Krishna's

# B.C.A. Mathematics-III

As per U.P. UNIFIED Syllabus (*w.e.f.* 2012–2013)

(*for B.C.A. IVth Sem. Students of all Colleges Affiliated to Universities in Uttar Pradesh*)

*By*


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# PREFACE

**DEDICATED TO LORD KRISHNA** — ***Publishers & Authors***

It gives me pleasure to place the book **'B.C.A. Mathematics-III' (All Universities in U.P.)** which has been completely revised and made upto date in the light of suggestions received from the students and the learned teachers of various universities.

The following are the main features of third edition:

- More than 150 new problems and examples have been added.
- Includes recent elegant and elementary proof.
- All chapters are arranged according to the latest **U.P. Unified Syllabus** implemented from 2012 – 2013, keeping in view the requirement of the students of B.C.A. IVth semester of all Universities in Uttar Pradesh (U.P.).
- Errors and omissions have been corrected.
- Up-to-date yearwise references from various universities examination papers have been given throughout the book.
- Each step is made clear by giving reasons with clarification.

We hope that the book in its present form will prove beneficial to the students. We shall also forward to receive your suggestions and comments about the book so that it could be further improved.

— ***J.P. Chauhan***

# SYLLABUS

## B.C.A. Mathematics-III

U.P. UNIFIED (*w.e.f.* 2012-2013)

**B.C.A. – S 210** **B.C.A. IV$^{th}$ Semester**

### UNIT-1

**Complex Variables:** Complex Number System, Algebra of Complex Numbers, Polar Form, Powers and Roots, Functions of Complex Variables, Elementary Functions, Inverse Trigonometric Function.

### UNIT-2

**Sequence, Series and Convergence:** Sequence, Finite and Infinite Sequences, Monotonic Sequence, Bounded Sequence, Limit of a Sequence, Convergence of a Sequence, Series, Partial Sums, Convergent Series, Theorems on Convergence of Series (statement, alternating series, conditional convergent), Leibnitz Test, Limit Comparison Test, Ratio Test, Cauchy's Root Test, Convergence of Binomial and Logarithmic Series, Raabe's Test, Logarithmic Test, Cauchy's Integral Test (without proof).

### UNIT-3

**Vector Calculus:** Differentiation of Vectors, Scalar and Vector Fields, Gradient, Directional Derivatives, Divergence and Curl and their Physical Meaning.

### UNIT-4

**Fourier Series:** Periodic Functions, Fourier Series, Fourier Series of Even and Odd Functions, Half Range Series.

### UNIT-5

**Ordinary Differential Equations of First Order:** Variable-Separable Method, Homogeneous Differential Equations, Exact Differential Equations, Linear Differential Equations, Bernoulli's Differential Equations, Differential Equations of First Order and First Degree by Integrating Factor.

### UNIT-6

**Ordinary Differential Equations of Second Order:** Homogenous Differential Equations with Constant Coefficients, Cases of Complex Roots and Repeated Roots, Differential Operator, Solutions by Methods of Direct Formulae for Particular Integrals, Solution by Undetermined Coefficients, Cauchy Differential Equations, (only Real and Distinct Roots) Operator Method for Finding Particular Integrals, (Direct Formulae).

# BRIEF CONTENTS

# DETAILED CONTENTS

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Chapter 1

# Complex Numbers and Their Geometrical Representation

## 1.1 Complex Numbers

A number of the form $x + iy$ where $i = \sqrt{(-1)}$ and $x, y$ are both real number, is called a complex number. A complex number is also defined as an ordered pair $(x, y)$ of real numbers. It is represented by $z = (x + iy)$ or $(x, y)$ then $x$ is called **real part** and $y$ is called the **imaginary part** of the complex number $z$ *i.e.* $x = R(z)$ and $y = I(z)$.

Therefore in the complex number $z = a + ib$ we have

$$R(z) = \text{real part of } z = a$$

$$I(z) = \text{imaginary part of } z = b$$

**Note:**

1. A complex number is said to be purely real if its imaginary part is zero.
2. A complex number is said to be purely imaginary if its real part is zero.
3. The complex number $2 + 0\,i$ may be written as 2.
4. The set of complex number is denote by $C$.

## 1.2 Algebra of Complex Numbers

### 1.2.1 Equality of Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$, or $(x_1, y_1)$ and $z_2 = x_2 + iy_2$, or $(x_2, y_2)$ are said to be equal if $x_1 = x_2$ and $y_1 = y_2$. Hence two complex numbers are equal if and only if the real part of one is equal to the real part of other and the imaginary part of one is equal to the imaginary part of the other.

### 1.2.2 Addition of Complex Numbers

If $z_1 = (x_1 + iy_1)$ or $(x_1, y_1)$ and $z_2 = (x_2 + iy_2)$ or $(x_2, y_2)$ are the two complex numbers, then the sum of $z_1$ and $z_2$ is written as $z_1 + z_2$ and defined as

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

or $$(z_1 + z_2) = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

### 1.2.3 Properties of the Addition of Complex Numbers

**Theorem 1: (Commutativity of addition in C).**

**To show that**

$$z_1 + z_2 = z_2 + z_1$$

**where $z_1$ and $z_2$ are any complex numbers.** **[B.C.A. (Avadh) 2009, 07]**

**Proof:** Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

where $x_1, x_2, y_1, y_2$ are real numbers

Then
$$\begin{aligned} z_1 + z_2 &= (x_1, y_1) + (x_2, y_2) \\ &= (x_1 + x_2, y_1 + y_2) && \text{[by addition in C]} \\ &= (x_2 + x_1, y_2 + y_1) && \text{[addition of real number is commutative]} \\ &= (x_2, y_2) + (x_1, y_1) \\ &= z_2 + z_1. \end{aligned}$$

**Theorem 2: (Associativity of addition in C).**

**To show that $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all complex numbers $z_1$, $z_2$ and $z_3$.**

**[B.C.A. (Bhopal) 2003]**

**Proof:** Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $z_3 = (x_3, y_3)$

where $x_1, x_2, x_3, y_1, y_2, y_3$ are real numbers.

Then $$(z_1 + z_2) + z_3 = [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$$

$$= [(x_1 + x_2, y_1 + y_2)] + (x_3, y_3)$$

$$= [(x_1 + x_2) + x_3 + (y_1 + y_2) + y_3]$$

$$= [x_1 + (x_2 + x_3) + y_1 + (y_2 + y_3)]$$

$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) = (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$$

$$= z_1 + (z_2 + z_3).$$

**NOTE:**

1. The complex number (0, 0) or $0 + i0$ is the additive identity or zero element of complex number.
2. The complex number $(-a, -b)$ is the additive inverse of the complex number $(a, b)$ since
$$(a, b) + (-a, -b) = (a - a, b - b) = (0, 0).$$
3. The complex number $(-a, -b)$ is called the negative of complex number $(a, b)$ and we denote
$$(-a, -b) = -(a, b).$$
4. If $z_1, z_2$ be two complex number then subtraction of $z_1$ and $z_2$ is defined by
$$z - z_2 \text{ if } z_1 = (x_1, y_1) \text{ and } z_2 = (x_2, y_2)$$
$$\therefore \quad z_1 - z_2 = (x_1, y_1) - (x_2, y_2)$$
$$= (x_1 - x_2, y_1 - y_2).$$
5. If $u, v$ and $w$ be any complex numbers then
$$u + w = v + w \Rightarrow u = v$$
cancellation law hold in addition in $C$.

## 1.2.4 Multiplication of Complex Numbers

If $z_1 = (x_1 + iy_1)$ or $(x_1, y_1)$ and $z_2 = (x_2 + iy_2)$ or $(x_2, y_2)$ are any two complex numbers then product is

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

## 1.2.5 Properties of Multiplication of Complex Numbers

Theorem 3: **(Commutativity of multiplication in C).**

**To show that $z_1 z_2 = z_2 z_1$ for all complex number $z_1$ and $z_2$.**

**[B.C.A. (Kashi) 2010, 07, 05]**

**Proof :** Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ where $x_1, x_2, y_1, y_2$ are real numbers.

We have
$$z_1 z_2 = (x_1, y_1)(x_2, y_2)$$
$$= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$
$$= (x_2 x_1 - y_2 y_1), y_2 x_1 + y_1 x_2$$
$$= (x_2, y_2)(x_1, y_1) = z_2 z_1$$

$\because$ $z_1 z_2 = z_2 z_1$ for all complex number $z_1$ and $z_2$.

**Theorem 4: (Associativity of multiplication in C).**

**To show that $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ for all complex numbers $z_1$, $z_2$ and $z_3$.**

**[B.C.A. (Lucknow) 2010]**

**Proof:** Let $z_1 = (a, b), z_2 = (c, d), z_3 = (e, f)$ where $a, b, c, d, e, f$ are real numbers.

We have $(z_1 z_2) z_3 = \{(a, b)(c, d)\}(e, f)$
$$= (ac - bd, ad + bc)(e, f) \quad \text{by definition of multiplication in C}$$
$$= [(ac - bd)e - \{(ad + bc)f\}, (ac - bd)f + \{ad + bc\}e]$$
$$= (ace - bde - adf - bcf, acf - bdf + ade + ace) \quad ...(1)$$

Again $z_1 (z_2 z_3) = (a, b)\{(c, d)(e, f)\}$
$$= (a, b)(ae - bf, cf + de)$$
$$= a\{ce - df\} - b\{cf + de\}, a\{cf + de\} + b\{ac - df\}$$
$$= (ace - adf - bcf - bde, acf + ade + bce - bdf)$$
$$= (ace - bde - adf - bcf, acf - bdf + ade + ace) \quad ...(2)$$

From (1) and (2) we find
$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

**NOTE:**

1. The complex number $(1, 0)$ or $1 + 0i$ is multiplicative identity.
2. The complex number $(x, y)$ is called the multiplicative inverse of the complex number $(a, b)$ if
$$(x, y)(a, b) = (1, 0).$$
3. If $z$ is a non-zero complex number, the multiplicative inverse of $z$ is $\frac{1}{z}$ or $z^{-1}$.
4. If $z_1, z_2$ and $z_3$ be three complex number and $z_3 \neq 0$ then $z_1 z_3 = z_2 z_3 \Rightarrow z_1 = z_2$
Hence cancellation law hold for multiplication.

## 1.3 Division in C

A complex number $(a, b)$ is said to be divisible by a complex number $(c, d)$ if there exists a complex number $(x, y)$ such that

$$(x, y)(c, d) = (a, b)$$

$$\Rightarrow \quad (xc - yd, xd + yc) = (a, b)$$

$$\Rightarrow \quad xc - yd = a \text{ and } xd + yc = b$$

Solve these equations we get

$$x = \frac{ac + bd}{c^2 + d^2}, \; y = \frac{bc - ad}{c^2 + d^2}$$

If $z_1, z_2$ be two complex number then division is $\frac{z_1}{z_2} = z_1(z_2)^{-1}$.

## 1.4 Symbol i and Its Powers

We denote the complex number $(0, 1)$ by $i$. Then

$$i^2 = (0,1)(0,1) = (0.0 - 1.1, 0.1 + 1.0) = (-1, 0)$$

$$\therefore \quad i^2 = (-1, 0)$$

$$\Rightarrow \quad i^3 = -i, i^4 = 1, i^6 = 1, \ldots\ldots$$

## 1.5 Conjugate of a Complex Number

The conjugate of complex number $z = x + iy$ is obtain by putting $i = -i$ and $-i = i$ it is denoted by $\bar{z} = x - iy$

Thus if $\quad z = -3 + 4i \Rightarrow \bar{z} = -3 - 4i$

The following results are for conjugate:

1. Two complex numbers are equal if and only if their conjugates are equal *i.e.*,
$$z_1 = z_2 \Leftrightarrow \overline{z_1} = \overline{z_2}$$
2. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

   and $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ provided $z_2 \neq 0$
3. $(\bar{\bar{z}}) = z$
4. Let $z = x + iy, \bar{z} = x - iy \Rightarrow z + \bar{z} = 2x$

   Thus, sum of two conjugate complex numbers is a real number, equal to twice the real part of each.

5. A complex number $z = x + iy$ is purely imaginary then

$$z + \bar{z} = 0$$

and $$z - \bar{z} = 2\,iy = 2i \text{ (imaginary part)}$$

6. If $z = x + iy \Rightarrow \bar{z} = x - iy$

$$\Rightarrow z\bar{z} = x^2 + y^2 = \text{real number}$$

Thus the product of two conjugate complex number is a purely real number which is never negative.

## 1.6 Separation of Real and Imaginary Part of Complex Number

Let the complex number $= \dfrac{a + ib}{c + id}$

$$= \frac{(a + ib)\,(c - id)}{(c + id)\,(c - id)} \quad \text{[Multiplying } N_r \text{ and } D_r \text{ by conjugate of } D_r]$$

$$= \frac{(ac + bd) + i\,(bc - ad)}{c^2 + d^2}$$

$$= \frac{(ac + bd)}{(c^2 + d^2)} + i\,\frac{(bc - ad)}{(c^2 + d^2)}$$

$$= A + iB \text{ where } A = \frac{ca + bd}{c^2 + d^2}, B = \frac{bc - ad}{c^2 + d^2}$$

$A$ is real part and $B$ is imaginary part of $\left(\dfrac{a + ib}{c + id}\right)$.

**NOTE:**

For separation of real and imaginary part of complex number multiplying $N_r$ and $D_r$ by conjugate of $D_r$.

## 1.7 Modulus of a Complex Number

If $z = (x, y)$ or $z = (x + iy)$ be any complex number, then the non-negative real number $\sqrt{x^2 + y^2}$ is called the modulus or absolute value of complex number $z$ and it is denoted by $|z|$ or mod $(z)$. Therefore

$$|2 + 3i| = \sqrt{(2)^2 + (3)^2} = \sqrt{4 + 9} = \sqrt{13}$$

$$|2-3i| = \sqrt{(2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$$

$$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

Hence, the modulus of a complex number is equal to the positive square root of the sum of the squares of the real and imaginary parts of that complex number.

### 1.7.1 Some Result About the Modulus of Complex Number

1. If $z$ be any complex number, then $|z| = |\bar{z}|$

$\therefore$ $$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$$

$$\bar{z} = x - iy \Rightarrow |\bar{z}| = \sqrt{x^2 + y^2}$$

$\Rightarrow$ $$|z| = |\bar{z}|$$

2. If $z$ is any complex number, then $z\bar{z} = |z|^2$

if $$z = x + iy$$

$$\bar{z} = x - iy \quad \text{multiplying}$$

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$$

3. $$|z_1 z_2| = |z_1||z_2|$$

*i.e.*, $$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2 |z_2|^2$$

or $$|z_1 z_2| = |z_1|\,|z_2|$$

Similarly

$$|z_1, z_2 \ldots\ldots z_n| = |z_1||z_2|\ldots\ldots|z_n|$$

4. If $z_1, z_2$ be any two complex number and $z_2 \neq 0$ then

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

## 1.8 Modulus-Argument Form or Polar Standard Form or Trigonometric Form of a Complex Number

Let $$z = x + iy,$$

put $$x = r\cos\theta \qquad \ldots(1)$$

$$y = r\sin\theta \qquad \ldots(2)$$

Square and add we get $$x^2 + y^2 = r^2$$

or $$r = +\sqrt{x^2 + y^2} \qquad ...(3)$$

or $$r = |z|$$

This $r$ is known and is equal to the modulus of the complex number $z$.

Put $r$ in (1) and (2) we get

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

Divide (2) by (1) we get

$$\tan\theta = \frac{y}{x}$$

or $$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

This $\theta$ is called **argument or amplitude** of complex number $z$. *i.e.*, $\theta = \text{Arg } z$ or amp $(z)$

Then modulus amplitude form of complex number

$$z = r\cos\theta + ir\sin\theta$$

$\Rightarrow$ $$z = r(\cos\theta + i\sin\theta)$$

$\Rightarrow$ $$z = re^{i\theta}.$$

## 1.9 Geometrical Representation of Complex Numbers Argand Diagram

A complex number $z = x + iy$ can be represented by a point $P$ in the cartesian plane whose co-ordinates are $(x, y)$ with respect to rectangular axes $OX$ and $OY$ which is called real and imaginary axes respectively. This representation of complex numbers as points in the plane is due to Argand and is called **Argand diagram** or **Argand plane** or **Complex plane.**

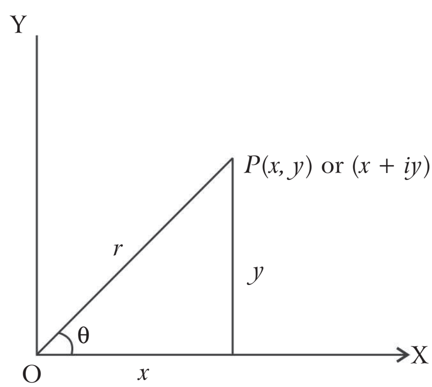


Fig. 1.1

The polar co-ordinates of the point $P$ are $(r, \theta)$ where $OP = r = \sqrt{x^2 + y^2}$ is modulus and $\theta = \angle POX = \tan^{-1}\left(\frac{y}{x}\right)$ is the argument of the complex number $z$. The complex number $z$ is known as the affix of the point $(x, y)$ which represents it.

**Theorem 5: The modulus of the product of two complex numbers is the product of their moduli.** **[B.C.A. (Meerut) 2006]**

**Proof:** We have $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2})$

$$= (z_1 z_2)(\overline{z_1}\,\overline{z_2})$$

$$= (z_1 \overline{z_1})(z_2 \overline{z_2})$$

$$= |z_1|^2 |z_2|^2$$

So that $\quad |z_1 z_2|^2 = |z_1|^2 |z_2|^2$

or $\quad |z_1 z_2| = |z_1||z_2|$

In general, $|z_1 z_2 \ldots\ldots z_n| = |z_1||z_2|\ldots\ldots|z_n|$.

**Theorem 6: The modulus of the sum of two complex numbers is always less than or equal to the sum of their moduli**

**or** $\quad |z_1 + z_2| \le |z_1| + |z_2|$. **[B.C.A. (Kanpur) 2011, 07; B.C.A. (Meerut) 2008]**

**Proof:** We have to show that

$$|z_1 + z_2| \le |z_1| + |z_2|$$

Now $\quad |z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$

$$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= z_1 \overline{z_1} + z_1 \overline{z_2} + z_2 \overline{z_1} + z_2 \overline{z_2}$$

$$= (z_1 \overline{z_1} + z_2 \overline{z_2}) + (z_1 \overline{z_2} + \overline{z_1} z_2)$$

$$= |z_1|^2 + |z_2|^2 + 2R(z_1 \overline{z_2})$$

$$\le |z_1|^2 + |z_2|^2 + 2|z_1 \overline{z_2}| \qquad [\because z_1 \overline{z_1} = |z_1|]$$

Also $\quad z_1 \overline{z_2} + \overline{z_1} z_2 = 2R(z_1 \overline{z_1}) \le 2|z_1 \overline{z_2}|$

$\therefore \quad |z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1 \overline{z_2}|$

$$\le |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \qquad [|\overline{z_2}| = |z_2|]$$

$$|z_1 + z_2|^2 \le (|z_1| + |z_2|)^2$$

$\Rightarrow \quad |z_1 + z_2| \le |z_1| + |z_2|$.

**Theorem 7:** **The modulus of the difference of two complex numbers is greater than or equal to the difference of their moduli.**

**or**

**To show** $|z_1 - z_2| \geq |z_1| - |z_2|$. **[ B.C.A. (Rohilkhand) 2010; B.C.A. (Lucknow) 2009]**

**Proof:** We have to show that

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

Now
$$|z_1 - z_2|^2 = (z_1 - z_2)\,(\overline{z_1 - z_2}) = (z_1 - z_2)\,(\overline{z_1} - \overline{z_2})$$
$$= z_1\overline{z_1} + z_2\overline{z_2} - (z_1\overline{z_2} + \overline{z_1}z_2)$$
$$= |z_1|^2 + |z_2|^2 - 2Re(z_1\overline{z_2})$$
$$\geq |z_1|^2 - 2|z_1\overline{z_2}| + |z_2|^2 \qquad [\,|\overline{z_2}| = |z_2|\,]$$
$$\geq |z_1|^2 - 2|z_1||z_2| + |z_2|^2$$
$$|z_1 - z_2|^2 \geq (|z_1| - |z_2|)^2$$

or
$$|z_1 - z_2| \geq |z_1| - |z_2|.$$

**Theorem 8:** **The argument of the product of two complex numbers is equal to the sum of their arguments.** **[B.C.A. (Rohilkhand) 2008]**

**Proof:** Let
$$z_1 = r_1 e^{i\theta_1}$$
$$z_2 = r_2 e^{i\theta_2}$$

Then
$$\arg(z_1) = \theta_1,\ \arg(z_2) = \theta_2$$

$\therefore$
$$z_1 z_2 = r_1 r_2\, e^{i(\theta_1 + \theta_2)}$$

Then
$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

In general, $\arg(z_1\, z_2\, z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$.

**Theorem 9:** **The argument of the quotient of two complex numbers is equal to the difference of their arguments.** **[B.C.A. (Agra) 2004]**

**Proof:** We have

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$\therefore$
$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2).$$

## Solved Examples

**Example 1:** ***Express*** $\frac{2+3i}{4+5i}$ ***in the form*** $x + iy$.

**Solution:** Multiplying the numerator and denominator of the given fraction by the conjugate complex of the denominator, we have

$$\Rightarrow \quad \frac{2+3i}{4+5i} = \frac{(2+3i)}{(4+5i)} \times \frac{(4-5i)}{(4-5i)} = \frac{8-10i+12i-15i^2}{16-25i^2}$$

$$= \frac{23+2i}{16+25} = \frac{23+2i}{41} = \frac{23}{41} + \frac{2}{41}i$$

$\because$ the real part $x = \frac{23}{41}$ and the imaginary part $y = \frac{2}{41}$.

**Example 2:** ***Find real numbers x and y, if***

$$x + iy = \frac{2-3i}{4+7i}.$$

**Solution:** We have

$$\frac{2-3i}{4+7i} = \frac{(2-3i)(4-7i)}{(4+7i)(4-7i)} = \frac{8-14i-12i+21i^2}{16-49i^2} = \frac{(8-21)-26i}{16+49}$$

$$= \frac{-13-26i}{65} = \frac{-13}{65} - \frac{26}{65}i = -\frac{1}{5} - \frac{2}{5}i$$

$$\therefore \quad x + iy = -\frac{1}{5} - \frac{2i}{5} \Rightarrow x = -1/5, \; y = -2/5.$$

**Example 3:** ***Express*** $1 - i$ ***in the modulus amplitude form.***

**Solution:** Let $1 - i = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ Equating real and imaginary part on both side,

$$\Rightarrow \quad r\cos\theta = 1 \qquad \ldots(1)$$

$$\Rightarrow \quad r\sin\theta = -1 \qquad \ldots(2)$$

$\Rightarrow$ Squaring and adding (1) and (2), we have

$$1^2 = 1 + 1 = 2, \quad \therefore \quad r = +\sqrt{2}$$

$\Rightarrow$ Divided (2) by (l)

$\Rightarrow \qquad \tan\theta = -1$

$\Rightarrow \qquad \theta = -\dfrac{\pi}{4}$

$\Rightarrow$ Hence, $\quad 1 - i = \sqrt{2}\,\{\cos(-\pi/4) + i\sin(-\pi/4)\}.$

***Example 4:*** ***Reduce to the form r*** $(\cos\theta + i\sin\theta)$ ***the quantity*** $(-1 + i\sqrt{3})$.

**[B.C.A. (Avadh) 2007]**

***Solution:*** Let $\quad -1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ Equating real and imaginary parts, we have

$\Rightarrow \qquad r\cos\theta = -1 \qquad ...(1)$

$\Rightarrow \qquad r\sin\theta = \sqrt{3} \qquad ...(2)$

$\Rightarrow$ Squaring and added (1) and (2), we have

$\Rightarrow \qquad r^2 = 1 + 3 = 4; \qquad \therefore \quad r = 2$

$\Rightarrow$ Dividing (2) by (1), we have $\tan\theta = -\sqrt{3}$

$\Rightarrow \qquad \theta = \dfrac{2\pi}{3}$

$\Rightarrow$ Choosing the values of $\theta$ lying between $-\pi$ and $\pi$ for which $\cos\theta$ is negative and $\sin\theta$ is positive.

