BCA 3rd Year

Numerical Methods

Page No :

UNIT - 1
Roots Of Equations

★ **Numerical Methods** — The limitation of analytical method in practical applications have lead scientific and engineers to evaluate numerical methods with the help of CBOT (Computer Based Optimization Techniques) (Numerical analysis), we can solve large problem with the accuracy result.

The aim of numerical analysis is therefore to provide constructive methods for obtaining answers to such problems in a numerical form.

★ **Accuracy of numbers** :-

1- Approximate numbers — There are two types of numbers

(i) Exact numbers like — 2, 4, 9, $\frac{7}{2}$

Approximate numbers like — 3.1444, 2.788

2- Round of / Significant numbers — Significant numbers like — 3456, 3.4546.

★ **Bisection Method** :-

Ques 1. Find a root of the equation $x^3 - 4x - 9 = 0$ using Bisection method in four stages.

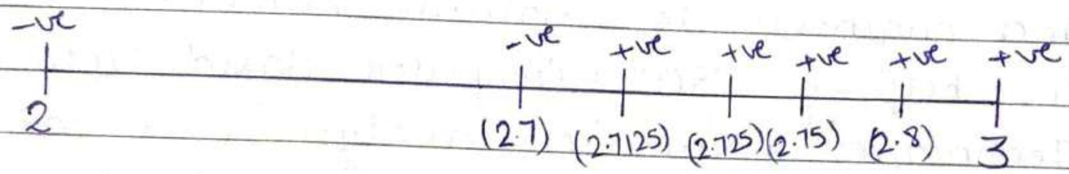
Sol :-

$$f(x) = x^3 - 4x - 9$$
$$f(1) = 1^3 - 4 - 9 = -12 \ (-ve)$$

$f(2) = 2^3 - 8 - 9 = -9$ (-ve)

$f(3) = 3^3 - 12 - 9 = 6$ (+ve)

The root lies between 2 and 3. We find nearest root with the help of points 2 and 3 then we find the value.



$f(2.5) = (2.5)^3 - 4 \times 2.5 - 9 = -3.375$ (-ve)

$f(2.6) = (2.6)^3 - 4 \times 2.6 - 9 = -1.824$ (-ve)

$f(2.7) = (2.7)^3 - 4 \times 2.7 - 9 = -0.117$ (-ve)

$f(2.8) = (2.8)^3 - 4 \times 2.8 - 9 = 1.752$ (+ve)

We find first approximation to the root is -

$$x_1 = \frac{2.7 + 2.8}{2} = 2.75$$

$f(2.75) = (2.75)^3 - 4 \times 2.75 - 9 = 0.7968$ (+ve)

The root lies between 2.7 and 2.75. We find second approximation to the root is

$$x_2 = \frac{2.7 + 2.75}{2} = 2.725$$

$f(2.725) = (2.725)^3 - 4 \times 2.725 - 9 = 0.3348$ (+ve)

The root lies between 2.7 and 2.725. We find third approximation to the root is

$$x_3 = \frac{2.7 + 2.725}{2} = 2.7125$$

$f(2.7125) = (2.7125)^3 - 4 \times 2.7125 - 9 = 0.1076$ (+ve)

The root lies between 2.7 and 2.7125. We find fourth approximation to the root is

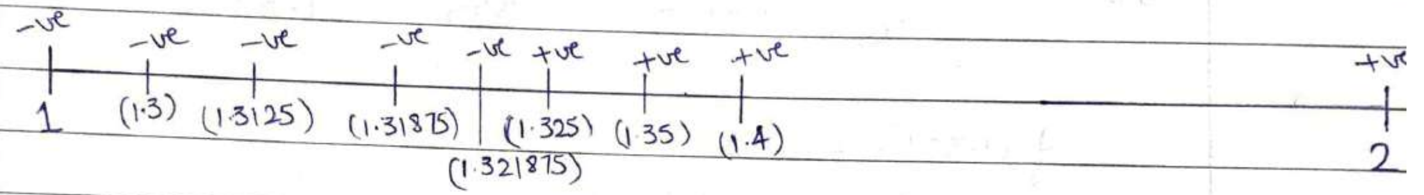
$$x_4 = \frac{2.7 + 2.7125}{2} = 2.70625, \text{ Ans.}$$

$$f(2.706) = (2.706)^3 - 4 \times 2.706 - 9 = -5 \ (-ve)$$

Ques2- $x^3 - x - 1 = 0$ (five stages)

$$f(x) = x^3 - x - 1$$
$$f(1) = 1^3 - 1 - 1 = -1 \ (-ve)$$
$$f(2) = (2)^3 - 2 - 1 = 5 \ (+ve)$$

The root lies between 1 and 2.



$$f(1.2) = (1.2)^3 - 1.2 - 1 = -0.472 \ (-ve)$$
$$f(1.3) = (1.3)^3 - 1.3 - 1 = -0.103 \ (-ve)$$
$$f(1.4) = (1.4)^3 - 1.4 - 1 = 0.344 \ (+ve)$$

We find first approximation to the root is-

$$x_1 = \frac{1.3 + 1.4}{2} = 1.35$$

$$f(1.35) = (1.35)^3 - 1.35 - 1 = 0.110 \ (+ve)$$

The root lies between 1.3 and 1.35. We find second approximation to the root is-

$$x_2 = \frac{1.3 + 1.35}{2} = 1.325$$

$$f(1.325) = (1.325)^3 - 1.325 - 1 = 1.203 \ (+ve)$$

The root lies between 1.3 and 1.325. We find third approximation to the root is-

$$x_3 = \frac{1.3 + 1.325}{2} = 1.3125$$

$$f(1.3125) = (1.3125)^3 - 1.3125 - 1 = -0.0515 \ (-ve)$$

The root lies between 1.3125 and 1.325. We find fourth approximation to the root is -

$$x_4 = \frac{1.3125 + 1.325}{2} = 1.31875$$

$$f(1.31875) = (1.31875)^3 - 1.31875 - 1 = -0.0253 \ (-ve)$$

The root lies between 1.31875 and 1.325. We find fifth approximation to the root is

$$x_5 = \frac{1.31875 + 1.325}{2} = 1.321875$$

$$f(1.321) = (1.321)^3 - 1.321 - 1 = -0.015 \ (-ve)$$

Ques 3-

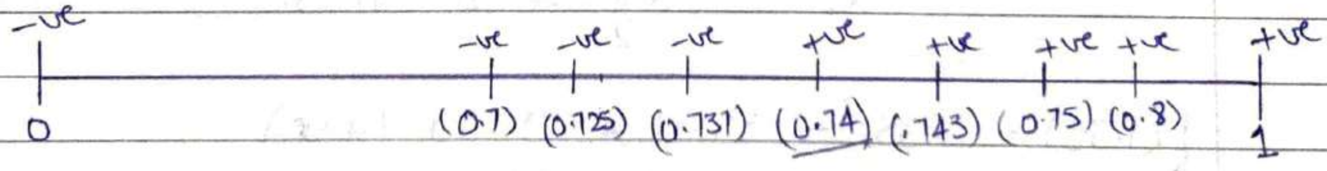
$$x - \cos x = 0$$

$$f(x) = x - \cos x$$

$$f(0) = 0 - \cos 0 = -1 \quad (-ve)$$

$$f(1) = 1 - \cos 1 = 1 \quad (+ve)$$

The root lies between 0 and 1.



$$f(0.5) = 0.5 - \cos 0.5 = -0.377 \ (-ve)$$

$$f(0.6) = 0.6 - \cos 0.6 = -0.225 \ (-ve)$$

$$f(0.7) = 0.7 - \cos 0.7 = -0.064 \ (-ve)$$

$$f(0.8) = 0.8 - \cos 0.8 = 0.103 \ (+ve)$$

First approximation $x_1 = \frac{0.7 + 0.8}{2} = 0.75$

$f(0.75) = 0.75 - \cos 0.75 = 0.0183$ (+ve)

Second approximation –

$$x_2 = \frac{0.7 + 0.75}{2} = 0.725$$

$f(0.725) = 0.725 - \cos 0.725 = -0.023$ (–ve)

Third approximation –

$$x_3 = \frac{0.725 + 0.75}{2} = 0.7375$$

$f(0.7375) = 0.7375 - \cos 0.7375 = -3.488$ (–ve)

Fourth approximation –

$$x_4 = \frac{0.7375 + 0.75}{2} = 0.74375$$

$\cancel{f(0.743) = 0.743 - \cos 0.743 = 6.557 \text{ (+ve)}}$

Fifth approximation –

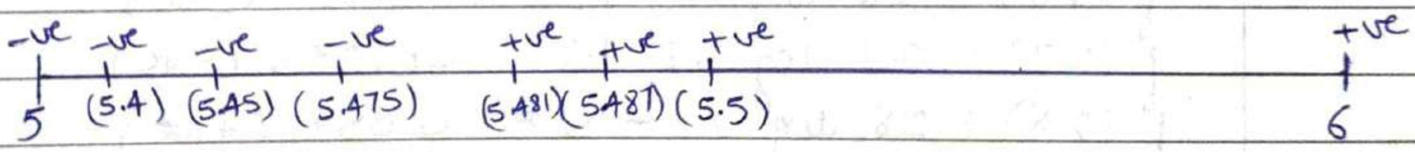
$$x_5 = \frac{0.7375 + 0.743}{2} = 0.74$$

$f(0.74) = 0.74 - \cos 0.74 = 1.53$ (+ve)

ues4 $f(x) = x^2 - 30$

$f(5) = (5)^2 - 30 = -5$ (–ve)

$f(6) = (6)^2 - 30 = 6$ (+ve)



$f(5.3) = (5.3)^2 - 30 = -1.91$ (–ve)

$f(5.4) = (5.4)^2 - 30 = -0.84$ (–ve)

$f(5.5) = (5.5)^2 - 30 = 0.25$ (+ve)

First approximation –

$$x_1 = \frac{5.4 + 5.5}{2} = 5.45$$

$f(5.45) = (5.45)^2 - 30 = -0.297$ (–ve)

Second approximation –

$$x_2 = \frac{5.45 + 5.5}{2} = 5.475$$

$f(4.7$ $f(5.475) = (5.475)^2 - 30 = -0.024$ (−ve)

Third approximation−

$$x_3 = \frac{5.475 + 5.5}{2} = 5.4875$$

$$f(5.4875) = (5.4875)^2 - 30 = 0.112 \text{ (+ve)}$$

Fourth approximation −

$$x_4 = \frac{5.475 + 5.4875}{2} = \boxed{5.48125}$$ Ans.

