...(2)}$$

Now, adding m origin constraints given in (1), we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \qquad \text{...(3)}$$

Also, adding $(n - 1)$ destination constraints given in (2), we get

$$\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \qquad \text{...(4)}$$

Subtracting (4) from (3), we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n-1} b_j$$

or $$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij}\right) = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j \qquad [\because\ \Sigma a_i = \Sigma b_j]$$

or $$\sum_{i=1}^{m} x_{in} = b_n,$$ which is the nth destination-constraint.

It follows that if $m + n - 1$ constraints are satisfied then the $(m + n)$th constraint will be automatically satisfied due to the condition $\Sigma a_i = \Sigma b_j$. Thus we have only $(m + n - 1)$ linearly independent equations. Out of $(m + n)$ equations, one (any) is redundant.

Hence, the theorem is proved.

> **NOTE:**
>
> It indicates that a B.F.S. will contain atmost $m + n - 1$ positive variables, others being zero.

3.2.3 Existence of an Optimal Solution

Theorem 3: There always exists an optimal solution to a balanced transportation problem.

Proof: We have $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

$\Rightarrow$ feasible solution exists of the problem *i.e.,* $x_{ij} \geq 0$ for all i and j.

From the constraints of the problem each $x_{ij} \leq \min.(a_i, b_j)$.

Thus, $0 \leq x_{ij} \leq \min(a_i, b_j)$ *i.e.,* the feasible region of the problem is non-empty, closed and bounded.

Hence, there exists an optimal solution.

3.3 Methods to Solve Transportation Problem

[B.B.A. (Meerut) 2004]

The solution procedure for the transportation problem consists of two parts:

1. Finding the initial basic feasible solution.
2. Optimization of the initial basic feasible solution obtained in part (i) There are three techniques to find the initial basic feasible solution. The solution from these techniques may not be optimal. The three techniques are:
   (i) North-west corner cell method
   (ii) Least cost entry method
   (iii) Vogal's approximation method (VAM) or penalty method.

   These techniques are mentioned in order of their solution accuracy *i.e.,* the VAM method gives the most accurate solution than the above two methods.

3.3.1 Algorithm for North-West Corner Cell Method

[B.B.A. (Meerut) 2008]

Step 1: Find the minimum of the supply and demand values with respect to the current north-west corner cell of the cost matrix.

Step 2: Allocate this minimum value to the current north-west cell and subtract this minimum from the supply and demand value of the current north-west corner cell.

Step 3: Check whether exactly one of the row or column correspondence to north-west corner cell has zero supply or demand, respectively. If so, go to step 4 else go to step 5.

Step 4: Delete that row or column of the current north-west cell which has zero supply or demand and go to step 6.

Step 5: Delete both the row and column with respect to the current north-west corner cell.

Step 6: Check whether exactly one row or column is left out. If yes, go to step 7, else go to step 1.

Step 7: Match the supply or demand of that row or column with the remaining demands or supplies of the undeleted columns or rows.

Step 8: Go to phase 2 for optimization of solution obtained above.

3.3.2 Transportation Problem as L.P. Problem

[B.B.A. (Meerut) 2006]

Simplex algorithm can be used to solve any linear programming model. But this algorithm is lengthy and laborious, due to their specialized structure wherever possible, we try to simplify the calculations. One such model requiring simplified calculations is called transportation model. The model is applicable to the sub-class of linear programming problems in which resources and requirements are expressed in terms of only one kind of unit. The name of this model is derived from transport to which it was first applied. This model can be used for machine assignment, plant location, product mixed problems and many others, so that the model is really not confined to transportation or distribution only.

3.3.3 Objective of Transportation Problem

[B.B.A. (Meerut) 2002, 2004, 2006]

The transportation problem deals with the transportation of a single product from several sources to several destinations. Transportation models which involve a number of shipping sources and a number of destinations, each shipping source has a certain capacity and each destination has a certain requirement associated with a certain cost of shipping from the sources to the destinations. The object is to minimize the cost of transportation while meeting the requirements at the destinations.

The distinct feature of transportation problems is that sources and jobs must be expressed in terms of only one kind of unit.

Solved Examples

Example 1: *Consider the following transportation problem and obtain basic feasible solution using north-west corner cell method.*

Destination

		1	2	3	4	*Supply*
	1	3	1	7	4	300
Source	2	2	6	5	9	400
	3	8	3	3	2	500
	Demand	250	350	400	200	1200

[B.C.A. (Avadh) 2008]

Solution: In the given table, the sum of supplies and the sum of demands are equal. Hence, the given transportation problem is a balanced one. We see that the supply and demand values corresponding to the north-west corner cell (1, 1) are 300 and 250 respectively. The minimum of these value is 250. Hence, we allocate 250 units to the cell (1, 1) and subtract the same from the supply and demand of the cell (1, 1). Now, the supply to the destination 1 is fully met.

Hence, delete this column and the resultant data is shown in table:

Destination

		1	2	3	4	**Supply**
	1	250 3	1	7	4	50
Source	2	2	6	5	9	400
	3	8	3	3	2	500
	Demand	0	350	400	200	950

Now, deleting column 1, then

Destination

Source		2	3	4	Supply
	1	50 1	7	4	0
	2	6	5	9	400
	3	3	3	2	500
	Demand	300	400	200	900

The supply and demand values corresponding to the north-west corner cell are 50 and 350 respectively. The minimum of these is 50. Hence, we allocate 50 units to the cell (1, 2) and subtract the same from the supply and demand values of the cell (1, 2). In this process, the supply of row 1 is fully exhausted. Hence, we delete this row and the resultant data.

Destination

Source		2	3	4	Supply
	2	300 6	5	9	100
	3	3	3	2	500
	Demand	0	400	200	600

The supply and demand values corresponding the north-west corner cell are 400 and 300 respectively. We allocate 300 to the cell (2, 2) and subtract the same from the supply and demand values of the cell (2, 2). Now, the demand of destination 2 is fully met. Hence, we delete this column and the resultant data.

Again, the supply and demand values to the north-west corner cell are 100 and 400. We allocate 100 to the cell (2, 3) and subtract the same from the supply and demand values of the cell (2, 3).

Destination

		3	4	Supply
Source	2	100 5	9	0
	3	3	2	500
	Demand	300	200	500

In this process, the supply of source 2 is fully exhausted. Hence, we delete this row and resultant data.

Destination

		3	4	
Source	3	300 3	200 2	0
		0	0	

In this table, only one source is left out. Hence, the demands of the destination 3 and 4 need to be matched with the supply of source 3.

The initial basic feasible solution for the given problem is shown in the following table:

Destination

		1	2	3	4	Supply
	1	250 3	50 1	7	4	300
Source	2	2	300 6	100 5	9	400
	3	8	3	300 3	200 2	500
	Demand	250	350	400	200	1200

A basic cell is the one which has a positive allocation. Thus, total cost

$$= 3 \times 250 + 1 \times 50 + 6 \times 300 + 5 \times 100 + 3 \times 300 + 2 \times 200$$

$$= 750 + 50 + 1800 + 500 + 900 + 400$$

$$= ₹\ 4{,}400.$$

Example 2: *Find the initial basic feasible solution of the following transportation problem.*

To

		w_1	w_2	w_3	*Available*
	F_1	2	7	4	5
From	F_2	3	3	1	8
	F_3	5	4	7	7
	F_4	1	6	2	14
Requirement		7	9	18	34

[B.C.A. (Bhopal) 2006, 2012]

Solution:

	w_1	w_2	w_3	Available
F_1	2 (5)	7	4	5
F_2	3 (2)	3 (6)	1	8
F_3	5	4 (3)	7 (4)	7
F_4	1	6	2 (14)	14
Requirement	7	9	18	34

Thus, Total cost $= 2 \times 5 + 3 \times 2 + 3 \times 6 + 4 \times 3 + 7 \times 4 + 2 \times 14 = 102.$

$$= 10 + 6 + 18 + 12 + 28 + 28$$

$$= ₹\ 102$$

The method of this example is explained in example (1) and in algorithm for north-west corner cell.

3.3.4 Algorithm for Least Cost Cell Method

[B.B.A. (Meerut) 2008]

Step 1: Find the minimum of the values or matrix minimum in the cost matrix.

Step 2: Find the minimum of the supply and demand values with respect to the cell corresponding to the minimum value obtained in step 1.

Step 3: Allocate the minimum of the supply and demand to the cell having matrix minimum. Also subtract the same from the supply and demand values corresponding to the cell with matrix minimum.

Step 4: Check whether exactly one of the row or column corresponding to the cell with matrix minimum has zero supply or zero demand respectively. If yes, go to step 5 else go to step 6.

Step 5: Delete the row or column with respect to the cell with matrix minimum which has the zero supply or zero demand and go to step 7.

Step 6: Delete both the row and the column with respect to the cell with the matrix minimum.

Step 7: Check whether exactly one row or column is left out. If yes, go to step 8, else go to step 1.

Step 8: Match the supply or demand of that row or column with the remaining demands or supplies of the undeleted columns or rows.

Step 9: Go to phase 2 for optimization of the solution obtained above.

Example 3: *Consider the following transportation problem and obtain the initial basic feasible solution using least cost cell method.*

Destination

		1	2	3	4	*Supply*
	1	3	1	7	4	300
Source	2	2	6	5	9	400
	3	8	3	3	2	500
	Demand	250	350	400	200	1200

Solution: The matrix minimum belongs to the cell (1, 2) and the corresponding supply and demand values are 300 and 350 respectively. We allocate minimum of these two in the cell (1, 2) and subtract the same from the supply and demand corresponding to the cell (1, 2), so we have

Destination

		1	2	3	4	Supply
	1	3	300 1	7	4	0
Source	2	2	6	5	9	400
	3	8	3	3	2	500
Demand		250	50	400	200	900

In this process, the supply of source 1 is completely exhausted. Hence, row 1 is deleted. Then,

Destination

	1	2	3	4	Supply
2	250 2	6	5	9	150
3	8	3	3	2	500
Demand	0	50	400	200	650

Again, the matrix minimum is 2 which occur at the cell (2, 1) and (3, 4). The cell (2, 1) is randomly selected for allocation. The supply and demand values corresponding to cell (2, 1) are 400 and 250 respectively. The minimum of these values is 250. Hence allocate 250 to cell (2, 1) and subtract the same from the supply and demand of the corresponding cell. In this process the demand of destination is completely met. Therefore, we delete thus column from the above table. Then,

Destination

		2	3	4	Supply
Source	2	6	5	9	150
	3	8	3	200 2	300
	Demand	50	400	0	450

Again, the matrix minimum is 2 at the cell (3, 4). The supply and the demand values corresponding to this cell are 500 and 200 respectively. We allocate the minimum of the two *i.e.,* 200 to the cell (3, 4) and subtract the same from the supply and demand values of the cell (3, 4). In this process the demand of destination 4 is fully met. After deleting this column, we get

Destination

		2	3	Supply
Source	2	6	5	150
	3	3	300 3	0
	Demand	50	100	150

Now, the matrix minimum is 3 which occurs at cells (3, 2) and (3, 3). The cell (3, 3) is randomly chosen for allocation. The supply and demand values corresponding to this cell are 300 and 400 respectively. The minimum of these two is 300. Hence, we allocate 300 to the cell (3, 3) and subtract the same from the supply and demand values of the cell (3, 3). In this process the supply of source 3 is fully exhausted. Hence, this row is deleted. Then,

	2	3	Supply
2	6 [50]	5 [100]	0
Demand	0	0	0

In this table only one source is felt out. Therefore, we should match the demands of destination 2 and 3 with supply of source 2. We allocate 100 and 50 to the cells (2, 3) and (2, 2) respectively. Hence, initial basic feasible solution using L.C. cell method.

Destination

		1	2	3	4	Supply
	1	3	1 [300]	7	4	300
Source	2	2 [250]	6 [50]	5 [100]	9	400
	3	8	3	3 [300]	2 [200]	500
Demand		250	350	400	200	1200

The total cost is calculated by adding the products of the cost of transportation per unit in each cell and the corresponding number of units allocated to it shown below:

$$\text{Total cost} = 1\times 300 + 2\times 250 + 6\times 50 + 5\times 100 + 3\times 300 + 2\times 200$$
$$= 300 + 500 + 300 + 500 + 900 + 400$$
$$= ₹\ 2900$$

3.3.5 Algorithm for Vogal's Approximation Method

[B.B.A. (Meerut) 2008]

Step 1: Find row penalties *i.e.,* the difference between the first minimum and second minimum in each row. If the two minimum values are equal, then the row penalty is zero.

Step 2: Find column penalties *i.e.,* the difference between first minimum and second minimum in each column. If two minimum values are equal, then the column penalty is zero.

Step 3: Find the maximum amongst row penalties and column penalties and identify whether it occurs in a row or in column. If the maximum penalty is in row go to step 4 otherwise go to step 7.

Step 4: Identify the cell for allocation which has the least cost in that row.

Step 5: Find the minimum of the supply and demand values with respect to the selected cell.

Step 6: Allocate this minimum values to that cell and subtract this minimum from the supply and demand values with respect to the selected cell and go to step 10.

Step 7: Identify the cell for allocation which has the least cost in that column.

Step 8: Find the minimum of the supply and demand values with respect to the selected cell.

Step 9: Allocate this minimum value to the selected cell and subtract this minimum from the supply and demand values with respect to the selected cell.

Step 10: Check whether exactly one of the rows and the columns corresponding to the selected cell has zero supply or demand, respectively if yes, go to step 11, otherwise go to step 12.

Step 11: Delete the row or column which has zero supply or demand and revise the corresponding row or column penalties. Then go to step 13.

Step 12: Delete both the row and the column with respect to the selected cell. Then, revise the row and the column penalties.

Step 13: Check whether exactly one row or column is left out, if yes, go to step 14, otherwise go to step 3.

Step 14: Match the supply or demand of the left out row or column with remaining demand or supplies of the undeleted columns or rows.

Step 15: Go to phase 2 for the optimization of the solution obtained above.

Example 4: *Consider the following transportation problem and obtain the initial basic feasible solution using Vogal's approximation method (VAM).*

Destination

		1	2	3	4	*Supply*
	1	3	1	7	4	300
Source	2	2	6	5	9	400
	3	8	3	3	2	500
	Demand	250	350	400	200	1200

Solution: First, find row penalties and column penalties. The maximum of these is 3 which is in row 2. Hence, the cell with the least cost in the row 2 is to be identified. This occurs at the cell (2, 1). The supply and demand values corresponding to the cell (2, 1) are 400 and 250 respectively. The minimum of these two is 250. Therefore, 250 units are allocated to the cell (2, 1) and the same is subtracted from the supply and demand values of the cell (2, 1).

Destination

		1	2	3	4	Supply	Penalty
Source	1	3	1	7	4	300	2
	2	2 (250)	6	5	9	150	3
	3	8	3	3	2	500	1
Demand		0	350	400	200	950	
Penalty		1	2	2	2		

In this process, the demand at destination 1 is fully met. Hence, this column is deleted and the resultant data is shown in next table.

Destination

		2	3	4	Supply	Penalty
Source	1	1 (300)	7	4	0	3
	2	6	5	9	150	1
	3	3	3	2	500	1
Demand		50	400	200	650	
Penalty		2	2	2		

As the column 1 deleted, we again revise the penalty. The maximum of these penalties is 3 which is in row 1. The cell with least cost is identified in row 1 as (1, 2). The supply and demand value corresponding to cell (1, 2) are 300 and 350 respectively. The minimum of these values is 300. Hence, we allocate 300 units to cell (1, 2). Thus, the supply at the row 1 is fully met. Hence, we delete row 1 and the resultant data is shown is next table.

	2	3	4	Supply	Penalty
2	6	5	9	150	1
3	3	3	200 2	300	1
Demand	50	400	0	450	
Penalty	3	2	7		

Similarly, working in the same way, we find that the column 4 is deleted and the resultant data is shown in next table.

	2	3	Supply	Penalty
2	6	5	150	1
3	50 3	3	250	0
Demand	0	400	400	
Penalty	3	2		

Similarly, we find that column 2 is deleted and only one column is left out. The supplies of the source 2 and 3 are matched with the demand of the destination 3 as follows:

	3	Supply
2	150 5	150
3	250 3	250
Demand	400	400

The initial basic feasible solution of the problem using VAM is as follows:

Destination

Source	1	2	3	4	Supply
1	3	300 / 1	7	4	300
2	250 / 2	6	150 / 5	9	400
3	8	50 / 3	250 / 3	200 / 2	500
Demand	250	350	400	200	1200

The total cost of the solution is obtained by adding the products of the cost of transportation per unit in the each and every basic all and the corresponding units allocated to it. Thus,

$$\text{Total cost} = 1 \times 300 + 2 \times 250 + 3 \times 50 + 5 \times 150 + 3 \times 250 + 2 \times 200$$

$$= 300 + 500 + 150 + 750 + 750 + 400$$

$$= ₹\ 2850$$

Thus, we can see that VAM provides the optimal transportation cost when the same is compared with other two methods.

Example 5: *Explain the transportation problem. What is the difference between a transportation and assignment problem?*

Four factories (A, B, C, D) supply the requirements of three warehouses (E, F, G). The availability at the factories, the requirement of the warehouses and the unit transportation costs are presented in the following table:

Warehouse

Factory	*E*	*F*	*G*	*Available*
A	10	8	9	15
B	5	2	3	20
C	6	7	4	30
D	7	6	8	35
Required	25	26	49	100

Find an initial basic feasible solution of the transportation problem by using:

(i) North-West corner rule

(ii) Matrix minimum method

(iii) Vogal's Approximation method. **[B.B.A. (Meerut) 2004]**

Solution: (i) By North-West Corner Rule.

	E	F	G	Available
A	10 [15]	8	9	15
B	5 [10]	2 [10]	3	20
C	6	7 [16]	4 [14]	30
D	7	6	8 [35]	35
Required	25	26	49	100

Total transportation cost $= 10 \times 15 + 5 \times 10 + 2 \times 20 + 7 \times 16 + 4 \times 14 + 8 \times 35$

$= 150 + 50 + 40 + 112 + 56 + 280$

$=$ ₹ 688

NOTE:

The process is explained in above example and algorithm.

(ii) By Matrix-Minimum Method.

	E	F	G	Available
A	10	8	9 [15]	15
B	5	2 [20]	3	20
C	6	7	4 [30]	30
D	7 [25]	6 [6]	8 [4]	35
Required	25	26	49	100

Total transportation cost $= 2 \times 20 + 9 \times 15 + 4 \times 30 + 7 \times 25 + 6 \times 6 + 8 \times 4$

$= 40 + 135 + 120 + 175 + 36 + 32$

$=$ ₹ 538

NOTE:

The process is explained in above example and algorithm.

(iii) By Vogal's Approximation Method.

	E	*F*	*G*	Available	Penalty
A	10	8	9	15	1
B	5	20 2	3	20	1
C	6	7	4	30	2
D	7	6	8	35	1
Required	25	26/6	49	100	
Penalty	1	4	1		

Reduced Matrix is

	E	*F*	*G*	Available	Penalty
A	10	8	9	15	1
C	6	7	30 4	30	2
D	7	6	8	35	1
Required	25	6	49/19	80	
Penalty	1	1	4		

Reduced Matrix is

	E	*F*	*G*	Available	Penalty
A	10	8	9	15	1
D	25 7	6	8	35/10	1
Required	25	6	19	50	
Penalty	3	2	1		

Reduced Matrix is

	F	G	Available	Penalty
A	8	9	15	1
D	6 (6)	8	10/4	2
Required	6	19	25	
Penalty	2	1		

Again reduced Matrix is

	G	Available	Penalty
A	9 (15)	15	9
D	8 (4)	4	8
Penalty	1		

Hence,

	E	F	G	Available
A	10	8	9 (15)	15
B	5	2 (20)	3	20
C	6	7	4 (30)	30
D	7 (25)	6 (6)	8 (4)	35
	25	26	49	100

The total transportation cost $= 9 \times 15 + 2 \times 20 + 4 \times 30 + 7 \times 25 + 6 \times 6 + 8 \times 4$

$= 135 + 40 + 120 + 175 + 36 + 32$

$=$ ₹ 538

Example 6: *Determine an initial B.F.S. to the following transportation table using Vogel's approximation method.*

Destination

		D_1	D_2	D_3	D_4	*Supply*
Origin	O_1	1	2	1	4	30
	O_2	3	3	2	1	50
	O_3	4	2	5	9	20
	Demand	20	40	30	10	100

[B.C.A. (Lucknow) 2009; B.C.A. (Indore) 2010; B.B.A. (Delhi) 2004; (Meerut) 2008]

Solution: First, we write cost and requirement matrix and compute the penalties which is the difference between lowest and second lowest element in each row and each column

	D_1	D_2	D_3	D_4		Penalties (P)
O_1	1	2	1	4	30	1
O_2	3	3	2	10 (1)	50 – 10	1
O_3	4	2	5	9	20	2
	20	40	30	10 – 10	90	
Penalties (P)	2	1	1	3 ↑		

Since maximum penalty 3 is in column 4th, then allocate maximum possible amount 10 in the cell (1, 4) having minimum cost *i.e.,* $x_{14} = 10$. Thus the requirements of 4th column are completed, so leaving column 4th then, reduced matrix as follows:

	D_1	D_2	D_3		Penalties (P)
O_1	1	2	1	30	1
O_2	3	3	2	40	1
O_3	4	20 (2)	5	20 – 20	2 ←
	20	40 – 20	30	70	
Penalties (P)	2	1	1		

Since maximum penalty 2 is in Ist column and IIIrd row, so we select one of them, say row 3rd and allocate maximum possible amount 20 in the cell (3, 2) having minimum cost *i.e.,* $x_{32} = 20$.

Thus, the requirements of 3rd row are completed, so leaving row 3rd then reduced matrix as follows:

	D_1	D_2	D_3		Penalties (P)
O_1	20 (1)	2	1	30 – 20	1
O_2	3	3	2	40	1
	20 – 20	20	30	50	
Penalties (P)	2 ↑	1	1		

Since maximum penalty 2 is Ist column and allocate maximum possible amount 20 in the cell (1, 1) having minimum cost *i.e.*, $x_{11} = 20$.

Thus, the requirements of Ist column are completed, so leaving column Ist then reduced matrix as follows:

	D_2	D_3		Penalties (P)
O_1	2	10 (1)	10 – 10	1 ←
O_2	20 (3)	20 (2)	40	1
	20	30	40	
Penalties (P)	1	1		

The maximum penalty 1 which is in Ist, IInd rows and I, II column then select any *i.e.* we select Ist row and allocating maximum possible amount 10 in row Ist at (1, 3), we complete the remaining allocation.

Thus, the final table giving initial B.F.S. is as follows:

Destination

		D_1	D_2	D_3	D_4	
	O_1	20 (1)	2	10 (1)	4	30
Origin	O_2	3	20 (3)	20 (2)	10 (1)	50
	O_3	4	20 (2)	5	9	20
		20	40	30	10	100

The total minimum cost according to above transportation problem

$$= 20 \times 1 + 10 \times 1 + 20 \times 3 + 20 \times 2 + 10 \times 1 + 20 \times 2$$

$$= 20 + 10 + 60 + 40 + 10 + 40$$

$$= ₹\,180$$

Example 7: *Use north-west corner rule to determine an initial basic feasible solution to the following transportation problem:*

To

		I	*II*	*III*	*IV*	*Supply*
	A	13	11	15	20	2
From	*B*	17	14	12	13	6
	C	18	18	15	12	7
Demand		3	3	4	5	15

[B.B.A. (Rohilkhand) 2006; M.C.A. (Meerut) 2007]

Solution: This is the balance transportation problem as $\Sigma a_i = \Sigma b_i = 15$.

Start with cell (1, 1) and allocate it the maximum amount thus $x_{11} = 2$. Since minimum of $a_1 = 2$ and $b_1 = 3$ is 2. Now no amount is left at source A, so we move vertically downwards to the cell (2, 1) and allocate it as much as possible. Since column I still needs amount 1 and the amount 6 is available at source *B* in row 2, so we allocate the maximum amount 1 to the cell (2, 1). Thus, allocation 3 for column 1 is complete. Then move to right of the cell (2, 1) *i.e.,* at cell (2, 2) and allocate here as much as possible. Since the amount 5 is still available in row 2 and amount 3 is available in column 2, so we can allocate maximum amount 3 in the cell (2, 2) *i.e.,* $x_{22} = 3$. Since the allocation 3 in column 2 is also completed. So we move to the cell (2, 3) on right of the cell (2, 2). Since amount 2 is still available in row 2 and amount 4 is available in column 3, so we allocate maximum amount 2 in cell (2, 3) *i.e.,* $x_{23} = 2$. Now no amount is available in row 2, so we move downwards to the cell (3, 3) and allocate there are much amount as possible. Since the amount 2 is still required in column 3 and amount 7 is available in row 3, so the amount 2 is allocated in the cell (3, 3) *i.e.,* $x_{33} = 2$. Thus, the allocation 4 in the column 3 is complete. Now the amount 5 is still available in the row 3 and also the amount 5 is required in column 5, so the amount 5 is allocated in the cell (3, 4) *i.e.,* $x_{34} = 5$ which complete the allocation.

	I	II	III	IV	b_j
A	2 (13)				2
B	1 (17)	3 (14)	2 (12)		6
C			2 (15)	5 (12)	7
a_i	3	3	4	5	15

The total minimum cost according to above transportation problem

$$= 2 \times 13 + 1 \times 17 + 3 \times 14 + 2 \times 12 + 2 \times 15 + 5 \times 12$$
$$= 26 + 17 + 42 + 24 + 30 + 60$$
$$= 199 \text{ units}$$

Example 8: *Determine an initial basic feasible solution to the following transportation problem by using north-west corner rule.*

Destination

		D_1	D_2	D_3	D_4	*Supply*
	S_1	21	16	15	3	11
Source	S_2	17	18	14	23	13
	S_3	32	27	18	41	19
Demand		6	10	12	15	43

[B.C.A. (Agra) 2008; B.B.A. (Meerut) 2002]

Solution:

Destination

		D_1	D_2	D_3	D_4	Supply
	S_1	6 (21)	5 (16)	15	3	11
Source	S_2	17	5 (18)	8 (14)	23	13
	S_3	32	27	4 (18)	15 (41)	19
Demand		6	10	12	15	43

We start from N-W. corner rule which is corresponding to cell $(1, 1)$, we allocate maximum to units to this cell. Thus, allocation of 6 units leaves the surplus amount of 5 units for the first row. Now 5 units are allocated to the cell $C(1, 2)$. Now, the allocation of first row and first column is complete. Now, we move to cell $(2, 2)$ and allocate 5 units to this cell. Now, move to cell $(2, 3)$ and allocate 8 units to this cell. So that demand of second row is satisfied. Now 4 units are required by third column, so we allocate 4 units to cell $(3, 3)$ and 15 units to cell $(3, 4)$.

The total minimum cost according to above problem

$$= 6 \times 21 + 5 \times 16 + 5 \times 18 + 8 \times 14 + 4 \times 18 + 15 \times 41$$

$$= 126 + 80 + 90 + 112 + 72 + 615$$

$$= ₹\, 1095.$$

Example 9: *Find the initial basic feasible solution by least cost method.*

Warehouse

Factories		W_1	W_2	W_3	W_4	*Supply*
	F_1	48	60	56	58	140
	F_2	45	55	53	60	260
	F_3	50	65	60	62	360
	F_4	52	64	55	61	220
	Demand	200	320	250	210	980

Solution: Since the total demand is equal to total supply so it is balanced transportation problem.

	W_1	W_2	W_3	W_4	Supply
F_1	48	60	56	140 (58)	140
F_2	200 (45)	55	60 (53)	60	260
F_3	50	320 (65)	60	40 (62)	360
F_4	52	64	190 (55)	30 (61)	220
Demand	200	320	250	210	980

The least cost is 45 to the cell (2, 1), thus we allocate maximum of 200 units to this cell. We proceed for next least cost which is 53 corresponding to cell (2, 3), we allocate only 60 units to this cell as 200 units have already been allocated to (2, 1) cell. Now the allocation of IInd row is complete. We proceed for next least cost which is 55 corresponding to (4, 3) cell, we allocate 190 units to this cell as 60 units have already been allocated to cell (2, 3). We proceed to next least cost 58 corresponding to (1, 4) cell, we allocate maximum 140 units to this cell, the allocation of the F_1 row is complete. This process is repeated again and again till all demands and supply is not complete.

Total minimum cost according to above problem

$$= 45 \times 200 + 58 \times 140 + 65 \times 320 + 62 \times 40 + 55 \times 190 + 61 \times 30 + 60 \times 53$$

$$= 9000 + 8120 + 20{,}800 + 2{,}480 + 10450 + 1830 + 3180$$

$$= ₹\ 55{,}860$$

3.4 Transportation Algorithm (MODI) Method

Various steps involved in solving any transportation problem may be summarized in the following iteration procedure.

Step 1: Find the initial basic feasible solution by using any of the three methods.

Step 2: Check the number of occupied cells. If these are less than $m + n - 1$, there exists degeneracy and we introduce a very small positive assignment of $\in (\in \to 0)$ in suitable independent positions, so that the number of occupied cells is exactly equal to $m + n - 1$.

Step 3: For each occupied cell in the current solution, solve the system of equations

$$u_i + v_j = c_{ij}$$

Starting initially with some $u_i = 0$ or $v_j = 0$ and entering successively the values of u_i and v_j in the transportation table margins.

Step 4: Compute the net evaluations $d_{ij} = (u_i + v_j) - (c_{ij})$ for all unoccupied basic cells and enter them in the upper right corner of the corresponding cells.

Step 5: Examine the sign of each d_{ij} if all $d_{ij} \le 0$, then the current basic feasible solution is an optimum one. If at least one $d_{ij} \ge 0$ select the unoccupied cell having the largest positive net evaluation to enter the basis.

Step 6: Let the unoccupied cell (r, s) enter the basis. Allocate an unknown quantity say θ to the cell (r, s). Identify a loop that starts and enters at the cell (r, s) and connects some of the basic cells. Add and subtract interchangeably θ to and from the transition cells of the loop in such a way that the rim requirements remain satisfied.

Step 7: Assign a maximum value θ in such a way that the value of one basic variable becomes zero and other basic variables remain non-negative. The basic cell whose allocation has been reduced to zero leaves the basis.

Step 8: Return the step 3 and repeat the process untill an optimum basic feasible solution has been obtained.

NOTE:

1. If at least one $d_{ij} = 0$, there exists an alternative optimum solution.
2. For maximization transportation problems, replace each element of the transportation table by its difference from the maximum element of the table. Then apply the steps of minimization transportation problem on the revised transportation table.

Example 10: *Solve the following transportation problem.* **[B.B.A. (Meerut) 2002]**

To

		A	*B*	*C*	*Available*
	I	50	30	220	1
From	*II*	90	45	170	3
	III	250	200	50	4
Requirement		4	2	2	

Solution: Using Vogel's approximation method and initial basic feasible solution is obtained as follows:

	A	B	C	Available
I	(1) 50	30	220	1
II	(3) 90	45	170	3
III	250	(2) 200	(2) 50	4
Requirement	4	2	2	8

The Basic feasible solution (B.F.S.) $= 1 \times 50 + 3 \times 90 + 2 \times 200 + 2 \times 50$

$= 50 + 270 + 400 + 100$

$= 820.$

Now, to test the optimality by Modi method.

The number of basic cells $= m + n - 1.$

where m = number of rows, n = number of columns.

But $m = 3, n = 3, m + n - 1 = 3 + 3 - 1 = 5 \neq$ not equal to number of basic cells which is 4. Then, it is the problem of degeneracy, we have allocated a small quantity $\in (> 0)$ in the cell (3, 1) to overcome degeneracy as the number of occupied cells was only 4. Using Modi method an optimal solution is obtained as:

				u_i
	1 50	$(-v_e)$ 30	$(-v_e)$ 220	-200
	3 90	$(-v_e)$ 45	$(-v_e)$ 170	-160
	$\in$ 250	2 200	2 50	0
v_j	250	200	50	

where u_i and v_j is obtained by $u_i + v_j = c_{ij}$

Let $\quad u_3 = 0$ then, $u_3 + v_3 = 50$

$$0 + v_3 = 50, \; v_3 = 50$$

$$u_3 + v_1 = 250, \quad u_3 + v_2 = 200, \; \text{but } \; u_3 = 0$$

$$v_1 = 250, \; v_2 = 200, \; u_1 + v_1 = 50, \quad u_2 + v_1 = 90$$

$$u_1 + 250 = 50, \quad u_2 + 250 = 90$$

$$\Rightarrow \quad u_1 = -200 \;\; u_2 = -160$$

The net evaluation for each of the unoccupied cells are now determined

$$d_{12} = u_1 + v_2 - C_{12} = -200 + 200 - 30 = -30$$

$$d_{13} = u_1 + v_3 - C_{13} = -200 + 50 - 220 = -370$$

$$d_{22} = u_2 + v_2 - C_{22} = -160 + 200 - 45 = -5$$

$$d_{23} = u_2 + v_3 - C_{23} = -160 + 50 - 170 = -280$$

These are obtained by $\quad d_{ij} = (u_i + v_j - c_{ij})$

some author may use $\quad d_{ij} = c_{ij} - (u_i + v_j)$ (then all the net evaluation are positive)

Since all the net evaluations are non-positive, the current solution is an optimum one. The optimum allocation is given by $x_{11} = 1, x_{22} = 3, x_{31} = \in, x_{32} = 2$ and $x_{33} = 2$.

The transportation cost according to the above route is given by

$$= 1 \times 50 + 3 \times 90 + 2 \times 200 + 2 \times 50 + \in \times 250$$

$$= 50 + 270 + 400 + 100 + 250 \in$$

$$= ₹\, 820 + 250 \in = ₹\, 820 \;\; \text{as } \in \to 0$$

Example 11: *Find the optimal solution of transportation problem.*

Destination

		1	2	3	4	*Supply*
	1	3	1	7	4	300
Source	2	2	6	5	9	400
	3	8	3	3	2	500
Demand		250	350	400	200	1200

Solution: First, find the B.F.S. by north-west corner cell method.

First, find the B.F.S. by north-west cell method.

Destination

		1	2	3	4	Supply
	1	250 3	50 1	7	4	300
Source	2	2	300 6	100 5	9	400
	3	8	3	300 3	200 2	500
Demand		250	350	400	200	1200

The B.F.S. by north-west corner rule

$$= 3 \times 250 + 1 \times 50 + 6 \times 300 + 5 \times 100 + 3 \times 300 + 2 \times 200$$

$$= 750 + 50 + 1800 + 500 + 900 + 400$$

$$= ₹\ 4{,}400$$

Now, to test the optimality by Modi method. The number of basic cells is equal to $m + n - 1$, where m is the number of sources and n is the number of destinations column. Hence, $m = 3, n = 4$.

$\therefore$ $m + n - 1 = 3 + 4 - 1 = 6$ = number of basic cells entry.

Now, to find u_i and v_j by $c_{ij} = u_i + v_j$

i.e. $$u_1 + v_1 = 3,\ u_1 + v_2 = 1,\ u_2 + v_2 = 6,\ u_2 + v_3 = 5$$

$$u_3 + v_3 = 3,\ u_3 + v_4 = 2,\ \text{let } u_1 = 0 \text{ then}$$

$$v_1 = 3,\ u_2 = 5,\ v_3 = 0,\ v_4 = -1,\ v_2 = 1,\ u_3 = 3$$

Then, the penalty for each of the non-basic cells is computed using the following formula and summarized within the squares at the bottom left corners of the respective cell

$$d_{ij} = u_i + v_j - c_{ij}$$

Destination

Source						u_i
	250 3 −	50 + 2	7 $-v_e$	4 $-v_e$	0	u_1
	2 + 6	300 − 6	100 5	9 $-v_e$	5	u_2
	8 $-v_e$	3 1	300 3	200 2	3	u_3
	3	1	0	−1		
v_i	v_1	v_2	v_3	v_4		

$$d_{13} = u_1 + v_3 - C_{13} = 0 + 0 - 7 = -7 = -ve$$

$$d_{14} = u_1 + v_4 - C_{14} = 0 - 1 - 4 = -5 = -ve$$

$$d_{24} = u_2 + v_4 - C_{24} = 5 + (-1) - 9 = -5 = -ve$$

In the above table if all the $d_{ij} \le 0$ then optimality is reached. Otherwise, select the cell which has most positive penalty. Here, the cell (2, 1) has most positive penalty. So, the solution can be improved. This non-basic cell is to be converted into a basic cell without effecting the supply and demand restrictions. Hence, we construct a closed loop starting from this new basic cell and passing through the basic cell (2, 2), (1, 2) and (1, 1). Then alternately, the + sign and − sign are assigned in the basic cell on the closed loop commencing from the new basic cell. The minimum of the existing allocation amongst the negatively signed cells on the loop is identified, which is 250 in this case. This minimum allocation is now added to all the positively signed cells on the closed loop but is subtracted from all the negatively signed cells. The improve table is as follows:

Destination

Source					u_i
	300; 1	1	7; $-v_e$	4; $-v_e$	0
	3; $-v_e$				
	250; 2	50; 6 −	100; + 5	9; $-v_e$	5
	8; $-v_e$	+ 3; +1	300; − 3	200; 2	3
v_j	3	1	0	−1	

Repeated the above process we get another improved table since (3, 2) has positive penalty.

					Supply
	3; $-v_e$	300; 1	7; $-v_e$	4; $-v_e$	300
	250; 2	6; $-v_e$	150; 5	9; $-v_e$	400
	8; $-v_e$	50; 3	250; 3	200; 2	500
Demand	250	350	400	200	1200

It is found that all these penalties are less than or equal to zero. (Written as –*ve* in block). Hence, the optimality is reached. Therefore, the total optimal cost

$$= 1 \times 300 + 2 \times 250 + 5 \times 150 + 3 \times 50 + 2 \times 200 + 3 \times 250$$

$$= 300 + 500 + 750 + 150 + 400 + 750$$

$$= ₹\ 2850$$

Example 12: *Obtain an initial basic solution to the following transportation problem. Is this solution on optimal solution? If not, obtain the optimal solution.* **[B.B.A. (Meerut) 2003, 2004, 2009]**

	W_1	W_2	W_3	W_4	a_i
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
b_j	5	8	7	14	34

Solution: The different necessary tables in order to obtain basic feasible solution by Vogal's approximation method are as follows. First write cost and required matrix and compute the penalties which is the difference between lowest and second lowest element in each row and each column.

	W_1	W_2	W_3	W_4	a_i	Penalties (P)
F_1	19	30	50	10	7	9
F_2	70	30	40	60	9	10
F_3	40	8 (8)	70	20	18 – 8	12
b_j	5	8 – 8	7	14	26	
Penalties (P)	21	22 ↑	10	10		

Since maximum penalties is 22 in column w_2, then allocate maximum possible amount 8 in the cell $(3, 2)$. Thus, the requirements of w_2 column are completed, so leaving the column w_2 then reduce matrix as follows:

	W_1	W_3	W_4		P
F_1	5 (19)	50	10	7 – 5	9
F_2	70	40	60	9	20
F_3	40	70	20	10	20
	5 – 5	7	14	21	
P	21 ↑	10	10		

Since maximum penalty is 21 which is in column w_1 and allocate maximum possible amount 5 in cell (1, 1) having maximum cost *i.e.*, $x_{11} = 5$. Thus, the requirements of w_1 column are completed so leave the column w_1 then reduce matrix as follows:

	W_3	W_4		P
F_1	50	10	2	40
F_2	40	60	9	20
F_3	70	10 (20)	10 – 10	50 ←
	7	14 – 10		
P	10	10		

	W_3	W_4		P
F_1	50	2 (10)	2	40
F_2	7 (40)	2 (60)	9	20
	7	4		
P	10	50 ↑		

Thus, the final giving the initial B.F.S. by VAM is as follows:

	W_1	W_2	W_3	W_4	
F_1	5 (19)	30	50	2 (10)	7
F_2	70	30	7 (40)	2 (60)	9
F_3	40	8 (8)	70	10 (20)	18
	5	8	7	14	34

The total minimum cost according to above transportation problem

$$= 5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20$$
$$= 95 + 20 + 280 + 120 + 64 + 200$$
$$= ₹\ 779$$

To Test the Optimality: Since $m + n - 1 = 3 + 4 - 1 = 6$ = number of independent positions

Where m = number of rows

n = number of columns

Now, we determine a set of u_i and v_j for each occupied cell (r, s) such that

$$c_{rs} = u_r + v_s$$

					u_i ↓
	(19) 5	(30) (– 2) (32)	(50) (– 10) (60)	(10) 2	10 (u_1)
	(70) (69) (1)	(30) (48) (– 18)	(40) 7	(60) 2	60 (u_2)
	(40) (29) (11)	(8) 8	(70) 0 (70)	(20) 10	20 (u_3)
v_j →	9 (v_1)	– 12 (v_2)	– 20 (v_3)	0 (v_4)	

Take any u_i and v_j to be zero which has maximum number of allocations *i.e.,* $v_4 = 0$ and other entry is obtain by $c_{rs} = u_r + v_s$.

$$c_{14} = 10 = u_1 + v_4 = u_1 + 0, c_{24} = 60 = u_2 + v_4 = u_2 + 0$$

$$c_{34} = 20 = u_3 + v_4 = u_3 + 0$$

$$\Rightarrow \quad u_1 = 10, u_2 = 60, u_3 = 20$$

$$c_{11} = 19 = u_1 + v_1 = 10 + v_1, c_{23} = 40 = u_2 + v_3 = 60 + v_3$$

$$c_{32} = 8 = u_3 + v_2 = 20 + v_2$$

$$\Rightarrow \quad v_1 = 9, v_3 = -20, v_2 = -12$$

Now, we have to find $u_i + v_j$ for each unoccupied cell and enter at the upper right corner of the corresponding cell then find $d_{ij} = c_{ij} - (u_i + v_j)$ for each unoccupied cell and enter at the lower right corner of the corresponding cell. By completing all entries, we find $d_{22} = -18 < 0$.

Therefore this solution is not optimal. So we proceed to improve this solution. Since minimum $d_{ij} = d_{22} - 18 < 0$, we shall allocate to this cell (2, 2) as much as possible. This allotment is shown in table (a) and table (b).

5			2
	+ 2	7	2 − 2
	8 − 2		10 + 2

				u_i ↓
(19) **5**	(30) (− 2) (32)	(50) (8) (42)	(10) **2**	0 (u_1)
(70) (51) (19)	(30) **2**	(40) **7**	(60) 42 18	32 (u_2)
(40) (29) (11)	(8) **6**	(70) (18) (52)	(20) **12**	10 (u_3)
$v_j \rightarrow$ 19 (v_1)	− 2 (v_2)	8 (v_3)	10 (v_4)	

For this solution, total transportation cost

$$= 5 \times 19 + 2 \times 10 + 2 \times 30 + 7 \times 40 + 6 \times 8 + 12 \times 20$$
$$= 95 + 20 + 60 + 48 + 280 + 240$$
$$= ₹\,743$$

For this solution is optimal we again test the optimality till all $d_{ij} > 0$. Therefore, this solution is optimal as all $d_{ij} > 0$.

Example 13: *Solve the transportation problem.*

	D_1	D_2	D_3	D_4	*Available*
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Required	20	40	30	10	100

[B.C.A. (Purvanchal) 2010; B.C.A. (Meerut) 2008, 2010]

Solution: By "Lowest cost entry method', we get by usual process the following B.F.S. of this problem.

	D_1	D_2	D_3	D_4	
O_1	(1) **20**	(2)	(1) **10**	(4)	30
O_2	(3)	(3) **20**	(2) **20**	(1) **10**	50
O_3	(4)	(2) **20**	(5)	9	20
	20	40	30	10	100

Total transportation cost

$$= 1 \times 20 + 1 \times 10 + 3 \times 20 + 2 \times 20 + 1 \times 10 + 2 \times 20$$
$$= 20 + 10 + 60 + 40 + 10 + 40$$
$$= ₹\,180$$

To Test the Optimality: Since $m + n - 1 = 3 + 4 - 1 = 6$ = number of independent positions, then applying Modi method.

Now, we have to find set of u_i and v_j by

$$c_{rs} = u_r + v_s \text{ where } c_{rs} \text{ is occupied cell}$$

Taking $u_2 = 0$ as row IInd has maximum number of allocations.

				$u_i \downarrow$
(1) **20**	(2) (2)	(1) **10** (0)	(4) (0) (4)	$-1\,(u_1)$
(3) (2) (1)	(3) **20**	(2) **20**	(1) **10**	$0\,(u_2)$
(4) (1) (3)	(2) **20**	(5) (1) (4)	(9) (0) (9)	$-1\,(u_3)$
2 (v_1)	3 (v_2)	2 (v_3)	1 (v_4)	$\leftarrow v_j$

As
$$c_{22} = 3 = u_2 + v_2 = 0 + v_2 \Rightarrow v_2 = 3$$
$$c_{23} = 2 = u_2 + v_3 = 0 + v_3 \Rightarrow v_3 = 2$$
$$c_{24} = 1 = u_2 + v_4 = 0 + v_4 \Rightarrow v_4 = 1$$
$$c_{32} = 2 = u_3 + v_2 = u_3 + 3 \Rightarrow u_3 = -1$$
$$c_{13} = 1 = u_1 + v_3 = u_1 + 2 \Rightarrow u_1 = -1$$
$$c_{11} = 1 = u_1 + v_1 = -1 + u_1 \Rightarrow v_1 = 2$$

For each unoccupied cell, we find cell evaluations $u_i + v_j$ and enter at the upper right corner of the corresponding cell. Then we find cell $d_{ij} = c_{ij} - (u_i + v_j)$ which is the difference of upper right corner entry from the upper left corner entry for all unoccupied cells and enter at the lower right corner of the corresponding cell. We see all $d_{ij} \geq 0$ then B.F.S. is also optimal solution *i.e.,* minimum transportation cost = ₹ 180.

Example 14: *The cost-requirement table for the transportation problem is given below:*

		To					
		W_1	W_2	W_3	W_4	W_5	*Available*
	F_1	4	3	1	2	6	40
From	F_2	5	2	3	4	5	30
	F_3	3	5	6	3	2	20
	F_4	2	4	4	5	3	10
Required		30	30	15	20	5	

Obtain the optimal solution of the problem. **[B.B.A. (Delhi) 2004, 2007; (Gwalior) 2008]**

Solution: By Vogel's approximation method, an initial B.F.S. of the problem is given in the following table:

(4) **5**	(3)	(1) **15**	(2) **20**	(6)	40
(5)	(2) **30**	(3)	(4)	(5)	30
(3) **15**	(5)	(6)	(3)	(2) **5**	20
(2) **10**	(4)	(4)	(5)	(3)	10
30	30	15	20	5	

For this solution total transportation cost

$$= 5 \times 4 + 15 \times 1 + 20 \times 2 + 30 \times 2 + 3 \times 15 + 5 \times 2 + 10 \times 2$$
$$= 20 + 15 + 40 + 60 + 45 + 10 + 20$$
$$= ₹\ 210$$

Here number of allocations is 7 which is one less than the number $m + n - 1 = 8$.

So the solution is degenerate solution.

To Test the Solution for Optimality: To make the number of allocations equal to 6 we allocate a small positive amount $\in$ to the cell $(2, 3)$ having minimum cost in empty cells. To test the solution for optimality, we find the set of u_i and v_j so that for each occupied cell (r, s), $c_{rs} = u_r + v_s$, in the usual manner, by considering $u_1 = 0$.

						$u_i \downarrow$
	(4) **5**	(3) (0) (3)	(1) **15**	(2) **20**	(6) (3) (3)	0 (u_1)
	(5) (6) (−1)	(2) **30**	(3) $\in$	(4) (4) (0)	(5) (5) (0)	2 (u_2)
	(3) **15**	(5) (−1) (6)	(6) (0) (6)	(3) (1) (2)	(2) **5**	−1 (u_3)
	(2) **10**	(4) (−2) (6)	(4) (−1) (5)	(5) (0) (5)	(3) (1) (2)	−2 (u_4)
$v_j \to$	4 v_1	0 v_2	1 v_3	2 v_4	3 v_5	

Now, in each unoccupied cell we enter the values of $u_i + v_j$ and $d_{ij} = c_{ij} - (u_i + v_j)$ at the appropriate corners and find that $d_{21} < 0$. So this solution is not optimal. Taking the cell evaluation $\in$ from cell (2, 3) to (2, 1) and proceeding similarly, we get the following table for the test of optimality of this solution:

					$u_i \downarrow$
(4) **5**	(3) (1) (2)	(1) **15**	(2) **20**	(6) (3) (3)	0 (u_1)
(5) $\in$	(2) **30**	(3) (2) (1)	(4) (3) (1)	(5) (4) (1)	1 (u_2)
(3) **15**	(5) (0) (5)	(6) (0) (6)	(3) (1) (2)	(2) **5**	−1 (u_3)
(2) **10**	(4) (−1) (5)	(4) (−1) (5)	(5) (0) (5)	(3) (1) (2)	−2 (u_4)
$v_j \rightarrow$ 4	1	1	2	3	

In this table all $d_{ij} \geq 0$.

Hence, this solution is optimal. Hence the optimal solution is B.F.S. by VAM

and minimum transportation cost = ₹ 210.

Example 15: *A company has factories A, B and C which supply warehouses at D, E, F and G. Monthly factory capacities are* 160, 150 *and* 190 *units respectively. Monthly warehouse requirements are* 80, 90, 110 *and* 160 *units respectively. Unit shipping costs (in rupees) are as follows:*

		To			
		D	*E*	*F*	*G*
	A	42	48	38	37
From	*B*	40	49	52	51
	C	39	38	40	43

Determine the optimum distribution for company to minimize shipping costs.

[B.C.A. (Kurukshetra) 2010; B.C.A. (Meerut) 2007, 2011]

Solution: Here, total monthly capacities = 160 + 150 + 190 = 500 units and total monthly requirements = 80 + 90 + 110 + 160 = 440 units.

Since two are not equal, so it is an unbalanced T.P. To convert the problem to a balanced one, we introduce a fictitious warehouse H with requirement of 60 units and having all the transportation costs to zero. Thus, the balanced transportation problem is as follows:

		To					
		D	E	F	G	H	Capacities
	A	(42)	(48)	**100** (38)	**60** (37)	0	160
From	B	**80** (40)	(49)	**10** (52)	(51)	**60** (0)	150
	C	(39)	**90** (38)	(40)	**100** (43)	(0)	190
Requirements		80	90	110	160	60	500

By Vogel's approximation method, an initial B.F.S. of the problem is given in the above table.

For this solution, total transportation cost

$$= 100 \times 38 + 60 \times 37 + 80 \times 40 + 10 \times 52 + 60 \times 0 + 90 \times 38 + 100 \times 43$$

$$= 3800 + 2220 + 3200 + 520 + 0 + 3420 + 4300$$

$$= ₹\ 17460.$$

Test of the Solution for Optimality: Here number of allocations is $7 = m + n - 1 (= 7)$ and are at independent positions. So, to test this solution for optimality, all necessary entries are made in the following table. In this table $d_{33} = -4 < 0$, so this solution is not optimal. Therefore we proceed to modify this solution.

						$u_i \downarrow$
	(42) (26) (16)	(48) (32) (16)	(38) **100 − θ**	(37) **60 + θ**	(0) (−14) (14)	$-14\ (u_1)$
	(40) **80**	(49) (46) (3)	(52) **10**	(51) (51) (0)	(0) **60**	$0\ (u_2)$
	(39) (32) (7)	(38) **90**	(40) (44) θ (−4)	(43) **100 − θ**	(0) (−8) (8)	$-8\ (u_3)$
$v_j \rightarrow$	40	46	52	51	0	
	v_1	v_2	v_3	v_4	v_5	

Modification of the Solution (First): Since minimum $d_{ij} = d_{33} = -4 < 0$, so we allocate maximum possible allocation θ to this cell $(3, 3)$. See the above table.

Here we observe that minimum allocation containing $-\theta$ is $100 - \theta$. Equating it to zero, we get $\theta = 100$.

Thus, making necessary changes in the allocations, the new solution is given in the following table, in which the number of allocations is 6 which is one less than the number $m + n - 1 (= 7)$.

So the solution is degenerate. To make this solution non-degenerate, we allocate a small amount $\in$ to the cell $(1, 3)$.

Total transportation cost for this solution

$$= 160 \times 37 + 80 \times 40 + 10 \times 52 + 90 \times 38 + 100 \times 40$$
$$= 5920 + 3200 + 520 + 3420 + 4000$$
$$= ₹\,17060$$

which is less than the cost for previous solution.

						$u_i \downarrow$
	(42) (26) (16)	(48) (34) (14)	(38) $\in$	(37) **160**	(0) (−14) (14)	−14 (u_1)
	(40) **80**	(49) (50) θ (−1)	(52) **10** − θ	(51) (51) (0)	(0) **60**	0 (u_2)
	(39) (28) (11)	(38) **90** − θ	(40) **100** + θ	(43) (39) (4)	(0) (−12) (12)	−12 (u_3)
$v_j \rightarrow$	40	50	52	51	0	
	v_1	v_2	v_3	v_4	v_5	

Now, to test the optimality of this solution all necessary entries are made in the above table and we observe that $d_{22} = -1 < 0$.

So this solution is also not optimal.

Thus, we proceed to modify this solution also.

Modification of the Solution (Second): Since minimum $d_{ij} = d_{22} = -160$, so we allocate as much as possible to this cell $(2, 2)$. Allocating θ to this cell and making necessary changes in the other allocations, the minimum allocation containing $-\theta$ is $10 - \theta$, equating it to zero, we get $\theta = 10$.

Thus, making necessary changes in the allocations, the new solution is given in the table given below in which the allocations are at independent positions.

In this case the total transportation cost

$$= 160 \times 37 + 80 \times 40 + 10 \times 49 + 80 \times 38 + 110 \times 40$$
$$= 5920 + 3200 + 490 + 3040 + 4400$$
$$= ₹\,17050$$

which is less than the cost after first modification.

Now, to test the optimality of this solution all necessary entries are made in this table and we observe that all $d_{ij} \geq 0$. Therefore this solution is optimal.

	D	E	F	G	H	$u_i \downarrow$
A	(42) (27) (15)	(48) (36) (12)	(38) $\in$	(37) **160**	(0) (–13) (13)	$-13\ (u_1)$
B	(40) **80**	(49) **10**	(52) (51) (1)	(51) (50) (1)	(0) **60**	$0\ (u_2)$
C	(39) (29) (10)	(38) **80**	(40) **110**	(43) (39) (4)	(0) (–11) (11)	$-11\ (u_3)$
$v_j \rightarrow$	40 v_1	49 v_2	51 v_3	50 v_4	0 v_5	

Hence, the optimal solution to the problem is

$$x_{14} = 160,\ x_{21} = 80,\ x_{22} = 10,\ x_{32} = 80,\ x_{33} = 110.$$

60 units are left with factory B which cannot be despatched as all requirements have been fulfilled. In this case minimum transportation cost = ₹ 17050.

Example 16: *Solve the following transportation problem (cell entries represent unit cost).*

	To						*Available*
From	5	3	7	3	8	5	3
	5	6	12	5	7	11	4
	2	1	2	4	8	2	2
	9	6	10	5	10	9	8
Requirement	3	3	6	2	1	2	17

[B.C.A. (Rohtak) 2011; B.B.A. (Raipur) 2008]

Solution: By Vogel's approximation method, an initial B.F.S. of the problem is given in the following table:

(5)	(3) **1**	(7)	(3)	(8)	(5) **2**	3
(5) **3**	(6)	(12)	(5)	(7) **1**	(11)	4
(2)	(1)	(2) **2**	(4)	(8)	(2)	2
(9)	(6) **2**	(10) **4**	(5) **2**	(10)	(9)	8
3	3	6	2	1	2	

The total minimum transportation cost of given problem

$$= 1 \times 3 + 2 \times 5 + 3 \times 5 + 1 \times 7 + 2 \times 2 + 2 \times 6 + 4 \times 10 + 2 \times 5$$

$$= 3 + 10 + 15 + 7 + 4 + 12 + 40 + 10$$

$$= ₹\ 101$$

To Test the Solution for Optimality: Here number of allocations is 8 which is one less than $m + n - 1 = 4 + 6 - 1 = 9$. m = number of rows, n = number of columns.

So this solution is degenerate solution.

$\therefore$ To resolve degeneracy we allocate a small amount $\in$ to the cell (3, 6) at independent position. We cannot take $\in$ in cell (3, 2) having minimum cost among unoccupied cells, as it is not the independent position.

							$u_i \downarrow$
	(5) (7) (−2)	(3) **1**	(7) (7) (0)	(3) (2) (1)	(8) (9) (−1)	(5) **2**	−3 (u_1)
	(5) **3**	(6) (1) (5)	(12) (5) (7)	(5) (0) (5)	(7) **1**	(11) (3) (8)	−5 (u_2)
	(2) $\in$	(1) (−2) (3)	(2) **2**	(4) (−3) (7)	(8) (4) (4)	(2) (0) (2)	−8 (u_3)
	(9) (10) (−1)	(6) **2**	(10) **4**	(5) **2**	(10) (12) (−2)	(9) (8) (1)	0 (u_4)
$v_j \rightarrow$	10	6	10	5	12	8	

Now, to test the optimality of the solution we find the set of u_i and v_j so that for each occupied cell $(r, s), c_{rs} = u_r + v_s$, in the usual manner, taking $u_4 = 0$.

Now in each unoccupied cell, we enter the values of $u_i + v_j$ and $d_{ij} = c_{ij} - (u_i + v_j)$ at the appropriate corner and observe that $c_{11}, c_{15}, c_{41}, c_{45} < 0$. So this solution is not optimal.

Since minimum $d_{11} = -2 < 0$, so we allocate $\in$ in this cell (1, 1) in place of c_{31}.

						$u_i \downarrow$
(5) $\in$	(3) **1**	(7) (7) (0)	(3) (2) (1)	(8) (7) (1)	(5) **2**	0
(5) **3**	(6) (3) (3)	(12) (7) (5)	(5) (2) (3)	(7) **1**	(11) (5) (6)	0
(2) (0) (2)	(1) (−2) (3)	(2) **2**	(4) (−3) (7)	(8) (2) (6)	(2) (0) (2)	−5
(9) (8) (1)	(6) **2**	(10) **4**	(5) **2**	(10) (10) (0)	(9) (8) (1)	3
$v_j \rightarrow$ 5	3	7	2	7	5	

Let $u_1 = 0$, we find the set of u_i and v_j and enter in all unoccupied $u_i + v_j$ and $d_{ij} = c_{ij} - (u_i + v_j)$ and see that all $d_{ij} > 0$.

Thus, this solution is optimal.

				$u_i \downarrow$
(4) 2 2	(3) **5**	(2) 0 2	(0) **5**	0
(2) **8**	(5) 3 2	(0) **4**	(0) **1**	0
(3) 2 1	(8) 3 5	(6) 0 6	(0) **12**	0
$v_j \rightarrow$ 2	3	0	0	

Since all $d_{ij} \geq 0$, so the solution under test is optimal.

Thus, the solution of the given problem is $x_{12} = 5, x_{21} = 8, x_{23} = 4$ and minimum transportation cost $= 3 \times 5 + 0 \times 5 + 2 \times 8 + 0 \times 4 + 0 \times 1 + 12 \times 0$

$$= 15 + 0 + 16 + 0 + 0 + 0$$

$$= ₹\ 31.$$

Example 17: *Consider the following unbalanced transportation problem:*

To

		1	2	3	*Supply*
	1	5	1	7	10
From	2	6	4	6	80
	3	3	2	5	15
Demand		75	20	50	

Since there is not enough supply, some of the demands at these destinations may not be satisfied. Suppose there are penalty costs for every unsatisfied demand unit which are given by 5, 3 and 2 for destinations 1, 2 and 3 respectively. Find the optimal solution. **[B.B.A. (Meerut) 2005, 2012]**

Solution: In this problem, demand $= 75 + 20 + 50 = 145$

and supply $= 10 + 80 + 15 = 105.$

i.e., demand > supply, so we introduce a dummy source 4 with supply $145 - 105 = 40$ and costs 5, 3, 2 respectively.

By VAM an initial B.F.S. to the problem is shown in the table.

	1	2	3	
1	(5)	(1) **10**	(7)	10
2	(6) **60**	(4) **10**	(6) **10**	80
3	(3) **15**	(2)	(5)	15
4	(5)	(3)	(2) **40**	40
	75	20	50	145

Here, number of allocations $= 6 = m + n - 1 = (4 + 3 - 1)$
and they are at independent positions.

Total transportation cost

$$= 10 \times 1 + 60 \times 6 + 10 \times 4 + 10 \times 6 + 15 \times 3 + 40 \times 2$$
$$= 10 + 360 + 40 + 60 + 45 + 80$$
$$= ₹\,595.$$

To Test the Solution for Optimality: To test the optimality of the solution we find the set of u_i and v_j for each occupied cell (r, s) s.t. $c_{rs} = u_r + v_s$, by considering $u_2 = 0$, in the usual manner.

				$u_i \downarrow$
	(5) (3) (2)	(1) **10**	(7) (3) (4)	$-3\ (u_1)$
	(6) **60**	(4) **10**	(6) **10**	$0\ (u_2)$
	(3) **15**	(2) (1) (1)	(5) (3) (2)	$-3\ (u_3)$
	(5) (2) (3)	(3) (0) (3)	(2) **40**	$-4\ (u_4)$
$v_j \rightarrow$	6 v_1	4 v_2	6 v_3	

Now in each unoccupied cell, we enter $u_i + v_j$ and $d_{ij} = c_{ij} - (u_i + v_j)$ at the appropriate corners and see that all $d_{ij} \geq 0$.

Thus, this solution is optimal.

Hence, the optimal solution is B.F.S. *i.e.,* ₹ 595.

Example 18: *A steel company has three open hearth furnaces and five rolling mills. Transportation costs (rupees per quintal) for shipping steel from furnaces to rolling mills are shown in the following table:*

		Mills					
		M_1	M_2	M_3	M_4	M_5	*Available*
	F_1	4	2	3	2	6	8
Furnaces	F_2	5	4	5	2	1	12
	F_3	6	5	4	7	3	14
Required		4	4	6	8	8	

What is an optimal shipping schedule? **[B.C.A. (Meerut) 2000; (Kanpur) 2007]**

Solution: Here requirement $= 4 + 4 + 6 + 8 + 8 = 30$

and availability $= 8 + 12 + 14 = 34,$

i.e., total requirement is less than availability.

To convert the problem to a balanced one, we introduce a fictitious mill M_6 with requirement $34 - 30 = 4$ and having all the transportation costs equal to zero. Thus, the balanced transportation problem is as follows:

		Mills						
		M_1	M_2	M_3	M_4	M_5	M_6	Available
	F_1	**4** (4)	**4** (2)	3	2	6	0	8
Furnaces	F_2	5	4	5	**8** (2)	**4** (1)	0	12
	F_3	6	5	**6** (4)	7	**4** (3)	**4** (0)	14
Required		4	4	6	8	8	4	34

By Vogel's approximation method, an initial B.F.S. of the problem is given in the above table, for which total transportation cost

$$= 4\times4 + 4\times2 + 8\times2 + 4\times1 + 6\times4 + 4\times3 + 4\times0$$
$$= 16 + 8 + 16 + 4 + 24 + 12 + 0$$
$$= ₹\ 80.$$

To Test the Solution for Optimality: Here total number of allocations is 7 which is one less than the number $m + n - 1\ (= 8)$. So the solution is degenerate.

							$u_i \downarrow$
F_1	(4) **4**	(2) **4**	(3) (2) (1)	(2) $\in$	(6) (1) (5)	(0) (−2) (2)	−2 (u_1)
F_2	(5) (4) (1)	(4) (2) (2)	(5) (2) (3)	(2) **8**	(1) **4**	(0) (−2) (2)	−2 (u_2)
F_3	(6) (6) (0)	(5) (4) (1)	(4) **6**	(7) (4) (3)	(3) **4**	(0) **4**	0 (u_3)
	6	4	4	4	3	0	
$v_j \rightarrow$	v_1	v_2	v_3	v_4	v_5	v_6	

To convert it to non-degenerate we introduce a small amount $\in$ in the cell (1, 4), so that the allocations remain at independent positions.

To test the optimality of the solution, we find the set of u_i and v_j s.t. for each occupied cell (r, s), $c_{rs} = u_r + v_s$ and enter the values of $u_i + v_j$ and $d_{ij} = c_{ij} - (u_i + v_j)$ in each unoccupied cell at the appropriate corner, and observe that all $d_{ij} \geq 0$.

Thus, this solution is optimal.

Hence, the optimal solution is B.F.S. by VAM = ₹ 80.

Problem Set

1. A company has three plants and four warehouses. The supply and demand in units and the transportation costs are given. The solution of the problem is given below:

		To				
		w_1	w_2	w_3	w_4	Supply
	O_1	5	10	4	5	10
From	O_2	6	8	7	5	25
	O_3	4	2	5	7	20
	Demand	25	10	15	5	

Answer the following questions:

(i) Is the solution feasible?

(ii) Is the solution degenerate?

(iii) Is this solution optimal?

(iv) Does this problem have more than one optimum solution? If so, show all of them.

(v) If the cost of route O_2w_3 is reduced from ₹ 7 to ₹ 6 per unit. What will be optimum solution? **[B.C.A. (Meerut) 2006, 2008, 2012]**

2. Use north-west corner cell to determine an initial basic feasible solution to the following transportation problem.

(i)

		To				
		I	II	III	IV	Supply
	A	13	11	15	20	2
From	B	17	14	12	13	6
	C	18	18	15	12	7
Demand		3	3	4	5	15

[B.C.A. (Indraprastha) 2012]

(ii)

		Destination				
		D_1	D_2	D_3	D_4	Supply
	O_1	6	4	1	5	14
Origin	O_2	8	9	2	7	16
	O_3	4	3	6	2	5
Demand		6	10	15	4	35

3. Determine an initial B.F.S. to the following transportation table using (i) matrix minima method (ii) VAM:

Destination

		D_1	D_2	D_3	D_4	Supply
	O_1	1	2	1	4	30
Origin	O_2	3	3	2	1	50
	O_3	4	2	5	9	20
Demand		20	40	30	10	100

[B.C.A. (Meerut) 2008]

4. Find the initial B.F.S. of the following transportation problem using (i) north-west corner rule (ii) matrix minima method (iii) VAM:

Warehouse

		w_1	w_2	w_3	w_4	Capacity
	F_1	19	30	50	10	7
Factory	F_2	70	30	40	60	9
	F_3	40	8	70	20	18
Requirement		5	8	7	14	

5. Determine the optimal B.F.S. of the following (T.P.):

To

		D_1	D_2	D_3	D_4	a_i
	O_1	5	3	6	2	19
From	O_2	4	7	9	1	37
	O_3	3	4	7	5	34
b_j		16	18	31	25	

6. Solve the following Transportation Problem (T.P.):

To

		D_1	D_2	D_3	a_i
	O_1	7	4	0	5
From	O_2	6	8	0	15
	O_3	3	9	0	9
b_j		15	6	8	29

7. Find the optimal solution of the following (T.P.):

(i)

To

		w_1	w_2	w_3	w_4	a_i
	F_1	19	30	50	10	7
From	F_2	70	30	40	60	9
	F_3	40	8	70	20	18
b_j		5	8	7	14	34

[B.C.A. (Meerut) 2003, 2004]

(ii) To

				Available
From	7	3	4	2
	2	1	3	3
	3	4	6	5
Demand	4	1	5	

(iii) To

		D_1	D_2	D_3	D_4	D_5	Available
	O_1	5	5	6	4	2	9
From	O_2	6	9	7	8	5	13
	O_3	5	6	4	6	3	9
Demand		3	7	8	5	8	

(iv) To

		D_1	D_2	D_3	D_4	D_5	a_i
	O_1	73	40	9	79	20	8
	O_2	62	93	96	8	13	7
From	O_3	96	65	80	50	65	9
	O_4	57	58	29	12	87	3
	O_5	56	23	87	18	12	5
b_j		6	8	10	4	4	

8. Define transportation problem. **[B.C.A. (Meerut) 2004, 2008, 2012]**

9. A company has three plants at locations *A, B* and *C* which supply to warehouses located of *D, E, F, G* and *H.* Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation cost (in Rupees) are given below:

To

		D	*E*	*F*	*G*	*H*
	A	5	8	6	6	3
From	*B*	4	7	7	6	5
	C	8	4	6	6	4

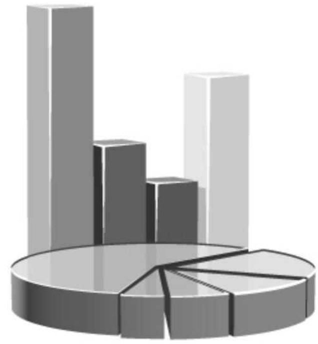
Determine an optimum distribution for the company in order to minimize the total transportation cost. **[B.C.A. (Meerut) 2007, 2009]**

Answers

1.	(i) Yes, solution is feasible. (ii) Solution is non-degenerate. (iii) Solution is optimum. (iv) No. (v) The solution is not feasible.
2.	(i) $x_{11}=2, x_{21}=1, x_{22}=3, x_{23}=2, x_{33}=2, x_{34}=5.$ (ii) $x_{11}=6, x_{12}=8, x_{22}=2, x_{23}=14, x_{33}=1, x_{34}=4.$
3.	For (i) and (ii) both: $x_{11}=20, x_{13}=10, x_{22}=20, x_{23}=20, x_{24}=10, x_{32}=20.$
4.	(i) $x_{11}=5, x_{12}=2, x_{22}=6, x_{23}=3, x_{33}=4, x_{34}=14.$ (ii) $x_{14}=7, x_{21}=2, x_{23}=7, x_{31}=3, x_{32}=8, x_{34}=7.$ (iii) $x_{11}=5, x_{14}=2, x_{23}=7, x_{24}=2, x_{32}=8, x_{34}=10.$
5.	$x_{12}=18, x_{13}=1, x_{21}=12, x_{24}=25, x_{31}=4, x_{33}=30.$
6.	$x_{12}=5, x_{21}=6, x_{22}=1, x_{23}=8, x_{31}=9.$
7.	(i) $x_{11}=5, x_{14}=2, x_{22}=2, x_{23}=7, x_{32}=6, x_{34}=12.$ (ii) $x_{13}=2, x_{22}=1, x_{23}=2, x_{31}=4, x_{33}=1.$ (iii) $x_{12}=4, x_{14}=5, x_{21}=3, x_{22}=2, x_{25}=8, x_{32}=1, x_{33}=8.$ (iv) $x_{13}=8, x_{24}=4, x_{25}=3, x_{31}=5, x_{32}=4, x_{41}=1, x_{43}=2, x_{52}=4, x_{55}=1.$
9.	Total cost = ₹ 141.00.