Hence $\quad -1 + i\sqrt{3} = 2\,[\cos(2\pi/3) + i\sin(2\pi/3)].$

***Example 5:*** ***Express*** $\dfrac{(\sqrt{3}-1) + i(\sqrt{3}+1)}{2\sqrt{2}}$ ***in the Trigonometric form.***

***Solution:*** Let $\dfrac{(\sqrt{3}-1)}{2\sqrt{2}} + i\,\dfrac{(\sqrt{3}+1)}{2\sqrt{2}} = r(\cos\theta + i\sin\theta)$

Equating real and imaginary part, we have

$$r\cos\theta = \frac{\sqrt{3}-1}{2\sqrt{2}} \qquad ...(1)$$

$$r\sin\theta = \frac{\sqrt{3}+1}{2\sqrt{2}} \qquad ...(2)$$

Squaring (1), (2) and adding these, we get

$$r^2 = \frac{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2}{8} = 1 \text{ i.e., } r = 1$$

Now putting $r = 1$ in (1) and (2), we have

$$\cos\theta = \frac{\sqrt{3}-1}{2\sqrt{2}}, \sin\theta = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\theta = 75° = \frac{5\pi}{12}.$$

Hence $\dfrac{(\sqrt{3}-1)+i(\sqrt{3}+1)}{2\sqrt{2}} = \cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}.$

***Example 6:*** ***If*** $\boldsymbol{x + iy = \dfrac{3}{2+\cos\theta + i\sin\theta}}$***, prove that***

$$\boldsymbol{(x-1)(x-3) + y^2 = 0.}$$ **[B.C.A. (Bhopal) 2012]**

***Solution:*** We have

$$x + iy = \frac{3}{2+\cos\theta + i\sin\theta}$$

$$\Rightarrow \quad x + iy = \frac{3(2+\cos\theta - i\sin\theta)}{[(2+\cos\theta)+i\sin\theta][(2+\cos\theta)-i\sin\theta]}$$

$$\Rightarrow \quad x + iy = \frac{3(2+\cos\theta - i\sin\theta)}{(2+\cos\theta)^2 + \sin^2\theta}$$

$$\Rightarrow \quad x + iy = \frac{6+3\cos\theta - i3\sin\theta}{5+4\cos\theta}$$

Equating real and imaginary part on both side,

$$x = \frac{6+3\cos\theta}{5+4\cos\theta}, \quad y = \frac{-3\sin\theta}{5+4\cos\theta}$$

$$\therefore \quad (x-1)(x-3) = \left(\frac{6+3\cos\theta}{5+4\cos\theta} - 1\right)\left(\frac{6+3\cos\theta}{5+4\cos\theta} - 3\right)$$

$$= \left(\frac{1-\cos\theta}{5+4\cos\theta}\right)\left(\frac{-9-9\cos\theta}{5+4\cos\theta}\right)$$

$$= -9\,\frac{(1-\cos\theta)(1+\cos\theta)}{(5+4\cos\theta)^2}$$

$$= -\frac{9\sin^2\theta}{(5+4\cos\theta)^2} \qquad ...(1)$$

Also $$y^2 = \frac{9\sin^2\theta}{(5+4\cos\theta)^2} \qquad ...(2)$$

Adding (1) and (2), we have

$$(x-1)(x-3)+y^2=0.$$

**Example 7:** ***If $z_1$ and $z_2$ are two complex number, prove that $|z_1+z_2|^2=|z_1|^2+|z_2|^2$, if and only if, $z_1\overline{z_2}$ is purely imaginary.*** **[B.C.A. (Meerut) 2008]**

**Solution:** We know that $z\bar{z}=|z|^2$

$\therefore$

$$|z_1+z_2|^2=|z_1+z_2).(\overline{z_1+z_2})=(z_1+z_2)(\overline{z_1}+\overline{z_2})$$

$$=z_1\overline{z_1}+z_2\overline{z_2}+z_1\overline{z_2}+z_2\overline{z_1}$$

$$=|z_1|^2+|z_2|^2+z_1\overline{z_2}+z_2\overline{z_1}$$

Now $\quad |z_1+z_2|^2=|z_1|^2+|z_2|^2$

$\Leftrightarrow \quad |z_1|^2+|z_2|^2+z_1\overline{z_2}+z_2\overline{z_1}=|z_1|^2+|z_2|^2$

$\Leftrightarrow \quad z_1\overline{z_2}+z_2\overline{z_1}=0$

$\Leftrightarrow \quad z_1\overline{z_2}+(\overline{z_1\overline{z_2}})=0, \qquad [\because \overline{(\bar{z})}=z,]$

Now if $z=x+iy$, then $z+\bar{z}=0$

$\Leftrightarrow \quad (x+iy)+(x-iy)=0\Leftrightarrow 2x=0$

$\Leftrightarrow \quad x=0\Leftrightarrow z$ is purely imaginary.

$\therefore \quad z_1\overline{z_2}+(\overline{z_1\overline{z_2}})=0\Leftrightarrow z_1\overline{z_2}$ is purely imaginary.

Hence $|z_1+z_2|^2=|z_1|^2+|z_2|^2$, if and only if $z_1\overline{z_2}$ is purely imaginary.

**Example 8:** ***Show that the representative points of the complex number $6i$, $2+2i$, $-1+8i$, and $6-6i$ are collinear.*** **[B.C.A. (Rohilkhand) 2009]**

**Solution:** Let the representative point of the complex number $6i, 2+2i, -1+8i$ and $6-6i$ in the Argand diagram be $A, B, C$ and $D$ respectively. Then the cartesian coordinates of $A, B, C, D$ are $A(0,6), B(2,2), C(-1,8), D(6,-6)$.

Four points will be collinear if we find equation of straight line through any two points and remaining point satisfying.

The equation of the line $AB$ is

$$y-6=-2(x-0) \qquad i.e., \qquad y+2x=6.$$

Put $C(-1,8)$ and $D(6,-6)$ in $y=2x+6$ and satisfying. So given points are collinear.

***Example 9:*** *If* $\left|\dfrac{z-1}{z+1}\right| = 2$, *prove that the locus of z on the Argand plane is a circle, whose centre has affix* $(-5/3, 0)$ *and whose radius is* $4/3$. **[B.C.A. (Meerut) 2009, 07]**

***Solution:*** We have $$\left|\frac{z-1}{z+1}\right| = 2$$

$\therefore$ $$\frac{|z-1|}{|z+1|} = 2, \quad \text{or} \quad |z-1| = 2\,|z+1|$$

$\Rightarrow$ $$|x+iy-1| = 2\,|x+iy+1|$$

$\Rightarrow$ $$|(x-1)+iy| = 2\,|(x+1)+iy|$$

$\Rightarrow$ $$\sqrt{(x-1)^2+y^2} = 2\sqrt{(x+1)^2+y^2}$$

Squaring both sides

$\Rightarrow$ $$(x-1)^2+y^2 = 4\,\{(x+1)^2+y^2\}$$

$\Rightarrow$ $$x^2+y^2-2x+1 = 4\,(x^2+y^2+2x+1)$$

$\Rightarrow$ $$(x^2+y^2)4-(x^2+y^2)+10x+3=0$$

$\Rightarrow$ $$3\,(x^2+y^2)+10x+3=0$$

$\Rightarrow$ $$x^2+y^2+\frac{10}{3}x+1=0$$

Which is the cartesian equation of a circle.

***Example 10:*** ***Find the radius and centre of the circle***

$$z\bar{z}-(2+3\,i)z-(2-3\,i)\,\bar{z}+9=0.$$ **[B.C.A. (Kashi) 2009; B.C.A. (Avadh) 2007]**

***Solution:*** The given equation can be written as

$$z\bar{z}-(2+3i)\,z-(2-3i)\,\bar{z}+9=0$$

We have $z\bar{z} = (x+iy)\,(x-iy) = x^2+y^2$

$\Rightarrow$ $z+\bar{z} = (x+iy)+(x-iy) = 2x$

$\Rightarrow$ $z-\bar{z} = 2\,i\,y$

$\therefore$ the given equation can be written as

$\Rightarrow$ $$z\bar{z}-2\,(z+\bar{z})-3i\,(z-\bar{z})+9=0 \qquad \ldots(1)$$

$\therefore$ the equation (1) becomes.

$\Rightarrow$ $$x^2+y^2-4x-3i\,(2iy)+9=0$$

$$\Rightarrow \qquad x^2 + y^2 - 4x + 6y + 9 = 0,$$

which is the cartesian equation of a circle. With centre $(2, -3)$

and radius $= \sqrt{2 + 9 - 9} = \sqrt{2}.$

**Example 11:** ***Prove that, if $z_1$ and $z_2$ are any two complex numbers, then***

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1)^2 + |z_2|^2).$$ **[B.C.A. (Meerut) 2008]**

***Solution:*** We have

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)\,(\overline{z_1 + z_2}) + (z_1 - z_2)\,(\overline{z_1 - z_2})$$

$$= (z_1 + z_2)\,(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)\,(\overline{z_1} - \overline{z_2})$$

$$= 2z_1\overline{z_1} + 2z_2\overline{z_2} = 2|z_1|^2 + 2|z_2|^2 = 2\,(|z_1|^2 + |z_2|^2).$$

**Example 12:** ***If $z_1, z_2, z_3$ are the vertices of an isosceles triangle, right angle at the vertex $z_2$, prove that***

$$z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3).$$

**[B.C.A (Kurukshetra) 2009; B.C.A. (Lucknow) 2007]**

***Solution:*** Since $\angle ABC = 90°$; therefore

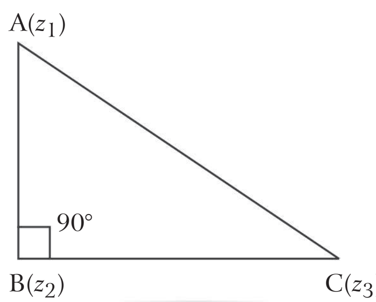


**Fig. 1.2**

$$\arg\left(\frac{z_2 - z_1}{z_2 - z_3}\right) = \frac{\pi}{2}$$

Then $\left(\dfrac{z_2 - z_1}{z_2 - z_3}\right)$ is purely imaginary

$$\therefore \qquad Re\left(\frac{z_2 - z_1}{z_2 - z_3}\right) = 0$$

or $$\frac{1}{2}\left[\frac{z_2 - z_1}{z_2 - z_3} + \frac{\overline{z_2} - \overline{z_1}}{\overline{z_2} - \overline{z_3}}\right] = 0$$

or $$\frac{(z_2 - z_1)}{(z_2 - z_3)} = -\frac{(\overline{z_2} - \overline{z_1})}{(\overline{z_2} - \overline{z_3})} \qquad ...(1)$$

Also $\Delta\,ABC$ is isosceles then we have

$$|z_2 - z_1| = |z_2 - z_3| \quad \text{or} \quad |z_2 - z_1|^2 = (z_2 - z_3)^2$$

or $$(z_2 - z_1)\,(\overline{z_2 - z_1}) = (z_2 - z_3)\,(\overline{z_2 - z_3}) \qquad ...(2)$$

Multiplying (1) and (2), we get

$$\frac{(z_2 - z_1)^2}{(z_2 - z_3)}(\overline{z_2} - \overline{z_1}) = -(\overline{z_2} - \overline{z_1})(z_2 - z_3)$$

or $\quad (z_2 - z_1)^2 + (z_2 - z_3)^2 = 0 \quad$ or $\quad z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$.

**Example 13:** ***Prove that the area of the triangle whose vertices are the points $z_1, z_2, z_3$ on the Argand diagram is $\sum \left\{\frac{(z_2 - z_3)|z_1|^2}{4iz_1}\right\}$.***

***Show also that the triangle is equilateral if***

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$ **[B.C.A. (I.G.N.O.U.) 2012]**

**Solution:** Let $z_1, z_2, z_3$ be the points $A, B$ and $C$ on the complex plane.

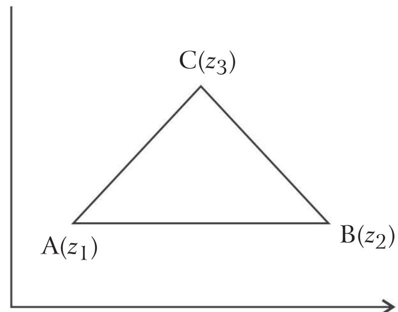


**Fig. 1.3**

Let
$$z_1 = x_1 + iy_1$$
$$z_2 = x_2 + iy_2$$
$$z_3 = x_3 + iy_3$$

Then the area of triangle $ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2i}\begin{vmatrix} x_1 & x_1 + iy_1 & 1 \\ x_2 & x_2 + iy_2 & 1 \\ x_3 & x_3 + iy_3 & 1 \end{vmatrix} = \frac{1}{2i}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2i}\sum x_1(z_2 - z_3) = \frac{1}{2i}\sum \frac{1}{2}(z_1 + \overline{z_1})(z_2 - z_3)$$

$$= \frac{1}{4i}\sum z_1(z_2 - z_3) + \frac{1}{4}\sum \overline{z_1}(z_2 - z_3)$$

$$= 0 + \frac{1}{4}\sum \frac{z_1\overline{z_1}}{z_1}(z_2 - z_3) = \sum \frac{|z_1|^2(z_2 - z_3)}{4iz_1}$$

Now, the triangle $ABC$ will be equilateral if

$$AB = BC = CA$$

*i.e.*, if $\quad |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$

*i.e.*, if $\quad |z_1 - z_2|^2 = |z_2 - z_3|^2 = |z_3 - z_1|^2$

*i.e.*, if $$(z_1 - z_2)(\overline{z_1 - z_2}) = (z_2 - z_3)(\overline{z_2 - z_3}) = (z_3 - z_1)(\overline{z_3 - z_1}) \qquad ...(1)$$

Taking first and second pair of (1), we get

$$\frac{z_1 - z_2}{\overline{z_2} - \overline{z_3}} = \frac{z_2 - z_3}{\overline{z_1} - \overline{z_2}} = \frac{(z_1 - z_2) + (z_2 - z_3)}{(\overline{z_2} - \overline{z_3}) + (\overline{z_1} - \overline{z_2})}$$

or $$\frac{(z_1 - z_2)}{(\overline{z_2} - \overline{z_3})} = \frac{(z_1 - z_3)}{(\overline{z_1} - \overline{z_3})} \qquad ...(2)$$

Again from last two pair of (1), we get

$$(z_2 - z_3)(\overline{z_2} - \overline{z_3}) = (z_3 - z_1)(\overline{z_3} - \overline{z_1}) \qquad ...(3)$$

Multiplying (2) and (3), we get

$$(z_1 - z_2)(z_2 - z_3) = (z_1 - z_3)^2$$

or $$z_1 z_2 - z_1 z_3 - z_2^2 + z_2 z_3 = z_1^2 + z_3^2 - 2z_1 z_3$$

or $$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

***Example 14:*** ***Find the loci of the points z satisfying the following conditions:***

***(i)*** $\arg \dfrac{z-1}{z+1} = \dfrac{\pi}{6}$ ***(ii)*** $|z-1| \geq 2$

***(iii)*** $\left|\dfrac{z-i}{z+i}\right| \geq 2$ ***(iv)*** $|z^2 - 1| < 1.$

***Solution:*** (i) Here

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy}$$

$$= \frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

$\therefore$ $$\arg\frac{z-1}{z+1} = \tan^{-1}\frac{2y}{x^2+y^2-1}$$

Therefore $$\tan^{-1}\frac{2y}{x^2+y^2-1} = \frac{\pi}{6}$$

$\Rightarrow$ $$\frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$\Rightarrow$ $$x^2 + y^2 - 2\sqrt{3y} - 1 = 0$$

which represents a circle.

(ii) Here $$|z-1| \geq 2 \quad \text{or} \quad |z-1|^2 \geq 4$$

$$\Rightarrow \quad (z-1)(\bar{z}-1) \geq 4$$

$$\Rightarrow \quad z\bar{z} - (z+\bar{z}) + 1 \geq 4$$

$$\Rightarrow \quad x^2 + y^2 - 2x - 3 \geq 0$$

which represent the exterior and frontier (or boundary) of the circle

$$x^2 + y^2 - 2x - 3 = 0.$$

(iii) Here $$\left|\frac{z-i}{z+i}\right| \geq 2 \quad \text{or} \quad \left|\frac{z-i}{z+i}\right|^2 \geq 4$$

$$\Rightarrow \quad \frac{(z-i)(\bar{z}+i)}{(z+i)(\bar{z}-i)} \geq 4$$

$$\Rightarrow \quad z\bar{z} + i(z-\bar{z}) + 1 \geq 4z\bar{z} + 4i(\bar{z}-z) + 4$$

$$\Rightarrow \quad 3z\bar{z} + 5i(\bar{z}-z) + 3 \leq 0$$

$$\Rightarrow \quad 3(x^2+y^2) + 5i(-2iy) + 3 \leq 0$$

$$\Rightarrow \quad 3x^2 + 3y^2 + 10y + 3 \leq 0$$

which represent the interior and frontier (or boundary) of the circle

$$3x^2 + 3y^2 + 10y + 3 = 0.$$

(iv) Here $|z^2 - 1| < 1$

$$\Rightarrow \quad (z^2-1)(\bar{z}^2-1) < 1$$

$$z^2\bar{z}^2 - (z^2 + \bar{z}^2) + 1 < 1$$

$$\Rightarrow \quad (x^2+y^2)^2 - 2(x^2-y^2) < 0$$

Changing into polar co-ordinates, we obtain

$$r^4 - 2r^2 \cos 2\theta < 0$$

or $$r^2 < 2\cos 2\theta$$

which represents the interior of the lemniscate

$$r^2 = 2\cos 2\theta.$$

***Example 15:*** ***Determine the regions of Argand diagram defined by***

***(i)*** $|z| \geq 1,$ ***(ii)*** $|z^2 - z| < 1,$ ***(iii)*** $|z-1| + |z+1| \leq 4.$

***Solution:*** (i) Here $|z| \geq 1$ or $|z|^2 \geq 1$

$$\Rightarrow \quad |x+iy|^2 \geq 1$$

$$\Rightarrow \quad x^2 + y^2 \geq 1$$

Here the region represented is exterior and boundary of the circle

$$x^2 + y^2 = 1.$$

(ii) Here $|z^2 - z| < 1$

$$\Rightarrow \quad |r^2(\cos 2\theta + i \sin 2\theta) - r(\cos\theta + i\sin\theta)| < 1$$

$$\Rightarrow \quad (r^2 \cos 2\theta - r\cos\theta)^2 + (r^2 \sin 2\theta - r\sin\theta)^2 < 1$$

$$\Rightarrow \quad r^4 - 2r^3(\cos 2\theta \cos\theta + \sin 2\theta \sin\theta) + r^2 < 1$$

$$\Rightarrow \quad r^4 - 2r^3 \cos\theta + r^2 - 1 \leq 0.$$

(iii) Here $|z-1| + |z+1| \leq 4$

$$\Rightarrow \quad |(x-1) + iy| + |(x+1) + iy| \leq 4$$

$$\Rightarrow \quad \sqrt{[(x-1)^2 + y^2]} + \sqrt{[(x+1)^2 + y^2]} \leq 4$$

$$\Rightarrow \quad \sqrt{[(x-1)^2 + y^2]} - 4 \leq -\sqrt{[(x+1)^2 + y^2]}$$

$$\Rightarrow \quad (x-1)^2 + y^2 - 8\sqrt{[(x-1)^2 + y^2]} + 16 \leq (x+1)^2 + y^2$$

$$\Rightarrow \quad -8\sqrt{[(x-1)^2 + y^2]} \leq 4x - 16$$

$$\Rightarrow \quad -2\sqrt{[(x-1)^2 + y^2]} \leq x - 4$$

$$\Rightarrow \quad 4[(x-1)^2 + y^2] \leq (x-4)^2$$

$$\Rightarrow \quad 3x^2 + 4y^2 - 12 \leq 0$$

$$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{3} \leq 1$$

Hence the region represented is the boundary and interior of the ellipse

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

whose foci are (1, 0), (–1, 0) major axis is 4 and minor axis is 3.

**Example 16:** ***Find regions of the Argand plane for which***

$$\left|\frac{z-a}{z+\bar{a}}\right| < 1, = 1 \text{ or } > 1$$

***where the real part of 'a' is positive.***

**Solution:** Here

$$|z-a| <, = \quad \text{or} \quad > |z+\bar{a}|$$

$$\Rightarrow \quad |z-a|^2 <, = \quad \text{or} \quad > |z+\bar{a}|^2$$

$$\Rightarrow \quad (z-a)(\bar{z}-\bar{a}) <, = \quad \text{or} \quad > |z+\bar{a}||\bar{z}+a|$$

$$\Rightarrow \quad z\bar{z} - (a\bar{z}+\bar{a}z) <, = \quad \text{or} \quad > z\bar{z} + za + \bar{z}\,\bar{a}$$

$$\Rightarrow \quad z(a+\bar{a}) + \bar{z}(a+\bar{a}) >, = \quad \text{or} \quad < 0$$

$$\Rightarrow \quad (z+\bar{z})(a+\bar{a}) >, = \quad \text{or} \quad < 0$$

$$\Rightarrow \quad 2x.2R(a) >, = \quad \text{or} \quad < 0$$

$$\text{which implies } x >, = \quad \text{or} \quad < 0 \qquad [\because R(a) \text{ is positive}]$$

The required regions are therefore the right half of the z-plane, the imaginary axis and the left half of the z-plane respectively.

**Example 17:** ***Prove that*** $\left|\frac{z-1}{z+1}\right|$ ***= const and amp*** $\left(\frac{z-1}{z+1}\right)$ ***= const are orthogonal circles.***

**[B.C.A. (Rohilkhand) 2008]**

**Solution:** Here

$$\left|\frac{z-1}{z+1}\right| = \text{const} = \lambda, \text{ say}$$

$$\Rightarrow \quad \left|\frac{x-1+iy}{x+1+iy}\right|^2 = \lambda^2$$

$$\Rightarrow \quad \frac{(x-1)^2+y^2}{(x+1)^2+y^2} = \lambda^2$$

or $\quad x^2+y^2+2\,\dfrac{\lambda^2+1}{\lambda^2-1}\,x+1=0$ this is circle of the form

$$x^2+y^2+2gx+1=0 \qquad ...(1)$$

Again, we have

$$\text{amp}\left(\frac{z-1}{z+1}\right) = \text{const}$$

$$\Rightarrow \quad \text{amp}(z-1) - \text{amp}(z+1) = \text{const}$$

$\Rightarrow$ $\quad \text{amp}\,(x-1+iy) - \text{amp}\,(x+1+iy) = \text{const}$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = \text{const}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\dfrac{y}{x-1} - \dfrac{y}{x+1}}{1 + \dfrac{y}{x-1}\cdot\dfrac{y}{x+1}}\right\} = \text{const}$$

or $\dfrac{2y}{x^2+y^2-1} = \mu$ where $\mu$ is constant; that is, $x^2 + y^2 - \dfrac{2}{\mu}y - 1 = 0$, this is of the form

$$x^2 + y^2 + 2fy - 1 = 0 \qquad \text{...(2)}$$

which also represents, circle.

Now the circles (1) and (2) with the usual notation satisfy the condition of orthogonality; *i.e.*, $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

Then the circles represented by equations (1) and (2) are orthogonal.

## Exercise

1. Find the locus of the point $z$ satisfying the condition

$$\arg\frac{z-1}{z+1} = \frac{\pi}{3}.$$

**[B.C.A. (Meerut) 2004]**

2. Show that the two lines joining the points $z = a$, $z = b$ and $z = c, z = d$ are perpendicular if

$$\arg\left(\frac{a-b}{c-d}\right) = \pm\frac{\pi}{2}$$

*i.e.*, if the complex number $\dfrac{a-b}{c-d}$ is purely imaginary.

3. If $z_1, z_2, z_3$ are the vertices of an isosceles triangle, right angled at the vertex $z_2$, prove that

$$z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_2).$$

**[B.C.A. (Agra) 2010]**

4. Show that the two triangles whose vertices in the Argand diagram have affixes $a, b, c,$ and $\alpha, \beta, \gamma$ will be similar if $(b\gamma - c\beta) + (c\alpha - a\gamma) + (a\beta - b\alpha) = 0$.