$$f(5.481) = (5.481)^2 - 30 = 0.041 \text{ (+ve)}$$

Ques5−

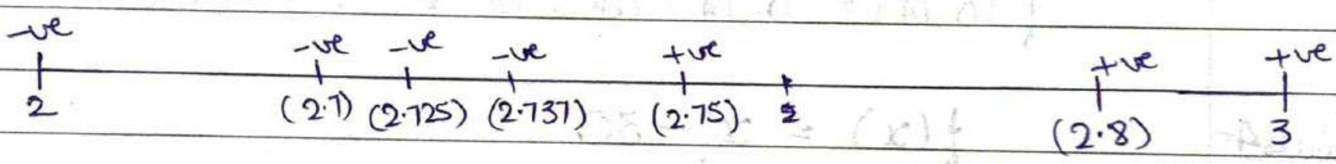
$$x \log x - 1.2 = 0$$

$f(x) = x \log x - 1.2$

$f(1) = 1 \log 1 - 1.2 = -1.2$ (−ve)

$f(2) = 2 \log 2 - 1.2 = -0.59$ (−ve)

$f(3) = 3 \log 3 - 1.2 = 0.23$ (+ve)



$f(2.5) = 2.5 \log 2.5 - 1.2 = -0.205$ (−ve)

$f(2.6) = 2.6 \log 2.6 - 1.2 = -0.121$ (−ve)

$f(2.7) = 2.7 \log 2.7 - 1.2 = -0.035$ (−ve)

$f(2.8) = 2.8 \log 2.8 - 1.2 = 0.052$ (+ve)

First approximation −

$$x_1 = \frac{2.7 + 2.8}{2} = 2.75$$

$$f(2.75) = 2.75 \log 2.75 - 1.2 = 0.08 \text{ (+ve)}$$

Second approximation−

$$x_2 = \frac{2.7 + 2.75}{2} = 2.725$$

$$f(2.725) = 2.725 \log 2.725 - 1.2 = -0.01 \text{ (−ve)}$$

Third approximation –

$$x_3 = \frac{2.725 + 2.75}{2} = 2.7375$$

$$f(2.7375) = 2.737 \log 2.737 - 1.2 = -0.003 \ (-ve)$$

Fourth approximation –

$$x_4 = \frac{2.737 + 2.75}{2} = 2.7435$$

$$f(2.743) = 2.743 \log 2.743 - 1.2 = 2.7435 \ (+ve)$$

★ False Position Method :-

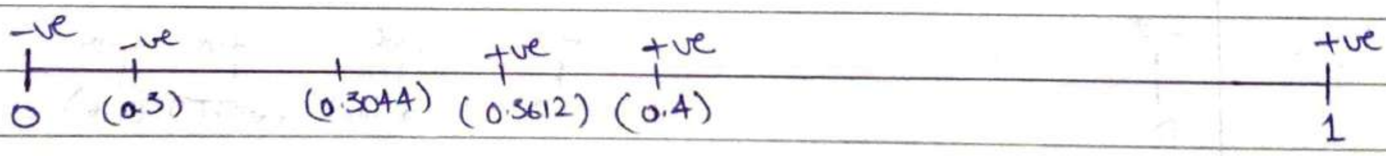
Ques:- Find a root of the eqⁿ $3x + \sin x - e^x = 0$ by the method of false position correct to four decimal places.

Sol:-

$$f(x) = 3x + \sin x - e^x$$

$$f(0) = 0 + \sin 0 - e^0 = -1 \ (-ve)$$

$$f(1) = 3 + \sin 1 - e^1 = 2.84 \ (+ve)$$



$$f(0.5) = 3 \times 0.5 + \sin 0.5 - e^{0.5} = 0.330 \ (+ve)$$

$$f(0.6) = 3 \times 0.6 + \sin 0.6 - e^{0.6} = 0.542 \ (+ve)$$

$$f(0.4) = 3 \times 0.4 + \sin 0.4 - e^{0.4} = 0.0975 \ (+ve)$$

$$f(0.3) = 3 \times 0.3 + \sin 0.3 - e^{0.3} = -0.154 \ (-ve)$$

where $x_0 (0.3)$ is negative and $x_1 (0.4)$ is positive then root lies between 0.3 and 0.4.

Using Regular falsi method –

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0)$$

$$= 0.3 + \frac{0.4 - 0.3}{0.0975 + 0.154} (0.154)$$

$$x_2 = 0.3612$$

$$f(0.3612) = 3 \times 0.3612 + \sin(0.3612) - e^{0.3612} = 1.946 \ (+ve)$$

The root lies between $\underset{x_1}{0.3}$ and $\underset{x_0}{0.3612}$

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0.3612 - \left(\frac{0.3 - 0.3612}{-0.154 - 1.946}\right) (1.946)$$

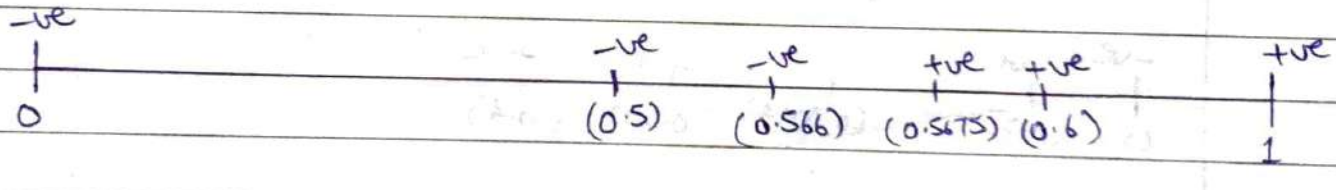
$$x_3 = 0.3044 \quad \text{Ans}$$

Ques2. $x - e^{-x} = 0$ Solve by Regular Falsi method.

Let $f(x) = x - e^{-x}$

$f(0) = 0 - e^{-0} = -1 \ (-ve)$

$f(1) = 1 - e^{-1} = 0.632 \ (+ve)$



$f(0.5) = 0.5 - e^{-0.5} = -0.106 \ (-ve)$

$f(0.6) = 0.6 - e^{-0.6} = 0.051 \ (+ve)$

The root lies between 0.5 and 0.6 where $x_0 = 0.5$, $x_1 = 0.6$

Using regular falsi method we have

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$x_2 = 0.5 - \left(\frac{0.6 - 0.5}{0.051 + 0.106}\right) (-0.106)$$

$$= 0.5 + 0.067$$

$$x_2 = 0.5675$$

$$f(0.5675) = 0.5675 - e^{-0.5675} = 0.00055 \ (+ve)$$

The root lies between $x_0\,(0.567)$ and $x_1\,(0.567)$

$$x_3 = 0.5 - \left(\frac{0.567 - 0.5}{0.00055 + 0.106}\right)(-0.106)$$

$$= 0.5 + 0.066$$

$$x_3 = 0.566$$

$$f(0.566) = 0.566 - e^{-0.566} = -0.0017 \ (-ve)$$

Two values are matched to decimal two places. The answer is 0.566.

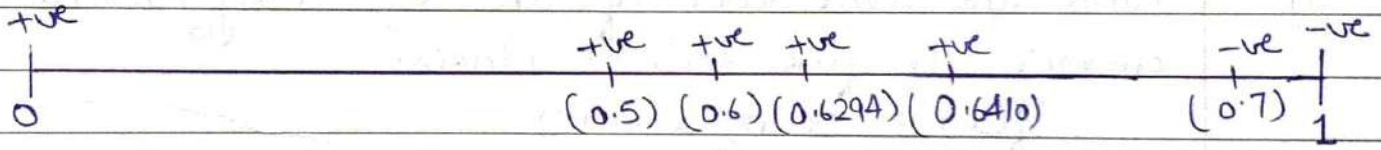
Ques3- $x^3 - 5x + 3 = 0$

$$f(x) = x^3 - 5x + 3$$

$$f(0) = 0 - 0 + 3 = 3 \ (+ve)$$

$$f(1) = (1)^3 - 5\times 1 + 3 = -1 \ (-ve)$$

The root lies between 0 and 1.



$$f(0.5) = (0.5)^3 - 5\times 0.5 + 3 = 0.625 \ (+ve)$$

$$f(0.6) = (0.6)^3 - 5\times 0.6 + 3 = 0.216 \ (+ve)$$

$$f(0.7) = (0.7)^3 - 5\times 0.7 + 3 = -0.517 \ (-ve)$$

The root lies between 0.6 and 0.7 where $x_0 = 0.6$, $x_1 = 0.7$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0.6 - \left(\frac{0.7 - 0.6}{-0.517 - 0.216}\right)(0.216)$$

$$= 0.6 + 0.0294$$

$$x_2 = 0.6294$$

$$f(0.6294) = (0.6294)^3 - 5\times 0.6294 + 3 = 0.1023 \ (+ve)$$

The root lies between 0.6294 and 0.7 where $x_0 = 0.6294, x_1 = 0.7$

$$x_3 = 0.6294 - \left(\frac{0.7 - 0.6294}{-0.517 - 0.1023}\right)(0.1023)$$

$$= 0.6294 + 0.0116$$

$$x_3 = 0.6410$$

$f(0.6410) = (0.6410)^3 - 5 \times 0.6410 + 3 = 0.0583$ (+ve)

The root lies between 0.6410 and 0.7

$$x_4 = 0.6410 - \left(\frac{0.7 - 0.6410}{-0.517 - 0.0583}\right)(0.0583)$$

$$= 0.6410 + 0.00597$$

$$= 0.6469$$

Two values are matched to decimal two points. The answer is 0.6469.

* Newton - Raphson Method (Newton iterative method)

Q1- Find the real root of the eq^n $x \log_{10} x = 1.2$ correct to five decimal places.

$$f(x) = x \log_{10} x - 1.2$$

$$f(1) = -1.2$$

$$f(2) = -0.597 \quad (-ve)$$

$$f(3) = 0.2313 \quad (+ve)$$

The root lies between 2 and 3

$$f(2.5) = -0.205$$

$$f(2.7) = -0.0353 \quad (-ve)$$

$$f(2.8) = 0.0520 \quad (+ve)$$

The root lies between 2.7 and 2.8.

$$f'(x) = x \log_{10} e \cdot \frac{1}{x} + \log x \times 1$$

$$f'(x) = \log_{10} e + \log x$$

$$f'(x) = 0.43429 + \log_{10} x$$

Let taking $x = 2.7$

By Newton iterative method –

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \left[\frac{x_n \log x_n - 1.2}{\log_{10} x_n + 0.43429}\right]$$

$$= \frac{\cancel{x_n \log_{10} x_n} + 0.43429 x_n - \cancel{x_n \log_{10} x_n} + 1.2}{\log_{10} x_n + 0.43429}$$

$$x_{n+1} = \frac{0.43429 x_n + \cancel{x} 1.2}{\log_{10} x_n + 0.43429}$$

First iteration, put $n=0$ then

$$x_1 = \frac{0.43429\, x_0 + 1.2}{\log_{10} x_0 + 0.43429} \qquad [\text{where } x_0 = 2.7]$$

$$= \frac{0.43429(2.7) + 1.2}{\log_{10}(2.7) + 0.43429} = 2.74079$$

Second iteration, we have $n=1$, $x_1 = 2.74079$

$$x_2 = \frac{0.43429(2.74079) + 1.2}{\log_{10}(2.74079) + 0.43429}$$

$$= 2.74064$$

Third iteration,

$$x_3 = \frac{0.43429(2.74064) + 1.2}{\log_{10}(2.74064) + 0.43429}$$

$$= 2.74064$$

The value has matched upto 5 decimal points.