❍❍❍

CHAPTER 4



Assignment Problem

4.1 Assignment Problem

[B.C.A. (Lucknow) 2010; B.C.A. (Avadh) 2008; B.C.A. (Agra) 2008; B.C.A. (Meerut) 2002, 2006]

The assignment problem can be stated in the form of $(n \times n)$ square cost matrix $[c_{ij}]$ of real number. Suppose there are n- jobs to be performed and n- persons are available for doing these jobs. Assume that each person can do each job at a time, through with varying degree of efficiency. Let c_{ij} be the cost (payment) if the i th person is assigned the j th job, the problem is to find an assignment *i.e.* which job should be assigned to which person, so that the total cost for performing all jobs is minimum.

$$
\begin{array}{cc}
 & \textbf{Jobs} \\
 & \begin{array}{ccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \cdots & \boldsymbol{j} & \cdots & \boldsymbol{n} \end{array} \\
\textbf{Persons} \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \vdots \\ \boldsymbol{i} \\ \vdots \\ \boldsymbol{n} \end{array} &
\left[\begin{array}{ccccccc}
c_{11} & c_{12} & c_{13} & \cdots & c_{1j} & \cdots & c_{1n} \\
c_{21} & c_{22} & c_{23} & \cdots & c_{2j} & \cdots & c_{2n} \\
c_{31} & c_{32} & c_{33} & \cdots & c_{3j} & \cdots & c_{3n} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
c_{i1} & c_{i2} & c_{i3} & \cdots & c_{ij} & \cdots & c_{in} \\
 & & & & & & \\
c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nj} & \cdots & c_{nn}
\end{array}\right]
\end{array}
$$

4.1.1 Mathematical Formulation of Assignment Problem

[B.C.A. (Meerut) 2009, 2010; B.C.A. (Agra) 2007; B.C.A. (Lucknow) 2007; B.B.A. (Meerut) 2002, 2006]

Mathematically, the assignment problem can be stated as:

Minimize the total cost:

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}, \qquad \ldots(1)$$

where $\begin{cases} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, n \end{cases}$

Subject to restrictions of the form

$$x_{ij} = \begin{cases} 1, & \text{if } i\text{th person is assigned } j\text{th job} \\ 0, & \text{if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ (one job is done by the } i\text{th person } i = 1, 2, 3 \ \ldots \ n)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ (only one person should be assigned the } j \text{ th job, } j = 1, 2, 3, \ldots \ n)$$

where x_{ij} denotes the j th job is to be assigned to the ith person.

4.1.2 Difference between Assignment and Transportation Problem

[B.C.A (Agra) 2004, 2006, 2010; B.C.A. (Kanpur) 2008; B.C.A. (Lucknow) 2005, 2009; B.C.A. (I.G.N.O.U.) 2012; B.C.A. (Meerut) 2008, 2011]

1. Assignment is the square matrix but transportation is not necessary square matrix if we put $m = n$ in transportation then it becomes the problem of assignment. Hence, we can say that assignment problem is the special case of transportation problem.

2. The numerical evaluations of such association are called 'effectiveness' instead of 'transportation costs'.

3. Mathematically, all a_i and b_j are unity and each x_{ij} is limited to one of the two values 0 and 1. In such circumstances, exactly n of the x_{ij} can be non-zero. One of each origin and one for each destination.

Remark. Give the mathematical formulation of assignment. How does it differ from the transportation problem? **[B.C.A. (Meerut) 2009, 2011]**

4.1.3 Fundamental Theorems

Theorem 1: (Reduction Theorem). In an assignment problem if we add (or subtract) a constant to every element of a row (or column) of the cost matrix $[c_{ij}]$, then an assignment which minimizes the total cost for one matrix also minimizes the total cost for the other matrix.

or

Mathematical Statement of Reduction Theorem: If $x_{ij} = X_{ij}$

$$\text{minimizes } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \text{ over all } x_{ij} = 0 \text{ or } 1 \text{ such that}$$

$$\sum_{i=1}^{n} x_{ij} = 1, \sum_{j=1}^{n} x_{ij} = 1, \text{ then } x_{ij} = X_{ij} \text{ also}$$

$$\text{minimizes } Z' = \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij}\, x_{ij} \text{ where } c_{ij}' = c_{ij} \pm a_i \pm b_j$$

a_i, b_j are some real numbers for i, $j = 1, 2, \ldots, n$. **[B.B.A. (Delhi) 2006, 2008, 2012]**

Proof: We have $Z' = \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij}\, x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} (c_{ij} \pm a_i \pm b_j)\, x_{ij}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \pm \sum_{i=1}^{n} \sum_{j=1}^{n} a_i x_{ij} \pm \sum_{i=1}^{n} \sum_{j=1}^{n} b_j\, x_{ij}$$

$$= Z \pm \sum_{i=1}^{n} a_i \sum_{j=1}^{n} x_{ij} \pm \sum_{j=1}^{n} b_j \sum_{i=1}^{n} x_{ij} \qquad \left[\because\ Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}\right]$$

$$= Z \pm \sum_{i=1}^{n} a_i \,.\, 1 \pm \sum_{j=1}^{n} b_j \,.\, 1 \qquad \left[\because\ \sum_{i=1}^{n} x_{ij} = 1 = \sum_{j=1}^{n} x_{ij}\right]$$

$$= Z \pm \sum_{i=1}^{n} a_i \pm \sum_{j=1}^{n} b_j$$

Since, terms $\sum_{i=1}^{n} a_i, \sum_{j=1}^{n} b_j$ are independent of x_{ij} it follows that Z' is minimized whenever Z is minimized and conversely.

Hence, $x_{ij} = X_{ij}$ which minimizes Z will also minimize Z'.

Theorem 2: If all $c_{ij} \geq 0$ and there exists a solution $x_{ij} = X_{ij}$ which satisfies

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} = 0$$

then this solution is an optimal solution for the problem (*i.e.* minimizes the objective function).

Proof: Since all $c_{ij} \geq 0$ and all $x_{ij} \geq 0$, the objective function

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \text{ cannot be negative.}$$

The minimum possible value that Z can attain is 0.

Hence, the solution $x_{ij} = X_{ij}$ for which $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} = 0$ is an optimal solution.

4.2 Assignment Algorithm or Hungarian Method or Reduced Matrix Method

[B.C.A. (Meerut), 2005, 2009; B.C.A. (Rohilkhand) 2007, 2009]

Various steps of the computational procedure for obtaining an optimum assignment may be summarized as follows:

Step 1: Check whether the number of rows and columns in the cost matrix are equal. If not, add dummy rows (columns) to form a square matrix.

Step 2: In the square cost matrix:

(i) Reduce each row element the lowest element of that row

(ii) Reduce each column element of the lowest element of that column.

Step 3: In the reduced matrix, search for optimum solution as follows:

(i) Examine the rows successively until a row with exactly single zero is found. Mark this zero by enrectangling ( ) and cross out (×) all others zeros of the corresponding column. Proceed in this manner until all rows have been examined.

(ii) Examine the columns successively until a column with exactly single zero is found. Mark this zero by enrectangling ( ) and cross out (×) all other zero of the corresponding row. Proceed in this manner until all columns have been examined.

(iii) If each row and each column has one and only one marked zero, the optimum allocation is attained which is indicated by the marked positions, otherwise go to next step.

Step 4: Draw the minimum number of lines passing through all the zeros as follows:

(i) Tick (✓) rows that do not have assignments.

(ii) Tick (✓) columns that have zeros in ticked rows.

(iii) Tick (✓) rows that have assignment in ticked columns.

(iv) Repeat (ii) and (iii) until the chain is completed.

(v) Draw straight lines through all unticked rows and ticked columns.

Step 5: If the minimum number of lines passing through all the zeros is equal to the number of rows or columns, the optimum solution is attained by an arbitrary allocation in the positions of the zeros not crossed in step 3. Otherwise go to next step.

Step 6: Revise the costs matrix as follows:

(i) Find the smallest element not covered by any of the lines of step 4.

(ii) Subtract this from all the uncrossed elements and add the same at the point of intersection of the two lines.

(iii) Other elements crossed by the line remain unchanged.

Step 7: Go to step 4 and repeat the procedure till an optimum solution is attained.

Solved Examples

Example 1: *Solve the following assignment problem:*

Machines

Jobs		M_1	M_2	M_3
	J_1	8	7	6
	J_2	5	7	8
	J_3	6	8	7

[B.C.A. (Meerut) 2004, 2006, 2010]

Solution:

Step 1: Subtract the smallest element of each row from every element of the corresponding row, we get

	M_1	M_2	M_3
J_1	2	1	0
J_2	0	2	3
J_3	0	2	1

Step 2: Subtract the smallest element of each column from every element of the corresponding column, we get

	M_1	M_2	M_3
J_1	2	0	0
J_2	0	1	3
J_3	0	1	1

Step 3: Reduced matrix, make assignments in rows and columns having single zeros.

$$\begin{array}{c|ccc|c}
 & M_1 & M_2 & M_3 & \\
\hline
J_1 & 2 & \boxed{0} & \cancel{0} & \\
J_2 & \boxed{0} & 1 & 3 & \checkmark \\
J_3 & \cancel{0} & 1 & 1 & \checkmark \\
\hline
 & \checkmark & & &
\end{array}$$

Step 4: Draw the minimum number of lines to cover all the zeros of the reduced matrix now subtract, from uncovered elements and adding the same at the intersection of two lines. The lowest element from uncovered.

	M_1	M_2	M_3
J_1	3	$\cancel{0}$	$\boxed{0}$
J_2	$\boxed{0}$	$\cancel{0}$	2
J_3	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$

Since, the number of assignments is equal to the order of the matrix, we have the optimum assignment schedule.

$$J_1 \to M_3,\ J_2 \to M_1,\ J_3 \to M_2$$

$$\text{Minimum cost } = 6 + 5 + 8$$
$$= 19$$

Example 2: *Solve the minimal assignment problem whose effectiveness matrix is:*

	I	*II*	*III*	*IV*
A	*2*	*3*	*4*	*5*
B	*4*	*5*	*6*	*7*
C	*7*	*8*	*9*	*8*
D	*3*	*5*	*8*	*4*

[B.C.A. (Lucknow) 2007; B.C.A.(Meerut) 2002, 2011; B.C.A. (Delhi) 2007]

Solution:

Step 1: Subtracting the minimum element of each row from every element of the corresponding row, the matrix reduces to

	I	II	III	IV
A	0	1	2	3
B	0	1	2	3
C	0	1	2	1
D	0	2	5	1

Step 2: Now subtracting the minimum element of each column from every element of the corresponding column, the matrix reduces to

$$
\begin{array}{c|cccc|}
 & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV} \\
\hline
\boldsymbol{A} & 0 & 0 & 0 & 2 \\
\boldsymbol{B} & 0 & 0 & 0 & 2 \\
\boldsymbol{C} & 0 & 0 & 0 & 0 \\
\boldsymbol{D} & 0 & 1 & 3 & 0 \\
\hline
\end{array}
$$

Step 3: Now to test whether it is possible to make an assignment using only zeros.

Here none of the rows or columns contain exactly one zero, therefore we start with row, searching two zeros. While examining rows successively, it is observed that row 4 has two zeros. Now, arbitrarily make an assignment ( ) one of these two zeros, say zero in the column 1 and cross other zeros in row 4 and column 1. Now we examine the columns and first column 4 which contains only one unmarked zero in row 3. We make assignment ( ) at this zero and cross all other zeros of this row. Now again we check the rows and columns for one unmarked zero. There is no such row or column. So we start with row 1 searching two unmarked zeros and find the row 1 containing two such zeros. We mark ( ) at any one of these zeros. Let zero at column 2 and cross other zeros of row 1 and column 2. Now the second row contains only one unmarked zero in third column where we can make an assignment.

At this stage all zeros have been either assigned or cross out. We observe that every row and column have one assignment, so we have the complete zero assignment.

Table (*a*), (*b*), (*c*) show the necessary steps for reaching the optimal assignment.

(a)

$$
\begin{array}{c|cccc|}
 & \mathbf{I} & \mathbf{II} & \mathbf{III} & \mathbf{IV} \\
\hline
\boldsymbol{A} & \xcancel{0} & 0 & 0 & 2 \\
\boldsymbol{B} & \xcancel{0} & 0 & 0 & 2 \\
\boldsymbol{C} & \xcancel{0} & 0 & 0 & 0 \\
\boldsymbol{D} & \boxed{0} & 1 & 3 & \xcancel{0} \\
\hline
\end{array}
$$

(b)

	I	II	III	IV
A	$\cancel{0}$	0	0	2
B	$\cancel{0}$	0	0	2
C	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
D	$\boxed{0}$	1	3	$\cancel{0}$

(c)

	I	II	III	IV
A	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
B	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
C	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$
D	$\boxed{0}$	1	3	$\cancel{0}$

Thus, we get the following optimal assignment $A \to II,\ B \to III,\ C \to IV,\ D \to I$.

$$\text{Minimum cost } = 3 + 6 + 8 + 3 = 20$$

NOTE:

The other optimal solutions are also possible. Each will have the cost 20.

Example 3: *A departmental head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and tasks differ in their intrinsic difficulty, this estimate of the time each man would take to perform each task is given in the matrix below.*

Subordinates

		I	*II*	*III*	*IV*
	A	18	26	17	11
Tasks	*B*	13	28	14	26
	C	38	19	18	15
	D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man hours?

[B.C.A.(Agra) 2005; B.C.A. (Avadh) 2007; B.C.A. (Rohilkhand) 2004, 2006, 2008, 2011]

Solution:

Step 1: Subtract the smallest element of each row from every element of the corresponding row, we get the reduced matrix :

	I	II	III	IV
A	7	15	6	0
B	0	15	1	13
C	23	4	3	0
D	9	16	14	0

Step 2: Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column. We get the following reduced matrix :

	I	II	III	IV
A	7	11	5	0
B	0	11	0	13
C	23	0	2	0
D	9	12	13	0

Step 3: Starting with row 1 we enrectangled ( ) to a single zero, if any and cross (×) all other zeros in the column of zero so marked. Thus, we get

	I	II	III	IV
A	7	11	5	$\boxed{0}$
B	$\boxed{0}$	11	$\cancel{0}$	13
C	23	$\boxed{0}$	2	$\cancel{0}$
D	9	12	13	$\cancel{0}$

In the above matrix, we arbitrarily enrectangled a zero in column 1, because row 2 had two zeros.

It may be noted that column 3 and row 4 do not have any assignment. So we move on to the next step.

Step 4:

(i) Since row 4 does not have any assignment, we tick this row (✓).

(ii) Now there is a zero in the fourth column of the ticked row. So we tick fourth column (✓).

(iii) Further there is an assignment in the first row of the ticked column. So we tick first row (✓).

(iv) Draw straight lines through all unticked rows and ticked columns. Thus, we have

$$\begin{array}{c|cccc|c} & \text{I} & \text{II} & \text{III} & \text{IV} & \\ \hline A & 7 & 11 & 5 & \boxed{0} & \checkmark \\ B & \boxed{0} & 11 & \cancel{0} & 13 & \\ C & 23 & \boxed{0} & 2 & \cancel{0} & \\ D & 9 & 12 & 13 & \cancel{0} & \checkmark \\ \hline & & & & \checkmark & \end{array}$$

Step 5: In step 4, we observe that the minimum number of lines so drawn is 3, which is less than the order of the cost matrix, indicating that the current assignment is not optimum.

To increase the minimum number of fives, we generate new zeros in the modified matrix.

Step 6: The smallest element not covered by the lines is 5 subtracting this element from all the uncovered elements and adding the same to all the elements lying at the intersection of lines, we obtain the following new modified cost matrix.

	I	II	III	IV
A	2	6	0	0
B	0	11	0	18
C	23	0	2	5
D	4	7	8	0

Step 7: Repeating step 4 on the modified matrix, we get

$$
\begin{array}{c|cccc|}
 & \text{I} & \text{II} & \text{III} & \text{IV} \\
\hline
A & 2 & 6 & \boxed{0} & \cancel{0} \\
B & \boxed{0} & 11 & \cancel{0} & 18 \\
C & 23 & \boxed{0} & 2 & 5 \\
D & 4 & 7 & 8 & \boxed{0} \\
\hline
\end{array}
$$

Now, since each row and each column has one and only one assignment, an optimum solution is reached.

The optimum assignment is

$$A \to III,\ B \to I,\ C \to II,\ D \to IV$$

The minimum total time for this assignment is $17 + 13 + 19 + 10 = 59$ man hours.

Example 4: *Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:*

Job

		1	2	3	4	5
	A	8	4	2	6	1
	B	0	9	5	5	4
Persons	***C***	3	8	9	2	6
	D	4	3	1	0	3
	E	9	5	8	9	5

Determine the optimum assignment schedule.

[B.C.A. (Bundelkhand) 2008; B.C.A. (Kanpur) 2005, 2007]

Solution:

Step 1: To minimize the row

	1	**2**	**3**	**4**	**5**
A	7	3	1	5	0
B	0	9	5	5	4
C	1	6	7	0	4
D	4	3	1	0	3
E	4	0	3	4	0

Step 2: To minimize the column, in row and column that have single zeros, we get the optimum assignment table.

	1	**2**	**3**	**4**	**5**
A	7	3	$\cancel{0}$	5	$\boxed{0}$
B	$\boxed{0}$	9	4	5	4
C	1	6	6	$\boxed{0}$	4
D	4	3	$\boxed{0}$	$\cancel{0}$	3
E	4	$\boxed{0}$	2	4	$\cancel{0}$

Optimum assignment schedule is $A \to 5, B \to 1, C \to 4, D \to 3$ and $E \to 2$

The minimum cost of assignment is $1 + 0 + 2 + 1 + 5 = 9$

Example 5 : *A car hire company has one car at each of five depots a, b, c, d and e. A customer requires a car in each town namely A, B, C, D and E. Distance (in kms) between depots (origins) and towns (destinations) are given in the following distance matrix :*

	a	***b***	***c***	***d***	***e***
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

[B.C.A. (Bhopal) 2006, 2012; B.C.A. (Rohilkhand) 2004, 2007]

How should cars be assigned to customers so as to minimize the distance travelled?

Solution:

Step 1: Subtracting the minimum element of each row from every element of the corresponding row, the matrix reduces to

	a	*b*	*c*	*d*	*e*
A	30	0	45	60	70
B	15	0	10	40	55
C	30	0	45	60	75
D	0	0	30	30	60
E	20	0	35	45	70

Now subtracting the minimum element of each column from every element of a corresponding column, the matrix reduces to

	a	*b*	*c*	*d*	*e*
A	30	0	35	30	15
B	15	0	0	10	0
C	30	0	35	30	20
D	0	0	20	0	5
E	20	0	25	15	15

Step 2: Now we give the zero assignments in our usual manner. Row 1 has a single zero in column 2. Make an assignment by marking ' ' around it and delete other zeros (if any) in column 2 by marking '×'. Examining the set of rows completely, an identical procedure is applied successively to columns.

	a	*b*	*c*	*d*	*e*
A	30	$\boxed{0}$	35	30	15
B	15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$
C	30	$\cancel{0}$	35	30	20
D	$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5
E	20	$\cancel{0}$	25	15	15

Now column 1 has a single zero in row 4. Make an assignment by marking '   ' at this zero and cross the other zero of row 4 which is not yet crossed. Column 3 has a single zero in row 2, make an assignment at this zero by putting '   ' and cross the other zero of row which is not yet crossed. At this stage all zeros have been either assigned or crossed out. It is observed that row 3, row 5, column 4 and column 5 each has no assignment. Hence, the required solution cannot be obtained at this stage. So we proceed to the next step.

Step 3: In this step we draw minimum number of lines to cover all zeros at least once. For this we proceed as follows :

(i) Tick (✓) row 3 and row 5 as they have no assignments.

(ii) Tick (✓) column 2 as having zeros in the marked rows 3 and 5.

(iii) Tick (✓) row 1 as it contains assignment in the marked column 2.

No further rows or columns will be required to mark during this procedure.

(iv) Now draw line L_1 through marked column 2. Then draw lines L_2 and L_3 through unmarked rows 2 and 4.

The required lines will be L_1, L_2 and L_3. No zero is left uncovered.

		a	*b* L_1	*c*	*d*	*e*	
	A	30	$\boxed{0}$	35	30	15	✓ (4)
L_2	***B***	15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
	C	30	$\cancel{0}$	35	30	20	✓ (1)
L_3	***D***	$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
	E	20	$\cancel{0}$	25	15	15	✓ (2)
			✓ (3)				

Step 4: In this step we select the smallest element among all uncovered elements of the matrix of step 3.

Here, this element is 15. Subtracting this element 15 from all the elements that do not have a line through them and adding to every element that lies at the intersection of two lines and leaving the remaining elements unchanged we get the following matrix.

	a	b	c	d	e
A	15	0	20	15	0
B	15	15	0	10	0
C	15	0	20	15	5
D	0	15	20	0	5
E	5	0	10	0	0

Step 5: Now again performing the step 2 we make the zero assignments. It is observed that there are no remaining zeros and every row (column) has an assignment as shown in the table.

	a	b	c	d	e
A	15	$\cancel{0}$	20	15	$\boxed{0}$
B	15	15	$\boxed{0}$	10	$\cancel{0}$
C	15	$\boxed{0}$	20	15	5
D	$\boxed{0}$	15	20	$\cancel{0}$	5
E	5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

Thus, the complete optimal assignment plan is given by

$$A \to e, B \to c, C \to b, D \to a, E \to d$$

From the original matrix, the minimum cost (distance travelled)

$$= (200 + 130 + 110 + 50 + 80) \text{ kms.} = 570 \text{ kms}.$$

Example 6: *Solve the assignment problem represented by the following matrix* :

	I	II	III	IV	V	VI
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	32
E	36	11	57	22	25	18
F	3	56	53	31	17	28

[B.C.A. (Indore) 2012; B.C.A. (Agra) 2001, 2004, 2006,2009]

Solution:

Step 1: Subtracting the minimum element of each row from every element of the corresponding row and then subtracting the minimum element of each column from every element of the corresponding column, the matrix reduces to

	I	II	III	IV	V	VI
A	0	13	49	0	0	13
B	0	35	29	5	10	0
C	13	0	63	7	7	0
D	47	15	0	20	2	0
E	25	0	46	9	4	2
F	0	53	50	26	4	20

Step 2: Make the 'zero assignments' in usual manner. The illustration is shown in the table. Since row 3 and column 5 have no assignments so we proceed to the next step.

	I	II	III	IV	V	VI
A	$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13
B	$\cancel{0}$	35	29	5	10	$\boxed{0}$
C	13	$\cancel{0}$	63	7	7	$\cancel{0}$
D	47	15	$\boxed{0}$	20	2	$\cancel{0}$
E	25	$\boxed{0}$	46	9	4	2
F	$\boxed{0}$	53	50	26	4	20

Step 3: Draw minimum number of lines to cover all zeros at least once. For this we proceed as follows:

(i) Tick (✓) row 3 as having no assignment.

(ii) Tick (✓) columns 2 and 6 as having zeros in marked row 3.

(iii) Tick (✓) rows 5 and 2 as having assignments in the marked columns 2 and 6.

(iv) Tick (✓) column 1 (not already marked) as having zero in the marked row 2.

(v) Then tick (✓) row 6 as having assignment in the marked column 1.

		L_1	L_2				L_3	
		I	**II**	**III**	**IV**	**V**	**VI**	
L_4	***A***	$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13	
	B	$\cancel{0}$	35	29	5	10	$\boxed{0}$	✓ (5)
	C	13	$\cancel{0}$	63	7	7	$\cancel{0}$	✓ (1)
L_5	***D***	47	15	$\boxed{0}$	20	2	$\cancel{0}$	
	E	25	$\boxed{0}$	46	9	4	2	✓ (4)
	F	$\boxed{0}$	53	50	26	4	20	✓ (7)
		✓	✓				✓	
		(6)	(2)				(3)	

Now draw lines L_1, L_2, L_3 through marked columns 1, 2, 6 respectively and L_4, L_5 through unmarked rows 1, 4 respectively. This way minimum set of five lines (5 < 6) to cover all the zeros is obtained.

Step 4: Now the smallest element among all uncovered elements is 4. Subtracting this element 4 from all the uncovered elements, adding to every element that lies at the intersection of two lines and leaving the remaining elements unchanged the matrix of step 3 reduces to the new form as shown in the table.

	I	**II**	**III**	**IV**	**V**	**VI**
A	4	17	49	0	0	17
B	0	35	25	1	6	0
C	13	0	59	3	3	0
D	51	19	0	20	2	4
E	25	0	42	5	0	2
F	0	53	46	22	0	20

Step 5: Repeating the step 2, make the ‘zero assignments as shown in the following table. Thus, exactly one marked ‘ ’ zero in each row and each column of the matrix is obtained.

	I	II	III	IV	V	VI
A	4	17	49	$\boxed{0}$	$\xcancel{0}$	17
B	$\boxed{0}$	35	25	1	6	$\xcancel{0}$
C	13	$\xcancel{0}$	59	3	3	$\boxed{0}$
D	51	19	$\boxed{0}$	20	2	4
E	25	$\boxed{0}$	42	5	$\xcancel{0}$	2
F	$\xcancel{0}$	53	46	22	$\boxed{0}$	20

Thus, the optimal assignment is

$$A \to IV, B \to I, C \to VI, D \to III, E \to II, F \to V$$

From the original matrix, minimum cost $= 11 + 43 + 33 + 27 + 11 + 17 = 142$.

NOTE:

Another optimal solution of this assignment problem is shown in the following table *i.e.* $A \to IV, B \to VI, C \to II, D \to III, E \to V, F \to I$.

	I	II	III	IV	V	VI
A	4	17	49	$\boxed{0}$	$\xcancel{0}$	17
B	$\xcancel{0}$	35	25	1	6	$\boxed{0}$
C	13	$\boxed{0}$	59	3	3	$\xcancel{0}$
D	51	19	$\boxed{0}$	20	2	4
E	25	$\xcancel{0}$	42	5	$\boxed{0}$	2
F	$\boxed{0}$	53	46	22	$\xcancel{0}$	20

From the original matrix, minimum cost $= 11 + 48 + 28 + 27 + 25 + 3 = 142$.

Example 7: *Solve the following assignment problem having the following cost elements.*

	1	2	3	4	5	6	7
A	35	22	60	41	27	52	44
B	51	39	42	33	65	47	58
C	25	32	53	41	50	36	43
D	32	28	40	46	3	55	49
E	43	36	45	63	57	49	42
F	27	18	31	46	35	42	34
G	48	50	72	59	43	64	58

[B.C.A. (Agra) 2004; B.C.A. (Lucknow) 2008; B.B.A. (I.G.N.O.U.) 2004, 2008, 2009]

Solution:

Step 1: Subtract the smallest element of each row (column) from all elements of the respective row (column).

	1	2	3	4	5	6	7
A	13	0	38	19	5	30	22
B	18	6	9	0	32	14	25
C	0	7	28	16	25	11	18
D	29	25	37	43	0	52	46
E	7	0	9	27	21	13	6
F	9	0	13	28	17	24	16
G	5	7	29	16	0	21	15

and

	1	2	3	4	5	6	7
A	13	0	29	19	5	19	16
B	18	6	0	0	32	3	19
C	0	7	19	16	25	0	12
D	29	25	28	43	0	41	40
E	7	0	0	27	21	2	0
F	9	0	4	28	17	13	10
G	5	7	20	16	0	10	9

Step 2: To make assignment in each row and each column with usual method.

	1	2	3	4	5	6	7	
A	13	$\boxed{0}$	29	19	5	19	16	✓
B	18	6	$\cancel{0}$	$\boxed{0}$	32	3	19	
C	$\boxed{0}$	7	19	16	25	$\cancel{0}$	12	
D	29	25	28	43	$\boxed{0}$	41	40	✓
E	7	$\cancel{0}$	$\cancel{0}$	27	21	2	$\boxed{0}$	
F	9	$\cancel{0}$	4	28	17	13	10	✓
G	5	7	20	16	$\cancel{0}$	10	9	✓
		✓			✓			

Now subtracting the element 4 from all the elements not covered by lines and adding the same at the intersection of two lines.

	1	2	3	4	5	6	7	
A	9	$\boxed{0}$	25	15	5	15	12	
B	18	10	$\cancel{0}$	$\boxed{0}$	36	3	19	
C	$\boxed{0}$	11	19	16	29	$\cancel{0}$	12	
D	25	25	24	39	$\boxed{0}$	37	36	✓
E	7	4	$\cancel{0}$	27	25	2	$\boxed{0}$	
F	5	$\cancel{0}$	$\boxed{0}$	24	17	9	6	
G	1	7	16	12	$\cancel{0}$	6	5	✓
					✓			

Final Iteration: Subtracting 1 from all the elements not covered by lines and add the same at the intersection of two lines.

	1	2	3	4	5	6	7
A	9	$\boxed{0}$	25	15	6	15	12
B	18	10	$\cancel{0}$	$\boxed{0}$	37	3	19
C	$\cancel{0}$	11	19	16	30	$\boxed{0}$	12
D	24	24	23	38	$\boxed{0}$	36	35
E	7	4	$\cancel{0}$	27	26	2	$\boxed{0}$
F	5	$\cancel{0}$	$\boxed{0}$	24	18	9	6
G	$\boxed{0}$	6	15	11	$\cancel{0}$	5	4

The optimum assignment is $A \to 2, B \to 4, C \to 6, D \to 5, E \to 7, F \to 3$ and $G \to 1$. The minimum cost $22 + 33 + 36 + 3 + 42 + 31 + 48 = 215$.

4.3 Unbalanced Assignment Problem

[B.C.A. (Meerut) 2005, 2008]

If the cost matrix of an assignment problem is not square matrix, the assignment problem is called unbalanced assignment problem. In such cases add dummy rows or columns with zero costs. Then the usual assignment algorithm can be applied to this resulting balanced problem.

4.4 Maximal Assignment Problem

[B.B.A. (Kanpur) 2007]

Sometimes, the assignment problem deals with the maximization of an objective function rather than to minimize it. For example, it may be required to assign persons to jobs in such a way that the expected profit is maximum. Such problem may be solved easily by first converting it to a minimization problem and then applying the usual procedure of assignment algorithm. This conversion can be very easily done by subtracting from the highest element, all the elements of the given profit matrix or by placing minus sign before each element of the profit matrix in order to make it cost matrix.

4.5 Restrictions on Assignment Problem

[B.B.A. (Delhi) 2008]

Sometimes due to some restrictions the assignment of a particular facility to a particular job is not permitted. To overcome such difficulty we assign a very high cost (∞) to the corresponding cell, so that the activity will be automatically excluded from the optimal solution.

4.6 Application of Assignment Problem

[B.B.A. (Meerut) 2004, 2005]

The assignment problem which finds many applications in allocation and scheduling, for example, in assigning planes or crews to commercial airline flights, trucks or drivers to different routes, man to offices and space to departments. If there are more jobs to do than can be done, we can decide either which job to leave undone or what resources to add. There are more number of decision making situations where assignment models can be used. For example, assignment of machines to jobs; assignment of salesman to sales territories; assignment of workers to various tasks, vehicles to routes; contracts to bidder and so on.

4.7 Objective of Assignment Problem

[B.B.A. (Meerut) 2002, 2003, 2004, 2006]

The assignment problem is a special type of allocation problem. In both cases the objective is to fulfil the targets by means of available resources which are available in specified amounts. But the operating conditions are different. While in the allocation problem each target can be obtained in one way only. In assignment problem the individual targets can be attained in different ways, we find that a transportation problem is degenerated, but in assignment problem each resource can be assigned to only one job and each job requires only one resource. Hence, the assignment problem is completely degenerated from the transportation problem.

Example 8: *Five jobs are to be processed and five machines are available any machine can process any job with the resultant profit (in rupees) as follows:*

	A	*B*	*C*	*D*	*E*
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

What is the maximum profit that may be expected if an optimum assignment is made?

Solution: It is maximization problem. To convert it into a minimization one, we multiply each element of the given matrix by –1. Thus, the resulting matrix is :

	A	*B*	*C*	*D*	*E*
1	– 32	– 38	– 40	– 28	– 40
2	– 40	– 24	– 28	– 21	– 36
3	– 41	– 27	– 33	– 30	– 37
4	– 22	– 38	– 41	– 36	– 36
5	– 29	– 33	– 40	– 35	– 39

We shall solve this minimization problem by usual assignment algorithm.

Step 1: Subtract the smallest element of each row from every element of the corresponding row then subtracting the smallest element of each column from each element of the corresponding column, we get the following table.

	A	*B*	*C*	*D*	*E*
1	8	0	0	7	0
2	$\boxed{0}$	14	12	14	4
3	0	12	8	6	4
4	19	1	0	0	5
5	11	5	0	0	1

Step 2: To make assignment in each row and each column by usual manner.

	A	*B*	*C*	*D*	*E*	
1	8	$\boxed{0}$	$\cancel{0}$	7	$\cancel{0}$	
2	$\boxed{0}$	14	12	14	4	✓
3	$\cancel{0}$	12	8	6	4	✓
4	19	1	$\boxed{0}$	$\cancel{0}$	5	
5	11	5	$\cancel{0}$	$\boxed{0}$	1	
	✓					

Step 3: Now, to draw the minimum number of lines to cover all the zeros at least once.

Step 4: Select the smallest element from uncovered elements *i.e.* 4. Subtract this element 4 from all the uncovered elements, adding to every element that lies at the intersection of two lines and leaving remaining elements unchanged.

	A	*B*	*C*	*D*	*E*
1	12	0	0	7	0
2	0	10	8	10	0
3	0	8	4	2	0
4	23	1	0	0	5
5	15	5	0	0	1

Step 5: Given zero assignments in the usual manner we see that each row and each column have an assignment.

	A	*B*	*C*	*D*	*E*
1	12	$\boxed{0}$	~~0~~	7	~~0~~
2	$\boxed{0}$	10	8	10	~~0~~
3	~~0~~	8	4	2	$\boxed{0}$
4	23	1	$\boxed{0}$	~~0~~	5
5	15	5	~~0~~	$\boxed{0}$	1

Hence the optimal solution for maximum sale to the problem is

$$1 \to B, 2 \to A, 3 \to E, 4 \to C, 5 \to D$$

i.e., Maximum sales $= 38 + 40 + 37 + 41 + 35$

$= 191$ *i.e.,* ₹ 19,100

Example 9: *A company has 4 machines to do 3 jobs. Each can be assigned to one and only one machine. The cost of each job on each machine is given in the following table.*

Machine

		W	*X*	*Y*	*Z*
	A	18	24	28	32
Job	***B***	8	13	17	19
	C	10	15	19	22

What are the job assignments which will minimize the cost? **[B.C.A. (Meerut) 2003, 2008, 2010]**

Solution: Since the matrix is not square. It is an unbalanced assignment problem, then adding dummy row *D* to get the square matrix.

	W	*X*	*Y*	*Z*
A	18	24	28	32
B	8	13	17	19
C	10	15	19	22
D	0	0	0	0

Now the problem can be solved by usual method.

Step 1: Subtracting the smallest element in each row from every element of the corresponding row and then subtracting the smallest element in each column from every element of the corresponding column, the matrix reduces to

$$\begin{array}{c|cccc|}
 & W & X & Y & Z \\
\hline
A & 0 & 6 & 10 & 14 \\
B & 0 & 5 & 9 & 11 \\
C & 0 & 5 & 9 & 12 \\
D & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$

Step 2: To make assignment in each row and each column by usual manner.

$$\begin{array}{c|cccc|c}
 & W & X & Y & Z & \\
\hline
A & \boxed{0} & 6 & 10 & 14 & \checkmark \\
B & \cancel{0} & 5 & 9 & 11 & \checkmark \\
C & \cancel{0} & 5 & 9 & 12 & \checkmark \\
D & \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} & \\
\hline
 & \checkmark & & & &
\end{array}$$

Step 3: Here, we draw minimum number of lines to cover all the zeros at least once the smallest elements among all the uncovered elements is 5. Subtracting this elements from all uncovered elements, adding to every element that lies at the intersection of two lines and leaving remaining elements unchanged the table thus, reduces as follows :

$$\begin{array}{c|cccc|}
 & W & X & Y & Z \\
\hline
A & 0 & 1 & 5 & 9 \\
B & 0 & 0 & 4 & 6 \\
C & 0 & 0 & 4 & 7 \\
D & 5 & 0 & 0 & 0 \\
\hline
\end{array}$$

Step 4: To make assignments in the usual manner.

$$\begin{array}{c|cccc|c}
 & W & X & Y & Z & \\
\hline
A & \boxed{0} & 1 & 5 & 9 & \checkmark \\
B & \cancel{0} & \boxed{0} & 4 & 6 & \checkmark \\
C & \cancel{0} & \cancel{0} & 4 & 7 & \checkmark \\
D & 5 & \cancel{0} & \boxed{0} & \cancel{0} & \\
\hline
 & \checkmark & \checkmark & & &
\end{array}$$

Step 5: Repeat the step 4 *i.e.,* to select the smallest element from uncovered elements *i.e.,* 4 then subtract to uncovered, covered remain same and add to all elements that lies at the intersection of two lines:

	W	*X*	*Y*	*Z*
A	0	1	1	5
B	0	0	0	2
C	0	0	0	3
D	9	4	0	0

Step 6: To make assignments in each rows and each columns.

	W	*X*	*Y*	*Z*
A	$\boxed{0}$	1	1	5
B	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
C	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	3
D	9	4	$\cancel{0}$	$\boxed{0}$

	W	*X*	*Y*	*Z*
A	$\boxed{0}$	1	1	5
B	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
C	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	3
D	9	4	$\cancel{0}$	$\boxed{0}$

Hence, the optimal assignments of jobs to machine for minimum cost are as

$$\text{Job } A \to W, B \to X, C \to Y, D \to Z$$

and

$$\text{Job } A \to W, B \to Y, C \to X, D \to Z$$

$$\text{The minimize cost } (Z) = 18 + 13 + 19 + 0 = 50.$$

Example 10: *Four engineers are available to design four projects. Engineer 2 is not competent to design the project B. Given the following time estimates needed to each engineer to design a given project, find how should be engineers be assigned to projects so as to minimize the total design time of four projects?* **[B.C.A. (Rohilkhand) 2003, 2007]**

		Projects			
		A	*B*	*C*	*D*
Engineers	**1**	*12*	*10*	*10*	*8*
	2	*14*	*Not suitable*	*15*	*11*
	3	*6*	*10*	*16*	*4*
	4	*8*	*10*	*9*	*7*

Solution: To avoid the assignment $2 \rightarrow B$, we take its time to be very large (say ∞). Then the cost matrix of the resulting assignment problem is shown in the following table :

	A	***B***	***C***	***D***
1	12	10	10	8
2	14	∞	15	11
3	6	10	16	4
4	8	10	9	7

Now we apply the assignment technique in the usual manner.

Step 1: Subtracting the minimum element of each row from every element of the corresponding row and then subtracting minimum element of each column from every element of the corresponding column, the reduced matrix is

	A	***B***	***C***	***D***
1	3	0	0	0
2	2	∞	2	0
3	1	4	10	0
4	0	1	0	0

Step 2: To make assignments in each rows and each columns by usual manner.

$$\begin{array}{cc|cccc|c}
 & & A & B & C & D & \\
\hline
L_2 & 1 & 3 & \boxed{0} & \cancel{0} & \cancel{0} & \\
 & 2 & 2 & \infty & 2 & \boxed{0} & \checkmark\ (3) \\
 & 3 & 1 & 4 & 10 & \cancel{0} & \checkmark\ (1) \\
L_3 & 4 & \boxed{0} & 1 & \cancel{0} & \cancel{0} & \\
\hline
 & & & & & \checkmark & \\
 & & & & & L_1 &
\end{array}$$

Step 3: In the above table, the smallest of the uncovered elements is 1. Subtracting this element 1 from all uncovered elements, adding to each element that lies at the intersection of two lines and leaving remaining elements unchanged, we get the following matrix

	A	***B***	***C***	***D***
1	3	0	0	1
2	1	∞	1	0
3	0	3	9	0
4	0	1	0	1

Step 4: Giving zero assignments in the usual manner, we observe that each row and each column have a zero assignment.

	A	*B*	*C*	*D*
1	3	$\boxed{0}$	$\cancel{0}$	1
2	1	∞	1	$\boxed{0}$
3	$\boxed{0}$	3	9	$\cancel{0}$
4	$\cancel{0}$	1	$\boxed{0}$	1

Hence, the optimal assignment is

Engineer → Project: $1 \to B, 2 \to D, 3 \to A, 4 \to C$

From the given matrix total minimum time $= 10 + 11 + 6 + 9 = 36$.

Example 11: *Find the optimal solution for the assignment problem with the following cost matrix :*

	I	*II*	*III*	*IV*	*V*
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

[B.C.A. (Meerut) 2002, 2008]

Solution: **Step 1:** Subtracting the smallest element of each row from every element of the corresponding row and then in the resulting matrix subtracting the smallest element of each column from every element of the corresponding column, we get the following matrix:

	I	II	III	IV	V
A	2	9	0	8	8
B	2	1	6	0	5
C	0	4	3	0	0
D	3	7	0	11	5
E	3	0	2	1	1

Step 2: Reduced matrix, make assignments in row and column having single zero

	I	II	III	IV	V
A	2	9	$\boxed{0}$	8	8
B	2	1	6	$\boxed{0}$	5
C	$\boxed{0}$	4	3	$\cancel{0}$	$\cancel{0}$
D	3	7	$\cancel{0}$	11	5
E	3	$\boxed{0}$	2	1	1

Since the 4th row and 5th column have no assignments, then applying tick rule.

Step 3: Draw the minimum number of lines to cover all zeros at least one. For this applying the following rules:

(i) Mark (✓) row 4 having no assignment

(ii) Mark (✓) column 3 having zero in marked row 4

(iii) Mark (✓) row 1 having assignment in the marked column

	I	II	III	IV	V	
A	2	9	$\boxed{0}$	8	8	✓
B	2	1	6	$\boxed{0}$	5	
C	$\boxed{0}$	4	3	$\cancel{0}$	$\cancel{0}$	
D	3	7	$\cancel{0}$	11	5	✓
E	3	$\boxed{0}$	2	1	1	
			✓			

Draw the line through unmarked row 2, 3 and 5 and marked column.

Step 4: Select the smallest element among all uncovered elements is 2. Subtract this element from all uncovered elements, adding to every element that lies at the intersection of two lines and leaving remaining elements unchanged, the matrix of step 3 reduces to the form.

	I	II	III	IV	V
A	0	7	0	6	6
B	2	1	8	0	5
C	0	4	5	0	0
D	1	5	0	9	3
E	3	0	4	1	1

Step 5: Repeating the step 2, make "zero assignment" we get the following table:

	I	II	III	IV	V
A	$\boxed{0}$	7	$\cancel{0}$	6	6
B	2	1	8	$\boxed{0}$	5
C	$\cancel{0}$	4	5	$\cancel{0}$	$\boxed{0}$
D	1	5	$\boxed{0}$	9	3
E	3	$\boxed{0}$	4	1	1

Thus, we get assignment **0** in each row and each column of the matrix.

Hence, the optimal assignment is

$$A \to I, B \to IV, C \to V, D \to III, E \to II$$

From the original matrix

Minimum $\text{cost } = 11 + 6 + 16 + 17 + 10 = 60$

Example 12: *The number of man-hours needed to complete a job for each job-man combination are given below*

Jobs

Men	*A*	*B*	*C*	*D*
1	*5*	*3*	*1*	*8*
2	*7*	*9*	*2*	*6*
3	*6*	*4*	*5*	*7*
4	*5*	*7*	*7*	*6*

Find the optimal assignment that will result in minimum man-hours needed.

[B.C.A. (Kanpur) 2004; B.C.A. (Meerut) 2006, 2011]

Solution: **Step 1:** Subtracting the smallest element each row from every element of the corresponding row and then subtracting the smallest element of each column from every element of the corresponding column, we get the following matrix :

Jobs

		A	*B*	*C*	*D*
	1	4	2	0	6
Men	**2**	5	7	0	3
	3	2	0	1	2
	4	0	2	2	0

Step 2: Make assignments $\boxed{0}$ in each row and each column by usual manner, we get the following matrix:

Job

		A	*B*	*C*	*D*
	1	4	2	$\boxed{0}$	6
Men	**2**	5	7	$\cancel{0}$	3
	3	2	$\boxed{0}$	1	2
	4	$\boxed{0}$	2	2	$\cancel{0}$

Since row 2 and column 4 have no zero assignment, then applying the following marked or tick (✓) rule.

Step 3: We draw the minimum number of lines to cover all the zeros at least once by usual manner.

	A	*B*	*C*	*D*	
1	4	2	$\boxed{0}$	6	✓
2	5	7	$\cancel{0}$	3	✓
3	2	$\boxed{0}$	1	2	
4	$\boxed{0}$	2	2	$\cancel{0}$	
			✓		

Step 4: The smallest element among all uncovered element is 2. Then subtract this element from all the uncovered elements, adding to every element that lies at the intersection of two lines and leaving remaining elements unchanged, the matrix in step 3 reduces to

	A	*B*	*C*	*D*
1	2	0	0	4
2	3	5	0	1
3	2	0	3	2
4	0	2	4	0

Step 5: Again repeat the step 2. We get the following matrix:

	A	*B*	*C*	*D*
1	2	$\cancel{0}$	$\cancel{0}$	4
2	3	5	$\boxed{0}$	1
3	2	$\boxed{0}$	3	2
4	$\boxed{0}$	2	4	$\cancel{0}$

Step 6: Again repeat step 3 of drawing minimum number of lines to cover all zero at least once then we find the following matrix:

$$\begin{array}{c|cccc|c} & A & B & C & D & \\ \hline 1 & 2 & \cancel{0} & \cancel{0} & 4 & \checkmark \\ 2 & 3 & 5 & \boxed{0} & 1 & \checkmark \\ 3 & 2 & \boxed{0} & 3 & 2 & \checkmark \\ 4 & \boxed{0} & 2 & 4 & \cancel{0} & \\ \hline & & \checkmark & \checkmark & & \end{array}$$

Step 7: Now repeat the step 4 where smallest element from uncovered element is 1 we get the following matrix :

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline 1 & 1 & 0 & 0 & 3 \\ 2 & 2 & 5 & 0 & 0 \\ 3 & 1 & 0 & 3 & 1 \\ 4 & 0 & 3 & 5 & 0 \\ \hline \end{array}$$

Step 8: Make zero assignments in each row and column by usual manner, we get the following matrix:

$$\begin{array}{c|cccc|} & A & B & C & D \\ \hline 1 & 1 & \cancel{0} & \boxed{0} & 3 \\ 2 & 2 & 5 & \cancel{0} & \boxed{0} \\ 3 & 1 & \boxed{0} & 3 & 1 \\ 4 & \boxed{0} & 3 & 5 & \cancel{0} \\ \hline \end{array}$$

Hence the optimal assignment for minimum cost of the problem is

$$1 \to C, 2 \to D, 3 \to B, 4 \to A$$

$\because$ Minimum cost $= 1 + 6 + 4 + 5 = 16.$

Example 13: *Explain the objective of an assignment problem. Five men are available to do five different jobs. From the past records, the time (in hours) that each man takes to do each job is known and given in the following table:*

		I	II	III	IV	V
	A	2	9	2	7	1
Men	B	6	8	7	6	1
	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

Find the assignment of men to jobs that will minimize the total time taken.

[B.C.A. (Lucknow) 2002, 2004, 2006]

Solution: **Step 1:** Subtracting the smallest element each row from every element of the corresponding row, we get the following matrix :

	I	II	III	IV	V
A	1	8	1	6	0
B	5	7	6	5	0
C	3	5	4	2	0
D	3	1	6	2	0
E	4	2	8	4	0

Step 2: Subtracting the smallest element each column from every element of the corresponding column, we get the following matrix :

	I	II	III	IV	V
A	0	7	0	4	0
B	4	6	5	3	0
C	2	4	3	0	0
D	2	0	5	0	0
E	3	1	7	2	0

Step 3: Reduced matrix, make assignments in row and column having single zero.

	I	II	III	IV	V
A	$\boxed{0}$	7	$\cancel{0}$	4	$\cancel{0}$
B	4	6	5	3	$\boxed{0}$
C	2	4	3	$\boxed{0}$	$\cancel{0}$
D	2	$\boxed{0}$	5	$\cancel{0}$	$\cancel{0}$
E	3	1	7	2	$\cancel{0}$

Since row 5^{th} and column 3^{rd} having no assignments. Then applying tick (✓) rule.

Step 4: Draw minimum number of lines to cover all zeros at least once. For this we apply following rules:

(i) Tick (✓) the row 5 having no assignment.

(ii) Tick (✓) the column 5^{th} having zero in tick row.

(iii) Tick (✓) row 2^{nd} having assignment in tick (✓) column.

	I	II	III	IV	V	
A	$\boxed{0}$	7	$\cancel{0}$	4	$\cancel{0}$	
B	4	6	5	3	$\boxed{0}$	✓
C	2	4	3	$\boxed{0}$	$\cancel{0}$	
D	2	$\boxed{0}$	5	$\cancel{0}$	$\cancel{0}$	
E	3	1	7	2	$\cancel{0}$	✓
					✓	

Draw the line through unticked row and tick column 5^{th}.

Step 5: The smallest element among all uncovered element is 1. Then subtract this element from all the uncovered elements, adding to every element that lies at the intersection of two lines and leaving remaining elements unchanged the matrix in step 4 reduces to.

	I	II	III	IV	V
A	0	7	0	4	1
B	3	5	4	2	0
C	2	4	3	0	1
D	2	0	5	0	1
E	2	0	6	1	0

Step 6: Repeat the step 3 in step 6.

	I	II	III	IV	V
A	$\boxed{0}$	7	$\cancel{0}$	4	1
B	3	5	4	2	$\boxed{0}$
C	2	4	3	$\boxed{0}$	1
D	2	$\boxed{0}$	5	$\cancel{0}$	1
E	2	$\cancel{0}$	6	1	$\cancel{0}$

Step 7: Repeat the step 4

	I	II	III	IV	V	
A	$\boxed{0}$	7	$\cancel{0}$	4	1	
B	3	5	4	2	$\boxed{0}$	✓
C	2	4	3	$\boxed{0}$	1	✓
D	2	$\boxed{0}$	5	$\cancel{0}$	1	✓
E	2	$\cancel{0}$	6	1	$\cancel{0}$	✓
		✓		✓	✓	

Step 8: Select the smallest element 2 from uncovered then repeat step 5.

	I	II	III	IV	V
A	$\cancel{0}$	9	$\boxed{0}$	6	3
B	1	5	2	2	$\boxed{0}$
C	$\cancel{0}$	4	1	$\boxed{0}$	1
D	$\boxed{0}$	$\cancel{0}$	3	$\cancel{0}$	1
E	$\cancel{0}$	$\boxed{0}$	4	1	$\cancel{0}$

Hence the optimal assignment for minimum cost of the problem is

$$A \to III, B \to V, C \to IV, D \to I, E \to II$$

$$\text{Minimum cost} \quad = 2 + 1 + 3 + 4 + 3 = 13$$

Problem Set

1. Consider the problem of assigning five operators to five machines. The assignment costs are given below :

Operators

		I	II	III	IV	V
	A	10	5	13	15	16
	B	3	9	18	3	6
Machines	***C***	10	7	2	2	2
	D	5	11	9	7	12
	E	7	9	10	4	12

Assign the operators to different machines so that total cost is minimized.

[B.C.A.(Indore) 2011; B.C.A. (Meerut) 2010; B.B.A. (Meerut) 2002]

2. A national car rental service has a surplus of one car in each of the cities 1, 2, 3, 4, 5, 6 and a deficit of one car in each of the cities 7, 8, 9, 10, 11, 12. The distances in miles between cities with a surplus and cities with a deficit are displayed in matrix below.

To

		7	8	9	10	11	12
	1	41	72	39	52	25	51
	2	22	29	49	65	81	50
From	**3**	27	39	60	51	32	32
	4	45	50	48	52	37	43
	5	29	40	39	26	30	33
	6	82	40	40	60	51	30

[B.C.A. (Bhopal) 2005, 2007]

3. Six machines M_1, M_2, M_3, M_4, M_5 and M_6 are to be located in six places P_1, P_2, P_3, P_4, P_5 and P_6. C_{ij} the cost of locating machine M_i at place P_j is given in the matrix below :

	P_1	P_2	P_3	P_4	P_5	P_6
M_1	20	23	18	10	16	20
M_2	50	20	17	16	15	11
M_3	60	30	40	55	8	7
M_4	6	7	10	20	25	9
M_5	18	19	28	17	60	70
M_6	9	10	20	30	40	55

Formulate the L.P. model to determine an optimal assignment. Write the objective function and the constraints in detail. Define any symbol used. Find an optimal layout by assignment technique of linear programming.

[B.C.A. (Kashividhyapeeth) 2008]

4. Solve the following assignment problems.

(a)

Machines \ Jobs	1	2	3	4	5
1	2.5	5	1	6	2
2	2	5	1.5	7	3
3	3	6.5	2	8	3
4	3.5	7	2	9	4.5
5	4	7	3	9	6
6	6	9	5	10	6

(b)

Machines \ Place	1	2	3	4	5
1	15	10	25	25	10
2	1	8	10	20	2
3	8	9	17	20	10
4	14	10	25	27	15
5	10	8	25	27	12

(c)

	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

[B.B.A. (Delhi) 2007; B.B.A. (Meerut) 2008]

(d)

Rider

Horse	R_1	R_2	R_3	R_4	R_5
H_1	5	3	4	7	1
H_2	2	3	7	6	5
H_3	4	1	5	2	4
H_4	6	8	1	2	3
H_5	4	2	5	7	1

(e)

	1	2	3	4	5
1	– 2	– 4	– 8	– 6	– 1
2	0	– 9	– 5	– 5	– 4
3	– 3	– 8	– 9	– 2	– 6
4	– 4	– 3	– 1	0	– 3
5	– 9	– 5	– 8	– 9	– 5

5. Assign four trucks 1, 2, 3 and 4 to vacant spaces 7, 8, 9, 10, 11 and 12 so that the distance travelled is minimized. The matrix below shows the distance.

	1	2	3	4
7	4	7	3	7
8	8	2	5	5
9	4	9	6	9
10	7	5	4	8
11	6	3	5	4
12	6	8	7	3

[B.C.A. (Bundelkhand) 2003]

6. A company has 5 jobs to be done. The following matrix shows the return in rupees on assigning ith $(i = 1, 2, 3, 4, 5)$ machines to the j th job $(j = A, B, C, D, E)$. Assign the five jobs to the five machines so as to maximize, the total expected profit.

	A	*B*	*C*	*D*	*E*
1	5	11	10	12	4
2	2	4	6	3	5
3	3	12	5	14	6
4	6	14	4	11	7
5	7	9	8	12	5

7. Alpha Corporation has four plants each of which can manufacture any of the four products. Production costs differ from plant to plant as do sales revenue. From the following data, obtain which product each plant should produce to maximize profit?

Sales Revenue (₹ 1000)

Product

Plant ↓	**1**	**2**	**3**	**4**
A	50	68	49	62
B	60	70	51	74
C	55	67	53	70
D	58	65	54	69

Production Cost (₹ 1000)

Product

Plant	**1**	**2**	**3**	**4**
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

8. There are 3 persons P_1, P_2 and P_3 and 5 jobs $J_1, J_2, \ldots, J_5$. Each person can do only one job and a job is to be done by one person only using Hungarian method. Find which 2 jobs should be left undone in the following cost minimizing assignment problem.

(a)

	J_1	J_2	J_3	J_4	J_5
P_1	7	8	6	5	9
P_2	9	6	7	6	10
P_3	8	7	9	5	6

(b)

	J_1	J_2	J_3	J_4	J_5	J_6
P_1	10	9	11	12	8	5
P_2	12	10	9	11	9	4
P_3	8	11	10	7	12	6
P_4	10	7	8	10	10	5

9. A company is faced with the problem of assigning 4 machines to 6 different jobs. The profits are estimated as follows:

Machines

		A	*B*	*C*	*D*
	1	3	6	2	6
	2	7	1	4	4
Jobs	**3**	3	8	5	8
	4	6	4	3	7
	5	5	2	4	3
	6	5	7	6	4

Solve the maximize problem. **[B.B.A. (I.G.N.O.U.) 2001, 2003, 2006, 2007]**

10. Define assignment problem. or What is the assignment problem?

[B.C.A. (Avadh) 2008; B.B.A. (Meerut) 2002, 2006]

11. What is an unbalanced assignment problem?

[B.C.A. (Kanpur) 2006; B.B.A. (Garhwal) 2001, 2005; B.B.A. (Agra) 2002, 2005, 2006]

12. Explain the objective of an assignment problem.

[B.C.A. (Lucknow) 2006; B.B.A. (Meerut) 2002, 2003]

13. Discuss practical application of an assignment problem. **[B.B.A. (Meerut) 2004]**

14. What do you mean by an assignment problem ? What is the objective in each problem ? **[B.B.A. (Meerut) 2006]**

15. Explain the difference between a transportation problem and an assignment problem.

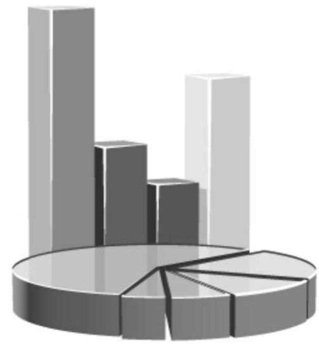
[B.C.A. (Agra) 2004, 2006, 2011; B.C.A. (Rohilkhand) 2008; B.C.A. (Lucknow) 2005, 2010; B.B.A. (Meerut) 2008]

Answers

1.	$A \to II, B \to V, C \to III, D - I, E \to IV$.
2.	$1 \to 11,\ 2 \to 8,\ 3 \to 7,\ 4 \to 9,\ 5 \to 10,\ 6 \to 12$, Min. $z = 185$ miles.
3.	$M_1 \to P_4, M_2 \to P_6, M_3 \to P_5, M_4 \to P_3, M_5 \to P_1, M_6 \to P_2$, Min $(z) = 67$.
4.	(a) $1 \to 4, 2 \to 1, 3 \to 5, 4 \to 3, 5 \to 2$, 6-Dummy, Min $(z) =$ ₹ 2,000.
	(b) $1 \to 5, 2 \to 3, 3 \to 4, 4 \to 2, 5 \to 1$, Min. $(z) = 60$.
	(c) $1 \to I, 2 \to IV, 3 \to V, 4 \to III, 5 \to II$, Min $(z) = 60$.
	(d) $H_1 \to R_5, H_2 \to R_1, H_3 \to R_4, H_4 \to R_3, H_5 \to R_2$, Min. $(z) = 8$.
	(e) $1 \to 3, 2 \to 2, 3 \to 5, 4 \to 1, 5 \to 4$, Min $(z) = -36$.
5.	$7 \to 3, 8 - 2, 9 \to 1, 12 \to 4$, Min $(z) = 12$.
6.	$1 \to C, 2 \to E, 3 \to D, 4 \to B, 5 \to A$, Max. $(z) =$ ₹ 50.
7.	$A \to 2, B \to 4, C \to 1, D \to 3$, Max. $(z) = 22{,}000$.
8.	(a) (*i*) $P_1 \to J_3, P_2 \to J_2, P_3 \to J_4$: Jobs J_1 and J_5 left undone.
	(*ii*) $P_1 \to J_4, P_2 \to J_2, P_3 \to J_5$: Jobs J_1 and J_3 left undone.
	In both cases minimum cost is 17.
	(b) $P_1 \to J_5, P_2 \to J_6, P_3 \to J_4, P_4 \to J_2$;
	Jobs J_1 and J_3 left undone.
9.	$2 \to A, 3 \to B, 4 \to D, 6 \to C$, Job 1 and 5 are left undone, Max $(z) = 28$.

❍❍❍

CHAPTER 5



Linear Programming and Graphical Methods

5.1 Linear Programming

[B.B.A. (Meerut) 2003]

The terms 'Linear programming' has been defined by various authors as given below:

According to **Dantzig** "*Linear programming is a programming of inter-dependent activities in a linear structure*".

According to **Richard H. Leftwich** "*Linear programming is the simplest and most widely used mathematical technique that has come into vague since world war II*".

5.1.1 Important Applications of Linear Programming

[B.C.A. (Agra) 2005, 2008; B.B.A. (Meerut) 2003]

The main areas of the applications of linear programming technique are as follows:

1. Linear programming approach is being used in five year plans in the area such as foodgrain storage, transportation, multi-level planning etc.
2. Air lines are also using linear programming in the selection of routes and allocation of aircrafts to various routes.
3. Linear programming is also being used in Production Management, Marketing Management, Human Resources etc.

4. Indian Railways have applied linear programming in a number of areas.
5. The oil refineries, steel industry use programming technique for blending decisions and for the moment of finished products.
6. Linear programming approach is being used in agriculture for crop rotation, mix of cash crops and food crops and fertilizer mix.
7. **The main problem of linear programming are:** Diet problem, Production Line Problem, Transportation Problem, Investment Problem, Staffing Problem, Product Mix Problem, Advertising Media Selection Problem, Production Plan Problem, Agricultural Problem, Food Management Problem etc.

5.1.2 Objectives of Using Linear Programming

[B.B.A. (Meerut) 2005]

The primary reason for using linear programming is to achieve the best allocation of some resources. For example:

1. To select an investment portfolio from a variety at stock and bond investment alternatives that maximize the return of investment.
2. To allocate the available machine, time and labour hours in each department along with the raw material to the activities of producing different products which have been scheduled and to determine the number of units of each of the products so as to maximize the profit.
3. To determine which warehouse should ship how much product to which market so that the total transportation costs are minimized.

   In linear programming terminology, the maximization or minimization of a quantity is referred to as the **objective of the problem**.

5.1.3 Main Characteristics of Linear Programming

1. **Decision Variable:** The decision variables refer to limited resources and are denoted by x, y etc.
2. **Objective Function:** The linear function between decision variables which is to be maximized or minimized is known as **objective function** of linear programming problem.

   [B.B.A. (Meerut) 2006]
3. **Constraints:** The restrictions or limits which are determined to make the solution optimum (maximum or minimum) are called constraints.

   [B.B.A. (Meerut) 2006]
4. **Linearity Proportionality and Non-Negativity Conditions:** The objective function and all the constraints must be linear and the values of all the decision variables should be non-negative. If any of the decision variables is **unrestricted** in sign, a trick can be employed which will enforce the non-negativity condition without changing the original information of the problem.

5. **Additivity:** Two or more variables under consideration are additive. For example, if one unit of product A required 12 kg of material in its manufacture and product B requires 19 kg of material, then one unit of A and one unit of B require (12+19) kg of material.
6. **Divisibility:** Divisibility means solutions may take any fractional value.
7. **Deterministic or Certainly:** All the coefficients used in a L.P.P. are completely known. For example, profit per unit, quantity of resource, available etc.
8. **Finiteness:** The number of decision variables and constraints is finite.

5.1.4 Mathematical Form of Linear Programming Problem

[B.C.A. (Meerut) 2009,2011; B.B.A. (Meerut) 2003, 2009]

Let $x_1, x_2, x_3, \ldots, x_n$ be n decision variables. Then the mathematical form of **linear programming problem** or so called **general linear programming** is as follows :

Determine the values of $x_1, x_2, \ldots, x_n$ which optimize

$$z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \qquad \ldots(1)$$

Subject to the constraints

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n \ (\leq, =, \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n \ (\leq, =, \geq)\ b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n \ (\leq, =, \geq)\ b_m \end{cases} \qquad \ldots(2)$$

and non-negative restrictions

$$x_1, x_2, \ldots, x_n \geq 0 \qquad \ldots(3)$$

where all c's b's, a's are known constants.

The equation (1) is the **objective function**, the equation (2) are **constraints** and equation (3) is **non-negativity conditions** $(\leq, =, \geq)$ means that any one of the three signs may be there. The constraints which do not affect the possible solutions are known as **redundant constraints**.

Or

A general linear programming problem can be stated as

$$\text{Optimize } (z) = \sum_{j=1}^{n} c_j x_j$$

Subject to $\quad \Sigma a_{ij} x_j \ (\leq, =, \geq)\ b_i, \ i = 1, 2, \ldots, m$

and $\quad x_j \geq 0, \ j = 1, 2, \ldots, n$

Or

The general linear programming problem can be written in matrix form as

Optimize $z = CX$

such that $AX\ (\leq, =, \geq)\ B$

and $X \geq 0$

where $C = [c_1, c_2, \ldots, c_n]$ a row vector

$X' = (x_1, x_2, \ldots, x_n)$ a row vector and X is a column vector

$A = [a_{ij}]_{m \times n}$ be the matrix of the coefficient a_{ij} of order $m \times n$,

$i = 1, 2, 3, \ldots m,\ \ j = 1, 2, 3, \ldots, n$

$B' = [b_1, b_2, \ldots, b_n]$ a row vector and B is column vector.

5.1.5 Formulation of Linear Programming Problem

While formulating a linear programming problem we proceed as follows:

Step 1: Identify the decision variables (the variables whose values are to be found out by solving the L.P.P.) and assign the symbols ($x_1, x_2, x_3, \ldots$ or $x, y, z, \ldots$) to them.

Step 2: Identify the objective function (to be optimized) and represent it as a linear function of the decision variables.

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots$$

Step 3: Identify all the restrictions or constraints in the problem regarding the resources which are available upto or beyond certain limit and express them as linear equations and/or inequalities in terms of decision variables in the following manner.

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n\ (\leq, =, \geq)\ b_1$$

Step 4: Since the negative values of decision variables do not have any valid physical interpretation.

$$x_i \geq 0\ \forall\ \ i = 1, 2, 3, \ldots.$$

Step 5: In the end write the objective function, linear constraints and non-negativity conditions together to form a model of linear programming problem.

5.1.6 Limitations of Linear Programming

[B.B.A. (Meerut) 2005]

1. The basic assumption that objective function and constraints are linear may not hold good in many practical situations.
2. When number of variables in given problem is large, then procedure become complicated, then use of computer is necessary.
3. Sometime it may be possible that the solutions are in terms of fractions.
4. The technique of L.P.P. does not consider the problems related to uncertainity.

Solved Examples

Example 1: *A company manufactures two types of bulbs A and B by using two machines M_1 and M_2. One bulb of type A requires 2 hours at machine M_1 and 1 hour at machine M_2. One bulb of type B requires 1 hour at machine M_1 and 2 hours at M_2. The profit contributions from each bulb of type A is ₹ 3 and from each bulb of type B is ₹ 2. The number of hours available per week on machine M_1* and *M_2 are 20 hours and 30 hours respectively. Formulate the above problem as a linear programming problem, the aim of the company is to earn maximum profit.*

Solution:

Step 1: The aim of the company is to earn maximum profit so the company desires to know the number of bulbs to be produced per week to achieve this aim.

Let

x_1 = The number of bulbs of type A to be produced per week

x_2 = The number of bulbs of type B to be produced per week

Then total profit $z = 3x_1 + 2x_2$

Step 2: Objective Function: The objective function is

$z = 3x_1 + 2x_2$ which is to be maximized. ...(1)

Step 3: Constraints: Let us arrange the given information in tabular form to get the constraints:

Machine	Time in Production of one bulb (in hours)		Time available per week (in hours)
	A	***B***	
M_1	2	1	20
M_2	1	2	30

Constraints for Machine M_1: The time consumed on machine M_1 for x_1 bulbs of type A and x_2 bulbs of type B is $2x_1 + x_2$.