5. Find the radius and centre of the circle

$$\left|\frac{z-i}{z+i}\right| = 5.$$

**6.** Find the locus of the point $z$ satisfying the condition

$$\arg\frac{z-1}{z+1}=\frac{\pi}{3}.$$

**7.** Show that the representative point of the complex numbers $i, -2-5i, 1+4i$ and $3+10i$ are collinear.

**8.** If $l^2+m^2+n^2=1$, and $(m+in)=(1+l)\,z$, show that

$\dfrac{1+im}{1+n}=\dfrac{1+iz}{1-iz}$, where $l, m, n$ are real number and $z$ is a complex number.

**9.** Express $\dfrac{1+7i}{(2-i)^2}$ in the modulus-amplitude form.

**10.** Express $(1+\cos\alpha-i\sin\alpha)$ in the form $r(\cos\theta+i\sin\theta)$.

**11.** Express $-5+12i$ in the modulus – argument form.

**12.** Express $-1-\sqrt{-3}$ in the polar form.

**13.** Express $(1+\sqrt{-3})$ in the modulus – amplitude form.

**14.** Express $-1-i$ in the form $r(\cos\theta+i\sin\theta)$.

**15.** Find the modulus and principal argument of $1+i$.

**16.** Express $1-i$ in the modulus amplitude form.

**17.** Find real numbers $x$ and $y$, if

$$x+iy=\frac{2-3i}{4+7i}.$$

**18.** Find real number $A$ and $B$ if

$$A+iB=\frac{3-2i}{7+4i}.$$

**19.** If $z_1, z_2$ are any complex number then,

$$|z_1+z_2|^2+|z_1-z_2|^2=2\{|z_1|^2+|z_2|^2\}.$$ **[B.C.A. (Lucknow) 2004]**

**20.** The modulus of the sum of two complex number can never exceed the sum of their moduli.

**21.** The point $A, B, C$ in the Argand plane represent the complex number $z_1, z_2, z_3$ respectively. Show that the triangle $ABC$ is equilateral if and only if.

$$\frac{1}{z_1-z_2}+\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}=0.$$

**22.** The vertices of a triangle are represented in Argand diagram by the complex number $z_1, z_2, z_3$. Interpret the modulus and argument of $\dfrac{z_2-z_1}{z_3-z_1}$ in terms of the sides and angle of the triangle.

**23.** Show that the equation of a circle in the Argand plane can be put in the form

$z\bar{z} + b\bar{z} + \bar{b}z + c = 0,$

where $c$ is a real and $b$, $a$ complex constant. **[B.C.A. (Meerut) 2002]**

**24.** Show that the representative points of the complex number $1 + i, 2 + i, 2 + 3i, 1 + 3i$ form a rectangle.

**25.** A variable complex $z = x + iy$ is such that the argument of the fraction $(z - 1)/(z + 1)$ is always equal to $\pi/4$ show that $x^2 + y^2 - 2y = 1$. **[B.C.A. (Avadh) 2005]**

## Answers

| | |
|---|---|
| **1.** | $x^2 + y^2 - (2/\sqrt{3})y - 1 = 0.$ |
| **5.** | radius $= 5/12$, centre of the circle $(0, -13/12)$. |
| **9.** | $\sqrt{2}\,(\cos 3\pi/4 + i \sin 3\pi/4).$ |
| **10.** | $2 \cos \alpha/2\,[\cos \alpha/2 - i \sin \alpha/2].$ |
| **11.** | $13\,[\cos \alpha + i \sin \alpha]$, where $\cos \alpha = -5/13, \sin \alpha = 12/13.$ |
| **12.** | $2\,[\cos (2\pi/3) - i \sin (2\pi/3)].$ |
| **13.** | $2\,[\cos (2\pi/3) + i \sin (2\pi/3)].$ |
| **14.** | $\sqrt{2}\,[\cos (3\pi/4) - i \sin (3\pi/4)].$ |
| **15.** | $\sqrt{2}\,[\cos (\pi/4) + i \sin (\pi/4)].$ |
| **16.** | $\sqrt{2}\,[\cos (-\pi/4) + i \sin (-\pi/4)].$ |
| **17.** | $x + iy = -\frac{1}{5} - \frac{2}{5}i,\ x = -1/5,\ y = -2/5.$ |
| **18.** | $A = 1/5, B = -2/5.$ |
| **22.** | $\left\lvert \frac{z_2 - z_1}{z_3 - z_1} \right\rvert = \frac{AB}{AC}$ and $\arg \frac{z_2 - z_1}{z_3 - z_1} = \angle BAC.$ |

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# Chapter 2

# Elementary Functions

## 2.1 Elementary Functions of a Complex Variable

We will discuss some elementary properties of certain functions of a complex variable for example, the exponential, the logarithmic and the trigonometrical functions to all of which is given the general title: Elementary functions.

## 2.2 The Exponential Function

The exponential function $f(z)$ of a complex variable is defined as the solution of differential equation.

$$f'(z) = f(z) \qquad \text{....(1)}$$

with initial value $f(0) = 1$

We have to solve by taking

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots + a_n z^n \qquad \text{...(2)}$$

$$\Rightarrow \qquad f'(z) = a_1 + 2a_2 z + \ldots\ldots + n a_n z^{n-1} \qquad \text{...(3)}$$

Put (2) and (3) in (1), we get

$$a_1 + 2a_2 z + \ldots\ldots + a_n n z^{n-1} = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

Compare the coefficient of like power of $z$ on both sides, we get

$$
\begin{aligned}
a_0 &= a_1 \\
a_1 &= 2a_2 \\
\vdots\ & \quad \vdots \quad \vdots \\
a_{n-1} &= na_n
\end{aligned}
$$

From these relation, we have

$$a_1 = 1, a_2 = \frac{1}{2},\ a_3 = \frac{a_2}{3} = \frac{1}{2.3} = \frac{1}{3!}$$

In general, $$a_n = \frac{1}{n!}$$

We shall denote the solution by exp $(z)$ or $e^z$. Thus,

$$\exp(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{\lfloor n} \qquad ...(4)$$

Now, we shall show that the series (4) converges. It converges absolutely.

Now, $$|u_n| = \frac{|z|^{n-1}}{(n-1)!}$$

$$|u_{n+1}| = \frac{|z|^n}{n!}$$

$\therefore$ $$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|z|^n}{n!}\times\frac{(n-1)!}{|z|^{n-1}} = \lim_{n\to\infty}\frac{|z|}{n} = 0 < 1\ \forall\ z$$

Hence, the series (4) converges absolutely in whole complex plane by D- Alembart's ratio test.

Now, $$\frac{d}{dz}\exp(z) = \frac{d}{dz}(e^z) = e^z = \exp(z)$$

Then, we can say that for all value of $z$, $e^z$ is an analytic function of $z$ in the whole Argand plane.

## Theorems

**Theorem 1: Prove that:**

$$\mathbf{\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)}.$$ **[B.C.A. (Bundelkhand) 2010]**

**Proof:** We have

$$\exp(z_1) = 1 + \sum_{n=1}^{\infty}\frac{z_1^n}{n!} \text{ and } \exp(z_2) = 1 + \sum_{n=1}^{\infty}\frac{z_2^n}{n!}$$

$\therefore$ $$\exp(z_1)\exp(z_2) = \left(1 + \sum_{n=1}^{\infty}\frac{z_1^n}{n!}\right)\left(1 + \sum_{n=1}^{\infty}\frac{z_2^n}{n!}\right)$$

$$= 1 + \frac{(z_1 + z_2)}{1!} + \frac{z_1^2 + 2z_1z_2 + z_2^2}{2!} + \ldots\ldots$$

$$= 1 + \frac{(z_1 + z_2)}{1!} + \frac{(z_1 + z_2)^2}{2!} + \ldots\ldots$$

$$= \exp(z_1 + z_2)$$

**Remark :** 1. $\exp(z_1)\exp(z_2).\exp(z_3)\ldots\ldots\exp(z_n) = \exp(z_1 + z_2 + \ldots,\ldots + z_n)$

2. $\exp(z)\exp(-z) = \exp(0) = 1$

## 2.3 The Trigonometrical Functions

The functions sin $z$ and cos $z$ for complex number $z$ are defined, as in the case of a real variable by means of the formulae.

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad \ldots.(1)$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \ldots(2)$$

We know $$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots. + \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad \ldots(3)$$

and $$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots.. + \frac{(-1)^n z^{2n}}{(2n)!} \quad \ldots(4)$$

Since each of these power series has an infinite radius of convergence. Then, sin $z$ and cos $z$ are regular (analytic) in every bounded domain of the $z$-plane.

Now, $$\tan z = \frac{\sin z}{\cos z},\ \cot z = \frac{\cos z}{\sin z},\ \sec z = \frac{1}{\cos z},\ \text{cosec}\,(z) = \frac{1}{\sin z}$$

Then, $$\frac{d}{dz}(\sin z) = \cos z,\ \frac{d}{dz}(\cos z) = -\sin z$$

and $$\sin(-z) = -\sin z,\ \cos(-z) = \cos z,\ \sin 0 = 0,\ \cos 0 = 1.$$

### 2.3.1 Euler's Equation

We have $$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i},\ \cos z = \frac{e^{iz} - e^{-iz}}{2}$$

Then, $$\cos z + i\sin z = \left\{\frac{e^{iz} + e^{-iz}}{2} + i\frac{e^{iz} - e^{-iz}}{2i}\right\}$$

$$= e^{iz}$$

$\therefore$ $$e^{iz} = \cos z + i\sin z.$$

**Theorem 2: Prove that:**

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

and $$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$ **[B.C.A. (Avadh) 2009]**

**Proof:** We have

$$\exp(i(z_1 + z_2)) = \exp(iz_1)\exp(iz_2) \quad ...(1)$$

Then from (1), we get

$$\cos(z_1 + z_2) + i\sin(z_1 + z_2) = (\cos z_1 + i\sin z_2)\ (\cos z_2 + i\sin z_2)$$

$$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) \quad ....(2)$$

Put $z_1 = -z_1$ and $z_2 = -z_2$ in (2), we get

$$\cos(z_1 + z_2) - i\sin(z_1 + z_2) = (\cos z_1 - i\sin z_1)(\cos z_2 - -i\sin z_2)$$

$$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) \quad ....(3)$$

Adding (2) and (3), we get

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \quad ...(4)$$

## 2.4 Hyperbolic Functions sinh(z) and cosh(z)

The hyperbolic functions of a complex variable are defined in the same way as for real variables. The fundamental formulae are:

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \quad ...(1)$$

and $$\cosh(z) = \frac{e^z + e^{-z}}{2} \quad ...(2)$$

Also $$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad ...(3)$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad ...(4)$$

## 2.5 Relation between Hyperbolic and Trigonometric Function

$$\sin(iz) = i\sinh z, \quad \cos(iz) = \cosh(z)$$

$$\sinh(iz) = i\sin z, \quad \cosh(iz) = \cos(z).$$

## 2.6 Logarithmic Function (Inverse of Exponential Function)

The logarithm of a complex variable $w$ denoted by log $w$ is defined as the solution of the equation.

$$\exp(z) = w$$

$$\Rightarrow \qquad z = \log w$$

$$\therefore \qquad \exp(z) = w$$

$$\Rightarrow \qquad e^{x+iy} = w \qquad (z = x + iy)$$

$$\Rightarrow \qquad e^x e^{iy} = w$$

**Theorem 3: If $w_1$ and $w_2$ are two complex number, then:**

$$\log (w_1 w_2) = \log w_1 + \log w_2$$

**and** $$\arg (w_1 w_2) = (\arg w_1).(\arg w_2).$$ **[B.C.A. (Rohilkhand) 2011]**

**Proof:** Let $\log w_1 = z_1$ and $\log w_2 = z_2$. Then by definition, we have

$$\exp(z_1) = w_1 \text{ and } \exp(z_2) = w_2$$

Therefore,
$$\log (w_1 w_2) = \log(\exp z_1 \exp z_2)$$
$$= \log(\exp(z_1 + z_2))$$
$$= z_1 + z_2$$

Thus,
$$\log (w_1 w_2) = \log w_1 + \log w_2$$

The second result *i.e.,* $\arg (w_1 w_2) = \arg (w_1). \arg (w_2)$ follows in the usual sense.

## 2.7 The Function of $a^z$ and $z^a$

We define the principal value of the function $a^z$ as the number uniquely determined by the equation

$$a^z = e^{z \log a},$$

where log a is the principal value of log a and we permit both $a$ and $z$ to be complex.

Let a is real or complex. Then, we defined $z^a$ by the equation

$$z^a = \exp(a \log z).$$

## Solved Examples

**Example1:** ***Prove that*** $\sin(\alpha+n\theta)-e^{i\alpha}\sin n\theta = e^{-in\theta}\sin\alpha$.

**Solution:** **L.H.S.** $= \sin(\alpha+n\theta)-(\cos\alpha+i\sin\alpha)\sin n\theta$

$$= \sin\alpha\cos n\theta+\cos\alpha\sin n\theta-\cos\alpha\sin n\theta-i\sin\alpha\sin n\theta$$

$$= \sin\alpha(\cos n\theta - i\sin n\theta)$$

$$= \sin\alpha\, e^{-in\theta}$$

$$= \text{R.H.S.}$$

**Example 2:** ***Prove that:***

$$(\sin(\alpha-\theta)+e^{-i\alpha}\sin\theta)^n = \sin^{n-1}\alpha\{\sin(\alpha-n\theta)+e^{-i\alpha}\sin n\theta\}.$$

**Solution:** **L.H.S.** $= (\sin\alpha\cos\theta-\cos\alpha\sin\theta+(\cos\alpha-i\sin\alpha)\sin\theta)^n$

$$= (\sin\alpha\cos\theta - i\sin\alpha\sin\theta)^n$$

$$= \sin^n\alpha(\cos\theta - i\sin\theta)^n$$

$$= \sin^n\alpha(\cos n\theta - i\sin n\theta) \quad [\text{ by De-Molver's thorem}].$$

$$\text{R.H.S.} = \sin^{n-1}(\alpha)\{\sin(\alpha-n\theta)+e^{-i\alpha}\sin n\theta\}$$

$$= \sin^{n-1}(\alpha)\{\sin\alpha\cos n\theta-\cos\alpha\sin n\theta+(\cos\alpha-i\sin\alpha)\sin n\theta\}$$

$$= \sin^{n-1}(\alpha)\{\sin\alpha\cos n\theta - i\sin\alpha\sin n\theta\}$$

$$= \sin^{n-1}\alpha\sin\alpha(\cos n\theta - i\sin n\theta)$$

$$= \sin^n\alpha\,(\cos n\theta - i\sin n\theta)$$

$$\text{L.H.S.} = \text{R.H.S.}$$

**Example 3:** ***Show that*** :

*(i)* $\cos^2 hx - \sin^2 hx = 1.$

*(ii)* $\sin h2x = 2\sin hx\cos hx.$

*(iii)* $\cos h2x = \cos^2 hx + \sin^2 hx.$

*(iv)* $\sec^2 hx = 1-\tan^2 hx.$

*(v)* $\tan h2x = \dfrac{2\tan hx}{1+\tan^2 hx}.$

**Solution:** (i) To show $\cos^2 hx - \sin^2 hx = 1$

$$\text{L.H.S.} = \cos^2 hx - \sin^2 hx$$

$$= \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$= \frac{1}{2} + \frac{1}{2} = 1 = \textbf{R.H.S.}$$

(ii) To show $\sin h 2x = 2 \sin hx \cos hx$

**L.H.S.** $= \sin h 2x$

$$= \left(\frac{e^{2x} - e^{-2x}}{2}\right)$$

**R.H.S.** $= 2 \sin hx \ \cos hx$

$$= 2\left(\frac{e^{x} - e^{-x}}{2}\right)\left(\frac{e^{x} + e^{-x}}{2}\right) = \left(\frac{e^{2x} - e^{-2x}}{2}\right)$$

**L.H.S. = R.H.S.**

(iii) To show $\cos h 2x = \cos^2 hx + \sin^2 hx$

**L.H.S.** $= \cos h 2x = \dfrac{e^{2x} + e^{-2x}}{2}$

**R.H.S.** $= \left(\dfrac{e^{x} + e^{-x}}{2}\right)^2 + \left(\dfrac{e^{x} - e^{-x}}{2}\right)^2$

$$= \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4}$$

$$= \frac{2(e^{2x} + e^{-2x})}{4} = \left(\frac{e^{2x} + e^{-2x}}{2}\right)$$

(iv) To show $\sec^2 hx = 1 - \tan^2 hx$

we known $\cos^2 hx - \sin^2 hx = 1$

Divide by $\cos^2 hx$ on both sides, we get

$$1 - \tan^2 hx = \sec^2 hx$$

(v) To show $\tan h 2x = \dfrac{2 \tan hx}{1 + \tan^2 hx}$

we know $i \tan h 2x = \tan(2ix)$

$\therefore$ $\tan h 2x = \dfrac{1}{i}\tan(2ix) = \dfrac{1}{i}\tan 2(ix)$

$$= \frac{1}{i}\frac{2 \tan ix}{1 - \tan^2(ix)} = \frac{1}{i}\ \frac{2i \tan hx}{1 - i^2 \ \tan^2 hx}$$

$$= 2\,\frac{\tan hx}{1 + \tan^2 hx}$$

# Exercise

1. Show that $\exp\left(\pm\frac{i\pi}{2}\right) = \pm i$.

2. Show that $\cos h2x + \cos h5x + \cos h\,8x + \cos h11x$

$$= 4\cos h\left(\frac{13x}{2}\right)\cos h\left(\frac{3x}{2}\right)\cos h3x.$$

3. Prove that $\frac{1}{2}(\cos hx + \cos x) = 1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \ldots.$

4. Show that $\tan h(x+y) = \frac{\tan hx + \tan hy}{1 - \tan hx \tan hy}$.

5. Prove that $\log(1+i) = \frac{1}{2}\log 2 + i\left(2n\pi + \frac{\pi}{2}\right)$.

6. Show that $\log(1+e^{i\theta}) = \log\left(2\cos\frac{\theta}{2}\right) + \frac{i\theta}{2}$.

7. Prove that $\log\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}\left(\frac{b}{a}\right)$.

8. Show that $i\log\left(\frac{x-i}{x+i}\right) = \pi - 2\tan^{-1} x$.

9. To show that $\tan\left(i\log\left(\frac{a-ib}{a+ib}\right)\right) = \frac{2ab}{a^2-b^2}$.

10. Prove that $(i)^i = e^{-(4n+1)\pi/2}$.

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Chapter 3

# Inverse Trigonometrical (Circular) Functions

## 3.1 Inverse Circular Functions

If $\sin\theta = x$ then $\theta = \sin^{-1} x$. If $\cos\theta = x$, then $\theta = \cos^{-1} x$. The quantities $\sin^{-1} x, \cos^{-1} x$, $\tan^{-1} x, \ldots$ etc. are called **inverse circular functions.** Sometime $\sin^{-1} x$ can be written as 'arc sin $x$'.

> **NOTE:**
>
> $$\sin^{-1} x \neq (\sin x)^{-1} \quad \text{as} \quad (\sin x)^{-1} = \frac{1}{\sin x}.$$

## 3.2 General and Principal Values of Inverse Circular Functions

Among the values of $\theta$ satisfying the equation $\theta = \sin^{-1} x$, the value which is numerically the smallest one is called the **principal value** of $\sin^{-1} x$. If $\theta$ is the principal value of $\sin^{-1} x$, then we must have $-\pi/2 \leq \theta \leq \pi/2$. If $x$ is zero, the principal value of $\sin^{-1} x$ is 0, if $x$ is negative, the principal value of $\sin^{-1} x$ lies between $-\pi/2$ and 0; if $x$ is positive, the principal value of $\sin^{-1} x$ lies between 0 and $\pi/2$.

If $\theta$ is principal value of $\sin^{-1} x$, then all the angles given by $n\pi + (-1)^n \theta$, where $n$ is any integer, have their sines equal to $x$. The value $n\pi + (-1)^n\theta$ is called **general value of $\sin^{-1} x$** and denoted by $\text{Sin}^{-1}x$. Generally, if the first letter of an inverse circular function is small, we consider the principal value, while if the first letter is capital, it means the general value.

***For example:*** $\sin^{-1}(1/2) = \pi/6$ while $\text{Sin}^{-1}(1/2) = n\pi + (-1)^n \dfrac{\pi}{6}$

where $n$ is any integer.

$$\therefore \qquad \text{Sin}^{-1}x = n\pi + (-1)^n \sin^{-1} x$$

Similarly, we can write the relation in general value and principal value for other circular functions.

$$\text{Cos}^{-1}(x) = 2n\pi \pm \cos^{-1} x$$

$$\text{Tan}^{-1}x = n\pi + \tan^{-1} x$$

$$\text{Cosec}^{-1}x = n\pi + (-1)^n \text{cosec}^{-1} x$$

$$\text{Sec}^{-1} x = 2n\pi \pm \sec^{-1} x$$

$$\text{Cot}^{-1}x = n\pi + \cot^{-1} x$$

where $n$ is any integer, positive or zero.

## 3.3 Relation between Inverse Functions

1. We have $\sin^{-1} x = \text{cosec}^{-1}(1/x)$, $\cos^{-1} x = \sec^{-1}(1/x)$ and $\tan^{-1} x = \cot^{-1}(1/x)$.

   We know $\quad \sin\theta = x \Rightarrow \theta = \sin^{-1} x$

   and if $\quad \sin\theta = x$, then $\text{cosec}\,\theta = 1/x \Rightarrow \theta = \text{cosec}^{-1}(1/x)$

   $\therefore \quad \sin^{-1} x = \text{cosec}^{-1}(1/x)$

   Similarly, we can show that $\cos^{-1}(x) = \sec^{-1}(1/x)$ and $\cot^{-1}(x) = \tan^{-1}(1/x)$

2. From the definition of an inverse circular function,

   we have $\quad \theta = \sin^{-1}(\sin\theta)$ and $x = \sin(\sin^{-1} x)$

   For, if $\quad \sin\theta = x \Rightarrow \theta = \sin^{-1} x = \sin^{-1}(\sin\theta)$

Similarly, $\theta = \cos^{-1}(\cos\theta)$ and $x = \cos(\cos^{-1} x)$

$\theta = \tan^{-1}(\tan\theta)$ and $x = \tan(\tan^{-1} x)$ etc.

3. To express one inverse circular function in terms of the others.

If $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

Then $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

*i.e.,* $\cos\theta = \sqrt{1-x^2} \Rightarrow \theta = \cos^{-1}\sqrt{1-x^2}$

and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{x}{\sqrt{1-x^2}} \Rightarrow \theta = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$

Hence, $\theta = \sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left\{\dfrac{x}{\sqrt{1-x^2}}\right\}$

Also by principle of reciprocity, we have

$$\theta = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

Hence, $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\left\{\dfrac{x}{\sqrt{1-x^2}}\right\}$

$$= \operatorname{cosec}^{-1}(1/x) = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

Similarly, $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left\{\dfrac{\sqrt{1-x^2}}{x}\right\}$

and $\tan^{-1} x = \sin^{-1}\left\{\dfrac{x}{\sqrt{1+x^2}}\right\} = \cos^{-1}\left\{\dfrac{1}{\sqrt{1+x^2}}\right\}$ etc.