The correct root is ~~2.70~~ 2.74064.

UNIT-2

Interpolation and Extrapolation

★ Finite difference formula

Forward   Backward

Q1- Find difference forward formula-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	9962	-114	-75	2	$\boxed{-1}$
10	9848	-189	$\boxed{-73}$	1	2
15	9659	-262	-72	3	
20	9397	-334	-69		
25	9063	-403			
30	8660				

Find the value of $\Delta^2 y_{10}$, $\Delta^4 y_5$

$$\Delta^2 y_{10} = -73$$
$$\Delta^4 y_5 = -1$$

Ques2- If $y = x^3 + x^2 - 2x + 1$. Calculate values of y for $x = 0, 1, 2, 3, 4, 5, 6$ from the difference table. Solve by foward difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1	0	8	6	0	0
1	1	8	14	6	0	0
2	9	22	20	6	0	
3	31	42	26	6		
4	73	68	32			
5	141	100				
6	241					

$\Delta^3 y_3 = 6$
$\Delta^2 y_3 = 26$
$\Delta^2 y_0 = 8$

Ques 3- Construct a backward difference formula for $y = \log x$ given that $x = 10, 20, 30, 40, 50$, $\nabla^3 \log_{40}$, $\nabla^4 \log_{50} = ?$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	1	−0.3010	−0.1249	−0.0737	−0.0505
20	1.3010	−0.1761	−0.0512	−0.0232	
30	1.4771	−0.1249	−0.028		
40	1.6020	−0.0969			
50	1.6989				

$\nabla^3 \log_{40} = -0.0232$ $\qquad$ $\nabla^4 \log_{50} = -0.0505$

★ **Newton forward interpolation formula** -

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{\underline{|2}}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{\underline{|3}}\Delta^3 f(a)$$

$$+ \dots \frac{u(u-1)(u-2)\dots(u-n+1)}{\underline{|n}}\Delta^n f(a)$$

Ques 1- Find the value of $\sin 52°$ from the given table -

θ	$\sin\theta$ $f(a)$	$\Delta f(a)$	$\Delta^2 f(a)$	$\Delta^3 f(a)$
45°	$\boxed{.7071}$ →	$\boxed{589}$ →	$\boxed{-57}$ →	$\boxed{-7}$
50°	.7660	532	−64	
55°	.8192	468		
60°	.8660			

$h = (50-45) = 5$ $\qquad$ $x = 52°$

$a = 45°$ $\qquad$ $u = \frac{x-a}{h} = 1.4$

$$f(45 + 5 \times .4) = 7071 + 1.4 \times 589 + \frac{1.4(1.4-1)}{\lfloor 2}(-57) + \frac{1.4(1.4-1)(1.4-2)}{\lfloor 3} \times (-7)$$

$$f(52) = 7071 + 824.6 + \frac{0.56}{2}(-57) - \frac{0.336}{6}(-7)$$

$$= 7071 + 824.6 - 15.96 + 0.392$$

$$= 7880.032$$

$\sin 52^\circ \Rightarrow 0.7880$ Ans.

Ques- The population of a town in the decimal places was given below estimated the population of the year 1895.

Year (x)	Population (Y)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
1891	46	20	-5	2	-3
1901	66	15	-3	-1	
1911	81	12	-4		
1921	93	8			
1931	101				

$h = 10$

$a = 1891$

$x = 1895$

$u = \frac{x-a}{h} = 0.4$

$$f(1891 + 10 \times 0.4) = 46 + (0.4 \times 20) + \frac{0.4(0.4-1)}{\lfloor 2}(-5) + \frac{0.4(0.4-1)(0.4-2)}{\lfloor 3} \times (2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{\lfloor 4}(-3)$$

$$f(1895) = 46 + 8 + 0.6 + 0.128 + 0.1248$$

$$f(1895) = 54.85$$ Ans.

Ques3- From the following table, estimate the number of student marks between 40 and 45.

Marks	30–40	40–50	50–60	60–70	70–80
No. of students	31	42	51	35	31

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$\boxed{40}$	$\boxed{31}$	$\boxed{42}$	$\boxed{9}$	$\boxed{-25}$	$\boxed{37}$
50	73	51	−16	12	
60	124	35	−4		
70	159	31			
80	190				

$h = 10$ $\qquad\qquad u = \frac{x-a}{h} = \frac{45-40}{10} = 0.5$

$x = 45$

$a = 40$

$$f(40 + 10\times 0.5) = 31 + 0.5\times 4.2 + \frac{0.5(0.5-1)}{\lfloor 2}\times 9 + \frac{0.5(0.5-1)(0.5-2)}{\lfloor 3}\times(-25)$$

$$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{\lfloor 4}\times 37$$

$$= 31 + 21 + (-1.125) + (-1.5625) + (-1.4455)$$

$$= 47.8672$$ Ans.

Ques4-

Year	1891	1901	1911	1921	1931
Population	46	66	81	93	101

Solve by Newton backward method for the population of the year 1925.

Year	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46	20	−5	2	$\boxed{-3}$
1901	66	15	−3	$\boxed{-1}$	
1911	81	12	$\boxed{-4}$		
1921	93	$\boxed{8}$			
1931	$\boxed{101}$				

$x = 1925$ $\quad u = \frac{x-a}{h} = -0.6$

$a = 1931$

$h = 10$

$$f(a+hu) = f(a) + u\nabla f(a) + \frac{u(u+1)}{\lfloor 2} \nabla^2 f(a) + \frac{u(u+1)(u+2)}{\lfloor 3} \nabla^3 f(a)$$

$$+ \dots \frac{u(u+1)(u+2)\dots(u+n-1)}{\lfloor n} \nabla^n f(a)$$

$$f[1931+10(0.6)] = 101 - (-0.6)\times 8 + \frac{(-0.6)(-0.6+1)}{\lfloor 2}(-4) +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)}{\lfloor 3}(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{\lfloor 4}(3)$$

$$= 101 - 4.8 + 0.48 + 0.056 + 0.1000$$

$$= 96.836 \text{ Ans.}$$

Imp.

★ Newton divident difference formula :-

$$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0$$

$$+ \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1})\Delta^n y_0$$

Ques-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22	$\left(\frac{30-22}{2-1}\right) = 8$	$\left(\frac{26-8}{4-1}\right) = 6$	$\left(\frac{-3.6-6}{7-1}\right) = -1.6$	$\left(\frac{0.535+1.6}{12-1}\right) = 0.1940$
2	30	$\left(\frac{82-30}{4-2}\right) = 26$	$\left(\frac{8-26}{7-2}\right) = -3.6$	$\left(\frac{1.75+3.6}{12-2}\right) = 0.535$	
4	82	$\left(\frac{106-82}{7-4}\right) = 8$	$\left(\frac{22-8}{12-4}\right) = 1.75$		
7	106	$\left(\frac{216-106}{12-7}\right) = 22$			
12	216				

Ques-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4	26	15	1
4	56	131	23	
9	711	269		
10	980			

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Ques –

	x	$f(x)$ (y_0)	$\Delta f(x)$ (y_1)	$\Delta^2 f(x)$ (y_2)	$\Delta^3 f(x)$ (y_3)	$\Delta^4 f(x)$ (y_4)
x_0	-4	$\boxed{1245}$	$\boxed{-404}$	$\boxed{94}$	$\boxed{-14}$	$\boxed{3}$
x_1	-1	33	-28	10	13	
x_2	0	5	2	88		
x_3	2	9	442			
x_4	5	1335				

Hence find $f(1)$.

$$y = f(x) = f(1) = 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)(x+0)(-14) + (x+4)(x+1)(x+0)(x-2)(3)$$

$$= 1245 + (1+4)(-404) + (1+4)(1+1)(94) + (1+4)(1+1)(1+0)(-14) + (1+4)(1+1)(1+0)(1-2)(+3)$$

$$= 1245 - 2020 + 940 + (-140) + (-30)$$

$$= -5 \quad \text{Ans.}$$

Ques – Find the value of $f(8)$ and $f(15)$.

	x	$f(x)$ (y_0)	$\Delta f(x)$ (y_1)	$\Delta^2 f(x)$ (y_2)	$\Delta^3 f(x)$ (y_3)	$\Delta^4 f(x)$ (y_4)
x_0	4	$\boxed{48}$	$\boxed{52}$	$\boxed{15}$	$\boxed{1}$	0
x_1	5	100	97	21	1	0
x_2	7	294	202	27	1	
x_3	10	900	310	33		
x_4	11	1210	409			
x_5	13	2028				

$$f(8) = 48 + (8-4)52 + (8-4)(8-5)\cdot 15 + (8-4)(8-5)(8-7)\cdot 1$$
$$= 48 + 208 + 180 + 12$$
$$= 448$$

$$f(15) = 48 + (15-4)52 + (15-4)(15-5)15 + (15-4)(15-5)(15-7)\cdot 1$$
$$= 48 + 572 + 1650 + 880$$
$$= 3150 \quad \text{Ans.}$$

Imp.
★ <u>Lagrange's Interpolation formula :-</u>

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

Ques1- Using Lagrange interpolation formula find the value of $y(10)$.

	x_0	x_1	x_2	x_3
x	5	6	9	11
y	12	13	14	16
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

$$y(10) = \left[\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\right] \cdot 12 + \left[\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\right] 13 + \left[\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\right]$$

$$\times 14 + \left[\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\right] 16$$

$$= \left(\frac{-4}{-24} \times 12\right) + \left(\frac{-5}{15} \times 13\right) + \left(\frac{-20}{-24} \times 14\right) + \left(\frac{20}{60} \times 16\right)$$

$$= 2 - 4.33 + 11.666 + 5.333$$

$$= 14.666 \quad \text{Ans.}$$

Ques2-

	x_0	x_1	x_2	x_3
x	1	2	3	4
f(x)	1	8	27	64

find $f(x) = 2.5$

$$y(2.5) = \left[\frac{(2.5-2)(2.5-3)(2.5-4)}{(1-2)(1-3)(1-4)}\right] 1 + \left[\frac{(2.5-1)(2.5-3)(2.5-4)}{(2-1)(2-3)(2-4)}\right] 8 +$$

$$\left[\frac{(2.5-1)(2.5-2)(2.5-4)}{(3-1)(3-2)(3-4)}\right] 27 + \left[\frac{(2.5-1)(2.5-2)(2.5-3)}{(4-1)(4-2)(4-3)}\right] 64$$