This time cannot exceed the time available on machine M_1.

Hence, $2x_1 + x_2 \leq 20$ ...(2)

Constraints for Machine M_2 : The time consumed on machine M_2 for x_1 bulbs of type A and x_2 bulbs of type B is $x_1 + 2x_2$.

This time cannot exceed the time available on machine M_2.

Hence, $x_1 + 2x_2 \leq 30$ ...(3)

Step 4: Condition of Non-Negativity: Since x_1 and x_2 are the numbers of bulbs to be produced these values can not be negative, hence $x_1 \geq 0,\ x_2 \geq 0$ ... (4)

Step 5: Thus, the mathematical form of the linear programming problem is

$$\text{Maximize } z = 3x_1 + 2x_2$$

such that
$$2x_1 + x_2 \leq 20$$
$$x_1 + 2x_2 \leq 30$$
$$x_1, x_2 \geq 0$$

Example 2: *A goldsmith manufactures necklace and bracelets. The total number of necklaces and bracelets that he can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. It is assumed that he can work for a maximum of 16 hours a day. Further the profit on a bracelet is ₹ 300 and the profit on a necklace is ₹ 100. Formulate this problem as a linear programming problem so as to maximize the profit.*

Solution:

Step 1: The aim of a goldsmith is to earn maximum profit so that the goldsmith desires to know the number of necklaces and bracelets to be handled per day.

Let x_1 = The number of necklace to be manufactured per day

x_2 = The number of bracelet to be manufactured per day.

Then total profit $z = 100\, x_1 + 300\, x_2$

Step 2: Objective Function: The objective function is

$z = 100\, x_1 + 300\, x_2$, which is to be maximized ...(1)

Step 3: Constraints: Since it takes half an hour to make one necklace, so the time required to make x_1 necklaces $= \frac{1}{2} x_1$ hours.

Again it takes one hour to make one bracelet, so the time required to make x_2 bracelets $= x_2$ hours.

Therefore, total time required to make x_1 necklaces and x_2 bracelets

$$= \left(\frac{1}{2} x_1 + x_2\right) \text{hours} \quad ...(2)$$

Since total available per day is 16 hours, therefore

$$\frac{1}{2} x_1 + x_2 \leq 16$$

or
$$x_1 + 2\, x_2 \leq 32 \quad ...(3)$$

The total number of necklaces and bracelets that the goldsmith can manufactures in a day is at most 24

$$\therefore \quad x_1 + x_2 \leq 24 \quad ...(4)$$

Step 4: Condition of Non-Negativity: Also the number of necklaces and bracelets manufactured can never be negative, therefore

$$x_1 \geq 0, \; x_2 \geq 0 \qquad ...(5)$$

Step 5: Thus, the linear programming problem formulated from the given problem is:

$$\text{Maximize } z = 100\, x_1 + 300\, x_2$$

Subject to the constraints

$$x_1 + 2x_2 \leq 32$$

$$x_1 + x_2 \leq 24$$

$$x_1 \geq 0, \; x_2 \geq 0$$

Example 3: *A manufacturer of medicines is proposed to prepare a production plan for medicines A and B. There are sufficient ingredients available to make* 20,000 *bottles of medicine A and* 40,000 *bottles of medicines B, but there are only* 45,000 *bottles into which either of the medicine can be filled. Further, it takes three hours to prepare, enough material to fill,* 1,000 *bottles of medicine A and one hour for medicine B and there are* 66 *hours available for this operation. The profit is* ₹ 8 *per bottles of medicine A and* ₹ 7 *per bottle of medicine B. Formulate this problem as L.P.P.*

Solution:

Step 1: The decision variables are:

x_1 = number of bottles of medicines A

and x_2 = number of bottles of medicines B.

produced by the manufacturer.

The profit on medicine A = ₹ 8 per bottle.

The profit on medicine B = ₹ 7 per bottle.

Hence, the total profit is $z = 8x_1 + 7x_2$.

Step 2: Objective Function: The objective function is $z = 8x_1 + 7x_2$ which is to be maximized.

Step 3: Constraint

(i) Time Constraint: Since it takes 3 hours to prepare 1,000 bottles of medicine.

A, hence the time required for x_1 bottles of medicines A is $\frac{3}{1000} x_1$ hours.

Similarly, time required for x_2 bottles of medicines B is $\frac{1}{1000} x_2$ hours.

$\therefore$ The total time required $= \frac{3}{1000} x_1 + \frac{1}{1000} x_2$

But the total time available is 66 hours.

Hence, the constraint is $\frac{3}{1000} x_1 + \frac{1}{1000} x_2 \leq 66$

or $3x_1 + x_2 \le 66{,}000$

(ii) Number Constraints: There are only 45,000 bottles available for filling medicines A and B. Hence, the constraints is

$$x_1 + x_2 \le 45{,}000$$

(iii) Ingredients Constraints: Since there are sufficient ingredients available to make 20,000 bottles of medicines A and 40,000 bottles of B, hence the constraints are

$$x_1 \le 20{,}000, \quad x_2 \le 40{,}000$$

Step 4: The condition of non-negativity:

$$x_1, x_2 \ge 0$$

Step 5: Thus, the formulation of L.P.P. is:

$$\text{Maximize } z = 8x_1 + 7x_2$$

subject to

$$3x_1 + x_2 \le 66{,}000$$
$$x_1 + x_2 \le 45{,}000$$
$$x_1 \le 20{,}000$$
$$x_2 \le 40{,}000$$

and

$$x_1, x_2 \ge 0$$

Remark: We may also take the number of bottles produced as x_1 thousands and x_2 thousand respectively, then we have:

$$\text{Maximize } z = 8x_1 + 7x_2$$

subject to

$$3x_1 + x_2 \le 66$$
$$x_1 + x_2 \le 45$$
$$x_1 \le 20$$
$$x_2 \le 40$$

and

$$x_1, x_2 \ge 0$$

Example 4: *A firm can produce three types A, B and C of cloths using three kinds, red, green and blue wool. One unit length of type A cloth need 2 metres of red wool and 3 metres of blue wool. One unit length of type B cloth need 3 metres of red wool, 2 metres of green wool and 2 metres of blue wool. One unit of type C cloth need 5 metres of green wool and 4 metres of blue wool. The firm has only a stock of 8 metres of red wool, 10 metres of green wool and 15 metres of blue wool. It is assumed that the income obtained from unit length of type A cloth is ₹ 3, of type B cloth is ₹ 5 and of type C cloth is ₹ 4. Formulate this problem as a L.P.P. to have maximum income from, the cloth produced.*

Solution:

Step 1: Construct a table for clear understanding of the problem

Kinds of wool	Requirement of wool (in metres) for cloth of type			Wool available (in metres)
	A	***B***	***C***	
Red	2	3	0	8
Green	0	2	5	10
Blue	3	2	4	15
Income from one unit length of cloth	3	5	4	

Step 2: Objective Function: Let the decision variable be

x_1 = the quantity produced of type A cloth (in metres)

x_2 = the quantity produced of type B cloth (in metres)

x_3 = the quantity produced of type C cloth (in metres)

one unit length is taken as one metre

Hence, income (in rupees) of the firm, (*i.e.,* objective function) is

$$z = 3x_1 + 5x_2 + 4x_3$$

Step 3: Constraint

(i) **Constraint for Red Wool:** The quantity of red wool required to prepare x_1 metres of type A, x_2 metres of type B, x_3 metres of type C is

$$2x_1 + 3x_2 + 0\,x_3$$

Since, the stock of red wool is 8 metres so the constraint is

$$2x_1 + 3x_2 + 0x_3 \le 8$$

(ii) **Constraint of Green Wool:** The quantity of green wool required to prepare x_1 metres of type A, x_2 metres of type B and x_3 metres of type C is

$$0.\,x_1 + 2x_2 + 5x_3$$

Since the stock of green wool is 10 metres so the constraint is

$$0.\,x_1 + 2x_2 + 5x_3 \le 10$$

(iii) **Constraint for Blue Wool:** The quantity of blue wool required to prepare x_1 metres of type A, x_2 metres of type B and x_3 metres of type C is

$$3x_1 + 2x_2 + 4x_3$$

Since the stock of blue wool is 15 metres so the constraint is

$$3x_1 + 2x_2 + 4x_3 \le 15$$

Step 4: The Condition of Non-Negativity: $x_1, x_2, x_3 \geq 0$

Since firm can not produce negative quantity

Step 5: The Mathematical form of L.P.P.

$$\text{Maximize } z = 3x_1 + 5x_2 + 4x_3$$

such that
$$2x_1 + 3x_2 \leq 8$$
$$2x_2 + 5x_3 \leq 10$$
$$3x_1 + 2x_2 + 4x_3 \leq 15$$
and
$$x_1, x_2, x_3 \geq 0$$

Example 5: *A tyre factory produces three types of tyres T_1, T_2, T_3. Three different types of chemicals say C_1, C_2, C_3 are required for production. One T_1 tyre needs 2 units of C_1, 3 units of C_3; one T_2 tyre needs 3 units of C_1, 2 units of C_2 and 2 units of C_3; and one T_3 tyre needs 5 units of C_2 and 4 units of C_3. The factory has only stock of 20 units of C_1, 25 units of C_2 and 30 units of C_3. Further the profit from the sale of one tyre T_1 is ₹ 6, one tyre T_2 is ₹10 and of one tyre T_3 is ₹8. Assuming that the factory can sell all that it produces, formulate a linear programming problem to maximize its profits.* **[B.C.A. (Rohilkhand) 2007]**

Solution:

Step 1: Let the factory produce x_1 tyres of type T_1, x_2 tyres of type T_2 and x_3 tyres of type T_3.

The given information can be put in a tabular form as given below:

	Tyre T_1	Tyre T_2	Tyre T_3	Total chemical available
Chemical C_1	2	3	0	20
Chemical C_2	0	2	5	25
Chemical C_3	3	2	4	30
Profit in ₹ (per tyre)	6	10	8	

Step 2: Objective Function: The total profit Z in ₹ is given by

$$Z = 6x_1 + 10x_2 + 8x_3 \qquad ...(1)$$

Step 3: Constraints: The total quantity of chemical C_1 required

$$= (2x_1 + 3x_2) \text{ units.}$$

Since the factory has a stock of 20 units of chemical C_1, therefore we have

$$2x_1 + 3x_2 \leq 20 \qquad ...(2)$$

Similarly, considering the total quantity of the chemicals C_2 and C_3 required, we have

$$2x_2 + 5x_3 \le 25 \quad ...(3)$$

and $$3x_1 + 2x_2 + 4x_3 \le 30 \quad ...(4)$$

Step 4: Condition of Non-Negativity:

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \quad ...(5)$$

Since the factory cannot produce negative quantities.

Step 5: Hence, the linear programming problem formulated from the given problem is as follows:

$$\text{Maximize } Z = 6x_1 + 10x_2 + 8x_3$$

s.t. $$2x_1 + 3x_2 \le 20,\ 2x_2 + 5x_3 \le 25,$$

$$3x_1 + 2x_2 + 4x_3 \le 30$$

and $$x_1 \ge 0,\ x_2 \ge 0$$

Example 6: *The objective of a diet problem is to ascertain the quantities of certain foods that should be eaten to meet certain nutritional requirement at a minimum cost. The consideration is limited to milk, green vegetables, eggs, and to vitamins A, B, C. The number of milligrams of each of these vitamins contained within a unit of each food and their daily minimum requirement along with the cost of each food is given in the table below:*

Vitamin	*Litres of milk*	*Kg. of vegetables*	*Dozen of eggs*	*Minimum daily requirement*
A	1	1	10	1 mg.
B	100	10	10	50 mg.
C	10	100	10	10 mg.
Cost in ₹	20	10	8	

Formulate a linear programming problem for this diet problem. **[B.C.A. (Kanpur) 2009]**

Solution: Let the daily diet consist of x_1 litres of milk, x_2 kg. of vegetables and x_3 dozens of eggs.

∴ the total cost Z per day in ₹ is given by

$$Z = 20x_1 + 10x_2 + 8x_3 \quad ...(1)$$

Total amount of vitamin A in the daily diet is

$$(x_1 + x_2 + 10x_3) \text{ mg}$$

which should be at least equal to 1 mg., therefore

$$x_1 + x_2 + 10x_3 \ge 1 \quad ...(2)$$

Similarly, considering the total amounts of vitamins B and C in the daily diet, we have

$$100x_1 + 10x_2 + 10x_3 \geq 50 \qquad ...(3)$$

and $$10x_1 + 100x_2 + 10x_3 \geq 10 \qquad ...(4)$$

Since the quantities of different food items to be consumed cannot be negative, therefore we have

$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0$$

Hence, the linear programming problem formulated from the given diet problem is:

$$\text{Minimize } Z = 20x_1 + 10x_2 + 8x_3$$

subject to the constraints

$$x_1 + x_2 + 10x_3 \geq 1$$

$$100x_1 + 10x_2 + 10x_3 \geq 50$$

$$10x_1 + 100x_2 + 10x_3 \geq 10$$

and the non-negative restrictions

$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0$$

Example 7: *According to the medical experts it is necessary for an adult to consume at least 75 gm of proteins, 85 gm of fats and 300 gm of carbohydrates daily. The following table gives the analysis of the food items readily available in the market with their respective costs.*

Food Type	***Food value (in gm.) per 100 gm***			***Costs in per kg.***
	Proteins	***Fats***	***Carbohydrates***	
A	18.0	15.0	—	3.0
B	16.0	4.0	7.0	4.0
C	4.0	20.0	2.5	2.0
D	5.0	8.0	40.0	1.5
Minimum daily requirement	75.0	85.0	300.0	

Formulate a linear programming problem for an optimum diet.

Solution: Let the daily diet consist of x_1 kg of food A, x_2 kg of food B, x_3 kg of food C and x_4 kg of food D.

Then the total cost per day in ₹ is

$$Z = 3x_1 + 4x_2 + 2x_3 + 1.5x_4 \qquad ...(1)$$

Total amount of proteins in the daily diet is

$$(180x_1 + 160x_2 + 40x_3 + 50x_4)$$

Since the minimum daily requirement of proteins is 75 gm, therefore we have

$$180x_1 + 160x_2 + 40x_3 + 50x_4 \geq 75 \quad \text{...(2)}$$

Similarly, considering the total amounts of fats and carbohydrates in the diet, we have

$$150x_1 + 40x_2 + 200x_3 + 80x_4 \geq 85 \quad \text{...(3)}$$

and

$$70x_2 + 25x_3 + 400x_4 \geq 300 \quad \text{...(4)}$$

Since, the daily diet cannot contain quantities with negative values of any food item, therefore

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \quad \text{...(5)}$$

Hence, the linear programming problem formulated for the given diet problem is :

$$\text{Minimize } Z = 3x_1 + 4x_2 + 2x_3 + 1.5x_4$$

subject to the constraints

$$180x_1 + 160x_2 + 40x_3 + 50x_4 \geq 75$$

$$150x_1 + 40x_2 + 200x_3 + 80x_4 \geq 85$$

$$70x_2 + 25x_3 + 400x_4 \geq 300$$

and the non-negative restrictions

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

Example 8: *A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below:*

Machine	***Time per unit (minutes)***			***Machine capacity (minutes/day)***
	Product 1	***Product 2***	***Product 3***	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is ₹ 4, ₹ 3 and ₹ 6 respectively. It is assumed that all the amounts produced are consumed in the market. **[B.C.A. (Meerut) 2006]**

Solution: **Formulation of Linear Programming Model**

Step 1: From the study of the situation find the key-decision to be made. In this connection looking for variables helps considerably. In the given situation key decision is to decide the extent of products 1, 2 and 3 as the extents are permitted to vary.

Step 2: Assume symbols for variable quantities noticed in step 1. Let the extent (amounts) of products 1, 2 and 3 manufactured daily be x_1, x_2 and x_3 respectively.

Step 3: Express the feasible alternatives mathematically in terms of variables. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of x_1, x_2 and x_3.

Where $$x_1, x_2, x_3 \geq 0 \qquad ...(1)$$

Since negative production has no meaning, is not feasible.

Step 4: Mention the objective quantitatively and express it as a linear function of variables. In the present situation objective is to maximize the profit

i.e., $$\text{maximize } Z = 4x_1 + 3x_2 + 6x_3 \qquad ...(2)$$

Step 5: Put into the words the influencing factors or constrains. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equalities/inequalities in terms of variables.

Here, constraints are on the machine capacities and can be mathematically expressed as

$$2x_1 + 3x_2 + 2x_3 \leq 440$$
$$4x_1 + 0x_2 + 3x_3 \leq 470$$
$$2x_1 + 5x_2 + 0x_3 \leq 430 \qquad ...(3)$$

Example 9: *A company has two grades of inspectors,* 1 *and* 2 *to undertake quality control inspection. At least* 1,500 *pieces must be inspected in an* 8*-hour day. Grade* 1 *inspector can check* 20 *pieces in an hour with an accuracy of* 96%.

Grade 2 *inspector checks* 14 *pieces in an hour with an accuracy of* 92%.

The daily wages of grade 1 *inspector are ₹ 5 per hour while those of grade* 2 *inspector are ₹* 4 *per hour. Any error made by an inspector costs ₹* 3 *to the company. If there are, in all,* 10 *grade* 1 *inspector and* 15 *grade* 2 *inspectors in the company, find the optimal assignment of inspectors that minimize the daily inspection cost.*

Solution: **Formulation of L.P. Model**.

Step 1: The key decision to be made is to determine the number of grade 1 and grade 2 inspectors for assignments.

Step 2: Let x_1 and x_2 represent the number of these inspectors.

Step 3: Feasible alternatives are sets of values of x_1 and x_2, where $x_1 \geq 0$, $x_2 \geq 0$.

Step 4: The objective to minimize the daily cost of inspection. Now the company has to incur two types of costs wages paid to the inspectors and the cost of their inspection errors. The cost of grade 1 inspector/hour is

$$₹\,(5 + 3 \times 0.04 \times 20) = ₹\,7.40$$

Similarly, cost of grade 2 inspector/hour is

$$₹\,(4 + 3 \times 0.08 \times 14) = ₹\,7.36$$

∴ The objective function is minimize

$$z = 8\,(7.40x_1 + 7.36x_2) = 59.20x_1 + 58.88x_2$$

Step 5: Constraints are

on the number of grade 1 inspectors : $x_1 \le 10$,

on the number of grade 2 inspectors : $x_2 \le 15$,

on the number of pieces to be inspected daily:

$$20 \times 8x_1 + 14 \times 8x_2 \ge 1{,}500$$

or

$$160x_1 + 112x_2 \ge 1{,}500$$

Example 10: *A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fats and carbohydrates at the minimum cost. The Choice is to be made from four different types of foods. The yield per unit of these foods are given in table:*

Food Type	*Yield per unit*			*Cost per unit* (₹)
	Proteins	*Fats*	*Carbohydrates*	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate linear programming model for the problem.

Solution: **Formulation of L.P. Model**:

Step 1: Key decision is to determine the number of units of food of type 1, 2, 3 and 4 to be used.

Step 2: Let these units be x_1, x_2, x_3 and x_4 respectively.

Step 3: Feasible alternatives are sets of values of x_j

where $$x_j \ge 0,\ \ j = 1, 2, 3, 4 \qquad ...(1)$$

Step 4: Objective is to minimize the cost *i.e.,*

$$\text{minimize}\, z = ₹\,(45x_1 + 40x_2 + 85x_3 + 65x_4) \qquad ...(2)$$

Step 5: Constraints are on the fulfilment of the daily requirements of the various constituents:

$$\left.\begin{array}{rl} i.e., \text{ for proteins} & 3x_1 + 4x_2 + 8x_3 + 6x_4 \ge 800 \\ \text{for fats} & 2x_1 + 2x_2 + 7x_3 + 5x_4 \ge 200 \\ \text{and for Carbohydrates,} & 6x_1 + 4x_2 + 7x_3 + 4x_4 \ge 700 \end{array}\right] \quad \ldots(3)$$

Thus, the L.P. Model is to determine the number of units of x_1, x_2, x_3 and x_4 that minimize equation (2) subject to constraints (3) and non-negativity restrictions (1).

Example 11: *A firm manufactures two types of products A and B and sells them at a profit of ₹ 2 on type A and ₹ 3 on type B. Each product is processed on two machines E* and *F. Type A requires one minutes of processing time on E and two minutes on F. Type B requires one minute on E and one minute on F. The machine E is available for not more than 6 hours 40 minutes while machine F is available for 10 hours during any working day. Formulate the problem as a linear programming problem.* **[B.C.A. (Meerut) 2010]**

Solution: Let the manufacturer produce x_1 units of the product of type A and x_2 units of the product of type B.

The given information can be systematically arranged in the form of the following table:

Machine	**Time of processing (minutes per unit)**		**Time available (minutes)**
	Type *A* (x_1 units)	**Type *B* (x_2 units)**	
E	1	1	400
F	2	1	600
Profit per unit	₹ 2	₹ 3	

Since, the machine E takes 1 minute time for processing a unit of type A and 1 minute time for processing a unit of type B, therefore the total time required on machine E is $(x_1 + x_2)$ minutes.

But the machine E is available for not more than 6 hours and 40 minutes $= 400$ minutes, therefore we have

$$x_1 + x_2 \le 400$$

Similarly, the total time (in minutes) required on machine F is $2x_1 + x_2$.

Since, the machine F is available for not more than 10 hours $= 600$ minutes, therefore we have

$$2x_1 + x_2 \le 600$$

Since, it is not possible to produce negative quantities, so we have $x_1 \ge 0,\ x_2 \ge 0$.

The profit on type A is ₹ 2 per unit so the profit on selling x_1 units of type A will be ₹ $2x_1$. Similarly, the profit on selling x_2 units of type B will be ₹ $3x_2$.

Therefore, the total profit (in ₹) on selling x_1 units of type A and x_2 units of type B is

$$Z = 2x_1 + 3x_2$$

Hence, the required linear programming problem formulated for the given problem is

$$\text{Maximize } Z = 2x_1 + 3x_2$$

subject to the conditions

$$x_1 + x_2 \le 400,$$

$$2x_1 + x_2 \le 600$$

and $$x_1 \ge 0,\ x_2 \ge 0$$

Example 12: *Sun chemical company is producing two products A and B. The processing times are 3 and 4 hours per unit for A an operations one and two respectively and 4 hours and 5 hours per unit of B an operations on one and two respectively. The available time is 18 hours and 21 hours for operation one and two respectively. The product A can be sold for ₹ 3 profit per unit and B of ₹ 8 profit per unit, solve for maximum program only formulate the problem.*

[B.B.A. (Meerut) 2006]

Solution: We have from given problem.

	Time		
	A	B	**Available time**
Operation I	3	4	18 hrs
Operation II	4	5	21 hrs
Profit per unit ₹	3	8	

Let x_1 and x_2 be the product of type A and B. Then objective function will be

$$\text{Max}\,(Z) = 3x_1 + 8x_2$$

subject to the constraints:

$$3x_1 + 4x_2 \le 18$$

$$4x_1 + 5x_2 \le 21$$

$$\forall\ x_1, x_2 \ge 0$$

Problem Set

1. A firm manufactures two types of products A and B and sells them at a profit of ₹2 on type A and ₹3 of type B. Each product is produced on two machines M and N. Type A requires one minute of processing time of M and 2 minutes of processing time of N. Type B requires one minute of processing time of M and one minute of processing time of N. The machine M is available for not more than 400 minutes whereas machine N is available for 10 hours. What is the linear programming formulation for this problem?

2. One kind of hens can be bought for ₹4.00 each but another kind cost ₹10.00. First kind of hens lay 3 eggs per week and another 5 eggs per week each egg worth being 60 paise. A hen cost ₹2.00 per week to feed. A person has only ₹160 to spend for hens and desire to gain a profit of more than ₹12 per week, while he cannot house more than 20 hens. Formulate this problem as a L.P.P to determine the number of each kind of hens the person should buy.

3. A company has undertaken a contract to supply a customer with at least 260 units in total of two products X and Y during the next month. At least 50% of the total output must be units of X. The products are each made by 2 grades of labour as follows:

	'X'	'Y'
Grade A labour	4 hours	6 hours
Grade B labour	4 hours	2 hours

Although additional labour can be made available at short notice, the company wishes to make use of 1200 hours of Grade A labour and 800 hours of Grade B labour which has already been assigned to working on the contract next month. The total variable cost per unit is ₹120 for X and ₹100 per Y. The company wishes to minimize expenditure on the contract next month. How much of X and Y should be supplied in order to meet the terms of the contract? Formulate linear programming model for this problem.

4. A seller buys some tables and chairs. He has ₹5,000 to invest and a space to store at most 60 pieces. A table costs him ₹250 and a chair ₹50. He can sell a table at a profit of ₹50 and a chair at a profit of ₹15. Assuming that he can sell all the pieces that he buys, prepare a mathematical formulation of this linear programming problem to determine the number of pieces of each type to gain maximum profit.

5. A toy company manufactures two types of toys A and B, say. Each toy of type B takes twice as long to produce as one of type A. Company can produce a maximum of 2,000 toys per day if it produces only A type toys. The supply of plastic is sufficient to produce 1,500 toys per day of both type. Type B toy requires a dress of which there only 600 per day available. If the company makes a profit of ₹3.00 and

₹5.00 per toy respectively on toy of type A and B, then how many of each of toy should be produced per day in order to maximize the profit? Formulate this problem as linear programming problem.

6. A man makes two types of furniture, chairs and tables. Profits are ₹20 per chair and ₹30 per table. Both products are processed on three machines M_1, M_2, M_3. The time required for each product in hours and total time available in hours for each machine are as follows:

Machine	Time required to produce a chair (hour)	Time required to produce a table (hour)	Available time (hour)
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

Formulate this problem as a linear programming problem to maximize the profit.

7. A dairy has two plants each of which produces and supplies two products : milk and butter. Each plant can work upto 16 hours a day. In plant-I, it takes 3 hours to prepare from powder and pack 1,000 litres of milk and 1 hour to prepare and pack 100 kg of butter. In plant-II, it takes 2 hours to prepare and pack 1,000 litres of milk and 1.5 hours to prepare and pack 100 kg of butter. In plant-I it costs ₹15,000 to prepare and pack 100 litres of milk and ₹28,000 to prepare and pack a 100 kg of butter, whereas these costs are ₹18,000 and ₹26,000 respectively in plant-II. The dairy is obliged to produce daily at least 10,000 litres of milk and 800 kg of butter.

Formulate this as a L.P.P. to find out as to how should the company organise its production so that the required amount of the products be obtained at minimum cost.

8. A furniture dealer deals in two items, *viz.* tables and chairs. He has ₹10,000 to invest and a space to store at most 60 pieces (including both tables and chairs) A table cost to him ₹500 and a chair ₹100. He can sell all the items that he buys, earning a profit of ₹50 for each table and ₹15 for each chair. Formulate this problem as a L.P.P. so that he maximizes the profit.

9. A dietician decides a certain minimum intake of vitamins A, B and C for a family. The minimum daily needs of the vitamins A, B and C are 30, 20, 16 units respectively. For the supply of these, the dietician depends on two types of foods X and Y. The first one gives 7, 5, 2 units per gram of vitamins A, B and C respectively. The second one gives 2, 4, 8 units per gram of these vitamins respectively. The first food costs ₹2 per gram and the second ₹1 per gram.

How many grams of each food stuff should the family buy everyday to keep the food expense at a minimum? Formulate a linear programming problem for this problem.

10. A furniture firm manufactures chairs and tables, each requiring the use of three machines *A*, *B* and *C*. Production of one chair requires 2 hours on machine *A*, 1 hour on machine *B* and 1 hour on machine *C*. Each table requires 1 hour each on machines *A* and *B* and 3 hours on machine *C*. The profit realized by selling one chair is ₹ 30 while for a table the figure is ₹ 60. The total time available per week on machine *A* is 70 hours, on machine *B* is 40 hours and on machine *C* is 90 hours. How many chairs and tables should be made per week so as to maximize the profit? Formulate a mathematical model for the problem.

11. A diet is to contain at least 4000 units of carbohydrates, 500 units of fat and 300 units of protein. Two foods *A* and *B* are available. Food *A* costs ₹ 2 per unit and food *B* costs ₹ 4 per unit. A unit of food *A* contains 10 units of carbohydrates, 20 units of fat and 15 units of protein. A unit of food *B* contains 25 units of carbohydrates, 10 units of fat and 20 units of protein. Formulate the problem as a L.P.P. so as to find the minimized cost for a diet consists of a mixture of these two foods and also meet the minimum nutrition requirements.

12. A company manufactures two kinds of leather purses, *A* and *B*. A is a high quality purse and *B* is lower quality. The sales of each of these purses *A* and *B* earn profit of ₹ 4 and ₹ 3 respectively. Each purse of type *A* requires twice as much time as a purse of type *B*, and if all purses are of type *B*, the company could make 1000 purses per day. The supply of leather is sufficient for only 800 purses per day (both *A* and *B* combined). Purse *A* requires a fancy buckle and only 400 buckle per day are available. There are only 700 buckles available for purse *B*. What should be the daily production of each type of purse to get the maximum profit? Formulate the problem as a L.P.P.

[B.C.A. (Meerut) 2006]

13. A city hospital has the following minimal daily requirements for nurses:

Period	Clock time (24 hr day)	Minimum number of nurses required
1	6 A. M. — 10 A. M.	2
2	10 A. M. — 2 P. M.	7
3	2 P. M. — 6 P. M.	15
4	6 P. M. — 10 P. M.	8
5	10 P. M. — 2 A. M.	20
6	2 A. M. — 6 A. M.	6

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimum number of nurses available for each period. Formulate this problem as a L.P.P. by setting up appropriate constraints and objective function.

14. Write note on the use of linear programming problem in the business.

[B.C.A. (Rohilkhand) 2002, 2004, 2007]

Answers

1. Maximize $z = 2x_1 + 3x_2$

such that $x_1 + x_2 \le 400$

$2x_1 + x_2 \le 600$

and $x_1, x_2 \ge 0$

2. Maximize $z = -0.2x + y$

subject to $4x + 10y \le 160$

$x + y \le 20$

$-0.2x + y \ge 12$

and $x \ge 0,\ y \ge 0$

3. Minimize $z = 120x + 100y$

subject to $4x + 6y \ge 1{,}200$

$4x + 2y \ge 800$

$x + y \ge 260$

and $x, y \ge 0$

4. Maximize $z = 50x_1 + 15x_2$

such that $250x_1 + 50x_2 \le 5{,}000$

$x_1 + x_2 \le 60$

and $x_1, x_2 \ge 0$

5. Maximize $z = 3x_1 + 5x_2$

subject to $x_1 + 2x_2 \le 2000$

$x_1 + x_2 \le 1500$

$x_2 \le 600$

and $x_1 \ge 0,\ x_2 \ge 0$

6. Maximize $z = 20x + 30y$

subject to $3x + 3y \le 36$

$5x + 2y \le 50$

$2x + 6y \le 60$

$x, y \ge 0$

7. Plant-I $\begin{cases} x_{11} = \text{Number of times } 1{,}000 \text{ litre of milk is prepared} \\ x_{12} = \text{Number of times } 100 \text{ kg of butter is prepared} \end{cases}$

Plant-II $\begin{cases} x_{21} = \text{Number of times } 1{,}000 \text{ litre of milk is prepared} \\ x_{22} = \text{Number of times } 100 \text{ kg of butter is prepared} \end{cases}$

Minimize $z = 15000x_{11} + 28000x_{12} + 18000x_{21} + 26000x_{22}$

subject to $3x_{11} + x_{12} \le 16$

$2x_{21} + 5x_{22} \le 16$

$x_{11} + x_{21} \ge 10$

$x_{21} + x_{22} \ge 8$

and $x_{11}, x_{12}, x_{21}, x_{22} \le 0$

8. Maximize $z = 50x + 15y$,

subject to the constraints $5x + y \le 100$

$x + y \le 60$

and the non-negative restrictions $x \ge 0,\ y \ge 0$.

9. Minimize $z = 2x + y$

subject to the constraints $7x + 2y \ge 30$

$5x + 4y \ge 20$

$2x + 8y \ge 16$

and the non-negative restrictions $x \ge 0,\ y \ge 0$.

10.	Maximize $z = 30x + 60y$ subject to the constraints $2x + y \le 70$ $x + y \le 40$ $x + 3y \le 90$ and the non-negative restrictions $x \ge 0, \; y \ge 0.$
11.	Minimize $z = 2x + 4y$ subject to the constraints $10x + 25y \ge 4000$ $20x + 10y \ge 500$ $15x + 20y \ge 300$ and the non-negative restrictions $x, y \ge 0$
12.	Maximize $z = 4x + 3y$ subject to the constraints $2x + y \le 1000$ $x + y \le 800$ $x \le 400$ $y \le 700$ and the non-negative restrictions $x \ge 0, \; y \ge 0.$
13.	Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ subject to the constraints $x_1 + x_2 \ge 7, \; x_2 + x_3 \ge 15$ $x_3 + x_4 \ge 8, \; x_4 + x_5 \ge 20$ $x_5 + x_6 \ge 6, \; x_6 + x_1 \ge 2$ with non-negativity restrictions : $x_1, x_2, x_3, x_4, x_5, x_6 \ge 0.$

5.2 Solution of a Linear Programming Problem

To obtain a solution of a L.P.P. means to determine the values of the different decision variables, *viz*. x_1, x_2, x_3 ... depicted in its model.

Definition

"A set of values of decision variables which satisfies the linear constraints of a L.P.P. is known as its solution".

"A set of values of decision variables which satisfies the linear constraints and the non-negativity conditions of a L.P.P. is called its ***Feasible Solution (F.S.)****".* **[B.C.A. (Meerut) 2009,2011]**

Or

"A solution of a L.P.P. is called feasible solution if it satisfies the non-negativity conditions. A solution which does not satisfy the non-negativity condition of L.P.P. is called a ***non-feasible solution****".*

"Any feasible solution which optimizes (minimize or maximize) the objective function of a L.P.P. is called its optimum solution". **[B.C.A. (Meerut) 2009]**

*"If a feasible solution minimize the objective function it is called minimum feasible solution. If the feasible solution maximizes the objective function it is called **maximum feasible solution**"*.

*"If the value of objective function z can be increased or decreased indefinitely such solutions are called **unbounded solutions**"*.

> **NOTE:**
>
> In solving a linear programming problem the solutions may be non-feasible, multiple feasible and unbounded feasible.

5.2.1 Methods of Solution of a Linear Programming Problem

There are two methods of solving a linear programming problem, *viz.,* (1) Graphic method and (2) Simplex method.

5.2.1.1 Graphic Method

A linear programming problem which involves only two decision (or decisive) variables, *viz*., x_1 and x_2 can be easily solved by graphic method using a cartesian space (*i.e.* rectangular co-ordinate system). One decision variable is taken along the x-axis and another along the y-axis of the co-ordinate system. Due to non-negativity conditions the graph shows only zero and positive values of the decision variables in the first quadrant only.

1. **Procedure of Graphic Method:** To solve a L.P.P. by graphic method the following steps are to be taken up one after another:

Step 1: Formulate the Appropriate L.P.P.: (If data is given textual and/or tabular form).

Step 2: Convert the Linear Constraints: (If they are given in the form of inequalities) and the non-negativity conditions into equations corresponding to each equation draw to points arbitrary.

The values of each of the decision variables under each equation may be obtained by assuming the other variable to be zero. The two points may be obtained as the intercept of the equation with x-axis and y-axis.

Step 3: Draw the Graph: Consider a two-dimensional co-ordinate system by taking one decision variable along x-axis another along y-axis with a suitable scale. Draw the lines corresponding to each equation by joining the points obtained in step 2.

Step 4: Identify the Region in which the points satisfy the linear constraints and negativity conditions by making an arrow (→) on the corresponding line drawn on the graph.

(i) Let us consider a linear constraint as

$$2x + 5y \geq 40$$

The straight line corresponding to it, is

$$2x + 5y = 40$$

or $$\frac{x}{20} + \frac{y}{8} = 1$$

Put $x = 0$, then $y = 8$, so one point is (0, 8). Again put $y = 0$, then $x = 20$, so another point is (20, 0).

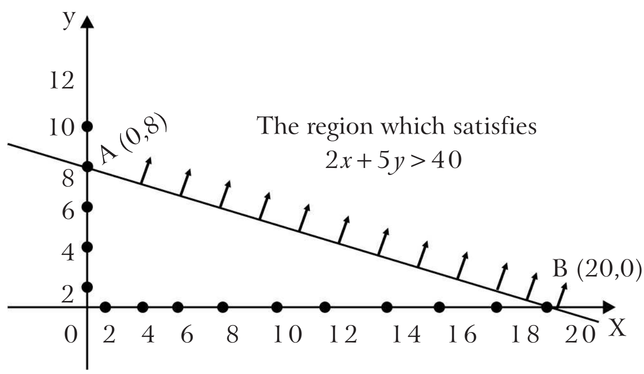


Fig. 5.1

Plot these two points on the graph with a suitable scale and draw a line AB, joining the points $A(0, 8)$ and B (20, 0). To make an arrow indicating the region corresponding to the inequality $2x + 5y > 40$, put $x = 0$, $y = 0$, which gives $0 > 40$, which is not true. This means the origin 0 (0, 0) does not satisfy this inequality, that is origin is not in the region represented by $2x + 5y > 40$. Thus, the region corresponding to the inequality $2x + 5y > 40$ is opposite the origin.

The region corresponding to the inequality $2x + 5y \geq 40$ includes the lines $2x + 5y = 40$ also.

(ii) Let us consider the linear constraint

$$4x + 3y \leq 24$$

The equation corresponding to this inequality is

$$4x + 3y = 24$$

or $$\frac{x}{6} + \frac{y}{8} = 1$$

Put $x = 0$, then $y = 8$, so one point is (0, 8)

Again put $y = 0$, then $x = 6$, so one point is (6, 0).

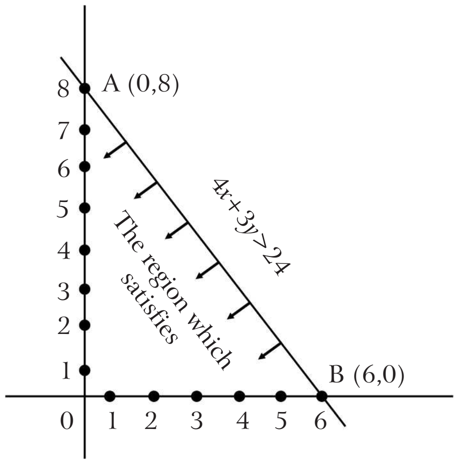


Fig. 5.2

Draw a line AB, joining the points A (0, 8) and B (6, 0). To make an arrow indicating the region corresponding to the inequality $4x + 3y < 24$, put $x = 0$, $y = 0$ which gives $0 < 24$ which is true. This means origin 0 (0, 0) satisfies this inequality. That is origin is, in the region represented by $4x + 3y < 24$. Thus, the region corresponding to the inequality, $4x + 3y > 24$ is towards the origin. The region corresponding to the inequality $4x + 3y \leq 24$ includes the line $4x + 3y = 24$ also.

(iii) Corresponding to the non-negativity conditions $x \geq 0$ and $y \geq 0$, the straight lines are $x = 0$ and $y = 0$ respectively, $x = 0$ represents y axis and $y = 0$ represents x-axis. The values of x and y are only in quadrant first.

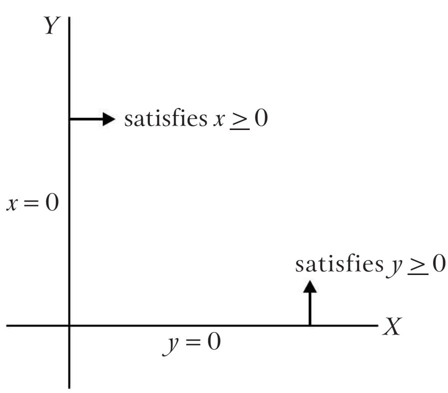


Fig. 5.3

Step 5: Identify the Feasible Region: The area containing all the feasible solutions to the L.P.P. is called a **feasible region** in a graph. In other words the area bounded by the lines drawn in step 3 is called a feasible region. The points in the feasible region satisfy all the linear constraints and the non-negatively conditions the feasible region is generally, shaded by drawing lines or otherwise.

Step 6: Find the Optimum Solution in the Feasible Region: For this make use of the theorem, "*At least one extreme point (corner point) of the feasible region will give the optimum (maximum or minimum) value of the objective function*".

In fact, optimum solution to an L.P.P. always at least on the two vertices of the feasible region (one for maximization and another for minimization).

It is possible for the objective function value of an L.P.P. to be the same at two distinct extreme points.

To determine optimum point (or corner point or extreme point) there are two methods:

(i) Locate the corner points on the graph or find them by solving the concerned linear equations taken two at a time. Calculate the value of the objective function at each corner point. The corner at which the objective function is optimum (minimum or maximum), will give the optimum solution of the L.P.P.

(ii) Draw the line $z = 0$ (the objective function line). Continue drawing lines parallel to it (*i.e.* draw the lines on the graph whose slopes are equal to the slope of the objective function line) such lines are often called iso-profit (or iso-cost) lines.

In case of maximization, the drawing of parallel lines will stop farthest from the origin and passes through a corner point of the feasible region. Find the co-ordinate of the point (farthest from or nearest to the origin), put these values in the objective function and obtain the optimum value of the objective function (if required).

Remark 1: When the feasible region is unbounded then solution to the L.P.P. is unbounded solution.

Remark 2: When there is no common region which satisfies all the constraints then, there is no solution to the L.P.P.

Remark 3: If there is more than one solution (*i.e.,* there are more than one pairs of x and y for which the objective function is optimum) then at least two must correspond to the vertices of the feasible region.

2. Indication of Common Region/Feasible Region

Example 13: *Draw the graph of the following linear inequalities and indicate the common region:*

$$4x + 5y \le 20$$

$$2 \le x \le 4$$

and $$y \ge 1.$$

[B.C.A. (Meerut) 2001]

Solution: Draw x-axis and y-axis and take a suitable scale (suppose 1 cm = 1 unit)

(i) For $4x + 5y \le 20$, the straight line is

$$4x + 5y = 20$$

$$\frac{x}{5} + \frac{y}{4} = 1$$

Hence, join the points (5, 0) and (0, 4) to draw this line. The region corresponding to the inequality is towards the origin.

(ii) For $x \le 4$, the straight line is $x = 4$, which is parallel to y-axis at a distance of 4 units from origin on the positive x-axis.

The region corresponding to the inequality $x \le 4$ is towards the origin.

(iii) For $x \ge 2$, the straight line is $x = 2$, which is parallel to y-axis at a distance of 2 units from origin on the positive x-axis.

The region corresponding to the inequality $x \ge 2$ is opposite the origin.

(iv) For $y \ge 1$, the straight line is $y = 1$, which is parallel to x-axis at distance of 1 unit above the origin on the positive y-axis.

The region corresponding to the inequality $y \ge 1$ is opposite the origin. The required graph with shaded region as common region (which is a bound region) is as follows:

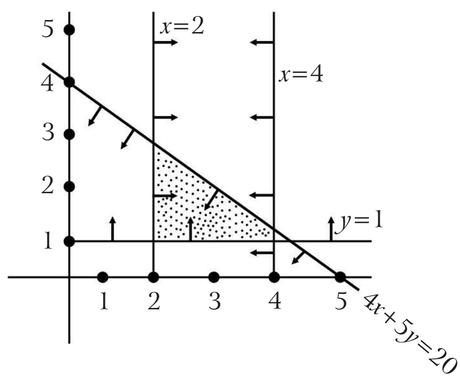


Fig. 5.4

Example 14: *Indicate the common region which satisfied.*

$$x_1 - 2x_2 \le 1$$

$$x_1 + 2x_2 \ge 3$$

and $$x_1 \ge 0, x_2 \ge 0$$

Solution: Take x_1 along x-axis and x_2 along y-axis with a suitable scale on a graph paper.

Consider the line

$$x_1 - 2x_2 = 1$$

Put $x_1 = 0$, then $x_2 = -\frac{1}{2}$, Put $x_2 = 0$, then $x_1 = 1$

Plot the points $\left(0, -\frac{1}{2}\right)$ and (1, 0) and join them to draw the line on the graph.

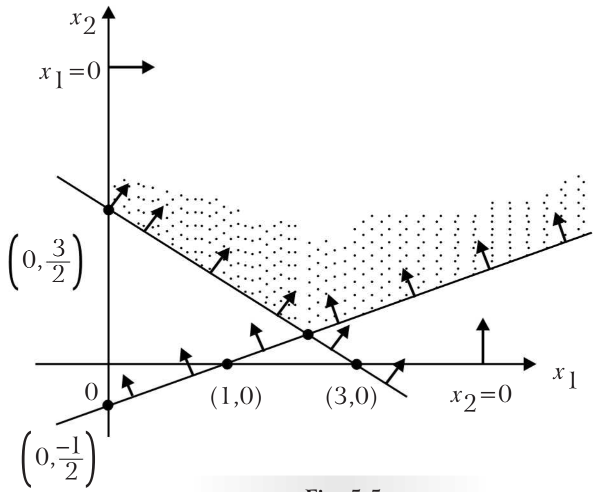


Fig. 5.5

Put $x_1 = 0, x_2 = 0$ in $x_1 - 2x_2 \le 1 \Rightarrow 0 - 0 \le 1$ or $0 \le 1$

which is true, hence origin is in the region, make the arrow accordingly.

Again consider the line $x_1 + 2x_2 = 3$

Put $x_1 = 0$, then $x_2 = \frac{3}{2}$, Put $x_2 = 0$, then $x_1 = 3$

Plot the point $\left(0, \frac{3}{2}\right)$ and (3, 0) and join them to draw the line on the graph.

Put $x_1 = 0, x_2 = 0$ in $x_1 + 2x_2 \ge 3 \Rightarrow 0 + 0 \ge 3$ or $0 \ge 3$

which is not true, hence origin is not in the region. Make the arrow accordingly.

The required graph is as follows:

Here the common region (**shaded area**) is unbounded.

Example 15: *Indicate the region satisfied by the inequalities.*

$$-2x_1 + 2x_2 \le 9$$

$$3x_1 - 2x_2 \le -20$$

and $$x_1 \ge 0, x_2 \ge 0$$

Solution: Take x_1 along x-axis and x_2 along y-axis with a suitable scale on a graph paper $x_1 = 0$ is the y-axis and $x_2 = 0$ is the x-axis.

(i) Consider the line $-2x_1 + 3x_2 = 9$

$$\text{Put } x_1 = 0, \text{ then } x_2 = \frac{9}{3} = 3$$

$$\text{Put } x_2 = 0, \text{ then } x_1 = -\frac{9}{2} = -4\frac{1}{2}$$

Plot the points (0, 3) and $\left(-4\frac{1}{2}, 0\right)$ and join to draw the line on the graph.

Put $x_1 = 0, x_2 = 0$ in $-2x_1 + 3x_2 \le 9 \Rightarrow 0 \le 9$ which is true, origin is in the region. Make the arrow accordingly.

(ii) Consider the $3x_1 - 2x_2 = -20$

$$\text{Put } x_1 = 0, \text{ then } x_2 = \frac{-20}{-2} = 10$$

$$\text{Put } x_2 = 1, \text{ then } x_1 = \frac{-20+2}{3} = -6$$

Plots the points (0, 10) and (–6, 1) and join them to draw the line on the graph. Put $x_1 = 0, x_2 = 0$ in $3x_1 - 2x_2 \le -20 \Rightarrow 0 - 0 \le -20$ or $0 \le -20$ which is not true, hence the origin is not in the region. Make the arrow accordingly.

The required graph (**with no common region**) is as follows:

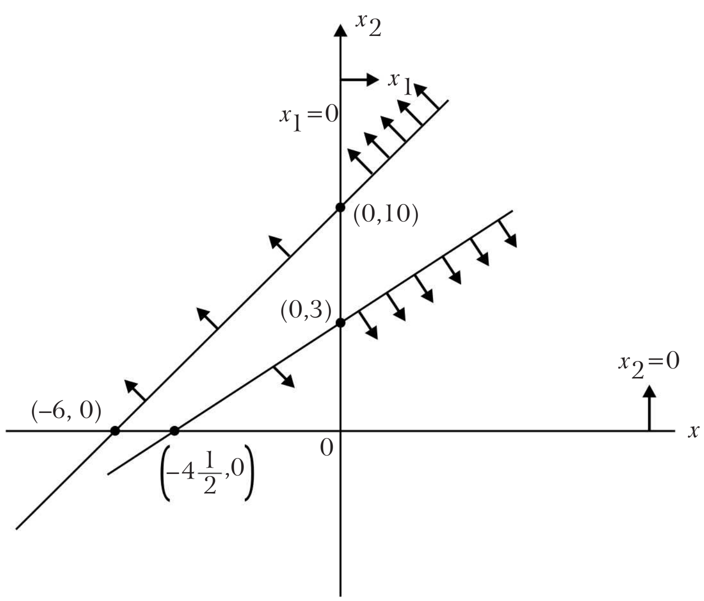


Fig. 5.6

Example 16: *Solve the following L.P. problem using graphical method.*

$$\text{Minimize } z = 2x_1 + 3x_2$$

$$\text{subject to } x_1 + x_2 \geq 6$$

$$7x_1 + x_2 \geq 14$$

and $x_1, x_2 \geq 0$ **[B.C.A. (Meerut) 2007,2010; B.B.A. (Meerut) 2004, 2006]**

Solution: Scale x-axis and y-axis : 1 cm = 1

(i) Obtain two points to draw the equation $x_1 + x_2 = 6$ on the graph corresponding to the inequality $x_1 + x_2 \geq 6$

If $x_1 = 0$ then $x_2 = 6$ point (0, 6)

If $x_2 = 0$ then $x_1 = 6$ point (6, 0)

Line l_1 represents $x_1 + x_2 = 6$ on the graph. To mark the arrow put $x_1 = 0, x_2 = 0$ in the inequality $x_1 + x_2 \geq 6$ which gives $0 > 6$ which is not true. This means origin (0, 0) does not satisfies the inequality $x_1 + x_2 > 6$. So make the arrow accordingly.

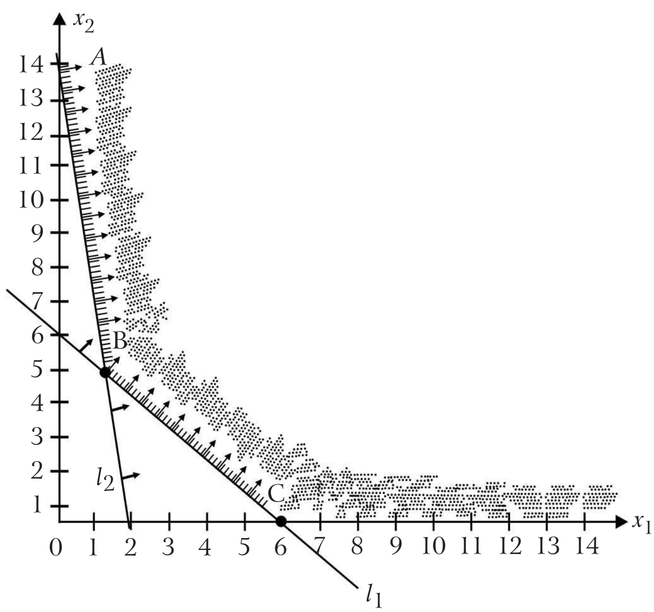


Fig. 5.7

(ii) The equation corresponding to the inequality

$$7x_1 + x_2 \geq 14 \text{ is } 7x_1 + x_2 = 14$$

on the graph corresponding to the inequality $7x_1 + x_2 \geq 14$ if $x_1 = 0$ then $x_2 = 14$ point (0, 14)

If $x_2 = 0$ then $x_1 = 2$ point (2, 0)

Line l_2 represents $7x_1 + x_2 = 14$ on the graph.

To mark the arrow put $x_1 = 0, x_2 = 0$ in the inequality $7x_1 + x_2 \geq 14 \Rightarrow 0 \geq 14$ which is not true. This means origin (0, 0) does not satisfies the inequality

$$7x_1 + x_2 \geq 14$$

So make the arrow accordingly.

Thus, the area bounded by these lines corresponding to the two inequalities is *ABC* which is shaded. This shaded area is the feasible region. The co-ordinate of corner point *A* are (0, 14). The co-ordinate of corner point *B* are (4/3, 14/3).

(Intersection of the lines $x_1 + x_2 = 6, 7x_1 + x_2 = 14$)

The co-ordinate of corner point *C* are (6, 0)

The values of objective function at the corner points of the feasible region are:

Corner points	Co-ordinates		$z = 2x_1 + 3x_2$
A	0	14	$2 \times 0 + 3 \times 14 = 42$
B	$\frac{4}{3}$	$\frac{14}{3}$	$2 \times \frac{4}{3} + 3 \times \frac{14}{3} = \frac{8}{3} + \frac{42}{3} = \frac{50}{3}$
C	6	0	$2 \times 6 + 3 \times 0 = 12$

The minimum value of z is 12 at the corner point $C\ (6, 0)$. The optimal solution to the given L.P.P. is $x_1 = 6,\ x_2 = 0,\ \text{Min}\ (z) = 12$.

Example 17: *Solve the following L.P.P. using graphical method.*

$$Min\ (z) = 3x + 5y$$

subject to constraints

$$-2x + y \leq 4$$
$$x + y \geq 3$$
$$x - 2y \leq 2$$
$$x, y \geq 0$$

[B.C.A. (Meerut) 2006; B.B.A. (Meerut) 2005]

Solution: Scale x-axis and y-axis. 1 cm = 1 unit

(i) Obtain two points to draw the equation

$$-2x + y = 4$$

on the graph corresponding to the inequality

$$-2x + y \leq 4$$

If $x = 0$ then $y = 4$ point (0, 4)

If $y = 0$ then $x = -2$ point (–2, 0)

Line l_1 represents $2x + y = 4$ on the graph. To mark the arrow, put $x = 0$, $y = 0$ in the inequality $-2x + y \leq 4$ which gives $0 \leq 4$ which is true. This means origin (0, 0) satisfies the inequality

$$-2x + y \leq 4$$

So make the arrow accordingly.

(ii) To obtain two points to draw the equation

$$x + y = 3$$

on the graph corresponding the inequality

$$x + y \geq 3$$

If $x = 0$ then $y = 3$ point (0, 3)

If $y = 0$ then $x = 3$ point (3, 0)

Line l_2 represents $x + y = 3$ on the graph. To mark the arrow.

(iii) To obtain two points to draw the equation

$$x - 2y = 2$$

on the graph corresponding to the inequality

$$x - 2y \leq 2$$

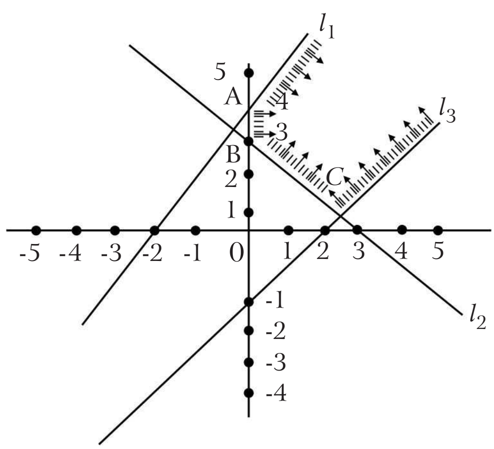


Fig. 5.8

If $x = 0$ then $y = -1$ point $(0, -1)$

If $y = 0$ then $x = 2$ point (2, 0)

Line l_3 represents $x - 2y = 2$ on the graph. To mark the arrow.

Thus, the area bounded by these three lines is ABC.

The co-ordinate of corner point A are (0, 4)

The co-ordinate of corner point B are (0, 3)

The co-ordinate of corner point C are $\left(\frac{8}{3}, \frac{1}{3}\right)$

(Intersection of the lines $x + y = 3,\ x - 2y = 2$)

The values of objective function at the corner points of feasible region are:

	Co-ordinates		
Corner points	x	y	$z = 3x + 5y$
A	0	4	$3 \times 0 + 5 \times 4 = 20$
B	0	3	$3 \times 0 + 5 \times 3 = 15$
C	$\frac{8}{3}$	$\frac{1}{3}$	$3 \times \frac{8}{3} + 5 \times \frac{1}{3} = 8 + \frac{5}{3} = \frac{29}{3}$

The minimum value of $z = \frac{29}{3}$ at the corner point $C\left(\frac{8}{3}, \frac{1}{3}\right)$. The optimal solution of the given L.P.P. is $x = \frac{8}{3}$, $y = \frac{1}{3}$ Min. (z) $= \frac{29}{3}$.

Example 18: *Solve by graphical method, the linear programming problem:*

$$\textit{Minimize } Z = 20x_1 + 10x_2$$

subject to the constraints, $x_1 + 2x_2 \le 40$

$$3x_1 + x_2 \ge 30$$

$$4x_1 + 3x_2 \ge 60$$

and the non-negative restrictions $x_1, x_2 \ge 0$. **[B.C.A. (Delhi) 2004,2007]**

Solution: **Step 1:** Considering the constraints as equations, we get the following equations

$$x_1 + 2x_2 = 40, 3x_1 + x_2 = 30, 4x_1 + 3x_2 = 60$$

Step 2: Draw lines corresponding to each equation.

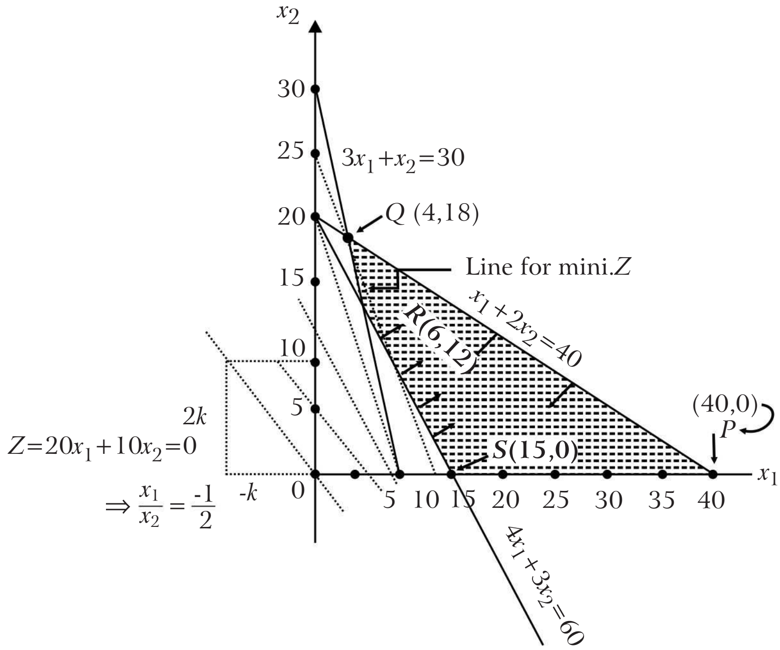


Fig. 5.9

Step 3: The shaded region PQRSP is the permissible region of the problem.

Step 4: By Corner-Point Method: Solving simultaneously the equations of the corresponding intersecting lines, the co-ordinates of the vertices of the convex polygon are

$$P\ (40,0), Q\ (4,18), R\ (6,12) \text{ and } S\ (15,0).$$

Now the values of the objective function Z at these vertices (corner point) are as given in the table below:

Point (x, y)	**Value of the objective function $Z = 20x_1 + 10x_2$**
P (40,0)	$Z = 20 \times 40 + 10 \times 0 = 800$
Q (4,18)	$Z = 20 \times 4 + 10 \times 18 = 260$
R (6,12)	$Z = 20 \times 6 + 10 \times 12 = 240$ (Mini)
S (15,0)	$Z = 20 \times 15 + 0 = 300$

Obviously, Z is minimum at R (6,12).

Hence, the optimal solution of the given L.P.P. is

$$x_1 = 6, x_2 = 12$$

and minimum $Z = 240$.

Step 5: By Iso-Profit Method: Here we draw the line through the origin corresponding to $Z = 0$, which is parallel to iso-profit line.

$$Z = 0 \Rightarrow 20x_1 + 10x_2 = 0 \Rightarrow \frac{x_1}{x_2} = \frac{(-1)}{2}$$

The dotted line through the origin is shown in the figure. Drawing parallel lines away from the origin (note) O, we see that the nearest line (since it is minimization problem) in the permissible region passes through the vertex R (6,12).

Hence, the optimal solution is

$$x_1 = 6, \ x_2 = 12$$

and minimum $Z = 20 \times 6 + 10 \times 12 = 240$

Example 19: *A farm is engaged in breeding hens. In view of the need to ensure certain nutrients (say x_1, x_2, x_3), it is necessary to buy two types of food, say A and B. One unit of food A contains 36 units of x_1, 3 units of x_2 and 20 units of x_3. One unit of food B contains 6 unit of x_1, 12 unit of x_2 and 10 unit of x_3. The minimum daily requirement of x_1, x_2 and x_3 is 108, 36 and 100 units respectively. The cost of food A is ₹20 per unit whereas food B costs ₹40 per unit. Find the minimum food cost so as to meet the minimum daily requirement of nutrients.*

[B.C.A. (Bundelkhand) 2010]

Solution: **Formulation of the Problem as L.P.P.**

Let x units of food A and y units of food B be bought to fulfill the minimum requirement of the nutrients x_1, x_2, x_3 and to minimize the cost.

If Z be the cost in ₹ of the two types of food bought, then $Z = 20x + 40y$.

According to the minimum daily requirements of x_1, x_2, x_3, we have

$$36x + 6y \geq 108, 3x + 12y \geq 36, 20x + 10y \geq 100$$

Also, since the quantity of each type of food bought cannot be negative, therefore $x \geq 0, \ y \geq 0$.

Hence, the mathematical form of the given problem as a L.P.P. is as follows:

$$\text{Minimize } Z = 20x + 40y$$

subject to the constraints $36x + 6y \geq 108, 3x + 12y \geq 36, 20x + 10y \geq 100$

and the non-negative restrictions $x \geq 0, y \geq 0$.

Solution of the Problem: Proceeding stepwise, the permissible region (the set of all points satisfying all the constraints and the non-negative restrictions) consists of the shaded region yLSTPx which is unbounded.

To find the minimum value of Z, we draw the dotted line through the origin

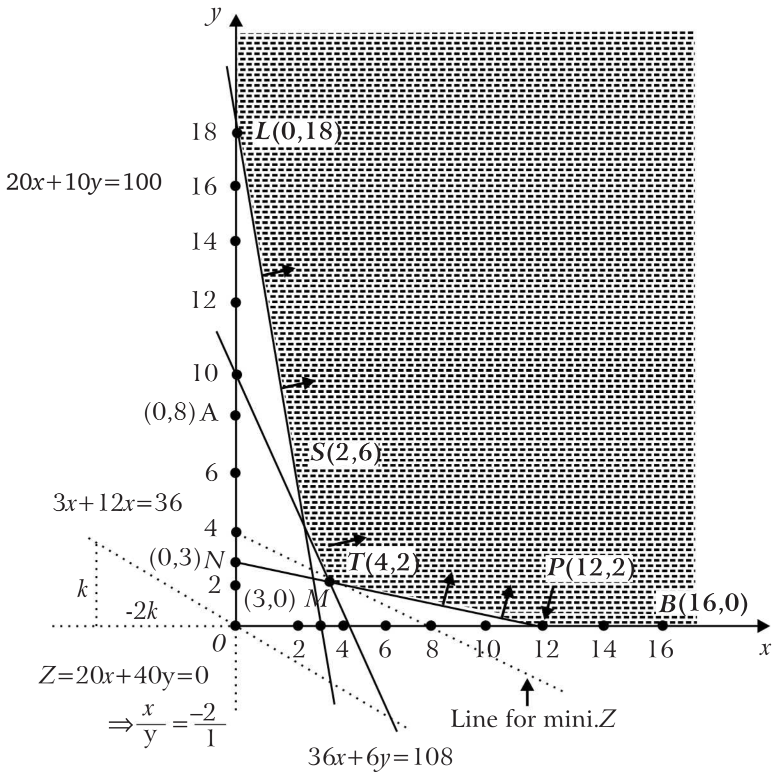


Fig. 5.10

corresponding to $Z = 20x + 40y$ which is parallel to iso-cost line. We now move this line parallel to itself so that it passes through only one point [here the corner point T $(4, 2)$] of the feasible region. This line is an iso-cost line having only one point *viz.* T in the feasible region which will give the minimum value of Z.

Therefore, Z is minimum for $x = 4$ and $y = 2$ and the minimum value of Z is $20 \times 4 + 40 \times 2 =$ ₹160.

Hence, 4 units of food A and 2 units of food B should be bought to fulfil the minimum requirements of x_1, x_2, x_3 at a minimum cost of ₹160.

Example 20: *A car manufacturing company manufactures cars of two models A and B. Model A requires* 150 *man hours for assembling* 50 *man hours for painting and* 10 *man hours for checking and testing. Model B requires* 60 *man hours for assembling* 40 *man hours for painting and* 20 *man hours for checking and testing. There are available* 30 *thousand man hours for assembling* 13 *thousand man hours for painting and* 5 *thousand man hours for checking and testing. Express this using linear inequalities. Draw graphs of these inequalities and then mark the feasible region.*

Solution: Suppose the company manufactures x_1 cars of model A and x_2 cars of model B, then decision variables are x_1 and x_2.

Inequality for man hours for assembling:

$$150x_1 + 60x_2 \leq 30{,}000$$

Inequality for man hours for painting:

$$50x_1 + 40x_2 \leq 13{,}000$$

Inequality for man hours for checking and testing:

$$10x_1 + 20x_2 \leq 5{,}000$$

clearly, $$x_1, x_2 \geq 0$$

To obtain the feasible region we proceed as follows:

(i) Consider the lines $x_1 = 0$ as y axis and $x_2 = 0$ as x-axis and a suitable scale.

(ii) Consider the line

$$150x_1 + 60x_2 = 30{,}000$$

$$\frac{x_1}{200} + \frac{x_2}{500} = 1$$

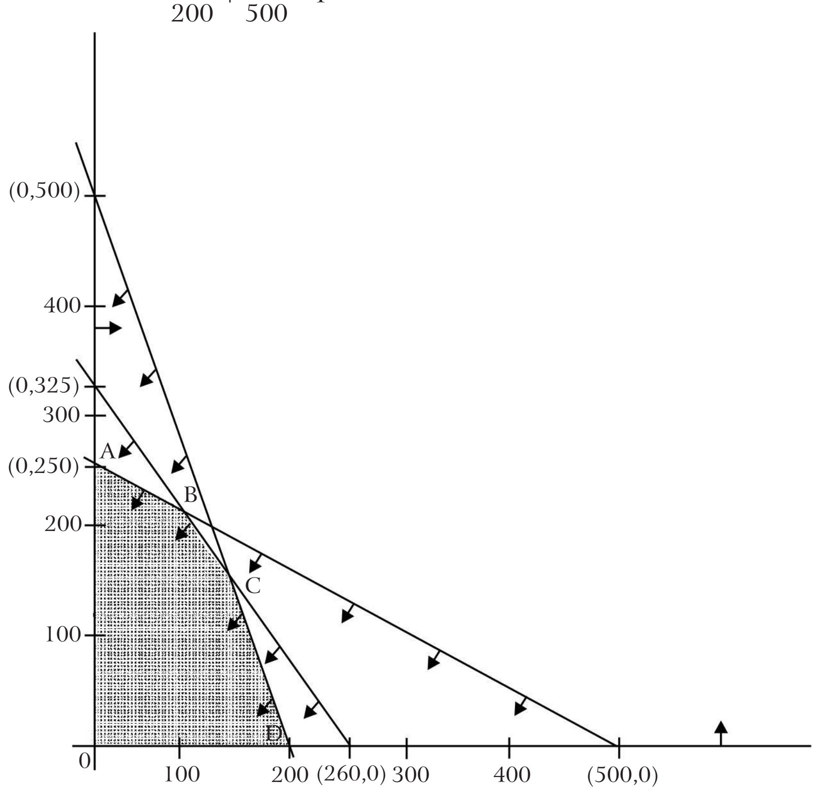


Fig. 5.11

The intercept on the x-axis is 200 and on the y-axis is 500. Draw this line l_1 and make the arrow to indicate the region given by $150x_1 + 6x_2 \leq 30000$.

(iii) Consider the line

$$50x_1 + 40x_2 = 13{,}000$$

$$\frac{x_1}{260} + \frac{x_2}{325} = 1$$

The intercept on the x-axis is 260 and on the y-axis is 325. Draw this line l_2 and make the arrow to indicate the region given by $50x_1 + 40x_2 \leq 13{,}000$.

(iv) Consider the line

$$10x_1 + 20x_2 = 5{,}000$$

$$\Rightarrow \quad \frac{x_1}{500} + \frac{x_2}{250} = 1$$

The intercept on the x-axis is 500 and on the y-axis is 250. Draw the line l_3 and make the arrow to indicate the region given by $10x_1 + 20x_2 \leq 5000$.