**4. To show that**

**(i) $\sin^{-1}(-x) = -\sin^{-1} x$** **(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$**

**(iii) $\tan^{-1}(-x) = -\tan^{-1} x$.**

**Proof:** (i) Put $-x = \sin\theta \Rightarrow \theta = \sin^{-1}(-x)$ ...(1)

Now, $-x = \sin\theta \Rightarrow x = -\sin\theta = \sin(-\theta)$

$\therefore$ $-\theta = \sin^{-1} x$ or $\theta = -\sin^{-1} x$ ...(2)

Equating the values of $\theta$ from (1) and (2), we get

$$\sin^{-1}(-x) = -\sin^{-1} x$$

(ii) Put $-x = \cos\theta \;\Rightarrow \theta = \cos^{-1}(-x)$ ...(3)

Now, $-x = \cos\theta \;\Rightarrow x = -\cos\theta = \cos(\pi - \theta)$

$\therefore$ $\pi - \theta = \cos^{-1} x$ or $\theta = \pi - \cos^{-1}(x)$ ...(4)

Equating the values of $\theta$ from (3) and (4), we get

$$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

(iii) By similar process, we can prove

$$\tan^{-1}(-x) = -\tan^{-1} x$$

## 3.4 Some Standard Results for Inverse Functions

1. **Complementary inverse functions:**

**(i) $\sin^{-1} x + \cos^{-1} x = \pi/2$** **(ii) $\tan^{-1} x + \cot^{-1} x = \pi/2$**

**(iii) $\sec^{-1} x + \mathrm{cosec}^{-1} x = \pi/2$.**

**[B.C.A. (I.G.N.O.U.) 2012, 08, 05; B.C.A. (Bundelkhand) 2011, 08]**

**Proof:** (i) Let $x = \sin\theta$. Then

$$\sin^{-1} x + \cos^{-1} x = \sin^{-1}(\sin\theta) + \cos^{-1}(\sin\theta)$$

$$= \sin^{-1}(\sin\theta) + \cos^{-1}(\cos \pi/2 - \theta)$$

$$= \theta + \pi/2 - \theta = \pi/2.$$

(ii) Let $x = \tan\theta$. Then

$$\tan^{-1} x + \cot^{-1} x = \tan^{-1}(\tan\theta) + \cot^{-1}(\tan\theta)$$

$$= \tan^{-1}(\tan\theta) + \cot^{-1}(\cot(\pi/2 - \theta))$$

$$= \theta + \pi/2 - \theta$$

$$= \frac{\pi}{2}.$$

(iii) Let $x = \sec^{-1} x$. Then

$$\sec^{-1} x + \mathrm{cosec}^{-1} x = \sec^{-1}(\sec^{-1} x) + \mathrm{cosec}^{-1}(\sec^{-1} x)$$

$$= x + \mathrm{cosec}^{-1}(\mathrm{cosec}^{-1}(\pi/2 - x) = x + \pi/2 - x = \pi/2.$$

**2. To prove that**

**(i)** $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ **[B.C.A. (I.G.N.O.U.) 2008]**

**(ii)** $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

**(iii)** $2\tan^{-1} x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

**(iv)** $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$.

**[B.C.A. (Bhopal) 2011, 06, 04]**

**Proof:** (i) We know

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\therefore \qquad A + B = \tan^{-1}\left\{\frac{\tan A + \tan B}{1 - \tan A \tan B}\right\} \qquad ...(1)$$

Let $A = \tan^{-1} x$ and $B = \tan^{-1} y$

$\Rightarrow$ $x = \tan A$ and $y = \tan B$

Put these values in (1), we get

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left\{\frac{x+y}{1-xy}\right\}$$

(ii) We know $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\therefore \qquad A - B = \tan^{-1}\left\{\frac{\tan A - \tan B}{1 + \tan A \tan B}\right\} \qquad ...(2)$$

Let $A = \tan^{-1} x \Rightarrow x = \tan A$ and $B = \tan^{-1} y \Rightarrow y = \tan B$

Put these values in (2), we get

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left\{\frac{x-y}{1+xy}\right\}.$$

(iii) Put $x = y$ in (1), we get

$$\tan^{-1} x + \tan^{-1} x = \tan^{-1}\left(\frac{x+x}{1-x.x}\right)$$

$$\Rightarrow \qquad 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

(iv) We have $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

$$= \tan^{-1}\left\{\frac{x+y}{1-xy}\right\} + \tan^{-1} z$$

$$= \tan^{-1}\left\{\frac{\frac{x+y}{1-xy}+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right\} = \tan^{-1}\left\{\frac{x+y+z-xyz}{1-xy-yz-zx}\right\}$$

Put $x = y = z$, we get

$$3\tan^{-1} x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

**3. To show that**

**(i)** $\cot^{-1} x + \cot^{-1} y = \cot^{-1}\left(\frac{xy-1}{x+y}\right)$

**(ii)** $\cot^{-1} x - \cot^{-1} y = \cot^{-1}\left(\frac{xy+1}{y-x}\right).$

**[B.C.A. (Indraprastha) 2012, 08, 06]**

**Proof:** (i) We know that

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$$

$$\Rightarrow \quad A + B = \cot^{-1}\left\{\frac{\cot A \cot B - 1}{\cot A + \cot B}\right\} \quad ...(1)$$

Let $A = \cot^{-1} x$ and $B = \cot^{-1}(y) \Rightarrow x = \cot A,\ y = \cot B$

Put these values in (1), we get

$$\cot^{-1} x + \cot^{-1} y = \cot^{-1}\left\{\frac{xy-1}{x+y}\right\}.$$

(ii) We know $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

$$\Rightarrow \quad A - B = \cot^{-1}\left\{\frac{\cot A \cot B + 1}{\cot B - \cot A}\right\}$$

Let $A = \cot^{-1} x \Rightarrow x = \cot A$ and $B = \cot^{-1} y \Rightarrow y = \cot B$

$$\therefore \quad \cot^{-1} x - \cot^{-1} y = \cot^{-1}\left\{\frac{xy+1}{y-x}\right\}.$$

## Solved Examples

**Example 1:** ***Prove that*** $\tan^{-1}(1/3) + \tan^{-1}(1/5) + \tan^{-1}(1/7) + \tan^{-1}(1/8) = \pi/4$.

**[B.C.A. (Meerut) 2000]**

**Solution:** Hence,

$$\textbf{L.H.S.} = \tan^{-1}(1/3) + \tan^{-1}(1/5) + \tan^{-1}(1/7) + \tan^{-1}(1/8)$$

$$= \tan^{-1}\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\cdot\frac{1}{5}} + \tan^{-1}\frac{1/7+1/8}{1-1/7\,.\,1/8}, \text{ by the formula}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1} 4/7 + \tan^{-1} 3/11$$

$$= \tan^{-1}\left(\frac{4/7+3/11}{1-4/7\,.\,3/11}\right) = \tan^{-1} 1 = \pi/4 = \textbf{R.H.S.}$$

**Example 2:** ***Prove that*** $\sin^{-1} 4/5 + \sin^{-1} 5/13 + \sin^{-1} 16/65 = \pi/2$.

**[B.C.A. (I.G.N.O.U) 2010]**

**Solution:** Hence,

$$\textbf{L.H.S.} = \sin^{-1} 4/5 + \sin^{-1} 5/13 + \sin^{-1} 16/65$$

$$= (\sin^{-1} 4/5 + \sin^{-1} 5/13) + \sin^{-1} 16/65$$

$$= \sin^{-1}\left[\frac{4}{5}\sqrt{\left(1-\left(\frac{5}{13}\right)^2\right)} + 5/13\sqrt{1-\left(\frac{4}{5}\right)^2}\right] + \sin^{-1} 16/65$$

$$[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}\}]$$

$$= \sin^{-1}\left[\frac{4}{5}\cdot\frac{12}{13} + \frac{5}{13}\cdot\frac{3}{5}\right] + \sin^{-1} 16/65$$

$$= \sin^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\right) + \cos^{-1}\sqrt{\left\{1-\left(\frac{16}{65}\right)^2\right\}} \qquad [\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}]$$

$$= \sin^{-1}\left(\frac{63}{65}\right) + \cos^{-1}\left(\frac{63}{65}\right) = \pi/2, \text{ for } \sin^{-1}x + \cos^{-1}x = \pi/2 = \textbf{R.H.S.}$$

**Example 3:** ***Prove that*** $2\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\frac{x}{2}\right\}=\cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right)$.

**[B.C.A. (Agra) 2005]**

**Solution:** Let $\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\frac{x}{2}\right\}=\theta,$

so that $$\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\frac{x}{2}=\tan\theta \qquad ...(1)$$

We have the **L.H.S.** $=2\theta=\cos^{-1}(\cos 2\theta)$

$$=\cos^{-1}\left[\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]=\cos^{-1}\left[\frac{1-\{(a-b)/(a+b)\}\tan^2 x/2}{1+\{(a-b)/(a+b)\}\tan^2 x/2}\right],$$

[from (1)]

$$=\cos^{-1}\left[\frac{(a+b)\cos^2 x/2-(a-b)\sin^2 x/2}{(a+b)\cos^2 x/2+(a-b)\sin^2 x/2}\right]$$

$$=\cos^{-1}\left[\frac{a(\cos^2 x/2-\sin^2 x/2)+b(\cos^2 x/2+\sin^2 x/2)}{a(\cos^2 x/2+\sin^2 x/2)+b(\cos^2 x/2-\sin^2 x/2)}\right]$$

$$=\cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)=\textbf{R.H.S.}$$

**Example 4:** ***If*** $\cos^{-1}(x/a)+\cos^{-1}(y/b)=\alpha$, ***prove that***

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

**[B.C.A. (Lucknow) 2006]**

**Solution:** $$\cos^{-1}(x/a)+\cos^{-1}(y/b)=\alpha$$

$$[\because \cos^{-1}x+\cos^{-1}y=\cos^{-1}[xy-\sqrt{1-x^2}\,.\sqrt{(1-y^2)}]$$

$$\Rightarrow \quad \cos^{-1}\left[\frac{x}{a}\cdot\frac{y}{b}-\sqrt{\left(1-\frac{x^2}{a^2}\right)}\cdot\sqrt{\left(1-\frac{y^2}{b^2}\right)}\right]=\alpha$$

$$\Rightarrow \quad \left[\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\cdot\sqrt{\left(1-\frac{y^2}{b^2}\right)}\right]=\cos\alpha$$

$$\Rightarrow \quad \left(\cos\alpha-\frac{xy}{ab}\right)^2=\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$$

$$\Rightarrow \qquad \cos^2\alpha + \frac{x^2y^2}{a^2b^2} - \frac{2xy\cos\alpha}{ab} = 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} + \frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy\cos\alpha}{ab} = 1 - \cos^2\alpha = \sin^2\alpha,$$ **Proved.**

***Example 5:*** ***If*** $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, ***prove that***

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

**[B.C.A. (Meerut) 2004]**

***Solution:*** We know that,

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\,.\sqrt{1-y^2}\,].$$

$$\Rightarrow \qquad \cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$$

$$\Rightarrow \qquad \cos^{-1}[xy - \sqrt{1-x^2}\,.\sqrt{1-y^2}\,] = \pi - \cos^{-1}z$$

$$\Rightarrow \qquad [xy - \sqrt{1-x^2}\,.\sqrt{1-y^2}\,] = \cos(\pi - \cos^{-1}z)$$

$$\Rightarrow \qquad [xy - \sqrt{1-x^2}\,.\sqrt{1-y^2}\,] = -\cos\cos^{-1}z$$

$$\Rightarrow \qquad [xy - \sqrt{1-x^2}\,.\sqrt{1-y^2}\,] = -z$$

$$\Rightarrow \qquad xy + z = \sqrt{1-x^2}\,.\sqrt{1-y^2}$$

Taking square on both sides

$$\Rightarrow \qquad (xy+z)^2 = (1-x^2)(1-y^2).$$

$$\Rightarrow \qquad x^2y^2 + z^2 + 2xyz = 1 - y^2 - x^2 + x^2y^2$$

$$\Rightarrow \qquad x^2 + y^2 + 2xyz = 1$$ **Proved.**

***Example 6:*** ***Solve*** $\tan^{-1}\dfrac{x-1}{x-2} + \tan^{-1}\dfrac{x+1}{x+2} = \pi/4.$

**[B.C.A. (Rohilkhand) 2006]**

***Solution:*** $$\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \pi/4$$

$$\because \qquad \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left[\frac{x+y}{1-xy}\right]$$

$\Rightarrow \qquad \tan^{-1}\left[\dfrac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\left(\dfrac{x-1}{x-2}\right)\left(\dfrac{x+1}{x+2}\right)}\right]=\pi/4$

$\Rightarrow \qquad \tan^{-1}\left[\dfrac{2x^2-4}{(x^2-4)-(x^2-1)}\right]=\pi/4$

$\Rightarrow \qquad \tan\dfrac{\pi}{4}=\dfrac{2x^2-4}{x^2-4-x^2+1}$

$\Rightarrow \qquad -3=2x^2-4$

$\Rightarrow \qquad 2x^2=1$

$\Rightarrow \qquad x^2=1/2, x=\pm\sqrt{1/2}.$

***Example 7:*** ***Solve the equation***

$$\mathbf{\cot^{-1} x+\cot^{-1}(n^2-x+1)=\cot^{-1}(n-1).}$$

***Solution:*** $\cot^{-1} x+\cot^{-1}(n^2-x+1)=\cot^{-1}(n-1)$

$\Rightarrow \qquad \cot^{-1}\left\{\dfrac{x(n^2-x+1)-1}{x+n^2-x+1}\right\}=\cot^{-1}(n-1)$

$\Rightarrow \qquad \dfrac{x(n^2-x+1)-1}{n^2+1}=n-1$

$\Rightarrow \qquad x(n^2-x+1)-1=(n^2+1)(n-1)$

$\Rightarrow \qquad x^2-(n^2+1)x+n(n^2-n+1)=0$

$\Rightarrow \qquad (x-n)\{x-(n^2-n+1)\}=0 \qquad \therefore\ x=n, n^2-n+1.$

Hence, $x=n,\ (n^2-n+1)$ are the solutions.

***Example 8:*** ***Solve*** $\mathbf{\sec^{-1}(x/a)-\sec^{-1}(x/b)=\sec^{-1} b-\sec^{-1} a.}$

**[B.C.A (Lucknow) 2008]**

***Solution:*** $\sec^{-1}(x/a)-\sec^{-1}(x/b)=\sec^{-1} b-\sec^{-1} a.$

$\Rightarrow \qquad \sec^{-1}(x/a)+\sec^{-1}a=\sec^{-1}(x/b)+\sec^{-1} b$

$\Rightarrow \qquad \cos^{-1}(a/x)+\cos^{-1}(1/a)=\cos^{-1}(b/x)+\cos^{-1}(1/b),$

$$\Rightarrow \quad \cos^{-1}\left\{\frac{a}{x}\cdot\frac{1}{a}-\sqrt{\left(1-\frac{a^2}{x^2}\right)}\cdot\sqrt{\left(1-\frac{1}{a^2}\right)}\right\}=\cos^{-1}\left\{\frac{b}{x}\cdot\frac{1}{b}-\sqrt{\left(1-\frac{b^2}{x^2}\right)}\cdot\sqrt{\left(1-\frac{1}{b^2}\right)}\right\}$$

$$\Rightarrow \quad \frac{1}{x}-\sqrt{\left(1-\frac{a^2}{x^2}\right)}\cdot\sqrt{\left(1-\frac{1}{a^2}\right)}=\frac{1}{x}-\sqrt{\left(1-\frac{b^2}{x^2}\right)}\cdot\sqrt{1-\frac{1}{b^2}}$$

Taking square on both sides

$$\Rightarrow \quad \left(1-\frac{a^2}{x^2}\right)\left(1-\frac{1}{a^2}\right)=\left(1-\frac{b^2}{x^2}\right)\left(1-\frac{1}{b^2}\right)$$

$$\Rightarrow \quad 1-\frac{1}{a^2}-\frac{a^2}{x^2}+\frac{1}{x^2}=1-\frac{1}{b^2}-\frac{b^2}{x^2}+\frac{1}{x^2}$$

$$\Rightarrow \quad \frac{1}{a^2}-\frac{1}{b^2}=\frac{b^2}{x^2}-\frac{a^2}{x^2}$$

$$\Rightarrow \quad \frac{x^2\,(b^2-a^2)}{a^2b^2}=\frac{(b^2-a^2)}{1}$$

$$\Rightarrow \quad x^2=a^2b^2$$

$$\Rightarrow \quad x=\pm\,ab.$$

If we take $x = -\,ab$, the **L.H.S.** of the given equation

$$\Rightarrow \quad \sec^{-1}(-\,b)-\sec^{-1}(-\,a)$$

$$\Rightarrow \quad (\pi-\sec^{-1}b)-(\pi-\sec^{-1}a)$$

$$\Rightarrow \quad \sec^{-1}a-\sec^{-1}b,$$

which is not equal to the **R.H.S.** of the given equation.

***Example 9:*** ***Find all the positive integral solutions of***

$$\mathbf{\tan^{-1}x+\cot^{-1}y=\tan^{-1}3.}$$

***Solution:*** The given equation is

$$\tan^{-1}x+\cot^{-1}y=\tan^{-1}3$$

$$\Rightarrow \quad \tan^{-1}x+\tan^{-1}(1/y)=\tan^{-1}3$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{x+1/y}{1-x/y}\right\}=\tan^{-1}3$$

$$\Rightarrow \quad \frac{x + 1/y}{1 - x/y} = 3$$

$$\Rightarrow \quad xy + 1 = 3y - 3x$$

$$\Rightarrow \quad x(3 + y) = 3y - 1$$

$$\Rightarrow \quad x = \frac{3y - 1}{3 + y} = 3 - \frac{10}{y + 3}, \text{ (by division)}$$

$\Rightarrow$ $\{3 - 10/(y + 3)\}$ should also be a positive integer. Obviously,

$\{3 - 10/(y + 3)\}$ will be a positive integer if

$$\frac{10}{y + 3} = 1 \text{ or } 2 \text{ } i.e., \text{ if } y = 7 \text{ or } 2$$

and then $x = 2$, or 1.

***Example 10:*** ***Solve*** $\mathbf{\sin^{-1}(5/x) + \sin^{-1}(12/x) = \pi/2}$.

***Solution:*** The given equation is

$$\Rightarrow \quad \sin^{-1}(5/x) = \frac{\pi}{2} - \sin^{-1}(12/x)$$

$$= \cos^{-1}(12/x), \qquad [\because \cos^{-1} x + \sin^{-1} x = \pi/2]$$

$$\Rightarrow \quad \sin^{-1}(5/x) = \sin^{-1}\sqrt{\left(1 - \left(\frac{12}{x}\right)^2\right)}, \qquad [\because \cos^{-1} x = \sin^{-1}\sqrt{1 - x^2}]$$

$$\therefore \quad 5/x = \sqrt{(1 - (12/x)^2))}$$

$$\Rightarrow \quad \frac{25}{x^2} = 1 - \frac{144}{x^2}$$

$$\Rightarrow \quad \frac{169}{x^2} = 1, \quad \text{or} \quad x = \pm 13.$$

***Example 11:*** ***Solve the equation***

$$\mathbf{\tan^{-1} 1/4 + 2\tan^{-1} 1/5 + \tan^{-1} 1/6 + \tan^{-1}(1/x) = \pi/4.}$$

**[B.C.A. (Bundelkhand) 2010, 08]**

***Solution:*** The given equation is

$$\Rightarrow \quad (\tan^{-1} 1/4 + \tan^{-1} 1/6) + (2\tan^{-1} 1/5) + \tan^{-1}(1/x) = \pi/4$$

$\Rightarrow$ $$\tan^{-1}\left(\frac{1/4+1/6}{1-1/4\,.1/6}\right)+\tan^{-1}\left(\frac{2/5}{1-\left(\frac{1}{5}\right)^2}\right)+\tan^{-1}(1/x)=\pi/4$$

$\Rightarrow$ $$(\tan^{-1}10/23+\tan^{-1}5/12)+\tan^{-1}(1/x)=\pi/4$$

$\Rightarrow$ $$\tan^{-1}\left(\frac{10/23+5/12}{1-10/23\,.5/12}\right)+\tan^{-1}(1/x)=\tan^{-1}(1)$$

$\Rightarrow$ $$\tan^{-1}(235/226)+\tan^{-1}(1/x)=\tan^{-1}(1)$$

$\Rightarrow$ $$\tan^{-1}(1/x)=\tan^{-1}(1)-\tan^{-1}(235/226)$$

$\Rightarrow$ $$\tan^{-1}(1/x)=\tan^{-1}\left(\frac{1-235/226}{1+1.235/226}\right)$$

$\Rightarrow$ $$\tan^{-1}(1/x)=\tan^{-1}(-9/461)$$

or $$1/x=-9/461$$

$\Rightarrow$ $x=-461/9$ is the solution of the equation.

***Example 12:* *Prove that***

$$\mathbf{(1/2)\tan^{-1}x=\cos^{-1}\left\{\frac{1+\sqrt{1+x^2}}{2\sqrt{(1+x^2)}}\right\}^{1/2}}.$$

**[B.C.A. (Bhopal) 2008]**

***Solution:*** Let $x=\tan\theta$, so that

$$\textbf{R.H.S.}=\cos^{-1}\left\{\frac{1+\sqrt{(1+\tan^2\theta)}}{2\sqrt{(1+\tan^2\theta)}}\right\}^{1/2}$$

$$=\cos^{-1}\left\{\frac{1+\sec\theta}{2\sec\theta}\right\}^{1/2}$$

$$=\cos^{-1}\left\{\frac{\cos\theta+1}{2}\right\}^{1/2}$$

$$=\cos^{-1}\left\{\frac{2\cos^2\theta/2}{2}\right\}^{1/2}$$

$$=\cos^{-1}(\cos\theta/2)=\theta/2=(1/2)\tan^{-1}x=\textbf{L.H.S.}$$

***Example 13:*** ***Prove that***

$$\tan^{-1}(bc/ar)+\tan^{-1}(ca/br)+\tan^{-1}(ab/cr)=\pi/2,$$

*where* $a^2+b^2+c^2=r^2$. **[B.C.A. (Purvanchal) 2008]**

***Solution:*** **L.H.S.** $=(\tan^{-1} bc/ar+\tan^{-1} ca/br)+\tan^{-1}(ab/cr)$

$$=\tan^{-1}\left\{\frac{bc/ar+ca/br}{1-(bc/ar\,.\,ca/br)}\right\}+\tan^{-1}(ab/cr)$$

$$=\tan^{-1}\left\{\frac{b^2cr+a^2cr}{ab\,(r^2-c^2)}\right\}+\tan^{-1}(ab/cr)$$

$$=\tan^{-1}\left\{\frac{(a^2+b^2)\,cr}{ab\,(a^2+b^2)}\right\}+\tan^{-1}(ab/cr) \qquad [\because\ r^2=a^2+b^2+c^2]$$

$$=\tan^{-1}\frac{cr}{ab}+a\tan^{-1}(ab/cr)$$

$$=\tan^{-1}\left\{\frac{(cr/ab)+ab/cr}{1-\{cr/ab\,.\,ab/cr\}}\right\}$$

$$=\tan^{-1}\infty$$

$$=\tan^{-1}(\tan\pi/2)$$

$$=\pi/2=\textbf{R.H.S.}$$

***Example 14:*** ***Prove that***

$$4\tan^{-1}1/5-\tan^{-1}1/70+\tan^{-1}1/99=\pi/4.$$

***Solution:*** **L.H.S.** $=4\tan^{-1}1/5-\tan^{-1}1/70+\tan^{-1}1/99$

$$=2\,(2\tan^{-1}1/5)-\tan^{-1}1/70+\tan^{-1}1/99$$

$$=2\,.\tan^{-1}\frac{2/5}{1-1/25}-\tan^{-1}\left(\frac{1/70-1/99}{1+1/70\,.1/99}\right)$$

$$=2\tan^{-1}5/12-\tan^{-1}29/6931$$

$$=\tan^{-1}\frac{2.5/12}{1-25/144}-\tan^{-1}29/6931$$

$$=\tan^{-1}120/119-\tan^{-1}1/239$$

$$= \tan^{-1} \frac{(120/119 - 1/239)}{(1 + 120/119 . 1/239)}$$

$$= \tan^{-1} (28561/28561) = \tan^{-1} 1 = \pi/4 = \textbf{R.H.S.}$$

***Example 15:*** ***Prove that***

$$\cot^{-1} \frac{ab+1}{a-b} + \cot^{-1} \frac{bc+1}{b-c} + \cot^{-1} \frac{ca+1}{c-a} = 0.$$

**[B.C.A. (Avadh) 2008]**

***Solution:*** **L.H.S.** $= \cot^{-1} \frac{ab+1}{a-b} + \cot^{-1} \frac{bc+1}{b-c} + \cot^{-1} \frac{ca+1}{c-a}$

$$= \tan^{-1} \frac{a-b}{1+ab} + \tan^{-1} \frac{b-c}{1+bc} + \tan^{-1} \frac{c-a}{1+ca}$$

$$= \tan^{-1} a - \tan^{-1} b + \tan^{-1} b - \tan^{-1} c + \tan^{-1} c - \tan^{-1} a$$

$$= 0 = \textbf{R.H.S.}$$

***Example 16:*** ***Prove that***

$$\tan \{2 \tan^{-1} \alpha\} = 2 \tan (\tan^{-1} \alpha + \tan^{-1} \alpha^3).$$