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$$= -0.0625 + 4.5 + 15.1875 - 4$$

$$f(2.5) = 15.625 \text{ Ans.}$$

★ <u>Gauss Forward Interpolation Formula (Centralized difference formula)</u> :-

$$y_P = y_0 + P\Delta y_0 + \frac{P(P-1)}{1.2}\Delta^2 y_{-1} + \frac{P(P-1)(P+1)}{1.2.3}\Delta^3 y_{-1} + \frac{P(P-1)(P+1)(P-2)}{1.2.3.4}\Delta^4 y_{-2} + \ldots$$

Q1- Interpolate by Gauss forward formula the population of a town for the year 1974 given that -

Year	P	Population (in thousands)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1939	-3	12	$\cancel{3}$				
			3				
1949	-2	15		2			
			5		0		
1959	-1	20		2		3	
			7		3		-10
1969	0	$\boxed{27}$		$\boxed{5}$		$\boxed{-7}$	
1974			$\boxed{12}$		$\boxed{-4}$		
1979	1	39		1			
			13				
1989	2	52					

Q2- Use forward formula to calculate y_{30} given that $y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$ and $y_{37} = 15.5154$

Sol^n-

x	P	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
21	−2	18.4708				
			−0.6564			
25	−1	17.8144		−0.370		
			−0.7074		−0.0074	
29	0	$\boxed{17.1070}$		$\boxed{-0.0564}$		$\boxed{-0.0022}$
30			$\boxed{-0.7638}$		$\boxed{-0.0076}$	
33	1	16.3432		−0.064		
			−0.8278			
37	2	15.5154				

$$P = \frac{x - x_0}{h} = \frac{30 - 29}{4} = 0.25$$

$$\boxed{y_P = y_0 + P\Delta y_0 + \frac{P(P-1)}{1.2}\Delta^2 y_{-1} + \frac{(P+1)P(P-1)}{1.2.3}\Delta^3 y_{-1} + \frac{(P+1)P(P-1)(P-2)}{1.2.3.4}\Delta^4 y_{-2} + \dots}$$

$$y_{30} = 17.1070 + [0.25 \times (-0.7638)] + \cancel{(0.25+1)}\left[\frac{(0.25)(0.25-1)}{1.2}\right](-0.0564)$$

$$+ \left[\frac{(0.25+1)(0.25)(0.25-1)}{1.2.3.}\right](-0.0076) + \left[\frac{(0.25+1)(0.25)(0.25-1)(0.25-2)}{1.2.3.4.}\right](-0.0022)$$

$= 17.1070 - 0.19095 + 0.00528 + 0.00029 - 0.000037$

$= 16.9215$ Ay.

Q3- Apply a central difference formula to obtain $f(32)$ given that: $f(25) = 0.2707$, $f(30) = 0.3027$, $f(35) = 0.3386$, $f(40) = 0.3794$

x	P	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
25	−1	0.2707			
			0.032		
30	0	$\boxed{0.3027}$	$\boxed{0.0359}$	$\boxed{0.0039}$	$\boxed{0.001}$
35	1	0.3386		0.0049	
			0.0408		
40	2	0.3794			

32

$$P = \frac{x - x_0}{h} = \frac{32-30}{5} = 0.4$$

$$f(32) = 0.3027 + (0.4 \times 0.0359) + \frac{(0.4)(0.4-1)}{2} \times (0.0039) + \frac{(0.4+1)(0.4)(0.4-1)}{3.2.1} \times (0.001)$$

$$= 0.3027 + 0.01436 + (-0.000468) + (-0.000056)$$

$$= 0.316536 \text{ Ay.}$$

Q4-

x	P	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	−2	1				
			−2			
2	−1	−1		4		
			2		−8	
3	0	$\boxed{1}$	$\boxed{-2}$	$\boxed{-4}$	$\boxed{8}$	$\boxed{16}$
4	1	−1		4		
			2			
5	2	1				

$$P = \frac{x - x_0}{h} = \frac{x-3}{1}$$

$$P = x - 3$$

$$f(P) = 1 + (x-3)(-2) + \frac{(x-3)(x-3-1)(-4)}{2} + \frac{(x-3+1)(x-3)(x-3-1)}{6} \times 8 +$$

$$\frac{(x-3+1)(x-3)(x-3-1)(x-3-2)}{1.\cancel{2}.3.\cancel{4}} \times \cancel{16}^{\,2}$$

$$= 1-2x+6+[(-2x+6)(x-4)] + \frac{4}{3}[(x-2)(x-3)(x-4)] + \frac{2}{3}[(x-2)(x-3)(x-4)(x-5)]$$

$$= 7-2x-2x^2+8x+6x-24+\frac{4}{3}[(x^2-3x-2x+6)(x-4)] + \frac{2}{3}[(x^2-5x+6)(x^2-9x+20)]$$

$$= 7-2x-2x^2+14x-24+\frac{4}{3}[x^3-4x^2-5x^2+20x+6x-24] +$$

$$\frac{2}{3}[x^4-9x^3+20x^2-5x^3+45x^2-100x+6x^2-54x+120]$$

$$= -2x^2+12x-17+\frac{4}{3}[x^3-9x^2+26x-24] + \frac{2}{3}[x^4-14x^3+71x^2-154x+120]$$

$$= -2x^2+12x-17+\frac{1}{3}(\cancel{4x^4}\ 4x^3-36x^2+104x-96+2x^4-28x^3-308x+142x^2+240)$$

$$= -2x^2+12x-17+\frac{1}{3}(2x^4-24x^3+106x^2-204x+144)$$

$$= -2x^2+12x-17+\frac{2}{3}x^4-8x^3+\frac{106}{3}x^2-\frac{204}{3}x+48$$

$$= \frac{2}{3}x^4-8x^3+\frac{100}{3}x^2-56x+31 \quad \text{Ans.}$$

Q5- The values of e^{-x} at $x = 1.72$ to 1.76. Find the value of $e^{-1.7425}$.

x	P	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.72	−2	0.17906				
			−178			
1.73	−1	0.17728		2		
			−176		−1	
1.74	0	$\boxed{0.17552}$		$\boxed{1}$		$\boxed{2}$
			$\boxed{-175}$		$\boxed{1}$	
1.75	1	0.17377		2		
			−173			
1.76	2	0.17204				

1.7425

$$P = \frac{x - x_0}{h} = \frac{1.7425 - 1.74}{0.01} = 0.25$$

$$y(1.7425) = 17552 + [0.25 \times (-175)] + \left[\frac{0.25(0.25-1) \times 1}{2}\right] + \left[\frac{(0.25+1)(0.25)(0.25-1)(1)}{6}\right] + \left[\frac{(0.25+1)(0.25)(0.25-1)(0.25-2)(2)}{24}\right]$$

$$= 17552 - 43.75 - 0.09375 - 0.03906 + 0.03417$$

$$= 17508.15136$$

$$e^{-17425} = 0.17508 \quad \text{Ans.}$$

★ Gauss Backward formula :-

$$f(u) = f(0) + u\Delta f(-1) + \frac{(u+1)u}{\underline{|2}} \Delta^2 f(-1) + \frac{(u+1)u(u-1)}{\underline{|3}} \Delta^3 f(-2) + \frac{(u+2)(u+1)u(u-1)}{\underline{|4}} \Delta^4 f(2) + \ldots$$

Q1- Find the value of cos 51°42'.

x	$f(x)$ cos x	P	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
50°	0.6428	-2				
			-135			
51°	0.6293	-1		-1		
			$\boxed{-136}$		$\boxed{-2}$	
52°	$\boxed{0.6157}$	0		$\boxed{-3}$		$\boxed{4}$
			-139		2	
53°	0.6018	1		-1		
			-14			
54°	0.5878	2				

$$u = \frac{x - x_0}{h} = \frac{51°42' - 52}{1} = \frac{-18'}{60} = -0.3$$

$$f(51°42') = 6157 + (-0.3)(-136) + \frac{(-0.3+1)(-0.3)}{2!}(-3) +$$

$$\frac{(-0.3+1)(-0.3)(-0.3-1)(-2)}{3!} + \frac{(-0.3+2)(-0.3+1)(-3)(-0.3-1)(4)}{4!}$$

$$= 6157 + 40.8 + 0.315 - 0.091 + 0.7735$$

$$= 6198.7975$$

$f(51°42') = 0.6198$ Ay.

★ Stirling formula :-

$$f(u) = f(0) + u\left\{\frac{\Delta f(0) + \Delta f(-1)}{2}\right\} + \frac{u^2}{2!}\Delta^2 f(-1) + \frac{(u+1)u(u-1)}{3!}\left\{\frac{\Delta^3 f(-1) + \Delta^3 f(-2)}{2}\right\}$$

$$\Delta^4 f(-2) + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}$$

$$+ \frac{u^2(u^2-1)}{4!}\left\{\frac{\Delta^4 f(-2) + \Delta^4 f(-3)}{2}\right\} + \ldots$$

Q1-

θ	$\tan\theta$	u	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
0°	0	-3						
			875					
15°	0.0875	-2		13				
			888		15			
10°	0.1763	-1		28		2		
			$\boxed{916}$		$\boxed{17}$		$\boxed{-2}$	
15°	0.$\boxed{2679}$	0		$\boxed{45}$		$\boxed{0}$		$\boxed{11}$
			$\boxed{961}$		$\boxed{17}$		$\boxed{9}$	
16 ← 20°	0.364	1		62		9		
			1023		26			
25°	0.4663	2		88				
			1111					
30°	0.5774	3						

Find the value of tan 16°. $u = \frac{16-15}{5} = 0.2$

$$f(16^\circ) = 2679 + (.2)\left[\frac{961+916}{2}\right] + \frac{(.2)^2}{\lfloor 2}(45) + \frac{(0.2+1)\,0.2\,(0.2-1)}{\lfloor 3}\left[\frac{17+17}{2}\right] +$$

$$+ 0 + \frac{(.2+2)(.2+1)0.2(.2-1)(.2-2)}{\lfloor 5}\left[\frac{-2+9}{2}\right] + \frac{(.2)^2[(.2)^2-1][(.2)^2-4]}{\lfloor 6}(11)$$

$$= 2679 + 187.7 + 0.9 + (-4.624) + 0.022176 + 0.0002323$$

$$= 2862.9983$$

$\tan 16^\circ = 0.28629983$ Ans

Q2- Find value of 1.22

x	u	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	-2	0				
			191			
0.5	-1	0.191		-41		
			$\boxed{150}$		$\boxed{-17}$	
1.22 ← 1.0	0	0.$\boxed{341}$		$\boxed{-58}$		$\boxed{27}$
			$\boxed{92}$		$\boxed{10}$	
1.5	1	0.433		-48		
			44			
2.0	2	0.477				

$u = \frac{1.22-1}{0.5} = 0.44$

$$f(1.22) = 341 + \frac{0.44}{2}[150+92] + \frac{(0.44)^2}{\lfloor 2}(-58) + \frac{(.44+1)(.44)(.44+1)}{\lfloor 3}$$

$$\left[\frac{17+10}{2}\right] + \frac{(0.44)^2(0.44^2-1)}{\lfloor 4}(-27)$$

$$= 341 + 53.44 - 5.6144 + 0.206976 - 0.175024$$

$= 388.657$

$f(1.22) = 0.388657$ Ans.