The required graph, marked with feasible region OABCDO (Shaded Area) is as shown in figure 5.11.

Example 21: *Draw the graph of inequality* $3x + y \geq 9$. **[B.C.A. (Meerut) 2003]**

Solution: Give the value of x find the value of y in the equation $3x + y = 9$.

x	0	1	2	3
y	9	6	3	0

The bounded region of this line is A, B, C, D.

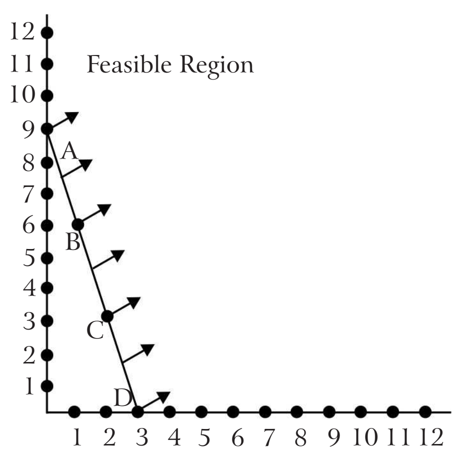


Fig. 5.12

The co-ordinate of $A\ (0, 9), B\ (1, 6), C\ (2, 3), D\ (3, 0)$.

Example 22: *Find two number x_1 and x_2 such that*

$$4x_1 + 2x_2 \geq 80$$
$$2x_1 + 5x_2 \leq 180$$
$$x_1 \geq 0,\ \ x_2 \geq 0$$

and maximize objective function

$$F = 3x_1 + 4x_2$$

Solution: Scale x-axis and y-axis : 5 cm = 1

(i) Obtain two points to draw the line $4x_1 + 2x_2 = 80$ on the given graph corresponding to the inequality

$$4x_1 + 2x_2 \geq 80$$

If $x_1 = 0$ then $x_2 = 40$ point (0, 40)

If $x_2 = 0$ then $x_1 = 20$ point (20, 0)

Line l_1 represents, $4x_1 + 2x_2 = 80$ on the graph. To mark the arrow accordingly.

(ii) Obtain two points to draw the line

$$2x_1 + 5x_2 = 180$$

on the given graph corresponding to the inequality

$$2x_1 + 5x_2 \leq 180$$

If $x_1 = 0$ then $x_2 = \frac{180}{5} = 36$ point (0, 36)

If $x_2 = 0$ then $x_1 = 90$ point (90, 0)

Line l_2 represents, $2x_1 + 5x_2 = 180$ on the graph. To mark the arrow accordingly.

The bounded region by these lines are ABC

The co-ordinate of A (90, 0)

The co-ordinate of B $\left(\frac{5}{2}, 35\right)$

The co-ordinate of C (20, 0)

The values of objective function at the corner points of feasible region are:

Corner points	Co-ordinates		$F = 3x_1 + 4x_2$
	x_1	x_2	
A	90	0	$F = 270$
B	$\frac{5}{2}$	35	$F = \frac{15}{2} + 140 = \frac{295}{2}$
C	20	0	$F = 60$

The maximum value of F is $\frac{295}{2}$ at the corner point $B\left(\frac{5}{2}, 35\right)$. The optimal solution to the given L.P.P. is $x_1 = \frac{5}{2}, x_2 = 35$

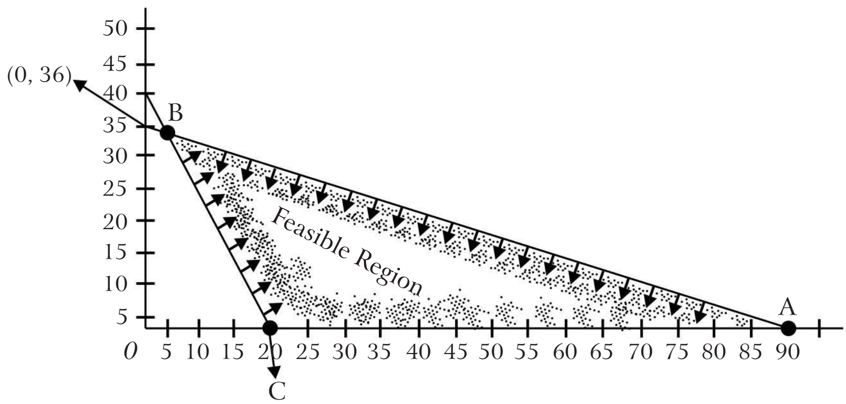


Fig. 5.13

$$\text{Max }(F) = \frac{295}{2}$$

Example 23: *Solve the following L.P.P. graphically.*

$$\textit{Maximize } (z) = 10x + 30y \textit{ such that}$$

$$x + 2y \le 20,\ x + 5y \le 35,\ x + 4y \le 48,\ x,\ y \ge 0$$

[B.B.A. (Meerut) 2002]

Solution: **Step 1:** To convert the inequalities in equations:

The equation corresponding to $x + 2y \le 20$ is $x + 2y = 20$.

The equation corresponding to $x + 5y \le 35$ is $x + 5y = 35$.

The equation corresponding to $x + 4y \le 48$ is $x + 4y = 48$.

The equation corresponding to $x \ge 0$ is $x = 0$.

The equation corresponding to $y \ge 0$ is $y = 0$.

Step 2: To draw straight lines:

Scale : x-axis : 1 cm = 5

y-axis : 1 cm = 2

(i) Equation $x + 2y = 20$

If $x = 0$ then $y = 10$

If $y = 0$ then $x = 20$.

Plot the points M_1 (20, 0) and N_1 (0, 10) where $OM_1 = 20$ and $ON_1 = 10$. Draw a straight line M_1N_1 joining the two points M_1 and N_1 which represents the equations $x + 2y = 20$ and various sets of (x, y). Each points below M_1N_1 will satisfy the inequality $x + 2y < 20$ (and including this line will satisfy the inequality $x + 2y \le 20$) which has been indicated by the arrow.

(ii) To represent the equation $x + 5y = 35$ on the graph,

if $x = 0$ then $y = 7$, point (0, 7)

if $y = 0$, then $x = 35$, point (35, 0)

Plot the points M_2 (35, 0) and N_2 (0, 7), where $OM_2 = 35$ and $ON_2 = 7$. Draw straight line M_2N_2 joining the two points M_2 and N_2 which represents the equation $x + 5y = 35$ and the various sets of (x, y). Each points below M_2N_2 will satisfy the inequality $x + 5y < 35$ (and including this line will satisfy the inequality $x + 5y \le 35$) which has been indicated by the arrow.

(iii) To represent the equation $x + 4y = 48$ on the graph,

if $x = 0$, then $y = 12$

if $y = 0$, then $x = 48$

Plot the points M_3 (48, 0) and N_3 (0, 12), where $OM_3 = 48$ and $ON_3 = 12$.

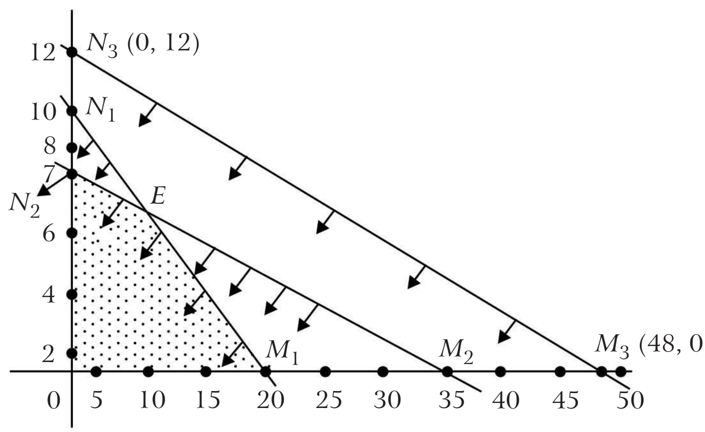


Fig. 5.14

Draw straight line M_3N_3 joining the two points M_3 and N_3 which represents the equation $x + 4y = 48$ and the various sets of (x, y). Each points below M_3N_3 will satisfy the inequality $x + 4y < 48$ (and including this line will satisfy the inequality $x + 4y \le 48$) which has been indicated by the arrow.

It is obvious that the effect of inequality $x + 4y < 48$ is a redundant constraint.

(iv) The area between the line OX (on x-axis) and the line OY (on y-axis) satisfied the non-negativity conditions ($x \ge 0$, $y \ge 0$). The area OM_1EN_2 bounded by the lines corresponding to the inequalities is the region of feasible solutions, where E (10, 5) is the intersection of the lines M_1N_1 and M_2N_2.

Step 3: To find optimum solution. The optimal solution is any corner point of the feasible region OM_1EN_2. The values of objective function at the corner points are as follows:

Corner point	Co-ordinates		$Z = 10x + 30y$
	x	y	
O	0	0	$Z = 10 \times 0 + 30 \times 0 = 0$
M_1	20	0	$Z = 10 \times 20 + 30 \times 0 = 200$
E	10	5	$Z = 10 \times 10 + 30 \times 5 = 250$
N_2	0	7	$Z = 10 \times 0 + 30 \times 7 = 210$

It is clear from the above table that the value of Z is maximum at the corner point (10, 5). Hence, the optimum solution of the linear programming problem is $x = 10$, $y = 5$ and the maximum value of Z is 250.

Example 24: *Solve the following L.P.P. by graphical method:*

$$\text{Minimize } Z = 3x + 2y$$

$$\text{such that } x + y \leq 5$$

$$3x + y \geq 6$$

$$x + 4y \geq 4$$

$$0 \leq x \leq 3$$

and $$0 \leq y \leq 3$$ **[B.B.A. (Meerut) 2005]**

Solution: **Step 1:** Draw two perpendicular lines ox and oy with a suitable scale (x-axis : 1 cm=1, y-axis : 1 cm=1) which represent $y = 0$ and $x = 0$. Since $x \geq 0, y \geq 0$ hence the feasible region will be in the first quadrant.

Step 2: The equation corresponding to $x + y \leq 5$ is $x + y = 5$.

The equation corresponding to $3x + y \geq 6$ is $3x + y = 6$.

The equation corresponding to $x + 4y \geq 4$ is $x + 4y = 4$.

The equation corresponding to $x \leq 3$ is $x = 3$.

The equation corresponding to $y \leq 3$ is $y = 3$.

The line M_1N_1 represents the equation $x + y = 5$ or $\frac{x}{5} + \frac{y}{5} = 1$ which passes through the points M_1 (5, 0) and N_1 (0, 5). The line M_2N_2 represents the equation $3x + y = 6$ or $\frac{x}{2} + \frac{y}{6} = 1$ which passes through the points M_2 (2, 0) and N_2 (0, 6).

The line M_3N_3 represents the equation $x + 4y = 4$ or $\frac{x}{4} + \frac{y}{1} = 1$ which passes through the points M_3 $(4, 0)$ and N_3 $(0, 1)$.

The line M_4N_4 represents the equation $x = 3$.

The line M_5N_5 represents the equation $y = 3$.

Step 3: To find feasible region:

(i) Since the co-ordinates of the origin $O(0, 0)$ satisfy the inequality $x + y < 5$, the feasible region will be in that side of M_1N_1 in which side the origin falls. This is indicated by the arrow on it.

(ii) Since the co-ordinates of the origin $O(0, 0)$ do not satisfy the inequality $3x + y > 6$, the feasible region will be in that side of M_2N_2 in which side the origin does not fall. This is indicated by the arrow on it.

(iii) Since the co-ordinates of the origin $O(0, 0)$ do not satisfy the inequality $x + 4y > 4$ the feasible region will be in that side of M_3N_3 in which side the origin does not fall. This is indicated by the arrow on it.

(iv) Since the co-ordinates of the origin $O(0, 0)$ satisfy the inequality $x < 3$, the feasible region will be in that side of M_4N_4 in which side the origin falls. This is indicated by the arrow on it.

(v) Since the co-ordinates of the origin $O(0, 0)$ satisfy the inequality $y < 3$, the feasible region will be in that side of M_5N_5 in which side the origin falls. This is indicated by the arrow on it.

Thus, the required feasible region (shaded area) is the polygon $ABCDE$.

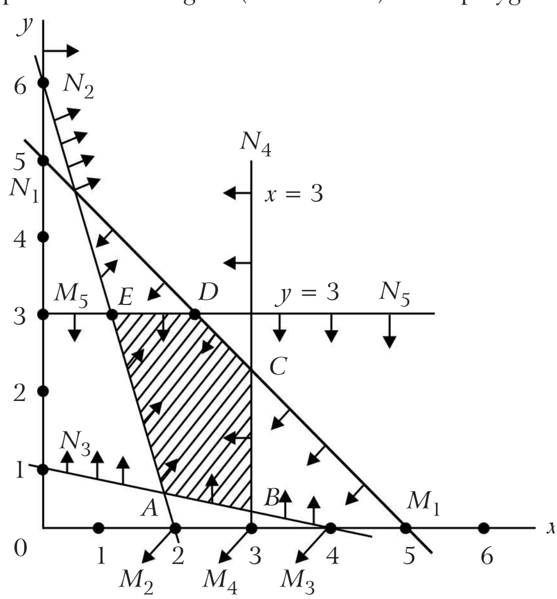


Fig. 5.15

Corner points	Co-ordinates		Objective function	Remark
	x	y	$Z = 3x + 2y$	
A	$\frac{20}{11}$	$\frac{6}{11}$	$Z = 3 \times \frac{20}{11} + 2 \times \frac{6}{11} = \frac{72}{11}$	point of intersection of $3x + y = 6$ and $x + 4y = 4$
B	3	$\frac{1}{4}$	$Z = 3 \times 3 + 2 \times \frac{1}{4} = \frac{19}{2}$	point of intersection of $x = 3$ and $x + 4y = 4$
C	3	2	$Z = 3 \times 3 + 2 \times 2 = 13$	point of intersection of $x = 3$ and $x + y = 5$
D	2	3	$Z = 3 \times 2 + 2 \times 3 = 12$	point of intersection $y = 3$ and $x + y = 5$
E	1	3	$Z = 3 \times 1 + 2 \times 3 = 9$	point of intersection of $y = 3$ and $3x + y = 6$

It is clear from the above table that the minimum value of Z is $\frac{72}{11}$ corresponding to the point $A\left(\frac{20}{11}, \frac{6}{11}\right)$. Hence, the optimal solution of linear programming problem is $x = \frac{20}{11}$, $y = \frac{6}{11}$.

Example 25: *Old hens can be bought at ₹2 each and young ones at ₹5 each. Old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen (young or old) cost ₹1 per week to feed. I have only ₹80 to spend for hens. How many of each kind should I buy to give me a maximum profit, assuming that I can not house more than 20 hens ?*

[B.C.A. (Meerut) 2003,2011]

Solution: **Formulation of the Problem as L.P.P.**

Let, x_1 = Total numbers of old hens

x_2 = Total numbers of young hens.

Since old hens lay 3 eggs per week and the young hens lay 5 eggs per week,

Total number of eggs I have per week $= 3x_1 + 5x_2$

The cost of each egg = 30 paise = ₹0.3

Total income per week = ₹ $0.3\,(3x_1 + 5x_2)$

$= ₹\,(.9x_1 + 1.5x_2)$

Total number of hens $= x_1 + x_2$

Expenditure on feeding = ₹ $1(x_1 + x_2)$

$$= ₹\ (x_1 + x_2)$$

Total profit per week $= [(.9x_1 + 1.5x_2) - (x_1 + x_2)] = -0.1x_1 + 0.5x_2$

Since the cost of old hen is ₹ 2 and the cost of one young hen is ₹ 5 and I have only ₹ 80 to spend for hens

$\therefore$ $$2x_1 + 5x_2 \le 80$$

Since I cannot house more than 20 hens

$\therefore$ $$x_1 + x_2 \le 20$$

But hens cannot be negative

$\therefore$ $$x_1 \ge 0,\ x_2 \ge 0$$

Hence general L.P.P. of given example is

$$\text{Maximize } z = -0.1x_1 + 0.5x_2$$

subject to condition $$2x_1 + 5x_2 \le 80$$

$$x_1 + x_2 \le 20$$

$$x_1 \ge 0,\ x_2 \ge 0$$

Graphical Solution: Scale x-axis and y-axis 4 cm = 1

(i) Obtain two points to draw the equation

$$2x_1 + 5x_2 = 80$$

on the graph corresponding to the inequality

$$2x_1 + 5x_2 \le 80$$

If $x_1 = 0$ then $x_2 = 16$ point (0, 16)

If $x_2 = 0$ then $x_1 = 40$ point (40, 0)

Line l_1 represents $2x_1 + 5x_2 = 80$ on the graph. To mark the arrow accordingly.

(ii) Obtain two points to draw the equation

$$x_1 + x_2 = 20$$

on the graph corresponding to the inequality

$$x_1 + x_2 \le 20$$

If $x_1 = 0$ then $x_2 = 20$ point (0, 20)

If $x_2 = 0$ then $x_1 = 20$ point (20, 0)

Line l_2 represents $x_1 + x_2 = 20$ on the graph to mark the arrow accordingly.

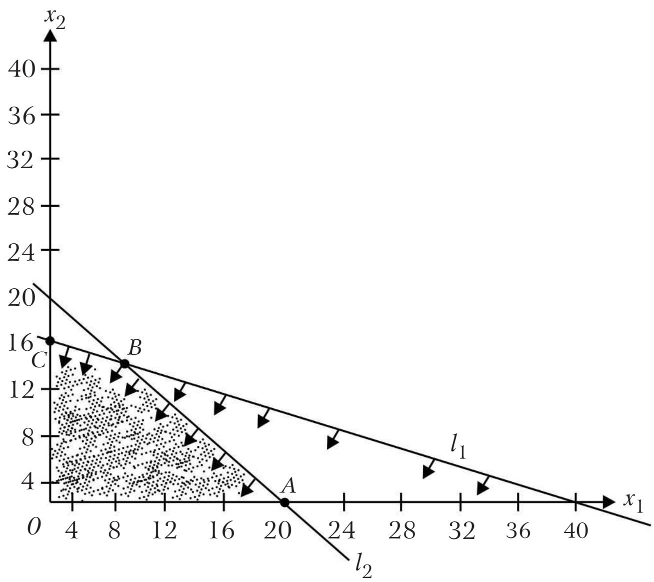


Fig. 5.16

The common feasible region in all lines is $OABC$.

The co-ordinate of corner point are $O\,(0, 0)$

The co-ordinate of corner point A are (20, 0)

The co-ordinate of corner point B are $\left(\dfrac{20}{3}, \dfrac{40}{3}\right)$

(Intersection of the lines $2x_1 + 5x_2 = 80,\ x_1 + x_2 = 20$)

The co-ordinate of corner point C are (0, 16)

The value of objective function at the corner points of the feasible region are:

Corner points	**Co-ordinates**		$Z = -0.1x_1 + 0.5x_2$
	x_1	x_2	
O	0	0	$-0.1\times 0 + .5\times 0 = 0$
A	20	0	$-0.1\times 20 + .5\times 0 = -2$
B	$\dfrac{20}{3}$	$\dfrac{40}{3}$	$-0.1\times\dfrac{20}{3} + .5\times\dfrac{40}{3} = 6$
C	0	16	$-0.1\times 0 + .5\times 16 = 8$

The maximum $Z = 8$ at $x_1 = 0, x_2 = 16$

Example 26: *Solve the following L.P.P. graphically*

$$\text{Maximize } Z = 3x_1 + 5x_2$$

$$\text{such that } x_1 + 2x_2 \le 20$$

$$x_1 + x_2 \le 15$$

$$x_2 \le 6$$

and $$x_1, x_2 \ge 0$$

Solution: Scale x-axis and y-axis : 1 cm = 5

(i) Obtain two points to draw the equation

$$x_1 + 2x_2 = 20$$

on the graph corresponding to the inequality

$$x_1 + 2x_2 \le 20$$

If $x_1 = 0$ then $x_2 = 10$ point (0, 10)

If $x_2 = 0$ then $x_1 = 20$ point (20, 0)

Line l_1 represents $x_1 + 2x_2 = 20$ on the graph. To make the arrow put $x_1 = 0, x_2 = 0$ in the inequality $x_1 + 2x_2 \le 20$ which gives $= 0 + 2 \times 0 \le 20$ *i.e.,* $0 < 20$ which is true. This means origin (0, 0) satisfies the inequality $x_1 + 2x_2 \le 20$. So make the arrow accordingly.

(ii) The equation corresponding to the inequality

$$x_1 + x_2 \le 15 \text{ is } x_1 + x_2 = 15$$

If $x_1 = 0$ then $x_2 = 15$ point (0, 15)

If $x_2 = 0$ then $x_1 = 15$ point (15, 0)

Line l_2 represents $x_1 + x_2 = 15$ on the graph. To make the arrow put $x_1 = 0, x_2 = 0$ in $x_1 + x_2 \le 15 \Rightarrow 0 \le 15$ which is true. This means origin (0, 0) satisfies the inequality $x_1 + x_2 \le 15$. So make the arrow accordingly.

(iii) The equation corresponding to the inequality $x_2 \le 6$ is $x_2 = 6$.

The line l_3 represents this equation on the graph. The arrow on it indicates the region in which all points satisfy the inequality $x_2 \le 6$.

(iv) $x_1 = 0$ is y-axis and $x_2 = 0$ is x-axis. In the first quadrant $x_1 \ge 0, x_2 \ge 0$.

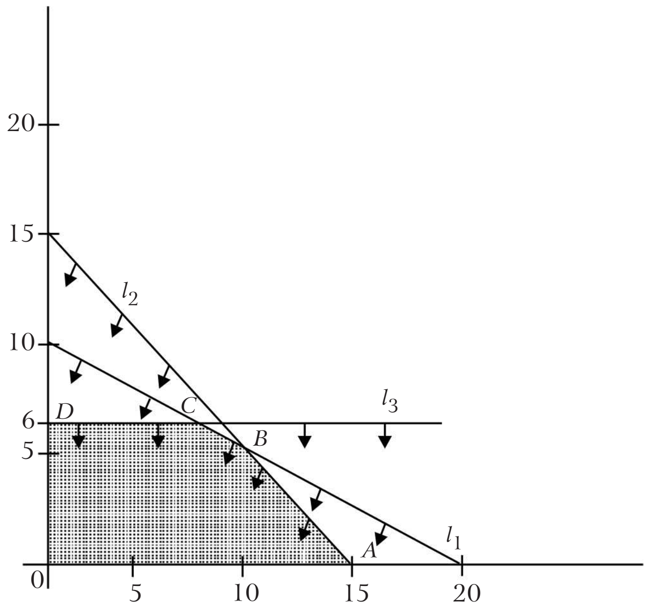


Fig. 5.17

Thus, the area bounded by five lines corresponding to the five inequalities is *OA BCD* which is shaded. This shaded area is the feasible region. The co-ordinates of corner points $O\,(0,0)$, $A\,(15,0)$, $B\,(10,5)$, $C\,(8,6)$, $D\,(0,6)$.

The values of objective function at corner points of the feasible region are:

Corner points	Co-ordinates		$Z = 3x_1 + 5x_2$
	x_1	x_2	
O	0	0	$Z = 3\times 0 + 5\times 0 = 0$
A	15	0	$Z = 3\times 15 + 5\times 0 = 45$
B	10	5	$Z = 3\times 10 + 5\times 5 = 55$
C	8	6	$Z = 3\times 8 + 5\times 6 = 54$
D	0	6	$Z = 3\times 0 + 5\times 6 = 30$

The maximum value of z is 55 at the corner point $B\,(10,5)$. Hence, Z is maximum for $x_1 = 10$ and $x_2 = 5$. This is optimal solution.

Example 27: *A company owns two flour mills, A and B, which have different production capacities for high, medium and low grade flour. This company has entered a contract to supply flour to a firm every week with at least* 12, 8 *and* 24 *quintals of high, medium and low grade respectively. It costs the company* ₹ 1,000 *and* ₹ 800 *per day to run mills A and B respectively. On a day, mill A produces* 6, 2 *and* 4 *quintals of high, medium and low grade flour respectively. Mill B produces* 2, 2 *and* 12 *quintals of high, medium and low grade flour respectively. How many days per week should each mill be operated in order to meet the contract order most economically? Solve graphically to L.P.P.* **[B.C.A. (Meerut) 2006]**

Solution:

Formulation of the Problem as L.P.P.

Let x_1 = Total product of mill A per day

x_2 = Total product of mill B per day

$$\text{Min } (Z) = 1000\, x_1 + 800\, x_2$$

subject to condition
$$6x_1 + 2x_2 \geq 12$$
$$2x_1 + 2x_2 \geq 8$$
$$4x_1 + 12x_2 \geq 24$$
$$x_1, x_2 \geq 0$$

Graphical Solution: Scale x-axis and y-axis, 1 cm = 1

(i) Obtain two points to draw the equation

$$6x_1 + 2x_2 = 12$$

on the graph corresponding to the inequality

$$6x_1 + 2x_2 \geq 12$$

If $x_1 = 0$ then $x_2 = 6$ point (0, 6)

If $x_2 = 0$ then $x_1 = 2$ point (2, 0)

Line l_1 represent $6x_1 + 2x_2 = 12$ on the graph. To mark the arrow accordingly.

(ii) Obtain two points to draw the equation

$$2x_1 + 2x_2 = 8$$

on the graph corresponding to the inequality

$$2x_1 + 2x_2 \geq 8$$

If $x_1 = 0$ then $x_2 = 4$ point (0, 4)

If $x_2 = 0$ then $x_1 = 4$ point (4, 0)

Line l_2 represent $2x_1 + 2x_2 = 8$ on the graph. To mark the arrow accordingly.

(iii) Obtain two points to draw the equation

$$4x_1 + 12x_2 = 24$$

on the graph corresponding to the inequality

$$4x_1 + 12x_2 \geq 24$$

If $x_1 = 0$ then $x_2 = 2$ point (0, 2)

If $x_2 = 0$ then $x_1 = 6$ point (6, 0)

Line l_3 represent $4x_1 + 12x_2 = 24$ on the graph. To mark the arrow accordingly.

The common feasible region is *ABCD*

The co-ordinate of corner point *A* are (6, 0)

The co-ordinate of corner point *B* are (3, 1)

The co-ordinate of corner point *C* are (1, 3)

The co-ordinate of corner *D* are (0, 6).

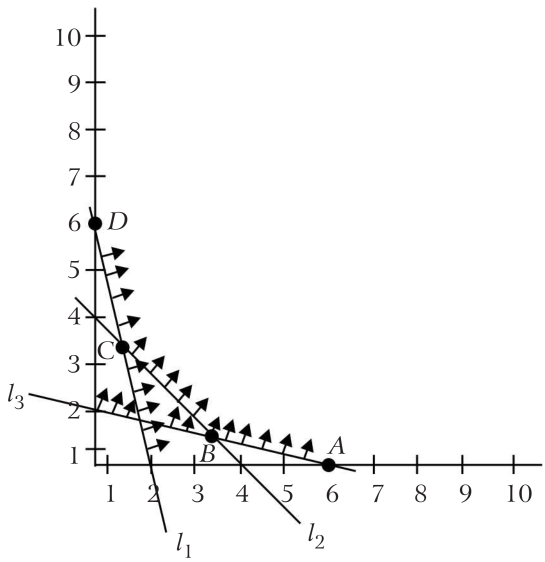


Fig. 5.18

The value of objective function at corner points of the feasible region are:

Corner points	Co-ordinates		$Z = 1000x_1 + 800x_2$
	x_1	x_2	
A	6	0	$6000 + 0 = 6000$
B	3	1	$1000 \times 3 + 800 = 3800$
C	1	3	$1000 \times 1 + 800 \times 3 = 3400$
D	0	6	$1000 \times 0 + 800 \times 6 = 4800$

The minimum value of the objective function $Z = 3400$ at the corner point C (1, 3) where $x_1 = 1, x_2 = 3$.

Example 28: *Multiple Optimal Solutions. Solve the following problem by graphical method:*

$$\text{Maximize } Z = x_1 + 1.5x_2 + 30$$

$$\text{subject to } 2x_1 + 3x_2 \leq 6$$

$$x_1 + 4x_2 \leq 4$$

and $$x_1, x_2 \geq 0$$

Solution: The constant (*i.e.,* 30) included in the objective function is deleted in the beginning and finally adjusted in the optimum value of Z.

(i) Draw the line $x_1 = 0$ as y-axis and mark the region represented by these lines.

(ii) Draw the line $2x_1 + 3x_2 = 6$ taking the points (0, 2) and (3, 0) or $\frac{x_1}{3} + \frac{x_2}{2} = 1$, taking the intercept 3 and 2 on x-axis and y-axis respectively and mark the region represented by the inequality

$$2x_1 + 3x_2 \leq 6$$

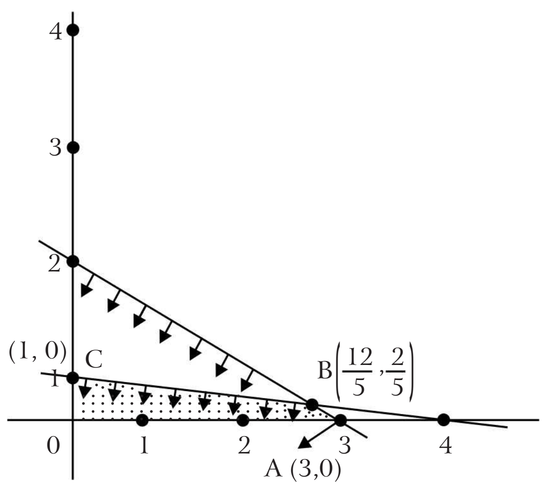


Fig. 5.19

(iii) Draw the line $x_1 + 4x_2 = 4$ taking the points (0, 1) and (4, 0)

or $$\frac{x_1}{4} + \frac{x_2}{1} = 1$$

taking the intercept 4 and 1 on x-axis and y-axis respectively and mark the region represented by the inequality

$$x_1 + 4x_2 \leq 4$$

(iv) Shade the feasible region *OABCO.*

Draw the line $Z = x_1 + 1.5x_2 = 0$, *i.e.,* $x_1 = -1.5x_2$ *i.e.,* $\frac{x_1}{x_2} = \frac{-1.5}{1}$.

Draw lines parallel to this line. One of these lines is passing through the points B and A, *i.e.,* is one edge of the feasible region ($OABCO$). This meant there is no unique points for which the value of Z is maximum. There are many points on the line AB for which the same maximum value of Z is obtained.

So this problem has infinitely many solutions with maximum Z.

(v) To find optimum solution we proceed as follows:

Corner points	Co-ordinates	$Z = x_1 + 1.5x_2$
O	(0, 0)	$Z = 0 + 1.5 \times 0 = 0$
A	(3, 0)	$Z = 3 + 1.5 \times 0 = 3$
B	$\left(\frac{12}{5}, \frac{2}{5}\right)$	$Z = \frac{12}{5} + 1.5 \times \frac{2}{5} = 3$
C	(0, 1)	$Z = 0 + 1.5 \times 1 = 1.5$

Z is maximum of $A(3,0)$ and at $B\left(\frac{12}{5}, \frac{2}{5}\right)$. That is $Z_{\max} = 3$. Finally, $Z_{\max} = 3 + 30 = 33$.

Example 29: *A firm produces three different products viz. R, S and T through two different plants viz., P_1 and P_2, the capacities of which in number of products per day are as follows:*

Plant	***Product***		
	R	***S***	***T***
P_1	3,000	1,000	2,000
P_2	1,000	1,000	6,000

The operating cost per day of running the plants P_1 and P_2 are ₹ 600 *and* ₹ 400 *respectively. The expected minimum demands during any month for the products R, S and T are* 24000 *units,* 16000 *units and* 48000 *units respectively.*

Show by graphic method how many days should the firm run each plant during a month so that the production cost is minimized while still meeting the market demand. **[B.C.A. (Lucknow) 2008]**

Solution: **The formulation of L.P.P.** : Let

x_1 = number of days plant P_1 runs

and x_2 = number of days plant P_2 runs

obviously, $x_1, x_2 \geq 0$.

$\therefore$ The cost of running the plants P_1 and P_2 is $600x_1 + 400x_2$

Thus, objective function is:

$$\text{Minimize } Z = 600x_1 + 400x_2$$

Constraints for Product *R*: The product of R is $(3000x_1 + 1000x_2)$ units and the minimum extracted demand is 24,000.

Hence, $3000x_1 + 1000x_2 \geq 24000$ units

Constraints for Product *S*: The product of *S* is $(1000x_1 + 1000x_2)$ units and the expected minimum demand is 16000 units.

Hence, $1000x_1 + 1000x_2 \geq 16000$

Constraint for Product *T*: The production of *T* is $(2000x_1 + 6000x_2)$ units and expected minimum demand is 48000 units.

Hence, $2000x_1 + 6000x_2 \geq 48000$

Thus, the required L.P.P. is as follows:

$$\text{Minimize } Z = 600x_1 + 400x_2$$

subject to
$$3000x_1 + 1000x_2 \geq 24000$$
$$1000x_1 + 1000x_2 \geq 16000$$
$$2000x_1 + 6000x_2 \geq 48000$$

and
$$x_1, x_2 \geq 0$$

Construction of graph

(i) The equation corresponding to $3000x_1 + 1000x_2 \geq 24000$ is:

$$3000x_1 + 1000x_2 = 24000$$
$$3x_1 + x_2 = 24$$
$$\frac{x_1}{8} + \frac{x_2}{24} = 1$$

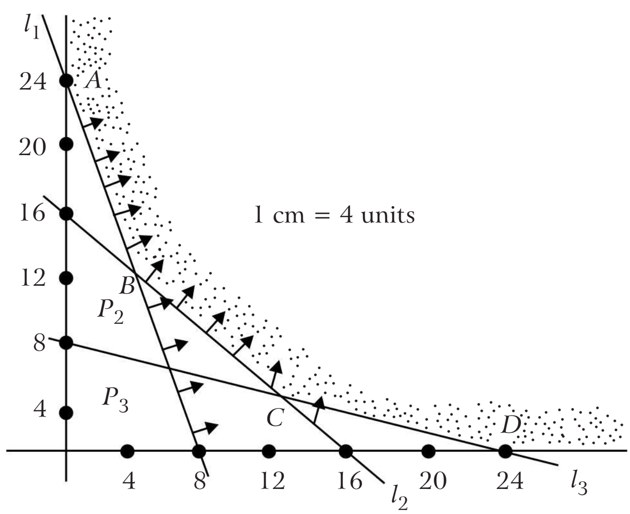


Fig. 5.20

Draw this line l_1 by joining the points (8, 0) and (0, 24). The region corresponding to this in equally is opposite the origin.

(ii) The equation corresponding to $1000x_1 + 1000x_2 \geq 16{,}000$ is:

$$1000x_1 + 1000x_2 = 16000 \qquad \begin{cases} \text{when } x_1 = 0, \text{ then } x_2 = 16 \\ \text{when } x_2 = 0, \text{ then } x_1 = 16 \end{cases}$$

or $$x_1 + x_2 = 16$$

Draw this line l_2 by joining the points (16, 0) and (0, 16). The region corresponding to this inequality is opposite the origin.

(iii) The equation corresponding to the equally $2000x_1 + 6000x_2 \geq 48000$ is $2000x_1 + 6000x_2 = 48000$

or $$x_1 + 3x_2 = 24$$

or $$\frac{x_1}{24} + \frac{x_2}{8} = 1$$

Draw the line l_3 by joining the points (24, 0) and (0, 8). The region corresponding to this inequality is opposite the origin.

According to the non-negativity conditions $x \geq 0$, $y \geq 0$, the region is in the first quadrant.

The required graph and the feasible region (shaded area) is as given above.

Computation of Production at Corner Points of the Feasible Region

(i) Co-ordinates of A are (0, 24).

(ii) Co-ordinates of B are obtained by solving the equations l_1 and l_2, *i.e.,*

$$3x_1 + x_2 = 24$$

and $$x_1 + x_2 = 16 \quad \text{(given in table)}$$

(iii) Co-ordinates of C are obtained by solving the equations l_2 and l_3, *i.e.,*

$$x_1 + x_2 = 16$$

and $$x_1 + 3x_2 = 24 \quad \text{(given in table)}$$

(iv) The co-ordinates of D are (24, 0).

Corner points	**Co-ordinates**		**Objective function $Z = 600x_1 + 400x_2$**
	x_1	x_2	
A	0	24	$600 \times 0 + 400 \times 24 = 9600$
B	4	12	$600 \times 4 + 400 \times 12 = 7200$
C	12	4	$600 \times 12 + 400 \times 4 = 8800$
D	24	0	$600 \times 24 + 400 \times 0 = 14400$

It is clear from the above table that the value of Z, the objective function is minimum at the corner point B (4,12). Hence, the firm should run the plant P_1 for 4 days and plant P_2 for 12 days to minimize the cost while still meeting the market demand.

Example 30: *The optimal solution to the following linear programming problem*

$$\text{Maximize } (Z) = 3x_1 + 2x_2$$

subject to the constraints

$$x_1 + x_2 \le 20$$

$$x_1 \le 15$$

$$x_1 + 3x_2 \le 45$$

$$-3x_1 + 5x_2 \le 60$$

and $$x_1, x_2 \ge 0$$

is $Z = \ldots$ *and is attained at* $x_1 = \ldots.\ x_2 = \ldots$ *you may use graphical method to fill in the blanks.*

[B.C.A. (Agra) 2002, 2004, 2006, 2010]

Solution: Draw the lines

$$x_1 + x_2 = 20 \text{ points } (0, 20), (20, 0)$$

$$x_1 = 15 \text{ point } (15, 0)$$

$$x_1 + 3x_2 = 45 \text{ points } (0, 15), (45, 0)$$

$$-3x_1 + 5x_2 = 60 \text{ points } (0, 12), (-20, 0)$$

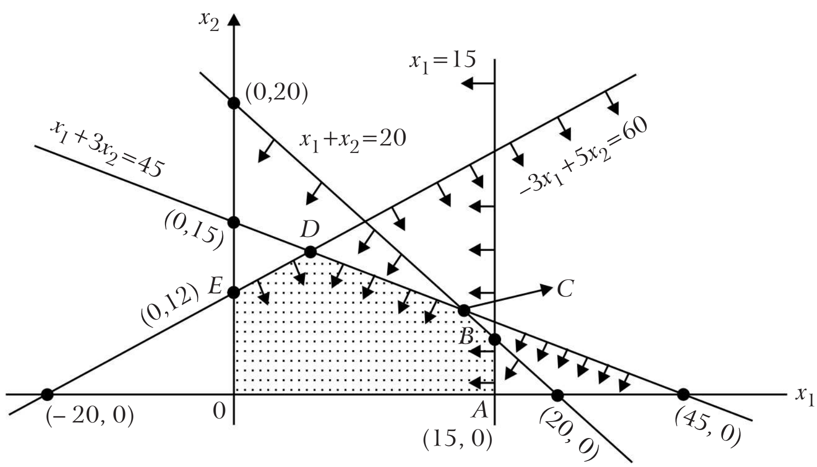


Fig. 5.21

The common feasible region *OABCDEO*

Corner points	Co-ordinates		$Z = 3x_1 + 2x_2$
	x_1	x_2	
O	0	0	$Z = 3\times 0 + 2\times 0 = 0$
A	15	0	$Z = 3\times 15 + 2\times 0 = 45$
B	15	5	$Z = 3\times 15 + 2\times 5 = 55$
C	$\frac{15}{2}$	$\frac{25}{2}$	$Z = 3\times\frac{15}{2} + 2\times\frac{25}{2} = 47.5$
D	3	14	$Z = 3\times 3 + 2\times 14 = 37$
E	0	12	$Z = 3\times 0 + 2\times 12 = 24$

Hence maximize $(z) = 55$ at corner point B (15, 5) where $x_1 = 15, x_2 = 5$.

Example 31: *The manager of a courier company wishes to hire extra helpers during the Christmas season, because of a large increase in the volume of mail handling and delivery. Keeping in view the limited office space and the budgetary conditions, the number of temporary helpers must not exceed 10. According to past experience, a man can handle 300 letters and 80 packages per day and a woman can handle 400 letters and 50 packages per day. It is believed that the daily volume of extra mail and packages will be no less than 3400 and 680 respectively. A man receives ₹25 a day and a woman receives ₹22 a day. How many men and women helpers should be hired to keep the pay-roll at a minimum ?*

Solution: **Formulation of the Problem as L.P.P.**

Let x_1 men and x_2 women be hired by the company to keep the pay-roll at a minimum. If Z be the daily pay-roll in ₹ of the extra helpers, then

$$Z = 25x_1 + 22x_2$$

Since the maximum number of helpers can be only 10, therefore $x_1 + x_2 \le 10$.

Given that a man can handle 300 letters daily and a woman can handle 400 letters daily and that the number of extra letters expected daily is not less than 3400, we must have

$$300x_1 + 400x_2 \ge 3400$$

Similarly $$80x_1 + 50x_2 \ge 680$$

Since the number of men and women hired cannot be negative, we have

$$x_1 \ge 0,\ x_2 \ge 0$$

Hence the L.P.P. formulated for the given problem is as follows:

$$\text{Minimize } Z = 25x_1 + 22x_2$$

subject to the constraints $x_1 + x_2 \le 10$

$$300x_1 + 400x_2 \ge 3400$$

$$80x_1 + 50x_2 \geq 680$$

and the non-negative restrictions $x_1 \geq 0, x_2 \geq 0$.

Solution of the Problem: Proceeding stepwise, the permissible region (the set of all points satisfying all the constraints and the non-negative restrictions) consists of the point $G(6, 4)$ only.

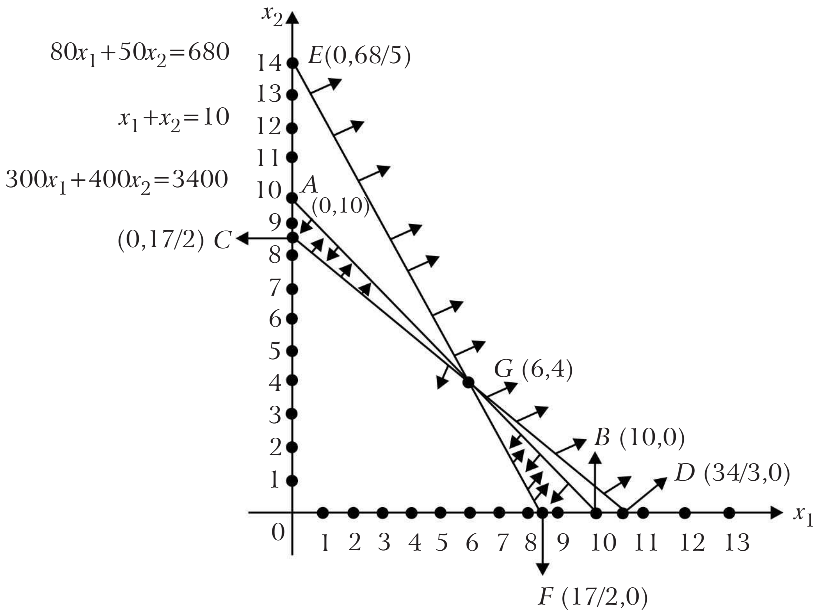


Fig. 5.22

Therefore the optimal solution is $x_1 = 6, x_2 = 4$ and the optimal value of

$$Z = 25 \times 6 + 22 \times 4 = 238$$

Hence, 6 men and 4 women helpers should be hired by the courier company to meet its seasonal requirements and keep the pay-roll at a minimum of ₹ 238.

Example 32: *Solve by graphical method the L.P.P.*

$$\text{Minimize } Z = 5x_1 + 6x_2$$

subject to the constraints

$$x_1 + x_2 \geq 50$$
$$x_1 + 2x_2 \leq 40$$
$$3x_1 + 4x_2 \leq 100$$

and the non-negative restrictions $x_1 \geq 0$ *and* $x_2 \geq 0$. **[B.C.A. (Lucknow) 2004, 2010]**

Solution: Solving simultaneously the inequations of the constraints and then non-negative restrictions by graphical method, we see that there exist no values of x_1 and x_2 that simultaneously satisfy all the constraints and the non-negative restrictions.

Hence, the given problem does not have any feasible solution.

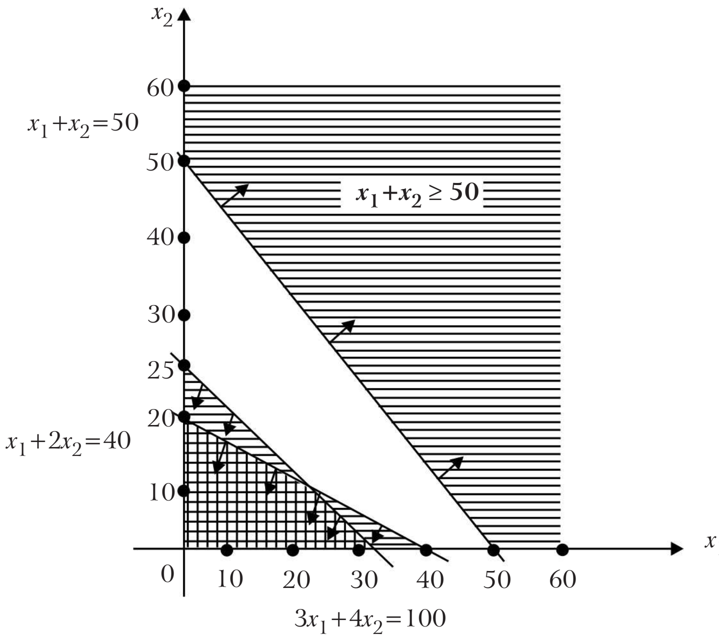


Fig. 5.23

Problems Having Unbounded Solution

Example 33: *Solve graphically the following L.P.P.*

$$\text{Maximize } Z = 3x_1 + 2x_2$$

subject to
$$x_1 - x_2 \leq 1$$
$$x_1 + x_2 \geq 3$$

and
$$x_1, x_2 \geq 0$$

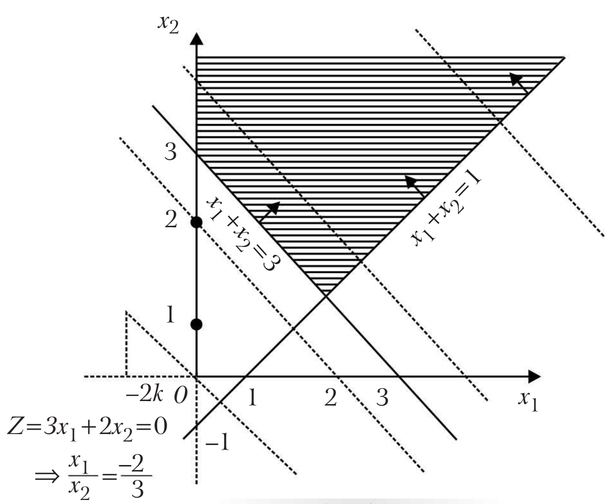


Fig. 5.24

Solution: Proceeding stepwise, the permissible region is shaded in the figure which is unbounded. From the figure it is clear that the dotted line through the origin representing $Z=0$ can be moved parallel to itself in the direction of Z increasing and still have some points in the permissible region.

Thus, Z can be made arbitrarily large and so the problem has no finite maximum value of Z. Hence, the problem has unbounded solution.

Example 34: *Solve the following L.P.P. graphically.*

$$\text{Maximize } (z) = 40x_1 + 35x_2$$

subject to $\quad 2x_1 + 3x_2 \leq 60$

$$4x_1 + 3x_2 \leq 96$$

$$x_1, x_2 \geq 0$$

[B.B.A. (Meerut) 2003]

Solution: Change inequality to equations

$$2x_1 + 3x_2 = 60 \qquad \text{...(1)}$$

$$4x_1 + 3x_2 = 96 \qquad \text{...(2)}$$

The co-ordinate of line (1) is $(0, 20), (30, 0)$

The co-ordinate of line (2) is $(0, 32), (24, 0)$

Plotting these two lines on graph with corresponding region

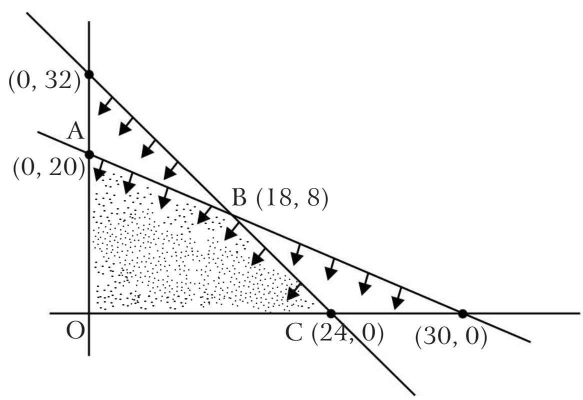


Fig. 5.25

we get common feasible region OABC. The co-ordinate of $O=(0,0)$ $A=(0,20)$, $C=(24,0)$, $B=$ point of intersection of line (1) and (2)

i.e., $\quad B=(18,8)$

Points	Value of objective function $Z = 40x_1 + 35x_2$
$O\ (0, 0)$	$Z = 0$
$A\ (0, 20)$	$Z = 40 \times 0 + 35 \times 20 = 700$
$B\ (18, 8)$	$Z = 40 \times 18 + 35 \times 8 = 1000$
$C\ (24, 0)$	$Z = 40 \times 24 + 0 = 960$

The maximum value of

$$Z = 1000 \text{ at point B}$$

where $\quad x_1 = 18, x_2 = 8.$

Problem Set

Solve Graphically the Following L.P.P.

1. Find the values of x_1 and x_2 which satisfy the constraint $x_1 + 2x_2 \le 2000$, $x_1 + x_2 \le 15000, x_2 \le 600, x_{1,}\ x_2 \ge 0$ and which maximize $Z = 3x_1 + 5x_2$.

[B.C.A. (Rohilkhand) 2007]

2. Minimize $Z = 1.5x + 2.5y$

subject to $x + 3y \ge 3$

$x + y \ge 2$

and $x, y \ge 0$ **[B.B.A. (Meerut) 2008]**

3. Maximize $Z = 5x_1 + 3x_2$

subject to $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

and $x_1, x_2 \ge 0$ **[B.C.A. (Kashi) 2002, 2008]**

4. Minimize $Z = 2x_1 + x_2$

subject to $5x_1 + 10x_2 \le 53$

$x_1 + x_2 \ge 1$

$x_1 \le 4$, and $x_1, x_2 > 0$

5. Maximize $Z = 7x_1 + 5x_2$

subject to $x_1 + 2x_2 \le 6$

$4x_1 + 3x_2 \le 12$

and $x_1, x_2 \ge 0$ **[B.C.A. (Aligarh) 2007, 2012]**

6. Minimize $Z = 4x_1 + 2x_2$

subject to $x_1 + 2x_2 \ge 2$

$3x_1 + x_2 \ge 3$

$4x_1 + 3x_2 \ge 6$

and $x_1, x_2 \ge 0$ **[B.C.A. (Delhi) 2006, 2012]**

7. Maximize $Z = 3x_1 + 2x_2$

subject to $2x_1 - x_2 \ge 2$

$x_1 + x_2 \le 8$

and $x_1, x_2 \ge 0$

[B.C.A. (Kanpur) 2007]

8. Maximize $Z = 3x_1 + 2x_2$

subject to $-2x_1 + x_2 \le 1$

$x_1 \le 2\ x_1 + x_2 \le 3$

and $x_1, x_2 \ge 0$ **[B.C.A. (Rohilkhand) 2010]**

9. Maximize $Z = 5x_1 + 7x_2$

subject to
$$x_1 + x_2 \le 4$$
$$3x_1 + 8x_2 \le 24,$$
$$10x_1 + 7x_2 \le 35$$
$$x_1, x_2 \ge 0$$

[B.B.A. (Meerut) 2008]

10. Minimize $Z = 3x + 5y$

subject to
$$-3x + 4y \le 12$$
$$x \le 4$$
$$2x - y \ge -2$$
$$y \ge 2$$
$$2x + 3y \ge 12$$
$$x, y \ge 0$$

[B.C.A. (Kurukshetra) 2012]

11. Maximize $Z = 3x_1 + 4x_2$

subject to
$$5x_1 + 4x_2 \le 200$$
$$3x_1 - 5x_2 \le 150$$
$$5x_1 + 4x_2 \ge 100$$
$$3x_1 + 4x_2 \ge 80$$

and
$$x_1, x_2 \ge 0$$

[B.C.A. (Agra) 2005, B.C.A. (I.G.N.O.U) 2012]

12. A manufacturer makes two types of products (A and B). The two products are manufactured by using two machines (I and II). It requires two hours on each machine to manufacture one unit of product A whereas one unit of B requires 3 hours on machine I and 1 hour on machine II, 12 and 8 machine hours are available respectively on machine I and II. The profit contributions from product A and B are ₹6 and ₹7 respectively. Find the optimal product mix and optimal profit using graphical method.

13. A home decorator manufactures two types of lamps A and B. Both lamps go through two technicians (i) a cutter and (ii) a finisher. Lamp A requires 2 hours of the cutters time and 1 hour of the finishers time. Lamp B requires 1 hour of cutters time and 2 hours of finishers time. The cutter has 104 hours and the finisher has 76 hours available time each month. Profit on one lamp A is ₹6 and on one lamp B is ₹11. Formulate a mathematical model and solve by graphic method.

14. A manufacturer produces two types of products A and B. Each unit of a requires 4 hours on the machine G and 2 hours on the machine P. Each unit of B requires 2 hours on the machine G and 4 hours on the machine P. The manufacturer has 2 machines G and 3 machines P. Each machine G works for 40 hours a week and each machine P works for 60 hours a week. Profit on product A is ₹30,000 and on product B is ₹40,000. The manufacturer wants to make the maximum profit in a week.

Formulate the above problem as a L.P.P. and solve it by the graphical method.

15. Max $(Z) = 5x_1 + 7x_2$

subject to
$$x_1 + x_2 \le 4,$$
$$3x_1 + 8x_2 \le 24$$
$$10x_1 + 7x_2 \le 35$$
$$x_1, x_2 \ge 0$$

16. Minimize $Z = 3x_1 + 5x_2$

subject to
$$-3x_1 + 4x_2 \le 12,\ 2x_1 - x_2 \ge -2$$
$$2x_1 + 3x_2 \ge 12,\quad x_1 \le 4,\ x_2 \ge 2$$

and
$$x_1, x_2 \ge 0$$

17. Maximize $Z = 3x_1 + 2x_2$

subject to
$$x_1 + x_2 \le 4,\ x_1 - x_2 \le 2,\ x_1, x_2 \ge 0$$

18. Maximize $(Z) = 3x_1 + 4x_2$

subject to
$$5x_1 + 4x_2 \le 200,\ 3x_1 + 5x_2 \le 150,\ 5x_1 + 4x_2 \ge 100$$
$$8x_1 + 4x_2 \ge 80,\ x_1, x_2 \ge 0$$

19. A company produces two types of leather belts say type *A* and *B*. Belt *A* is of superior quality and belt *B* is of a lower quality. Profits on the two types of belts are 40 and 30 paise per belt, respectively. Each belt of type *A* requires twice as much time as required by a belt of type *B*. If all belts were of type *B*, the company would produce 1000 belts per day. But the supply of leather is sufficient only for 800 per day. Belt *A* requires a fancy buckle and 400 fancy buckles are available for this per day. For belt of type *B*, only 700 buckles are available per day. How should the company manufacture of two types of belts in order to have maximum over all profit?

[B.C.A. (Rohilkhand) 2010]

20. Give an account of linear programming problem with its important components. What does the non-negativity restriction mean ?

[B.B.A. (Meerut) 2002, 2003]

21. Solve the following L.P.P. graphically.

Maximize $Z = 40x_1 + 35x_2$

subject to
$$2x_1 + 3x_2 \le 60$$
$$4x_1 + 3x_2 \le 96$$
$$x_1, x_2 \ge 0$$

[B.B.A. (Meerut) 2003]

22. Write a note on the following terms used in a L.P.P.

(i) Objective function coefficient

(ii) Objective function

(iii) Set of constraints

(iv) Slack and surplus variables in L.P.P.

[B.B.A. (Meerut) 2004, 2006]

23. Solve the following L.P.P. by using graphical method.

Min $(z) = 2x_1 + 3x_2$

subject to
$$x_1 + x_2 \geq 6$$
$$7x_1 + x_2 \geq 14$$
$$x_1, x_2 \geq 0$$

[B.C.A. (Meerut) 2005]

24. Use the graphical method to solve the following L.P.P. :

Max $(z) = 3x_1 + 6x_2$

subject to
$$3x_1 + 4x_2 \geq 12$$
$$-2x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

[B.B.A. (Meerut) 2007]

25. Solve the following using graphical method.

Maximize $z = 9x + 10y$

subject to
$$11x + 9y \leq 9900$$
$$7x + 12y \leq 8400$$
$$6x + 16y \leq 9600$$
$$x \geq 0, \quad y \geq 0$$

[B.B.A. (Meerut) 2008]

26. Explain the following terms in relation to L.P.P.

(i) Feasible solution (ii) Infeasible solution

(iii) Optimum solution (iv) Alternative solution

(v) Unbounded solution

[B.B.A. (Meerut) 2007]

Answers

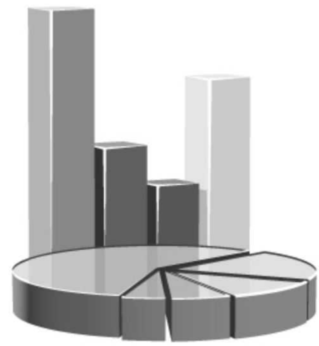
1.	$x_1 = 1000, x_2 = 500, Z_{max} = 5500.$	**2.**	$Z_{min} = 3.5, x = \frac{3}{2}, y = \frac{1}{2}.$
3.	$x_1 = \frac{20}{19}, x_2 = \frac{45}{19}, Z_{max} = \frac{235}{19}.$	**4.**	$x_1 = 0, x_2 = 1, Z_{min.} = 1.$
5.	$x_1 = 3, x_2 = 0, Z_{max.} = 21.$	**6.**	$x_1 = 0.6, x_2 = 1.2, Z_{min.} = 4.8$
7.	L.P.P. has no solution.	**8.**	$x_1 = 2, x_2 = 1, Z_{max.} = 8.$
9.	$x_1 = 1.6, x_2 = 2.4, Z_{max.} = 24.8.$	**10.**	$x = 3, y = 2, Z_{min.} = 19.$
11.	$x_1 = \frac{400}{13}, x_2 = \frac{150}{13}, Z_{max.} = \frac{1800}{13}.$	**12.**	Maximize $Z = 6x + 7y$ subject to $2x + 3y \le 12$ $2x + y \le 8$ $x, y \ge 0$ $Z_{max.} =$ ₹ 32 at $x = 3, y = 2$
13.	Maximize $Z = 6x + 11y$ subject to $2x + y ,= 104$ $x + 2y \le 76$ $x, y \ge 0$ $Z_{max.} = 440$ at $x = 44, y = 16$	**14.**	Maximize $Z = 30{,}000x_1 + 10{,}000x_2$ subject to $4x_1 + 2x_2 \le 80$ $2x_1 + 4x_2 \le 180$ and $x_1, x_2 \ge 0.$
15.	$x_1 = 1.6, x_2 = 2.4, Z = 24.8$	**16.**	$x_1 = 3, x_2 = 2, Z = 19.$
17.	$x_1 = 3, x_2 = 1, Z = 11.$	**18.**	$x_1 = \frac{400}{13}, x_2 = \frac{150}{13}, Z = \frac{1800}{13}.$

19.	200 belts of type A, 600 belts of type B, Profit = ₹ 260.
21.	Maxi $(z) = 1{,}000$ at $x_1 = 18$, $x_2 = 8$
23.	Min $(z) = 12$ at $x_1 = 6$, $x_2 = 0$
24.	Min $(z) = 12$ at $x_1 = 4$, $x_2 = 0$
25.	Max $(z) = 898.63$ at $x = 626.67$, $y = 334.8$

❍❍❍

CHAPTER

6



Queuing Theory

6.1 Introduction

[B.B.A. (Meerut) 2002, 2004]

Waiting lines or queues are omnipresent. Businesses of all types, industries, schools, hospitals, book stores, libraries, bank, post offices, petrol pump, all have queuing problems.

Waiting line problems arise either because:

1. There is too much demands on the facilities so that we say that there is an excess of waiting time or inadequate number of service facilities.
2. There is too less demand, in which case there is too much idle facility time or too many facilities.

In either case, the problem is to either **schedule arrivals** or **provide facilities or both** so as to obtain an optimum balance between the costs associated with waiting time and idle time.

A group of items waiting to receive service, including. Those receiving the service is known as a ***waiting line*** *or* ***a queue.***

[B.C.A. (Lucknow) 2011]

6.2 The Basic Queuing Process and its Characteristics

[B.C.A. (I.G.N.O.U.) 2012; B.B.A. (Meerut) 2002, 2003, 2004, 2005,2007,2011]

The basic queuing process can be described as a process in which the customers arrive for service at a service counter or station, wait for their turn in the queue if the server is busy in the service of the customer and are served when the server gets free and the customer leave the system as soon as he is served.

6.2.1 Characteristics of Queuing System

[B.C.A. (Agra) 2004, 2006,2009; B.B.A. (Meerut) 2002, 2003, 2004]

A queuing modal or system is specified completely by six main characteristics:

1. **Arrival Distribution:** Its represents the pattern in which the number of customers arrive at the system. Arrivals may also be represented by the **inter-arrival time**, which is the period between two successive arrivals.

   The rate at which customers arrive to be serviced, *i.e.* number of customers arriving per unit of time is called **arrival rate.** When the arrival rate is random, the customers arrive in no logical pattern. This represents most cases in the business world.

2. **Service Distribution:** It represents the **pattern** in which the number of customers leave the system. Departures may also be represented by the service time which is the time between two successive services. Service time may be constant or variable but known or random.

3. **Service Channels:** The queuing system may have a single service channel. Arriving customers may form one line and get serviced, as in a doctor's clinic. The system may have a number of service channels which may be arranged in parallel or in series or complex combination of both. In case of parallel channels, several customers may be serviced simultaneously as in a barber shop. For series channels, a customer must pass successively through all the channels before service is completed. A queuing model is called **one service model** when the system has one server only and a **multi-server model** when the system has a number of parallel channels each with one server.

4. **Service Discipline:** The service discipline refers to the manner in which the members in the queue are chosen for service. The following service disciplines are seen in common practice.

   (i) First Come, First Served (FCFS): According to this discipline the customers are served in the order of their arrival. This service discipline may be seen at a railway ticket window etc.

   (ii) Last Come, First Served (LCFS): According to this discipline the items arriving last are taken out first. This discipline may be seen in big godowns where the units (items) which come last are taken out first.

   (iii) Service in Random Order.

   (iv) Service on Some Priority-Procedure: Some customers are served before the others without considering their order of arrival.

5. **Maximum Number of Customers Allowed in the System:** Maximum number of customers in the system can be either finite or infinite. In some facilities, only a limited number of customers are allowed in the system and new arriving customers are not allowed to join the system unless the number becomes less than the limiting value.

6. **Service Mechanism:** The service mechanism refers to
   (i) the pattern according to which the customers are served
   (ii) facilities given to the customers:
      (a) Single-channel: Here the customers are served by one counter only.
      (b) Multi-channel: Here the customers are served by more than one counter.

6.2.2 Customers Behavior in a Queue

Customers generally behave in the following ways:

1. **Balking:** A customer may not like to wait in a queue due to lack of time or space or otherwise.
2. **Reneging:** A customer may leave the queue due to impatience.
3. **Collusion:** Some customers may collaborate and only one of them may join the queue.
4. **Jockeying:** If there are more than one queue then one customer may leave one queue and joining the other. This occurs generally in the super market.

6.3 Important Definitions in Queuing Problem

Here we give the definitions of various terms used in this chapter.

1. **Queue Length:** Queue length is defined by the number of persons (customers) waiting in the line at any time.
2. **Average Length of Line:** Average length of line (or Queue) is defined by the number of customers in the queue per unit time.
3. **Waiting Time:** It is time upto which a unit has to wait in the queue before it is taken into service.
4. **Servicing Time:** The time taken for servicing of a unit is called its servicing time.
5. **Busy Period:** Busy-period of a server is the time during which he remains busy in servicing. Thus, it is the time between the start of service of the first unit to the end of service of the last unit in the queue.
6. **Idle Period:** When all the units in the queue are served. The idle period of the server begins and it continues upto the time of arrival of the unit (customer). The ideal period of a server is the time during which he remains free because there is no customers present in the system.
7. **Mean Arrival Rate:** The mean arrival rate in a waiting-line situation is defined as the expected number of arrivals occurring in a time interval of length unity.

8. **Mean Servicing Rate:** The mean servicing rate for a particular servicing station is defined as the expected number of services completed in a time interval of length unity, given that the servicing is going on throughout the entire time unit.

9. **Transient State:** A system is said to be transient state when its operating characteristics are dependent on time. Thus, a queuing system is in transient state when the probability distributions of arrivals, waiting time and servicing time of the customers are dependent on time. This state occurs at the beginning of the operation of the system.

10. **Steady State:** A system is said to be in steady state when its operating characteristics becomes independent of time.

11. **Traffic Intensity:** **[B.B.A. (Meerut) 2002]**

In case of a simple queue the traffic intensity is the ratio of mean arrival rate and the mean servicing rate.

i.e. $$\text{Traffic intensity} = \frac{\text{Mean arrival rate}}{\text{Mean servicing rate}}$$

6.4 Poisson Process

Theorem 1: In Poisson process the probability of *n* arrivals during time intervals of length *t* is given by.

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \qquad \text{...(1)}$$

where λ, t is the parameter.

[B.C.A. (Kanpur) 2009; B.C.A. (Avadh) 2008; B.C.A. (Lucknow) 2010]

Proof : Case I: When $n = 0$ then from (1), the probability of no arrival in time Δt is given by

$$P_0(\Delta t) = \frac{(\lambda \Delta t)^{\circ} e^{-\lambda \Delta t}}{0!} = e^{-\lambda \Delta t}$$

$$= 1 - \lambda \Delta t + \frac{\lambda^2}{2!}(\Delta t)^2 + \ldots = 1 - \lambda \Delta t + O(\Delta t)$$

where $O(\Delta t)$ = smaller order of magnitude then Δt

If Δt is very small then $O(\Delta t) \to 0$

$\therefore$ $$P_0(\Delta t) = 1 - \lambda \Delta t$$

i.e., the probability of no arrival in time $\Delta t = 1 - \lambda \Delta t$

Case II: When $n = 1$, then from (1)

$$P_1(\Delta t) = \frac{(\lambda \Delta t)^1 e^{-\lambda \Delta t}}{1!}$$

$$= \lambda \Delta t \left\{1 - \lambda \Delta t + \frac{\lambda^2}{2!}(\Delta t)^2 + \ldots\right\}$$

$$= \lambda \Delta t + O(\Delta t)$$

$$\therefore \qquad P_1(\Delta t) = \lambda \Delta t$$

i.e., the probability of one arrival in time $\Delta t = \lambda \Delta t$

Case III: When $n = m > 1$, then from (1), we have

$$P_m(\Delta t) = \frac{(\lambda \Delta t)^m e^{-\lambda \Delta t}}{m!}$$

$$= \frac{\lambda^m (\Delta t)^m}{m!} \left\{1 - \lambda \Delta t + \frac{\lambda^2 (\Delta t)^2}{2!} + \dots\right\}$$

$$= \frac{\lambda^m}{m!} \{(\Delta t)^m - \lambda (\Delta t)^{m+1} + \dots\}$$

$$= 0$$

Thus, the probability of more than one arrival in time $\Delta t = 0$

6.5 Some Distributions

Here, we give some distributions which we are used in queuing theory.

1. **Exponential Distribution:** If T is a random variable representing the inter-arrival time (the time between consecutive arrivals), then T obey on exponential distribution *i.e.* if $f(T)$ denotes the probability function of T, then

$$f(T) = \lambda e^{-T}$$

where λ is parameter.

2. **Regular Distribution:** In regular distribution the inter-arrival time is taken as constant and defined as

$$f(T) = \begin{cases} 0 & \text{for } T < a \\ 1 & \text{for } T \geq a \end{cases}$$

a is constant.

3. **Erlang Service Time Distribution with k Phases (E_k):** The probability density function for the Erlang distribution is given by

$$f(t, \mu, k) = \frac{(\mu k)^k}{(k-1)!} t^{k-1} e^{-k\mu t}$$

$$t \geq 0, k \geq 1$$

where μ and k are positive parameters of the distribution.

6.6 Kendall's Notation for Representing Queuing Models

D.G. Kendall and later A. Lee introduced useful notation for queuing models. The complete notation can be expressed as

$$(a/b/c):(d/e/f)$$

where a = arrival or inter arrival distribution

b = departure or service time distribution

c = number of parallel service channels in the system

d = service discipline

e = maximum number of customers allowed in the system

f = calling source or population.

The following conventional codes are generally used to replace the symbol a, b and d

Symbols for *a* and *b*

M = Markovian (Poisson) arrival or departure distribution (or exponential inter arrival or service time distribution).