***Solution:*** **L.H.S.** $= \tan \{2 \tan^{-1} \alpha\}$

$$= \tan \tan^{-1} \left( \frac{2\alpha}{1-\alpha^2} \right)$$

$$= 2\alpha / 1 - \alpha^2$$

**R.H.S.** $= 2 \tan (\tan^{-1} \alpha + \tan^{-1} \alpha^3)$

$$= 2 \tan \tan^{-1} \frac{(\alpha + \alpha^3)}{1 - \alpha^4}$$

$$= \frac{2\alpha (1+\alpha^2)}{(1+\alpha^2)(1-\alpha^2)}$$

$$= 2\alpha / 1 - \alpha^2$$

$\because$ **L.H.S.** $=$ **R.H.S.**

***Example 17:*** ***If*** **sin (π cos θ) = cos (π sin θ),** ***show that***

$$\theta = \pm (1/2) \sin^{-1} (3/4).$$ **[B.C.A. (I.G.N.O.U.) 2007]**

***Solution:*** We have $\sin (\pi \cos \theta) = \cos (\pi \sin \theta)$

*i.e.,* $\sin (\pi \cos \theta) = \sin \left(\frac{\pi}{2} \pm \pi \sin \theta\right)$

$\Rightarrow$ $\pi \cos \theta = \pi/2 \pm \pi \sin \theta$

$\Rightarrow$ $\cos \theta = 1/2 \pm \sin \theta$

$\Rightarrow$ $\cos \theta \pm \sin \theta = 1/2$

$\Rightarrow$ Taking square on both sides

$\Rightarrow$ $(\cos \theta \pm \sin \theta)^2 = 1/4$

$\Rightarrow$ $\cos^2 \theta + \sin^2 \theta \pm 2 \sin \theta \cos \theta = 1/4$

$\Rightarrow$ $1 \pm \sin 2\theta = 1/4$

$\Rightarrow$ $\sin 2\theta = \pm 3/4$

$\Rightarrow$ $2\theta = \sin^{-1} (\pm 3/4)$

$\Rightarrow$ $\theta = \pm 1/2 \sin^{-1} (3/4).$

***Example 18:*** ***Solve*** $\cot^{-1} x + \sin^{-1} (1/\sqrt{5}) = \pi/4.$

***Solution:*** We have $\cot^{-1} x + \sin^{-1} 1/\sqrt{5} = \pi/4$

$\Rightarrow$ *i.e.,* $\tan^{-1} (1/x) + \tan^{-1} \frac{1}{\sqrt{(\sqrt{5})^2 - 1}} = \pi/4$

$\Rightarrow$ $\tan^{-1} (1/x) + \tan^{-1} 1/2 = \pi/4$

$\Rightarrow$ $\tan^{-1} \frac{1/x + 1/2}{1 - 1/x \,.\, 1/2} = \pi/4$

$\Rightarrow$ $\tan \pi/4 = \frac{2 + x}{2x - 1}$

$\Rightarrow$ $2x - 1 = 2 + x$

$\Rightarrow$ $x = 3.$

***Example 19:*** ***Solve*** $\tan^{-1} 1/(2x+1) + \tan^{-1} 1/(4x+1) = \tan^{-1}(2/x^2)$.

***Solution:*** We have

$\Rightarrow \quad \tan^{-1} 1/(2x+1) + \tan^{-1} 1/(4x+1) = \tan^{-1} 2/x^2$

$\Rightarrow \quad \tan^{-1}\left\{\dfrac{1/(2x+1)+1/(4x+1)}{1-1/(2x+1).1/(4x+1)}\right\} = \tan^{-1} 2/x^2$

$\Rightarrow \quad \dfrac{(4x+1)+(2x+1)}{(4x+1)(2x+1)-1} = 2/x^2$

$\Rightarrow \quad \dfrac{6x+2}{8x^2+6x} = 2/x^2$

$\Rightarrow \quad x^2(3x+1) = 8x^2+6x$

$\Rightarrow \quad 3x^3+x^2 = 8x^2+6x$

$\Rightarrow \quad x(3x^2-7x-6) = 0$

$\Rightarrow \quad x(3x+2)(x-3) = 0$

$\Rightarrow \quad x = 0, -2/3, 3$ are the solutions.

***Example 20:*** ***Solve***

$$\sin^{-1} 2a/(1+a^2) + \sin^{-1} 2b/(1+b^2) = 2\tan^{-1} x.$$

***Solution:*** Let $a = \tan\alpha, b = \tan\beta,$

$\Rightarrow \quad \sin^{-1}\left\{\dfrac{2\tan\alpha}{1+\tan^2\alpha}\right\} + \sin^{-1}\left\{\dfrac{2\tan\beta}{1+\tan^2\beta}\right\} = 2\tan^{-1} x$

$\Rightarrow \quad \sin^{-1}\sin 2\alpha + \sin^{-1}\sin 2\beta = 2\tan^{-1} x$

$\Rightarrow \quad 2\alpha + 2\beta = 2\tan^{-1} x$

$\Rightarrow \quad \alpha + \beta = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1} a + \tan^{-1} b = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\dfrac{a+b}{1-ab} = \tan^{-1} x$

$\Rightarrow \quad \dfrac{a+b}{1-ab} = x$

$\Rightarrow \quad \therefore \quad x = (a+b)/(1-ab)$ is the solution.

***Example 21:*** ***Solve*** $\tan^{-1} 1/(a-1) = \tan^{-1} 1/x + \tan^{-1} 1/(a^2 - x + 1)$.

**[B.C.A. (Agra) 2008]**

***Solution:*** We have

$$\Rightarrow \quad \tan^{-1}\frac{1}{a-1} = \tan^{-1}\frac{\frac{1}{x}+\frac{1}{a^2-x+1}}{1-\frac{1}{x}\times\frac{1}{a^2-x+1}}$$

$$\Rightarrow \quad \tan^{-1}\frac{1}{a-1} = \tan^{-1}\frac{(a^2-x+1+x)}{(a^2-x+1)\,x-1}$$

$$\Rightarrow \quad \frac{1}{a-1} = \frac{a^2+1}{(a^2-x+1)\,x-1}$$

$$\Rightarrow \quad \frac{1}{a-1} = \frac{a^2+1}{a^2x-x^2+x-1}$$

$$\Rightarrow \quad a^2x - x^2 + x - 1 = a^3 + a - a^2 - 1$$

$$\Rightarrow \quad x^2 - x\,(a^2+1) + a^3 - a^2 + a = 0$$

$$\Rightarrow \quad (x^2 - a^2) - x\,(a^2+1) + a\,(a^2+1) = 0$$

$$\Rightarrow \quad (x^2 - a^2) - x\,(a^2+1) + a\,(a^2+1) = 0$$

$$\Rightarrow \quad (x+a)\,(x-a) - (a^2+1)\,(x-a) = 0$$

$$\Rightarrow \quad (x-a)\,(x+a-a^2-1) = 0$$

$x = a, x = a^2 - a + 1$ are the solutions.

***Example 22:*** ***Prove that*** $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$. **[B.C.A. (Kanpur) 2008]**

***Solution:*** Let $\tan^{-1} x = A, \tan^{-1} y = B, \tan^{-1} z = C$

$\Rightarrow$ So that $\tan A = x, \tan B = y, \tan C = z$

$\Rightarrow$ Therefore, the given relation $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$

$$\Rightarrow \quad A + B + C = \pi$$

$$\Rightarrow \quad \tan(A+B+C) = \tan\pi$$

$$\Rightarrow \quad \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \,.\, \tan B - \tan B \tan C - \tan C \,.\, \tan A} = 0$$

$$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$\Rightarrow \quad x + y + z = xyz.$$

## Exercise

1. Prove that

$$\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}=\tan^{-1}a-\tan^{-1}c.$$

**[B.C.A. (Rohilkhand) 2010]**

2. Prove that

$$\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1).$$

3. Prove that

$$\tan^{-1}x=2\tan^{-1}[\operatorname{cosec}(\tan^{-1}x)-\tan(\cot^{-1}x)].$$

**[B.C.A. (Kanpur) 2009]**

4. If $\sin^{-1}(x/a)+\sin^{-1}(y/b)=\alpha$, prove that

$$\frac{x^2}{a^2}+\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

5. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi/2$, prove that

$$xy+yz+zx=1.$$ **[B.C.A. (Avadh) 2010]**

6. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, prove that

$$x\sqrt{(1-x^2)}+y\sqrt{(1-y^2)}+z\sqrt{(1-z^2)}=2xyz.$$

7. Prove that

$$2\tan^{-1}[\tan(45°-\alpha)\tan\beta/2]=\cos^{-1}\left[\frac{\sin 2\alpha+\cos\beta}{1+\sin 2\alpha\,.\cos\beta}\right].$$

8. Prove that

$$\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]=\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}].$$

9. Prove that

$$\tan\left[\frac{1}{2}\sin^{-1}\frac{2a}{1+a^2}+1/2\cos^{-1}\frac{1-a^2}{1+a^2}\right]=2a/1-a^2.$$

**[B.C.A. (Meerut) 2006]**

**10.** Prove that

$$2b/a = \tan(\pi/4 + 1/2 \cos^{-1} a/b) + \tan(\pi/4 - 1/2 \cos^{-1} a/b).$$

**11.** Solve the equation

$$\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x.$$

**12.** Solve the equation

$$\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \tan^{-1}(-7).$$

**[B.C.A. (Agra) 2009]**

**13.** Solve

$$2\tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x).$$

**14.** Solve

$$\sin^{-1} x + \sin^{-1} x/2 = \pi/4.$$

**[B.C.A. (Indore) 2009]**

**15.** Solve

$$\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}\frac{2x}{1-x^2}.$$

**16.** Solve

$$\sin[2\cos^{-1}\cot(2\tan^{-1} x)] = 0.$$

**17.** Show that

(i) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

(ii) $3\cos^{-1} x = \cos^{-1}(2x^2 - 1)$.

**18.** Show that

(i) $2\tan^{-1} x = \sin^{-1}\dfrac{2x}{1+x^2}$

(ii) $2\tan^{-1} x = \cos^{-1}\dfrac{1-x^2}{1+x^2}$

(iii) $2\tan^{-1} x = \tan^{-1}\left\{\dfrac{2x}{1-x^2}\right\}$.

**19.** Prove that

$$\tan^{-1}(1/2 \tan 2A) + \{\tan^{-1}(\cot A) + \tan^{-1} (\cot 3A) = 0.$$

**[B.C.A. (Lucknow) 2008]**

**20.** Prove that

$$2 \tan^{-1}[\tan \alpha/2 \tan (\pi/4 - \beta/2)] = \tan^{-1} \left(\frac{\sin \alpha \cos \beta}{\sin \beta + \cos \alpha}\right).$$

**21.** Prove that

$$\text{if } \frac{1}{2} \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{1}{2} \cos^{-1}\left(\frac{1-y^2}{1+y^2}\right) + \frac{1}{3} \tan^{-1}\left(\frac{3z - z^3}{1-3z^2}\right) = 5\pi.$$

Prove that $x + y + z = xyz$.

**22.** Solve the equation

$$\tan^{-1} \frac{\sqrt{(1+x^2)} - \sqrt{(1-x^2)}}{\sqrt{(1+x^2)} + \sqrt{(1-x^2)}} = \beta.$$

**23.** Prove that

$$\frac{a^3}{2} \operatorname{cosec}^2 \left(\frac{1}{2} \tan^{-1} \frac{a}{b}\right) + \frac{b^3}{2} \sec^2 \left(\frac{1}{2} \tan^{-1} \frac{b}{a}\right) = (a+b)(a^2 + b^2).$$

**[B.C.A. (Lucknow) 2006]**

## Answers

| | |
|---|---|
| **11.** | $x = 0, \pm 1/2$. |
| **12.** | $x = 2$. |
| **13.** | $x = n\pi + \pi/4$. |
| **14.** | $x^2 = -\frac{2}{17}(5+2\sqrt{2})$ means $x^2 > 1$ which is not possible because then $\sin^{-1} x$ will have no meaning. Thus, $x^2 \neq \frac{-2}{17}(5+2\sqrt{2})$. |
| **15.** | $x = \frac{(a-b)}{(1+ab)}$. |
| **16.** | $x = \pm 1$ or $\pm (1 \pm \sqrt{2})$. |
| **22** | $x = \pm \sqrt{\sin 2\beta}$. |

OOO

# Chapter 4

# Sequences

## 4.1 Sequence

**[B.C.A. (Delhi) 2012, 09, 08; B.C.A. (Kanpur) 2011, 07; B.C.A. (Agra) 2010, 06, 04]**

Let $S$ be any non-empty set. A function whose domain is the set $N$ of natural numbers and whose range is a subset of $S$, is called a **sequence** in the set $S$.

*Or*

A sequence in a set $S$ is a rule which assigns to each natural number a unique element of $S$.

### 4.1.1 Real Sequence

A sequence whose range is a subset of $R$ is called a **real sequence** or a sequence of real number. A sequence is denoted by $< s_n >$ or $\{s_n\}$, where $s_n$ is the $n$th term of sequence.

*e.g.*, The sequence $< 1, 8, 27, 64, \ldots\ldots n^3 \ldots\ldots >$ or $< n^3 > \forall\, n \in N$.

**NOTE:**

The sequence can be denoted by Recursion formula.

Let $a_1 = 1, a_{n+1} = 3a_n$, for all $n \geq 1$

***Illustration 1:*** $\left\langle \frac{1}{n} \right\rangle$ is the sequence $< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots\ldots\ldots >$.

***Illustration 2:*** $\left\langle \frac{n}{2n+1} \right\rangle$ is the sequence $< \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots . \frac{n}{2n+1} \ldots . >$.

***Illustration 3:*** Let $s_1 = 1$, $s_2 = 1$ and $s_{n+2} = s_{n+1} + s_n, \forall n \geq 1$.

### 4.1.2 Range of Sequence

**[B.C.A. (Meerut) 2012, 09, 00]**

The set of all distinct terms of a sequence is called its **range.**

$\therefore$ The range of sequence $<s_n>$ = The set $\{s_1, s_2, s_3 .....\}$

***Illustration 1:*** The range of sequence $<(-1)^n> = \{-1, 1\}$, a finite set.

***Illustration 2:*** The range of sequence $\left\langle \frac{1}{n+1} \right\rangle = \left\{ \frac{1}{n+1} : n \in N \right\}$ is an infinite set.

### 4.1.3 Constant Sequence

**[B.C.A. (Agra) 2011, 07]**

A sequence $<s_n>$ defined by $s_n = a$, $\forall n \in N$ is called a **constant sequence.**

Thus, the sequence $<s_n> = <a, a, a, .....>$ is a constant sequence. The range of $<s_n> = a$.

### 4.1.4 Equality of Two Sequences

Two sequences $<s_n>$ and $<t_n>$ are said to be equal if $s_n = t_n, \forall n \in N$.

### 4.1.5 Operations of Sequences

Let $<s_n>$ and $<t_n>$ be two sequences. Then:

1. Sum of sequences = $<s_n + t_n>$
2. Difference of sequences = $<s_n - t_n>$
3. Product of sequence = $<s_n t_n>$
4. Quotient of sequences = $\left\langle \frac{s_n}{t_n} \right\rangle$
5. Reciprocal sequences of $<s_n> = \left\langle \frac{1}{s_n} \right\rangle$.

## 4.2 Sub-Sequences

**[B.C.A. (Kanpur) 2009, 07]**

Let $<s_n>$ be any sequence. If $<n_1, n_2, n_3, .. n_k ....>$ be strictly increasing sequence of positive integer *i.e.*, $i > j \Rightarrow n_i > n_j$ then the sequence $\left\langle s_{n_1}, s_{n_2}, s_{n_3}, .... s_{n_k}, ..... \right\rangle$ is called a **sub sequence** of $<s_n>$.

***Illustration 1:*** Let $<s_n> = <1, 0, 1, 0, 1, 0, ......>$

*i.e.*, $s_1 = 1,\ s_2 = 0,\ s_3 = 1,\ s_4 = 0 ......$

If $n_1 = 1,\ n_2 = 3,\ n_3 = 5, \ldots$. Then $< n_r >$ is a sequence of positive integer such that

$$n_1 < n_2 < n_3 \ldots\ldots .$$

Hence $< 1, 1, 1, \ldots\ldots >$ is a sub sequence of $< s_n >$.

***Illustration 2:*** The sequence $< 7^2, 3^2, 15^2, 11^2, 19^2, \ldots >$ is not a subsequence of the sequence. $< 1^2, 2^2, 3^2, 4^2 \ldots\ldots >$.

Here $n_1 = 7,\ n_2 = 3,\ n_3 = 15,\ n_4 = 11, \ldots\ldots$ but $< 7, 3, 15, 11, 19, \ldots\ldots >$ is not strictly increasing sequence of positive integers.

## 4.3 Bounded Sequences

**[B.C.A. (Meerut) 2003, 02]**

### 4.3.1 Bounded Above

A sequence $< s_n >$ is said to be bounded above if the range set of $< s_n >$ is **bounded above** *i.e.,* if there exists a real number $(k_1)$ such that

$$s_n \le k_1 \forall n \in N$$

The number $k_1$ is called an **upper bound** of the sequence $< s_n >$.

### 4.3.2 Bounded Below

A sequence $< s_n >$ is said to be bounded below if the range set of $< s_n >$ is **bounded below** *i.e.,* if there exists a real number $k_2$ such that

$$s_n \ge k_2,\ \forall n \in N$$

The number $k_2$ is called a **lower bound** of the sequence $< s_n >$.

### 4.3.3 Bounded Sequence

**[B.C.A. (Meerut) 2012, 08; B.C.A. (Lucknow) 2010, 04]**

A sequence $< s_n >$ is said to be bounded if the range set of $< s_n >$ is both bounded above and bounded below *i.e.,* if there exists two real numbers $k_1$ and $k_2$ such that

$$k_2 \le s_n \le k_1,\quad \forall n \in N$$

*Or*

A sequence $< s_n >$ is bounded if and only if there exists a real number $k > 0$ such that

$$|s_n| \le k,\quad \forall n \in N$$

It is not necessary that a sequence be bounded above or bounded below.

## 4.4 Unbounded Sequence

**[B.C.A. (Purvanchal) 2012, 09, 07; B.C.A. (Meerut) 2011, 03]**

A sequence $<s_n>$ is said to be **unbounded** if it is either unbounded below or unbounded above.

### 4.4.1 Supremum (Sup) or Least Upper Bound (l. u. b.)

**[B.C.A. (Bundelkhand) 2010; B.C.A. (Agra) 2009]**

The least number say $M$, if exists, of set of the upper bounds of $<s_n>$ is called the **least upper bound** $(l, u, b)$ or the **supremum** (sup) of the sequence $<s_n>$.

### 4.4.2 Infimum (inf) or Greatest Lower Bound (g. l. b.)

**[B.C.A. (Bundelkhand) 2010; B.C.A. (Agra) 2009]**

The greatest number say $m$, if it exists of the set of the lower bounds of $<s_n>$ is called the greatest lower bound $(g.l.b.)$ or the infimum $(inf)$ of the sequence $<s_n>$.

## 4.5 Finite and Infinite Sequence

If the range of a sequence is finite, then it is called **finite sequence**. If the range of sequence is infinite then it is called **infinite sequence**.

***Illustration 1:*** The sequence $\left\langle \frac{1}{n} \right\rangle$ is bounded, since. $\left|\frac{1}{n}\right| \le 1,\ \forall n \in N$.

***Illustration 2:*** The sequence $< n^2 >$ is bounded below by 1 but not bounded above.

***Illustration 3:*** The sequence $<s_n> = <(-1)^n n>$ is neither bounded below nor bounded.

***Illustration 4:*** The sequence $<s_n> = 1 + (-1)^n,\ \forall n \in N$ is bounded, since the range of the sequence is $(0, 2)$ which is finite set.

***Illustration 5:*** The sequence $\left\langle \frac{n}{n+1} \right\rangle$ is bounded, since $\frac{1}{2} \le \frac{n}{n+1} < 1,\ \forall n \in N$.

## *Theorems*

**Theorem 1: A sequence $<s_n>$ is bounded iff, there exists $m \in N$, $l \in R$ and $a > 0$ such that**

$$|s_n - l| < a, \quad \forall n \ge m.$$

**[B.C.A. (Kashi) 2010, 07]**

**Proof:** Let $<s_n>$ be a bounded sequence. Then there exist two real numbers $k_1, k_2$ such that

$$k_1 < s_n < k_2,\ \forall n \in N \qquad ...(1)$$

Subtracting $\frac{k_1+k_2}{2}$ in (1) inequality

$$\Rightarrow k_1 - \frac{k_1+k_2}{2} < s_n - \frac{k_1+k_2}{2} < k_2 - \frac{k_1+k_2}{2} \quad \forall n \in N$$

$$\Rightarrow \frac{k_1-k_2}{2} < s_n - \frac{k_1+k_2}{2} < \frac{k_2-k_1}{2} \quad \forall n \in N$$

$$\Rightarrow -a < s_n - l < a \quad \forall n \in N, \quad \text{where } l = \frac{k_1+k_2}{2}, \; a = \frac{k_2-k_1}{2}$$

$$\Rightarrow |s_n - l| < a \quad \forall n \in N$$

$$\Rightarrow |s_n - l| < a \quad \forall n \geq m, \text{ where } m = 1 \in N, l \in R \text{ and } a > 0$$

**Conversely**, Let there exist $l \in R, a > 0$ and $m \in N$ such that

$$|s_n - l| < a, \; \forall n \geq m$$

$$\Rightarrow \quad l - a < s_n < l + a \quad \forall n \geq m$$

Select $k_1 = \text{Min}\{s_1, s_2, \ldots. s_{m-1}, l-a\}$

then $k_1 \leq s_n, \forall n \in N$

Again select $k_2 = Max\{s_1, s_2, \ldots.. s_{m-1}, l+a\}$

$$\Rightarrow s_n \leq k_2, \forall n \in N$$

$$\therefore \quad k_1 \leq s_n \leq k_2, \quad \forall n \in N$$

$\Rightarrow$ $<s_n>$ is bounded sequence.

## 4.6 Convergent Sequences

**[B.C.A. (Meerut) 2012, 09, 07, 02]**

A sequence $<s_n>$ is said to converge to a number $l$, if for any given $\in > 0$ there exists a positive integer $m$ such that

$$|s_n - l| < \in, \forall n \geq m$$

The number $l$ is called the limit of the sequence $<s_n>$. It can be represented as

$$\lim_{n \to \infty} s_n = l \text{ or } \lim s_n = l$$

The positive integer $m$ depends on $\in$.

**NOTE:**

$$|s_n - l| < \in \Rightarrow l - \in < s_n < l + \in$$

$$\Rightarrow s_n \in ]l - \in, l + \in[ .$$

**Theorem 2: If $< s_n >$ is a sequence of non-negative numbers such that $\lim s_n = l$, then**

$$l \geq 0.$$

**Proof:** Suppose if possible $l < 0 \Rightarrow -l > 0$

Since $l = \lim s_n$, then for every $\in = \frac{-l}{2} > 0$, there exists $m \in N$ such that

$$| s_n - l | < \frac{-l}{2}, \ \forall n \geq m$$

In particular, $| s_m - l | < -\frac{l}{2}$

$\Rightarrow$ $l + \frac{l}{2} < s_m < l - \frac{l}{2}$

$\Rightarrow$ $s_m < \frac{l}{2} \quad or \quad s_m < 0, \quad as \ l < 0$

But $s_m \geq 0$, which is contradiction so $l \geq 0$.

**Theorem 3: A sequence cannot converge to more than one limit *i.e.*, the limit of a sequence is unique.**

**[B.C.A. (Meerut) 2012, 09, 08, 07, 05, 02, 01; B.C.A. (Agra) 2012, 09, 08, 06, 04; B.C.A. (Rohilkhand) 2012, 09; B.C.A. (Kanpur) 2010, 07]**

**Proof:** Let us consider the sequence $< s_n >$ converge to two different number $l$ and $l_1$. Since $l \neq l_1$, therefore $| l - l_1 | > 0$.