★ **Bessel's formula :-**

$$f(u) = \left\{\frac{f(0) + f(1)}{2}\right\} + \left(u - \frac{1}{2}\right)\Delta f(0) + \frac{u(u-1)}{\lfloor 2}\left\{\frac{\Delta^2 f(0) + \Delta^2 f(-1)}{2}\right\}$$

$$+ \frac{(u-1)(u-1/2)u}{\lfloor 3}\Delta^3 f(-1) + \frac{(u+1)u(u-1)(u-2)}{\lfloor 4}\left\{\frac{\Delta^4 f(-1) + \Delta^4 f(-2)}{2}\right\}$$

$$+ \frac{(u-2)(u-1)(u-1/2)u(u+1)}{\lfloor 5}\Delta^5 f(-2) + \cdots$$

Ques. Find the value of 27.4.

u	x	$f(x)$	$10^3\Delta f(x)$	$10^3\Delta^2 f(x)$	$10^3\Delta^3 f(x)$	$10^3\Delta^4 f(x)$	$10^3\Delta^5 f(x)$
−2	25	4.000					
			−154				
−1	26	3.846		12			
			−142		−3		
0	27	3.704		9		4	
			−133		1		−7
1	28	3.571		10		−3	
			−123		−2		
2	29	3.448		8			
			−115				
3	30	3.338					

$$u = \frac{x - x_0}{h} = \frac{27.4 - 27}{1} = 0.4$$

$$10^3 f(0.4) = \left(\frac{3704 + 3571}{2}\right) + (.4 - .5)(-133) + \frac{(.4)(.4-1)}{2}\left(\frac{9+10}{2}\right) +$$

$$\frac{(.4-1)(.4-.5)(.4)}{6}(1) + \frac{(.4+1)(.4)(.4-1)(.4-2)}{24}\left(\frac{4-3}{2}\right)$$

$$+ \frac{(.4-2)(.4-1)(.4-.5)(.4)(.4+1)}{120}(-7)$$

$= 3637.5 + 13.3 - 1.14 + 0.004 + 0.0112 - 0.003136$

$= 3649.672$

$f(27.4) = 3.649672$ Ans.

Q2- Find the value of $f(9)$

u	x	$f(x)$	$10^4\Delta f(x)$	$10^4\Delta^2 f(x)$	$10^4\Delta^3 f(x)$	$10^4\Delta^4 f(x)$	$10^4\Delta^5 f(x)$
-2	4	3.5460	15293				
-1	6	5.0753	13879	-1414	120		
0	8	$\boxed{6.4632}$	$\boxed{12585}$	$\boxed{-1294}$	$\boxed{125}$	$\boxed{5}$	$\boxed{-24}$
1	10	$\boxed{7.7217}$	11416	$\boxed{-1169}$	106	$\boxed{-19}$	
2	12	8.8633	10353	-1063			
3	14	9.8986					

$$u = \frac{9-8}{2} = 0.5$$

$$10^4 f(0.5) = \left(\frac{64632+77217}{2}\right) + 0 + \frac{(.5)(.5-1)}{2}\left(\frac{-1294-1169}{2}\right) + 0 +$$

$$\frac{(.5+1).5(.5-1)(.5-2)}{24}\left(\frac{5-19}{2}\right) + 0$$

$$= 70924.5 + 153.9375 - 0.1640625$$

$$= 71078.27344$$

$$f(9) = 7.107827344 \text{ Ans}$$

★ <u>Laplace Everett's formula</u>:-

$$f(u) = \left\{ u f(1) + \frac{(u+1)u(u-1)}{\lfloor 3} \Delta^2 f(0) + \frac{(u+2)(u+1)u(u-1)(u-2)}{\lfloor 5} \Delta^4 f(-1) \ldots \right\}$$

$$+ \left\{ w f(0) + \frac{(w+1)w(w-1)}{\lfloor 3} \Delta^2 f(-1) + \frac{(w+2)(w+1)w(w-1)(w-2)}{\lfloor 5} \Delta^4 f(-2) \ldots \right\}$$

Q1- Calculate $f(30)$ by Laplace Everett's formula -

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	20	2854	308		
0	28	$\boxed{3162}$	3926	$\boxed{3618}$	-6648
1	36	$\boxed{7088}$	896	$\boxed{-3030}$	
2	44	7984			

$$u = \frac{30-28}{8} = 0.25$$

$\omega = 1 - u = 0.75$

$$f(0.25) = \left\{(0.25)(7088) + \frac{(1.25)(.25)(.25-1)}{6}(-3030)\right\} +$$

$$\left\{(.75)(3162) + \frac{(1.75)(.75)(.75-1)}{6}(3618)\right\}$$

$= 1772 + 118.359375 + 2371.5 - 197.859375$

$= 4064$ Ans.

Q2- Find the value of 27.4.

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
−2	25	4.000					
			−154				
−1	26	3.846		12			
			−142		−3		
0	27	$\boxed{3.704}$		$\boxed{9}$		$\boxed{4}$	
			−133		1		−7
1	28	$\boxed{3.571}$		$\boxed{10}$		$\boxed{-3}$	
			−123		−2		
2	29	3.448		8			
			−115				
3	30	3.333					

$u = 0.4$

$\omega = 1 - 0.4 = 0.6$

$$10^3 f(0.4) = \left\{(0.4)(3571) + \frac{(1.4).4(.4-1)}{6}(10) + \frac{(2.4)(1.4).4(.4-1)(.4-2)(-}{120}\right.$$

$$+\left\{(0.6)(3704) + \frac{(1.6).6(.6-1)}{6}(9) + \frac{(2.6)(1.6).6(.6-1)(.6-2)(4}{120}\right.$$

$= 1428.4 - 0.56 - 0.032256 + 2222.4 - 0.576 + 0.0465$

$= 3649.678336$

$f(27.4) = 3.649678336$ Ans.

UNIT - 3

Numerical differentiation and integration

★ **Formula for derivative :-**

Newton forward formula -

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0 + \ldots\ldots$$

Diff. this equation in respect to 'u'

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \ldots\ldots$$

$$u = \frac{x-a}{h}$$

$$\frac{du}{dx} = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \ldots\ldots\right]$$

Now find the formula for the derivative equation

Put the $u = 0$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\right]}$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \ldots\ldots\right]}$$

$$\boxed{\left(\frac{d^3y}{dx^3}\right)_{x=a} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots\right]}$$

Backward formula -

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right)$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left(\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \dots\right)$$

Ques1. x : .1 .2 .3 .4

y : .9975 .9900 .9776 .9604

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
.1	.9975			
		$\boxed{-75}$		
.2	.9900		$\boxed{-49}$	
		-124		$\boxed{1}$
.3	.9776		-48	
		-172		
.4	.9604			

$\frac{dy}{dx}$, $x = 0.1$ $h = 0.1$

$$\frac{dy}{dx} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right)$$

$$= \frac{1}{0.1}\left(-75 - \frac{1}{2}(-49) + \frac{1}{3}(1)\right)$$

$$= 10\,(-75 + 24.5 + 0.3333)$$

$$= -501.666$$

$$\Rightarrow -0.050166$$

Ques2- The table given below the velocity v of the body during the time t. Find the acceleration at $t = 1.1$.

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
1.0	43.1				
		4.6			
1.1	47.7		-0.2		
		$\boxed{4.4}$		0.1	
1.2	52.1		$\boxed{-0.1}$		0.1
		4.3		$\boxed{0.2}$	
1.3	56.4		0.1		
		4.4			
1.4	60.8				

$h = 0.1$

$$\frac{dv}{dt} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0\right]$$

$$= \frac{1}{0.1}\left[4.4 - \frac{1}{2}(-0.1) + \frac{1}{3}(0.2)\right]$$

$$= \frac{1}{0.1}\left[4.4 + 0.05 + 0.0666\right]$$

$$= 45.166 \quad \underline{Ans}.$$

★ 1- Trapezoidal Rule ($n=1$)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots y_{n-1})\right]$$

2- Simpson's one-third rule ($n=2$)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\dots y_{n-1}) + 2(y_2+y_4+\dots y_{n-2})\right]$$

3- Simpson's three-eight rule ($n=3$)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\dots y_{n-2}+y_{n-1}) + 2(y_3+y_6+\dots y_{n-3})\right]$$

Q- Evaluate $\int_0^6 \frac{1}{1+x^2}\,dx$ by using

(i) Simpson's one-third rule

(ii) Simpson's three-eight rule (iii) Trapezoidal rule

x	$f(x)$	
0	1	y_0
1	0.5	y_1
2	0.2	y_2
3	0.1	y_3
4	0.0588	y_4
5	0.03846	y_5
6	0.02702	y_6

(i) $\int_0^6 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$

$= \frac{1}{3}\left[(1+0.02702)+4(0.5+0.1+0.03846)+2(0.2+0.058\right.$

$= 1.36617$

(ii) $\int_0^6 \frac{1}{1+x^2}\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$