E_k = Erlangian or gamma inter arrival or service time distribution with parameter k.

GI = general independent arrival distribution.

G = general departure distribution.

D = deterministic inter arrival or service times.

Symbol for *d*

$ECFS$ = first come, first served

$LCFS$ = last come, first served

$SIRO$ = service in random order

GD = general service discipline

The symbols *e* and *f* represent a finite (N) or infinite (∞) number of customers in the system and calling source respectively.

$(M/E_k/1)$ = Poisson arrival, Erlangian departure, single server.

$(FCFS/N/\infty)$ = first come, first served, discipline, maximum allowable customer N in the system and infinite population model.

6.7 A List of Symbols

The following notations will be used throughout the chapter:

T = The inter arrival time between two successive customers.

n = Number of customers in the queuing system.

$P_n(t)$ = Transient state-probability that there are exactly n units in the system at any time t.

P_n = Steady state probability of having n units in the system $\left(\lim_{t\to\infty} P_n(t) = P_n\right)$.

λ_n = Mean arrival rate of customers when there are n units presents in the system.

μ_n = Mean service rate when there are n units in the system.

λ = Mean arrival rate of customer (independent of n).

μ = Mean service rate (independent of n).

s or c = Number of parallel service stations.

$\rho = \dfrac{\lambda}{\mu}$ = Traffic intensity.

$O(\Delta t)$ =A quantity which is of smaller order of magnitude than Δt.

$FCFS$ = First come first served (service discipline).

$\psi(\omega)$ = Probability density function of waiting time in the system.

$E(L_s)$ or L_s = Expected number of customers in the system (waiting and in service) *i.e.,* expected or mean line length.

$E(L_q)$ or L_q = Expected number of customers in the queue *i.e.,* expected or mean queue length (excluding the number of units in service).

$E(W_s)$ or W_s = Expected waiting time per customer in the system (including service time).

$E(W_q)$ or W_q = Expected waiting time per customer in the queue. (excluding service time).

$E(L / L > 0)$ = Expected length of the non-empty queue.

$E(W / W > 0)$ = Expected waiting time of a customer who has to wait.

$$\binom{m}{n} = {}^mC_n = \frac{m!}{n!(m-n)!}$$

The value of $\rho = \lambda / \mu$ must always be less than 1, so that the steady state condition may be obtained. If $\rho > 1$ *i.e.,* $\lambda > \mu$, then extremely large queues will be obtained.

The Queuing Problems:

6.8 Model 1 $(M/M/1):(\infty/FCFS)$ (Birth and Death Model)

[B.B.A. (Meerut) 2002, 2004, 2005, 2006, 2008, 2009, 2012]

The basic characteristics of this queuing problem (valid only when $\frac{\lambda}{\mu} < 1$) are:

1. Probability of no customer in the system is

$$P_0 = 1 - \rho \qquad \text{where} \quad \rho = \lambda/\mu$$

2. Probability of n customers in the system is

$$P_n = (1-\rho)\,\rho^n \qquad \text{where} \quad \rho = \frac{\lambda}{\mu} \quad \text{and} \quad n \geq 0$$

3. Probability that there are more than n customers in the system

$$P\,(>n) = \rho^{(n+1)}$$

4. Probability that are more than n customers in the queue

$$P\,(>n+1) = \rho^{(n+2)}$$

5. The average (expected) number of customers in the system is

$$L \text{ or } L_s \text{ or } E(n) = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1-\rho}$$

6. Average (expected) queue length is

$$L_q \quad \text{or} \quad E(m) = E(n) - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

or

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{\rho^2}{1-\rho}$$

7. Average (expected) waiting time of customer in queue is

$$W_q \quad \text{or} \quad W \quad \text{or} \quad E(W) = \frac{1}{\lambda}\,E(m) = \frac{\lambda}{\mu(\mu-\lambda)}$$

8. Average (expected) waiting time that a customer spends in the system is

$$W_s \quad \text{or} \quad E(v) = \frac{1}{\lambda}\,E(n) \qquad \text{or} \quad \frac{1}{\lambda}(L_q) = \frac{1}{\mu-\lambda}$$

9. Average (expected) waiting time of an arrival who has to wait is

$$E\,(W/W>0) = \frac{1}{\mu-\lambda}$$

10. Average (expected) length of a non-empty queue is

$$(L/L>0) \text{ or } E\,(m/m>0) = \frac{\mu}{\mu-\lambda} = \frac{1}{1-\rho}$$

11. The probability of arrivals during the service time of any given customer.

$$K_r = \left(\frac{\lambda}{\lambda+\mu}\right)^r \frac{\mu}{\lambda+\mu}$$

where r the probability of r arrivals.

12. The variance of queue length.

$$\text{Var}(n) = \frac{\rho}{(1-\rho)^2} \qquad \text{where} \qquad \rho = \frac{\lambda}{\mu} < 1$$

13. Probability [queue size $\geq N$] $= \left(\frac{\lambda}{\mu}\right)^N$.

6.8.1 Inter-relationship between L_s, L_q, W_s, W_q

$$L_s = \lambda W_s \qquad \text{...(1)}$$

$$L_q = \lambda W_q \qquad \text{...(2)}$$

$$W_q = W_s - \frac{1}{\mu} \qquad \text{...(3)}$$

$$L_q = L_s - \frac{\lambda}{\mu} \qquad \text{...(4)}$$

Solved Examples

Example 1: *A TV repairman finds that time spend on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they come in, and if the arrival of sets is approximately Poisson with an average rate 10 per 8-hour day, what is repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?*

[B.C.A. (Kashi) 2008, 2011; B.C.A. (Delhi) 2006, 2010; B.C.A. (Rohilkhand) 2007, 2009]

Solution: Here, $\mu = \frac{1}{30}, \lambda = \frac{10}{8 \times 60} = \frac{1}{48}$

Therefore, expected number of jobs are

$$L_s = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda} = \frac{1/48}{1/30 - 1/48} = \frac{5}{3}$$

Since the fraction of the time the repairman is busy is equal to λ/μ, the number of hours for which the repairman remains busy in a 8 hour day is

$$= 8\left(\frac{\lambda}{\mu}\right) = 8 \times \frac{30}{48} = 5 \text{ hours.}$$

Example 2: *At what average rate must a clerk at a supermarket work in order to ensure a probability of 0.90 that the customer will not have to wait longer than 12 minutes? It is assume that there is only one counter at which costumers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.*

[B.C.A. (Agra) 2003; B.B.A. (Lucknow) 2007; B.C.A.(Garhwal) 2008, 2012; B.C.A. (I.G.N.O.U.) 2011]

Solution: Here $\lambda = \frac{15}{60} = \frac{1}{4}$ customer/minute, $\mu = ?$

$$\text{Prob [Waiting time} \geq 12] = 1 - 0.90 = 0.10$$

Therefore,
$$\int_{12}^{\infty} \lambda \left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu - \lambda) w} \, dw = 0.10$$

or
$$\lambda \left(1 - \frac{\lambda}{\mu}\right) \left[\frac{e^{-(\mu - \lambda) w}}{-(\mu - \lambda)}\right]_{12}^{\infty} = 0.10$$

or
$$e^{(3 - 12\mu)} = 0.4\mu$$

or
$$\frac{1}{\mu} = 2.48 \text{ minute per service.}$$

Example 3: *In super market the average arrival rate of customers is 5 every 30 minutes. The average time it takes to list and calculate the customers purchases at the cash desk is 4.5 minutes, and this time is exponentially distributed.*

(i) How long will the customer expect to wait for service at the cash desk?

(ii) What is the chance that the queue length will exceeds 5?

(iii) What is the probability that the cashier is working? **[B.C.A. (I.G.N.O.U.) 2004, 2012]**

Solution: Here

$$\lambda = 5 \text{ every 30 minutes or } \frac{1}{6} \text{ per minute.}$$

$$\mu = \frac{2}{9} \text{ per minute}$$

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{1}{6} \times \frac{9}{2} = \frac{3}{4}$$

(i)
$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\frac{1}{6}}{\frac{2}{9}\left(\frac{2}{9} - \frac{1}{6}\right)} = \frac{\frac{1}{6}}{\frac{2}{9} \times \frac{3}{9 \times 6}} = \frac{9 \times 9 \times 6}{2 \times 3 \times 6} = \frac{27}{2} = 13.5$$

(ii) $P[> n + 1] = \rho^{5+2} = (0.75)^7$ or $.133$, since $n = 5$

(iii) Probability that cashier is working

$= $ Probability of one or more customers in the system

$= 1 -$ probability of no customer in the system

$= 1 - P_0 = \rho = 0.75$, Since $P_0 = 1 - \rho$

Example 4: *On an average 96 patients per 24-hour day require the service of an emergency clinic. Also on average, a patient requires 10 minutes of active attention. Assume that facility can handle only one emergency at a time. Suppose that it costs the clinic ₹ 100 per patient treated to obtain an average servicing time of 10 minutes, and that each minute of decrease in this average time would cost ₹ 10 per patient treated. How much would to be budgeted by the clinic to decrease the average size of the queue from* $1\frac{1}{3}$ *patients to* $\frac{1}{2}$ *patient ?*

[B.C.A. (Lucknow) 2006]

Solution: Here $\lambda = \frac{96}{24 \times 60} = \frac{1}{15}$ patient/minute

$$\mu = \frac{1}{10} \text{ patient/minute}$$

$$\therefore \qquad \rho = \lambda/\mu = 2/3$$

Expected number of patients in the waiting line

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\left(\frac{1}{15}\right)^2}{\left(\frac{1}{10}\right)\left(\frac{1}{10} - \frac{1}{15}\right)} = \frac{4}{3}$$

Fraction of the time for which there are no patients is given by

$$P_0 = 1 - \rho = 1 - \frac{2}{3} = \frac{1}{3}$$

Now, when the average queue size is decreased from $\frac{4}{3}$ patient to $\frac{1}{2}$ patient, we are to determine the value of μ. So, we have

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} \Rightarrow \frac{1}{2} = \frac{\left(\frac{1}{15}\right)^2}{\mu\left(\mu - \frac{1}{15}\right)}$$

$$\Rightarrow \mu = \frac{2}{15} \text{ patients per minutes}$$

$\therefore$ Average rate of treatment required $= \frac{1}{\mu} = \frac{15}{2} = 7.5$ minutes.

Decrease in the average rate of treatment $= (10 - 7.5) = 2.5$ minutes

Budget per patient = ₹ $(100 + 2.5 \times 10)$ = ₹ 125

Hence, in order to get the required size of the queue, the budget should be increase from ₹ 100 per patient to ₹ 125 per patient.

Example 5: *In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average 36 minutes. Calculate the following:*

(i) The mean queue size (line length)

(ii) The probability that the queue size exceeds 10.

If the input of trains increase to an average 33 per day, what will be the change in (i) and (ii)?

Solution: Here

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48}$$

$$\mu = \frac{1}{36} \text{ trains per minutes}$$

$\therefore$
$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{48}}{\frac{1}{36}} = \frac{36}{48} = \frac{3}{4} = 0.75$$

(i)
$$L_q = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3 \text{ trains}$$

(ii)
$$P[\geq 10] = \rho^{10} = (0.75)^{10} = 0.06$$

when the input increases to 33 trains per day, we have

$$\lambda = \frac{33}{60 \times 24} = \frac{11}{480} \text{ and } \mu = \frac{1}{36} \text{ trains per minutes}$$

$\therefore$
$$\rho = \frac{\lambda}{\mu} = \frac{11}{480} \times 36 = 0.83, \text{ hence}$$

(i) $L_s = \dfrac{0.83}{0.16} = 5$ trains

(ii) Prob (queue size ≥ 10) $= (0.83)^{10} = 0.2$ (approximation)

Example 6: *A self-service store employs one cashier at its counter. Nine customers arrive on an average every 5 minutes while the cashier can serve 10 customers in 5 minutes. Assuming Poisson distribution for arrival rate and exponential distribution for service rate, find*

(i) Average number of customers in the system.

(ii) Average number of customers in queue or average queue length.

(iii) Average time a customer spends in the system.

(iv) Average time a customer waits before being served.

Solution: Arrival rate $\lambda = 9/5 = 1.8$ customers/minute

Service rate $\mu = 10/5 = 2$ customers/minute

(i) Average number of customers in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1.8}{2 - 1.8} = 9$$

(ii) Average number of customers in the queue

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu - \lambda} = \frac{1.8}{2} \times \frac{1.8}{2 - 1.8} = 8.1$$

(iii) Average time a customer spends in the system

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{2 - 1.8} = 5 \text{ minutes.}$$

(iv) Average time a customer spends in the queue

$$W_q = \frac{\lambda}{\mu}\left(\frac{1}{\mu - \lambda}\right) = \frac{1.8}{2}\left(\frac{1}{2 - 1.8}\right) = 4.5 \text{ minutes.}$$

Example 7: *A person repairing radios finds that the time spend on the radio sets has been exponential distribution with mean 20 minutes. If the radios are repaired in the order in which they come in and their arrival is approximately Poisson with an average rate of 15 for 8-hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in?* **[B.B.A. (Meerut) 2004]**

Solution: Arrival rate $\lambda = \frac{15}{8 \times 60} = \frac{1}{32}$ units/minute service rate

$$\mu = \frac{1}{20} \text{ units/minute}$$

Number of jobs ahead to the set brought in = Average number of jobs in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{32}}{\frac{1}{20} - \frac{1}{32}} = 5/3$$

Number of hours for which the repairman remains busy in an 8-hours day

$$= 8\frac{\lambda}{\mu} = \frac{8 \times 1/32}{1/20} = 8 \times \frac{20}{32} = 5 \text{ hours.}$$

$\therefore$ Time for which repairman remains idle in an 8-hours day $= 8 - 5 = 3$ hours.

Example 8: *A branch of Punjab National Bank has only one typist. Since the typing work varies in length (Number of pages to be typed), the typing rate is randomly distributed approximating a Poisson distribution with mean service rate of 8 letters per hour. The letters arrive at a rate of 5 per hour during the entire 8 -hour work day. If the typewriter is valued at ₹ 1.50 per hour, determine*

(i) Equipment utilization.

(ii) The percent time that an arriving letter has to wait.

(iii) Average system time.

(iv) Average cost due to waiting on the part of typewriter. **[B.B.A. (Rohilkhand) 2005, 2007]**

Solution: Arrival rate, $\lambda = 5$ per hour

Service rate, $\mu = 8$ per hour

(i) Equipment utilization, $\rho = \frac{\lambda}{\mu} = \frac{5}{8} = 0.625$

(ii) The percent time an arriving letter has to wait = percent time type writer remains busy $= 62.5\%$

(iii) Average system time, $W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-5} = \frac{1}{3}$ hr $= 20$ minutes

(iv) Average cost due to waiting on the part of the type writer per day

$$= 8 \times \left(1 - \frac{5}{8}\right) \times ₹\,1.50 = ₹\,4.50$$

Example 9: *Arrival rate of telephone calls at a telephone booth are according to Poisson distribution, with an average time of 9 minutes between two consecutive arrivals. The length of telephone call is assumed to be exponentially distributed, with mean 3 minutes.*

(i) Determine the probability that a person arriving at the booth will have to wait.

(ii) Find the average queue length that is formed from time to time.

[B.B.A. (Delhi) 2005, 2008]

(iii) The telephone company will install a second booth when convinced that an arrival would expect to have to wait at least four minutes for a phone. Find the increase in the flow of arrivals which will justify a second booth.

(iv) What is the probability that an arrival will have to wait for more than 10 minutes before the phone is free? **[B.C.A. (I.G.N.O.U.) 2010]**

(v) Find the fraction of a day that the phone will be in use.

Solution: Here, arrival rate $\lambda = \frac{1}{9}$ per minute

Service rate $\mu = \frac{1}{3}$ per minute

(i) Probability that a person will have to wait $= \frac{\lambda}{\mu} = \frac{1/9}{1/3} = \frac{1}{3} = 0.33$

(ii) Average queue length $= \frac{\mu}{\mu - \lambda} = \frac{1/3}{1/3 - 1/9} = 1.5$ persons

(iii) Average waiting time in the queue $= \frac{\lambda_1}{\mu(\mu - \lambda_1)}$

$$4 = \frac{\lambda_1}{\frac{1}{3}\left(\frac{1}{3} - \lambda_1\right)}$$

or
$$\frac{1}{9} - \frac{\lambda_1}{3} = \frac{\lambda_1}{4} \text{ or } \lambda_1 \times \frac{7}{12} = \frac{1}{9}$$

or
$$\lambda = \frac{4}{21} \text{ arrivals/minute}$$

$\therefore$ increase in the flow of arrivals $= \frac{4}{21} - \frac{1}{9} = \frac{5}{63}$ per minute

(iv) Probability [waiting time ≥ 10] $= \int_{10}^{\infty} \left(1 - \frac{\lambda}{\mu}\right) \lambda\, e^{(\lambda - \mu)t}\, dt$

$$= \lambda\left(1-\frac{\lambda}{\mu}\right)\left[\frac{e^{(\lambda-\mu)t}}{\lambda-\mu}\right]_{10}^{\infty}$$

$$= \frac{\lambda\,(\mu-\lambda)}{\mu}\left[0-\frac{1}{\lambda-\mu}\,e^{(\lambda-\mu)10}\right]$$

$$= \frac{\lambda}{\mu}\,e^{(\lambda-\mu)10}$$

$$= \frac{1/9}{1/3}\,e^{\left(\frac{1}{9}-\frac{1}{3}\right)10} = \frac{1}{3}\,e^{-20/9} = \frac{1}{30}$$

(v) The expected fraction of a day that the phone will be in use $= \lambda/\mu = 0.33$

Example 10: *Customers arrive at a one window drive in bank according as a Poisson distribution with mean* 10 *per hour. Service time per customer is exponential with mean* 5 *minutes. The space in front of the window, including that for the serviced car can accommodate a maximum of three cars. Others cars can wait outside this space.*

(i) What is the probability that an arriving customer can derive directly to the space in front of the window?

(ii) What is the probability that an arriving customer will have to wait outside the indicated space?

(iii) How long is an arriving customer expected to wait before starting service?

[B.C.A. (Rohtak) 2006, 2009, 2011]

Solution: Here, $\lambda = 10$ per hours, $\mu = \frac{60}{5} = 12$ per hour

(i) The probability that an arriving customer can derive directly to the space in front of the window.

$$= p_0 + p_1 + p_2$$

$$= \left(1-\frac{\lambda}{\mu}\right) + \frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left[1+\frac{\lambda}{\mu}+\frac{\lambda^2}{\mu^2}\right]$$

$$= \left(1-\frac{10}{12}\right)\left[1+\frac{10}{12}+\frac{100}{144}\right] = 0.42$$

(ii) The probability that an arriving customer has to wait outside the indicated space

$$= 1 - 0.42 = 0.58.$$

(iii) Average waiting time of a customer in the queue

$$= \frac{1}{\mu}\,\frac{\lambda}{\mu-\lambda} = \frac{10}{12\,(12-10)} = 0.417 \text{ hours.}$$

Example 11: *A repairman is to be hired to repair machine which breakdown at an average rate of 6 per hour. The breakdown follow Poisson distribution. The non-productive time of a machine is considered to cost ₹ 20 per hour. Two repairman, Mr. X and Mr. Y have been interviewed for this purpose. Mr. X charge ₹ 10 per hour and he services breakdown machines at the rate of 8 per hour. Mr. Y demands ₹ 14 per hour and he services at an average rate of 12 per hour. Which repairman should be hired? (Assume 8 hours shift per day).*

Solution: We have, average down time (non-productive time) of a machine per hour

= Average waiting time of a breakdown machine in the system per hour

$= W_s = 1/(\mu - \lambda)$

Here mean arrival rate (breakdown rate), $\lambda = 6$ machines per hour.

For repairman Mr. X : Mean service rate, $\mu = 8$ machines per hour.

$$\therefore \qquad W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2} \text{ hour.}$$

So the total cost per hour = Down time cost of 6 breakdown machines per hour

+ Charges paid to Mr. X per hour

$= ₹\ [20 \times 6 \times (1/2) + 10] = ₹\ 70$.

$\therefore$ the total cost in 8 hours shift per day.

$= ₹\ 70 \times 8 = ₹\ 560$.

For repairman Mr. Y : Mean service rate.

$\mu = 12$ machines per hour.

$$\therefore \qquad W_s = \frac{1}{\mu - \lambda} = \frac{1}{12-6} = \frac{1}{6} \text{ hour.}$$

So, the total cost per hour

= down time cost of 6 breakdown machine per hour

+ charges paid to Mr. Y per hour

$= ₹\ [20 \times 6 \times (1/6) + 14] = ₹\ 34$

$\therefore$ the total cost in 8 hours shift per day $= ₹\ 34 \times 8 = ₹\ 272$.

Since, per day cost of Mr. Y is less than that of Mr. X. Hence, the repairman Mr. Y should be hired.

Example 12: *A warehouse has only one loading dock manned by a three persons crew. Trucks arrive at the loading dock at an average rate of 4 trucks per hour and the arrival rate is Poisson distributed. The loading of a truck takes 10 minutes on an average and can be assumed to be exponentially distributed. The operating cost of a truck is ₹ 20 per hour and the members of the loading crew are paid ₹ 6 each per hour. Would you advise the truck owner to add another crew of three persons ?* **[B.C.A. (Delhi) 2012]**

Solution: It is $(M/M/1):(\infty/FCFS)$ problem

Here, mean arrival rate,

$$\lambda = 4 \text{ trucks per hour.}$$

For present crew of 3 persons.

$$\text{Mean service rate, } \mu = 60/10 = 6 \text{ trucks per hour.}$$

$\because$ $$W_s = \text{Average waiting time per truck, per hour}$$

$$= \frac{1}{\mu - \lambda} = \frac{1}{6-4} = \frac{1}{2} \text{ hour.}$$

$\therefore$ Total cost per hour

$$= \text{Loading crew cost + waiting time cost}$$

$$= ₹\, 6 \times 3 + W_s \times \text{number of trucks arrive in one hour}$$

$$\times \text{waiting time cost per hour}$$

$$= ₹\, 18 + ₹\, \{(1/2) \times 4 \times 20\} = ₹\, 58.$$

After the addition of another crew of 3 persons.

$$\text{Mean service rate } \mu = 2 \times 6 = 12 \text{ trucks per hour.}$$

$\therefore$ $$W_s = \frac{1}{\mu - \lambda} = \frac{1}{12-4} = \frac{1}{8} \text{ hour}$$

$\therefore$ Total cost per hour, (after the addition of another crew)

$$= \text{Loading crew cost + waiting time cost}$$

$$= ₹\, 6 \times 6 + ₹\, \{(1/8) \times 4 \times 20\} = ₹\, 46.$$

Since total cost per hour is the second case is less than that in the previous case, hence we advice the truck owner to add another crew of three persons.

Example 13: *Problems arrive at a computing centre in a Poisson fashion at an average rate of five per day. The rules of the computing centre are that man waiting to get his problem solved must aid the man whose problem is being solved. If the time to solve a problem with one man has an exponential distribution with mean time of 1/3 day, and if the average solving time is inversely proportional to the number of peoples working on the problem, approximate the expected time in the centre of a person entering the line.* **[B.C.A. (I.G.N.O.U.) 2010]**

Solution: This problem is base on $(M/M/1):(\infty/FCFS)$ model.

Here, $\lambda = 5$ problems/day

and μ (mean service rate with one unsolved problem)

$$= 3 \text{ problems/day.}$$

and $$P_n = \frac{e^{-\lambda/\mu}}{n!} \cdot \left(\frac{\lambda}{\mu}\right)^n.$$

$\therefore$ Expected number of persons working on the problem at any instant

$$L_s = \sum_{n=0}^{\infty} n.P_n$$

$$= \sum_{n=0}^{\infty} n.\frac{e^{-\lambda/\mu}}{n!}.\left(\frac{\lambda}{\mu}\right)^n$$

$$= e^{-\lambda/\mu}.\left(\frac{\lambda}{\mu}\right)\sum_{n=1}^{\infty}\left\{\frac{(\lambda/\mu)^{n-1}}{(n-1)!}\right\}$$

$$= e^{-\lambda/\mu}.\left(\frac{\lambda}{\mu}\right).e^{\lambda/\mu} = \frac{\lambda}{\mu} = \frac{5}{3} \text{ persons}$$

Average solving time (which is inversely proportional to the number of peoples working on the problem)

$$= 1/5 \text{ days/problems.}$$

$\therefore$ Expected time in the centre for a person entering the line

$$= \frac{1}{5} L_s = \frac{1}{5} \times \frac{5}{3} = \frac{1}{3} \text{ days} = 8 \text{ hours.}$$

Example 14: *Customers arrive at a box window, being manned by single individual, according to Poisson input process with a mean rate of 30 per hour. The time required to serve a customer has an exponential distribution with a mean of 90 seconds. Find the average waiting time of a customer. Also determine the average number of customers in the system and average queue length.*

[B.C.A. (Kurukshetra) 2007, 2011]

Solution: It is $(M/M/1):(\infty\ FCFS)$ problem queuing model

Here, $\lambda = \frac{30}{60} = 0.5$

and $\mu = \frac{60}{90} = 0.67$ per minute

$\therefore$ Average waiting time of a customer is given by

$$W_q = E(w) = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{0.5}{0.67(0.67 - 0.5)} = 4.5 \text{ minute per customer}$$

Average number of customers in the system is given by

$$= \frac{\lambda}{\mu - \lambda} = \frac{0.50}{0.67 - 0.50} = \frac{0.50}{0.17} = 3$$

Average queue length is given by

$$E(m) = L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(0.5)^2}{(0.67)(0.67 - 0.5)} = 2.27$$

Example 15: *A departmental store has a single cashier. During the rush hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. Assume that the conditions for use of the single-channel queuing model apply.*

(i) What is the probability that the cashier is idle?

(ii) What is the average number of customers in the queuing system?

(iii) What is the average time a customer spends in the system?

(iv) What is the average number of customers in the queue?

(v) What is the average time a customer spends in the queue waiting for service?

[B.C.A. (Kanpur) 2007]

Solution: We are given $\lambda = 20$ and $\mu = 24$ customers per hour

(i) Probability that the cashier is idle is

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{20}{24} = \frac{1}{6} = 0.67$$

(ii) Average number of customers in the system is

$$E(n) = L_q = L = \frac{\lambda}{\mu - \lambda} = \frac{20}{24 - 20} = \frac{1}{4} = 0.25$$

(iii) Average time a customer spends in the system

$$E(v) = W_q = \frac{1}{\mu - \lambda} = \frac{1}{24 - 20} = \frac{1}{4} = 0.25$$

(iv) Average number of customers in the queue is

$$E(m) = L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{20 \times 20}{24(24 - 20)} = 4.17$$

(v) Average time a customer spends in the queue is

$$E(W) = W_q = W = E(W) = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{5}{24} = 0.21$$

6.9 Model II (M/M/1) : (N/FCFS) Finite Queue Length Model

[B.B.A. (Meerut) 2008]

In this queuing system, the number of customers is limited to N

1. The probability of no customer in the system is

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

where $\rho = \dfrac{\lambda}{\mu}$

λ = mean arrival rate or number of arrivals per unit of time.

μ = mean service rate per busy server or number of customers served. per unit of time

2. The probability of n customers in the system

$$P_n = \frac{1-\rho}{1-\rho^{N+1}} . \rho^n$$

3. Average number of customers in the system

$$L_s = \frac{\rho\,[1-(1+N)\,\rho^N + N\,\rho^{N+1}}{(1-\rho)\,(1-\rho^{N+1})}$$

where $\rho = \dfrac{\lambda}{\mu}$

4. Average number of customers in the queue.

$$L_q = \frac{1-N\,\rho^{N-1} + (N-1)\,\rho^N}{(1-\rho)\,(1-\rho^{N+1})}\,\rho^2 = \sum_{n=0}^{N} n P_n$$

5. Average time a customer spends in the system

$$W_s = \frac{L_s}{\lambda'} \qquad \text{where} \qquad \lambda' = \lambda\,(1-\rho^N)$$

6. Average waiting time in the queue

$$W_q = \frac{L_q}{\lambda'}, \qquad \text{where} \qquad \lambda' = \lambda\,(1-\rho^N)$$

Example 16: *Trains arrive at the yard every 15 minutes and the service time is 33 minutes. If the line capacity of the yard is limited to 4 trains, find*

(i) The probability that the yard is empty.

(ii) The average number of trains in the system. **[B.C.A. (Agra) 2009]**

Solution: Here

$$\lambda = \frac{1}{15} \text{ per minute}$$

$$\mu = \frac{1}{33} \text{ per minute}$$

$$\therefore \qquad \rho = \frac{\lambda}{\mu} = \frac{33}{15} = 2.2$$

$$N = 4$$

(i) $$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{\rho - 1}{\rho^{N+1} - 1} = \frac{2.2 - 1}{(2.2)^5 - 1} = \frac{1.1}{51.15 - 1}$$

$$= \frac{1.1}{50.15} = 0.021934 = 0.237 \text{ (Appro.)}$$

(ii) Average number of trains in the system

$$L_s = \sum_{n=0}^{N} n P_n = 0 + P_1 + 2P_2 + 3P_3 + 4P_4, N = 4$$

$$= P_0 (\rho + 2\rho^2 + 3\rho^3 + 4\rho^4)$$

$$= 0.237 (2.2 + 2 \times (2.2)^2 + 3(2.2)^3 + 4(2.2)^4$$

$$= 0.237 (2.2 + 9.68 + 31.94 + 93.70)$$

$$= 32.6$$

Example 17: *Consider a single server queuing system with a Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour the expected service time is 0.25 hours, and the maximum number permissible number calling units in the system is two. Derive the steady state probability distribution of the number of calling units in the system, and then calculate the expected number in the system.* **[B.C.A. (Garhwal) 2008]**

Solution: Here

$$\lambda = 3 \text{ units per hour}$$

and $$\frac{1}{\mu} = .25 \quad \text{or} \quad \mu = 4 \text{ units per hours}$$

$$\therefore \qquad \rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75 \text{ which implies that}$$

$$1 - \rho = 1 - 0.75 = 0.25$$

Also, we are given $N = 2$

The steady-state probability distribution of the number of calling units in the system is given by

$$P_n = \frac{(1-\rho)\,\rho^n}{1-\rho^{N+1}} = \frac{(0.25)\,(0.75)^n}{1-(0.75)^3} = 0.43\,(0.75)^n$$

and
$$P_0 = \frac{1-\rho}{1-\rho^{2+1}} = \frac{(0.25)}{1-(0.75)^3} = 0.43$$

Expected number in the system is given by

$$E(n) \text{ or } E(L_q) \text{ or } L_q = \sum_{n=0}^{N} n P_n$$

$$= \sum_{n=0}^{2} n\,(0.43)\,(0.75)^n = (0.43) \sum_{n=0}^{2} n (0.75)^2$$

$$= (0.43)\,[1 \times (0.75) + 2 \times (0.75)^2] = 0.81$$

Example 18: *Assume that the goods trains are coming in a yard at the rate of 30 trains per day and suppose that the inter-arrival times follow the exponential distribution. The service time for each train is assumed to be exponential with an average of 36 minutes. If the yard can admit 9 trains at a time (there being 10 lines, one of which is reserved for shunting purpose) calculate the probability that the yard is empty and find the average queue length.* **[B.C.A. (I.G.N.O.U.) 2009]**

Solution: Here

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48},\ N = 9$$

and
$$\mu = \frac{1}{16} \text{ trains per minute}$$

∴
$$\rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75$$

The probability that the yard is empty is given by

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^{10}}$$

$$= \frac{0.25}{0.90} = 0.28$$

Average queue length is given by

$$E(m) \text{ or } E(L_q) \text{ or } L_q = \frac{\rho^2\,[1 - N\rho^{N-1} + (N-1)\,\rho^N]}{(1-\rho)\,(1-\rho^{N+1})}$$

$$= \frac{(0.75)^2\,[1 - 9 \times (0.75)^8 + 8\,(0.75)^9]}{0.25\,(1-(0.75)^{10})}$$

$$= 2.22\,\frac{[1-0.303]}{[1-0.005]} = 1.55$$

Example 19: *The railway marshalling yard is sufficient only for trains (there being 10 lines, one of which earmarked for the shunting engine to reverse itself from the crest of the hump to the rear of train). Trains arrive at the rate 30 trains per day, inter-arrival time follows on exponential distribution and service time distribution is also exponential with an average of 36 minutes calculate the following:*

(i) The probability that the yard is empty

(ii) The average line length

Solution: Here

$$\lambda = 30 \text{ trains per day or } \frac{1}{48} \text{ trains per minute}$$

$$\mu = \frac{1}{36} \text{ trains per minute}$$

$$N = 9$$

$$\therefore \qquad \rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75$$

(i) $$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^{10}} = 0.28$$

(ii) $$L_q = P_0 \sum_{n=0}^{N} n\rho^n = 0.28 \times \sum_{n=0}^{9} n(0.75)^n$$

$$= 0.28 \times 9.58 = 2.68 \quad \text{or} \quad 3 \text{ trains}$$

6.10 Multi-Channel Queuing Theory Model III (M / M / C : ∞ / FCFS)

Multi-channel queuing theory treats the condition in which there are several service stations in parallel and each element in the waiting line can be served by more than one station. The arrival rate λ and service rate μ are mean values from Poisson distribution and exponential distribution respectively, service discipline is first come, first served and customers are taken for single queue

Let

n = number of customers in the system

P_n = probability of n customers in the system

c = number of parallel service channels ($c > 1$)

λ = arrival rate of customers

μ = service rate of individual channel

1. $$P_n = \begin{cases} \dfrac{1}{n!}\left(\dfrac{\lambda}{\mu}\right)^n P_0 & \text{for } 1 \le n < c-1 \\ \dfrac{1}{c^{n-c}\, c!}\left(\dfrac{\lambda}{\mu}\right)^n P_0 & \text{for } n \ge c \end{cases}$$

2. $$P_0 = \left[\sum_{n=0}^{c-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{c!}\left(\frac{\lambda}{\mu}\right)^c \frac{\mu c}{\mu c - \lambda}\right]^{-1}$$

3. Expected (average) number of customers in the system

$$L_s = \frac{\lambda \,.\, \mu \left(\dfrac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu - \lambda)^2} P_0 + \frac{\lambda}{\mu}$$

4. Expected (average) number of customers waiting in the queue

$$L_q = \frac{\lambda \mu \left(\dfrac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu - \lambda)^2} P_0$$

5. Average time a customer spends in the system

$$W_s = \frac{L_s}{\lambda} = \frac{\mu \left(\dfrac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu - \lambda)^2} P_0 + \frac{1}{\mu}$$

6. Average waiting time of a customer in the queue

$$W_q = \frac{L_q}{\lambda} = \frac{\mu \left(\dfrac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu - \lambda)^2} P_0$$

7. Probability that a customer has to wait

$$P\,(n \ge c) = \frac{\mu \left(\dfrac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu - \lambda)} P_0$$

8. Probability that a customer enters the service without waiting

$$1 - P\,(n \ge c) = 1 - \frac{\mu \left(\dfrac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu - \lambda)} P_0$$

9. Average number of idle servers

$= c -$ (average number of customers served)

10. Utilisation rate $\rho = \dfrac{\lambda}{c\mu}$

Example 20: *A tax consulting firm has 3 counters in its office to receive people who have problems concerning their income, wealth and sales taxes. On the average 48 persons arrive in an 8-hour day. Each tax adviser spends 15 minutes on an average on an arrival. If the arrivals are Poissonly distributed and service time are according to exponential distribution, find*

(i) The average number of customers in the system.

(ii) Average number of customers waiting to be served.

(iii) Average time a customers spends in the system.

(iv) Average waiting time for a customer.

(v) The number of hours each week a tax adviser spends performing his job.

(vi) The probability that a customer has to wait before he gets service.

Solution: It is $(M/M/C):(\infty\ FCFS)$ problem.

Here $$\lambda = \frac{48}{8} = 6 \text{ per hour, } c = 3$$

$$\mu = \frac{1}{15} \times 60 = 4 \text{ per hour}$$

$$\frac{\lambda}{\mu} = \frac{6}{4} = \frac{3}{2}$$

Now to find the probability P_0 which is the probability of having no customer in the system.

$$P_0 = \frac{1}{\displaystyle\sum_{n=0}^{c-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^c}{c\,!} \cdot \frac{c\mu}{c\mu - \lambda}}$$

$$= \frac{1}{\displaystyle\sum_{n=0}^{2} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^3}{3\,!} \times \frac{3\mu}{3\mu - \lambda}}$$

$$= \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{\left(\frac{\lambda}{\mu}\right)^3}{6} \frac{3\mu}{3\mu - \lambda}}$$

$$= \frac{1}{\left(1 + \frac{3}{2} + \frac{9}{8}\right) + \frac{27}{48} \cdot \frac{12}{12-6}} = \frac{1}{\frac{29}{8} + \frac{9}{8}} = \frac{8}{38} = \frac{4}{19}$$

$$= 0.21$$

(i) Average number of customers in the system.

$$L_s = \frac{\lambda \mu \left(\frac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu - \lambda)^2} P_0 + \frac{\lambda}{\mu}$$

$$= \frac{6 \times 4 \times \left(\frac{3}{2}\right)^3}{2!\,(12-6)} \times 0.021 + \frac{3}{2} = 1.74$$

(ii) Average number of customers waiting to be served.

$$L_q = L_s - \frac{\lambda}{\mu} = 1.74 - \frac{3}{2} = 0.24$$

(iii) Average time a customer spends in the system.

$$W_s = \frac{L_s}{\lambda} = \frac{1.74}{6} = 0.29 \text{ hour}$$

$$= 17.4 \text{ minutes.}$$

(iv) Average waiting time for a customer.

$$W_q = \frac{L_q}{\lambda} = \frac{0.24}{6} = 0.04 \text{ hour} = 2.4 \text{ minutes}$$

(v) Utilisation factor $\rho = \frac{\lambda}{c\mu} = \frac{6}{3 \times 4} = \frac{1}{2}$

Therefore number of hours each day a tax, adviser spends doing his job

$$= \frac{1}{2} \times 8 = 4 \text{ hours}$$

$\therefore$ on an average, a tax adviser is busy $4 \times 5 = 20$ hours based on 5-working day week.

(vi) Probability that a customer has to wait,

$$P(n > c) = \frac{\mu \left(\frac{\lambda}{\mu}\right)^c}{(c-1)!\,(\mu c - \lambda)} P_0$$

$$= \frac{4 \times \left(\frac{3}{2}\right)^3}{2!\,(12-6)} \times 0.21$$

$$= 0.236$$

Example 21: *A telephone exchange has two long distance operators. The telephone company finds that, during the peak load, long distance calls arrive in Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 minutes.*

(i) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?

(ii) If the subscribers will wait and serviced in turn, what is the expected waiting time?

[B.C.A. (Bhopal) 2006, 2008, 2012]

Solution: It is $(M/M/C):(\infty/FCFS)$ problem

Here, $$\lambda = \frac{15}{60} = \frac{1}{4} \text{ calls/minute}$$

$$\mu = \frac{1}{5} \text{ calls/minute, } c = 2$$

$$\rho = \frac{\lambda}{\mu c} = \frac{1/4}{(1/5) \times 2} = \frac{5}{8}$$

Now to find P_0 by

$$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!} \cdot \frac{\mu c}{c\mu - \lambda}}$$

$$= \frac{1}{\sum_{n=0}^{1} \frac{\left(\frac{5}{4}\right)^n}{n!} + \frac{\left(\frac{5}{4}\right)^2}{2!} \cdot \frac{\frac{1}{5} \times 2}{\left(2 \times \frac{1}{5} - \frac{1}{4}\right)}}$$

$$= \frac{1}{1 + \frac{\frac{5}{4}}{1!} + \frac{25 \times 8}{32 \times 3}} = \frac{3}{13}$$

(i) Probability that a subscriber will have to wait for his long distance call

$$= \text{Probability } (n \geq 2) = \sum_{n=2}^{\infty} P_n$$

$$= \sum_{n=0}^{\infty} P_n - \sum_{n=0}^{1} P_n = 1 - P_0 - P_1$$

$$= 1 - \frac{3}{13} - \left[\frac{1}{1!}\left(\frac{5}{4}\right)^1 P_0\right], \qquad \left[P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, n \leq c\right]$$

$$= 1 - \frac{3}{13} - \frac{5}{4} \times \frac{3}{13}$$

$$= \frac{25}{52}$$

$$= 0.48$$

(ii) Expected waiting time $= w_q$

$$= \frac{L_q}{\lambda}$$

$$= \frac{\mu\left(\frac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu-\lambda)^2} P_0$$

$$= \frac{\frac{1}{5}\left(\frac{5}{4}\right)^2}{1!\left(2\times\frac{1}{5}-\frac{1}{4}\right)^2}$$

$$= \frac{125}{39} = 3.2 \text{ minutes.}$$

Example 22: *Four counters are being run on the frontier of a country to check the passports and necessary papers of the tourists. The tourists choose counter at random. If the arrivals at frontier is Poisson at the rate λ and the service time is exponential with parameter $\frac{\lambda}{2}$, what is the steady-stage average queue at each counter?* **[B.C.A. (Indore) 2010; B.B.A. (Meerut) 2005]**

Solution: This problem is of $(M/M/C):(\infty/FCFS)$ model

we have $\lambda = \lambda,\ \mu = \frac{\lambda}{2},\ c = 4,\ \rho = \frac{\lambda}{\mu c} = \frac{1}{2}$

$$P_0 = \frac{1}{\sum_{n=0}^{c-1}\frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!}\cdot\frac{\mu c}{\mu c-\lambda}}$$

$$= \frac{1}{\sum_{n=0}^{3}\frac{(2)^n}{n!} + \frac{(2)^4}{4!}\,\frac{\frac{\lambda}{2}\times 4}{\frac{\lambda}{2}\times 4-\lambda}}$$

$$= \frac{1}{1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{4}{3}} = \frac{3}{23}$$

Average queue length $= L_q$

$$= \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^c}{(c-1)!\,(c\mu-\lambda)^2} P_0$$

$$= \frac{\lambda \times \frac{\lambda}{2}(2)^4}{3!\left(4 \times \frac{\lambda}{2} - \lambda\right)^2} \times \frac{3}{23}$$

$$= \frac{\lambda^2 \times 2^3}{6 \times (\lambda^2)} \times \frac{3}{23} = \frac{4}{23}$$

Example 23: *A super market has two girls ringing sales at the counters. If the service time for each customer is exponential with mean 4 minutes, and if people arrive in a Poisson fashion at the counter at the rate of 10 an hour:*

(i) What is the probability of having to wait for service?

[B.C.A. (I.G.N.O.U.) 2008; B.C.A. (Agra) 2008]

(ii) What is the expected percentage of idle time for each girl?

Solution: This problem is base on $(M/M/C):(\infty/FCFS)$ model.

Here, $$\lambda = \frac{10}{60} = \frac{1}{6} \text{ people/minute}$$

$$\mu = \frac{1}{4} \text{ people/minute}$$

$$\rho = \frac{\lambda}{\mu c} = \frac{1}{3}, c = 2$$

Now we have to find P_0

$$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!} \times \frac{\mu c}{(\mu c - \lambda)}}$$

or $$P_0 = \frac{1}{\sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!(1-\rho)}}$$

$$= \frac{1}{\left[\sum_{n=0}^{1} \frac{\left(\frac{2}{3}\right)^n}{n!} + \frac{\left(\frac{2}{3}\right)^2}{2!\left(1 - \frac{1}{3}\right)}\right]}$$

$$= \frac{1}{1 + \frac{(2/3)}{1!} + \frac{1}{3}\Big]} = \frac{1}{2}$$

(i) Probability to having to wait for service.

$$= P\ (n \geq 2) = \sum_{n=2}^{\infty} P_n = 1 - P_0 - P_1$$

$$= 1 - \frac{1}{2} - \frac{(2/3)}{1!} \times P_0 \qquad \left[P_n = \frac{(c\rho)^n}{n!} P_0,\ s \leq n \right]$$

$$= \frac{1}{2} - \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6} = 0.167$$

(ii) Fraction of time the service remains busy $= \rho$ (traffic intensity) $= \frac{1}{3}$

The fraction of the time the service remains idle

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

i.e. expected percentage of idle time for each girl

$$= \frac{2}{3} \times 100\% = 67\%$$

6.11 Model IV $(M / E_k / 1) : (\infty / FCFS)$

This is a queuing model with Poisson arrivals, Erlang service time with k phase, single channel, first-come-first served discipline and infinite production. In this model one unit is served in k phases. The arrival or departure of one unit, therefore, means an increase or decrease of k phases in the system and the completion of service of one phase of a unit will mean the decrease of one phase in the system.

If at any time there are m units waiting in the queue with one unit in service which has to wait pass through s phase, then the total number of phases n in the system at that time will be given by

$$\boxed{n = nk + s}$$

Now, μ = the number of units served per unit time.

μk = the number of phases served per unit time.

$\therefore \qquad \lambda_n = \lambda,\ \mu_n = \mu$

The characteristic properties of this model are as follows:

1. Average number of units in the system

$$= L_s = \frac{(k+1)}{2k}\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu-\lambda}\right)+\frac{\lambda}{\mu}$$

2. Average number of units in the queue

$$L_q = \frac{(k+1)}{2k}\cdot\left(\frac{\lambda}{\mu}\right)\cdot\left(\frac{\lambda}{\mu-\lambda}\right)$$

3. Average time spend by a unit in the system

$$W_s = \frac{(k+1)}{2k}\cdot\frac{\lambda}{\mu(\mu-\lambda)}+\frac{1}{\mu}$$

4. Average waiting time of a unit in the queue

$$W_q = \frac{(k+1)}{2k}\cdot\frac{\lambda}{\mu(\mu-\lambda)}$$

5. When $k \to \infty$ then

$$L_s = \frac{1}{2}\cdot\frac{\lambda^2}{\mu(\mu-\lambda)}+\frac{\lambda}{\mu}$$

$$L_q = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)\cdot\frac{\lambda}{(\mu-\lambda)}$$

$$W_s = \frac{1}{2}\cdot\frac{\lambda}{\mu(\mu-\lambda)}+\frac{1}{\mu}$$

$$W_q = \frac{1}{2\mu}\cdot\frac{\lambda}{(\mu-\lambda)}$$

When $k = 1$ then the Erlang service time distribution reduces to exponential distribution and values of L_s, L_q, W_s and W_q are same as Model I.

6. Probability that there are no unit in the system

$$P_0 = (1-\rho k)$$

where $\rho = \frac{\lambda}{\mu}, k = \text{phase}$

Example 24: *In a factory cafeteria the customers have to pass through three counters. The customers busy coupons at the first counter, select and collect the snacks at the second counter and collect tea at the third. The server at each counter takes on an average* 1.5 *minutes although the distribution of service time is approximately exponential. If the arrival of customers to the cafeteria is approximately Poisson at an average rate of* 6 *per hour, calculate*

(i) The average time a customer spends waiting in the cafeteria.

(ii) The average time of getting the service.

(iii) The most probable time in getting the service.

Solution: This problem is of $(M / E_k / 1) : (\infty / FCFS)$ model.

Here, number of phase $k = 3$

Service time per phase $= 1.5$ minutes

$\therefore$ Service time per customer $= 1.5 \times 3 = 4.5$ minutes

$\because$ $$\mu = \frac{1}{4.5} \text{ customers/minute} = \frac{40}{3} \text{ customer/hours}$$

$$\lambda = 6 \text{ customers/hour}$$

(i) Average waiting time

$$W_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu}\right)\left(\frac{1}{\mu - \lambda}\right)$$

$$= \left(\frac{3+1}{2 \times 3}\right)\left(\frac{6 \times 3}{40}\right)\left(\frac{1}{\frac{40}{3} - 6}\right)$$

$$= \frac{9}{220} \text{ hours} = \frac{27}{11} = 2.45 \text{ minutes}$$

(ii) Average time of getting the service *i.e.* in collecting coupons, snacks etc is the mean of t when it is following the IIIrd member of Erlang family.

$\therefore$ $$\text{Average time spent} = \frac{1}{\mu} = \frac{3}{40} \text{ hours} = 4.5 \text{ minutes}$$

(iii) The most probable time spend in getting the service is the model value of t for the IIIrd member of Erlang family.

$\therefore$ $$\text{Most probable time spend} = \frac{k-1}{\mu k}$$

$$= \frac{3-1}{\frac{40}{3} \times 3}$$

$$= \frac{1}{20} \text{ hour} = 3 \text{ minutes.}$$

Example 25: *A hospital clinic has a doctor examining patients brought in for a general check up. The doctor averages 4 minutes on each phase of the check up although the distribution of time spent on each phase is approximately exponential. If each patient goes through four phases in the check up and if the arrivals of the patients is the doctors office are approximately Poisson at an average rate of 3 per hour, what is the average time spent by a patient waiting in the doctor's office? What is the average time spent in the examination? What is the most probable time spent in the examination?*

[B.C.A. (Kurukshetra) 2006]

Solution: This problem is base on $(M / E_k / 1) : (\infty / FCFS)$ model.

Here, we have

$$\lambda = \text{mean arrival rate} = \frac{3}{60} = \frac{1}{20} \text{ patients/minutes}$$

$$k = \text{number of phase} = 4$$

$$\mu = \text{mean service time per phase} = 4 \times 4 = 16 \text{ minutes}$$

(i) Average time spent by a patient waiting in the doctor's office

$$W_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right) = \frac{4+1}{8}\left(\frac{1/20}{\frac{1}{16}\left(\frac{1}{16}-\frac{1}{20}\right)}\right) = 40 \text{ minutes}$$

(ii) Average time spent in the examination

$$= \frac{1}{\mu} = 16 \text{ minutes}$$

(iii) Most probable time spent in the examination

$$= \frac{k-1}{\mu k} = \frac{4-1}{\left(\frac{1}{16}\right)\times 4} = 12 \text{ minutes}$$

Example 26: *Repairing a certain type of machine which breaks down in a given factory consists of 5 basic steps that must be performed sequentially. The time taken to perform each of the 5 steps is found to have an exponential distribution with mean 5 minutes and is independent of other steps. If these machines breakdown in a Poisson fashion at an average rate of two per hour and if there is only one repairman, what is the average idle time for each machine that has broken down?*

Solution: This problem is base on $(M / E_k / 1) : (\infty / FCFS)$ model

Here, we have

$$k = \text{number of phase} = 5$$

Service time per phase $= 5 \times 5 = 25$ minutes

$$\because \quad \mu = \frac{1}{25} \text{ minutes/minute} = \frac{12}{5} \text{ units/hour}$$

$$\lambda = 2 \text{ units/hour}$$

Average ideal time of the machine = Average time spent by the machine in the system

$$W_s = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu}\right)\left(\frac{1}{\mu-\lambda}\right) + \frac{1}{\mu}$$

$$W_s = \frac{5+1}{2\times 5}\left(\frac{2}{\frac{12}{5}}\right)\left(\frac{1}{\frac{12}{5}-2}\right) + \frac{1}{\frac{12}{5}} = \frac{5}{3} \text{ hours}$$

$$= 100 \text{ minutes.}$$

Example 27: *A colliery working one shift per day uses a large number of locomotives which breakdown at random intervals, an average one failing per 8-hour shift. The fitter carries out a standard maintenance schedule on each faulty locomotive. Each of the 5 main parts of this schedule takes an average of $\frac{1}{2}$ hour but the time varies widely. How much time will be fitter have for other tasks and what is the average time a locomotive is out of service?*

Solution: This problem is base on $(M / E_k / 1) : (\infty / FCFS)$ model.

Here, we have

$$k = 5, \lambda = \frac{1}{8} \text{ per hours}$$

$$\text{Service time per part} = \frac{1}{2} \text{ hour}$$

$$\therefore \quad \text{Service time per locomotive} = \frac{5}{2} \text{ hours}$$

$$\therefore \quad \mu = \frac{2}{5} \text{ per hours}$$

Fraction of time the fitter will have for other tasks

= Fraction of time for which the fitter is idle

$$= 1 - \frac{\lambda}{\mu} = 1 - \frac{1/8}{2/5} = 1 - \frac{5}{16} = \frac{11}{16}$$

$$\therefore \quad \text{Time the fitter will have for other tasks in a day} = \frac{11}{16} \times 8 = 5.5$$

Average time a locomotive is out of service

= Average time spent by the locomotive in the system

$$= \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu}\right)\left(\frac{1}{\mu - \lambda}\right) + \frac{1}{\mu} = \frac{5+1}{2 \times 5} \times \frac{(1/8)}{(2/5)} \cdot \left(\frac{1}{\frac{2}{5} - \frac{1}{8}}\right) + 5/2$$

$$= \frac{6}{10} \times \frac{5}{16} \times \frac{40}{11} + \frac{5}{2} = 3.18 \text{ hours}$$

Example 28: *A barber with one-man taken exactly 25 minutes to complete one hair cut. If customers arrive in a Poisson fashion at an average rate of one every 40 minutes, how long on the average must a customer wait for service?* **[B.B.A. (Rohilkhand) 2007]**

Solution: This problem is based on $(M / E_k / 1) : (\infty / FCFS)$ model.

Here, we have

$$\lambda = \text{mean arrival rate} = \frac{1}{40} \text{ customers/minutes}$$

Since service rate is constant then $k = \infty$

$$\mu = \text{mean service rate} = \frac{1}{25} \text{ customers/minute}$$

Average waiting time of a customer in queue

$$W_q = \lim_{k \to \infty} \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right)$$

$$= \frac{\lambda}{2\mu(\mu-\lambda)}$$

$$= \frac{1/40}{2 \times \frac{1}{25}\left(\frac{1}{2} - \frac{1}{40}\right)} = \frac{126}{6} = 20.8 \text{ minutes.}$$

Example 29: *In a car manufacturing plant, a loading crane takes exactly* 10 *minutes to load a car into wagon and again come back to the position of loading another car. If the arrivals of a cars is a Poisson stream at an average of one every* 20 *minutes. Calculate the average waiting time of a car.*

Solution: Here, we have $k = \infty, \lambda = 3, \mu = 6$

Average waiting time of a car is given by

$$W_q = \frac{1}{2}\frac{\lambda}{\mu} \cdot \frac{1}{\mu - \lambda} = \frac{1}{2} \times \frac{3}{6} \times \frac{1}{6-3}$$

$$= \frac{1}{12} \text{ hour} = 5 \text{ minutes.}$$

Example 30: *At a certain airport it takes exactly* 5 *minutes to land an air plane, once it is given the signal to land. Although in coming planes have scheduled, arrival times, the wide variability in arrival times products an effect which makes the incoming planes appear to arrive in a Poisson fashion at an average rate of six per hour. This produces occasional stock-ups at the airport which can be dangerous and costly. Under these circumstances, how much time will a pilot expect to spent circling the field waiting to land?*

Solution: This problem is base on $(M / E_k / 1) : (\infty / FCFS)$ model.

Here, we have

$$\lambda = \text{mean arrival rate} = \frac{6}{60} = \frac{1}{10} \text{ airplanes/minute}$$

$$\mu = \text{mean service rate} = \frac{1}{5} \text{ airplanes/minutes}$$

Since service rate is constant then $k = \infty$

$$\text{Average waiting time } (W_q) = \frac{\lambda}{2\mu(\mu-\lambda)}$$

$$= \frac{1/10}{2 \times \frac{1}{5}\left(\frac{1}{5} - \frac{1}{10}\right)} = 2.5 \text{ minutes.}$$

Problem Set

1. Explain, what you mean by a Poisson process ?

[B.C.A. (Avadh) 2008]

2. Give some important properties of the Poisson process.

3. Write short note on queuing models.

[B.B.A. (Meerut) 2002, 2003, 2004]

4. Give the essential characteristics of queuing process.

[B.B.A. (Meerut) 2005]

5. In a public telephone booth the arrivals are on the average 15 per hour. A call on the average takes 3 minutes. If there is just one phone.
   (i) What is the expected number of callers in the booth at any time?
   (ii) For what proportion of time is the booth expected to be idle?

[B.C.A. (Lucknow) 2005]

6. Customers arrive at a box office window, being manned by single individual, according to a Poisson input process with a mean rate of 20 per hour. The time required to serve a customer per on exponential distribution with mean of 90 seconds find the average waiting time of a customer. Also determine the average number of customers in the system and average queue length.

   [Hint: $\lambda = 0.5, \mu = 0.67$ then used W_q, L_q and L_s]

7. At a one-man barber shop, customers arrive according to Poisson distribution with a mean arrival rate of 5 per hour and his hair cutting time was exponentially distributed with an average hair cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following:
   (i) Average number of customers in the shop and the average number of customers waiting for a hair cut.
   (ii) The percentage of time an arrival can walk right in without having to wait.
   (iii) The percentage of customers who have to wait prior to getting into the barber's chair.

[B.C.A. (Rohilkhand) 2006]

[Hint: $\lambda = 1/2$ customers/minutes, $\mu = \frac{1}{10}$ customers per minutes $\rho = 5/6$

(i) $L_s = \dfrac{\lambda}{\mu - \lambda} = 5,\qquad L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \dfrac{26}{5}$

(ii) $P_0 = 1 - \rho = \dfrac{1}{6}$ Required % $= \dfrac{1}{6} \times 100 = 16.7\%$

(iii) $P(n \geq 0) = 1 - P_0 = 1 - \dfrac{1}{6} = \dfrac{5}{6} = .833$

Required % $= 83.3\%$]

8. Customers arrive at a first class ticket counter of a theater in a Poisson distributed arrival rate of 25 per hour. Service time is constant at 2 minutes. Calculate
   (i) the mean number in the waiting line.
   (ii) the mean waiting time.

   [Hint: $\lambda = \frac{25}{60} = \frac{5}{12}$ customers per minutes, $\mu = \frac{1}{2}$ cost/min, $\rho = \frac{5}{6}$

   (i) $L_s = 5$ customers, $L_q = \frac{25}{6} = 4$ customers

   (ii) $W_q = \frac{\lambda}{\mu(\mu - \lambda)} = 10$ min, $W_s = \frac{\lambda}{\mu - \lambda} = 12$ min]

9. In a bank, cheques are cashed at a single teller counter. Customers arrive at the counter in a Poisson manner at an average rate of 30 customers per hour. The teller takes, on an average a minute and a half to cash cheque. The service time has been shown to be exponentially distributed.
   (i) Calculate the percentage of time the teller is busy.
   (ii) Calculate the average time a customer is expected to wait.

   [Hint: $\lambda = 30$ cus/h, $\mu = 40$ cus/h

   (i) $1 - P_0 = \rho$ (ii) $W_s = \frac{1}{\mu - \lambda}$]

10. Data have been accumulated at a banking facility regarding a waiting time for delivery trucks to be loaded. The data show that the average arrival rate for the trucks at the loading dock is 2/hour. The average time to load a truck, using 2 loaders is 10 minutes so that the service rate is 3 trucks per hour. The management is considering hiring another loader at ₹ 5 per hour to reduce the loading time. Drivers are paids ₹ 4 per hour and truck utilisation is valued at ₹ 3 per hour. Should the additional loader be hired if an increase in the service rate to 4 trucks per hour would result?

    [Hint: Without additional loader $L_s = \frac{\lambda}{\mu - \lambda} = \frac{2}{3-2} = 2$

    ∴ cost/hour = ₹ 2 (3 + 4) = ₹ 14

    with additional loader $L_s = \frac{2}{4-2} = 1$

    ∴ cost/hour = ₹ [1 (3 + 4) + 5] = ₹ = 12]

11. In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follow as exponential distribution and the service time distribution is also exponential with an average 36 minutes, calculate the following:
    (i) Average number of customers in the system.
    (ii) The probability that the queue size exceeds 10.
    (iii) If the input of trains increase to an average 33 per day. What will change in (i) and (ii)?

12. The tool room company's quality control department is manned by a single clerk who takes an average of 5 minutes in checking parts of each of the machine coming for inspection. The machines arrive once in every 8 minutes on the average. One hour of the machine is valued at ₹ 15 and a clerk's time is valued at ₹ 4 per hour. What are the average hourly queuing system costs associated with the quality control department?

[B.C.A. (Kashi) 2007]

[**Hint:** $\lambda = \frac{60}{8} = 7.5/\text{hr}, \mu = \frac{60}{2} = 12/\text{hr}$

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{12 - 7.5} = \frac{2}{9}\text{ hr}$$

$$\text{Average queuing cost/machine} = ₹\left(15 \times \frac{2}{9}\right) = ₹\ \frac{10}{3}$$

$$\therefore \quad \text{Average queuing cost/hr} = ₹\ \frac{10}{3} \times 7.5 = ₹\ 25$$

$$\text{Average cost of a clerk/hr} = ₹\ 4$$

$\therefore$ Total cost for the department/hr = ₹ 29]

13. Cars arrive at a toll gate on a frequency according to Poisson distribution with mean 90/hour. Average time for passing through the gate is 38 seconds. Drivers complain of long waiting time. Authorities are willing to decrease the passing time through the gate to 30 seconds by introducing new automatic devices. This can be justified only if under the old system, the number of waiting cars exceeds 5. In addition, the percentage of the gate's idle time under the new system should not exceed 10% can the new device be justified?

[B.C.A. (Kanpur) 2004]

[**Hint:** $\lambda = 90/\text{hr} = 1.5$ minutes, $\quad \mu = \frac{60}{38} = \frac{30}{19}$ per minutes

$$\therefore \quad \lambda_q = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu - \lambda} = \frac{1.5 \times 19}{30} \times \frac{15}{\frac{30}{19} - 1.5} = 18\ (> 5)$$

$$P_0 = 1 - \frac{\lambda}{\mu'} = 1 - \frac{15}{\frac{60}{30}} = 0.25 > 0.11 \Bigg]$$

14. Problems arrive at computer centre in a Poisson fashion at an average rate of 5/day. The rules of the computer centre are that any man waiting to get his problem solved must aid the man whose problem is being solved. If the time to solve a problem with one man has an exponential distribution with mean time of $\frac{1}{3}$ day and if the average solving time is inversely proportional to the number of people working on the problem, find the expected time for person entering the line.

[B.B.A. (Meerut) 2011]

[**Hint:** $L_s = \sum_{n=0}^{\infty} n\, P_n = \sum_{n=0}^{\infty} n \frac{\rho^n}{n!} e^{-\rho} = e^{-\rho}\, \rho \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!}$

$$= e^{-\rho}\,\rho\, e^{\rho} = \rho = 5/3,\; W_s = \frac{5}{3} \times \frac{1}{5} = \frac{1}{3} \text{ day} = 8 \text{ hrs}\Big]$$

15. A super marks has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean 4 minutes and if people arrive in a Poisson fashion at the rate of 10 per hours

(i) What is the probability of having to wait for the service?

(ii) What is the expected percentage of idle time for each girl?

(iii) Find the average queue length and the average number of units in the system.

[Hint: $\rho = \frac{\lambda}{c\mu} = \frac{10}{2 \times 5} = 1/3,$

Fraction of time the service channels remain idle $= 1 - \rho = \frac{2}{3}$

$\therefore$ Expected percentage of idle time for each girl $= \frac{2}{3} \times 100 = 67\%$ **]**

16. A warehouse in a small state receives orders for a certain item and sends them by a truck as soon as possible to the customer. The orders arrive in a Poisson fashion at a mean rate of 0.9 per day. Only one item at a time can be shipped by a truck from the warehouse, which is located in the control part of a state. Because the customers are located in various places in the state, the distribution of service time in days has a distribution with probability density $4te^{-2t}$. What is the expected delay between the arrival of an order and the arrival of the item to the customer? Service time here implies the time the truck takes to load, gets to the customer, unloads and returns to the warehouse loading and unloading times are small as compared to the travel time. **[B.C.A. (I.G.N.O.U.) 2007]**

[Hint: $\mu = 1, \lambda = 0.9, k = 2, w_q = \frac{k+1}{2k} \frac{\lambda}{\mu(\mu - \lambda)} = 6.75$ days

$W_s = \frac{k+1}{2k} \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu} = 7.75$ days **]**

17. In a Bhawan cafeteria it was observed that there is only one bearer who takes exactly 4 minutes to serve a cup of coffee once the order has been placed with him. If the students arrive in the cafeteria at an average rate 10 per hour, how must time one is expected to spend waiting for his tern to place the order?

18. A petrol pump station has two pumps. The service time follow the exponential distribution with a mean of 4 minutes and cars arrive for service in a Poisson process at the rate 10% per hour. Find the probability that a customer have to wait for service. What proportion of time the pumps remain idle?

19. A barber with one man shop takes exactly 40 minutes to complete one hair cut. If the inter-arrival time of the customer follows in exponential distribution with average one every 50 minutes, how long a customer must wait for service?

20. Explain the different types of queuing models. Why must the service rate be greater than the arrival rate in a single channel queuing system? **[B.B.A. (Meerut) 2008]**

21. Discuss $M/M/1$ queue system with its important properties.

[B.B.A. (Meerut) 2002, 2004, 2005, 2006, 2008]

22. Discuss the basics of a queuing model.

[B.B.A. (Meerut) 2006]

23. Explain the objectives of a queuing model. Explain properties of $M/M/1:(FCFS)$ system.

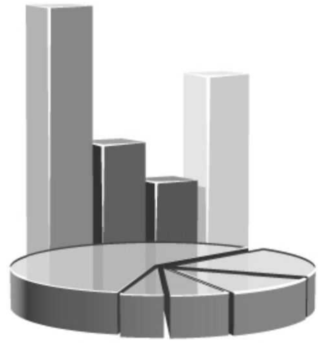
[B.B.A. (Meerut) 2002, 2003, 2011]

Answers

5.	(i) 3 (ii) $\frac{1}{4}$
6.	$W_q = 4.5$ min/customer, $L_q = 2.25$, $L_s = 3$
7.	(i) 5 customers in shop and $4\frac{1}{6}$ customers waiting for hair cut (ii) 16.7% (iii) 83.3%
8.	(i) $L_s = 5$ customers, $L_q = 4$ customers (ii) $W_q = 10$ min, $W_s = 12$ minutes
9.	(i) 75 % (ii) 6 minutes
10.	₹ 2 per hour saving with new loader
11.	(i) 3 (ii) $(0.75)^{10} = 0.06$ (iii) 4.8, $(0.83)^{10} = 0.2$
12.	$W_s = 2/9$ hr, Total cost for deptt/hr = ₹ 29
13.	$L_q = 18$, $P_0 = 0.25\ (> 0.10)$
14.	$L_s = 5/3$, $W_s = 8$ hours
15.	(i) 0.167 (ii) 67%
16.	$W_q = 6.75$ days $W_s = 7.75$ days.
17.	4 minutes.
18.	0.167, 67% for each pump.
19.	1 hour and 20 minutes.

❍❍❍

CHAPTER 7



Replacement Problems

7.1 The Replacement Problems

[B.C.A. (Bhopal) 2008, 2012; B.C.A. (Lucknow) 2011; B.C.A. (Agra) 2004,2012]

The efficiency of all industrial and military equipments deteriorate with time. Some times the equipment fails completely and effects the whole system. For example, a machine requires higher operating cost, a transport vehicle such as car or air plane requires more and more maintenance cost. The ever increasing repair and maintenance cost necessitates the replacement of equipment.

The replacement problems are concerned with situations that arise when some items such as men, car, truck, air plane etc need replacement due to their decreased efficiency or working capacities.

There are four type of replacement problem

1. Replacement of old items has become in bad condition and their working efficiency decrease with time or require expensive maintenance.
2. Replacement of items when the system is completely fail due to accident or otherwise.
3. Problem in mortality or staffing.
4. Replacement of an equipment, when a better or more efficient design of machine or equipment has become available in the market. For example, an equipment may have an economic life of 20 years, yet it may become obsolete after 10 year because of better technical developments.

7.2 Replacement of Items that Deteriorate with Time

It is found that repair and maintenance costs of items increase with time and a stage may become when these costs become so high that it is more economical to replace the item by a new one. Thus, the problem of replacement in this case is to find the best time at which the old machine should be replaced by new one.

Now, we shall consider a few cases of items that deteriorate with time and it will be assumed that suitable expressions for maintenance costs are available.
To find the best replacement age (time) of a machine when

1. Its maintenance cost is given by a function increase with time.
2. Its scrap value is constant and
3. The money value is not considered.

7.2.1 Theorem

The cost of maintenance of a machine is given as a function increasing with time and its scrap value is constant.

1. **If time is measured continuously, then the average annual cost will be minimized by replacing the machine when the average cost becomes equal to the maintenance cost.**
2. **If time is measured in discrete units, then the average annual cost will be minimized by replacing the machine when the next periods, maintenance cost become greater than the current average cost.**

[B.C.A.(Kanpur) 2006; B.C.A. (Rohilkhand) 2011]

Proof: 1. When time 't' is a continuous variables.