Let $\in = \frac{1}{2} | l - l_1 |$, then $\in > 0$

Since $< s_n >$ converges to $l$ then for every $\in > 0$ there exists $m_1 \in N$ such that

$$| s_n - 1 | < \in, \ \forall n \geq m_1 \quad \text{...(1)}$$

Again, since $< s_n >$ converges to $l_1$ then for every $\in > 0$ there exists $m_2 \in N$ such that

$$| s_n - l_1 | < \in, \ \forall n \geq m_2 \quad \text{...(2)}$$

Let $m = \max(m_1, m_2)$ then $n \geq m$ for (1) and (2)

$\therefore$ $| l - l_1 | = |(s_n - l) - (s_n - l_1)| \leq | s_n - l | + | s_n - l_1 |$

$< \in + \in = | l - l_1 |$

Thus $| l - l_1 | < | l - l_1 |$ which is not possible so $l = l_1$ *i.e.*, the limit of sequence is unique.

**Theorem 4: If $<s_n>$ converge to l, then any subsequence of $<s_n>$ also converge to l.**

**Proof:** Let $<s_{n_k}>$ be any subsequence of $<s_n>$. Then by definition as subsequence $n_1, n_2, \ldots\ldots n_k, \ldots\ldots\ldots$ are positive integers such that $n_1 < n_2 \ldots\ldots\ldots < n_k < \ldots\ldots$

Now $$n_1 \geq 1 \Rightarrow n_k \geq k \text{ (by induction)}$$

Since $<s_n>$ converge to $l$, so given $\in > 0$, there exist a positive integer $m$ such that

$$|s_k - l| < \in, \forall k \geq m$$

For $k \geq m$, we have $n_k \geq k \geq m$

$\therefore$ $$|s_{n_k} - l| < \in, \forall n_k \geq m$$

$\Rightarrow$ $$<s_{n_k}> \text{ converge to } l.$$

**Theorem 5: If the subsequences $<s_{2n-1}>$ and $<s_{2n}>$ of the sequence $<s_n>$ converge to the same limit l, then the sequence $<s_n>$ converge to l.** **[B.C.A. (Lucknow) 2012, 10]**

**Proof:** Let $\in > 0$ be given, then, since limit $(s_{2n-1}) = l$, there exists $m_1 \in N$ such that

$$|s_{2n-1} - l| < \in \quad \forall n \geq m_1$$

Similarly $$\lim s_{2n} = l$$

$\Rightarrow$ $$|s_{2n} - l| < \in \quad \forall n \geq m_2$$

Let $$m = \max(m_1, m_2),$$

Then $$|s_{2n-1} - l| < \in, \text{ and } |s_{2n} - l| < \in, \quad \forall n \geq m$$

$\therefore$ $$|s_n - l| < \in \quad \forall n \geq 2m - 1$$

Hence, $<s_n>$ converges to $l$.

**Theorem 6: Every convergent sequence is bounded.**

**[B.C.A. (Meerut) 2010, 08, 05, 04, 02, 00; B.C.A. (Rohilkhand) 2012, 09, 05, 00; B.C.A. (Agra) 2009]**

*Or*

**Show that every convergent sequence is bounded. Is converse true ? Give reasons for your answer.** **[B.C.A. (Kanpur) 2012, 08, 04, 01; B.C.A. (Meerut) 2008]**

**Proof:** Let $<s_n>$ be a sequence which converges to $l$.

Let $\in = 1 > 0$, then there exists a positive integer $m$ such

$$|s_n - l| < 1, \forall n \geq m$$

$\Rightarrow$ $$l - 1 < s_n < l + 1, \forall n \geq m$$

Let $$k = \min \{s_1, s_2, \ldots s_{m-1}, l-1\}$$

$$K = \max \{s_1, s_2, \ldots s_{m-1}, l+1\}$$

$\therefore$ $$k \le s_n \le K, \quad \forall n \in N$$

Hence, the sequence $< s_n >$ is bounded.

The converse of the above theorem need not be true. That is, a bounded sequence need not be convergent. For example, the sequence $< (-1)^n >$ is bounded but is not convergent.

## 4.7 Null Sequence

A sequence $< s_n >$ is called **null sequence** if $\lim s_n = 0$.

### Solved Examples

***Example 1:*** ***If*** $s_n = \dfrac{n^3 - 2n + 1}{n^3 + 2n^2 - 1}$, ***prove that*** $\lim s_n = 1$.

**[B.C.A. (Delhi) 2012, 10, 08, 06, 04; B.C.A. (Kashi) 2012; B.C.A. (Purvanchal) 2010]**

***Solution:*** We have $l$ is the limit of $s_n$ if $\lim\limits_{n\to\infty} s_n = l$

$\therefore$ $$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left\{\frac{n^3 - 2n + 1}{n^3 + 2n^2 - 1}\right\}$$

$$= \lim_{n\to\infty} \left\{\frac{n^3\left(1 - \dfrac{2}{n^2} + \dfrac{1}{n^3}\right)}{n^3\left(1 + \dfrac{2}{n} - \dfrac{1}{n^3}\right)}\right\}$$

$$= \lim_{n\to\infty} \left\{\frac{1 - \dfrac{2}{n^2} + \dfrac{1}{n^3}}{1 + \dfrac{2}{n} - \dfrac{1}{n^3}}\right\}$$

$$= \frac{1 - 2\cdot 0 + 0}{1 + 0 - 0} = 1.$$

***Example 2:*** ***If*** $s_n = \dfrac{(3n-1)(n^4 - n)}{(n^2 + 2)(n^3 + 1)}$, ***prove that*** $\lim s_n = 3$.

**[B.C.A. (Agra) 2008, 06]**

***Solution:*** We know $l$ is the limit of $s_n$ if $\lim\limits_{n\to\infty} s_n = l$.

$$\therefore \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left\{\frac{(3n-1)(n^4-n)}{(n^2+2)(n^3+1)}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{3n\left(1-\frac{1}{3n}\right)n^4\left(1-\frac{1}{n^3}\right)}{n^2\left(1+\frac{2}{n^2}\right)n^3\left(1+\frac{1}{n^3}\right)}\right\}$$

$$= \frac{3\,(1-0)\,(1-0)}{(1+0)\,(1+0)}$$

$$= 3.$$

***Example 3:*** ***If*** $s_n = \left\langle \dfrac{n^2+3n+5}{2n^2+5n+7} \right\rangle$ ***then show*** $<s_n>$ ***converge to*** $\dfrac{1}{2}$.

**[B.C.A. (Rohilkhand) 2011, 08, 05, 02]**

***Solution:*** We know $l$ is the limit of $s_n$ if $\lim\limits_{n\to\infty} s_n = l$.

$$\therefore \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left\{\frac{n^2+3n+5}{2n^2+5n+7}\right\}$$

$$= \lim_{n\to\infty}\frac{n^2\left(1+\frac{3}{n}+\frac{5}{n^2}\right)}{2n^2\left(1+\frac{5}{2n}+\frac{7}{2n^2}\right)}$$

$$= \frac{1+0+0}{2\,(1+0+0)}$$

$$= \frac{1}{2}.$$

***Example 4:*** ***Prove that the sequence*** $<s_n>$ ***where*** $s_n = \dfrac{n}{n^2+1}$ ***is convergent.***

**[B.C.A. (Rohtak) 2012]**

***Solution:*** The sequence converge to $l$ if $\lim\limits_{n\to\infty} s_n = l$.

$$\therefore \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left\{\frac{n}{n^2+1}\right\} = \lim_{n\to\infty}\left\{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{n\left(1+\frac{1}{n^2}\right)}\right\}$$

$$= 0.$$

**Another Solution:** By using the definition of the limit of a sequence we shall show the given sequence converge to zero.

We have $\quad |s_n - 0| < \in$

$\Rightarrow \quad |s_n| < \in$

$\Rightarrow \quad \left|\frac{n}{n^2+1}\right| < \in$

$\Rightarrow \quad \frac{n}{n^2+1} < \frac{n}{n^2} = 1/n < \in$, provided $n > \frac{1}{\in}$

If we take $m \in N$ such that $m > \frac{1}{\in}$, then $|s_n - 0| < \in \quad \forall n \geq m$.

$\therefore \quad \lim s_n = 0.$

**Example 5:** ***Show that the sequence*** $< s_n >$ ***where*** $s_n = \frac{3n}{n + 5n^{1/2}}$ ***has limit*** 3.

**[B.C.A. (Bundelkhand) 2012; B.C.A. (Meerut) 2010, 08, 06, 05]**

***Solution:*** We know $l$ is the limit of $s_n$ if $\lim\limits_{n\to\infty} s_n = l$.

$$\therefore \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left\{\frac{3n}{n+5n^{1/2}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{3n}{n(1+5n^{-1/2})}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{3}{1+\frac{5}{\sqrt{n}}}\right\}$$

$$= \frac{3}{1+0} = 3.$$

**Another Solution:** Let $\in > 0$ be given

We have $\quad |s_n - 3| = \left|\frac{3n}{n+5n^{1/2}} - 3\right|$

$$= \left|\frac{3n - 3n - 15n^{1/2}}{n+5n^{1/2}}\right|$$

$$= \frac{15n^{1/2}}{n+5n^{1/2}} < \frac{15n^{1/2}}{n}$$

$\therefore$ $$= |s_n - 3| < \frac{15}{n^{1/2}}$$

If we take $\dfrac{15}{n^{1/2}} < \in$ *i.e.,* if $n > \dfrac{225}{\in^2}$

If we choose a positive integer $m > \dfrac{225}{\in^2}$, then

$$|s_n - 3| < \in \quad \forall n \ \geq m.$$

$\Rightarrow$ $$\lim_{n\to\infty} s_n = 3$$

$\Rightarrow$ $$\lim_{n\to\infty} \left\{\frac{3n}{n + 5n^{1/2}}\right\} = 3.$$

**Example 6:** ***Show that the sequence*** $< s_n >$ ***where*** $s_n = \dfrac{2n^2+1}{2n^2-1}$ ***converge to*** **1.**

**[B.C.A. (I.G.N.O.U. ) 2012, 10, 07]**

***Solution:*** We know $l$ is the limit of $s_n$, if $\lim\limits_{n\to\infty} s_n = l$.

$\therefore$ $$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left\{\frac{2n^2+1}{2n^2-1}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{2n^2\left(1+\frac{1}{2n^2}\right)}{2n^2\left(1-\frac{1}{2n^2}\right)}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1+\frac{1}{2n^2}}{1-\frac{1}{2n^2}}\right\} = \frac{1+0}{1-0} = 1.$$

**Another Solution:** Let $\in > 0$ be given

We have $$|s_n - 1| = \left|\frac{2n^2+1}{2n^2-1} - 1\right| = \left|\frac{2}{2n^2-1}\right|$$

$$= \frac{2}{2n^2-1}$$

If we take $\dfrac{2}{2n^2-1} < \in$ *i.e.,* $n > \sqrt{\dfrac{2+\in}{2\in}}$

and $m > \sqrt{\dfrac{2+\in}{2\in}}$, then for all $n \geq m$.

$\therefore \qquad \left| s_n - 1 \right| < \in \quad \forall n \geq m$

$\Rightarrow \qquad \lim s_n = 1.$

**Example 7:** ***Show that sequence*** $< \frac{1}{n} >$ ***has the limit*** **0.**

**Solution:** We know $l$ is the limit of $s_n$, if $\lim\limits_{n\to\infty} s_n = l$

$$\therefore \qquad \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

**Another Solution:** Let $\in > 0$ be given

We have $\qquad |s_n - 0| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \in$ where $\frac{1}{n} < \in$

When $n > \frac{1}{\in}$ if we take $m > \frac{1}{n}$: Then $n \geq m$. We have $\left| \frac{1}{n} - 0 \right| < \in \;\; \forall n \geq m.$

$\Rightarrow \qquad < \frac{1}{n} >$ converge to 0.

**Example 8:** ***The sequence*** $< s_n >$ ***where*** $s_n = \frac{1}{2^n}$ ***converges to*** **0.**

**[B.C.A. (Indore) 2008]**

**Solution:** We know $l$ is the limit of $s_n = \frac{1}{2^n}$ if $\operatorname{Lim}\limits_{n\to\infty} s_n = l.$

$$\therefore \qquad \operatorname*{Lim}_{n\to\infty} s_n = \operatorname*{Lim}_{n\to\infty} \frac{1}{2^n} = \frac{1}{\infty} = 0.$$

**Another Solution :** We have

$$|s_n - 0| = \left| \frac{1}{2^n} - 0 \right| = \frac{1}{2^n}$$

If we take $\frac{1}{2^n} < \in$ *i.e.*, $\in > \frac{1}{2^n}$

$$\Rightarrow \qquad 2^n > \frac{1}{\in} \Rightarrow n \log 2 > \log(1/\in)$$

$$\Rightarrow \qquad n > \frac{1}{\log 2} \log(1/\in)$$

Let us consider the positive integer $m > \frac{\log(1/\in)}{\log 2}$, then

$$|s_n - 0| < \in, \forall n \geq m.$$

***Example 9:*** ***Find $m \in N$ such that*** $\left| \dfrac{2n}{n+3} - 2 \right| < \dfrac{1}{5} \quad \forall n \geq m.$

***Solution:*** We have $\left| \dfrac{2n}{n+3} - 2 \right| < \dfrac{1}{5}$

$$\Rightarrow \qquad \left| \frac{2n-2n-6}{n+3} \right| < \frac{1}{5}$$

$$\Rightarrow \qquad \frac{6}{n+3} < \frac{1}{5} \Rightarrow \frac{n+3}{6} > 5 \quad \Rightarrow n > 27$$

If we take a positive integer $m > 27$, we have

$$\left| \frac{2n}{n+3} - 2 \right| < \frac{1}{5}, \quad \forall n \geq m$$

Hence for $\in = \dfrac{1}{5}$, the required least value of $m = 28$.

***Example 10:*** ***Show that the sequence $<s_n>$ defined by $s_n = \sqrt{n+1} - \sqrt{n}$, $\forall n \in N$ convergent.***

**[B.C.A. (Rohtak) 2010, 08, 04; B.C.A. (Meerut) 2009, 07, 06]**

***Solution:*** We have
$$s_n = \sqrt{n+1} - \sqrt{n}$$
$$= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$
$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\therefore \qquad \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1}{(\sqrt{n+1} + \sqrt{n})} = \frac{1}{\infty} = 0$$

Hence, $<s_n>$ converge to zero.

**Another Solution:** Let $\in > 0$ be given

We have
$$s_n = \frac{1}{\sqrt{n} + \sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

$$\therefore \qquad | s_n - 0 | < \in$$

$$\Rightarrow \qquad \frac{1}{\sqrt{n}} < \in$$

Provided
$$\sqrt{n} > \frac{1}{\in} \Rightarrow n > \frac{1}{\in^2}$$

If $m$ is a positive integer greater than $\dfrac{1}{\in^2}$, then

$$| s_n - 0 | < \in \quad \forall n \geq m$$

$$\therefore \qquad \text{Lim } s_n = 0.$$

**Example 11:** ***Show that Lim*** $n\sqrt{n} = 1.$

**[B.C.A. (Lucknow) 2010, 08; B.C.A. (Kanpur) 2009, 06, 05]**

***Solution:*** We have $s_n = n\sqrt{n} = (n)^{1/n}$

Taking log on both sides

$$\log s_n = \frac{1}{n}\log n$$

$$\log s_n = \frac{\log n}{n}$$

Taking limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} \log(s_n) = \lim_{n\to\infty} \frac{\log n}{n}, \qquad \left(\frac{\infty}{\infty}\right) \text{forms}$$

Then applying L Hospital rule

$$\therefore \quad \lim_{n\to\infty} \log(s_n) = \lim_{n\to\infty} \left(\frac{1/n}{1}\right) = \frac{1}{\infty} = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} s_n = e^0$$

$$\Rightarrow \quad \lim_{n\to\infty} s_n = 1$$

$$\Rightarrow \quad \lim_{n\to\infty} (n)^{1/n} = 1.$$

**Another Solution**: Let $(n)^{1/n} = 1 + h_n$ where $h_n \geq 0$

$$\therefore \quad n = (1+h_n)^n = 1 + nh_n + \frac{n(n-1)}{1\cdot 2}h_n^2 + \ldots\ldots + h_n^n$$

$$> \frac{n(n-1)}{1\cdot 2}h_n^2 \text{ for all } n.$$

$$\Rightarrow \quad h_n^2 < \frac{2}{n-1}, \text{ for } n \geq 2$$

$$\therefore \quad |h_n| < \sqrt{\frac{2}{n-1}} \text{ for } n \geq 2$$

Let $\in > 0$ be given then $|h_n| < \sqrt{\frac{2}{n-1}} < \in$

*i.e.,* $$\frac{2}{n-1} < \in^2 \Rightarrow n > \frac{2}{\in^2} + 1$$

If we take $m > \frac{2}{\in^2} + 1$, then we have

$$|h_n| < \in \quad \forall n \geq m$$

$$|n\sqrt{n} - 1| < \in \quad \forall n \geq m$$

$\therefore$ $$\text{Lim } n\sqrt{n} = 1.$$

***Example 12:*** ***Show that the sequence*** $< s_n >$ ***defined by*** $s_n = r^n$ ***converge to zero if*** $|r| < 1$.

***Solution:*** If $|r| < 1$ then $|r| = \frac{1}{1+h}, \quad h > 0$

$\Rightarrow$ $$(1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \ldots + h^n \geq 1 + nh$$

Now $$|s_n - 0| = |r^n| = |r|^n = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh}, \ \forall n$$

Let $\in > 0$ be given. Then $|s_n - 0| < \in$ is $\frac{1}{1+nh} < \in$

*i.e.,* $$n > \left(\frac{1}{\in} - 1\right)/h$$

If we take positive integer $m > \left(\frac{1}{\in} - 1\right)/h, \ \forall n \geq m.$

$\therefore$ $$|s_n - 0| < \in, \quad \forall n \geq m$$

$\Rightarrow$ $< s_n >$ converge to zero.

***Example 13:*** ***Show that the sequence*** $< s_n >$, ***where*** $s_n = \frac{(-1)^{n-1}}{n}$ ***converge to*** **0.**

***Solution:*** Let $\in > 0$ be given

We have $$|s_n - 0| = |s_n| = \left|\frac{(-1)^{n-1}}{n}\right| = \frac{1}{n} < \in, \text{ if } n > \frac{1}{\in}$$

If we take $m > \frac{1}{\in}$, then

$$|s_n - 0| < \in \quad \forall n \geq m$$

Hence, $< s_n >$ converges to zero.

**Example 14:** ***Prove that if x be any real number, then***

$$\lim_{n\to\infty} \frac{x^n}{(n)!} = 0.$$

**[B.C.A. (Meerut) 2007]**

**Solution:** Let $s_n = \dfrac{x^n}{(n)!}$

If $x = 0$, then $s_n = 0 \ \ \forall n \in N$ and so $\lim s_n = 0$. So let $x \neq 0$

Now we know that if $s_n \neq 0$ for any $n$ and $\lim \dfrac{s_{n+1}}{s_n} = l$ where $|l| < 1$, then $\lim s_n = 0$

Now $x \neq 0 \Rightarrow s_n \neq 0, \ \forall n \in N$

We have $\dfrac{s_{n+1}}{s_n} = \dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{(n)!}{x^n} = \dfrac{x}{n+1}$

$$\therefore \quad \lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to\infty} \frac{x}{n+1} = 0$$

and $|0| = 0 < 1$ then $\lim s_n = 0$.

**Example 15:** ***If*** $s_n = \dfrac{n}{2^n}$***, prove that*** $< s_n > \Rightarrow 0$.

**[B.C.A. (Indraprastha) 2012]**

**Solution:** Here $s_n \neq 0 \ \ \forall n \in N$

Also $\dfrac{s_{n+1}}{s_n} = \dfrac{(n+1)}{2^{n+1}} \times \dfrac{2^n}{n} = \dfrac{1}{2} \dfrac{(n+1)}{n}$

$$= \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

$$\therefore \quad \lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}(1+0) = \frac{1}{2} \text{ and } \left|\frac{1}{2}\right| = \frac{1}{2} < 1,$$

Then $\lim\limits_{n\to\infty} s_n = 0.$

**Example 16:** ***Find the value of*** $\lim\limits_{n\to\infty} \dfrac{3 + 2\sqrt{n}}{\sqrt{n}}$.

**[B.C.A. (Meerut) 2003]**

**Solution:** We have

$$\lim_{n\to\infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}\left(\dfrac{3}{\sqrt{n}} + 2\right)}{\sqrt{n}}$$

$$= \lim_{n\to\infty}\left(\frac{3}{\sqrt{n}}+2\right)$$

$$=0+2$$

$$=2.$$

**Example 17:** ***Show that*** $< s_n >$ ***where*** $s_n = n^{-n-1}(n+1)^n$ ***converges. Find its limit also.***

**Solution:** We have **[B.C.A. (Garhwal) 2006; B.C.A. (Delhi) 2008]**

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} n^{-n-1}(n+1)^n = \lim_{n\to\infty}\frac{(n+1)^n}{n^{n+1}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\frac{1}{n}.\left(1+\frac{1}{n}\right)^n$$

$$=0\times e=0$$

$\Rightarrow$ limit of the sequence $< s_n >$ exists

$\Rightarrow$ $< s_n >$ is convergent.

**Example 18:** ***Show that*** $\left\langle \frac{3n}{3n+1}\right\rangle$ ***is a subsequence of*** $\left\langle \frac{n}{n+1}\right\rangle$.

**[B.C.A. (Agra) 2009, 05; B.C.A. (Rohilkhand) 2011, 08, 00]**

**Solution:** We have $\left\langle \frac{n}{n+1}\right\rangle = \left\langle \frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots\ldots\ldots\right\rangle$

Also $\left\langle \frac{3n}{3n+1}\right\rangle = \left\langle \frac{3}{4},\frac{6}{7},\frac{9}{10},\ldots\ldots\right\rangle$

In general the sequence, $\left\langle \frac{n}{n+1}\right\rangle$ contains terms of every $n \in N$, while $\left\langle \frac{3n}{3n+1}\right\rangle$ contains the terms obtained by putting number 3 $n$ only multiple of 3. Hence the sequence $\left\langle \frac{3n}{3n+1}\right\rangle$ contains $s_3, s_6, s_9\ldots\ldots$ terms of the sequence $< s_n > = \left\langle \frac{n}{n+1}\right\rangle$.

**Example 19:** ***Show that the sequence*** $\left\langle \frac{1}{2n+1}\right\rangle$ ***is subsequence of the sequence*** $\left\langle \frac{1}{n}\right\rangle$.

**[B.C.A. (Garhwal) 2005]**

**Solution:** We know that sequence $< t_n >$ is a subsequence of a sequence $< s_n >$ if there exists a sequence $< a_n >$ such that $a_n < a_{n+1}$ and $t_n = s_{a_n}\ \forall n \in N$.

Here, let $s_n = \frac{1}{n}, t_n = \frac{1}{2n-1}$ $\forall n \in N$. Then

$$t_n = \frac{1}{2n-1} = s_{2n-1} = s_{a_n}, \text{ where } a_n = 2n-1 \quad \forall\, n \in N$$

Now, $a_n = 2n-1$, $a_{n+1} = 2(n+1)-1 = 2n+1 \Rightarrow a_n < a_{n+1}$, $\forall n \in N$

Thus, we have got a sequence $< a_n >$ such that $a_n \in N$ and $a_n < a_{n+1}$.

Therefore, the sequence $\left\langle \frac{1}{2n-1} \right\rangle$ is a subsequence of the sequence $\left\langle \frac{1}{n} \right\rangle$.

***Example 20:*** ***Show that the sequence*** $< s_n >$ ***where*** $s_n = \frac{(-1)^n}{n}$ ***converges.***

***Solution:*** We have $s_n = \frac{1}{n}$ if $n$ is even and $s_n = \frac{-1}{n}$ if $n$ is odd. The sequences $< s_{2n} >$ and $< s_{2n-1} >$ are subsequence of sequence $< s_n >$.

Now $$< s_{2n} > = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, .....\right\}$$

And $$< s_{2n-1} > = \left\{-1, \frac{-1}{3}, \frac{-1}{5}, ....\right\}$$

$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} \frac{(-1)^{2n}}{2n} = \lim_{n\to\infty} \left(\frac{1}{2n}\right) = 0$$

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} \frac{(-1)^{2n+1}}{2n+1} = \lim_{n\to\infty} \frac{-1}{(2n+1)} = 0$$

$$\therefore \quad \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n+1} = 0$$

But $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, .....\right\}$ and $\left\{-1, \frac{-1}{3}, .......\right\}$ are subsequences of sequence $\left\langle \frac{1}{n} \right\rangle$. Since $< s_{2n} >$ and $< s_{2n-1} >$ both converge to the same limit 0. Then the sequence $< s_n >$ also converges to 0.