$= \frac{3}{8}\left[(1+0.02707)+3(0.5+0.2+0.0588+0.03846)+2(0.1)\right]$

≈ 1.3570

(iii) $\int_0^6 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$

$= \frac{1}{2}\left[(1+0.02707)+2(0.5+0.2+0.1+0.0588+0.03846\right.$

$= 1.41077$

Q- Use trapezoidal rule to evaluate $\int_0^1 x^3\,dx$ consider 5 sub-intervals

$$\frac{1-0}{5} = 0.2$$

x	0	0.2	0.4	0.6	0.8	1.0
$f(x)$	0	0.008	0.064	0.216	0.512	1

$$\int_0^1 x^3\,dx = \frac{0.2}{2}\left[(0+1)+2(0.008+0.064+0.216+0.512)\right]$$

$$= 0.26 \quad \text{Ans.}$$

Q- Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using

(A) Simpson's $\frac{1}{3}$ taking $h=\frac{1}{4}$

(B) Simpson's $\frac{3}{8}$ taking $h=\frac{1}{6}$

(A)

x	0	1/4	2/4	3/4	1
$f(x)$	1	0.9411	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4

(A) $$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right]$$

$$= \frac{1}{12}\left[(1+0.5)+4(0.9411+0.64)+2(0.8)\right]$$

$$= 0.7853$$

(B)

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$f(x)$	1	0.9729	0.9	0.8	0.6923	0.5901	0.5

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{1}{16}\left[(1+0.5)+3(0.9729+0.9+0.6923+0.5901)+2(0\right.$$

$$= 0.7853 \quad \underline{Ans.}$$

Ques–

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.2	$\boxed{0.9320}$	$\boxed{0.0316}$	$\boxed{-0.0097}$		
1.3	0.9636	0.0219	-0.0099	-0.0002	0.0002
1.4	0.9855	0.0120	-0.0099	0	
1.5	0.9975	0.0021			
1.6	0.9996				

$y_0 = 0.9320$ and $a = 1.2$, $h = 1.3 - 1.2 = 0.1$

Newton's forward difference formula.

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \ldots\ldots$$

$$y = 0.9320 + 0.0316u + \frac{u(u-1)}{2}(-0.0097)$$

$$\frac{dy}{du} = 0 + 0.0316 + \left(\frac{2u-1}{2}\right)(-0.0097)$$

At a max, $\frac{dy}{dx} = 0$

$$0 = 0.0316 + \left(\frac{2u-1}{2}\right)(-0.0097)$$

$$0.0316 = \left(\frac{2u-1}{2}\right)(0.0097)$$

$$0.0632 = (2u-1)(0.0097)$$

$$0.0632 = 0.0194u - 0.0097$$

$$u = 3.757$$

$$u = 3.76$$

$x = a + hu$

$x = 1.2 + (0.1)(3.76)$

$x = 1.576$

To find ymax, use backward diff. formula

$x = x_n + hu$

$x = 1.6 + (0.1)u$

$1.576 = 1.6 + 0.1u$

$u = -0.24$

$$y(a+hu) = y_n + u\nabla y_n + \frac{u(u+1)}{2}\nabla^2 y_n + \frac{u(u+1)(u+2)}{6}\nabla^3 y_n$$

$$y(1.576) = 0.9996 + (-0.24)(0.0021) + (-0.0535)(-0.0099)$$

$$= 0.9996 - \cancel{0.0005} + \cancel{0.0005}$$

$$= 0.9996$$

UNIT – 4
Solution of linear equation

★ Gauss elimination method –

Ques – By Gauss elimination method solve this equation –

$$5x - y - 2z = 142 \quad \text{— (1)}$$
$$x - 3y - z = -30 \quad \text{— (2)}$$
$$2x - y - 3z = -50 \quad \text{— (3)}$$

Sol:- Step 1 – Dividing (1) by 5

$$x - \frac{y}{5} - \frac{2}{5}z = \frac{142}{5} \quad \text{— (4)}$$

Now eliminating x from (2) and (3) with the help of (4)

$$\cancel{x} - \frac{1}{5}y - \frac{2}{5}z = \frac{142}{5}$$
$$\underset{-}{\cancel{x}} \underset{+}{-3y} \underset{+}{-z} = \underset{+}{-30}$$
$$3y - \frac{1}{5}y + z - \frac{2}{5}z = \frac{142}{5} + 30$$
$$\frac{14}{5}y + \frac{3}{5}z = \frac{292}{5}$$

$$\cancel{x} - \frac{1}{5}y - \frac{2}{5}z = \frac{142}{5}$$
$$\underset{-}{\cancel{x}} \underset{+}{-\frac{1}{2}y} \underset{+}{-\frac{3}{2}z} = \underset{+}{-25}$$
$$\frac{1}{2}y - \frac{1}{5}y + \frac{3}{2}z - \frac{2}{5}z = \frac{142}{5} + 25$$
$$\frac{3}{10}y + \frac{11}{10}z = \frac{267}{5}$$

Ques - $x + 4y - z = -5$ —(1)

$x + y - 6z = -12$ —(2)

$3x - y - z = 4$ —(3)

Sol:- Eliminating x from the eq (2) and eq(3) with the help of eq(1), then we get -

$$
\begin{array}{r}
\cancel{x} + 4y - z = -5 \\
\cancel{x} + y - 6z = -12 \\
- \quad - \quad + \quad + \\
\hline
3y + 5z = 7
\end{array}
\quad —(4)
$$

$(x + 4y - z = -5) \times 3$

$$
\begin{array}{r}
3\cancel{x} + 12y - 3z = -15 \\
\cancel{3x} - y - z = 4 \\
- \quad + \quad + \quad - \\
\hline
13y - 2z = -19
\end{array}
\quad —(5)
$$

Taking eq (4) and (5) eliminating y, then find the value of z.

$$
\begin{array}{r}
\cancel{39y} + 65z = 91 \\
\cancel{39y} - 6z = -57 \\
- \quad + \quad + \\
\hline
71z = 148
\end{array}
$$

$$\boxed{z = \frac{148}{71}}$$

Put the value of z in eq(5)

$3y + 5z = 7$

$3y + \frac{740}{71}\cancel{z} = 7$

$$\boxed{y = \frac{-81}{71}}$$

Now from eq (1)

$x + 4y - z = -5$

$x + \left(\frac{-324}{71}\right) - \frac{148}{71} = -5$

$$\boxed{x = \frac{117}{71}}$$

Ques-
$$x + 2y + z = 3 \quad -(1)$$
$$2x + 3y + 3z = 10 \quad -(2)$$
$$3x - y + 2z = 13 \quad -(3)$$

Sol:- Eliminating x from eq (2) & (3) with the help of (1)

$$\begin{array}{l} \cancel{2x} + 4y + 2z = 6 \\ \cancel{2x} + 3y + 3z = 10 \\ - \quad\ - \quad - \quad\ - \\ \hline y - z = -4 \quad -(4) \end{array} \qquad \begin{array}{l} \cancel{3x} + 6y + 3z = 9 \\ \cancel{3x} - y + 2z = 13 \\ - \quad\ + \quad - \quad\ - \\ \hline 7y + z = -4 \quad -(5) \end{array}$$

Taking eq (4) and (5) and eliminating z.

$$\begin{array}{l} y - \cancel{z} = -4 \\ 7y + \cancel{z} = -4 \\ \hline 8y = -8 \end{array}$$
$$\boxed{y = -1}$$

From eq (4)
$$y - z = -4$$
$$-z = -4$$
$$\boxed{z = 3}$$

From eq (1)
$$x + 2y + z = 3$$
$$x - 2 + 3 = 3$$
$$\boxed{x = 2}$$

$x = 2,\ y = -1,\ z = 3$

★ Gauss Seidal Integration / Iterative method -

Ques-
$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Sol^n: We have
$$x = \frac{1}{20}(17 + 2z - y) \quad -(1)$$
$$y = \frac{1}{20}(-18 - 3x + z) \quad -(2)$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad — (3)$$

First approximation – Putting $y=0$ and $z=0$ in eq (1)

$$x_1 = \frac{1}{20}[17 + 0 - 0] = \frac{17}{20} = 0.85$$

Put $x = 0.85$, $z=0$ in eq (2)

$$y_1 = \frac{1}{20}[-18 - 3(0.85) + 0] = -1.0275$$

Put $y = -1.0275$ and $x = 0.85$ in eq (3)

$$z_1 = \frac{1}{20}[25 - 2(0.85) + 3(-1.0275)] = 1.0109$$

Second approximation –

Put $y = -1.0275$ and $z = 1.0109$ in eq (1)

$$x_2 = \frac{1}{20}[17 + 2(1.0109) + 1.0275] = 1.0024$$

Put $x = 1.0024$ and ~~y~~ $z = 1.0109$ in eq (2)

$$y_2 = \frac{1}{20}[-18 - 3(1.0024) + 1.0109] = -0.9998$$

Put $y = -0.9998$ and $x = 1.0024$ in eq (3)

$$z_3 = \frac{1}{20}[25 - 2(1.0024) + 3(-0.9998)] = 0.9997$$

Third approximation –

Put $y = -0.9998$ and $z = 0.9997$ in eq (1)

$$x_3 = \frac{1}{20}[17 + 0.9998 + 2(0.9997)] = 0.9999$$

Put $x = 0.9999$ and $z = 0.9997$ in eq (2)

$$y_3 = \frac{1}{20}[-18 - 3(0.9999) + 0.9997] = -1$$

Put $x = 0.9999$ and $y = -1$ in eq (3)

$$z_3 = \frac{1}{20}[25 - 2(0.9999) + 3(-1)] = 1.0000$$

Fourth approximation –

Put $y = -1$ and $z = 1$ in eq (1)

$$x_4 = \frac{1}{20}[17 + 1 + 2(1)] = 1$$

Put $x = 1$ and $z = 1$ in eq (2)

$$y_4 = \frac{1}{20}[-18 - 3(1) + 1] = -1$$

Put $x = 1$ and $y = -1$ and eq (3)

$$z_4 = \frac{1}{20}[25 - 2(1) + 3(-1)] = 1$$

Fifth approximation –

$$x_5 = \frac{1}{20}[17 + 1 + 2(1)] = 1$$

$$y_5 = \frac{1}{20}[-18 - 3(1) + 1] = -1$$

$$z_5 = \frac{1}{20}[25 - 2(1) + 3(-1)] = 1$$

$\boxed{x = 1, y = -1, z = 1}$ Ans.