Let C = Capital cost of the item or cost of the machine

S = Scrap value of the item

$C_m(t)$ = Maintenance cost of the machine at the time

$A(n)$ = Average annual total cost of the item

n = Number of years the item is to be in use

Annual cost of the item at any time

= capital cost – scrap value + maintenance cost at time t

Now total maintenance cost incurred during n years = $\int_0^n C_m(t)\,dt$

Total cost incurred during n years = $C - S + \int_0^n C_m(t)\,dt$

The average cost of the item per year during n year is given by

$$A(n) = \frac{1}{n}\left[C - S + \int_0^n C_m(t)\,dt\right]$$

or $$A(n) = \frac{C-S}{n} + \frac{1}{n}\int_0^n C_m(t)\,dt \qquad ...(1)$$

Now we shall find that value of n for which $A(n)$ is minimum, differentiating eq. (1) w.r.t.n, we find

$$\frac{d}{dn}(A(n)) = \frac{-(C-S)}{n^2} - \frac{1}{n^2}\int_0^n C_m(t)\,dt + \frac{1}{n} C_m(t)$$

For $$\frac{d}{dn} A(n) = 0, \text{ we have}$$

$$C_m(t) = \frac{(C-S) + \int_0^n C_m(t)\,dt}{n} = A(n) \text{ by } (1)$$

i.e., Maintenance cost of item at time t = Average cost of the item per year

Also $$\frac{d^2 A(n)}{dn^2} = \frac{2(C-S)}{n^3} + \frac{2}{n^3}\int_0^n C_m(t)\,dt > 0$$

Thus, the item should be replace when the average annual cost to date becomes equal to the current maintenance cost.

2. When time 't' is a discrete variable

In this case, the total maintenance cost of the machine during n years

$$= \sum_{m=1}^{n} C_m(t) \qquad ...(1)$$

$\therefore$ Average annual cost of item during n years is given by

$$A(n) = \frac{C - S + \sum_{m=1}^{n} C_m(t)}{n} \qquad ...(2)$$

But $A(n)$ is minimum for that value of n for which

$$\Delta A(n-1) < 0 < \Delta A(n)$$

Now $$\Delta A(n) = A(n+1) - A(n) \qquad ...(3)$$

$$= \left[\frac{C-S}{n+1} + \frac{1}{n+1}\sum_{m=1}^{n+1} C_m(t)\right] - \left[\frac{C-S}{n} + \frac{1}{n}\sum_{m=1}^{n} C_m(t)\right]$$

$$= (C-S)\left(\frac{1}{n+1} - \frac{1}{n}\right) + \left(\frac{1}{n+1}\right)\left[\sum_{m=1}^{n} C_m(t) + C_{n+1}(t)\right] - \frac{1}{n}\sum_{m=1}^{n} C_m(t)$$

$$= -\frac{(C-S)}{n(n+1)} + \left(\frac{1}{n+1} - \frac{1}{n}\right)\sum_{m=1}^{n} C_m(t) + \frac{1}{n+1} C_{n+1}(t)$$

$$= -\left[\frac{(C-S) + \sum_{m=1}^{n} C_m(t)}{n(n+1)}\right] + \frac{1}{n+1} C_{n+1}(t)$$

$$= -\frac{A(n)}{n+1} + \frac{1}{n+1} C_{n+1}(t) \qquad \text{(from 2)}$$

Similarly, $$\Delta A(n-1) = -\frac{A(n-1)}{n} + \frac{1}{n} C_n(t)$$

$\therefore$ $A(n)$ is minimum for value of n for which

$$\Delta A(n) > 0 \quad \text{and} \quad \Delta A(n-1) < 0$$

i.e., $$-\frac{A(n)}{n+1} + \frac{1}{n+1} C_{n+1}(t) > 0$$

and $$-\frac{A(n-1)}{n} + \frac{1}{n} C_n(t) < 0$$

or $$C_{n+1}(t) > A(n) \ \text{and} \ C_n(t) < A(n-1) \qquad ...(4)$$

Hence, we can say that

1. Do not replace if the next years maintenance cost is less than the previous years average total cost.
2. Replace if the next years maintenance cost is greater than the previous years average total cost.

This completes the proof.

Solved Examples

Example 1: *The cost of a machine in* ₹ 6100 *and its scrap value (resale value) is only* ₹ 100. *The maintenance costs are found from experience to be under*

Year	1	2	3	4	5	6	7	8
Maintenance cost in ₹	100	250	400	600	900	1250	1600	2000

when should the machine be replaced?

[B.C.A. (Agra) 2004; B.C.A. (Lucknow) 2006; B.B.A. (Rohilkhand) 2006; B.C.A. (Bhopal) 2008, 2009, 2011]

Solution: Given, C = ₹ 6100 and S = ₹ 100

Optimum replacement period is determine as below:

Year (n)	Maintenance cost $C_m(t)$	Total Maintenance cost or cumulative cost $\Sigma\, C_m\,(t)$	$C-S$	Total cost $= C - S + \Sigma\, C_m\,(t)$	Average cost $A\,(n) = \left(\frac{5}{n}\right)$
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6)
1	100	100	6000	6100	6100.00
2	250	350	6000	6350	3175.00
3	400	750	6000	6750	2250.00
4	600	1350	6000	7350	1837.50
5	900	2250	6000	8250	1650.00
6	1250	3500	6000	9500	1583.33←
7	1600	5100	6000	11100	1585.71
8	2000	7100	6000	13100	1637.50

This table shows that the value $A\,(n)$ during the 6^{th} year is minimum. Hence, the machine should be replaced after every 6^{th} year.

Example 2: *The cost of a machine is ₹ 6100 and its scrap value is ₹ 100. The maintenance costs found from experience are as follows:*

Year	1	2	3	4	5	6	7	8
Maintenance cost (in ₹)	100	250	400	600	900	1200	1600	2000

When should the machine be replaced ? **[B.C.A. (Indore) 2010; B.B.A. (Delhi) 2008, 2009]**

Solution: Since the scrap value of the machine is ₹ 100. Then resale value of machine after one year become constant throughout.

The optimum replacement period is determine as below:

Year (n)	Maintenance cost $C_m\,(t)$	Total Maintenance cost or cumulative cost $\Sigma\, C_m\,(t)$	$C-S$	Total cost $= C - S + \Sigma\, C_m\,(t)$	Average cost $A\,(n) = \left(\frac{5}{n}\right)$
(1)	(2)	(3)	(4)	(5) = (3) + 4	(6)
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837
5	900	2250	6000	8250	1650
6	1200	3450	6000	9450	1575 ←
7	1600	5050	6000	11050	1579
8	2000	7050	6000	13050	1631

This table shows that the value $A(n)$ during the 6^{th} year is minimum. Hence, the machine should be replaced after every 6^{th} year.

Example 3: *Following table give the running cost per year and resale price of a certain equipment whose purchase price is ₹ 5000*

Year	1	2	3	4	5	6	7	8
Running cost (in ₹)	1500	1600	1800	2100	2500	2900	3400	4000
Resale value (in ₹)	3500	2500	1700	1200	800	500	500	500

At what year is the replacement due ? **[B.B.A. (Meerut) 2007]**

Solution: Given $C =$ ₹ 5000

The optimum replacement period is determine as below:

Year (n) (1)	Running cost $C_m(t)$ (2)	Total running cost or cumulative cost $\Sigma C_m(t)$ (3)	Resale S	$C - S$ (4)	Total cost = $(C - S) + \Sigma C_m(t)$ (5) = (3) + (4)	Average cost $A(n) = \left(\frac{5}{n}\right)$ (6)
1	1500	1500	3500	1500	3000	3000
2	1600	3100	2500	2500	5600	2800
3	1800	4900	1700	3300	8200	2733.3
4	2100	7000	1200	3800	10800	2700 ←
5	2500	9500	800	4200	13700	2740
6	2900	12400	500	4500	16900	2816.6
7	3400	15800	500	4500	20300	2900
8	4000	19800	500	4500	24300	3037.5

This table shows that the value of $A(n)$ during the 4^{th} year is a minimum. Hence, the equipment should be replaced every 4^{th} year.

Example 4: *A fleet owner finds his past records that the costs per year of running a vehicle whose purchase price is ₹ 50,000, are as under:*

Year	1	2	3	4	5	6	7
Running cost (in ₹)	5000	6000	7000	9000	11500	16000	18000
Resale value (in ₹)	30000	15000	7500	3750	2000	2000	2000

Thereafter, running cost increases by ₹ 2000 but resale value remains constant at ₹ 2000.
At what age replacement due ? **[B.C.A. (Agra) 2003]**

Solution: Given $C = 50000$

Year (n)	Running cost C_m (t)	Total running cost or cumulative cost Σ C_m (t)	$C - S$	Total cost $= C - S$ $+ \Sigma$ C_m (t)	Average cost A (n) $= \left(\frac{5}{n}\right)$
(1)	(2)	(3)	(4)	(5) = (3) + (4)	(6)
1	5000	5000	20000	25000	25000
2	6000	11000	35000	46000	23000
3	7000	18000	42500	60500	20167
4	9000	27000	46250	73250	18312
5	11500	38500	48000	86500	17300
6	16000	54500	48000	102500	17083 ←
7	18000	72500	48000	120500	17214

This table shows that the value of A (n) during the 6^{th} year is a minimum. Hence, the vehicle should be replaced every 6^{th} year.

Example 5: *Fleet cars have increased their costs as they continue in service due to increased direct operating cost (gas and oil) and increased maintenance (repairs, tyres, batteries etc.) The initial cost is* ₹ 3,500 *and trade in value drops as time passes until it reaches a constant value of* ₹ 500. *Given cost of operating, maintaining and trade in value determine the proper length of service before cars should be replaced.*

Years of service	1	2	3	4	5
Year end trade in value	1900	1050	600	500	500
Annual operating cost	1500	1800	2100	2400	2700
Annual maintaining cost	300	400	600	800	1000

[B.B.A. (Garhwal) 2008]

Solution: Given C = ₹ 3500

S = year end trade in value

$C_m(t)$ = operating cost + maintenance cost

The optimum replacement period is determined in the table below:

Year (n)	C_m (t)	Σ C_m (t)	($C - S$)	Total cost = ($C - S$) + Σ C_m (t)	Average cost A (n) $= \left(\frac{5}{n}\right)$
(1)	(2)	(3)	(4)	(5) = (2) + (3)	(6)
1	1800	1800	1600	3400	3400
2	2200	4000	2450	6450	3225
3	2700	6700	2900	9600	3200 ←
4	3200	9900	3000	12,900	3225
5	3700	13600	3000	16, 600	3320

This table shows that the value of A (n) during the third year is minimum. Hence, the car should be replaced every third year.

Example 6: *(i) Machine A cost ₹ 9000. Annual operating costs are ₹ 200 for the first year and then increase by ₹ 2000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed. What will be the average yearly cost of owning and operating the machine ?*

(ii) Machine B costs ₹10000. Annual operating costs are ₹ 400 for the first year and then increase by ₹. 800 every year. You now have a machine of type A which is one year old. Should you replace it with B if, so when ? **[B.C.A. (Kurukshetra) 2007, 2011]**

Solution: (i) It is given that machine A has no resale value, when replaced. The average annual cost is computed as below:

Year (n) (1)	Maintenance cost $C_m(t)$ (2)	Total maintenance cost $\Sigma\, C_m(t)$ (3)	S	$(C - S)$ (4)	Total cost (5)=(4)+(3)	Average cost $A(n)$ = (5 / n) (6)
1	200	200	Nil	9000	9200	9200
2	2200	2400	Nil	9000	11400	5700
3	4200	6600	Nil	9000	15600	5200 ←
4	6200	12800	Nil	9000	21800	5450
5	8200	21000	Nil	9000	30000	6000

From this table we find that machine A should be replaced at the end of 3 year and average yearly cost of owning and operating the machine at this time replacement is ₹ 5200.

(ii) For machine B, the average cost per year can be calculated as follows:

Year (n) (1)	Maintenance cost C_m (t) (2)	Total maintenance cost $\Sigma\, C_m$ (t) (3)	S (4)	$C - S$ (5)	Total cost (6) =(5)+ (3)	Average cost $A(n)$ $= \left(\frac{6}{n}\right)$
1	400	400	Nil	10000	10400	10400
2	1200	1600	Nil	10000	11600	5800
3	2000	3600	Nil	10000	13600	4533
4	2800	6400	Nil	10000	16400	4100
5	3600	10000	Nil	10000	20000	4000 ←
6	4400	14400	Nil	10000	24400	4066

Since the minimum average cost for machine *B* is lower than that for machine *A*, machine should be replaced by machine *A*.

Now we have to determine as to when machine *A* should be replaced. Machine *A* should be replaced when the cost for next year of running this machine becomes more than the average yearly cost for machine *B*.

Now, total cost of machine *A* in the first year = ₹ 9200

total cost of machine *A* in the second year = ₹ 11400 – ₹ 9200

= ₹ 2200

total cost of machine *A* in the third year = ₹ 4200

total cost of machine *A* in the fourth year = 6200

As the cost of running machine *A* in third year (₹ 4200) is more than the average yearly cost for machine *B* (₹ 4000): machine *A* should be replaced at the end of two years. *i.e.,* one year after it is one year old (one year hence).

Example 7: *For a machine, for the following data are available:*

Year	0	1	2	3	4	5	6
Cost of spares (₹)	–	200	400	700	1000	1400	1600
Salary maintenance staff (₹)	–	1200	1200	1400	1600	2000	2600
Loss due to break- down (₹)	–	600	800	700	1000	1200	1600
Resale value (₹)	12000	6000	3000	1500	800	400	400

Determine the optimum period for replacement of the above machine.

Solution: The optimum period for replacement is determine in the table below:

Year (n)	**Cost of spares (₹)**	**Salary of maintenance staff (₹)**	**Losses due to break-down (₹)**	**$C - S$ depreciation**	**Total cost (1)**	**Average cost $A(n) =$ (1 / n)**
1	200	1200	600	6000	8000	8000
2	600	2400	1400	9000	13400	6700
3	1300	3800	2100	10500	17700	5900
4	2300	5400	3100	11200	22000	5500
5	3700	7400	4300	11600	27000	5400 ←
6	5300	10000	5900	11600	32800	5466.66

This table shows that the value of $A(n)$ during 5^{th} year is minimum. Hence, the machine should be replaced every 5^{th} year.

Example 8: (*i*) *An auto-rickshaw driver finds from his pervious records that the cost per year of running an auto-rickshaw whose purchase prices is* ₹ 7000 *is as given below*:

Year	1	2	3	4	5	6	7	8
Running cost **(₹)**	1100	1300	1500	1900	2400	2900	3500	4100
Resale Price **(₹)**	3100	1600	850	475	300	300	300	300

At what age is the replacement due?

(*ii*) *Another person has three auto rickshaws of the same purchase price and cost of running each as import* (*i*). *Two of these vehicles are* 2 *years old and the third one is* 1 *year old. He is considering a new type of auto-rickshaw with* 50% *more capacity than one of the old ones and at a unit price of* ₹ 9000. *He estimates that the running costs and resale price for the new vehicle will be as follows:*

Year	1	2	3	4	5	6	7	8
Running cost **(₹)**	1300	1600	1900	2500	3200	4100	5100	6200
Resale Price **(₹)**	4100	2100	1100	600	400	400	400	400

Assuming that the loss of flexibility due to fewer vehicles is of no importance, and that he will continue to have sufficient work for three of the old vehicles, what should be his policy?

[M.B.A. (Meerut) 2003]

Solution: (i) The average annual cost for old auto-rickshaw is determine as follows:

Year (n) (1)	**Annual maintenance cost $C_m(t)$ ₹** (2)	**Total maintenance cost or cumulative cost $\Sigma C_m(t)$ ₹** (3)	**Resale value S ₹** (4)	**Purchase price-resale value $(C-S)$ ₹** (5)	**Total cost $= [(C-S) + (\Sigma C_m(t)]$ ₹** (6) = (5) + (3)	**Average cost annual cost $A(n) = \left(\frac{6}{n}\right)$**
1	1100	1100	3100	3900	5000	5000
2	1300	2400	1600	5400	7800	3900
3	1500	3900	850	6150	10050	3350
4	1900	5800	475	6525	12325	3081
5	2400	8200	300	6700	14900	2980
6	2900	11100	300	6700	17800	2967 ←
7	3500	14600	300	6700	21300	3043
8	4100	18700	300	6700	25400	3175

This table shows that the value of $A(n)$ during 6^{th} year is minimum. Hence, the old auto-rickshaw should be replace at the end of every 6^{th} year.

(ii) Now let us determine the average annual cost of the new auto-rickshaw of larger capacity.

Year (n) (1)	Annual maintenance cost $C_m(t)$ (2)	Total maintenance cost or cumulative cost $\Sigma\, C_m(t)$ (3)	Purchase price-resale value = $(C - S)$ (4)	Resale value (S) (5)	Total cost $= (C - S) + (\Sigma\, C_m(t)$ (6) = (4) + (3)	Average annual cost $A(n) = \frac{(6)}{n}$
1	1300	1300	4900	4100	6200	6200
2	1600	2900	6900	2100	9800	4900
3	1900	4800	7900	1100	12700	4233
4	2500	7300	8400	600	15700	3925
5	3200	10500	8600	400	19100	3820 ←
6	4100	14600	8600	400	23200	3867
7	5100	19700	8600	400	28300	4043
8	6200	25900	8600	400	34500	4312

As the new auto-rickshaw has 50% more capacity than the old one, the minimum average annual cost of ₹ 3820 for the former is equivalent to

₹ $3820 \times \frac{2}{3} =$ ₹ 2547 for the latter. Since this amount is less than ₹ 2967 for it, the latter will be replaced by new auto-rickshaw.

The new vehicles will be purchased when the cost for the next year of running the three old vehicles becomes more than the average annual cost of the two new ones.

Total annual cost of one smaller auto-rickshaw during first year = ₹ 5000

annual cost of one smaller auto-rickshaw during second year = ₹ 7800 – ₹ 5000 = ₹ 2800

annual cost of one smaller auto-rickshaw during third year = ₹ 2250

annual cost of one smaller auto-rickshaw during fourth year = ₹ 2275

annual cost of one smaller auto-rickshaw during fifth year = ₹ 2575

annual cost of one smaller auto-rickshaw during sixth year = ₹ 2900 and so on.

Total cost during next first year for two smaller vehicles aged two years and one vehicle aged one year = $2 \times 2250 + 2800 =$ ₹ 7300

Similarly, total cost during next second year = $2 \times 2275 + 2250 =$ ₹ 6800

total cost during next third year = $2 \times 2575 + 2275 =$ ₹ 7425

total cost during next fourth year = $2 \times 2900 + 2575 =$ ₹ 8375

and so on.

But minimum average cost for two new vehicles = $2 \times 3820 =$ ₹ 7640

As the total cost of old vehicles during next third year is less than the minimum average cost of the new vehicles and becomes more only in the next fourth year. Hence, old auto-rickshaws should be replaced by the new larger ones after the next third year of this life.

Problem Set

1. Write short note on replacement problems.
2. Discuss some important replacement situations. **[B.B.A. (Meerut) 2012]**
3. What is replacement? Describe some important replacement situations.
4. A firm is considering when to replace its machine whose price is ₹ 12200. The scrap value of the machine is ₹ 200 only. From past experience the maintenance costs of the machine are as under:

Year	1	2	3	4	5	6	7	8
Maintenance cost in (₹)	200	500	800	1200	1800	2500	3200	4000

Find when the new machine should be purchased. **[B.B.A. (Delhi) 2007]**

5. The cost of a truck is ₹ 10000. The salvage value and the running costs are given below. Find the most economical age for replacement.

Year	1	2	3	4	5	6	7
Running cost (₹)	3000	3200	3600	4200	5000	5800	6800
Resale value (₹)	7000	5000	3400	2400	1600	1000	1000

6. Machine *A* costs ₹ 45000 and the operating costs are estimated at ₹ 1000 for the first year increasing by ₹ 10,000 per year in the second and subsequent years. Machine *B* costs ₹ 50000 and operating cost are ₹ 2000 for the first year increasing by ₹ 4000 per year in the second and subsequent years. If you now have a machine of type *A*, should you replace it with *B* ? If so, when ? Assume that both the machines have no resale value, and that the future costs are not discounted.
7. Explain with examples the failure mechanism of items. **[B.C.A. (I.G.N.O.U) 2012]**
8. What are the situations which makes the replacement of items necessary ?

Answers

4.	After 6^{th} years
5.	After 4^{th} years
6.	Machine *A* should be replaced by *B*, when its age is 2 years.

7.3 Money Value Present Worth Factor (P.W.F.) and Discount Rate

7.3.1 Money Value

The value of money change with time. Since has value over time. This can be explain with the help of following example:

If we borrow ₹ 100 at the rate of interest 10% per year and spend this amount today, then we have to pay ₹ 110 after one year

Therefore

₹ 110 after one year $= ₹\ 100$ today

$\therefore$ ₹ 1 after one year $= ₹\ \dfrac{100}{110}$

$$= \frac{10}{(1.1)} = (1.1)^{-1}$$

So, ₹ 1.00, n year after from now = ₹ $(1.1)^{-n}$ today

at the rate of 10%.

7.3.2 Present Value or Present Worth Factor

The quantity $(1.1)^{-n}$ is called the **present worth factor** (pwf) or present value of one rupee spend n years from now.

If r is the rate of interest then $(1+r)^{-n}$ is called the **present worth factor** (pwf) of one rupee spent in n years.

7.3.3 Discount Rate (Depreciation Value)

The present worth factor of unit amount to be spent after one year is given by $v = \dfrac{1}{(1+r)}$

where r the rate of interest.

Then v is called discount rate or depreciation value.

Example 9: *The yearly cost of two machines A and B, when money value is neglected is shown following table*:

Year	1	2	3
Machine A (₹)	1800	1200	1400
Machine B (₹)	2800	200	1400

[B.C.A. (I.G.N.O.U.) 2012]

Solution: The total expenditure for three years for machine A

$$= 1800 + 1200 + 1400 = ₹\ 4400$$

The total expenditure for three years for machine B

$$= 2800 + 200 + 1400 = ₹\ 4400$$

Thus, the two machines are equally good if the money has no value over time.

Now consider the money value at the rate of 10% per year, the discount rate

$$v = \frac{1}{1+\frac{10}{100}} = \frac{1}{1+0.10} = \frac{1}{1.1} = 0.9091$$

The discount cost patterns for machines A and B for three year is shown in table:

Year	1	2	3	Total cost (₹)
Machine A (Discounted cost in ₹)	1800	1200×0.9091 $= 1090.90$	$1400 \times (0.9091)^2$ $= 1157.04$	404794
Machine B (Discounted cost in ₹)	2800	200×0.9091 $= 181.82$	$1400 \times (0.9091)^2$ $= 1157.04$	4138.86

The data show that the cost for machine A is less than that for machine B, machine A is more economical.

Example 10: *Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years where as machine B is replaced after every 6 years. The yearly costs of both the machines are given as under*:

Year	1	2	3	4	5	6
Machine A (₹)	1000	200	400	1000	200	400
Machine B (₹)	1700	100	200	300	400	500

Determine which machine should be purchased.

Solution: The present worth factor is given by

$$v = \frac{1}{1+r} = \frac{1}{1+\frac{10}{100}} = \frac{100}{100+10} = \frac{100}{100+10} = \frac{100}{110} = \frac{10}{11}$$

$\therefore$ Total discount cost of A for three years

$$= ₹\left[1000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2\right]$$

$$= ₹\,1512 \text{ (approximately)}$$

$\therefore$ Total discount cost of B for six years

$$= ₹\left[1700 + 100 \times \frac{10}{11} + 200 \times \left(\frac{10}{11}\right)^2 + 300 \times \left(\frac{10}{11}\right)^3\right.$$

$$+ 400 \times \left(\frac{10}{11}\right)^4 + 500 \times \left(\frac{10}{11}\right)^5 \Bigg]$$

$$= ₹\, 2765$$

Average annual cost of $\quad A = \frac{1512}{3} = ₹\, 504 \quad ...(1)$

Average annual cost of $\quad B = \frac{2765}{6} = ₹\, 461 \quad ...(2)$

From (1) and (2), we find advantage with B but this comparison is unfair because the period of both machines are different.

If we consider 6^{th} year period for machine A, then total discount of

$$A = 1000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2 + 1000 \times \left(\frac{10}{11}\right)^3$$

$$+ 200 \times \left(\frac{10}{11}\right)^4 + 400 \times \left(\frac{10}{11}\right)^5 = ₹\, 2647$$

Which is ₹ 118 less than machine B over the same period so machine A should be purchased.

7.4 Replacement of Items whose Maintenance Costs Increase with Time and Value of Money also Change with Time

7.4.1 Theorem

The maintenance cost increases with time and the money value decreases with constant rate *i.e.,* depreciation value is given. Then replacement policy will be

1. **Replace if the next period's cost is greater than the weighted average of previous costs.**
2. **Do not replace if the next period's cost is less than the weighted average of previous costs.** **[B.C.A. (Rohilkhand) 2009; B.C.A. (Rohtak) 2008]**

Proof: Suppose that the item (may be a machine or equipment etc.) is available for use over a series of time periods of equal interval say one year.

Let A = purchase cost of a machine

C_r $(r = 1, 2, 3 \ldots n)$ be maintenance costs in year

i.e., $C_1, C_2, \ldots C_n$ be maintenance costs in years $1, 2, 3 \ldots n$ respectively.

r = rate of interest

$v = \frac{1}{1+r}$ is the present worth of a rupee to be spent a year

The present value of maintenance costs $C_1, C_2 \ldots C_n$ in the years $1, 2, 3 \ldots n$ are

$C_1, C_2\, v, C_3\, v^2, \ldots C_n\, v^{n-1}$ respectively

The present value of the total expenditure on the machine in n years will be given by

$$P(n) = A + C_1 + C_2\, v + C_3\, v^2 + \ldots + C_n\, v^{n-1} = A + \sum_{r=1}^{n} C_r\, v^{r-1}$$

If the machine is replaced after n years from now

Then, the present value of the total expenditures on the machines with a replacement policy of n years is given by

$$\begin{aligned} C(n) = {} & [A + C_1 + C_2\, v + C_3\, v^2 + \ldots + C_n\, v^{n-1}] \\ & + [(A + C_1)\, v^n + C_2\, v^{n+1} + C_3\, v^{n+2} + \ldots + C_n\, v^{2n-1}] \\ & + [(A + C_1)\, v^{2n} + C_2\, v^{2n+1} + \ldots + C_n\, v^{3n-1}] + \ldots \end{aligned}$$

$$= \left[A + \sum_{r=1}^{n} C_r\, v^{r-1}\right] + v^n \left[A + \sum_{r=1}^{n} C_r\, v^{r-1}\right] + v^{2n} \left[A + \sum_{r=1}^{n} C_r\, v^{r-1}\right] + \ldots$$

$$= \left[A + \sum_{r=1}^{n} C_r\, v^{r-1}\right][1 + v^n + v^{2n} + \ldots.] \qquad \left[\because 1 + v^n + v^{2n} + \ldots.. = \frac{1}{1 - v^n}\right]$$

$$C(n) = A + \sum_{r=1}^{n} C_r\, v^{r-1}$$

or

$$C(n) = \frac{A + \sum_{r=1}^{n} C_r\, v^{r-1}}{1 - v^n} = \frac{P(n)}{1 - v^n} \qquad \ldots(1)$$

Now we are interested to find that value of n for which $C(n)$ is minimum.

$C(n)$ is minimum for that value of n for which

$$\Delta C(n-1) < 0 < \Delta C(n) \qquad \ldots(2)$$

Now,

$$\Delta C(n) = C(n+1) - C(n)$$

$$= \frac{A + \sum_{r=1}^{n+1} C_r\, v^{r-1}}{1 - v^{n+1}} - \frac{A + \sum_{r=1}^{n} C_r\, v^{r-1}}{1 - v^n}$$

$$= \frac{\left(A + \sum_{r=1}^{n+1} C_r\, v^{r-1}\right)(1 - v^n) - \left(A + \sum_{r=1}^{n} C_r v^{r-1}\right)(1 - v^{n+1})}{(1 - v^{n+1})(1 - v^n)}$$

$$= \frac{A\,(v^{n+1} - v^n) + \left(\sum_{r=1}^{n+1} C_r\, v^{r-1}\right)(1 - v^n) - \left(\sum_{r=1}^{n} C_r v^{r-1}\right)(1 - v^{n+1})}{(1 - v^{n+1})(1 - v^n)}$$

$$= \frac{-A(1-v)v^n + \sum_{r=1}^{n} C_r v^{r-1} \{(1-v^n) - (1-v^{n+1})\} + C_{n+1} v^n (1-v^n)}{(1-v^{n+1})(1-v^n)}$$

$$= \frac{v^n}{(1-v^{n+1})(1-v^n)} \cdot \left[(1-v^n) C_{n+1} - (1-v) \cdot \left(A + \sum_{r=1}^{n} C_r v^{r-1} \right) \right]$$

or
$$\Delta C(n) = \frac{v^n (1-v)}{(1-v^{n+1})(1-v^n)} \cdot \left[\frac{(1-v^n)}{(1-v)} C_{n+1} - \left(A + \sum_{r=1}^{n} C_r v^{r-1} \right) \right]$$

Similarly,

$$\Delta C(n-1) = \frac{v^{n-1} (1-v)}{(1-v^n)(1-v^{n-1})} \cdot \left[\frac{(1-v^{n-1})}{(1-v)} C_n - \left(A + \sum_{r=1}^{n-1} C_r v^{r-1} \right) \right]$$

Now, from (2), $C(n)$ is minimum for that value of n for which

$$\frac{v^{n-1} (1-v)}{(1-v^n)(1-v^{n-1})} \left[\frac{1-v^{n-1}}{1-v} \cdot C_n - \left(A + \sum_{r=1}^{n-1} C_r v^{r-1} \right) \right] < 0$$

$$< \frac{v^n (1-v)}{(1-v^{n+1})(1-v^n)} \cdot \left[\frac{1-v^n}{1-v} C_{n+1} - \left(A + \sum_{r=1}^{n} C_r v^{r-1} \right) \right]$$

or
$$\frac{1-v^{n-1}}{1-v} C_n - \left(A + \sum_{r=1}^{n-1} C_r v^{r-1} \right) < 0 < \frac{1-v^n}{1-v} \cdot C_{n+1} - \left(A + \sum_{r=1}^{n} C_r v^{r-1} \right)$$

$$\left[\text{Since } \frac{v^{n-1}(1-v)}{(1-v^n)(1-v^{n-1})} \text{ and } \frac{v^n (1-v)}{(1-v^{n+1})(1-v^n)} \text{ are both positive as } v < 1 \right]$$

or
$$\frac{1-v^{n-1}}{1-v} \cdot C_n - P(n-1) < 0 < \frac{1-v^n}{1-v} \cdot C_{n+1} - P(n) \qquad \ldots(3)$$

Thus, the best replacement age of the machine is n years for which inequality (3) holds.

Again from (3), we find

$$\frac{1-v^n}{1-v} C_{n+1} - \left(A + \sum_{r=1}^{n} C_r \cdot v^{r-1} \right) > 0$$

and
$$\frac{1-v^{n-1}}{1-v} C_n - \left(A + \sum_{r=1}^{n-1} C_r \cdot v^{r-1} \right) < 0$$

or $$C_{n+1} > \frac{A + \sum_{r=1}^{n} C_r \,.\, v^{r-1}}{\frac{1 - v^n}{1 - v}} \quad \text{and} \quad C_n < \frac{A + \sum_{r=1}^{n-1} C_r \,.\, v^{r-1}}{\frac{1 - v^{n-1}}{1 - v}}$$

or $$C_{n+1} > \frac{A + \sum_{r=1}^{n} C_r \,.\, v^{r-1}}{1 + v + v^2 + \ldots v^{n-1}} \quad \text{and} \quad C_n < \frac{A + \sum_{r=1}^{n-1} C_r \,.\, v^{r-1}}{1 + v + v^2 + \ldots v^{n-1}}$$

Since $$\frac{1 - v^n}{1 - v} = 1 + v + v^2 + \ldots + v^{n-1}$$

or $$C_{n+1} > R(n) \quad \text{and} \quad C_n < R\,(n-1) \qquad \ldots(4)$$

where $R\,(n) = \dfrac{A + \sum_{r=1}^{n} C_r \,.\, v^{r-1}}{1 + v + v^2 + v^3 + \ldots + v^{n-1}}$ = weighted average of costs in n years.

Hence, we conclude,

1. Do not replace if the operating cost of the next period (year) is less than the weighted average of previous costs.
2. Replace if the operating cost of the next period (year) is greater than the weighted average of the previous costs.

7.4.2 Theorem

A discounted cost $P\,(n)$ is invested by taking loan at the interest rate r; and the loan is repaid by fixed annual payments say x, throughout the life of the machine. To find the minimum value of x for optimum period n at which to replace the machine

[B.C.A. (Lucknow) 2004]

Proof: The present worth of fixed annual payments x for n years must be equal to $P\,(n)$ the sum borrowed.

$\therefore$ $$P\,(n) = x + vx + v^2 x + \ldots\ldots + v^{n-1} \,.\, x \qquad \ldots(1)$$

[where $v = \dfrac{1}{1 + r}$. (Discount rate)]

$$= (1 + v + v^2 + \ldots + v^{n-1})\, x$$

$$= \frac{1 - v^n}{1 - v} \,.\, x$$

or $$x = \frac{(1 - v)}{1 - v^n} \,.\, P\,(n) \qquad \ldots(2)$$

Since $(1 - v)$ is constant, independent of n.

$\therefore$ x is minimum when

$$C(n) = \frac{P(n)}{1 - v^n} \qquad ...(3)$$

is minimum.

$C(n)$ given by (3) is minimum, when

$$\Delta C(n-1) < 0 < \Delta C(n) \qquad ...(4)$$

From (3),

$$\Delta C(n) = C(n+1) - C(n)$$

$$= \frac{P(n+1)}{1 - v^{n+1}} - \frac{P(n)}{1 - v^n} = \frac{(1 - v^n)\, P(n+1) - (1 - v^{n+1})\, P(n)}{(1 - v^{n+1})(1 - v^n)}$$

$$= \frac{P(n+1) - P(n) - v^n\, [P(n+1) - v.\, P(n)]}{(1 - v^{n+1})(1 - v^n)} \qquad ...(5)$$

But if A is the cost of the machine and $C_1, C_2, \ldots, C_n$ are the running costs in years $1, 2, \ldots, n$ then the present value of the total expenditure in n years must be equal to $P(n)$.

$$\therefore \qquad P(n) = A + C_1 + C_2\, v + C_3\, v^2 + \ldots + C_n\, v^{n-1}$$

$$\therefore \qquad P(n+1) = A + C_1 + C_2\, v + C_3\, v^2 + \ldots + C_n\, v^{n-1} + C_{n+1}\, v^n$$

$$= P(n) + C_{n+1}\, v^n.$$

$\therefore$ From (5), we get

$$\Delta C(n) = \frac{C_{n+1}\, v^n - v^n\, [P(n) + C_{n+1}\, v^n - v.\, P(n)]}{(1 - v^{n+1})(1 - v^n)}$$

$$= \frac{(1 - v^n)\, C_{n+1}\, v^n - (1 - v)\, P(n)\, .\, v^n}{(1 - v^{n+1})(1 - v^n)}$$

$$= \frac{(1 - v)\, v^n}{(1 - v^{n+1})(1 - v^n)} \cdot \left[\frac{1 - v^n}{1 - v}\, C_{n+1} - P(n)\right]$$

$\therefore$ $\Delta C(n) > 0$, gives

$$\frac{(1 - v)\, v^n}{(1 - v^{n+1})(1 - v^n)} \cdot \left[\frac{1 - v^n}{1 - v}\, C_{n+1} - P(n)\right] > 0$$

or $$\frac{1 - v^n}{1 - v}\, C_{n+1} - P(n) > 0 \quad \text{since} \quad \frac{(1 - v)\, v^n}{(1 - v^{n+1})(1 - v^n)} \text{ is positive as } 0 < v < 1$$

or $$C_{n+1} > \frac{P(n)}{\left(\dfrac{1 - v^n}{1 - v}\right)} \text{ or } C_{n+1} > \frac{A + C_1 + C_2 v + C_3 v^2 + \ldots + C_n v^{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} \qquad ...(6)$$

Similarly, put $n = n - 1$ in $C(n)$, we get

$$\Delta C(n-1) \quad \text{and} \quad \Delta C(n-1) < 0$$

∴ $$\frac{1 - v^{n-1}}{1 - v} C_n - P(n-1) < 0$$

or $$C_n < \frac{A + C_1 + C_2 v + C_3 v^2 + \ldots + C_{n-1} v^{n-2}}{1 + v + v^2 + \ldots + v^{n-2}} \qquad \ldots(7)$$

Hence, we conclude from (6) and (7) that

$$\frac{1 - v^{n-1}}{1 - v} C_n - P(n-1) < 0 < \frac{1 - v^n}{1 - v} C_{n+1} - P(n) \qquad \ldots(8)$$

From this, we find best replacement age n of machine for which $C(n)$ or $P(n)$ is minimum and minimum of x is obtain from (2) equation.

7.4.3 Procedure for Finding best Machine

Step 1: Find the present value of the maintenance cost for each of the years

i.e., $$\sum C_n v^{n-1} \text{ where } n = 1, 2, 3 \ldots \text{and } v = \frac{1}{1 + r}, \; r = \text{rate of interest}$$

Step 2: Calculate cost plus the accumulated present values obtained in step 1

i.e., $$A + \sum C_n v^{n-1} \text{ where } A = \text{purchase cost}$$

Step 3: Find the cumulative present value factors up to each of the year $n = 1, 2, 3 \ldots$ *i.e.,*

$$\sum v^{n-1}$$

Step 4: Determine annualized cost $w(n)$ by adding the entries obtained in step 2 by the corresponding entries obtained in step 3 *i.e.,*

$$\frac{\left[A + \sum C_n v^{n-1}\right]}{\sum v^{n-1}}$$

Once the annualized costs or weighted average costs for different years is obtained, the following sale are followed to decide on the replacement

1. Do not replace the equipments if the next period's cost is less than the weighted average of previous costs.
2. Replace the equipments if the next period's cost is greater than the weighted average of previous costs.

Example 11: *Let $v = 0.9$ and initial price is ₹ 5000. Running cost varies as follows:*

Year :	1	2	3	4	5	6	7
Running cost (₹) :	400	500	700	1000	1300	1700	2100

[B.C.A. (Avadh) 2010]

What would be the optimum replacement interval?

Solution: Given $v = 0.9$, A = Initial price = ₹ 5000, C_n = maintenance cost

The optimum replacement age n is obtained in the following table :

Year (n) (1)	Maintenance cost C_n (2)	v^{n-1} (3)	$C_n\ v^{n-1}$ (4)	$A + \sum C_n v^{n-1}$ (5)	$\sum v^{n-1}$ (6)	Weighted average $w(n)$ (7) =(5) / (6)
1	400	1.0	400	5400	1.00	5400
2	500	0.90	450	5850	1.90	3079
3	700	0.81	567	6417	2.710	2368
4	1000	0.729	729	7146	3.439	2083
5	1300	0.6556	853	7999	4.096	1953
6	1700	0.599	1016	9015	4.694	1921 ←
7	2100	0.540	1134	10149	5.234	1939

Since $C_5 < w(6) < C_7$ *i.e.,* $1300 < 1921 < 2100$.

Optimum replacement is just under 6 years.

Example 12: *Assume that present value of one rupee to be spent in a year's time is ₹ 0.9 and A = ₹ 3000 capital cost of equipment and the running cost are given in the table below. When should the machine be replaced?*

Year (n) :	1	2	3	4	5	6	7
Running cost (₹) :	500	600	800	1000	1300	1600	2000

Solution: We have $v = 0.9$, A = ₹ 3000, v^{n-1} = discount factor and C_n $(n = 1, 2, 3, \ldots 7)$ where $C_n\ v^{n-1}$ = discounted cost, C_n = maintenance cost for $n = 1, 2, 3, 4, 5, 6, 7$ years.

The optimum replacement age n is obtained in the following table:

Year (n) (1)	Maintenance cost C_n (2)	v^{n-1} (3)	$C_n v^{n-1}$ (4)	$A + \sum C_n v^{n-1}$ (5)	$\sum v^{n-1}$ (6)	$w(n) = \left(\frac{5}{6}\right)$ (7)
1	500	1.00	500	3500	1.00	3500
2	600	0.90	540	3540	1.90	1863
3	800	0.81	648	3648	2.71	1346
4	1000	0.73	730	3730	3.44	1084 ←
5	1300	0.66	858	3858	4.10	941
6	1600	0.59	944	3944	4.69	841
7	2000	0.53	1060	4060	5.22	778

Since $C_3 < W(4) < C_5$. Optimum replacement period is 4^{th} year.

Example 13: *The cost of a new machine is ₹ 5000. The maintenance cost of n^{th} year is given by $C_n = 500\,(n-1) : n = 1, 2, 3 \ldots.$ Suppose that the discount rate per year is 0.05. After how many years it will be economical to replace the machine by a new one?*

[B.C.A. (Kashi) 2009; B.B.A. (Indore) 2008]

Solution: We are given A=₹5000 and $C_n = 500\,(n-1), n = 1, 2, 3 \ldots$. Since the discount rate of money per year is 0.05, the present worth of the money to be spent over a period of one year is $v = \dfrac{1}{1 + 0.05} = 0.9523$

The optimum replacement line is determined in the following table :

Year (n) (1)	C_n Maintenance cost (2)	v^{n-1} (3)	$C_n v^{n-1}$ (4)	$A + \sum C_n v^{n-1}$ (5)	$\sum v^{n-1}$ (6)	$w(n) = \frac{(5)}{(6)}$ (7)
1	0	1.000	0	5000	1.00	5000
2	500	0.9523	476	6476	1.9523	3317
3	1000	0.9070	907	6383	2.8593	2232
4	1500	0.8638	1296	7679	3.7231	2062
5	2000	0.8227	1645	9924	4.5458	2083 ←
6	2500	0.7835	1959	11283	5.3293	2117

Since, $w(n)$ is minimum for $n = 5$ and $C_4 = 1500 < w(5)$ as well as $w(5) > C_6 = 2500$. It is economical to replace the machine by new one at the end of five years.

Example 14: *A machine costs ₹10000 operating costs are ₹100 per year for the first five year. In the sixth and succeeding years operating cost increases by ₹100 per year. Assuming a 10% discount rate of money per year, find the optimum length of time to hold the machine before we replace it.*

Solution: The discount rate of money per year is given as 10%. Therefore, the present worth of the money to be spent a period of one year is

$$v = \frac{1}{1+r} = \frac{1}{1+\frac{10}{100}} = \frac{100}{110} = 0.9091$$

The optimal replacement time is determined in the following table:

Year n (1)	C_n (2)	v^{n-1} (3)	$C_n v^{n-1}$ (4)	$A + \sum C_n v^{n-1}$ (5)	$\sum v^{n-1}$ (6)	$w(n)$ (7) = (5) / (6)
1	500	1.000	500	10500	1.000	10500
2	500	0.9091	456	10956	1.9091	5731
3	500	0.8264	413	11369	2.7355	4156
4	500	0.7513	376	11745	3.4868	3368
5	500	0.6830	342	12087	4.1698	2898
6	600	0.62090	373	12460	4.7907	2600
7	700	0.5645	395	12885	5.3552	2400
8	800	0.5133	411	13266	5.8684	2260
9	900	0.4665	420	13686	6.3349	2160
10	1000	0.4241	424	14110	6.7590	2087
11	1100	0.3856	424	14534	7.1446	2034
12	1200	0.3506	421	14955	7.4952	1995
13	1300	0.3187	414	15369	7.8139	1966
14	1400	0.2897	406	15775	8.1036	1946
15	1500	0.2637	396	16171	8.3673	1932
16	1600	0.2397	384	16555	8.6070	1923
17	1700	0.2179	370	16925	8.8249	1917
18	1800	0.1981	357	17282	9.0230	1915
19	1900	0.1801	342	17624	9.2031	1915 ←
20	2000	0.1637	327	17951	9.3668	1914

The machine should be replaced after 19 years of service.

Problem Set

1. Purchase price of a machine in ₹ 3000 and its running cost is given in the table below. If the discount rate is 0.90, find at what age the machine should be replaced.

Year	1	2	3	4	5	6	7
Running cost (₹)	500	600	800	1000	1300	1600	2000

[B.B.A. (Bhopal) 2007]

2. If you wish to have a return of 10% per annum on your investment, which of the following plans would your prefer ?

	Plan *A* (₹)	**Plan *B* (₹)**
Ist cost :	2,00,000	2,50,000
Scrap value after 15 years :	1,50,000	1,80,000
Excess of annual revenue :	25,000	30,000
Over annual disbursement		

3. A truck is priced at ₹ 60000 and running costs are estimated at ₹ 6000 for each of the first four years, increasing by ₹ 2000 per year in the fifth and subsequent years. If money is worth 10% per year, when should the truck be replaced? Assume that the truck will eventually be sold for scrap at a negligible price.

[B.B.A. (Delhi) 2008]

Answers

1.	After 3 years
2.	Plan *A*
3.	After 9 years

7.5 Replacement of Items that Fail Suddenly

There are many real life situations in which items do not deteriorate with time but fail suddenly. A system usually consists of a large number of low cost items that increasingly liable to failure with age some times, the failure of an item may cause a complete breakdown of the system. The costs of failure, in such a case will be quite higher than the cost of item itself. It is, therefore, to know as to when the failure is likely to take place so that item can be replaced before it actually fails. The problem, then is to find the optimal value of time t which minimizes the total cost involved in the system.

The following two policies are followed:

1. **Individual Replacement Policy:** In that case an item is replaced immediately after it fails.
2. **Group Replacement Policy:** In that case all items are replaced, irrespective of whether they have failed or not, with a provision that if any item fails before the optimal time, it may be individually replaced.

7.5.1 Individual Replacement Policy: Mortality Theorem

These problems are the special cases of the problems in industry where the failure of any item can be treated as death and the replacement of any item on failure can be taken to be a birth in the human populations. To determine the probability distribution of failure its mortality tables are used.

7.5.1.1 Mortality Theorem (Only Statement)

A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exits. Show that the age distribution ultimately becomes stables and that the number of deaths per unit time becomes constant. Which is equal to the size of total population divided by the mean age at death.

i.e.,

$$A_0 = \frac{\text{The size of population (N)}}{\text{Mean age at death}}$$

7.5.1.2 Mortality Tables

Mortality tables for any item can be used to obtain the probability distribution of its life span.

Let N = The total number of items in the system in the beginning

$N(t)$ = Number of survivors at any time t.

Then the probability that any item will fail in the time interval $(t-1, t)$ is given by

$$\frac{N(t-1) - N(t)}{N} \quad \text{...(1)}$$

and the probability that any item that survived upto the age $(t-1)$ will die in the next year is given by

$$\frac{N(t-1) - N(t)}{N(t-1)} \quad \text{...(2)}$$

7.5.2 Group Replacement Policy

Group replacement policy is defined in the following theorem and later it is explained in numerical problems.

7.5.2.1 Theorem (Group Replacement Policy)

1. **Group replacement should be made at the end of *n* th period if the cost of individual replacement for the *n* th period is greater than the average cost per period by the end of *n* periods.**
2. **Group replacement should not be made at the end of *n* th period if the cost of individual replacement at the end of $(n-1)$th period is not less than the average cost per period by the end of $(n-1)$ periods.**

Proof: Here it is proposed to replace all items at fixed interval 't' whether they have failed or not, and continue replacing failed items as and when they fail.

Let N = Total number of items in the system

$N(X)$ = Number of items failed during Xth period, $X = 1, 2, 3 \ldots (n-1)$

C_g = The cost per item when all the items are replaced as a group

C_i = The cost of replacing an individual item on its failure

$C(n)$ = Total cost in the interval t consisting of n periods.

$= N \cdot C_g + C_i$ [total number of failures in the periods $1, 2, 3, \ldots (n-1)$]

$$= N \cdot C_g + C_i \sum_{X=1}^{n-1} N(X) \qquad \ldots(1)$$

Average cost per period $A(n) = \dfrac{C(n)}{n}$ ...(2)

Now in order to determine the replacement age 't', the average cost per period should be minimum.

The condition for minimum of $A(n)$ is

$$\Delta A(n-1) < 0 < \Delta A(n) \qquad \ldots(3)$$

Now $\Delta A(n) = A(n+1) - A(n)$

$$= \frac{C(n+1)}{n+1} - \frac{C(n)}{n}$$

$$= \frac{C(n) + C_i N(n)}{n+1} - \frac{C(n)}{n}$$

$$= \frac{n C_i N(n) - C(n)}{n(n+1)} = \frac{C_i N(n) - \dfrac{C(n)}{n}}{(n+1)}$$

Similarly, $$\Delta A(n-1) = \frac{C_i N(n-1) - \dfrac{C(n-1)}{n-1}}{n}$$

Put $\Delta A(n)$ and $\Delta A(n-1)$ in (3), we get

$$\frac{C_i N(n-1) - \dfrac{C(n-1)}{n-1}}{n} < 0 < \frac{C_i N(n) - \dfrac{C(n)}{n}}{n+1}$$

or $$C_i N(n-1) - \frac{C(n-1)}{(n-1)} < 0 < C_i N(n) - \frac{C(n)}{n} \quad ...(4)$$

or $$C_i N(n) > \frac{C(n)}{n} \quad ...(5)$$

and $$C_i N(n-1) < \frac{C(n-1)}{(n-1)} \quad ...(6)$$

Thus, from (5) and (6) the group replacement policy is completely established.

Example 15: *The following mortality rates have been observed for a certain type of light bulb:*

Week	1	2	3	4	5
Percent failing by end of week	10	25	50	80	100

There are 1000 *bulbs in use and it costs* ₹ 1.00 *to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously it would cost* 25 *paise per bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out and to continue replacing burnt out bulbs as they fail. At what intervals should all the bulbs be replaced?* **[M.C.A. (Meerut) 2008]**

Solution: Let P_i be the probability that a light bulb, which was now when placed in position for use, fails during the *i*th week of its life. Thus, following frequency distribution is obtained assuming to replace burnt out bulbs as and when they fail.

P_1 = the probability of failure in the Ist week $= \dfrac{10}{100} = 0.10$

P_2 = the probability of failure in the IInd week $= \dfrac{25-10}{100} = 0.15$

P_3 = the probability of failure in the IIIrd week $= \dfrac{50-25}{100} = 0.25$

P_4 = the probability of failure in IVth week $= \dfrac{80-50}{100} = 0.30$

P_5 = the probability of failure in Vth week $= \dfrac{100-80}{100} = 0.20$

Here, $P_1 + P_2 + P_3 + P_4 + P_5 = 1$

Since the sum of all probabilities can never be greater than unity, therefore all further probabilities P_6, P_7, P_8 and soon will be zero.

Therefore, a bulb can not survive for more than five weeks *i.e.,* a bulb which has survived for four weeks is sure to fail in the fifth week. Further, we assume that the burnt out bulbs in any week are replaced just at the end of that week.

Let N_i = the number of replacement made at the end of i th week, while all 1000 bulbs were now initially.

Then, we have

N_0 = Number of bulbs in the beginning $= 1000$

N_1 = Number of burnt out bulbs replaced at the end of first week

$= N_0 P_1 = 1000 \times 0.10 = 100$

N_2 = Number of burnt out bulbs replaced at the end of second week

$= N_0 P_2 + N_1 P_1 = 1000 \times 0.15 + 100 \times 0.10 = 160$

N_3 = Number of burnt out bulbs replaced at the end of third week

$= N_0 P_3 + N_1 P_2 + N_2 P_1$

$= 1000 \times 0.25 + 100 \times 0.15 + 160 \times 0.1 = 281$

N_4 = Number of burnt out bulbs replaced at the end of fourth week

$= N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= 1000 \times 0.30 + 100 \times 0.25 + 160 \times 0.15 + 281 \times 0.10 = 377$

N_5 = Number of burnt out bulbs replaced at the end of the fifth week

$= N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$

$= 1000 \times 0.20 + 100 \times 0.30 + 160 \times 0.25 + 281 \times 0.15 + 377 \times 0.10 = 350$

N_6 = Number of burnt out bulbs replaced at the end of the sixth week

$= N_0 P_6 + N_1 P_5 + N_2 P_4 + N_3 P_3 + N_4 P_2 + N_5 P_1$

$= 0 + 100 \times 0.20 + 160 \times 0.15 + 281 \times 0.25 + 377 \times 0.15 + 350 \times 0.10 = 230$

N_7 = Number of burnt out bulbs replaced at the end of the seventh week

$= N_0 P_7 + N_1 P_6 + N_2 P_5 + N_3 P_4 + N_4 P_3 + N_5 P_2 + N_6 P_1$

$= 0 + 0 + 160 \times 0.20 + 281 \times 0.30 + 377 \times 0.25 + 350 \times 0.15 + 230 \times 0.10 = 286$

N_8 = Number of burnt out bulbs replaced at the end of eighth week

$= N_0 P_8 + N_1 P_7 + N_2 P_6 + N_3 P_5 + N_4 P_4 + N_5 P_3 + N_6 P_2 + N_7 P_1$

$= 0+0+0+281 \times 0.20+377 \times 0.30+350 \times 0.25+230 \times 0.15+289 \times 0.10=320$

and soon.

Thus, we find that the number of bulbs failing each week increases till the fourth week, then decreases and again increasing from seventh week. Thus, N_i will continue to oscillate till the system attains a steady state.

The average life of a bulb

$= \Sigma X_i P_i$, where X_i = week, P_i = Probability of failure in the i-th week

$= X_1 P_1 + X_2 P_2 + X_3 P_3 + X_4 P_4 + X_5 P_5 + X_6 P_6 + \ldots$

$= 1 P_1 + 2 P_2 + 3 P_3 + 4 P_4 + 5 P_5 + 0$

$= 0.10 + 2 \times 0.15 + 3 \times 0.25 + 4 \times 0.30 + 5 \times 0.20 = 3.35$

$\therefore$ Average number of replacement per week $= \dfrac{N}{\text{(Mean Age)}} = \dfrac{1000}{3.35} = 299$

$\therefore$ Average cost of weekly individual policy = ₹ 299

Now we consider the case of group replacement.

End of Week	Total cost of group replacement in ₹	Average cost per week in ₹
1	$1000 \times 0.25 + 100 \times 1 = 350$	350
2	$1000 \times 0.25 + (100 + 160) \times 1 = 510$	255
3	$1000 \times 0.25 + (100 + 160 + 281) \times 1 = 791$	263.66

Thus, the minimum cost per week is ₹ 255.00 if the bulbs are replaced as a group after every two weeks and this cost is also less than the average cost of weekly individual replacement policy. It is optimal to have a group replacement after every **two weeks.**

Example 16: *The following mortality rates have been observed for a certain type of light bulbs:*

End of week :	1	2	3	4	5	6
Probability of failure to date :	0.09	0.25	0.49	0.85	0.97	1.00

There are large number of such bulbs which are to be kept in working order. If a bulb fails in service, its costs ₹ 3 to replace but if all the bulbs are replaced in the same operation, it can be done for only ₹ 0.70 a bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing burnt out bulbs as they fail.

(i) *What is the best interval between group replacements?*

(ii) *At what group replacement price per bulb, would a policy of strictly individual replacement become preferable to the adopted policy ?*

Solution: Let the total number of bulbs in use be 1000. Let P_i be the probability that a new light bulb fails during the ith of its life.

Thus, we have

$$P_1 = 0.09$$

$$P_2 = 0.25 - 0.09 = 0.16$$

$$P_3 = 0.49 - 0.25 = 0.24$$

$$P_4 = 0.85 - 0.49 = 0.36$$

$$P_5 = 0.97 - 0.85 = 0.12$$

$$P_6 = 1.00 - 0.97 = 0.03$$

Since the sum of all the probabilities is unity, all probabilities higher than P_6 must be zero *i.e.,* $P_7 = P_8 = P_9$, etc. $= 0$. Thus all light bulbs are sure to burn out by the 6^{th} week.

Further we assume

(i) That light bulbs which fail during a week are replaced just before the end of that week.

(ii) That the actual percentage of the failures during a week for a sub-population of the bulbs with the same age is the same as the expected percentage of failures during the week for that sub-population.

Let N_i represent the number of replacements made at the end of ith week when all the 1000 bulbs are new initially. Then we have

$$N_0 = N_0 \qquad = 1000$$

$$N_1 = N_0\, P_1 = 1000 \times 0.09 \qquad = 90$$

$$N_2 = N_0\, P_2 + N_1\, P_1 = 1000 \times 0.16 + 90 \times 0.09 \qquad = 168$$

$$N_3 = N_0\, P_3 + N_1\, P_2 + N_2\, P_1$$

$$= 1000 \times 0.24 + 90 \times 0.16 + 168 \times 0.09 \qquad = 269$$

$$N_4 = N_0\, P_4 + N_1\, P_3 + N_2\, P_2 + N_3\, P_1$$

$$= 1000 \times 0.36 + 90 \times 0.24 + 168 \times 0.16 + 269 \times 0.09 \qquad = 432,$$

$$N_5 = N_0\, P_5 + N_1\, P_4 + N_2\, P_3 + N_3\, P_2 + N_4\, P_1$$

$$= 1000 \times 0.12 + 90 \times 0.36 + 168 \times 0.24 + 269 \times 0.16 + 432 \times 0.09 = 274$$

$$N_6 = N_0\, P_6 + N_1\, P_5 + N_2\, P_4 + N_3\, P_3 + N_4\, P_2 + N_5\, P_1$$

$$= 1000 \times 0.03 + 90 \times 0.12 + 168 \times 0.36 + 269 \times 0.24$$

$$+ 432 \times 0.16 + 274 \times 0.09 \qquad = 260$$

$$N_7 = 0 + N_1 P_6 + N_2 P_5 + N_3 P_4 + N_4 P_3 + N_5 P_2 + N_6 P_1$$

$$= 90 \times 0.03 + 168 \times 0.12 + 269 \times 0.36 + 432 \times 0.24$$

$$+ 274 \times 0.16 + 260 \times 0.09 \qquad = 291$$

and so on.

Thus, we find that the number of bulbs failing each week increases till the 4th week, then decreases and again increases from 7th week. Thus N_i, will continue to oscillate till the system attains a steady state.

Average (expected) life of light bulbs = $\sum_{i=1}^{6} X_i P_i$

$$= X_1 P_1 + X_2 P_2 + X_3 P_3 + X_4 P_4 + X_5 P_5 + X_6 P_6$$

$$= 1 \times 0.09 + 2 \times 0.16 + 3 \times 0.24 + 4 \times 0.36 + 5 \times 0.12 + 6 \times 0.03 = 3.35$$

$\therefore$ Average number of failures per week = $\dfrac{1000}{3.35} = 299.$

$\therefore$ Cost individual replacement of bulbs = ₹ $3 \times 299 =$ ₹ 897

Since the replacement of all the 1,000 bulbs in one operation costs ₹ 0.70 per bulb and replacement of an individual bulb cost ₹ 3. The total cost of replacement is

End of Week	Total cost of group replacement (₹)	Average cost per week (₹)
1.	$1000 \times 0.70 + 90 \times 3 = 970$	970.00
2.	$1000 \times 0.70 + 3\,(90 + 168) = 1474$	737.00
3.	$1000 \times 0.70 + 3\,(90 + 168 + 269) = 2281$	760.33

(i) As the average minimum cost is in the 2nd week, it is optimal to have a group replacement after every two weeks.

(ii) Let ₹ x be the group replacement price for bulb. Then

$$₹\ 897 < \frac{1,000\,x + 3\,(90 + 168)}{2}$$

$$\therefore \qquad x > ₹\ 1.02.$$

Example 17: *It has been suggested by a data processing firm that they adopt a policy of periodically replacing all the tubes in a certain piece of equipment. A given type of tube is known to have the mortality distribution shown in the table:*

Tube failures/week :	1	2	3	4	5
Probability of failure :	0.3	0.1	0.1	0.2	0.3

There are approximately 1000 *tubes of this type in all the combined equipment. The cost of replacing the tubes on an individual basis is estimated to be* ₹1.00 *per tube and the cost of a group replacement policy average* ₹ 0.30 *per tube. Compare the cost of preventive replacement with that of remedial replacement.*

Solution: Let P_i be the probability that a tube, which was new when placed in the position for use, fails during the ith week of its life. Then, we have

$$P_1 = 0.3,\ P_2 = 0.1,\ P_3 = 0.1,\ P_4 = 0.2, \text{ and } P_5 = 0.3$$

Let N_i = the number of replacements at the end of the ith week.

$$N_0 = \text{number of tubes in the beginning} \qquad = 1000$$

$$N_1 = N_0 P_1 = 1000 \times 0.3 \qquad = 300$$

$$N_2 = N_0 P_2 + N_1 P_1 = 1000 \times 0.1 + 300 \times 0.3 \qquad = 190$$

$$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$$

$$= 1000 \times 0.1 + 300 \times 0.1 + 190 \times 0.3 \qquad = 187$$

$$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$$

$$= 1000 \times 0.2 + 300 \times 0.1 + 190 \times 0.1 + 187 \times 0.3 \qquad = 305$$

$$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$$

$$= 1000 \times 0.3 + 300 \times 0.2 + 190 \times 0.1 + 187 \times 0.1 + 305 \times 0.3 \qquad = 489$$

and so on

Thus, we find that the number of tubes failing each week decrease till the third week, then increase in 4^{th} and 5^{th} week, Thus, N_i, will continue to oscillate till the system attains a steady state.

The expected life of each tubes $= \sum_{i=1}^{5} P_i X_i$

$$= P_1 X_1 + P_2 X_2 + P_3 X_3 + P_4 X_4 + P_5 X_5$$
$$= 0.3 \times 1 + 0.1 \times 2 + 0.1 \times 3 + 0.2 \times 4 + 0.3 \times 5 = 3.10$$

Average number of failures per week $= \dfrac{1000}{3.10}$

$$= 323 \text{ (approx.)}$$

Therefore, the cost of individual replacement

$$= ₹\,(323 \times 1) = ₹\,323$$

Now, since the replacement of all the 1000 tubes simultaneously cost ₹ 0.30 per tube and the replacement of an individual tube on failure costs ₹ 1.

Now consider the cost of group replacement

End of Week	Total (Group + individual) cost of group replacement in ₹	Average cost (₹)
1	$1000 \times 0.30 + 300 \times 1 = 600$	600
2	$1000 \times 0.30 + (300 + 190) \times 1 = 790$	395
3	$1000 \times 0.30 + (300 + 190 + 187) \times 1 = 977$	326
4	$1000 \times 0.30 + (300 + 190 + 187 + 305) \times 1 = 1282$	321 ←
5	$1000 \times 0.30 + (300 \times 190 + 187 + 305 + 489) \times 1 = 1771$	354

Thus, the minimum cost per week is ₹ 321 if the tubes are replaced as a group after every 4 week and this cost is also less than the average cost of weekly individual replacement policy. It is optimal to have a group replacement after every four weeks.

Example 18: *A large computer installation contains* 2000 *components of identical nature which are subject to failure as per probability distribution given below*:

Week end	1	2	3	4	5
Percentage failure to date	10	25	50	80	100

Components which fail have to be replaced for efficient functioning of the system. If they are replaced as and when failure occur, the cost of replacement per unit is ₹ 3. Alternatively if all components are replaced in one lot at periodical intervals and individually replaced only as such failures occur between group replacement, the cost of component replaced is ₹ 1

(*i*) *Assess which policy of replacement would be economical.*

(*ii*) *If group replacement is economical at current costs, then assess at what cost of individual replacement would group replacement be uneconomical.*

(*iii*) *How high can the cost per unit in group replacement be to make a preference for individual replacement policy?*

Solution: (i) Assume P_i is the probability that a component fails during the ith week of its life. Then, we have

$$P_1 = \frac{10}{100} = 0.10$$

$$P_2 = \frac{25-10}{100} = 0.15$$

$$P_3 = \frac{50-25}{100} = 0.25$$

$$P_4 = \frac{80-50}{100} = 0.30$$

$$P_5 = \frac{100-80}{100} = 0.20$$

Let N_i = the number of replacements made at the end of i th week. Then, we have

initial number of components.

N_0 = initial number of components. $= 2000$

$N_i = N_0 P_1 = 2000 \times 0.10 \qquad = 200$

$N_2 = N_0 P_2 + N_1 P_1 = 2000 \times 0.15 + 200 \times 0.10 \qquad = 320$

$N_3 = N_0 P_3 + N_1 P_2 + N_2 P_1$

$= 2000 \times 0.25 + 200 \times 0.15 + 320 \times 0.10 \qquad = 562$

$N_4 = N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= 2000 \times 0.30 + 200 \times 0.25 + 320 \times 0.15 + 562 \times 0.10 \qquad = 754$

$N_5 = N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$

$= 2000 \times 0.20 + 200 \times 0.30 + 320 \times 0.25 + 562 \times 0.15 + 754 \times 0.10$

$= 700$

Thus, we find that number of components which fail, increases till the fourth week and then start decreasing.

Thus, N_i will continue to oscillate till the system attains a steady state.

Expected life of each components $= \sum_{i=1}^{5} X_i P_i$

$= X_1 P_1 + X_2 P_2 + X_3 P_3 + X_4 P_4 + X_5 P_5$

$= 1 \times 0.10 + 2 \times 0.15 + 3 \times 0.25 + 4 \times 0.30 + 5 \times 0.20$

$= 3.35$ weeks

Average number of failures per week $= \frac{2000}{3.35}$

$= 597$ per week

Since, the cost of individual replacement is ₹ 3, the average cost of individual replacement $= ₹\,597 \times 3 = ₹\,1791$

Again, since the cost of 2000 components simultaneously is ₹ 1 per component, the average cost for different group replacement policies is obtain as below:

End of Week	Total (Group + individual) cost of group replacement in ₹	Average cost (₹)
1	$2000 \times 1 + 200 \times 3 = 2600$	2600
2	$2000 \times 1 + (200 + 320) \times 3 = 3560$	1780
3	$2000 \times 1 + (200 + 320 + 562) \times 3 = 5246$	1748
4	$2000 \times 1 + (200 + 320 + 562 + 754) \times 3 = 7508$	1877 ←
5	$2000 \times 1 + (200 + 320 + 562 + 754 + 700) \times 3 = 9605$	1921

Since the average cost is lowest against month 3, the optimal interval between group replacements is 3 months. Also, since the average cost is less than ₹ 1.791 for individual replacement policy, the policy of group replacement is better.

(ii) For a group replacement policy to be uneconomical, the average cost per month of this policy should be greater than the average cost per month of the policy of individual replacement. If k be the cost of an individual replacement (IR), we can determine its value in respect of each of the group replacement policies and then decide on the basis of the least of all the k value obtained.

Group replacement by month (I)	Average cost of group replacement		Average cost of IR		k value
1	$(2000 + 200k)$	$>$	597	$\Rightarrow$	$k < 5.04$
2	$[(2000 + 520k)/2]$	$>$	597	$\Rightarrow$	$k < 2.97$
3	$[(2000 + 1082k)/3]$	$>$	597	$\Rightarrow$	$k < 2.82$ ←
4	$[(2000 + 1836k)/4]$	$>$	597	$\Rightarrow$	$k < 3.62$
5	$[(2000 + 2535k)/5]$	$>$	597	$\Rightarrow$	$k < 4.45$

Since the least of the k values is 2.82, it follows that if the cost of the individual replacement is anything smaller than ₹ 2.82, the group replacement policy would be uneconomical.

(iii) As in (ii) a preference for individual replacement policy implies that the average cost per month of a group replacement policy should be greater then the average monthly case under this policy. With p as the per unit cost of item replacement under group replacement policy, we have

Group Replacement	**Average cost per month**		***p*-value**
	Group replacement IR		
1 month	$(2000p + 600) > 1791$	$\Rightarrow$	$p > 0.60$
2 month	$[(2000p + 1560)/2] > 1791$	$\Rightarrow$	$p > 1.01$
3 month	$[(2000p + 3246)/3] > 1791$	$\Rightarrow$	$p > 1.06^*$
4 month	$[(2000p + 5508.6)/4] > 1791$	$\Rightarrow$	$p > 0.83$
5 month	$[(2000p + 7607.76)/5] > 1791$	$\Rightarrow$	$p > 0.67$

Since the largest of the p value is 1.06, it follows that per unit cost (of replacement on a group basis) of any value greater than ₹ 1.06 would imply that an Individual Replacement Policy would be preferred to a Group Replacement Policy.

7.6 Recruitment and Promotion Problems

Problems concerning recruitment and promotion of staff can sometimes be analysed in a manner similar to that used in replacement problems in industry. In staffing problems, with fixed total staff and fixed size of staff groups, the proportion of staff in each group determines the promotion age, and conversely.

Example 19: *An airline requires* 250 *assistant hostesses,* 350 *hostesses and* 50 *supervisors. Girls are recruited at age* 21 *and, if in service, they retire at age* 60 . *Given the life table, determine*:

(i) *How many girls should be recruited each year?*

(ii) *At what age promotions should take place?*

Age (Years)	*No. in service*	*Age (Year)*	*No. in service*
21	1000	40	125
22	700	41	120
23	500	42	112
24	400	43	105
25	300	44	100
26	260	45	92
27	230	46	88
28	210	47	80
29	195	48	72
30	180	49	65
31	170	50	60
32	165	51	53
33	160	52	45
34	155	53	40
35	150	54	32
36	145	55	26
37	140	56	20
38	135	57	18
39	130	58	15
		59	10
		60	–

Solution: If 1000 girls had been recruited each year for the past 39 years, the total number of them serving upto the age of 59 years = 6603.