## Exercise 4.1

1. Write the formula for the nth terms $s_n$ for each of the following sequences:

   (i) $1, -4, 9, -16, 25, -36, .....$ (ii) $1, 0, 1, 0, 1, 0, .......$ (iii) $1, 3, 6, 10, 15, ......$

2. Find whether the following sequences are bounded above or below:

   (i) $< \frac{(-1)^n}{n} >$ (ii) $< 2^n >$ (iii) $< n! >$

3. Use the definition of the limit of a sequence to show that the limit of the sequence $< s_n >$ where $s_n = \dfrac{2n}{n+3}$ is 2.

4. Show that the sequence $< s_n >$ where $s_n = \dfrac{n}{n+1}$ converges to 1.

5. If $s_n = \sqrt{n+1} - \sqrt{n}$, prove that $\lim s_n = 0$. **[B.C.A. (Kurukshetra) 2012]**

6. If $s_n = \dfrac{2n}{n+4n^{1/2}}$, prove that $< s_n >$ is convergent.

7. Prove that the sequence $< s_n >$ where $s_n = \dfrac{n}{n^2+1}$ is convergent.

8. Define with examples:

   (i) Real sequence (ii) Range of sequence. **[B.C.A. (Meerut) 2012, 05]**

## Answers 4.1

| | | |
|---|---|---|
| **1.** | (i) $s_n = (-1)^{n-1} n^2$ | (ii) $s_n = 1$ if $n$ is odd, $s_n = 0$ if $n$ is even |
| | (iii) $s_n = \dfrac{n(n+1)}{2}$ | |
| **2.** | (i) Bounded above as well as bounded below | (ii) Bounded below but not above |
| | (iii) Bounded below but not above | |

## 4.8 Divergent Sequences

1. **Diverge to $+\infty$.** A sequence $< s_n >$ is said to diverge to $(+\infty)$ if for any given $k > 0$ (however large), there exists $m \in N$ such that $s_n > k$, $\forall n \geq m$.

   If $< s_n >$ diverges to infinity, we write $s_n \to \infty$ as $n \to \infty$ or $\text{Lim}\ s_n = +\infty$.

2. **Diverge to $-\infty$.** A sequence $< s_n >$ is said to diverge to $-\infty$ if for any given $k < 0$ (however small), there exists $m \in N$, such that $s_n < k$, $\forall n \geq m$.

   If $< s_n >$ diverges to $-\infty$, we write

   $s_n \to -\infty$ as $n \to \infty$ or $\lim s_n = -\infty$

***Illustration 1:*** $< 3, 3^2, 3^3 \ldots\ldots 3^n, \ldots\ldots\ldots >$ diverges to $(+\infty)$.

***Illustration 2:*** $< -2, -4, -6, \ldots\ldots -2n, \ldots\ldots\ldots >$ diverges to $(-\infty)$.

***Illustration 3:*** $< -x, -x^2, -x^3 \ldots\ldots -x^n, \ldots\ldots >$, $x > 1$ diverges to $(-\infty)$.

**Theorem 7: If a sequence $< s_n >$ diverges to infinity then any sequence of $< s_n >$ also diverges to infinity.**

**Proof:** Let $< s_{n_k} >$ be subsequence of the sequence $< s_n >$. Then by definition of a subsequence $< n_1, n_2, \ldots n_k, \ldots. >$ is a strictly increasing sequences of positive integers.

$\Rightarrow$ $$n_1 \geq 1 \Rightarrow n_k \geq k \text{ (by induction)}$$

Take any given positive real number $k_1$.

Now $< s_n >$ diverges to $\infty \Rightarrow k_1 > 0$ there exists $m \in N$ such that $s_n > k_1$, $\forall n \geq m$ *i.e.,* $s_k > k_1 \ \forall \, n \geq m$

For $k \geq m$, we have $n_k \geq k \geq m$ *i.e.,* $n_k \geq m$

$\therefore$ $$s_{n_k} > k_1 \quad \forall n_k \geq m$$

$\therefore$ $$< s_{n_k} > \text{ diverges to infinite.}$$

***Example 21:*** ***Prove that the sequence $< n^P >$ where $p > 0$ diverges to infinity.***

**[B.C.A. (Bhopal) 2008, 06]**

***Solution:*** Let $s_n = n^P$. Then $s_n > 0 \ \ \forall n \in N$ and $p > 0$

$\therefore$ The sequence $< \frac{1}{s_n} > = < \frac{1}{n^P} >$ exists

Since we know that $$\frac{1}{n^P} \to 0 \text{ as } n \to \infty$$

$\therefore$ $$n^P \to \infty \text{ as } n \to \infty$$

Hence $< n^P >$ diverges to $\infty$.

***Example 22:*** ***Show that the sequence $\left\langle \log \left(\frac{1}{n}\right) \right\rangle$ diverge to $-\infty$.***

**[B.C.A. (Indore) 2010]**

***Solution:*** Let $$s_n = \log\left(\frac{1}{n}\right), \text{ take any given } k < 0.$$

Then $$s_n < k \text{ if } \log\left(\frac{1}{n}\right) < k \ \ \textit{i.e.,} \text{ if } -\log n < k$$

*i.e.,* if $$\log n > -k \quad \textit{i.e,} \quad \text{if } n > e^{-k}$$

If we take $m \in N$ such that $m > e^{-k}$, then $s_n < k \quad \forall n \geq m$

Hence, $$s_n \to -\infty \text{ as } n \to \infty.$$

**Example 23:** ***If $< t_n >$ diverges to $\infty$ and $s_n > t_n$ $\forall n$, then $< s_n >$ diverges to $\infty$.***

***Solution:*** Let given $k > 0$

Since $< t_n >$ diverge to $\infty$, therefore for $k > 0$ there exists $m \in N$ such that

$$t_n > k \;\forall\; n \geq m$$

$\Rightarrow$ $$s_n > k \;\forall n \geq m$$

Hence, $< s_n >$ diverges to $\infty$.

**Theorem 8: (Sandwich Theorem)**

**If $< s_n >$, $< t_n >$ and $< u_n >$ are three sequences such that**

**(i) for some positive integer $k$, $s_n \leq u_n \leq t_n$, for $n \geq k$**

**(ii) $\lim s_n = \lim t_n$ then $\lim u_n = l$.**

**Proof:** Let given $\in > 0$ be given

Since $\lim s_n = l$, therefore, there exists $m_1 \in N$ such that

$$| s_n - l | < \in, \quad \forall n \geq m_1$$

$\Rightarrow$ $$l - \in < s_n < l + \in, \quad \forall n \geq m_1 \qquad \text{...(1)}$$

Again $\lim t_n = l$, therefore, there exists $m_2 \in N$ such that

$$| t_n - l | < \in, \quad \forall n \geq m_2$$

$\Rightarrow$ $$l - \in < t_n < l + \in, \forall n \geq m_2 \qquad \text{...(2)}$$

Let $m = \max\{m_1, m_2, k\}$. Then, for $n \geq m$, we have

$$l - \in < s_n \leq u_n \leq t_n < l + \in$$

or $$l - \in < u_n < l + \in$$

Thus $$| u_n - l | < \in, \quad \forall n \geq m$$

Hence, $$\lim u_n = l.$$

**Example 24:** ***Prove that the sequence $< s_n >$ where $s_n = \dfrac{n}{n^2+1}$ is convergent.***

***Solution:*** Let $$s_n = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

Thus $$0 < s_n < 1/n, \quad \forall n \in N$$

Since $\operatorname*{Lim}_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \dfrac{1}{n} = 0$, therefore by Sandwich theorem $\lim_{n\to\infty} s_n = 0$.

**Example 25:** ***Show that the sequence*** $< s_n >$ ***defined by***

$$s_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots\ldots + \frac{1}{\sqrt{n^2+n}} \text{ converges to } 1.$$

**[B.C.A. (Lucknow) 2011; B.C.A. (Kanpur) 2010; B.C.A. (Delhi) 2008, 06, 05, 03]**

***Solution:*** For all $n > 1$, we have,

$$s_n > \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \ldots\ldots + \frac{1}{\sqrt{n^2+n}} \qquad \text{[upto } n \text{ terms]}$$

$$= \frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+\frac{1}{n}}}$$

And,
$$s_n < \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots\ldots + \frac{1}{\sqrt{n^2+1}} \qquad \text{[upto } n \text{ terms]}$$

$$= \frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

Thus
$$\frac{1}{\sqrt{\left(1+\frac{1}{n}\right)}} < s_n < \frac{1}{\sqrt{\left(1+\frac{1}{n^2}\right)}}, \qquad \forall n > 1$$

Let
$$u_n = \frac{1}{\sqrt{1+\frac{1}{n}}}, t_n = \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$\therefore$
$$u_n < s_n < t_n$$

Since
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$$

And,
$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$

Therefore by Sandwich theorem $\lim\limits_{n\to\infty} s_n = 1$.

**Example 26:** ***Prove that***

$$\lim \left[\frac{1}{\sqrt{2n^2+1}} + \frac{1}{\sqrt{2n^2+2}} + \ldots\ldots + \frac{1}{\sqrt{2n^2+n}}\right] = \frac{1}{\sqrt{2}}.$$

***Solution:*** Let
$$s_n = \frac{1}{\sqrt{2n^2+1}} + \frac{1}{\sqrt{2n^2+2}} + \ldots . + \frac{1}{\sqrt{2n^2+n}}$$

Then for all $n > 1$, we have

$$\frac{n}{\sqrt{2n^2+n}} < s_n < \frac{n}{\sqrt{2n^2+1}}$$

Let $$u_n = \frac{n}{\sqrt{2n^2+n}}, \quad t_n = \frac{n}{\sqrt{2n^2+1}}$$

$\therefore$ $$u_n < s_n < t_n, \forall n > 1$$

Since $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left\{\frac{n}{\sqrt{2n^2+n}}\right\} = \frac{1}{\sqrt{2}}$$

And, $$\lim_{n\to\infty} t_n = \lim_{n\to\infty}\left\{\frac{n}{\sqrt{2n^2+1}}\right\} = \lim_{n\to\infty}\left\{\frac{1}{\sqrt{2+\frac{1}{n^2}}}\right\} = \frac{1}{\sqrt{2}}$$

Therefore by Sandwich theorem $\lim\limits_{n\to\infty} s_n = \frac{1}{\sqrt{2}}$.

**Theorem 9: (Cauchy's First Theorem on Limits)**

**If $\lim\limits_{n\to\infty} s_n = l$, then $\lim\limits_{n\to\infty} \frac{s_1+s_2+\ldots+s_n}{n} = l$.**

**Proof:** Define a sequence $<t_n>$ such that

$$s_n = l + t_n, \ \forall n \in N$$

$\therefore$ $$\lim \ t_n = 0$$

And $$\frac{s_1+s_2+\ldots..+s_n}{n} = l + \frac{t_1+t_2+\ldots\ldots+t_n}{n} \quad \ldots(1)$$

In order to prove the theorem we have to show that $\lim\limits_{n\to\infty} \frac{t_1+t_2+\ldots\ldots..+t_n}{n} = 0$.

Let $\in > 0$ be given, since $\lim t_n = 0$, therefore, these exists a positive integer $m$, such that

$$|t_n - 0| = |t_n| < \frac{\in}{2} \quad \forall n \geq m \quad \ldots(2)$$

Also, since every convergent sequence is bounded, hence there exists a real number $k > 0$ such that

$$|t_n| \leq k \ \forall \ n \geq \in N \quad \ldots(3)$$

$\therefore$ $$\left|\frac{t_1+t_2+\ldots..+t_n}{n}\right| = \left|\frac{t_1+t_2+\ldots\ldots..+t_m}{n}\right| + \left|\frac{t_{m+1}+t_{m+2}+\ldots\ldots..+t_n}{n}\right|$$

$$\leq \frac{|t_1|+|t_2|+......+|t_m|}{n} + \frac{|t_{m+1}|+|t_{m+2}|+......+|t_n|}{n}$$

$$\leq \frac{mk}{n} + \frac{n-m}{n} \cdot \frac{\in}{2} \qquad \text{[using (2) and (3)]}$$

$$< \frac{mk}{n} + \frac{\in}{2} \qquad ...(4)$$

$$\left[0 \leq n-m < n, \Rightarrow 0 \leq \frac{n-m}{n} < 1\right]$$

If we take $\frac{mk}{n} < \frac{\in}{2}$ *i.e.,* if $n > \frac{2mk}{\in}$

Let us consider a positive integer $p > \frac{2mk}{\in}$, ...(5)

Then $\qquad \frac{mk}{n} < \frac{1}{2} \in$ for $n \geq p$

Let $\quad M = \max\{m, p\}$; from (4) and (5) we get

$$\left|\frac{t_1 + t_2 + ....+ t_n}{n}\right| < \frac{\in}{2} + \frac{\in}{2} = \in \ \forall n \geq m$$

Thus $\qquad \lim \frac{t_1 + t_2 + ......+ t_n}{n} = 0$

$\Rightarrow \qquad \lim \frac{s_1 + s_2 + ......+ s_n}{n} = l.$

**Theorem 10: (Cauchy's Second Theorem on Limits). If $< s_n >$ is a sequence such that $s_n > 0$ for all n and $\lim s_n = l$, then $\lim (s_1 s_2 ........... s_n)^{1/n} = l$.**

**Proof:** Let us define a sequence $< t_n >$ such that $t_n = \log s_n$, for all $n$.

Since $\lim \ s_n = l$, therefore $\lim t_n = \log l$.

By Cauchy's first theorem on limits, we have

$$\lim \frac{t_1 + t_{2,} + .....+ t_n}{n} = \log l$$

$$\Rightarrow \qquad \lim \frac{\log s_1 + \log s_{2,} + .....+ \log s_n}{n} = \log l$$

$$\Rightarrow \qquad \lim \log \ (s_1 s_2 .....s_n)^{1/n} = \log l$$

Hence, $\qquad \lim (s_1, s_2 .........s_n)^{1/n} = l.$

**Theorem 11:** **If $< s_n >$ is a sequence such that $s_n > 0, \ \forall n \in N$ and $\lim\left(\frac{s_{n+1}}{s_n}\right) = l$, then $\lim \sqrt[n]{s_n} = l$.**

**Proof:** Let us define a sequence $< t_n >$ such that

$$t_1 = s_1,\ t_2 = \frac{s_2}{s_1},\ t_3 = \frac{s_3}{s_2},\ \ldots\ldots\ t_n = \frac{s_n}{s_{n-1}}$$

$\therefore \qquad t_1\, t_2 \ldots\ldots t_n = s_n$

Also $\qquad \lim\left(\frac{s_{n+1}}{s_n}\right) = l \ \Rightarrow\ \lim\left(\frac{s_n}{s_{n-1}}\right) = l \ \Rightarrow\ \lim\ (t_n) = l$

Since $s_n > 0, \ \forall n$, Hence $t_n > 0, \forall n$

Thus, we have a sequence $< t_n >$ such that $t_n > 0, \ \forall n$ and $\lim t_n = l$.

Hence, by Cauchy's first theorem, we have

$$\lim\ (t_1,\ t_2 \ldots. t_n)^{1/n} = l$$

$$\Rightarrow \qquad \lim\ (s_n)^{1/n} = l.$$

**Theorem 12:** **(Cesaro's theorem). If $\lim s_n = l$ and $\lim t_n = l'$**

**Then $\lim \frac{s_1 t_n + s_2 t_{n-1} + \ldots\ldots + s_n t_1}{n} = ll'$.**

**Proof:** Let $s_n = l + x_n$ and $|\, x_n \,| = X_n$. Then $\lim x_n = 0$ and hence $\lim X_n = 0$. Therefore by Cauchy's first theorem. We have

$$\lim \frac{(X_1 + X_2 + \ldots\ldots + X_n)}{n} = 0$$

Now $\qquad \frac{1}{n}(s_1 t_n + s_2 t_{n-1} + \ldots\ldots + s_n t_1) = \frac{1}{n}(t_1 + t_2 + \ldots\ldots + t_n) + \frac{1}{n}(x_1 t_n + x_2 t_{n-1} + \ldots\ldots + x_n t_1) \quad \ldots(1)$

By putting for $s_1, s_2 \ldots\ldots, s_n$

Now the sequence $< t_n >$ is convergent and every convergent sequence is bounded.

Therefore there exists a positive real number $k$ such that

$$|\, t_n \,| < k \ \forall n$$

$\therefore \qquad 0 \le \left| \frac{1}{n}(x_1 t_n + x_2 t_{n-1} + \ldots\ldots + x_n t_1) \right|$

$$\le \left| \frac{1}{n} [\, |x_1||t_n| + |x_2||t_{n-1}| + \ldots\ldots + |x_n||t_1| \,] \right.$$

$$\le \frac{k}{n} [\, |x_1| + |x_2| + \ldots\ldots\ldots + |x_n| \,]$$

$$= \frac{k}{n} [(X_1 + X_2 + \ldots\ldots\ldots + X_n)] \to 0$$

Since $k$ is fixed for all $n$.

$\therefore \qquad \lim \frac{1}{n}(x_1 t_n + x_2 t_{n-1} + \ldots\ldots\ldots + x_n t_1) = 0 \qquad$ (By Sandwich theorem)

Now, since $\lim t_n = l'$, we have by Cauchy's first theorem

$$\lim \frac{1}{n}(t_1 + t_2 + \ldots\ldots + t_n) = l'$$

Then from (1)

$$\lim \frac{1}{n}(s_1 t_n + s_2 t_{n-1} + \ldots\ldots\ldots + s_n t_1) = ll'.$$

***Example 27:*** ***Show that $< s_n >$ converges to e, where $s_n$ is***

***(i)*** $\left(1 + \frac{1}{n}\right)^{n+1}$ ***(ii)*** $\left(1 + \frac{1}{n+1}\right)^{n}$ ***(iii)*** $\left(1 - \frac{1}{n}\right)^{-n}$.

**[B.C.A. (Avadh) 2012; B.C.A. (Kurukshetra) 2011, 07, 05; B.C.A. (Rohilkhand) 2010]**

***Solution:***

(i) We have $s_n = \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right)^{n} \left(1 + \frac{1}{n}\right)$

$\Rightarrow \qquad \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left\{ \left(1 + \frac{1}{n}\right)^{n} \left(1 + \frac{1}{n}\right) \right\}$

But $\qquad \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n} = e$ and $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 + 0 = 1$

$\therefore \qquad \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n} . \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$

$$\lim_{n\to\infty} s_n = e . 1 = e.$$

(ii) We have $$s_n = \left(1+\frac{1}{n+1}\right)^n = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)}$$

$\therefore$ $$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)}$$

$$= \frac{e}{1} = e.$$

(iii) We have $$s_n = \left(1-\frac{1}{n}\right)^{-n} = \left(\frac{n-1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^n$$

$$= \left(\frac{n-1+1}{n-1}\right)^n = \left(1+\frac{1}{n-1}\right)^n$$

$$= \left(1+\frac{1}{n-1}\right)^{n-1}\left(1+\frac{1}{n-1}\right)$$

$\therefore$ $$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\left(1+\frac{1}{n-1}\right)^{n-1}\right]\left[1+\frac{1}{n-1}\right]$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{n-1} \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)$$

$$= e\,.1 = e.$$

***Example 28:*** ***Show that the sequence*** $< s_n >$***, where*** $s_n = \left(1+\frac{2}{n}\right)^n$ ***converge to*** $e^2$***.***

**[B.C.A. (Bundelkhand) 2008]**

***Solution:*** We have $$s_n = \left(1+\frac{2}{n}\right)^n = \left(1+\frac{1}{n+1}\right)^n.\left[1+\frac{1}{n}\right]^n$$

$$= \frac{\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n+1}\right)}$$

$\therefore$ $$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left\{\frac{\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n+1}\right)}\right\}$$

$$= \frac{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)} = \frac{e.e}{1} = e^2.$$

**Example 29:** ***Show that*** $\lim \frac{1}{n}\left(1+\frac{1}{3}+\ldots\ldots+\frac{1}{2n-1}\right)=0.$

**Solution:** Let $s_n=\frac{1}{2n-1}$, then $\lim s_n=\lim \frac{1}{(2n-1)}=0$

By Cauchy's first theorem on limits

$$\lim_{n\to\infty}\left\{\frac{s_1+s_2+\ldots\ldots+s_n}{n}\right\}=0$$

Since $s_1=1,\quad s_2=1/3,\ldots\ldots\ldots,\ s_n=\frac{1}{2n-1}$

$\therefore$ $$\lim \frac{1}{n}\left\{1+\frac{1}{3}+\frac{1}{5}+\ldots\ldots+\frac{1}{2n-1}\right\}=0.$$

**Example 30:** ***Prove that*** $\lim \frac{1}{n}(1+(2)^{1/2}+(3)^{1/3}+\ldots\ldots+(n)^{1/n})=1.$

**[B.C.A. (Meerut) 2003]**

**Solution:** Let $s_n=n^{1/n}$. Then we know that $\lim\limits_{n\to\infty}(n)^{1/n}=1$

Hence, by Cauchy's first theorem on limits.

$$\lim \frac{1}{n}(s_1+s_2+\ldots\ldots+s_n)=1$$

or $$\lim \frac{1}{n}(1+(2)^{1/2}+(3)^{1/3}+\ldots\ldots+(n)^{1/n})=1.$$

**Example 31:** ***Prove that*** $\lim\left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\ldots\ldots\left(\frac{n+1}{n}\right)^n\right]^{1/n}=e.$

**[B.C.A. (Meerut) 2008, 04]**

**Solution:** Let $s_n=\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n$. Then $\lim\limits_{n\to\infty}s_n=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

$\therefore$ $s_n>0$ for all $n$.

Then by Cauchy's second theorem,

$$\lim_{n\to\infty}(s_1,s_2\ldots\ldots s_n)^{1/n}=e$$

Since $s_1=\frac{2}{1},\ s_2=\left(\frac{3}{2}\right)^2,\ s_3=\left(\frac{4}{3}\right)^3\ldots\ldots\ldots s_n=\left(\frac{n+1}{n}\right)^n$

$\therefore$ $$\lim\left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\ldots\ldots\left(\frac{n+1}{n}\right)^n\right]^{1/n}=e.$$

**Example 32:** ***Prove that*** $\lim\left[\dfrac{(n!)^{1/n}}{n}\right]=\dfrac{1}{e}$.

**[B.C.A. (Rohtak) 2008; B.C.A. (Meerut) 2002]**

**Solution:** Let $s_n=\dfrac{n!}{n^n}$. Than $s_n>0\ \forall n\in N$

we know that if $s_n>0,\ \forall n\in N$, then $\lim (s_n)^{1/n}=\lim\left(\dfrac{s_{n+1}}{s_n}\right)$

We have $$\left(\frac{s_{n+1}}{s_n}\right)=\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\frac{n^n}{(n+1)^n}=\frac{1}{\left(1+\dfrac{1}{n}\right)^n}$$

$\therefore$ $$\lim\left(\frac{s_{n+1}}{s_n}\right)=\lim\frac{1}{\left(1+\dfrac{1}{n}\right)^n}=\frac{1}{e}$$

Then $$\lim (s_n)^{1/n}=\lim\left[\frac{(n)!^{1/n}}{n}\right]=\frac{1}{e}.$$

**Example 33:** ***Prove that*** $\lim\limits_{n\to\infty}\left[\dfrac{(3n)!}{(n!)^3}\right]^{1/n}=27$.

**[B.C.A. (Lucknow) 2010; B.C.A. (Kanpur) 2008; B.C.A. (Delhi) 2008; B.C.A. (Garhwal) 2007, 04]**

**Solution:** Let $s_n=\dfrac{(3n)!}{(n!)^3}$. Then $s_n>0\quad\forall n\in N,$

Now we know that if $s_n>0\quad\forall n\in N$, then
$$\lim\ (s_n)^{1/n}=\lim\left(\frac{s_{n+1}}{s_n}\right)$$

We have $$\frac{s_{n+1}}{s_n}=\frac{(3n+3)!}{((n+1)!)^3}\ \frac{(n)!^3}{(3n)!}$$
$$=\frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3}$$
$$=\frac{\left(3+\dfrac{3}{n}\right)\left(3+\dfrac{2}{n}\right)\left(3+\dfrac{1}{n}\right)}{\left(1+\dfrac{1}{n}\right)^3}$$

$\therefore$ $$\lim_{n\to\infty}\left(\frac{s_{n+1}}{s_n}\right)=\frac{(3+0)(3+0)(3+0)}{(1+0)^3}=27$$

$\therefore$ $$\lim\ (s_n)^{1/n}=\lim\left[\frac{(3n)!}{(n!)^3}\right]^{1/n}=27.$$

***Example 34:*** ***Show that***

***(i)*** $\lim \dfrac{n}{(n!)^{1/n}} = e.$

***(ii)*** $\lim [\{(n+1)(n+2)\ldots\ldots\ldots(n+n)\}^{1/n}/n] = 4/e.$ **[B.C.A. (Rohilkhand) 2008]**

***Solution:***

(i) Let $s_n = \dfrac{n^n}{n!}$ then $s_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)!}$

$$\therefore \quad \frac{s_{n+1}}{s_n} = \frac{(n+1)^{n+1}}{(n+1)} \cdot \frac{1}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$$

Then $$\lim_{n\to\infty} (s_n)^{1/n} = \lim \left(\frac{s_{n+1}}{s_n}\right) = \lim \left(1 + \frac{1}{n}\right)^n = e$$

*i.e.,* $$\lim_{n\to\infty} \frac{n!}{(n!)^{1/n}} = e.$$

(ii) Let $s_n = (n+1)(n+2)\ldots\ldots.(n+n)/n^n$

Then $$\frac{s_{n+1}}{s_n} = \frac{2(2n+1)}{(n+1)} \left(\frac{n}{n+1}\right)^n$$

So that $$\lim \left(\frac{s_{n+1}}{s_n}\right) = \lim \left[\frac{2(2n+1)}{(n+1)} \times \frac{1}{\left(1+\frac{1}{n}\right)^n}\right] = 4.\frac{1}{e} = \frac{4}{e}$$

$$\therefore \quad \lim_{n\to\infty} (s_n)^{1/n} = \lim [\{(n+1)(n+2)\ldots\ldots.(n+n)^{1/n}/n\}] = \frac{4}{e}.$$

## 4.9 Oscillatory Sequences

**[B.C.A. (Meerut) 2000]**

A sequence $< s_n >$ is said to be an **oscillatory sequence** if it is neither convergent nor divergent.