★ Newton Raphson / Iterative method –

Ques – Find a real root of a equation $x = e^{-x}$.

$$\cancel{f(x) =}\ x = e^{-x}$$

$$x = \frac{1}{e^x}$$

$$xe^x = 1$$

$$f(x) = xe^x - 1$$

$$f'(x) = x.e^x + e^x.1 - 0$$

$$f'(x) = e^x(1 + x)$$

$$f(x) = xe^x - 1$$

$$f(0) = -1 \ (-ve)$$

$$f(1) = 1.7182 \quad (+ve)$$

The root lies between 0 and 1.

$$f(0.5) = -0.1756 \quad (-ve)$$

$$f(0.6) = 0.0932 \quad (+ve)$$

The root lies between 0.5 and 0.6

By Newton Iterative method –

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(xe^x - 1)}{e^x(1+x)}$$

$$= \frac{xe^x(1+x) - xe^x + 1}{e^x(1+x)}$$

$$= \frac{\cancel{xe^x} + x^2e^x - \cancel{xe^x} + 1}{e^x(1+x)}$$

$$x = \frac{x^2e^x + 1}{e^x(1+x)}$$

$$x_{n+1} = \frac{x_n^2 e^{x_n} + 1}{e^{x_n}(1+x_n)}$$

$n=0$, Taking $x_0 = 0.5$

$$x_1 = \frac{x_0^2 e^{x_0} + 1}{e^{x_0}(1+x_0)}$$

$$= \frac{(0.5)^2 e^{(0.5)} + 1}{e^{(0.5)}(1+0.5)}$$

$$= 0.57102$$

For second iteration, we have $n=1$ and $x_1 = 0.57102$

$$x_2 = \frac{x_1^2 e^{x_1} + 1}{e^{x_1}(1+x_1)}$$

$$= \frac{(.57102)^2 e^{(.57102)} + 1}{e^{(.57102)}(1+0.57102)}$$

$$= 0.56715$$

For third iteration, $n=2$ and $x_2 = 0.56715$

$$x_3 = \frac{x_2^2 e^{x_2} + 1}{e^{x_2}(1+x_2)}$$

$$x_3 = \frac{(.56715)^2 e^{(.56715)} + 1}{e^{(.56715)}(1 + 0.56715)}$$

$$= 0.56714$$

So, the correct root is 0.5671 correct to 4 decimal places.

Q- $f(x) = x^3 - 3x - 5$

$f(0) = -5$

$f(1) = -7$

$f(2) = -3 \quad (-ve)$

$f(3) = 13 \quad (+ve)$

The root lies between 2 and 3.

$f(2.5) = 3.125$

$f(2.4) = 1.624$

$f(2.3) = 0.267 \quad (+ve)$

$f(2.2) = -0.952 \quad (-ve)$

The root lies between 2.2 and 2.3.

$f'(x) = 3x^2 - 3$

$$x_{n+1} = x_n - \left[\frac{x_n^3 - 3x_n - 5}{3x_n^2 - 3}\right]$$

$$= \frac{3x_n^3 - \cancel{3x_n} - x_n^3 + \cancel{3x_n} + 5}{3x_n^2 - 3}$$

$$x_{n+1} = \frac{2x_n^3 + 5}{3x_n^2 - 3}$$

For first iteration, $n=0$, $x_0 = 2.2$

$$x_1 = \frac{2(2.2)^3 + 5}{3(2.2)^2 - 3} = 2.2826$$

Second iteration, $n=1$, $x_1 = 2.2826$

$$x_2 = \frac{2(2.2826)^3 + 5}{3(2.2826)^2 - 3} = 2.2790$$

Third iteration, $n=2$ and $x_2 = 2.2790$

$$x_3 = \frac{2(2.2790)^3 + 5}{3(2.2790)^2 - 3} = 2.2790$$

The value has matched upto four decimal places. So the correct root is 2.2790.

UNIT – 5

★ **Euler Method :-** It is the simplest one-step method and has a limited application because of its low accuracy.

In this method, we determine

$$\frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0$$

In general, it can be shown that

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n)}$$

Ques1. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y=1$ for $x=0$. Find y approximately for $x = 0.1$ by Euler method.

Sol:- $x_0 = 0$, $y_0 = 1$, $h = 0.1$

Then we find the approximate value is –

$n=0$,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_{0+1} = y_0 + hf(x_0, y_0)$$

$$y_1 = y_0 + h\left[\frac{y_0 - x_0}{y_0 + x_0}\right]$$

$$y_1 = 1 + 0.1\left[\frac{1-0}{1+0}\right]$$

$$y_1 = 1.1$$

To find the better accuracy, we obtain by breaking upto the intervals 0 – 0.1 into five steps.

We can take the 5 steps 0, 0.02, 0.04, 0.06, 0.08, 0.10.

	x	y
x_0	0	$y_0 = 1$
x_1	0.02	$y_1 = 1.02$
x_2	0.04	$y_2 = 1.0392$
x_3	0.06	$y_3 = 1.0577$
x_4	0.08	$y_4 = 1.0755$
x_5	0.10	$y_5 = 1.0927$

$h = 0.02$

at $x = 0.02$

$$y_1 = y_0 + h(x_0, y_0)$$
$$y_1 = 1 + 0.02\left[\frac{1-0}{1+0}\right]$$

$$y_1 = 1.02$$

$x = 0.04$

$$y_2 = y_1 + h(x_1, y_1)$$
$$= 1.02 + 0.02\left[\frac{1.02 - 0.02}{1.02 + 0.02}\right]$$

$$y_2 = 1.0392$$

$x = 0.06$

$$y_3 = y_2 + h(\cancel{y}x_2, y_2)$$
$$= 1.0392 + 0.02\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right]$$

$$y_3 = 1.0577$$

$x = 0.08$

$$y_4 = y_3 + h(x_3, y_3)$$
$$= 1.0577 + 0.02\left[\frac{1.0577 - 0.06}{1.0577 + 0.06}\right]$$

$$= 1.0577 + 0.02\left[\frac{0.9977}{1.1177}\right]$$

$$y_4 = 1.0755$$

$x = 0.1$

$$y_5 = 1.0755 + 0.02\left[\frac{1.0755 - 0.08}{1.0755 + 0.08}\right]$$

$$y_5 = 1.0755 + 0.02\begin{bmatrix} 0.9955 \\ 1.1555 \end{bmatrix}$$

$$y_5 = 1.0927$$

★ Runge – Kutta Method :–

1- First order Runge – Kutta method –

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$

Euler method gives

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0'$$

Expanding by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{\underline{|2}} y_0'' + \ldots\ldots$$

2- Second order Runge – Kutta method –

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$\Delta y = \frac{1}{2}(k_1 + k_2) \quad \text{Taken in the given order.}$$

Then, $x_1 = x_0 + h$

$$y_1 = y_0 + \Delta y = y_0 + \frac{1}{2}(k_1 + k_2)$$

Similarly,

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf(x_1 + h, y_1 + k_1)$$

$$\Delta y = \frac{1}{2}(k_1 + k_2)$$

Similarly go to next step.

3- Third order Runge - Kutta method -

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$

$$y_1 = y_0 + \delta y$$

$$\delta y = \frac{h}{6}(k_1 + 4k_2 + k_3)$$

Here,

$$k_1 = f(x_0, y_0)$$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = f(x_0 + h, y_0 + k')$$

$$k' = hf(x_0 + h, y_0 + k_1)$$

4- Fourth order Runge - Kutta Method -

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$ taken in the given order.

Then,

$$x_1 = x_0 + h$$

$$y_1 = y_0 + \Delta y$$

Similarly,

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Ques - Solve the equation $\frac{dy}{dx} = x+y$ with initial condition $y(0)=1$ by Runge Kutta method from $x=0$ to 0.4 with $h = .1$

Sol:-
$$f(x,y) = x+y$$
$$x_0 = 0 \quad , \quad y_0 = 1$$

$$k_1 = hf(x_0, y_0)$$
$$= h[x_0 + y_0] = .1[0+1] = 0.1$$
$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$
$$= h[x+y]$$
$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2}\right]$$
$$k_2 = .1\left[0 + \frac{.1}{2} + 1 + \frac{.1}{2}\right] = 0.11$$
$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right]$$
$$= h[x+y]$$
$$= h\left[x_0 + \frac{h}{2} + y_0 + \frac{k_2}{2}\right]$$
$$= .1\left[0 + \frac{.1}{2} + 1 + \frac{.11}{2}\right]$$
$$k_3 = 0.1105$$
$$k_4 = hf[x_0 + h, y_0 + k_3]$$
$$= h[x+y]$$
$$= h[x_0 + h + y_0 + k_3]$$
$$= 0.1[0 + .1 + 1 + 0.1105]$$
$$= 0.12105$$
$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= \frac{1}{6}[.1 + 0.22 + 0.221 + 0.12105$$

$\Delta y = 0.11034$

Thus, $x_1 = x_0 + h = 0 + 0.1$

$x_1 = 0.1$

$y_1 = y_0 + \Delta y$

$y_1 = 1 + 0.11034 = 1.11034$

Now for the second interval we have,

$k_1 = hf(x_1, y_1)$

$= h[x_1 + y_1] = 0.1[0.1 + 1.11034]$

$= 0.121034$

$k_2 = h\left[x_1 + \frac{h}{2} + y_1 + \frac{k_1}{2}\right]$

$= 0.1\left[0.1 + \frac{0.1}{2} + 1.11034 + \frac{0.121034}{2}\right]$

$= 0.13208$

$k_3 = h\left[x_1 + \frac{h}{2} + y_1 + \frac{k_2}{2}\right]$

$= 0.1\left[0.1 + \frac{0.1}{2} + 1.11034 + \frac{0.13208}{2}\right]$

$= 0.13263$

$k_4 = h[x_1 + h + y_1 + k_3]$

$= 0.1[0.1 + 0.1 + 1.11034 + 0.13263]$

$= 0.14429$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}[0.121034 + 0.26416 + 0.26526 + 0.14429]$