Total number of girls required in the airline

$$= 250 + 350 + 50 = 650$$

(i) $\therefore$ Number of girls to be recruited every year in order to maintain a strength of 650

$$= \frac{1000}{6603} \times 650 = 98 \text{ (approximately)}$$

(ii) Let the assistant hostesses be promoted at the age x. Then upto age x-one year, number of assistant hostesses required $= 250$

Now out of 650 girls, 250 are assistant hostesses, therefore out of 1000, their number $= \frac{250}{650} \times 1000 = 385$ (approx.)

From table, this number is available upto the age of 24 years.

$\therefore$ Promotion of assistant hostesses is due in the 25^{th} year.

Now out of 650 girls, 350 are hostesses. Therefore, if we recruit 1000 girls.

The number of hostesses $= \frac{350}{650} \times 1000 = 538$ (approx.)

$\therefore$ Total number of assistant hostesses and hostesses in a recruitment of 1000

$$= 385 + 538 = 923$$

$\therefore$ Number of supervisors required $= 1000 - 923 = 77$

From the table, this number is available upto the age of 47 years.

$\therefore$ Promotion of hostesses is due in the 48^{th} year.

Example 20: *A faculty in a college is planned to rise to a strength of 50 staff members and then to remain at that level. The wastage of recruits depends upon their length of service and is as follows:*

Year	1	2	3	4	5	6	7	8	9	10
Total % who left up to the end of year	5	35	56	65	70	76	80	86	95	100

[B.C.A. (Kurukshetra) 2010]

(*i*) *Find the number of staff member to be recruited every year.*

(*ii*) *If there are seven posts of Head of department for which length of service is the only criterion of promotion. What will be average length of service after which a new entrant should expect promotion.*

Solution: Let us consider the recruitment per year is 100. Then it is clear that the 100 who join in the 1^{st} year will become zero in the 10^{th} year, the 100 who join in the 2^{nd} year will (serve for 9 years and) become 5 at the end of the 10^{th} year and the 100 who join in the 3^{rd} year will (serve for 8 years) and become 14 at the end of the 10^{th} year and so on. Thus, when the equilibrium is attained, the distribution of length of service of staff members will be as follows:

Year	No. of staff members
0	100
1	95
2	65
3	44
4	35
5	30
6	24
7	20
8	14
9	5
10	0
	Total = 432

(i) Thus, if 100 staff members are recruited every year, the total member of staff members after 10 years of service $= 432$

To maintain a strength of 50, the number to be recruited every year

$$= \frac{100}{432} \times 50 = 11.6$$

If it is assumed that they left immediately after completing *x* years service, the total number will become $= 432 - 100 = 332$

And required intake $= 50 \times \dfrac{100}{332} = 15$

But in practice they may leave at any time in the year so that reasonable number of recruitments per year $= \dfrac{11.6 + 15}{2} = 13$ (approx.)

(ii) If we recruit 13 persons every year then we want 7 seniors. Hence, if we recruit 100 every year then we shall require $\dfrac{7}{13} \times 100 = 54$ (approx.) seniors.

It can be soon from **table** and 54 seniors will available if we promote them during 6^{th} year of their service. $(\therefore\ 0 + 5 + 14 + 20 + 24 = 63 > 54)$

Therefore, the promotion of a newly recruited staff member will be due after completing 5 years and before putting in 6 years of service.

Example 21: *Calculate the probability of staff resignation in each year from the following table:*

Year (n)	0	1	2	3	4	5	6	7	8	9	10
No. of original staff in service at the end of the year	1000	940	820	580	400	28	190	130	70	30	0

[B.C.A. (Purvanchal) 2009]

Solution: Let P_t denote the probability of a staff resignation (failure) in the t-th year.

Thus, we have
$$P_t = \frac{N(t-1) - N(t)}{N}$$

where N = total number of staff members is the system

$N(t)$ = total number of staff members in the system at the end of t-th year

Here, $N = 1000$

$$N(0) = 1000, \quad N(1) = 940, \quad N(2) = 820, \quad N(3) = 580,$$
$$N(4) = 400, \quad N(5) = 280, \quad N(6) = 190, \quad N(7) = 130,$$
$$N(8) = 70, \quad N(9) = 30, \quad N(10) = 0.$$

$\therefore$ P_1 = Probability that a staff will resign in 1^{st} year

$$= \frac{N(0) - N(1)}{N} = \frac{1000 - 940}{1000} = 0.06$$

P_2 = Probability that a staff will resign in the 2^{nd} year

$$= \frac{N(1) - N(2)}{N} = \frac{940 - 820}{1000} = 0.12$$

P_3 = Probability that a staff will resign in the 3^{rd} year

$$= \frac{N(2) - N(3)}{N} = \frac{820 - 580}{1000} = 0.24$$

P_4 = Probability that a staff will resign in the 4^{th} year

$$= \frac{N(3) - N(4)}{N} = \frac{580 - 400}{1000} = 0.18$$

P_5 = Probability that a staff will resign in the 5^{th} year

$$= \frac{N(4) - N(5)}{N} = \frac{400 - 280}{1000} = 0.12$$

P_6 = Probability that a staff will resign in the 6^{th} year

$$= \frac{N(5) - N(6)}{N} = \frac{280 - 190}{1000} = 0.09$$

P_7 = Probability that a staff will resign in the 7^{th} year

$$= \frac{N(6) - N(7)}{N} = \frac{190 - 130}{1000} = 0.06$$

P_8 = Probability that a staff will resign in the 8^{th} year

$$= \frac{N(7) - N(8)}{N} = \frac{130 - 70}{1000} = 0.06$$

P_9 = Probability that a staff will resign in the 9^{th} year

$$= \frac{N(8) - N(9)}{N} = \frac{70 - 30}{1000} = 0.04$$

P_{10} = Probability that a staff will resign in the 10^{th} year

$$= \frac{N(9) - N(10)}{N} = \frac{30 - 0}{1000} = 0.03$$

Total $= 1.00$

Since the sum of all these probabilities is 1,

so $\quad P_{11} = P_{12} = P_{13} \ldots = 6.$

Example 22: *A research team is planned to rise to a strength of 50 chemists and then to remain at that level. The wastage of recruits depends on their length of service which is as follows:*

Year	1	2	3	4	5	6	7	8	9	10
Total % who have left upon the end of the year.	5	36	56	63	68	73	79	87	97	100

What is the recruitment per year necessary to maintain the required strength ? There are 8 senior posts for which length of service is the main criterium. What is the average length of service after which new entrant expect his promotion in one of these pasts.

Solution: We see from given table that none of the recruits is continues in the service for more than 10 years.

Now, we construct the following table for 100 recruitment every year.

Year (x) (1)	No. of person who leave at the end of the year (x) (2)	No. of persons in services at the end of year x (3) = 100 − (2)	Prob. of a person to be in service at the end of the year x (4) = (3) / 100, P_x
0	0	100	1.00
1	5	95	0.95
2	36	64	0.64
3	56	44	0.44
4	63	37	0.37
5	68	32	0.32
6	73	27	0.27
7	79	21	0.21
8	87	13	0.13
9	97	3	0.03
10	100	0	0.00
Total	**664**	**436**	

We find from the table that there are 436 chemists in the organisation if 100 are recruited every year.

Therefore, in order to maintain a strength of 50 chemists, one must recruit

$$\frac{100 \times 50}{436} = 12 \text{ chemists every year.}$$

If P_x is the probability of a person to be in service at the end of year x, then out of 12 recruits the total number of survivals at the end of year x will be equal to 12 P_x.

Now, we construct a table showing the total number of chemists in service at the end of each year.

Year x	No. of chemists in service at the end of xth year= 12 P_x
0	$12 \times 1.00 = 12$
1	$12 \times 0.95 = 11$
2	$12 \times 0.64 = 8$
3	$12 \times 0.44 = 5$
4	$12 \times 0.37 = 4$
5	$12 \times 0.32 = 4$
6	$12 \times 0.27 = 3$
7	$12 \times 0.21 = 2$
8	$12 \times 0.13 = 2$
9	$12 \times 0.03 = 0$
10	$12 \times 0.03 = 0$

There are 8 senior posts for which the length of service is the main criterion.

Since there are $3 + 2 + 2 = 7$ chemists in service whose lengths of services are from 6 years to 8 years which is less than the total number of senior posts.

Hence, the proportions of the new entrants will start by the end of 5^{th} year.

Problem Set

1. Explain how the theory of replacement is used in the following problems:

(i) Replacement of items whose maintenance cost varies with time.

(ii) Replacement of items that fail completely. **[B.C.A. (Rohilkhand) 2002, 2004]**

2. The following mortality rates have been observed for a certain type of light bulbs:

Week	1	2	3	4	5
Percent failing by week end	10	25	50	80	100

There are, 1000 bulbs in use and it costs ₹ 2 to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously, it would cost 50 paise per bulb, it is proposed to replace all the bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing burnt out bulbs as they fail. At what intervals should all the bulbs be replaced ?

3. The probability P_n of failure just before age n is shown below. If individual replacement cost ₹ 1.25 and group replacement cost ₹ 0.50 per item, find the optimal group replacement policy.

n	1	2	3	4	5	6	7	8	9	10	11
P_n	0.01	0.03	0.05	0.07	0.10	0.15	0.20	0.15	0.11	0.08	0.05

4. A computer contains 10000 resistors. When only resistor fails it is replaced. The cost of replacing a resistor individually is ₹ 1 only. If all the resistor are replaced at the same time, the per resistor would be reduced to 35 paise. The percent surviving at the end of month t is given below :

Month	0	1	2	3	4	5	6
% surviving at the end of month	100	97	90	70	30	15	0

What is the optimum replacement plan? **[B.B.A. (Meerut) 2010]**

5. The following mortality rates have been found for a certain type of coal cutter motor.

Weeks	10	20	30	40	50
Total % failure up to end of 10 weeks period	5	15	35	65	100

If the motors are replaced over the week end the total cost is ₹ 20. If they fail during the week the total cost is ₹ 100 per failure. Is it better to replace the motors failure and if so when?

6. The probability P_n of failure just before age n are shown below. If individual replacement costs ₹ 1.25 and group replacement costs ₹ 0.50 per item, find the optimal group replacement policy (assume that there are 1000 bulbs in use)

n	1	2	3	4	5	6	7	8	9	10	11
P_n	0.01	0.03	0.05	0.07	0.10	0.15	0.20	0.15	0.11	0.08	0.05

7. Discuss briefly the various type of replacement problems.

[B.C.A. (Agra) 2004, 2012; B.C.A (Lucknow) 2011]

8. State some of the simple replacement policies.
9. What are three strategies of replacement of solving replacement problems?

Answers

2.	Every two weeks
3.	After every 6 weeks
4.	Replace every 3 months
5.	After every 20 weeks
6.	After every 5 weeks

CHAPTER

8

Inventory Theory

8.1 Introduction

An inventory consists of a usable, but idle resources such as money, machines and men. When the resource involved is a material, the inventory is also called ‘stock’. An inventory problem is said to exist if either the resources are subjected to control or if there is at least, one such cost that decreases as inventory increases. The objective is to minimize total cost. However, in situations where inventory affects demand, the objective may also be to maximize profit.

8.2 Necessity for Maintaining Inventory

[B.C.A. (Rohtak) 2007, 2012; B.C.A. (Bhopal) 2011]

1. It helps in smooth and efficient running of an enterprise. It decouples the production from customers and venders and simplifies otherwise complex organization for manufacture and reduces the co-ordination effort.
2. It provides service to the customer at a short notice. Timely deliveries can fetch more goodwill and orders.
3. It helps in maintaining economy by absorbing some of the fluctuations when the demand for an item fluctuates or is seasonal.
4. Process and moment inventories are quite necessary in big enterprises where significant amounts of times are required to tranship items from one location to another.

5. It reduces product costs since there is an added advantage of batching and long, uninterrupted production runs.
6. It acts as a buffer stock when raw materials are received late and shop rejections are too many.

8.3 Inventory Costs

The three costs considered inventory control models are:

1. Inventory carrying cost or stock holding costs.
2. Procurement costs or set-up costs.
3. Storage costs.
4. Total inventory cost.

8.3.1 Inventory Carrying Costs or Stock Holding Costs

They arise an account of maintaining the stocks and the interest paid on the capital tied up with stocks. They vary directly with the size of the inventory as well as the time the item is held in stock. Some components of the stock holding cost are:

1. **Cost of Money or Capital Tied up in Inventories:** More borrowed from the banks may cost interest of about 18%. It is generally taken somewhere around 15% to 20% of the value of the inventories. **[B.C.A. (Agra) 2008]**
2. **Cost of Storage Space:** This consists of rent for space. Besides space expenses, this will also include heating, lighting and other atmospheric control expenses. Typical values may vary from 1 to 3%. **[B.C.A. (Agra) 2008]**
3. **Depreciation and Deterioration Costs:** They are important for fashion items or items undergoing chemical changes during storage. Fragile items such as crockery are liable to damage, breakage etc. 0.2% to 1% of the stock value may be lost due to damage and deterioration.
4. **Pil Ferage Cost:** It depends upon the nature of the item. Valuables such as gun metal bushes and expensive tools may be more tempting. While there is hardly any possibility of heavy costing or forging being stolen. While the former must be kept under lock and key, the latter may be simply dumped in the stockyard. It may be consider as 1% of stock value.
5. **Obsolescence Cost:** It depends upon the nature of the item in stock. Changes in design also lead to obsolescence. It may be taken as 5% of the stock value.
6. **Handling Costs:** Expenditure on stock holding is called **handling costs.** Such as cost of labour, overhead cranes, gantries and other machinery used for this purpose. **[B.C.A. (Kashi) 2006, 2008; B.C.A. (Avadh) 2009]**
7. **Taxes and Insurance:** Most organization have insurance cover and this may cost 1% to 2% of the invested capital.

8.3.2 Ordering or Set-up Cost

[B.C.A. (Lucknow) 2010]

Ordering cost includes all cost that do not vary with the size of the order but are incurred each time an order is placed for procuring items outside supplies. The cost per order generally includes:

1. Requisition cost of handling of invoices, stationery, payments etc.
2. Cost of services which includes cost of mailing, telephone calls, transportation and other follow up actions.
3. Materials handling cost in curred in receiving, sorting, inspecting and storing the items included in the order.
4. Accounting and auditing, etc.

When an item is produced internally, ordering cost is referred as **set-up cost** which includes both paper work costs and physical preparation costs.

Ordering or set-up cost is independent of the size of the order, if large number of orders are placed, more money will required for procuring the items. Thus,

Ordering cost = (cost per order or per set-up) × (number of orders or set-ups places in the planning period).

8.3.3 Shortage or Stock Out and Customer-service Cost

[B.C.A. (Rohilkhand) 2009]

The shortage of items occurs when items cannot be supplied on demand. Therefore, shortage costs are usually interpreted in two ways:

1. The supply of items is awaited by the customers *i.e.*, the items are back ordered.
2. Customers are not ready to wait.

This situation may lead to less of customer goodwill and therefore causes loss of sale. Therefore,

Shortage cost = (cost of being short one unit of an item) × (average number of units cost)

The average number of units short in a planning period is obtain by average number of units short

$$= \left(\frac{\text{minimum shortage} + \text{maximum shortage}}{2}\right) \times \text{period of shortage}$$

8.3.4 Total Inventory Cost

If unit of an item depends on the quantity purchase, then we should formulate an inventory policy which takes into consideration the purchase cost of the items held in stock also.

$\therefore$ Total inventory cost = purchase cost + ordering cost + carrying cost + shortage cost.

When price discounts are not offered, the purchase cost remains constant and is independent of the quality purchased. Then,

Total variable inventory cost = ordering cost + carrying cost + shortage cost.

8.4 Replenishment Lead Time

1. **Order Cycle:** The order cycle is the time period between two successive replenishments which can be determined in one of the two ways.
   (i) Continuous review
   (ii) Periodic review
2. **Lead Time or Delivery Lag:** When an order is placed, it may require sometime before delivery of the items ordered is reached. The time delay between placing an order and receipt of delivery is called **delivery lag** or **lead time**. This time may be deterministic or probabilistic.
3. **Stock Replenishment:** An inventory may operate with lead time, actual replenishment of stock may occur instantaneously or gradual, the instantaneous replenishment is possible when the stock is purchased from outside sources, while gradual replenishment is possible due to finite production rate within the firm.

8.5 Inventory Control Problem

The inventory control problem consists of determination of two basic factors.

1. **When to Order:** This is related to the lead time of an item.

   There should be sufficient stock for each item so that customers, order can be reasonably met from this stock until replenishment. This stock level known as reorder level. It is obtain by compromising the cost of maintaining these stocks and the dis-service to the customer if his orders are not filled in time.
2. **How must to Order:** We know each order is related with its the ordering cost. To maintain it low, the number of orders should be as few as possible. But large order size would imply high inventory carrying cost. Thus over problem is determine how must order is solved by compromising between the acquisition costs and inventory carrying cost.

8.6 Concept of Economic Ordering Quantity (E.O.Q.)

The concept is that management is confronted with a set of opposing costs as the lot size increases, the carrying charges will increase while the ordering cost will decrease or we can say that as the lot size decreases the carrying cost will decrease but the ordering costs will increase. Thus economic ordering quantity (E.O.Q.) is that size of order which minimizes total annual cost of carrying inventory and cost of ordering under the assumed conditions of certainly and that annual demands are known.

The E.O.Q. concept is applicable under the following conditions:

1. The item is replenished in lots or batches, either by purchasing or by manufacturing.
2. Consumption of items is uniform or continuous. E.O.Q. is that order quantity or optimal order size which minimizes the total cost.

   The model is describe under the following conditions:

   (i) Planning period is one year.

   (ii) Demand is deterministic.

   (iii) Cost of purchases or of one unit is C.

   (iv) Cost of ordering is C_3 and or C_0. For manufacturing goods it is known as set-up cost.

   (v) The cost of holding or inventory carrying cost is C_1 or C_h per unit per year expressed.

   (vi) Shortage cost or backorder cost is C_2 or C_s per unit per year.

   (vii) Leading time is L, expressed in unit time.

   (viii) Cycle period is replenishment is t.

   (ix) Order size is Q.

8.7 List of Symbols Used

The following symbols generally used in inventory theory.

C = purchase or manufacturing cost of an item (₹ per unit)

C_0 = ordering or set-up cost per order (₹ per order)

C_s = shortage cost per unit per time (₹ per unit time)

$C_h = C.r$ = cost of carrying one unit of an item in the inventory for a given length of time (₹ per item per unit time)

D = annual demand (requirement) of an item (units per unit time)

Q = order quantity *i.e.*, number of units ordered per order

R = demand rate

ROL = reorder level

LT = replenishment lead time

n = number of orders per time period

t = reorder cycle time

t_p = production period

r_p = production rate $= K$

T_c = total inventory cost (₹)

TVC = total variable inventory cost (₹)

L = lead time

q = quantity already present in the beginning

z = order level or stock level

t = scheduling time period which is variable.

> **NOTE:**
> Sometime the cost C_h, C_s and C_0 also denoted by C_1, C_2 and C_3.

8.8 Deterministic Models

8.8.1 Single Item Inventory Control Models without Shortages

Model (I): The economic lot size system with uniform demand.

Theorem:

Consider a period T during which R units of an item are required. The assumptions for this model are as follows:

1. Demand is deterministic and is at uniform rates r units of quantity, per unit of time.
2. The inventory is replenished as soon as the level of the inventory reaches to zero. Thus the shortages are not allowed, in other word, we can say that shortage cost is infinite so that it can not be considered in the cost of minimization problem.
3. The rate of replenishment of inventory is infinite *i.e.*, the production or supply of items to the inventory is instantaneous or say items are procured in one lot.
4. Lead time is zero.
5. The lot size is same for each cycle q *i.e.*, $0 < q < \infty$.
6. Set-up cost is C_3 per cycle.
7. Carrying cost is C_1 per unit of quantity per unit of time.

[B.C.A. (Lucknow) 2004, 2010; B.C.A. (Bhopal) 2009]

[B.C.A. (I.G.N.O.U.) 2007, 2012]

Proof: We have to find optimal value of q which is most economic *i.e.*, gives the minimum total cost.

Let after every time t, the quantity q is produced or ordered throughout the period T. Let q_0 and t_0 be optimal value of q and t respectively which minimize the total cost per unit of time. In this model, we consider only C_1 and C_3 costs.

The figure 8.1 shows the situation of inventory in the time period T.

This type of figure is known as time-inventory graph and is very much helpful in understanding the inventory situation at various times.

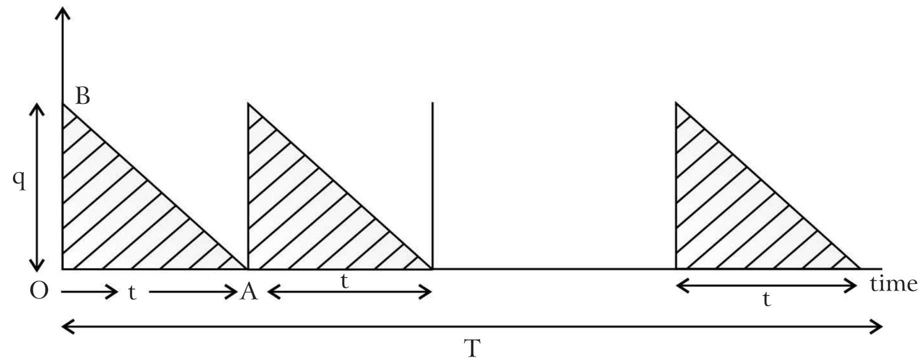


Fig. 8.1

The total inventory in one cycle *i.e.*, for t units of time = area of the $\Delta\, OAB = \frac{1}{2} qt$.

$\because$ The carrying cost for t units of time $= \frac{1}{2} q\, t\, C_1$.

The set-up cost for one cycle *i.e.*, for t units of time $= C_3$

$\therefore$ The total cost in t units of time $= \frac{1}{2} q\, C_1 + C_3$.

$\therefore$ The total cost for one unit of time $= C\,(q) = \left(\frac{1}{2} q\, C_1 + C_3\right)/t$

$$= \frac{1}{2} q\, C_1 + \frac{C_3}{t}.$$

But $$q = rt \quad \text{or} \quad t = \frac{q}{r}$$

Then $$C\,(q) = \frac{1}{2} q\, C_1 + \frac{r}{q} C_3 \qquad \ldots(1)$$

This $C\,(q)$ is called the **cost equation**.

This cost must be minimized, then $\frac{d\,C\,(q)}{dt} = 0$

$$\therefore \qquad \frac{dC\,(q)}{dq} = \frac{1}{2} C_1 - \frac{C_3\, r}{q^2}$$

$$\Rightarrow \qquad 0 = \frac{1}{2} C_1 - \frac{C_3\, r}{q^2}$$

or $$q_0 = \sqrt{\frac{3\, C_3\, r}{C_1}} \qquad \ldots(2)$$

Again $$\frac{d^2 C\,(q)}{dq^2} = \frac{2 C_3\, r}{q^3}$$

At $q = q_0$, we have $\left(\frac{d^2 C\,(q)}{dq^2}\right)_{q_0} = \frac{2 C_3\, r}{q_0^3} > 0$

$\Rightarrow\ q = q_0$ minimizes the cost $C(q)$.

Equation (2) is known as Harri's or Wilson's economic lot size formula or simply economic lot-size formula and q_0 is also taking as $E.O.Q.$

Again $$t_0 = \frac{q_0}{r} = \sqrt{\frac{2C_3}{C_1 r}} \quad \text{...(3)}$$

Also minimum cost $$= C_0(q_0) = \frac{1}{2}\sqrt{\frac{2C_3 r}{C_1}}\, C_1 + r\, C_3 \sqrt{\frac{C_1}{2C_3 r}}$$

$$= \frac{1}{2}\sqrt{2C_1C_3 r} + \frac{1}{2}\sqrt{2C_1C_3 r} = \sqrt{2C_1C_3 r} \quad \text{...(4)}$$

Remark: 1. Total inventory in one cycle *i.e.*, for t units of time

$$= \frac{1}{2} q\, t.$$

The average inventory at any time $= \frac{1}{2} qt / t = \frac{1}{2} q$.

Therefore throughout the period T, average level of inventory is $q/2$.

2. The optimum number of orders to be placed by the formula

$$N = r/q = \frac{\text{total annual quantity requirement}}{\text{economic ordering quantity}}.$$

3. The number of days, supply per optimum is obtain by formula $d = 365/N$.

Solved Examples

Example 1: *A particular item has a demand of 9,000 units per year. The cost of one procurement is ₹ 100 and the holding cost per unit is ₹ 2.40 per year. The replacement is instantaneous and no shortages are allowed. Determine*

(i) The economic lot size

(ii) The number of orders per year

(iii) The time between orders

(iv) The total cost per year if the cost of one unit is ₹ 1.

[B.C.A. (I.G.N.O.U.) 2006; B.C.A. (Kanpur) 2007]

Solution: Here $r = 9{,}000$ units / year

$C_3 =$ ₹100 / procurement

$C_1 =$ ₹2.40 / unit / year

(i) $$q_0 = \sqrt{\frac{2C_3 r}{C_1}} = \sqrt{\frac{2\times 100\times 9{,}000}{2.40}} = 866 \text{ units/procurment}$$

(ii) The number of orders per year

$$= N_0 = \frac{1}{t_0} = \sqrt{\frac{C_1 r}{2 C_3}}$$

$$= \sqrt{\frac{2.40 \times 9{,}000}{2 \times 100}} = \sqrt{108} = 10.4 \text{ orders / year}$$

(iii) The time between orders $= t_0 = \frac{1}{N_0} = \frac{1}{10.4} = 0.0962$ years between procurement.

(iv) The total cost per year

$$= 9{,}000 + \sqrt{2 C_1 C_3 r}$$

$$= 9{,}000 + \sqrt{2 \times 2.40 \times 100 \times 9{,}000}$$

$$= 9{,}000 + 2{,}080 = ₹11{,}080 \text{ / year.}$$

Example 2: *A precision engineering factory consumes 50,000 units of a component per year. The ordering, receiving and handling costs are ₹ 3 per order while the trucking costs are ₹ 12 per order. Further details are as follows:*

Interest cost ₹ 0.06 per unit per year.

Deterioration and obsolescence cost ₹ 0.004 per unit per year.

Storage cost ₹ 1,000 per year for 5,000 units; calculate the economic order quantity.

[B.C.A. (Indore) 2012]

Solution: Here,

$$C_3 = ₹3 + ₹12 = ₹15 \text{ / order}$$

$$r = 50{,}000$$

$$C_1 = ₹0.06 + ₹0.004 + ₹\frac{1000}{50{,}000} \text{ per unit}$$

$$= ₹0.084 \text{ / unit}$$

$$\therefore \quad \text{E.O.Q.} = \sqrt{\frac{2 C_3 r}{C_1}} = \sqrt{\frac{2 \times 50{,}000 \times 15}{0.084}} = 4226 \text{ units.}$$

Example 3: *A stockist has to supply 400 units of a product every Monday to his customers. He gets the product at ₹ 50 per unit from the manufacturer. The cost of ordering and transportation from the manufacturer is ₹ 75 per order. The cost of carrying inventory is 7.5% per year of the cost of the product. Find*

(i) The economic lot size

(ii) The total optimal cost (including the capital cost). **[B.C.A. (Avadh) 2003]**

Solution: Here

$$r = 400 \text{ units / week}$$

$$C_3 = ₹75 \text{ / per order}$$

$$C_1 = 7.5\% \text{ / per year of the cost of the product}$$

$$= ₹\left(\frac{7.5}{100}\times 50\right) \text{ per unit per year}$$

$$= ₹\left(\frac{7.5}{100}\times\frac{50}{52}\right) \text{ per unit per week}$$

$$= ₹\,\frac{3.75}{52} \text{ per unit per week}$$

(i) The economic lot size $q_0 = \sqrt{\dfrac{2\,C_3\,r}{C_1}}$

$$= \sqrt{\frac{2\times 75\times 400\times 52}{3.75}} = 912 \text{ units/order}$$

(ii) The total capital cost $= 400\times 50 + \sqrt{2\,C_1\,C_3\,r}$

$$= 20{,}000 + \sqrt{\frac{2\times 3.75}{52}\times 75\times 400}$$

$$= 20{,}000 + 65.80 = ₹\,20{,}065.80 \text{ per week.}$$

Example 4: *A manufacturer has to supply* 12,000 *units of a product per year to his customer. The demand is fixed and known and shortage cost is assumed is to be infinite. The inventory holding cost is* ₹ 0.20 *per unit per month and the set-up cost per run is* ₹ 350. *Determine*

(i) The optimum run size q_0

(ii) Optimum scheduling period t_0

(iii) Minimum total variable yearly cost.

Solution: Here

$$\text{Supply rate } r = \frac{12{,}000}{12} = 1{,}000 \text{ units / month}$$

$$C_1 = ₹\,0.20 \text{ per unit per month}$$

$$C_3 = ₹\,350 \text{ per run}$$

(i) $q_0 = \sqrt{\dfrac{2\,C_3\,r}{C_1}} = \sqrt{\dfrac{2\times 350\times 1{,}000}{0.20}} = 1{,}870$ units/run

(ii) $t_0 = \sqrt{\dfrac{2\,C_3}{C_1\,r}} = \sqrt{\dfrac{2\times 350}{0.20\times 1{,}000}} = 1.870$ units/run

$$= 8.1 \text{ weeks between runs.}$$

(iii) Minimum total variable yearly cost $C_0 = \sqrt{2\,C_1\,C_3\,r}$

$$= \sqrt{2\times 0.20\times 12\times 350\times (1{,}000\times 12)}$$

$$= ₹\,4{,}490 \text{ per year}$$

Example 5: *(i) Calculate the E.O.Q. in units and total variable cost for the following items, assuming an ordering cost of ₹ 5 and a holding cost of 10%.*

Item	*Annual Demand*	*Unit Price (₹)*
A	800 *units*	0.02
B	400 *units*	1.00
C	392 *units*	8.00
D	13, 800 *units*	0.20

(ii) *For the above problem, compute E.O.Q. in ₹, as well as in years of supply. Also calculate the E.O.Q. frequency for each of the four items.*

Solution: (i) **For item A**

$$\text{The E.O.Q.} \quad (q_0) = \sqrt{\frac{2C_3 r}{C_2}} = \sqrt{\frac{2 \times 5 \times 800}{0.02 \times \frac{10}{100}}} = \sqrt{\frac{8{,}000}{0.002}} = 2{,}000 \text{ units}$$

$$\text{Total variable cost } (C_0) = \sqrt{2\, C_1\, C_3\, r}$$

$$= \sqrt{2 \times 5 \times 800 \times 0.02 \times \frac{10}{100}} = ₹\, 4$$

For item B

$$q_0 = \sqrt{\frac{2 \times 5 \times 400}{1.00 \times \frac{10}{100}}} = 200 \text{ units}$$

$$C_0 = \sqrt{2 \times 5 \times 400 \times 1.00 \times \frac{10}{100}} = ₹\, 20$$

For item C

$$q_0 = \sqrt{\frac{2 \times 5 \times 392}{8.00 \times \frac{10}{100}}} = 70 \text{ units}$$

$$C_0 = \sqrt{2 \times 5 \times 392 \times 8.00 \times \frac{10}{100}} = ₹\, 56$$

For item D

$$q_0 = \sqrt{\frac{2 \times 5 \times 13{,}800}{0.20 \times \frac{10}{100}}} = 2{,}627 \text{ units}$$

$$C_0 = \sqrt{2 \times 5 \times 13{,}800 \times 0.20 \times \frac{10}{100}} = ₹\, 52.54$$

(ii) E.O.Q. in ₹

$$\text{For item } A = 2{,}000 \times 0.02 = 40$$

$$\text{For item } B = 200 \times 1 = 200$$

$$\text{For item } C = 70 \times 8 = 560$$

$$\text{For item } D = 2{,}627 \times 0.20 = 525.40$$

and E.O.Q. in the years of supply

$$\text{For item } A = \frac{2000}{800} = 2.5 \text{ years}$$

$$\text{For item } B = \frac{200}{400} = 0.5 \text{ year}$$

$$\text{For item } C = \frac{70}{392} = 0.18 \text{ year}$$

$$\text{For item } D = \frac{2627}{13{,}800} = 0.19 \text{ year}$$

E.O.Q. frequency (number of orders per year)

$$\text{For item } A = 1/2.5 = 0.4$$

$$\text{For item } B = 1/0.5 = 2$$

$$\text{For item } C = 1/0.18 = 5.6$$

$$\text{and For item } D = 1/0.19 = 5.25$$

8.9 Economic Lot Size with Different Rates of Demand in Different Cycles

In that case all assumptions are same as in first model only change that the demand is uniform with different rates in different cycles.

Let $t_1, t_2, \ldots, t_n$ are the time of successive cycles and $r_1, r_2, \ldots, r_n$ are the demand rates in these cycles respectively, then

$$t_1 + t_2 + \ldots + t_n = T$$

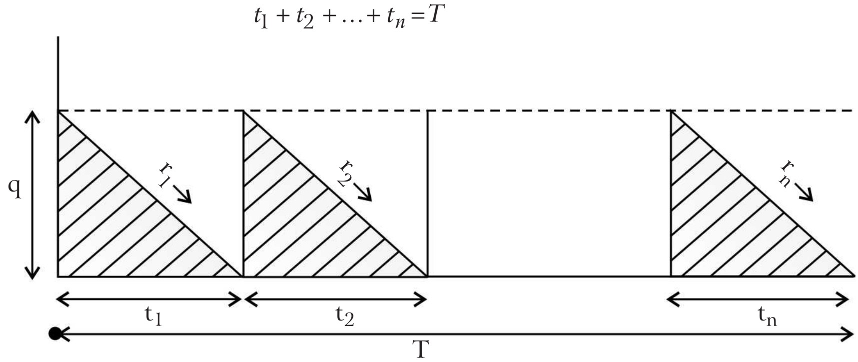


Fig. 8.2

Now, we have to minimize the total cost for time period T.

The inventory carrying cost for the time period T

$$= \frac{1}{2} C_1 \, q \, t_1 + \frac{1}{2} C_1 \, q \, t_2 + \ldots + \frac{1}{2} C_1 \, q \, t_n$$

$$= \frac{1}{2} C_1 \, q \, (t_1 + t_2 + \ldots + t_n) = \frac{1}{2} q \, C_1 \, T$$

Since each time q units are produced, the number of cycles in the total period T is $\frac{r}{q}$, r be the demand of period T.

Therefore the set-up cost for time $T = \frac{C_3 \, r}{q}$

Now, the total cost for the time period T is given by

$$C(q) = \frac{1}{2} C_1 \, q T + C_3 \, \frac{r}{q}$$

$$\Rightarrow \qquad \frac{d \, C(q)}{dt} = \frac{1}{2} C_1 \, T - \frac{C_3 \, r}{q^2} \qquad \ldots(1)$$

For minimum cost put $\frac{d \, C(q)}{dt} = 0$ we get $q_0 = \sqrt{\frac{2 \, C_3 \, r}{C_1 \, T}}$

This is optimal value of q.

The minimum cost $= C_0 \, (q_0) = \sqrt{\frac{2 \, C_1 \, C_3 \, r}{T}}$ ...(2)

8.9.1 Economic Lot Size with Finite Rate of Replenishment or Production Lot Size Model (PLS)

In both models, we consider replenishment time is zero *i.e.*, item order is produced in one lot. This model is similar to first model except that the rate of replenishment of inventory is finite say k unit of quantity per unit of time. We have to obtain an economic lot size q_0 and this model having C_1 and C_3 costs.

Each cycle consists of two parts as shown in figure 8.3. In the first part of time *i.e.*, t_1 the inventory is building up *i.e.*, items are assembled at constant rate of $(k - r)$ units per unit of time and in the second part t_2, there is only withdrawl of items from the inventory at a constant rate of r units per unit of time.

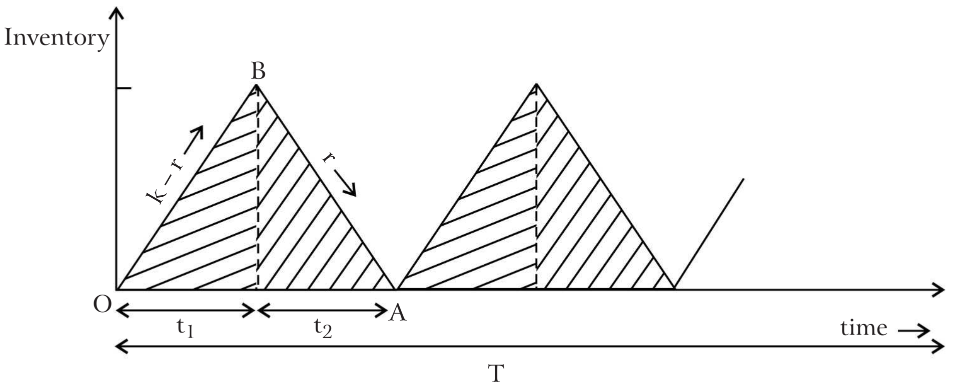


Fig. 8.3

Let at the end of first part of time t_1, the level of inventory is S. Then

$$S = t_1 (k - r) \quad \text{and} \quad S = t_2 r$$

or
$$t_1 = \frac{S}{k-r} \quad \text{and} \quad t_2 = \frac{S}{r} \qquad ...(1)$$

If q is the lot size, then $q - t_1 r = S$

or
$$q - \frac{S}{k-r} r = S \text{ or } S = \frac{k-r}{k} q \qquad ...(2)$$

The carrying cost for one cycle is

$$\left(\frac{1}{2} S t_1 + \frac{1}{2} S t_2\right) C_1 = \frac{1}{2}(t_1 + t_2) S C_1$$

Therefore, carrying cost per unit of time $= \frac{1}{2} \frac{(t_1 + t_2) S C_1}{t_1 + t_2} = \frac{1}{2} S C_1$

$$= \frac{1}{2} \frac{k-r}{k} q C_1$$

The set-up cost for one cycle $= C_3$

$\therefore$ Set-up cost for one unit of time $= \frac{C_3}{t_1 + t_2} = \frac{C}{\frac{S}{k-r} + \frac{S}{r}}$

$$= \frac{C_3}{S} \cdot \frac{r(k-r)}{k} = C_3 \frac{r(k-r)}{k} \cdot \frac{k}{(k-r) q} = \frac{C_3 r}{q}$$

Hence, the total cost for one unit of time is $C(q)$ which is given by

$$C(q) = \frac{1}{2}\left(\frac{k-r}{k}\right) C_1 q + \frac{C_3 r}{q}$$

Then for $C(q)$ is minimum $\Rightarrow \frac{dC(q)}{dq} = 0$

$\Rightarrow$ $$= \frac{1}{2}\left(1 - \frac{r}{k}\right) C_1 - \frac{C_3\, r}{q^2} = 0$$

or $$q = q_0 = \sqrt{\frac{2\, C_3\, r}{C_1} \cdot \frac{1}{\left(1 - \frac{r}{k}\right)}} \qquad ...(3)$$

This is the required optimal lot size.

Also optimum time for one cycle $= t_0 = \dfrac{q_0}{r}$

$$= \sqrt{\frac{2\, C_3}{r\, C_1 \left(1 - \frac{r}{k}\right)}} \qquad ...(4)$$

Putting this value of q, we can obtain the minimum value of cost $C_0\,(q_0)$.

Example 6: *A company has a demand of* 12,000 *units/year for an item and it can produce* 2,000 *such items per month.*

The cost of one set-up is ₹ 400 *and the holding cost/unit/month is* ₹ 0.15. *Find the optimum lot size and the total cost per year, assuming the cost of* 1 *unit at* ₹ 4. *Also find the maximum inventory manufacturing time and total time.* **[B.C.A. (Purvanchal) 2009]**

Solution: Here

$$r = 12{,}000 \text{ units / year}$$

$$k = 2{,}000 \times 12 = 24{,}000 \text{ units / year}$$

$$C_3 = ₹\,400 \text{ / set-up}$$

$$C_1 = ₹\,0.15 \times 12 = ₹\,1.80 \text{ / unit / year}$$

The optimum lot size $$q_0 = \sqrt{\frac{2\, C_3}{C_1} \frac{k\, r}{k - r}}$$

$$= \sqrt{\frac{2 \times 400}{1.8} \times \frac{12{,}000 \times 24{,}000}{12{,}000}} = 3{,}264 \text{ units / set-up}$$

The total cost per year $$= C_0 = 12{,}000 \times 4 + \sqrt{2\, C_1\, C_3\, r \left(\frac{k - r}{k}\right)}$$

$$= 48{,}000 + \sqrt{2 \times 1.8 \times 400 \times 12{,}000 \times \frac{12{,}000}{24{,}000}}$$

$$= 48{,}000 + 2{,}940 = ₹\,50{,}940 \text{ / year}$$

The maximum inventory $$I_{m_0} = \frac{k - r}{k}\, q_0$$

$$= \frac{24{,}000 - 12{,}000}{24{,}000} \times 3{,}264$$

$= 1{,}632$ units

The manufacturing time $t_1 = \dfrac{I_{m_0}}{k-r} = \dfrac{1{,}632}{12{,}000} = 0.136$ years

The total time $t_0 = \dfrac{q_0}{r} = \dfrac{3{,}264}{12{,}000} = 0.272$ years

Example 7: *An item is produced at the rate of 50 items per day. The demand occurs at the rate of 25 items per day. If the set-up cost is ₹ 100.00 per set-up and holding cost is ₹ 0.01 per unit of item per day, find the economic lot size for one run, assuming that the shortages are not permitted.*

Solution: Here, we have

$$k = 50 / \text{day}, r = 25 / \text{day}$$

$$C_1 = ₹ 0.01 / \text{day} \qquad \text{and} \qquad C_3 = ₹ 100 / \text{run}$$

$\therefore$
$$q_0 = \sqrt{\left\{\frac{2 \times 100 \times 25}{0.01 \times \left(1 - \frac{25}{50}\right)}\right\}} = \sqrt{\frac{2 \times 100 \times 25 \times 50}{0.01 \times 25}} = 1000 \text{ items}$$

$\therefore$
$$t_0 = \left(\frac{1000}{25}\right) = 40 \text{ days}$$

The minimum cost = ₹ 5 per day

Therefore, the total cost for one run = ₹ 200.00.

Example 8: *A contractor has to supply* 10, 000 *paper cones per day to a textile unit. He finds that, when he starts a production run he can produce* 25000 *paper cones per day. The cost of holding a paper cone is stock for one year is* 2 *paise and the set-up cost of a production run is* ₹ 18. *How frequently should production run be made ?*

[B.C.A. (Rohilkhand) 2004, 2008; B.C.A. (Kurukshetra) 2007, 2008]

Solution: Here, we have

$$r = 10{,}000 / \text{day}$$

$$k = 25{,}000 / \text{day}, \qquad C_3 = 18$$

$$C_1 = ₹ 0.02 / \text{year} = ₹ \left(\frac{0.02}{365}\right) / \text{day}$$

$\therefore$ The optimal lot size
$$q_0 = \sqrt{\frac{2\, C_3\, r}{C_1} \frac{1}{\left(1 - \frac{r}{k}\right)}}$$

$$= \sqrt{\frac{2 \times 10000 \times 18 \times 365}{0.02} \times \frac{25000}{25000 - 10000}}$$

$= 104447$ paper cones

Also
$$t_0 = \frac{q_0}{k} = \frac{104447}{25000} = 4 \text{ days (approximately).}$$

8.9.2 Limitations of E.O.Q. Formula

In that model the demand is neither exactly known nor uniform. If the fluctuations are mild, the formula serves the purpose while in case of high fluctuations it is not valid. If an inventory level reaches to zero, the instantaneous supply of items is not possible. The determination of ordering cost is difficult and also it is not linearly related to the number of orders. It may not be independent of the order size. Inventory carry charge is also difficult to determine. We generally use only a simple assumption of about 20% while in practice it may vary from 5% to 35%.

Example 9: *A contractor has to supply* 10,000 *bearings per day to an automobile manufacturer. He finds that, when he starts a production run, he can produce* 25,000 *bearings per day. The cost of holding a bearing in stock for one year is* 20 *paise and set-up cost of a production run is* ₹ 180.00.

How frequently should production run be made.

Solution: We have the optimum lot size

$$q_0 = \sqrt{\frac{2\,C_3\,rk}{C_1\,(k-r)}} = \sqrt{\frac{2\,C_3}{C_1}}\sqrt{\frac{rk}{k-r}} \qquad \text{...(1)}$$

and

$$t_0 = \sqrt{\frac{2\,C_3\,k}{r\,C_1\,(k-r)}} = \sqrt{\frac{2\,C_3}{r\,C_1}}\sqrt{\frac{k}{k-r}} \qquad \text{...(2)}$$

Here, we have

$$C_1 = ₹\,0.20 \text{ per bearing per year}$$

$$= ₹\left(\frac{0.20}{365}\right) \text{ per bearing per day}$$

$$= ₹\,0.00055$$

$$C_3 = ₹\,180.00 \text{ per production run}$$

$$r = 10{,}000 \text{ bearings per year}$$

$$k = 25{,}000 \text{ bearings per day}$$

Put these values in (1) and (2), we find

$$q_0 = \sqrt{\frac{2\times180\times10{,}000\times25{,}000}{0.00055\times(25{,}000-10{,}000)}} = \sqrt{1.09\times10^{10}} = 1{,}05{,}000 \text{ bearings}$$

$$t_0 = \sqrt{\frac{2\times180\times25{,}000}{10{,}000\times0.00055\times15{,}000}} = 0.3 \text{ day.}$$

8.10 IInd Model (The E.O.Q. Model with Shortage)

This model is the extension of model I allowing shortages. We discuss this model in three form.

8.10.1 Ist Form Single Item Inventory Control Models with Shortages

1. C_1 is holding cost per quantity unit per unit time.
2. C_2 is the shortage cost per quantity unit per unit time.
3. r quantity per unit time is the demand rate.
4. t_p is the scheduling time period which is constant.
5. q_p is the fixed lot size ($q_p = r\,t_p$).
6. z is the order level to which the inventory is raised in the beginning of each scheduling period. Shortages, if any, have to be made-up. Here z is the variable.
7. Production rate is infinte.
8. Load time is zero.

Then obtain optimal ordering level and minimum average cost. **[B.C.A. (Kanpur) 2010]**

Proof: When $0 \le z \le q_p$, then there exist inventory carrying cost C_1 and shortage cost C_2.

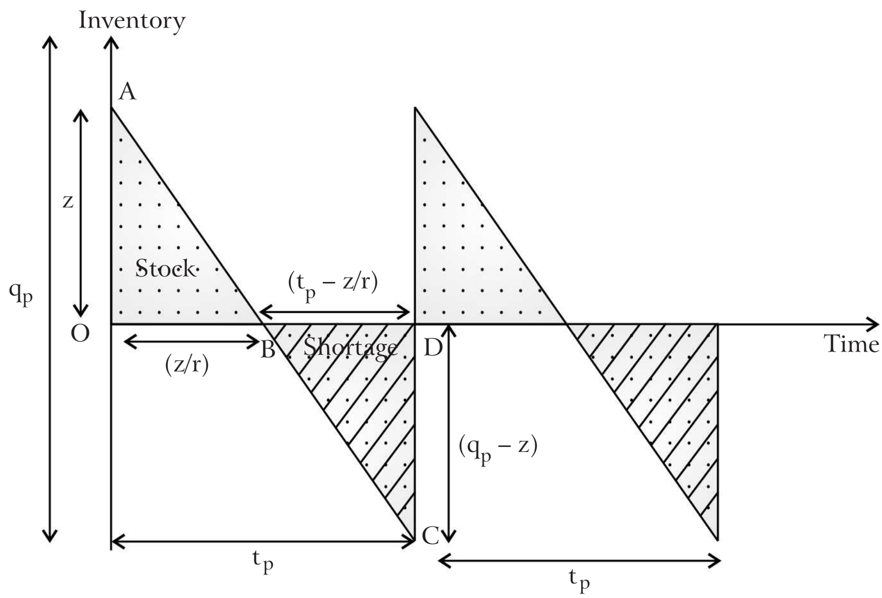


Fig. 8.4

The dotted area (ΔDBC) represents the failure to meet the demand and shady area (ΔAOB) show the inventory. Since q_p is the lot size required to meet the demand for time t_p, but q_p amount of stock is planned in order to meet the demand for time(z/r).

Shortage of amount $(q_p - z)$ will arise for the entire remaining period $(t_p - z/r)$.

The holding cost per unit time $= \dfrac{C_1 (\Delta OAB)}{t_p}$

$$= \frac{C_1}{t_p}\left(\frac{1}{2} \times z \times \frac{z}{r}\right) = \frac{1}{2}\frac{z^2 C_1}{q_p} \quad (\because q_p = r\, t_p) \qquad \text{...(1)}$$

and shortage cost per unit time $= \dfrac{C_2 (\Delta BDC)}{t_p}$

$$\frac{C_2\left(\frac{1}{2} \times BD \times DC\right)}{t_p} = \frac{C_2}{2\, t_p}\left(t_p - \frac{z}{r}\right)(q_p - z)$$

$$= \frac{C_2}{2\, r\, t_p}(r\, t_p - z)(q_p - z) = \frac{1}{2}\frac{C_2}{q_p}(q_p - z)^2 \qquad \text{...(2)}$$

Total average cost $C(z)$

$$C(z) = \frac{1}{2}\frac{z^2 C_1}{q_p} + \frac{1}{2}\frac{C_2}{q_p}(q_p - z)^2 \qquad \text{...(3)}$$

This is called **cost equation**.

Now to find the optimum level z for this put

$$\frac{dC(z)}{dz} = 0$$

$$\therefore \quad \frac{dC(z)}{dz} = \frac{1}{2}\frac{C_1}{q_p}(2z) + \frac{1}{2}\frac{C_2}{q_p}2(q_p - z)(-1)$$

$$\Rightarrow \quad 0 = \frac{1}{2}\frac{C_1}{q_p}(2z) + \frac{1}{2}\frac{C_2}{q_p}2(q_p - z)(-1)$$

$$\Rightarrow \quad z = \frac{C_2}{C_1 + C_2} q_p = \frac{C_2}{C_1 + C_2} r\, t_p \qquad \text{...(4)}$$

Again $\quad \dfrac{d^2 C(z)}{dz^2} = \dfrac{C_1}{q_p} + \dfrac{C_2}{q_p} > 0$

i.e., $C(z)$ is minimum, which is obtain by putting z from (4) in (3)

$$C_{\min} = \frac{1}{2}\frac{C_1}{q_p}\left(\frac{C_2\, q_p}{C_1 + C_2}\right)^2 + \frac{1}{2}\frac{C_2}{q_p}\left(q_p - \frac{C_2\, q_p}{C_1 + C_2}\right)^2$$

or $\quad C_{\min} = \dfrac{1}{2}\dfrac{C_1 C_2}{C_1 + C_2}.q_p = \dfrac{C_1 C_2}{2(C_1 + C_2)} r\, t_p \qquad \text{...(5)}$

8.10.2 IInd Form the E.O.Q. with Constant Rate of Demand Scheduling Time Variable

In a certain manufacturing situation:

1. R is demand per year.
2. The production is instantaneous.
3. $q = r\,t$ is the order quantity per run.
4. t is the scheduling time period which is variable.
5. z is the order level to which the inventory is raised.
6. Lead time is zero.

Show that the optimal order quantity q per-run which minimizes the total cost is

$$q = \sqrt{\frac{2\,r\,C_3\,(C_1 + C_2)}{C_1\,C_2}}$$

where C_1 = holding cost per unit per year

C_2 = shortage cost per unit per year

C_3 = set-up cost per run. **[B.C.A. (Agra) 2004, 2008]**

Proof: This form is same as first form only change with difference that the scheduling period t is not constant here, now to consider the average set-up cost (C_3/t) in the cost equation.

Thus the cost equation in this form becomes

$$C(t, z) = \frac{1}{t}\left\{\frac{C_1 z^2}{2r} + \frac{1}{2}\frac{C_2}{r}(r\,t - z)^2 + C_3\right\} \qquad \text{...(1)}$$

This cost must be minimized so that $\dfrac{\partial C}{\partial z} = 0, \dfrac{\partial C}{\partial t} = 0$

$$\therefore \qquad \frac{\partial C}{\partial z} = \frac{1}{t}\left\{\frac{C_1 z}{r} - \frac{C_2}{r}(rt - z)\right\} = 0$$

$$\Rightarrow \qquad z = \frac{C_2\,rt}{C_1 + C_2} \qquad \text{...(2)}$$

Also

$$\frac{\partial C}{\partial t} = -\frac{1}{t^2}\left\{\frac{C_1 z^2}{2r} + \frac{C_2}{2r}(rt - z)^2 + C_3\right\} + \frac{1}{t}\left\{0 + \frac{C_2}{2r}2(rt - z)\,r + 0\right\} = 0$$

$$\Rightarrow \qquad -\frac{1}{t^2}\left\{\frac{C_1 z^2}{2r} + \frac{C_2}{2r}(rt - z)^2 + C_3\right\} + \frac{C_2}{t}(rt - z) = 0$$

$$\Rightarrow \qquad -(C_1 + C_2)\, z^2 + C_2\, r^2\, t^2 = 2\, r\, C_3 \qquad \text{...(3)}$$

Put z from (2) in (3), we get

$$-\frac{r^2\, t^2\, C_2^2}{C_1 + C_2} + C_2\, r^2\, t^2 = 2\, r\, C_3$$

$$\Rightarrow \qquad C_2\, r^2\, t^2 \left\{1 - \frac{C_2}{C_1 + C_2}\right\} = 2 r\, C_3$$

or
$$t = \sqrt{\frac{2\, C_3\, (C_1 + C_2)}{r\, C_1\, C_2}} \qquad \text{...(4)}$$

Also
$$\frac{\partial^2 C}{\partial t^2} \cdot \frac{\partial^2 C}{\partial z^2} - \left(\frac{\partial^2 C}{\partial t\, \partial z}\right)^2 > 0$$

and
$$\frac{\partial^2 C}{\partial t^2} > 0, \frac{\partial^2 C}{\partial z^2} > 0$$

Optimal order quantity q is given by

$$q_0 = r\, t_0 = r \sqrt{\frac{2\, C_3\, (C_1 + C_2)}{r\, C_1\, C_2}}$$

or
$$q_0 = \sqrt{\frac{2\, r\, C_3\, (C_1 + C_2)}{C_1\, C_2}} \qquad \text{...(5)}$$

This is E.O.Q in this form.

For minimum cost put z from (2) in (1), we have

$$C_{\min} = \frac{C_1}{2 r t} \cdot \frac{C_2^2\, (rt)^2}{(C_1 + C_2)^2} + \frac{C_2}{2 rt}\left(rt - \frac{C_2\, rt}{C_1 + C_2}\right)^2 + \frac{C_3}{t}$$

$$= \frac{C_1\, C_2^2\, rt}{2\, (C_1 + C_2)^2} + \frac{C_2\, (rt)}{2\, (C_1 + C_2)^2}\, (C_1 + C_2 \;\; - C_2)^2 + \frac{C_3}{t}$$

$$C_{\min} = \frac{C_1\, C_2\, r t}{2\, (C_1 + C_2)} + \frac{C_3}{t} \qquad \text{...(6)}$$

Put t from (4), we get

$$C_{\min} = \frac{C_1\, C_2\, r}{2\, (C_1 + C_2)} \cdot \sqrt{\frac{2\, C_3\, (C_1 + C_2)}{r\, C_1\, C_2}} + C_3 \cdot \sqrt{\frac{r\, C_1\, C_2}{2\, C_3\, (C_1 + C_2)}}$$

$$= \sqrt{\frac{C_1\, C_2\, C_3\, r}{2\, (C_1 + C_2)}} + \sqrt{\frac{C_1\, C_2\, C_3\, r}{2\, (C_1 + C_2)}} = \sqrt{\frac{2\, C_1\, C_2\, C_3\, r}{(C_1 + C_2)}}$$

or
$$C_{\min} = \sqrt{2\, C_1\, C_3\, r}\, \sqrt{\frac{C_2}{C_1 + C_2}} \qquad \text{...(7)}$$

8.10.3 III[rd] Form the Production Lot Size Model with Shortages

To determine an economic lot-size formula for optimum production quantity q per cycle of a single product so as to minimize the total average variable cost per unit time, where

1. r units per unit time is the uniform demand rate.
2. Lead time is zero.
3. Production rate is finite (k units per unit time), $k > r$.
4. Inventory carrying cost in ₹ $C_1 = IP$ per quantity per unit time.
5. Shortages are allowed and back logged.
6. Shortage cost in ₹ C_2 per quantity per unit time.
7. Set-up cost in ₹ C_3 per set-up.

Proof: This figure 8.5 shows that there is an inventory cycle. Stock short at zero and increase for a period t_1 and decrease for period t_2 until they again reach zero at the point where a back-log piles up for the time t_3 at the end of time t_3 production starts and backlog is diminished for the time t_4 when the backlog reaches zero.

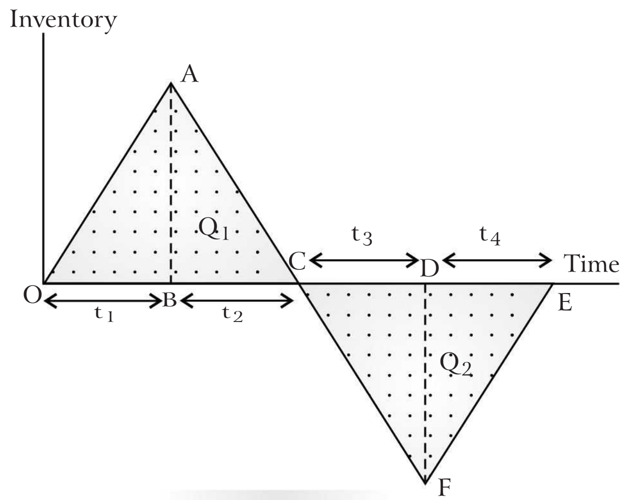


Fig. 8.5

This cycle repeated after time period $(t_1 + t_2 + t_3 + t_4)$.

Then,

Holding cost $= C_1 \times$ area of triangle OAC

$$= C_1 \times \frac{1}{2} Q_1 (t_1 + t_2)$$

Shortage cost $= C_2 \times$ area of triangle EFC

$$= C_2 \times \frac{1}{2} Q_2 (t_3 + t_4)$$

Set-up cost $= C_3$

Thus; average cost $$C = \frac{\frac{1}{2}[C_1 Q_1 (t_1 + t_2) + C_2 Q_2 (t_3 + t_4) + C_3]}{t_1 + t_2 + t_3 + t_4} \quad ...(1)$$

where C is the function of six variables $(Q_1, Q_2, t_1, t_2, t_3, t_4)$, but there are four relationships.

From given we see that the inventory is zero at O and during the period t_1 an account $k\,t_1$ is produced but order are being filled up at a rate r, the net increase Q_1 in inventory during t_1 is given by

$$Q_1 = k\,t_1 - r\,t_1 = t_1 (k - r) \quad ...(2)$$

After time t_1, the production is stopped and the stock Q_1 is used up during t_1 and the rate is r, then

$$Q_1 = r\,t_2 \quad ...(3)$$

From (2) and (3), we have $$t_1 = \frac{Q_1}{k-r} = \frac{r\,t_2}{k-r} \quad ...(4)$$

During period t_3 shortages accumulate at a rate r

$$Q_2 = r\,t_3 \quad ...(5)$$

During period t_4, production rate is k and demand rate is r, so that the net rate of reduction of shortage becomes $k - r$ and thus, we have

$$Q_2 = t_4 (k - r) \quad ...(6)$$

From (5) and (6), we have $$t_4 = \frac{Q_2}{k-r} = \frac{r\,t_3}{k-r} \quad ...(7)$$

$\therefore$ $$q = r(t_1 + t_2 + t_3 + t_4) \quad ...(8)$$

Put t_1 and t_4 from (4) and (7) in (8), we get

$$q = r\left(\frac{r\,t_2}{k-r} + (t_2 + t_3) + \frac{r\,t_3}{k-r}\right)$$

or $$q = \frac{(t_2 + t_3)\,k}{k-r} \quad ...(9)$$

Put t_1, t_4, Q_1, Q_2 in (1), we get

$$C = \frac{\frac{1}{2}\left\{C_1 (rt_2)\left(\frac{r\,t_2}{k-r} + t_2\right) + C_2\, r\,t_3\left(t_3 + \frac{r\,t_3}{k-r}\right) + C_3\right\}}{\frac{r t_3}{k-r} + t_2 + t_3 + \frac{r\,t_3}{k-r}}$$

$$C = \frac{\frac{1}{2}\left\{\frac{C_1\, t_2^2\, rk}{k-r} + \frac{C_2\, t_3^2\, k\,r}{k-r}\right\} + C_3}{(t_2 + t_3)\left(1 + \frac{r}{k-r}\right)}$$

$$C = \frac{\frac{1}{2}\{C_1 t_2^2 + C_2 t_3^2\}\frac{kr}{k-r} + C_3}{(t_2+t_3)\left(\frac{k}{k-r}\right)}$$

$$= \frac{\frac{1}{2}[C_1 t_2^2 + C_2 t_3^2]\,kr + C_3(k-r)}{k(t_2+t_3)}$$

$\therefore$
$$C(t_2, t_3) = \frac{\frac{1}{2}[C_1 t_2^2 + C_2 t_3^2]\,kr + C_3(k-r)}{k(t_2+t_3)} \qquad \text{...(10)}$$

To find best values of t_2^* and t_3^* of t_2 and t_3 for this put

$$\frac{\partial C}{\partial t_2} = 0 \quad \text{and} \quad \frac{\partial C}{\partial t_3} = 0, \text{ we get}$$

$$t_2^* = \sqrt{\frac{2C_3 C_2\left(1-\frac{r}{k}\right)}{r(C_1+C_2)C_1}} \qquad \text{...(11)}$$

$$t_3^* = \sqrt{\frac{2C_3 C_1\left(1-\frac{r}{k}\right)}{r(C_1+C_2)C_2}} \qquad \text{...(12)}$$

Then put $t_2 = t_2^*, t_3 = t_3^*$ in (9), we get

$$q^* = \sqrt{\frac{2rC_3(C_1+C_2)}{C_1 C_2}}\sqrt{\frac{1}{1-r/k}}$$

$$= \sqrt{\frac{2rC_1C_3\left(1-\frac{k}{r}\right)}{(C_1+C_2)C_2}} \qquad \text{...(13)}$$

Now, the minimum cost from (1), we get

$$C^* = \sqrt{\frac{2rC_1C_2C_3\left(1-\frac{r}{k}\right)}{C_1+C_2}} \qquad \text{...(14)}$$

when $k \to \infty$ then (10) becomes

$$C(t_2, t_3) = \left\{\frac{1}{2}r(C_1 t_2^2 + C_2 t_3^2) + C_3\right\} \Big/ (t_2+t_3)$$

and
$$q^* = \sqrt{\frac{2rC_3(C_1+C_2)}{C_1C_2}}, \quad Q^* = \sqrt{\frac{2rC_1C_3}{(C_1+C_2)C_2}},$$

$$C^* = \sqrt{\frac{2rC_1C_2C_3}{(C_1+C_2)}}$$

Example 10: *A sub-contractor undertakes a supply diesel engines to a truck manufacturer at the rate of* 25 *per day. There is a clause in contract penalizing him* ₹ 10 *per engine per day late for missing the scheduled delivery date. He finds that the cost of holding a completed engine in stock is* ₹ 16 *per month. This production process is such that each month* (30 *days*) *he starts a batch of engines through the shops and all these engines are available for delivery any time after the end of the month. What should his inventory level at the beginning of each month* (*i.e., immediately after taking in*) *to stock the engines made in the previous month and then shipping engines to fill unsatisfied demand from previous month ?* **[B.B.A. (Meerut) 2010]**

Solution: Here, $r = 25$ engines per day, $C_1 = ₹\,(16/30)$

$C_2 = ₹10$, $t_p = 30$ days, then inventory level (z) is given by

$$z = \frac{C_2}{C_1 + C_2}\, r t_p = \frac{10}{10 + \left(\dfrac{16}{30}\right)} \times 25 \times 30 = 712 \text{ engines}$$

Example 11: *The demand of an item is uniform at a rates of* 25 *units per month. The fixed cost is* ₹ 15 *each time a production run is made. The production cost is* ₹ 1 *per item and the inventory carrying cost is* ₹ 0.30 *per item per month. If the shortage cost is* ₹1.50 per *item per month, determine how often to make a production run and of what size it should be ?*

[B.C.A. (Bundelkhand) 2007]

Solution: Here, we have

$$C_1 = ₹0.30 \text{ per item per month}$$
$$C_2 = ₹1.5 \text{ per item per month}$$
$$C_3 = ₹15 \text{ per production run}$$
$$r = 25 \text{ units per month}$$
$$b = ₹1 \text{ per item}$$

Then optimum values of q and t are obtained by

$$q^* \text{ or } q_0 = \sqrt{\frac{2r\,C_3\,(C_1 + C_2)}{C_1\,C_2}} = \sqrt{\frac{2 \times 15 \times 25\,(0.30 + 1.50)}{0.30 \times 1.50}} = 54 \text{ items.}$$

and $$t^* \text{ or } t_0 = \frac{q_0}{r} = \frac{54}{25} \text{ months} = 2.16 \text{ months.}$$

Example 12: *The demand for an item is deterministic and constant over time and it is equal to* 600 *units per year. The unit cost of the item is* ₹ 50 *while the cost of placing an order is* ₹5. *The inventory carrying cost is* 20% *of the cost of inventory per annum and the cost of holding is* ₹ 1 *per month. Find the optimal ordering quantity when stock outs are permitted. If the stock outs are not permitted, what would be the loss to the company.*

Solution: We have, $r = 600$ units, $C_1 = 0.20 \times 50 = ₹10$, $C_3 = ₹5$ per order

and $$C_2 = ₹1 \text{ per unit / month} \quad \text{or} \quad ₹12 \text{ per year / unit}$$

When stock outs are permitted, the optimal ordering quantity is

$$q^* = \sqrt{\frac{2r\,C_3}{C_1}\left(\frac{(C_1+C_2)}{C_2}\right)} = \sqrt{\frac{2\times 600\times 5}{10}\left(\frac{10+12}{12}\right)} = 33 \text{ units}$$

Maximum number of back orders $z^* = q^*\left(\frac{C_1}{C_1+C_2}\right) = 33\left(\frac{10}{10+12}\right) = 15$ units

Total expected yearly cost without shortages is given by

$$C\,(q^*) = \sqrt{2r\,C_1C_3\left(\frac{C_2}{C_1+C_2}\right)}$$

$$= \sqrt{2\times 600\times 10\times 5\left(\frac{12}{10+12}\right)} = 180.91 = 181$$

If stock outs or back-orderings are not permitted, the optimal order quantity is

$$q_1^* = \sqrt{\frac{2r\,C_3}{C_1}} = \sqrt{\frac{2\times 600\times 5}{10}} = 24.5 \text{ units}$$

The total relevant cost is given by

$$C\,(q_1^*) = \sqrt{2\,C_1C_3\,r} = \sqrt{2\times 10\times 5\times 600} = \sqrt{60{,}000} = ₹\,245$$

Thus, the additional cost when back-ordering is not allowed

$$= ₹\,(245-181) = ₹\,64.$$

Example 13: *The demand for an item in a company is* 18,000 *units per year and the company can produce the item at a rate of* 3,000 *per month is* 15 *paise. The shortage cost of one unit is* ₹ 20 *per year. Determine the optimum manufacturing quantity and the number of shortages. Also, determine the manufacturing time and the time between set-ups.* **[B.C.A. (Kurukshetra) 2012]**

Solution: Here, we have $C_1 = ₹\,0.15,\ C_2 = ₹\,20,\ C_3 = ₹\,500$

$k = 3{,}000$ units/month, $r = 18{,}000/12$ units/month $= 15{,}00$ units/month

∴ The optimal manufacturing quantity

$$q^* = \sqrt{\frac{2\,C_3\,r\,(C_1+C_2)}{C_1\,C_2}}\cdot\sqrt{\frac{k}{k-r}}$$

$$= \sqrt{\frac{2\times 500\times 1500\times(0.15\times 12+20)}{(12\times 0.15)\times 20}}\cdot\sqrt{\frac{3000}{3000-1500}} = 1344 \text{ units}$$

The number of shortages are given by

$$S = \frac{C_1}{C_1+C_2}\,q^*\left(1-\frac{r}{k}\right) = \left(\frac{0.15\times 12}{0.15\times 12+20}\right)\times 1344\left(1-\frac{1500}{3000}\right)$$

$$= \frac{1.8}{21.8} \times 1344 \times 0.5 = 55.486 \text{ units}$$

$$\text{Manufacturing time } = \frac{q^*}{k} = \frac{1344}{3000 \times 12} = .037 \text{ year}$$

$$\text{Time between set-ups } = \frac{q^*}{r} = \frac{1344}{18{,}000} = .074 \text{ year.}$$

Problem Set

1. Find the E.O.Q. for the following data:

Annual usage = 1,000 pieces	Expediting cost = ₹ 4 per order
Cost per piece = ₹ 250	Inventory holding cost = 20% of average inventory
Ordering cost = ₹ 6 per order	Material holding cost = ₹ 1 per piece

2. You have to supply your customers 100 units of a certain product every Monday (and only then). You obtain the product from a local supplier at ₹ 60 per unit. The costs of ordering and transportation from the supplier are ₹ 150 per order. The cost of carrying inventory is estimated at 15% per year of the cost of the product carried.

   (i) Find the lot size which will minimize the cost of the system.

   (ii) Determine the optimal cost. **[B.C.A. (I.G.N.O.U.) 2004, 2010]**

3. An aircraft company uses rivets at an approximate customer rate of 2,500 kg per year. Each unit cost ₹ 30 per kg and the company personnel estimate that it costs ₹ 130 to place an order and that the carrying cost of inventory is 10% per year. How frequently should orders for rivets be placed ? Also determine the optimal size of each order.

4. An aircraft uses rivets at an approximately constant rate of 5,000 kg per year. The rivets cost ₹ 20 per kg and the company personnel estimate that it costs ₹ 200 to place an order and the carrying cost of inventory is 10% per year.

   (i) How frequently should orders for rivets be placed and what quantities should be order for ?

   (ii) If the actual costs are ₹ 500 to place an order and 15% for carrying cost, the optimal policy would change. How must is the company losing per year because of imperfect cost information ?

5. A contractor has to supply 10,000 bearings per day to an automobile manufacturer. He finds that, when he starts production run, he can produce 25,000 bearings per day. The cost of holding a bearing in stock for a year is ₹ 2 and the set-up cost of a production run is ₹ 1,800. How frequently should production run be made ?

6. A manufacturing company uses an E.O.Q. (Economic Order Quantity) approach in planning its production of gears. The following information is available. Each gear cost ₹250 per unit, annual demand is 60,000 gears, set-up cost are ₹4,000 per set-up and the inventory carrying cost per month is established at 2% of the average inventory value. When in production, these gears can be produced at the rate of 400 units per day and this company works only for 300 days in a year. Determine the economic lot size, the number of production runs per year and the total inventory costs. **[B.C.A. (Delhi) 2012]**

7. A dealer supplies you the following information with regard to a product dealt in by him: Annual demand : 10,000 unit : ordering cost : ₹ 10 per order price ₹ 20 per unit. Inventory carrying cost : 20% of the value of inventory per year. The dealer is considering the possibility of allowing some back order to occur. He has estimated that the annual cost of back ordering will be 25% of the value of inventory.
   (i) What should be the optimum number of units of the product he should busy in one let ?
   (ii) What quantity of the product should be allowed to be back ordered, if any ?
   (iii) What would be the maximum quantity of inventory at any time of the year ?
   (iv) Would you recommend to allow back-ordering ? If so, what would be the annual cost saving by adopting the policy of back-ordering ?

8. The annual demand for a product is 64,000 units or 1280 units per week. The buying cost per order is ₹ 10 and the estimated cost of carrying one unit in the stock for a year is 20%. The normal price of the product is ₹ 10 per unit. However, the supplier offers a quantity discount of 2% on an order of at least 1000 units of a time and a discount of 3% if the order is for at least 5,000 units.

9. An oil engine manufacturer purchases lubricants at the rate of ₹ 42 per piece from a vendor, the requirements of these lubricants is 1,800 per year. What should be the order quantity per order, if the cost per placement of an order is ₹ 16 and inventory carrying charges per rupee per year is only 20 paise. **[B.C.A. (Bhopal) 2008]**

10. A company uses 24,000 units of a raw material which costs ₹ 12.5 per unit. Planning each order costs ₹ 22.5 and the carrying cost is 5.4 percent per year of the average inventory. Find the economic quantity and the total inventory cost (including the cost of material).

11. A manufacturer has to supply his customers 24,000 units of his product per year. The demand is fixed and known. The customer has no storage space and so the manufacturer has to ship a day's supply each day. If the manufacturer fails to supply the penalty is ₹0.20 per unit per month. The inventory holding cost amounts to ₹0.10 per unit per month and the set-up cost is ₹350 per production run. Find the optimum lot size for the manufacturer.

12. A contractor undertakes to supply diesel engines to a truck manufacturer at rate of 25 per day. He finds that the cost of holding a completed engine in stock is ₹16 per month, and there is a clause in the contract penalizing him ₹10 per engine per day late for missing the scheduled delivery date. Production of engines is in batches, and each time a new batch is started there are set-up costs of ₹10,000. How frequently should batches be started and what should be the initial inventory level at the time each batch is completed. **[B.C.A. (Kashi) 2010]**

13. The demand for an item is 18,000 units per year. The holding cost is ₹1.20 per unit time and the cost of shortage is ₹5.00, the production cost is ₹400. Assume that replenishment rate is instantaneous, determine the optimal order quantity.

14. What are assumptions of economic lot size formula ? **[B.B.A. (Meerut) 2011]**

15. Write the note on economic lot size problem with uniform rate of demand.

16. Derive economic order quantity model for an inventory problem when shortages of costs are not allowed.

17. Describe the single item production inventory model with no shortages and derive the formula for optimum lot size for one run and the optimum time between two runs.

18. Derive an economic lot size with different rates of demand in different cycles. **[B.C.A. (Rohilkhand) 2007]**

19. Derive the optimal economic lot size formula

$$q = \sqrt{\frac{2\,C_3\,rk}{C_1\,(k-r)}}$$ in the usual notations when the rate of replenishment is finite. Also, derive the minimum cost formula $C_{\min} = \left\{2\,C_1\left(1-\frac{r}{k}\right)C_3\,r\right\}^{1/2}$.

20. Discuss various inventory costs. Derive economic lot size formula when shortage costs are allowed.

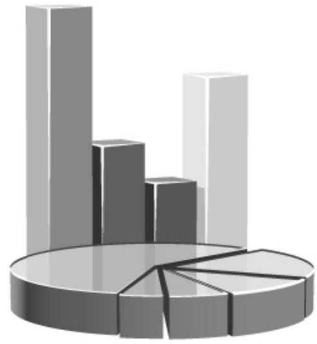
Answers

1.	20 [Hint, $D = 1{,}000$, $C_3 = 6 + 4 = 10$, $C_1 = C \times I = 250 \times 0.20 = 50$, $EOQ = \sqrt{\dfrac{2 \times 10 \times 1000}{250 \times 0.20}} = 20$]	**2.**	(i) 416 units; (ii) ₹ 6,072
3.	466 units, 2.16 months, $n = 5$ orders per year	**4.**	(i) ₹ 2,000; (ii) ₹ 1,873
5.	1,04,446 bearings, 10.44 days	**6.**	Economic lot size = 4,000 gears, Number of production = 15 Total inventory cost = ₹ 1,20,000
7.	(i) 223.6 units, 300 units (ii) 133 units (iii) 167 units (iv) T.C. (223.6) = ₹ 894.43, T.C. (300) = ₹ 666.67	**9.**	$q^* = 83$ lubricants
10.	$q^* = 4{,}000$ units, cost = ₹30,270	**11.**	4,744 units per run
12.	$q^* = 943$ engines, $t^* = q^*/38$ days	**13.**	(i) $q^* = 3{,}600$ (ii) 50 orders per year (iii) C_{min} = ₹6,000

❍❍❍

CHAPTER

9



Job Sequencing

9.1 Sequencing Problem

[B.C.A.(Agra) 2010]

The selection of an appropriate order for a series of jobs to be done on a finite number of a service facilities is called **sequencing**. In a sequencing problem, we have to determine the order (sequence) of performing the jobs in such a way that the total involved cost may be minimum. A general sequencing problem may be defined as follows. Let there be n-jobs $(1, 2, 3 \ldots n)$ which have to be processed one at a time at each of m machines $A, B, C \ldots.$. The order of machines for each job, in which it should be go to the machines is given. The time required by jobs on each of machines is also given. Then the problem is to find the sequence out of $(n!)^m$ sequences, which optimizes (minimizes) the total time elapsed from the start of the first job to the completion of the last job.

Mathematically,

Let A_i = time required for job i on machine A

B_i = time required for job i on machine B

T = total elapsed time for jobs $1, 2, 3 \ldots n$ *i.e.* time from start of the first job to completion of the last job.

The problem is to determine a sequence $(i_1, i_2, \ldots i_n)$ where $(i_1, i_2, \ldots i_n)$ is the permutation of integers which will minimize T.

Analytic methods have been developed for solving only four simple cases:

1. n jobs and two machines A and B: all jobs processed in the order AB.
2. n jobs and three machines A, B and C: all jobs processed in the order ABC.
3. n jobs and m machines $A, B, C\ldots.K$: all jobs processed in the order $ABC\ldots..K$.
4. two jobs and m machines: each job to be processed through the machines in a prescribed order.

9.2 General Assumptions

Following are the basic assumptions under lying a sequencing problem:

1. No machine can process more than one job at a time.
2. Processing times are independent of processing the jobs.
3. The time involved in moving a job from one machine to another is negligibly small.
4. Each job, once started on machine is to be continued till its completion on it.
5. All machines are of different types.
6. All jobs are completely known and are ready for processing before the period under consideration begins.
7. Only one machine of each type is available.
8. A job is processed as soon as possible, but only in the order specified.
9. The cost of in-process inventory for each jobs is same and negligibly small.

9.2.1 Terminology and Notations

1. **Number of Machines:** It means the service facilities through which a job must pass before it is completed.
2. **Processing Order:** The order in which various machines are required for completing the job.
3. **Processing Time:** It means the time required by each job on each machine.
4. **Idle Time on a Machine:** This is the time for which a machine remains idle during the total elapsed time.

   [B.B.A. (Garhwal) 2008]
5. **Total Elapsed Time:** This is the time between starting the first job and completing the last job. It is denoted by T.

   [B.B.A. (Meerut) 2007]

9.3 Sequencing Decision Problem for n-jobs on Two Machines

[B.C.A.(Rohilkhand) 2007]

Johnson's Method

Let there are n jobs to be processed on two machines, A and B in order AB. The processing time for each job on each machine is known and is given by

A_i = Processing time for i^{th} job on machine A

B_i = Processing time for i^{th} job on machine B

The procedure for the solution of the above problem, was developed by **Johnson and Bellman**. The method is based on minimizing an optimal sequence is as follows:

Step 1: Examine the $A_i's$ and $B_i's$ for $i = 1, 2, 3 \dots. n$ and select the minimum of these. If there are two or more minimum processing times, then select any one of them arbitrarily.

Step 2: (i) If the minimum processing time is for machine A, process (do) that job first and place it at the beginning of the sequence.

(ii) If the minimum processing time is for machine B, process (do) that job first and place it at the end of the sequence.

Step 3: Cross all the jobs already assigned and repeat step 1 and 2, placing the remaining jobs next to first or next to last, until all the jobs have been assigned.

Step 4: Calculate the time at which each job in the sequence will be processed on machine A. This time can be calculated as follows:

Time at which ith job in a sequence finishes on machine A = Time when the $(i-1)^{th}$ job in a sequence finishes on machine A plus the time of processing the i^{th} job on machine A_i $(i = 1, 2 \dots n)$ and the time for start of first job on machine A is zero.

Step 5. Calculate the time at which each job in the sequence will start and finish on machine B as follows:

(i) Time when first job in a sequence starts on machine B = time when the first job in a sequence finishes on machine A.

(ii) Time the i^{th} job in the sequence finishes at B = time when the i^{th} job in a sequence starts on machine B + the processing time of i^{th} job on machine B. $(i = 1, 2, 3 \dots n)$

(iii) Time at which the $(i+1)^{th}$ job in a sequence finishes on machine B = maximum (time when the $(i+1)^{th}$ job in a sequence finishes on machine at the time when the i^{th} job in a sequence finishes on machine B. $(i = 1, 2, 3 \dots n)$

Step 6: Calculate the total elapsed time to process all jobs through two machines *i.e.* time when the n^{th} job in a sequence finishes on machine B.

Step 7: Compute the idle time for machines A and B as follows:

(i) Idle time for machine A = time when the n^{th} job in a sequence finishes on machine B minus the time when the n^{th} job in a sequence finishes on machine B.

(ii) Idle time for machine B = time at which the first job in a sequence finishes on machine $A + \sum_{i=1}^{N}$ (time when the i^{th} job in a sequence starts on machine B – time when the $(i-1)^{\text{th}}$ job in sequence finishes on machine B.)

Solved Examples

Example 1: *A company has 3 jobs on hand. Each of these must be processed through two departments, the sequential order for which is*

Department A : Press shop

Department B : Finishing

The table below lists the number of days required by each job in each department:

	Job I	*Job II*	*Job III*
Department A →	8	6	5
Department B →	8	3	4

Find the sequence in which the three jobs should be processed so as to take minimum time to finish all the 3 jobs. **[B.C.A. (Lucknow) 2010]**

Solution:

Step 1: The smallest processing time in the two departments is 3.

Step 2: Since the smallest processing time corresponds to job II in department B. Job II will be processed in the last as shown

		II

Step 3: After job II which is put in the last. We are left with 2 jobs and their processing times as given below:

	Job I	Job III
Department *A*	8	5
Department *B*	8	4

Step 1 and 2: The minimum processing time in the reduced problem is 4 which corresponds to job III in department B. Thus job III will be processed next to last and job I will be processed first.

The optimal sequence, therefore is

I	III	II

Step 4 and 5: Now we can calculate the elapse time corresponding to the optimal sequence.

Job	Department *A*		Department *B*		Ideal time of *B*
	Time in	Time out	Time in	Time out	
I	0	8	8	16	8
III	8	13	16	20	
II	13	$\boxed{19}$	20	$\boxed{23}$	

Step 6: We see that job I starts at zero time in department A and is over by time 8 days, when it passes to department *B* to be worked on till 16 days. Job III then start in the department *A* at time 8 days as this department is free at that time. It is completed at time 13 and has to wait for 3 days before it is passing in *B*, starting at time 16 days when this department is free and completes its finishing by time 20 days. Job II starts in *A* at time 13 and is completed at the time 19. Then it has to wait for one day before passing on to *B*. It enters *B* at time 20 and works on till 23 days. Thus, the total minimum elapsed time = 23 days

Step 7: Ideal time in the two departments is given below:

Department *A* : 4 days = (23 – 19)

Department *B* : 8 days = (8 – 0)

Example 2: *There are 7 jobs, each of which has to go through the machines A and B in the order AB. Processing times hours are given as*

Job :	1	2	3	4	5	6	7
Machine A :	3	12	15	6	10	11	9
Machine B :	8	10	10	6	12	1	3

Determine a sequence of these jobs that will minimize the total elapsed time T.

[M.B.A. (Meerut) 2005, 2007, 2009]

Solution: The smallest processing time in given example is 1 hour for job 6 on machine *B*. Thus, we schedule job 6 last as shown below

						6

The reduced set of processing times becomes

Job :	1	2	3	4	5	7
Machine *A* :	3	12	15	6	10	9
Machine *B* :	8	10	10	6	12	3

There are two equal minimum values, processing time of 3 hours for job 1 on machine *A* and processing time of 3 hours for job seven on machine *B*. Job 1 is scheduled first and job 7 next to job 6 as shown below.

1					7	6

The reduced set of processing time becomes

Job :	2	3	4	5
Machine *A* :	12	15	6	10
Machine *B* :	10	10	6	12

Again there are two equal minimum value, processing time of 6 hours for job 4 on machine *A* as well as on machine *B*. We may choose arbitrarily to process job 4 next to job 1 or next to job 7 *i.e*.

1	4				7	6

The reduced set of processing times becomes

Job :	2	3	5
Machine *A* :	12	15	10
Machine *B* :	10	10	12

There are three equal minimal values, processing time of 10 hours for job 5 on machine *A* and for job 2 and 3 on machine *B*. Then job 5 is scheduled next to job 4 and job 2 next to job 7. The optimal sequence is shown below.

1	4	5	3	2	7	6

For minimum elapsed time, we have

Job	Machine A		Machine B		Ideal time for Machine B
	Time in	Time out	Time in	Time out	
1	0	3	3	11	3
4	3	9	11	17	0
5	9	19	19	31	2
3	19	34	34	44	3
2	34	46	46	56	2
7	46	55	56	59	0
6	55	$\boxed{66}$	66	$\boxed{67}$	7

From above information we get minimum elapsed time $T = 67$ hours

Ideal time for machine $A = 67 - 66 = 1$ hour

Ideal time for machine $B = 3 + 0 + 2 + 3 + 2 + 0 + 7 = 17$ hours

Example 3: *Six jobs go first over machine I and then over II. The order of the completion of jobs has no significance. The following table gives the machine times in hours for six jobs on the two machines*:

Job :	1	2	3	4	5	6
Time on machine I (A_i) :	5	9	4	7	8	6
Time on machine II (B_i) :	7	4	8	3	9	5

Find the sequence of the jobs that minimizes the total elapsed time to complete the jobs. Find the minimum time using Gantt 's chart or by other method.

[B.C.A. (Rohilkhand) 2006, 2010]

Solution: The minimum time in the given table is 3 which is of B_4 machine. Thus, we schedule job 4 last as shown below

					4

The reduced set of processing times becomes

Job :	1	2	3	5	6
Time on machine I (A_i) :	5	9	4	8	6
Time on machine II (B_i) :	7	4	8	9	5

Now, minimum time in this table is 4 which is A_3

Thus, we shall schedule job 3 first as shown below

3					4

The reduced set of processing times becomes

Job :	1	2	5	6
Time on machine I (A_i) :	5	9	8	6
Time on machine II (B_i) :	7	4	9	5

Now, minimum time in this table is 4 which is of B_2. Thus, we shall schedule job 2 just before the last job 4 as shown below

3				2	4

The reduced set of processing times becomes

Job :	1	5	6
Time on machine I (A_i) :	5	8	6
Time on machine II (B_i) :	7	9	5

Now, the minimum time in this table is 5 which is of A_1 and B_6. Thus we shall schedule job 1 just after the first job 3 and job 6 just before the last job 2 as shown below

3	1	5	6	2	4

For minimum elapsed time, we have

Job	**Machine I**		**Machine II**		**Ideal time for machine II**
	Time in	**Time out**	**Time in**	**Time out**	
3	0	4	4	12	4
1	4	9	12	19	0
5	9	17	19	28	0
6	17	23	28	33	0
2	23	32	33	37	0
4	32	$\boxed{39}$	39	$\boxed{42}$	2

From the above information we get minimum elapsed time $T = 42$ hours.

Ideal time for machine $I = (42 - 39) = 3$ hours

Ideal time for machine $II = 4 + 2 = 6$

Example 4: *In a factory, there are six jobs to perform, each of which should go through two machines A and B, in the order AB. The processing timings (in hours) for the jobs are given here. You are required to determine the sequence for performing the jobs that would minimize the total elapsed time T. What is the value of T ?*

Job :	J_1	J_2	J_3	J_4	J_5	J_6
Machine A :	1	3	8	5	6	3
Machine B :	5	6	3	2	2	10

Solution:

Step 1: The smallest processing time in the given table is 1 on machine A. So perform J_1 in the beginning. Now, delete this job from the given data.

Step 2: of all timings now, minimum is 2 which corresponds to J_4 and J_5 both on B. Since the corresponding processing time of J_4 on A is less than the corresponding processing time of J_5 on A, therefore, J_4 will be processing in the last and J_5 next to last.

Step 3: After deleting J_4 and J_5 also, the minimum time is 3 hours which corresponds to J_2 and J_6 on machine A, and to J_3 on machine B. Thus J_2 will be placed in the IInd sequence cell and J_3 is placed in the sequence cell before J_5.

The resulting optimized sequence of jobs is

J_1	J_2	J_6	J_3	J_5	J_4

The total elapsed time is calculated below.

Job	**Machine *A***		**Machine *B***		**Ideal time for machine *B***
	Time in	**Time out**	**Time in**	**Time out**	
J_1	0	1	1	6	1
J_2	1	4	6	12	0
J_6	4	7	12	22	0
J_3	7	15	22	25	0
J_5	15	21	25	27	0
J_4	21	$\boxed{26}$	27	$\boxed{29}$	0

From the above information, we get minimum elapsed time $T = 29$ hours.

Ideal time for machine $A = (29 - 26) = 3$ hours.

Ideal time for machine $B = 1$ hour.

Example 5: *A book binder has one printing press, one binding machine, and the manuscripts of number of different books. The time required to perform the printing and binding operations for each book are shown below. Determine the order in which books should be processed, in order to minimize the total time required to turn out all the books.*

Book :	1	2	3	4	5	6
Printing time (hrs) :	30	120	50	20	90	110
Binding time (hrs) :	80	100	90	60	30	10

[M.B.A. (Meerut) 2008, M.C.A. (Meerut) 2009]

Solution: Here, the books will first go to the printing press and then on to the binding machine. If

A_i $(i = 1, 2, 3, 4, 5, 6)$ = the time in hours on printing press

B_i $(i = 1, 2, 3, 4, 5, 6)$ = the binding time for book

Then using sequencing algorithm the following sequence is obtained :

4	1	3	2	5	6

The total elapsed time is calculated below.

Job	**Printing machine**		**Binding machine**		**Ideal time of binding machine**
	Time in	**Time out**	**Time in**	**Time out**	
4	0	20	20	80	20
1	20	50	80	160	0
3	50	100	160	250	0
2	100	220	250	350	0
5	220	310	350	380	0
6	310	$\boxed{420}$	420	$\boxed{430}$	40

From the above information, we get minimum elapsed time $T = 430$ hours

Ideal time for printing machine $= 430 - 420 = 10$ hours

Ideal time for binding machine $= 20 + 40 = 60$

Example 6: *Find the sequence that minimizes the total elapsed time required to complete the following tasks on two machines*:

Task	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**
Machine (I)	2	5	4	9	6	8	7	5	4
Machine (II)	6	8	7	4	3	9	3	8	11

[B.B.A. (Delhi) 2007, 2009; B.B.A. (Kanpur) 2008]

Solution: The minimum time in the table is 2 which is for job A on machine I.

∴ We must do the job *A* first. After assignment job *A* we are left with the remaining 8 jobs with their processing times. Now the minimum of the processing time for the remaining jobs is 3 which is for job *E* and *G* both on machine II. Therefore we must do the job *E* or *G* (either of them) in the last. Proceeding in this way we get the following optimal sequencings.

1.	*A*	*C*	*I*	*B*	*H*	*F*	*D*	*G*	*E*
2.	*A*	*I*	*C*	*H*	*B*	*F*	*D*	*G*	*E*
3.	*A*	*I*	*C*	*B*	*H*	*F*	*D*	*G*	*E*
4.	*A*	*C*	*I*	*H*	*B*	*F*	*D*	*G*	*E*
5.	*A*	*C*	*I*	*B*	*H*	*F*	*D*	*E*	*G*
6.	*A*	*I*	*C*	*H*	*B*	*F*	*D*	*E*	*G*
7.	*A*	*C*	*I*	*H*	*B*	*F*	*D*	*E*	*G*
8.	*A*	*I*	*C*	*B*	*H*	*F*	*D*	*E*	*G*

It may be seen that the total time elapsed for all sequencings are equal.

The minimum elapsed time may be computed as follows:

Task	Machine I		Machine II		Ideal time for Machine II
	Time in	Time out	Time in	Time out	
A	0	2	2	8	2
C	2	6	8	15	0
I	6	10	15	26	6
B	10	15	26	34	0
H	15	20	34	42	0
F	20	28	42	51	0
D	28	37	51	55	0
G	37	44	55	58	0
E	44	$\boxed{50}$	58	$\boxed{61}$	–

From the above information, we get minimum elapsed time = 61 hours

Ideal time for machine $(I) = 61 - 50 = 11$ hours.

Ideal time for machine $(II) = 2 + 0 = 2$ hours.

9.4 Sequence Decision Problem for n-jobs on Three Machines

Modified Johnson's Method

Let there are n-jobs each of which is to be processed through three machines A, B and C. This sequencing problem is completely described as follows:

1. Only three machines A, B and C are involved.
2. Each job is processed in the prescribed order ABC.
3. No passing of jobs is permitted.
4. The actual or expected processing times $A_1, A_2 \ldots A_n, B_1, B_2 \ldots B_n$ and $C_1, C_2, \ldots C_n$ are known.

The problem, again is to find the optimum sequence of jobs which minimizes the total time T.

No general solution is available at present for such a case. The method of previous article can be extended to the cases in which either one or both of the following conditions holds.

1. The minimum time on machine A is $\geq$ maximum time on machine B.

and

2. The minimum time on machine $C \geq$ maximum time on machine B.

The method, described here is to replace the problem by an equivalent problem involving n jobs and two machines. These two machines are denoted by G and H and corresponding processing time are given by

$$\begin{cases} G_i = A_i + B_i \\ H_i = B_i + C_i \end{cases}$$

Now find the optimal sequence of jobs in the order GH of these two machines G and H by the method of n-jobs and two machines. These resulting optimal sequence on two machines G and H will also be the optimal sequence on three machines A, B and C.

Example 7: *A machine operator has perform three operations: Turning, threading and knurling on a number of different jobs. The time required to perform these operations (in minutes) for each job is known. Determine the order in which the jobs should be processed in order to minimize the total time required to turn out all the jobs.*

Table

Jobs	*Time for Turning (A) (minutes)*	*Time for Threading (B) (minutes)*	*Time for Knurling (C) (minutes)*
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

Solution: Here, min $A_i = 2$, Max $B_i = 8$ and mini $(C_i) = 8$ since min C_i = maxi B_i. Then three machines can be converted in two machines G and H

Processing times for 6 jobs and two machines

Job	$G_i = A_i + B_i$	$H_i = B_i + C_i$
1	11	21
2	18	20
3	9	13
4	8	18
5	12	11
6	12	14

Now, to form job sequence, we find that the smallest value is 8 under operation G_i in row 4. Thus we schedule job 4 first as shown below.

4					

The reduced set of processing times becomes.

Job	G_i	H_i
1	11	21
2	18	20
3	9	13
5	12	11
6	12	14

The next smallest value is 9 under the column G_i for job 3 we can schedule job 3 as shown below.

4	3				

The reduced set of processing times becomes.

Job	G_i	H_i
1	11	21
2	18	20
5	12	11
6	12	14

There are two equal minimal values, processing time of 11 minutes under column G_i for job 1 and processing time of 11 minutes under column H_i for job 5. Then job 1 is placed next to job 3 and job 5 is placed last as shown below.

4	3	1			5

The reduced set of processing times becomes

Job	G_i	H_i
2	18	20
6	12	14

The smallest value is 12 under column G_i, for job 6. Hence we schedule job 6 next to job 1 and the optimal sequence becomes.

4	3	1	6	2	5

For the total elapsed time, we have

Job	**Turning operation (*A*)**		**Threading operation (*B*)**		**Knurling operation (*C*)**		**Ideal time for *B***	**Ideal time for *C***
	Time in	**Time out**	**Time in**	**Time out**	**Time in**	**Time out**		
4	0	2	2	8	8	20	2	8
3	2	7	8	12	20	29	0	0
1	7	10	12	20	29	42	0	0
6	10	21	21	22	42	55	1	0
2	21	33	33	39	55	69	11	0
5	33	$\boxed{42}$	42	$\boxed{45}$	69	$\boxed{77}$	3 + (77 – 45)	0

From the given information, we get minimum elapsed time $T = 77$ minutes

Ideal time for turning operation

$$(A) = (77 - 42) = 35 \text{ minutes}$$

Ideal time for threading operation

$$(B) = 2 + 0 + 0 + 1 + 11 + 3 + (77 - 45) = 49 \text{ minutes}$$

Ideal time for knurling operation

$$(C) = 8 + 0 + 0 + 0 + 0 + 0 = 8 \text{ minutes.}$$

Example 8: *We have five jobs, each of which must go through the machines A, B and C in the order ABC, processing times are:*

Job	*A*	*B*	*C*
1	4	5	8
2	9	6	10
3	8	2	6
4	6	3	7
5	5	4	11

Determine a sequence for the five jobs that will minimize the elapsed time T.

Solution: Here mini $A_i = 4$, Max. $B_i = 6$, Mini $C_i = 6$

since Max. $B_i \le$ Mini. C_i

$\therefore$ We may consider two fictitious machines G and H whose processing times are given in the following table.

Jobs	$G_i = A_i + B_i$	$H_i = B_i + C_i$
1	9	13
2	15	16
3	10	8
4	9	10
5	9	15

Using sequencing algorithms we get the following optimal sequencings

1	4	5	2	3

1	5	4	2	3

4	5	1	2	3

5	4	1	2	3

5	1	4	2	3

4	1	5	2	3

The minimum elapsed time can now be calculated corresponding to the first optimal sequencing using the individual machining times given in the problem.

Jobs	Machine (*A*)		Machine (*B*)		Machine (*C*)		Ideal time of machine (*C*)
	Time in	Time out	Time in	Time out	Time in	Time out	
1	0	4	4	9	9	17	9
4	4	10	10	13	17	24	0
5	10	15	15	19	24	35	0
2	15	24	24	30	35	45	0
3	24	$\boxed{32}$	32	$\boxed{34}$	45	$\boxed{51}$	0

From the given information, we get the minimum elapsed time $T = 51$ hours

Ideal time for machine $A = 51 - 32 = 19$ hours

Ideal time for machine $B = 4 + 1 + 2 + 5 + 2 + (51 - 34) = 31$ hours

Ideal time for machine $C = 9 + 0 + 0 + 0 + 0 = 9$ hours.

Example 9: *There are five jobs, each of which is to be processed through three machines A, B and C in the order ABC. Processing times in hours are*

Jobs	*A*	*B*	*C*
1	3	4	7
2	8	5	9
3	7	1	5
4	5	2	6
5	4	3	10

Determine the optimum sequence for the five jobs and the minimum elapsed time.

Solution: Here mini $A_i = 3$, maxi $B_i = 5$ and min $C_i = 5$. Since min C_i = max B_i, then three machines can be converted into two machines G and H.

Hence the equivalent problem become 5 jobs and two machines G and H.

Machines times for five jobs and two machines *G* and *H*

Jobs	G ($G_i = A_i + B_i$)	H ($H_i = B_i + C_i$)
1	7	11
2	13	14
3	8	6
4	7	8
5	7	13

Examining the columns G_i and H_i, we find that the smallest value of machining time is 7 hrs, under the column G_i for job 1, 4 and 5. Since all values are in the same column G_i, we schedule job 4 first since it has lower entry 8 in column H_i, job 1 next as it next height entry 11 in column H_i and job 5 next to job 1, since it has still higher entry 13 in column H_i then job sequence is shown below

4	1	5		

The reduced set of machining time becomes

Jobs	G_i	H_i
2	13	14
3	8	6

The smallest value is 6 hrs in column H_i for job 3. Thus we put job 3 last and the optimum sequence becomes

4	1	5	2	3

For total elapse time we have

Jobs	Machine *A*		Machine *B*		Machine *C*		Ideal time for machine *B*	Ideal time for machine *C*
	Time in	Time out	Time in	Time out	Time in	Time out		
4	0	5	5	7	7	13	5	7
1	5	8	8	12	13	20	1	0
5	8	12	12	15	20	30	0	0
2	12	20	20	25	30	39	5	0
3	20	$\boxed{27}$	27	$\boxed{28}$	39	$\boxed{44}$	2 +(44 – 28)	0

From the given information, we get the minimal elapsed time $T = 44$ hours

Ideal time for machine $A = 44 - 27 = 17$ hours.

Ideal time for machine $B = 5 + 1 + 0 + 5 + 2 + (44 - 28)$

$$= 13 + 16 = 29 \text{ hours}$$

Ideal time for machine $C = 7 + 0 + 0 + 0 + 0 = 7$ hours

9.5 Sequence Decision Problem for n Jobs on m Machines

Let there are n jobs each of which is to be processed through m machines say $M_1, M_2 \ldots M_m$ in the order $M_1 \; M_2 \ldots M_m$. Now to find an optimal sequence as follows:

Step 1. Find minimum M_{i1}, min $M_i \, m \; (i = 1, 2, 3 \ldots n)$ and maximum of each of

$$M_{i2}, M_{i3}, \ldots M_{i\,m-1} \quad \text{for} \quad i = 1, 2, 3 \ldots n$$

Step 2. Now to check either one or both conditions are satisfying

(i) min M_{i1} $(i = 1, 2 \ldots n) \geq \max M_{ij}$ for $j = 1, 2 \ldots (m-1)$

(ii) min M_{im} $(i = 1, 2 \ldots n) \geq \max M_{ij}$ for $j = 1, 2 \ldots (m-1)$

Step 3. If the inequality of step 2 is not satisfying then method fail. Otherwise go to next step.

Step 4. Convert the m machine problem into two machine problem G and H. Such that

$$M_i G = M_{i1} + M_{i2} + \ldots M_{i\,(m-1)}$$

and

$$M_i H = M_{i2} + M_{i3} + \ldots + M_{im}$$

Determine the optimal sequence of n jobs through 2 machines by using optimal sequence algorithm.

Step 5. In addition to conditions given in step 4, if

$$M_{i2} + M_{i3} + \ldots + M_{i\,(m-1)} = C$$

is a fixed positive constant for all $i = 1, 2, 3 \ldots n$, then determine the optimal sequence for n jobs and two machines M_1 and M_m in order $M_1 \; M_m$ by using the optimal sequence algorithm.

Example 10: *Solve the following sequencing problem when passing is not allowed.*

Items	*Machines*				
	A	*B*	*C*	*D*	*E*
I	9	7	4	5	11
II	8	8	6	7	12
III	7	6	7	8	10
IV	10	5	5	4	8

Solution: Here, min $A_i = 7$, min $E_i = 8$ and max $B_i = 8$, maxi $C_i = 7$ and max $D_i = 8$

Since min $E_i \geq \{\max B_i, \max c_i, \max D_i\}$, then given problem 4 jobs and 5 machines can be converted into 4 jobs and two machines G and H.

where $$G = A_i + B_i + C_i + D_i$$
$$H = B_i + C_i + D_i + E_i$$

Hence,

Jobs :	I	II	III	IV
Machine (*G*) :	25	29	28	24
Machine (*H*) :	27	33	31	22

The optimal sequence of jobs is as follows by usual process.

I	III	II	IV

Now, we may calculate the total elapsed time corresponding to optimal sequence, **using the individual processing time given in the original problem**.

Jobs	Machine *A*		Machine *B*		Machine *C*		Machine *D*		Machine *E*		Ideal time for *E*
	Time in	Time out	Time in	Time out	Time in	Time out	Time in	Time out	Time in	Time out	
I	0	9	9	16	16	20	20	25	25	36	25
III	9	16	16	22	22	29	29	37	37	47	1
II	16	24	24	32	32	38	38	45	47	59	
IV	24	$\boxed{34}$	34	$\boxed{39}$	39	$\boxed{44}$	45	$\boxed{49}$	59	$\boxed{67}$	

From the above information, we get minimum elapsed time $T = 67$ hours

Ideal time for machine $(A) = (67 - 34) = 33$ hours

Ideal time for machine $(B) = 9 + 0 + 2 + 2 + (67 - 39)$

$= 9 + 2 + 2 + 28$

$= 41$ hours

Ideal time for machine $(C) = 16 + 2 + 3 + 1 + (67 - 44)$

$= 22 + 23$

$= 45$ hours

Ideal time for machine $D = 20 + 4 + 1 + (67 - 49)$

$= 25 + 18$

$= 43$ hours

Ideal time for machine $E = 25 + 1 + 0 + 0 = 26$ hours.

Example 11: *We have 4 jobs each of which go through the machine M_j $(j = 1, 2, 3, 4, 5, 6)$ in order M_1 M_2 … M_6 processing time (in hours) is given below:*

Jobs	***Machines***					
	M_1	M_2	M_3	M_4	M_5	M_6
A	18	8	7	2	10	25
B	17	6	9	6	8	19
C	11	5	8	5	7	15
D	20	4	3	4	8	12

Determine a sequence of these four jobs that minimizes the total elapsed time T.

Solution: Here Min $(M_1) = 11$, Maxi $(M_2) = 8$, Max $(M_3) = 9$,

Max $(M_4) = 6$, Maxi $(M_5) = 10$, Min $(M_6) = 12$

Hence (i) Mini $(M_1) \geq \max \{M_2, M_3, M_4, M_5\}$

(ii) Min $(M_6) \geq \text{Max} \{M_2, M_3, M_4, M_5\}$

Now, both conditions are satisfying then 6-machines can be converted into two machines G and H

Jobs :	*A*	*B*	*C*	*D*
Machine *G* :	45	46	36	39
Machine *H* :	52	48	40	31

Where $G_i = M_1 + M_2 + M_3 + M_4 + M_5$

and $H_i = M_2 + M_3 + M_4 + M_5 + M_6$

Using the optimal sequence algorithm, the following optimal sequence is easily obtained

C	*A*	*B*	*D*

Total elapsed time is given in the following table

Job	Machine M_1		Machine M_2		Machine M_3		Machine M_4		Machine M_5		Machine M_6	
	Time in	Time out	Time in	Time out	Time in	Time out	Time in	Time out	Time in	Time out	Time in	Time out
C	0	11	11	16	16	24	24	29	29	36	36	51
A	11	29	29	37	37	44	44	46	46	56	56	81
B	29	46	46	52	52	61	61	67	67	75	81	100
D	46	$\boxed{66}$	66	$\boxed{70}$	70	$\boxed{73}$	73	$\boxed{77}$	77	$\boxed{85}$	100	$\boxed{112}$

This table shows that the minimal total elapse time $T = 112$ hours.

9.6 Processing Two Jobs Through m Machines

When two jobs are to be processed on *m* machines, we generally use graphical method to determine the total elapsed time as well as the optimal sequence. Let us consider the following situation.

1. There are m machines, denoted by $A, B, C, \ldots K$
2. Only two jobs are to be performed : job 1 and job 2.
3. The technological ordering of each of the two jobs through m machines is known.
4. The actual or expected processing time $A_1, A_2 \ldots K_i, A_2, B_2 \ldots K_2$ are known.
5. Each machine can work only one job at a time and storage space for in-process inventory is available.

Example 12: *Using graphical method, calculate the minimum time needed to process job 1 and 2 on five machines, A, B, C, D and E i.e. for each machine find the job which should be done first. Also calculate the total time needed to complete both jobs.*

Job 1		*Job 2*	
Sequence of machines	***Time***	***Sequence of machines***	***Time***
A	3	B	5
B	4	C	4
C	2	A	3
D	6	D	2
E	2	E	6

Solution: The graphical method is described with the help of the following steps:

Step 1: Draw two axes at right angles to each other, represent processing time on job 1 along horizontal axis and processing time on job 2 along vertical axis.

Step 2: Layout the machine time for the two jobs on corresponding axes in the given technological order.

Step 3: Machine A requires 3 hour for job 1 and 3 hours for job 2. A rectangle LMNP is, thus constructed for machine A. Similar rectangles are constructed for machines B, C, D and E as shown.

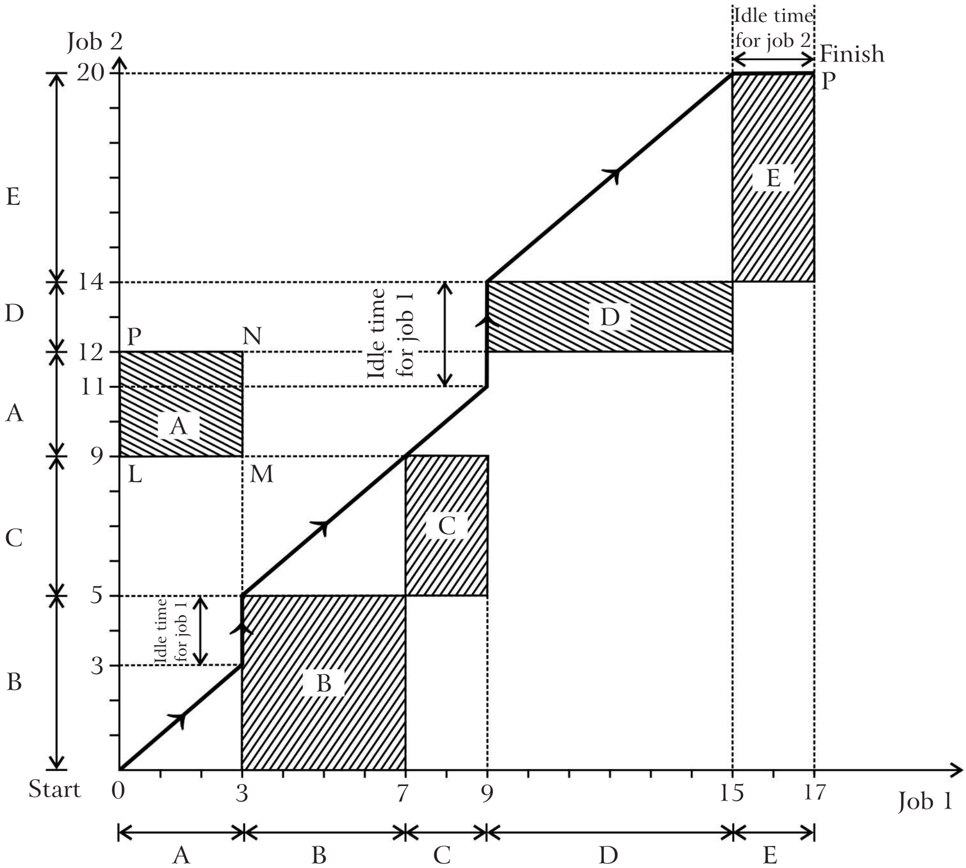


Fig. 9.1

Step 4: Draw the line by starting from origin (0) and moving through the various stages of completion (points) till the point marked finish is reached. Then take path consisting only of horizontal, vertical and 45° lines. A horizontal line represents work on job 1 while job 2 remains idle, a vertical line represents work on job 2 while job 1 remains idle and a 45° line to the base represents simultaneous work on both jobs.

Step 5: Find the optimal path. An optimal path is one that minimizes idle time for job 1. Likewise, an optimal path is one that minimizes idle time for job 2. Then the optimal path is one which coincides with 45° line to maximum extent.

Further, both jobs can not be processed simultaneously on one machine. Graphically it means that diagonal through the blocked out area is not allowed.

Step 6: Find the elapsed time *i.e.*

The elapsed time = processing time of job 1 + ideal time of job 1

$= 17 + (2 + 3) = 22$ hrs

or = Processing time of job 2 + ideal time of job 2

$= 20 + 2 = 22$ hrs.

The optimal processing sequence for the two jobs on various machines, as evident from the figure is

Job 1 before 2 on machine A

Job 2 before 1 on machine B

Job 2 before 1 on machine C

Job 2 before 1 on machine D

and Job 2 before 1 on machine E

Example 13: *Use graphical method to minimize the time required to process the following jobs on the machines i.e. for each machine specify the job which should done first. Also calculate the total elapsed time to complete both jobs.*

[M.B.A. (Meerut) 2007, 2009]

Job 1		*Job 2*	
Sequence of machines	***Time***	***Sequence of machines***	***Time***
A	6	***B***	10
B	8	***C***	8
C	4	***A***	6
D	12	***D***	4
E	4	***E***	12

Solution:

Step 1: Draw two axes at right angles to each other, represent processing time on job 1 along horizontal axis and processing time on job 2 along vertical axis.

Step 2: Layout the machine time for the two jobs on corresponding axes in the given technological order.

Step 3: Machine A requires 6 hour for job 1 and 6 hours for job 2. A rectangle LMNP is, thus constructed for machine A. Similar rectangles are constructed for machines B, C, D and E as shown.

Step 4: Draw the line by starting from origin (O) and moving through the various stages of completion (points) till the point marked finish is reached. Then take path consisting only of horizontal, vertical and 45° lines. A horizontal line represents work on job 1 while job 2 remains idle, a vertical line represents work on job 2 while job 1 remains idle and a 45° line to the base represents simultaneous work on both jobs.

Step 5: Find the optimal path. An optimal path is one that minimizes idle time for job 1. Likewise, an optimal path is one that minimizes idle time for job 2. Then the optimal path is one which coincides with 45° line to maximum extent.

Further, both jobs can not be processed simultaneously on one machine. Graphically it means that diagonal through the blocked out area is not allowed.

Step 6: Find the elasped time *i.e.*

The elapsed time = processing time of job 1 + ideal time of job 1

$= 34 + 10 = 44$ hrs.

The elapsed time = processing time of job 2 + ideal time of job 2

$= 40 + 4 = 44$ hrs.

The optimal processing sequence for the two jobs on various machines, as

Job 1 before job 2 on machine *A*

Job 2 before job 1 on machine *B*

Job 2 before job 1 on machine *C*

Job 2 before job 1 on machine *D* and

Job 2 before job 1 on machine *E*

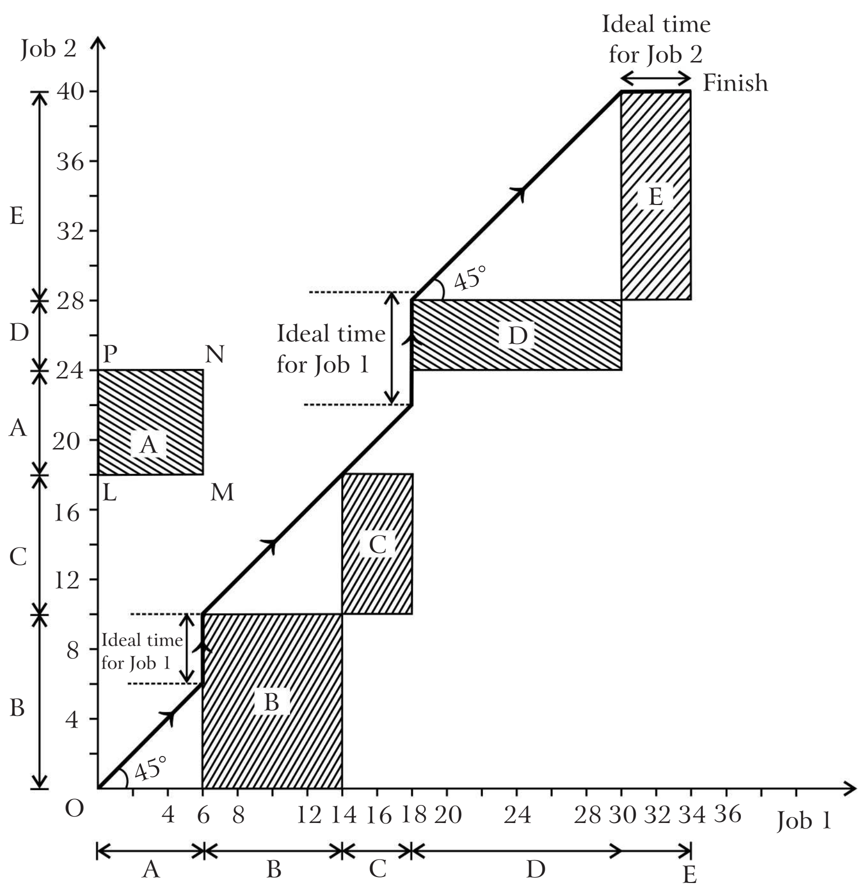


Fig. 9.2

Problem Set

1. What is sequencing problem ?
2. Write short note on sequencing problem. **[B.C.A. (Rohilkhand) 2008]**
3. What is 'no passing rule' in a sequencing algorithm ?
4. Give the Johnson's method for determining an optimal sequence for processing. **[B.C.A. (Agra) 2010]**
5. Explain how to process n jobs through m-machines.
6. State clearly the assumptions used in the study of sequencing problems.
7. Seven jobs each of which has to go through two machines M_1 and M_2 in order M_1 M_2 take time on the machine as follows : Find in which order the jobs should be performed to minimise the time.

Jobs :	**1**	**2**	**3**	**4**	**5**	**6**	**7**
M_1 :	3	12	15	6	10	11	9
M_2 :	8	10	10	6	12	1	3

8. Six jobs go over two machines I and II in that order. The order of the completion of the jobs has no significance. From the data given below, find the sequence that minimizes the total elapsed time, and also find the minimum time.

Jobs :		**1**	**2**	**3**	**4**	**5**	**6**
Time in hours :	**Machine I**	4	8	3	6	7	5
	Machine II	6	3	7	2	8	4

9. The following table given the machine times (in hours) for 5 jobs to be processed on two-different machines:

Jobs :	**1**	**2**	**3**	**4**	**5**
Machine *A* :	3	7	4	5	7
Machine *B* :	6	2	7	3	4

Passing time is not allowed. Find the optimal sequence in which jobs should be processed. **[B.C.A. (Kanpur) 2009]**

10. A readymade garment manufacturer has to process seven items through two stages of production *i.e.*, cutting and sewing. The time taken for each of these item at the different stages are given below in appropriate units:

Item No.	Processing Time Cutting	Processing Time Sewing
1	5	2
2	7	6
3	3	7
4	4	5
5	6	9
6	7	5
7	12	8

Find an order in which these items are to be processed through these stages so as to minimize the total processing time. **[B.C.A. (Avadh) 2010]**

11. We have 5 jobs, each of which must go through the two machine A and B in order AB. Processing times are given in the table below:

Processing time in hours

Job :	1	2	3	4	5
Machine A :	5	1	9	3	10
Machine B :	2	6	7	8	4

Determine a sequence for the five jobs that will minimize the elapsed time T.

12. A book-binder has one printing press, one binding machine and manuscripts of a number of different books. The time required in min. to perform the printing and binding operation of each book are known. We wish to determine the order in which books should be processed in order to minimize the total time required to turn out all the books.

Book :	1	2	3	4	5	6
Printing time :	30	120	50	20	90	110
Binding time :	80	100	90	60	30	10

13. Find the sequence, for the following eight jobs, that will minimize the total elapsed time for the completion of all the jobs. Each job is processed in the same order *CAB*. Entries give the time in hours on the machines.

Jobs		1	2	3	4	5	6	7	8
Times on Machines	*A*	4	6	7	4	5	3	6	2
	B	8	10	7	8	11	8	9	13
	C	5	6	2	3	4	9	15	11

14. There are five jobs, each of which must go through the machines *A*, *B* and *C* in the order *ABC*. Processing times are:

Job	A	B	C
1	4	5	8
2	9	6	10
3	8	2	6
4	6	3	7
5	5	4	11

Determine a sequence for the five jobs that will minimize the elapsed time *T*.

15. We have 5 jobs, each of which must go through machines, *A*, *B* and *C* in the order *ABC*. Processing times are given in the following table :

Jobs	1	2	3	4	5
Machines A (A_i)	8	10	6	7	11
Machines B (B_i)	5	6	2	3	4
Machines C (C_i)	4	9	8	6	5

Determine a sequence for the five jobs that will minimize the elapsed time *T*.

16. Find the time that minimizes the total required to complete the following tasks.

Task	A	B	C	D	E	F	G
Machines I	3	8	7	4	9	8	7
Machines II	4	3	2	5	1	4	3
Machines III	6	7	5	11	5	6	12

[B.B.A. (Rohilkhand) 2008]

17. Find the sequence and idle time of each machine that minimized the total elapsed time required to complete all the jobs on machines A, B, C in the order ABC.

Machines	Jobs				
	1	**2**	**3**	**4**	**5**
A	7	12	11	9	8
B	8	9	5	6	7
C	11	13	9	10	14

18. Solve the following sequencing problem giving an optimal solution when passing time is not allowed.

Machines	Jobs				
	A	***B***	***C***	***D***	***E***
M_1	10	12	8	15	16
M_2	3	2	4	1	5
M_3	5	6	4	7	3
M_4	14	7	12	8	10

[B.B.A. (Delhi) 2005, 2007, 2009; B.B.A. (Bhopal) 2008, 2009; B.B.A. (Agra) 2009]

19. Solve the following sequencing problem of jobs on six machines M_j $(j = 1, 2, 3, 4, 5, 6)$ in the order $M_1, M_2, M_3, M_4, M_5, M_6$. Processing times (in hours) are given below.

Jobs	Machines					
	M_1	M_2	M_3	M_4	M_5	M_6
A	18	8	7	2	10	25
B	17	6	9	6	8	19
C	11	5	8	5	7	15
D	20	4	3	4	8	12

[B.B.A. (Lucknow) 2008]

20. Use the graphical method to minimize the time needed to process the following two jobs on four machines A, B, C and D with technological orderings.

Job 1 :	A	B	C	D
Job 2 :	D	B	A	C

and the processing times:

	A	***B***	***C***	***D***
Job 1 :	2	4	5	1
Job 2 :	6	5	3	6

21. Use graphical method to minimize the time needed to process the following jobs on two machines shown below *i.e.* for each machine find the job which should be done first. Also calculate the total time needed to complete both jobs:

Job 1		**Job 2**	
Sequence of Machines	**Time**	**Sequence of Machines**	**Time**
A	2	***C***	4
B	3	***A***	5
C	4	***D***	3
D	6	***E***	2
E	2	***B***	6

[B.B.A. (Kanpur) 2007]

Answers

7.	$1-4-5-3-2-7-6$, $1-5-3-2-4-7-6$ $1-4-5-2-3-7-6$ and $1-5-2-3-4-7-6$
8.	$3-1-5-6-2-4$, Mini time $=35$ hrs.
9.	$1-3-5-4-2$
10.	$3-4-5-7-2-6-1$
11.	$2-4-3-5-1$; Mini time $T=30$ hrs.
12.	$4-1-3-2-5-6$; Mini time $T=430$ minutes
13.	$4-1-3-5-2-8-7-6$; Mini time $T=81$ hrs.
14.	$4-1-5-2-3$; $4-5-1-2-3$; $1-4-5-2-3$; $1-5-4-2-3$, $5-1-4-2-3$; $5-4-1-2-3$, Total elape time $=51$ hrs.
15.	$3-2-1-4-5$; $3-2-4-1-5$; $3-2-4-5-1$, $3-2-5-4-1$, $3-2-1-5-4$ and $3-2-5-1-4$
16.	$A-D-G-F-B-C-E$, $A-D-G-B-F-C-E$ Minimum time $T=59$ hours
17.	$1-5-4-2-3$, Total elapsed time $=72$ minutes Ideal times are for $A=25$ min, for $B=37m$ and for $C=15\ m$
18.	$C-A-E-D-B$
19.	$C-A-B-D$
20.	Minimum time $T=21$ hrs.
21.	Job 1 before Job 2 on machine A Job 1 before Job 2 on machine B Job 1 before Job 2 on machine C Job 2 before Job 1 on machine D Job 1 before Job 2 on machine E Total elapsed time 20 hrs.

9.7 The Travelling Salesman Problem

[B.B.A. (Meerut) 2012]

There are number of cities a salesman wants to visit. The distance (or time or cost) between every pair of cities is known. He starts from his home city, passes through each city once and only once or exactly one time and return to his home city. The problem is to find the routes shortest in distance (or time or cost).

If the distance or time or cost between every pair of cities is independent of the direction of travel, the problem is said to be **symmetrical** if for one or more pair of cities (or time or cost) varies with direction the problem is called **asymmetrical.** For example *i.e.* it take more time to go up a hill between two stations then come down the hill between them; similarly a flight may take more time against the wind direction compared to that in the direction of wind. If the salesman is to visit only two cities there is of course no choice. If the number of cities is three (A, B and C) of which starting base A, there are two possible routes $A \to B \to C$ and $A \to C \to B$. In general, for n cities there are $(n-1)!$ possible routes. The problem is to find the best route without trying each one. Such types of problems arise in the following areas of management:

1. Postal deliveries 2. Inspection 3. School bus routing 4. Television relays 5. Assembly lines etc. The travelling salesman problem appear to be related to sequencing problem but actually it is more similar to assignment problem with the difference that there is an additional constraint mathematically, the problem may be stated as follows; given C_{ij} as the cost of going from. City i to city j and $x_{ij} = 1$, for going directly from i to j and 0, otherwise,

$$\text{minimize} \sum_i \sum_j c_{ij}\, x_{ij}$$

Subject to the additional constraint that x_{ij} is to be so chosen that no city is visited twice before the tour of all the cities is completed. In particular, going from i to i is not permitted which means $c_{ij} = \infty$ it should be noted that there must be only one $x_{ij} = 1$ for each value of i and j. The travelling salesman problem can be written in the form of square assignment problem.

To

		1	**2**	**3**	…	***n***
	1	∞	c_{12}	c_{13}	…	c_{1n}
	2	c_{21}	∞	c_{23}	…	c_{2n}
From	**3**	c_{31}	c_{32}	∞	…	c_{3n}
	⋮	⋮	⋮	⋮	⋮	⋮
	n	c_{n1}	c_{n2}	c_{n3}	…	∞

The above assignment problem can be solved and it can be hoped that the solution will satisfy the additional constraint. If it does not satisfy it can be adjusted by inspection.

Example 14: *Solve the travelling salesman problem in the matrix shown below.*

	1	2	3	4	5
1	∞	6	12	6	4
2	6	∞	10	5	4
3	8	7	∞	11	3
4	5	4	11	∞	5
5	5	2	7	8	∞

[B.C.A. (Rohilkhand) 2004]

Solution: Consider the problem as an assignment problem. Solving the problem by usual assignment algorithm. We get the following tables:

	1	2	3	4	5	
1	∞	2	3	1	$\boxed{0}$	✓
2	1	∞	1	$\boxed{0}$	$\cancel{0}$	
3	4	4	∞	7	$\cancel{0}$	✓
4	$\boxed{0}$	$\cancel{0}$	2	∞	1	
5	2	$\boxed{0}$	$\cancel{0}$	5	∞	
					✓	

	1	2	3	4	5	
1	∞	1	2	$\cancel{0}$	$\cancel{0}$	✓
2	1	∞	1	$\boxed{0}$	1	✓
3	3	3	∞	6	$\boxed{0}$	✓
4	$\boxed{0}$	$\cancel{0}$	2	∞	2	
5	2	$\boxed{0}$	$\cancel{0}$	5	∞	
				✓	✓	

	1	2	3	4	5
1	∞	$\boxed{0}$	1	$\cancel{0}$	$\cancel{0}$
2	$\cancel{0}$	∞	$\cancel{0}$	$\boxed{0}$	1
3	2	2	∞	6	$\boxed{0}$
4	$\boxed{0}$	$\cancel{0}$	2	∞	3
5	2	$\cancel{0}$	$\boxed{0}$	6	∞

Hence optimal solution of the assignment problem is

$$1 \to 2, 2 \to 4, 3 \to 5, 4 \to 1, 5 \to 3$$

or $\quad 1 \to 2 \to 4 \to 1, 3 \to 5 \to 3$

So this is not the solution to the travelling salesman problem, as it is not allowed to go back to city 1 from city 4 without visiting cities 3 and 5.

Now we try to get next best solution which satisfy the extra restriction. The smallest element in the table other than zero is 1. So we try to bring 1 into the solution. Since 1 occurs at two places, so we shall consider both the cases separately until the acceptable solution is attained.

Now, if we put assignment in the cell (1, 3) having the element 1 instead of making 0 assignment in the cell (5, 3) and there will be no assignment in the first row and third column then make assignment in the cell (5, 2) having element 0, instead of assignment in (5, 3)

	1	2	3	4	5
1	∞	$\cancel{0}$	$\boxed{1}$	$\cancel{0}$	$\cancel{0}$
2	$\cancel{0}$	∞	$\cancel{0}$	$\boxed{0}$	1
3	2	2	∞	6	$\boxed{0}$
4	$\boxed{0}$	$\cancel{0}$	2	1	3
5	2	$\boxed{0}$	$\cancel{0}$	6	∞

Hence resulting feasible solution is $1 \to 3 \to 5 \to 2 \to 4 \to 1$ having the cost 1. If we put assignment in the cell (2, 5) in place (2, 4) and change the assignment to (3, 4) from (3,5) in that case the feasible solution is

$1 \to 2 \to 5 \to 3 \to 4 \to 1$, having cost 6 which is more than 1 from above. Hence the optimal route is $1 \to 3 \to 5 \to 2 \to 4 \to 1$ and minimum cost $12 + 3 + 2 + 5 + 5 = 27$

Example 15: *Solve the following travelling salesman problem so as to minimize the cost per cycle:*

To

		A	***B***	***C***	***D***	***E***
	A	–	3	6	2	3
	B	3	–	5	2	3
From	***C***	6	5	–	6	4
	D	2	2	6	–	6
	E	3	3	4	6	–

Solution: Reduce the cost matrix and make assignment in rows and columns having single zeroes. Then draw the minimum number of lines to cover all the zeros by usual assignment process, we get

	A	B	C	D	E	
A	M	1	3	0	1	✓
B	1	M	2	$\cancel{0}$	1	✓
C	2	1	M	2	$\boxed{0}$	
D	$\boxed{0}$	$\cancel{0}$	3	M	4	
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	3	M	
				✓		

	A	B	C	D	E
A	M	$\boxed{0}$	2	$\cancel{0}$	$\cancel{0}$
B	$\cancel{0}$	M	1	$\boxed{0}$	$\cancel{0}$
C	2	1	M	3	$\boxed{0}$
D	$\boxed{0}$	$\cancel{0}$	3	M	4
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	M

By subtracting the lowest element 1 from all the elements not covered by lines and adding the some at the intersection of two lines.

The optimum assignment is $A \to B, B \to D, C \to E, D \to A$ and $E \to C$ with minimum cost $= 3 + 2 + 4 + 2 + 4 = 15$ since this assignment schedule does not provide us the solution of travelling salesman problem. We try to find the next solution which satisfies the extra condition also make assignment at $(2, 3)$ instead of zero at $(2, 4)$.

The resulting table is

	A	B	C	D	E
A	M	$\cancel{0}$	2	$\boxed{0}$	$\cancel{0}$
B	$\cancel{0}$	M	$\boxed{1}$	$\cancel{0}$	$\cancel{0}$
C	2	$\cancel{0}$	M	3	$\boxed{0}$
D	$\cancel{0}$	$\boxed{0}$	3	M	4
E	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	4	M

	A	B	C	D	E
A	M	$\cancel{0}$	2	$\cancel{0}$	$\boxed{0}$
B	$\cancel{0}$	M	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$
C	2	$\boxed{1}$	M	3	$\cancel{0}$
D	$\boxed{0}$	$\cancel{0}$	3	M	4
E	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	4	M

The optimum assignment schedule is

$A \to D \to B \to C \to E \to A$ or $A \to E \to C \to B \to D \to A$

Total minimum cost per cycle in both the cases will be 16.

Example 16: *Solve the travelling salesman problem given by the following data:*

$$c_{12} = 20,\ c_{13} = 4,\ c_{14} = 10,\ c_{23} = 5,\ c_{34} = 6$$

$$c_{25} = 10,\ c_{35} = 6,\ c_{45} = 20,\ c_{ij} = c_{ji}$$

and there is no route between cities i and j if the value for c_{ij} is not given above.

[B.C.A. (Avadh) 2011]

Solution: The assignment of given problem can be written as

	1	2	3	4	5
1	∞	20	4	10	∞
2	20	∞	5	∞	10
3	4	5	∞	6	6
4	10	∞	6	∞	20
5	∞	10	6	20	∞

If there is no route between cities *i* and *j* then we have taking $c_{ij} = \infty$ for $i = j$

Now we will solve the assignment by usual process. we get

	1	2	3	4	5
1	∞	15	$\boxed{0}$	4	∞
2	15	∞	$\cancel{0}$	∞	3
3	$\boxed{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$
4	4	∞	$\cancel{0}$	∞	12
5	∞	3	$\cancel{0}$	12	∞

	1	2	3	4	5
1	∞	12	$\boxed{0}$	1	∞
2	12	∞	$\cancel{0}$	∞	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	∞	$\cancel{0}$	$\cancel{0}$
4	1	∞	$\cancel{0}$	∞	9
5	∞	$\boxed{0}$	$\cancel{0}$	9	∞

	1	2	3	4	5
1	∞	11	$\boxed{0}$	$\cancel{0}$	∞
2	12	∞	1	∞	$\boxed{0}$
3	$\cancel{0}$	$\cancel{0}$	∞	$\boxed{0}$	$\cancel{0}$
4	$\boxed{0}$	∞	$\cancel{0}$	∞	8
5	∞	$\boxed{0}$	1	9	∞

Hence, the optimal solution of the assignment problem is:

$$1 \to 3, 3 \to 4, 4 \to 1, 2 \to 5, 5 \to 2$$

But this is not the solution to the travelling salesman problem, as it is not allowed to go from city 4 to 1 without visiting the cities 2 and 5.

We try to find best solution which satisfying the additional condition. The smallest element other then zero is one which is in two places. We consider both the cases separately until the acceptable solution is attained.

Make assignment at 1 in cell (2, 3) instead of 0 in the cell (2, 5) making this assignment we note that there are two assignments in column 3 while there is no assignment in column 5. So we should shift the assignment at 0 in cell (1, 3) to cell (1, 5) which is not possible, as this route is not allowed.

	A	***B***	***C***	***D***	***E***
A	∞	$\boxed{11}$	0	0	∞
B	12	∞	1	∞	$\boxed{0}$
C	0	0	∞	$\boxed{0}$	0
D	$\boxed{0}$	∞	0	∞	8
E	∞	0	$\boxed{1}$	9	∞

Again we make assignment at 1 in cell (5, 3) instead 0 in cell (5, 2). Making this assignment the assignment in cell (1, 3) is shifted to cell (1, 2). So we get the following table.

This feasible solution is $1 \to 2 \to 5 \to 3 \to 4 \to 1$

In this case the cost is increased by $(11+1) - 0 = 12$. In comparision in previous cost 0 in the last table

The next element in the table greater than 1 is 8 in cell (4, 5). So we shift the assignment from 0 to cell (4, 1) to 8 of cell (4, 5) and then shift the assignment from cell (2, 5) to cell (2, 1). So we get the following table.

∞	11	$\boxed{0}$	0	∞
$\boxed{12}$	∞	1	∞	0
0	0	∞	$\boxed{0}$	0
0	∞	0	∞	$\boxed{8}$
∞	$\boxed{0}$	1	9	∞

This feasible solution is $1 \to 3 \to 4 \to 5 \to 2 \to 1$

In this case the cost is increase by $(2 + 8) - 0 = 20$

Again next greater element in this table is 9 in cell (5, 4). So we shift the assignment from cell (3, 4) to (3, 2). This we get the following table and the feasible solution is $1 \to 3 \to 2 \to 5 \to 4 \to 1$:

	1	2	3	4	5
1	∞	11	$\boxed{0}$	0	∞
2	12	∞	1	∞	$\boxed{0}$
3	0	$\boxed{0}$	∞	0	0
4	$\boxed{0}$	∞	0	∞	0
5	∞	0	1	$\boxed{9}$	∞

In this case the cost is increased by $(9 + 0) - 0 = 9$

Hence the optimal feasible solution of the problem *i.e.* the optimum route is

$$1 \to 3 \to 2 \to 5 \to 4 \to 1$$

The minimum cost $= 4 + 5 + 10 + 20 + 10 = 49$.

Problem Set

1. A salesman has to visit five cities A, B, C, D and E. The distance (in hundred miles) between the five cities are as follows:

	A	B	C	D	E
A	–	7	6	8	4
B	7	–	8	5	6
C	6	8	–	9	7
D	8	5	9	–	8
E	4	6	7	8	–

If the salesman starts from city A and has to come back to city A, which route should he select so that the total distance travelled is minimum.

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2. Solve the travelling salesman problem given by the following data. $c_{12} = 16,\ c_{13} = 4,\ c_{14} = 12,\ c_{23} = 6,\ c_{34} = 5,\ c_{25} = 8,\ c_{35} = 6,\ c_{45} = 20,\quad c_{ij} = c_{ji},$ $c_{ij} = \infty$ for all value of i and j not given in the data. Find the optimum sequence of products in order to minimize the total set up cost.

3. Solve the travelling salesman problem given by the following data $c_{12} = 4,$ $c_{13} = 7,\ c_{14} = 3,\ c_{23} = 6,\ c_{24} = 3,$ and $c_{34} = 7$ where $c_{ij} = c_{ji}$.

4. Solve the following travelling salesman problem.

	1	2	3	4	5	6
1	∞	20	23	27	29	34
2	21	∞	19	26	31	24
3	26	28	∞	15	36	26
4	25	16	25	∞	23	18
5	23	40	23	31	∞	10
6	27	18	12	35	16	∞

Answers

1.	$A \to C \to D \to B \to E \to A$ and $A \to E \to B \to D \to C \to A$. In both case total distance covered = 30 *i.e.* 3000 miles.
2.	$1 \to 2 \to 5 \to 3 \to 4 \to 1$, Minimum total setup cost = ₹ 47
3.	$1 \to 3 \to 2 \to 4 \to 1$, Minimum cost = ₹ 19
4.	$1 \to 5 \to 6 \to 3 \to 4 \to 2 \to 1$, Minimum cost = ₹ 103

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