≈ 0.132457

Then, $x_2 = x_1 + h$

$= 0.1 + 0.1 = 0.2$

$y_2 = y_1 + \Delta y$

$= 1.11034 + 0.132457$

$= 1.24279$

Now, for the third interval,

$$k_1 = hf(x_2, y_2)$$
$$= h(x_2 + y_2)$$
$$= 0.1\,(0.2 + 1.24279) \qquad = 0.14427$$

$$k_2 = h\left[x_2 + \frac{h}{2} + y_2 + \frac{k_1}{2}\right]$$

$$= 0.1\left[0.2 + \frac{0.1}{2} + 1.24279 + \frac{0.\cancel{14449}\ 14427}{2}\right]$$

$$= \cancel{0.15710}\ 0.15649$$

$$k_3 = h\left[x_2 + \frac{h}{2} + y_2 + \frac{k_2}{2}\right]$$

$$= 0.1\left[0.2 + \frac{0.1}{2} + 1.24279 + \frac{0.15649}{2}\right]$$

$$= 0.15710$$

$$k_4 = h\left[x_2 + h + y_2 + k_3\right]$$
$$= 0.1\left[0.2 + 0.1 + 1.24279 + 0.15710\right]$$
$$= 0.16998$$

$$\Delta y = \frac{1}{6}\left[0.14427 + 2(0.15649) + 2(0.15710) + 0.16998\right]$$

$$= 0.15690$$

$$x_3 = x_2 + h \quad = 0.2 + 0.1$$
$$= 0.3$$

$$y_3 = y_2 + \Delta y \quad = 1.3997$$

Now, for the fourth interval,

$$k_1 = h\left[x_3 + y_3\right] \quad = 0.1\left[0.3 + 1.3997\right]$$
$$= 0.16997$$

$$k_2 = h\left[x_3 + \frac{h}{2} + y_3 + \frac{k_1}{2}\right]$$

$$= 0.1\left[0.3 + \frac{0.1}{2} + 1.3997 + \frac{0.16997}{2}\right]$$

$$= 0.18346$$

$$k_3 = h\left[x_3 + \frac{h}{2} + y_3 + \frac{k_2}{2}\right]$$

$$= 0.1\left[0.3 + \frac{0.1}{2} + 1.3997 + \frac{0.18346}{2}\right] = 0.18414$$

$$k_4 = h[x_3 + h + y_3 + k_3]$$
$$= 0.1[0.3 + 0.1 + 1.3997 + 0.18414] = 0.19838$$

$$\Delta y = \frac{1}{6}[0.16997 + 2(0.18346) + 2(0.18414) + 0.19838]$$

$$= 0.18392$$

$$x_4 = x_3 + h$$
$$= 0.3 + 0.1 = 0.4$$

$$y_4 = y_3 + \Delta y$$
$$= 1.3997 + 0.18392$$

$$y_4 = 1.583825$$

$$\boxed{y_4 = 1.5836}$$

- Given $\frac{dy}{dx} = y - x$, $y(0) = 2$ find $y(0.1)$ and $y(0.2)$ using II order method.

$$\frac{dy}{dx} = y - x, \quad x_0 = 0, \quad y_0 = 2, \quad h = 0.1$$

$$k_1 = h f(x_0, y_0)$$
$$= h[y_0 - x_0]$$
$$= 0.1[2 - 0] = 0.2$$

$$k_2 = h f(x_0 + h, y_0 + k_1)$$
$$= h[y_0 + k_1 - x_0 - h]$$
$$= 0.1[2 + 0.2 - 0 - 0.1]$$
$$= 0.21$$

$$\Delta y = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.2 + 0.21) = 0.205$$

$$x_1 = x_0 + h$$
$$= 0 + 0.1 = 0.1$$
$$y_1 = y_0 + \Delta y$$
$$= 2 + 0.205 = 2.205$$

Similarly
$$k_1 = h[y_1 - x_1]$$
$$= 0.1[2.205 - 0.1]$$
$$= 0.2105$$
$$k_2 = h[y_1 + k_1 - x_1 - h]$$
$$= 0.1[2.205 + 0.2105 - 0.1 - 0.1]$$
$$= 0.22155$$
$$\Delta y = \frac{1}{2}(0.2105 + 0.22155) = 0.21602$$

$$x_2 = x_1 + h$$
$$= 0.1 + 0.1 = 0.2$$
$$y_2 = y_1 + \Delta y$$
$$= 2.205 + 0.21602 = \boxed{2.4210}$$

★ Picard Method :-

Ques - $\frac{dy}{dx} = x - y$ with initial condition $y = 1$ when $x = 0$. Use the picard method to obtain the value y for the point 0.2.

Sol:- $f(x,y) = x - y$, $x_0 = 0$, $y_0 = 1$

We have first approximation -
$$\boxed{y^{(1)} = y_0 + \int_0^x f(x, y_0)\, dx}$$
$$y^{(1)} = 1 + \int_0^x (x - y_0)\, dx$$
$$y^{(1)} = 1 + \int_0^x (x - 1)\, dx$$
$$y^{(1)} = 1 + \left[\frac{x^2}{2} - x\right]_0^x = 1 + \frac{x^2}{2} - x$$

We have second approximation –

$$y^{(2)} = y_0 + \int_0^x f(x, y_1)\, dx$$

$$= 1 + \int_0^x \left\{x - \left(1 + \frac{x^2}{2} - x\right)\right\} dx$$

$$= 1 + \left[\frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{2}\right]_0^x$$

$$y^{(2)} = 1 + x^2 - x - \frac{x^3}{6}$$

Third approximation –

$$y^{(3)} = y_0 + \int_0^x f(x, y_2)\, dx$$

$$= 1 + \int_0^x (x - y_2)\, dx$$

$$= 1 + \int_0^x \left(x - 1 - x^2 + x + \frac{x^3}{6}\right) dx$$

$$= 1 + \frac{x^2}{2} - x - \frac{x^3}{3} + \frac{x^2}{2} + \frac{x^4}{24}$$

$$y^{(3)} = 1 + x^2 - x - \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth approximation –

$$y^{(4)} = y_0 + \int_0^x (x - y_3)\, dx$$

$$= 1 + \int_0^x \left(x - 1 - x^2 + x + \frac{x^3}{3} - \frac{x^4}{24}\right) dx$$

$$= 1 + \frac{x^2}{2} - x - \frac{x^3}{3} + \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^5}{120}$$

$$y^{(4)} = 1 + x^2 - x - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

Fifth approximation –

$$y^{(5)} = y_0 + \int_0^x (x - y_4)\, dx$$

$$y^{(5)} = 1 + \int_0^x \left(x - 1 - x^2 + x + \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{120} \right) dx$$

$$= 1 + \frac{x^2}{2} - x - \frac{x^3}{3} + \frac{x^2}{2} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}$$

$$y^{(5)} = 1 + x^2 - x - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}$$

When $x = 0.2$, we get the value

$$y^{(1)} = 1 + \frac{(.2)^2}{2} - (.2) = .82$$

$$y^{(2)} = 1 + (.2)^2 - (.2) - \frac{(.2)^3}{6} = 0.8386$$

$$y^{(3)} = 1 + (.2)^2 - (.2) - \frac{(.2)^3}{3} + \frac{(.2)^4}{24} = 0.8374$$

$$y^{(4)} = 1 + (.2)^2 - (.2) - \frac{(.2)^3}{3} + \frac{(.2)^4}{24} - \frac{(.2)^5}{120} = 0.8374$$

$$y^{(5)} = 0.8374$$

Thus, $\boxed{y = 0.837}$ when $x = .2$.

Ques2- $\frac{dy}{dx} = 3x + y^2$, $y = 1$ at $x = 0$. Obtain y for $x = 0.1$.

Sol:- $f(x,y) = 3x + y^2$, $x_0 = 0$, $y_0 = 1$

First approximation-

$$y^{(1)} = y_0 + \int_0^x f(x, y_0)\, dx$$

$$= y_0 + \int_0^x (3x + y_0^2)\, dx$$

$$= 1 + \int_0^x (3x + 1)\, dx$$

$$= 1 + \frac{3x^2}{2} + x$$

Second approximation-

$$y^{(2)} = y_0 + \int_0^x (3x + y_1^2)\, dx$$

$$= 1 + \int_0^x \left\{ 3x + \left(1 + x + \frac{3x^2}{2}\right)^2 \right\} dx$$

$$= 1 + \int_0^x \left(3x + 1 + x^2 + 2x + \frac{9x^4}{4} + 3x^2 + 3x^3 \right) dx$$

$$= 1 + \frac{3x^2}{2} + x + \frac{x^3}{3} + x^2 + \frac{9x^5}{20} + \frac{3x^3}{3} + \frac{3x^4}{4}$$

$$y^{(2)} = 1 + x + \frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20}$$

Third approximation –

$$y^{(3)} = y_0 + \int_0^x \{3x + (y_2)^2\} dx$$

$$y^{(3)} = 1 + \int_0^x \left\{ 3x + \left(1 + x + \frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20}\right)^2 \right\} dx$$

$$y^{(3)} = 1 + \int_0^x \left\{ 3x + 1 + x^2 + 2x + \left(\frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20}\right)^2 + 2(1+x)\left(\frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20}\right) \right\}$$

$$y^{(3)} = 1 + \int_0^x \left\{ 1 + 5x + x^2 + \left(\frac{5x^2}{2} + \frac{4x^3}{3}\right)^2 + \left(\frac{3x^4}{4} + \frac{9x^5}{20}\right)^2 + 2\left(\frac{5x^2}{2} + \frac{4x^3}{3}\right)\left(\frac{3x^4}{4} + \frac{9x^5}{20}\right) + 2(1+x)\left(\frac{5x^2}{2} + \frac{4x^3}{3} + \frac{3x^4}{4} + \frac{9x^5}{20}\right) \right\} dx$$

$$y^{(3)} = 1 + \int_0^x \Big(1 + 5x + x^2 + \frac{25x^4}{4} + \frac{16x^6}{9} + \frac{20x^5}{3} + \frac{9x^8}{16} + \frac{81x^{10}}{400} + \frac{27x^9}{40} + \frac{15}{4}x^6 + \frac{9}{4}x^7 + 2x^7 + \frac{18}{15}x^8 + 5x^2 + \frac{8x^3}{3} + \frac{3x^4}{2} + \frac{9x^5}{10} + 5x^3 + \frac{8x^4}{3} + \frac{3x^5}{2} + \frac{9x^6}{10} \Big) dx$$

$$y^{(3)} = 1 + \int_0^x \Big(1 + 5x + 6x^2 + \frac{23}{3}x^3 + \frac{125}{12}x^4 + \frac{272}{30}x^5 + \frac{1157}{180}x^6 + \frac{17}{4}x^7 + \frac{423}{240}x^8 + \frac{27}{40}x^9 + \frac{81}{400}x^{10} \Big) dx$$

$$y^{(3)} = 1 + x + \frac{5x^2}{2} + 2x^3 + \frac{23}{12}x^4 + \frac{25}{12}x^5 + \frac{\cancel{272}^{\,136}}{\cancel{30}_{15} \times 6\,3}x^6 + \frac{17}{32}x^8 + \frac{47}{240}x^9 + \frac{27}{400}x^{10} + \frac{81}{4400}x^{11}$$

$$y^{(3)} = 1 + x + \frac{5x^2}{2} + 2x^3 + \frac{23}{12}x^4 + \frac{25}{12}x^5 + \frac{68}{45}x^6 + \frac{17}{32}x^8 + \frac{47}{240}x^9 + \frac{27}{400}x^{10} + \frac{81}{4400}x^{11}$$

When $x = 0.1$, we get the value

$y^{(1)} = 1.115$

$y^{(2)} = 1.1264$

$y^{(3)} = 1.1272$

Thus, $\boxed{y = 1.127}$ Ans.