***Illustration 1:*** The sequence $< (-1)^n >$ oscillates finitely.

***Illustration 2:*** The sequence $< (-1)^n n >$ oscillates infinitely. **[B.C.A. (Meerut) 2000]**

## 4.10 Monotonic Sequence

**[B.C.A. (Meerut) 2007]**

### 4.10.1 Definition

1. A sequence $< s_n >$ is said to be **monotonically increasing** or non-decreasing, if $s_n \le s_{n+1}$ for all $n$ *i.e.,* $s_n \le s_m$ for all $n < m$.

2. A sequence $< s_n >$ is said to be **monotonically strictly increasing** if $s_n < s_{n+1} \quad \forall n \in N$.

3. A sequence $< s_n >$ is said to be **monotonically decreasing** or non-increasing, if $s_n \ge s_{n+1}, \ \forall n$

   *i.e.,* $\quad s_n \ge s_m, \ \forall n < m$.

4. A sequence $< s_n >$ is said to be **strictly decreasing** if $s_n > s_{n+1}, \ \forall n \in N$.

5. A sequence $< s_n >$ is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.

***For Example:***

(i) The sequence $< 1, 2, 3, \ldots.. n, \ldots >$ is strictly increasing.

(ii) The sequence $< 2, 2, 4, 4, 6, 6, \ldots >$ is monotonically increasing.

(iii) The sequence $< -\frac{1}{n} >$ is strictly increasing.

(iv) The sequence $< \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots. >$ is strictly decreasing.

(v) The sequence $< -2, 2, -4, 4, -6, 6 \ldots >$ is not monotonic.

**Theorem 13: Every bounded monotonically increasing sequence converges.**

**[B.C.A. (Kanpur) 2011]**

**Or**

**Prove that a monotonically increasing and bounded sequence converge to its supremum.** **[B.C.A. (Meerut) 2010, 07; B.C.A. (Purvanchal) 2010, 07]**

**Proof:** Let $< s_n >$ be a bounded monotonically increasing sequence. Then we have to show $< s_n >$ converges to $l$.

Let $\in > 0$ be given. Then $l - \in < l$, so that $l - \in$ is not an upper bound of S. Hence there exists a positive integer $m$ such that $s_m > l - \in$. Since $< s_n >$ is monotonically increasing, therefore

$$s_n \ge s_m > l - \in, \quad \forall n \ge m \qquad \ldots(1)$$

Also, since $l$ is the supremum of $S$, therefore

$$s_n \le l < l + \in, \quad \forall n \in N \qquad ...(2)$$

From (1) and (2), we get $l - \in < s_n < l + \in, \forall n \ge m$

$\Rightarrow$ $\qquad |s_n - l| < \in \quad \forall n \ge m$

$\Rightarrow$ $\qquad \lim s_n = l.$

**Theorem 14: Every bounded monotonically decreasing sequence converges.**

**Or**

**A monotonic decreasing sequence is convergent iff it is bounded and converges to its infimum.** **[B.C.A. (Delhi) 2010, 06]**

> **NOTE:**
>
> We can prove this result by taking infimum of the set $S$.

**Theorem 15: A non-decreasing sequence which is not bounded above diverges to infinity.**

**Proof:** Let $< s_n >$ be a non–decreasing sequence which is not bounded above.

Let us consider any real number $k > 0$.

Now $< s_n >$ is not bounded above $\Rightarrow$ there exist $m \in N$ such that $s_m > k$, also $< s_n >$ is non–decreasing

$\Rightarrow$ $\qquad s_n \ge s_m,$ for $n \ge m$

$\therefore$ $\qquad s_n \ge s_m > k$ for $n > m$ or $s_n > k$ for $n > m$

Hence, $< s_n >$ diverges to $(\infty)$.

**Theorem 16: A non-increasing sequence which is not bounded below diverges to minus infinity.**

**Proof:** Let $< s_n >$ be a non-increasing sequence which is not bounded below.

Let us consider real number $k < 0$

Now $< s_n >$ is not bounded below $\Rightarrow$ there exists $m \in N$ such that $s_m < k$.

Also $< s_n >$ is non-increasing $\Rightarrow s_n \le s_m$, for $n > m$

$\therefore$ $\qquad s_n \le s_m < k$ for $n > m$ or $s_n < k$ for $n > m$

Hence, $< s_n >$ diverges to $(-\infty)$.

**Theorem 17: Every sequence has a monotonic subsequence.**

**Proof:** Let us consider the sequence. $a_0 = <s_n>$, Let $a_1, a_2, a_3, .....$

denote the subsequence

$$<s_2, s_3, ......>, <s_3, s_4, s_5, ........> <s_4, s_5, .....> ..... \text{respectively.}$$

There are two different cases :

1. Each of the sequences $a_0, a_1, a_2, ......$ has a greatest term. Let $s_{n_1}, s_{n_2}, s_{n_3}, .......$ be the greatest value of $a_0, a_1, a_2, .....$

   Then $n_1 \le n_2 \le n_3 .......$ and $s_{n_1} \ge s_{n_2} \ge s_{n_3} \ge .....$

   Hence, $< s_{n_1}, s_{n_2}, s_{n_3}, ......>$ is a monotonically decreasing subsequence of $< s_n >$.

2. At least one of the sequences $a_0, a_1, a_2, ......$ has no greatest term. Suppose $a_m$ has no greatest term. Then each term of $a_m$ that exceeds it. For if there is a term of $a_m$ which exceeds all terms following it, then it can be exceeded by finitely many terms at the most and hence, $a_m$ must have a greatest term. Now $s_{m+1}$ is the first term of $a_m$. Let $s_{n_2}$ be the first term of $a_m$ greater than $s_{m+1}$, $s_{n_3}$ the first term of $a_m$ follows $s_{n_2}$ and greater it, and so on.

   Thus, $<s_{m+1}, s_{n_2}, s_{n_3}, s_{n_4}, .......>$ is a monotonically increasing subsequence of $<s_n>$.

## 4.11 Limit Points of Sequence

A real number $p$ is said to be a **limit** of a sequence $<s_n>$ if every neighbourhood of $p$ contains infinite number of terms of the sequence,

*Or*

A real number $p$ is a limit point of a sequence $<s_n>$ iff given $\in > 0$, $s_n \in ]\, p-\varepsilon, p+\varepsilon\,[$ for infinitely many values of $n$.

***Illustration 1:*** The sequence $< \frac{1}{n} >$ has only one limit point, *i.e.,* 0 is only limit point.

***Illustration 2:*** The sequence $< 1, 2, 3, .......... n, ........ >$ has no limit point.

**Solution:** Let $p \in R$, whatever $\in$ we take, the neighbourhood $]\, p-\varepsilon, p+\varepsilon\,[$ of $p$ contains at the most a finite number of terms of this sequence. Hence $p$ is not limit point.

***Illustration 3:*** The sequence $< (-1)^n >$ has 1 and $-1$ as limit points. Here $s_n = -1$, if $n$ is odd and $s_n = 1$, if $n$ is even, any neighbourhood of $-1$ will contain all the odd terms of the sequence hence $-1$ is a limit point.

Similarly any neighbourhood of 1 will contain all the even terms of the sequence. So, 1 is limit point.

**Theorem 18: If $l$ is a limit point of the range of a sequence $< s_n >$, then $l$ is a limit point of the sequence $< s_n >$.**

**Proof:** Let $S$ be the range set of the sequence $< s_n >$. Since $l$ is the limit of $s_n$ therefore, every neighbourhood (*nbd*) of $l$ contains infinite number of distinct elements of the set $S$. But each element of the set $S$ is a term of the sequence $< s_n >$. Hence every *nbd* of $l$ contains infinite number of terms of the sequence $< s_n >$.
Then $l$ is a limit point of the sequence $< s_n >$.

**Theorem 19: If $s_n \to l$, then $l$ is the only limit point of $< s_n >$.**

**Proof:** First of all we shall show that $l$ is a limit point of $<s_n>$. Let $\in > 0$ be given. Since $s_n \to l$, therefore, there exists a positive integer $m$ such that

$$|s_n - l| < \in \text{ for all } n \geq m$$

*i.e,* $$|s_n - l| < \in \text{ for infinitely many values of } n.$$

This shows that $l$ is limit point of $< s_n >$. Next we shall show that if $l'$ be any limit point of $< s_n >$, then we must have $l' = l$.

Let $\in > 0$ be given. Since $l$ is the limit of $<s_n>$, therefore, there exists a positive integer $p$ such that

$$|s_n - l| < \frac{\in}{2} \quad \forall n \geq p \qquad ...(1)$$

Since $l'$ is a limit point of $<s_n>$, therefore, there must exists a positive integer $q > p$ such that

$$|s_q - l'| < \frac{\in}{2} \qquad ...(2)$$

Substituting $n = q$ in (1) we get

$$|s_q - l'| < \frac{\in}{2} \qquad ...(3)$$

$$\therefore \quad |l - l'| = |(s_q - l') - (s_q - l)| \leq |s_q - l'| + |s_q - l|$$

$$< \frac{\in}{2} + \frac{\in}{2} \qquad \text{[using (2), (3)]}$$

$$\Rightarrow \quad |l - l'| < \in$$

Since $\in$ is arbitrary, hence we must

$$|l - l'| = 0 \Rightarrow l = l'.$$

**Theorem 20: (Bolzano-Weierstrass Theorem) Every bounded sequence has at least one limit point.** **[B.C.A. (Meerut) 2008]**

**Proof:** Let $<s_n>$ be a bounded sequence and $S$ be its range set. Then $S = \{s_n : n \in N\}$. Since the sequence $<s_n>$ is bounded, therefore, $S$ is bounded set. There are two possibilities.

1. Let $S$ be a finite set. Then for infinitely many indices $n$, $s_n = p$, where $p$ is some real number. Then $p$ is a limit point of $<s_n>$.

2. Let $S$ be an infinite set. Since $S$ is bounded, by Bolzano-Weierstrass theorem for sets of real numbers. $S$ has limit point say $p$. Hence every *nbd* of $p$ contains infinitely many distinct point of $S$ or every *nbd* of $p$ contains infinitely many terms of the sequence $<s_n>$ and hence $p$ is a limit point of the sequence $<s_n>$.

**NOTE:**

1. If $F$ is a closed and bounded set of real numbers, then every sequence in $F$ has a limit point in $F$.
2. If I is a closed interval, then every sequence in I has a limit point in I.

**Theorem 21: If a sequence $<s_n>$ is bounded and has only one limit point, say $l$, then $s_n \to l$.**

**Proof:** Since $<s_n>$ is bounded, so it has at least one limit point. But $l$ is the only limit point of $<s_n>$. Hence, for any $\in\, > 0$, $]\, l - \in,\ l + \in\, [$ contains $s_n$ for all except a finite number of values of $n$.

Let $s_{m_1},\ s_{m_2}, \ldots\ldots. s_{m_p}$ be the finite number of terms of the sequence $<s_n>$ that lie out side $]\, l - \in,\ l + \in\, [$.

If $m = \max\ (m_1,\ m_2, \ldots\ldots. m_p)$ then

$$s_n \in ]\, l - \in,\ l + \in\, [ \qquad \forall n \geq m$$

Hence for any $\in\, > 0$, there exists $m \in N$ such that

$$|\, s_n - l\,| < \in \quad \forall n \geq m$$

$\therefore$ the sequence $<s_n>$ converges to $l$.

**Theorem 22: A real number $p$ is a limit point of a sequence $<s_n>$ iff there exists a subsequence of $<s_n>$ converge to $p$.**

**Proof:** Let $p$ be a limit point of a sequence $<s_n>$. We shall use the result that a real number $p$ is limit point of a sequence $<s_n>$ if given any $\in\, > 0$ and any positive integer $m$, there exists a positive integer $k > m$ such that $s_k \in ]\, p - \in,\ p + \in\, [$

We consider $\in = 1$ and $m = 1$, there must exist positive integer $n_1 > 1$, such that

$$| s_{n_1} - p | < 1 \qquad ...(1)$$

Again let $\in = \frac{1}{2}$, $m = n_1$, there must exist a positive integer $n_2 > n_1$, such that

$$| s_{n_2} - p | < \frac{1}{2} \qquad ...(2)$$

Continuing in this way, we can inductively define a sub-sequence $< s_{n_1}, \; s_{n_2}, \; ........ s_{n_k} ..... >$ such that

$$| s_{n_k} - p | < \frac{1}{k}.$$

In fact, if we assume that $s_{n_1}, \; s_{n_2}, ....... s_{n_k}$ has been obtained, by selecting.

$$\in = \frac{1}{k+1}, \; m = n_k.$$

We can get a positive integer $n_{k+1} > n_k$, such that

$$| s_{k+1} - p | < \frac{1}{k+1}$$

But $s_{n_1}$ has already been obtained. Thus, the construction of $< s_{n_k} >$ is complete by induction. We now claim that the sequence $< s_{n_k} > \to p$. In fact, for any $\in > 0$, we can choose a positive integer $j$, such that $1/j < \in$. For this choice of $j$, we get

$$| s_{n_k} - p | < 1/j < \in \;\; \forall k \geq j.$$

This shows that the sequence $< s_{n_k} > \to p$.

**Conversely**, let $< s_{n_k} >$ be a subsequence of $< s_n >$ converging to $p$, then to show $p$ is limit point of $< s_n >$.

Since $s_{n_k} \to p$, therefore, given any $\in > 0$ there exists positive integer such that

$$| s_{n_k} - p | < \in \;\; \forall \; k \geq j$$

Thus, every neighbourhood of $p$ contains infinitely many terms of $< s_{n_k} >$ then $p$ is limit point of $< s_n >$.

**Theorem 23: The set of limit points of a bounded sequence is bounded.**

**Proof:** Let $< s_n >$ be a bounded sequence. Then there exists $k_1, k_2 \in R$ such that

$$k_1 \leq s_n \leq k_2, \quad \forall \, n \in N$$

*i.e,* $\qquad s_n \notin ]-\infty, k_1[$ and $s_n \notin ] \; k_2, \infty[$

Hence if $l \in R$ and $l \in ]-\infty, k_1 \; [\cup] \; k_2, \infty[$, then $l$ is not a limit point of the sequence. Thus, if $l \in R$ is a limit point of the sequence, then $l \in [k_1, \; k_2]$. Hence the set of limit points of $< s_n >$ is bounded.

**Theorem 24:** **Every bounded sequence has the greatest and the least limit points.**

**Proof:** Let $< s_n >$ be bounded sequence. Then the set L of limit points of $< s_n >$ is bounded.

If $\inf(L) = u$ and $\sup(L) = v$, then we have to show that $u, v \in L$.

For $\in > 0, ]v - \in, v + \in[$ is a *nbd* of $v$

Since $v = \sup$ (L), therefore, there exists some $x \in L$ such that

$$v - \in < x \le v < v + \in$$

$\Rightarrow$ $x \in ]v - \in, v + \in[$

$\Rightarrow$ $]v - \in, v + \in[$ is *nbd* of $x$

Since $x$ is a limit point of $< s_n >$, hence $]v - \in, v + \in[$ contains infinite number of terms of the sequence.

$\therefore$ $v$ is a limit point of the sequence $< s_n >$

$\therefore$ $v \in L$

Similarly $u \in L$.

***Example 35:*** ***Show that the sequence*** $< x_n > = \left\langle \dfrac{2^n}{(n)!} \right\rangle$ ***is monotonic decreasing sequence. Also show that it is bounded and find its limit.***

**[B.C.A. (Delhi) 2010, 07, 04; B.C.A. (Meerut) 2004]**

***Or***

***Is the sequence*** $\left\langle \dfrac{2^n}{(n)!} \right\rangle$ ***a monotonic non-increasing sequence or non-decreasing sequence ?***

***Find the bounds of this sequence.*** **[B.C.A. (Meerut) 2008; B.C.A. (Garhwal) 2007]**

***Solution:*** We have $x_n = \dfrac{2^n}{(n)!}$ and $x_{n+1} = \dfrac{2^{n+1}}{(n+1)!}$

$$\therefore \quad \frac{x_n}{x_{n+1}} = \frac{2^n}{(n)!} \cdot \frac{(n+1)!}{2^{n+1}} = \frac{2^n (n+1)(n)!}{(n)! 2^n . 2}$$

$$= \frac{n+1}{2} \ge 1 \ \forall \ n \in N$$

$\Rightarrow$ $$\frac{x_n}{x_{n+1}} \geq 1 \text{ or } x_n \geq x_{n+1} \quad \forall n \in N$$

$\Rightarrow$ The given sequence is a monotonically decreasing sequence.

Putting $n = 1, 2, 3, 4, \ldots\ldots$ in $x_n = \dfrac{2^n}{(n)!}$, we get

$$< x_n > = \left\langle \frac{2}{(1)!}, \frac{2^2}{(2)!}, \frac{2^3}{(3)!}, \frac{2^4}{(4)!}, \ldots\ldots \right\rangle$$

Hence, it is evident that $< x_n >$ is bounded above by 2. also each term $x_n > 0$.

$\Rightarrow$ $< x_n >$ is bounded above by 0

$\Rightarrow$ sequence $< x_n >$ convergent

Again $$\lim_{n\to\infty} \frac{2^n}{(n)!} = \lim_{n\to\infty} \left\{\frac{2}{n}, \frac{2}{n-1}, \ldots\ldots \frac{2}{1}\right\}$$

$$= \lim_{n\to\infty} \frac{2}{n} \lim_{n\to\infty} \frac{2}{n-1} \ldots\ldots \lim_{n\to\infty} \frac{2}{-1} = 0.$$

***Example 36:*** ***Prove that the sequence***

$$< s_n > \textit{ where } s_n = 2 - \frac{1}{2^{n-1}} \textit{ converge.}$$

**[B.C.A. (Meerut) 2008; B.C.A. (Agra) 2007, 06, 04; B.C.A. (Rohilkhand) 2005]**

***Solution:*** We have $$s_n = 2 - \frac{1}{2^{n-1}}$$

$\Rightarrow$ $$s_{n+1} = 2 - \frac{1}{2^n}$$

$\Rightarrow$ $$s_{n+1} - s_n = \left(2 - \frac{1}{2^n}\right) - \left(2 - \frac{1}{2^{n-1}}\right)$$

$$= \frac{1}{2^{n-1}} - \frac{1}{2^n} > 0$$

$\Rightarrow$ $< s_n >$ is monotonically increasing

Again $$s_n = 2 - \frac{1}{2^{n-1}} < 2 \; \forall \, n \text{ as } \frac{1}{2^{n+1}} > 0$$

$\Rightarrow$ $< s_n >$ is bounded and monotonically increasing sequence, therefore, it converges

Now $$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2 - \frac{1}{2^{n-1}}\right) = 2 - 0 = 2$$

**_Example 37:_** ***Find the bounds of the sequence*** $< s_n >$

$$\textit{where} \qquad s_n = \frac{4n-1}{5n+2}.$$

**[B.C.A. (Garhwal) 2006; B.C.A. (Agra) 2005; B.C.A. (Meerut) 2003]**

**_Solution:_** We have
$$s_n = \frac{4n-1}{5n+2}$$

and
$$s_{n+1} = \frac{4n+3}{5n+7}$$

$\therefore$
$$s_{n+1} - s_n = \frac{4n+3}{5n+7} - \frac{4n-1}{5n+2}$$

$$= \frac{(4n+3)(5n+2)-(4n-1)(5n+7)}{(5n+7)(5n+2)}$$

$$= \frac{20n^2+8n+15n+6-20n^2-28n+5n+7}{(5n+7)(5n+2)}$$

$$= \frac{13}{(5n+7)(5n+2)} > 0$$

Therefore, the sequence is monotonic increase the lower bound is the first term (for $n = 1$) *i.e.*, $3/7$

Now
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{4n-1}{5n+2} = \lim_{n\to\infty} \frac{4n\left(1-\frac{1}{4n}\right)}{5n\left(1+\frac{2}{5n}\right)}$$

$$= \frac{4}{5}$$

$\Rightarrow$
$$\frac{4}{5} \text{ is upper bound}$$

These are respectively the greatest lower bound and the least upper bound of the sequence $< s_n > \to 4/5$.

**_Example 38:_** ***Prove that the sequence*** $< s_n >$***, where***

$s_n = \dfrac{2n-7}{3n+2}$ ***is a monotonic increasing sequence. Further show that it is bounded and tends to the limit*** $\left(\dfrac{2}{3}\right)$.

**[B.C.A. (Rohilkhand) 2008, 00; B.C.A (Agra) 2006;**
**B.C.A. (Meerut) 2006, 02, 00]**

**_Solution:_** We have
$$s_n = \frac{2n-7}{3n+2} \qquad \text{...(1)}$$

and
$$s_{n+1} = \frac{2(n+1)-7}{3(n+1)+2} = \frac{2n-5}{3n+5} \qquad \text{...(2)}$$

$$\Rightarrow \qquad s_n - s_{n+1} = \frac{2n-7}{3n+2} - \frac{2n-5}{3n+5}$$

$$= \frac{-25}{(3n+2)(3n+5)} < 0$$

$$\Rightarrow \qquad s_n - s_{n+1} < 0 \, \forall n \in N$$

$$\Rightarrow \qquad s_n < s_{n+1} \quad \forall n \in N$$

$\Rightarrow < s_n >$ is monotonic increasing sequence. Again putting $n = 1,2,3,4,5.......$in (1),

we get $$< s_n > = \left\langle -1, \frac{-3}{8}, \frac{-1}{11}, \frac{1}{14}, \frac{3}{17}, ...... \right\rangle$$

It is evident that $s_n \geq -1 \ \forall \ n \in N$

Also $$s_n = \frac{2n-7}{3n+2} = \frac{(3n+2)-(n+9)}{3n+2} = 1 - \frac{n+9}{3n+2} < 1$$

*i.e.,* $$s_n < 1 \, \forall n \in N$$

Hence, we find $-1 \leq s_n \leq 1 \ \forall \ n \in N$ *i.e.*, the sequence $< s_n >$ is bounded. Thus, we find that the sequence $< s_n >$ is monotonic increasing sequence bounded above, so it

is convergent. Also

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n-7}{3n+2} = \lim_{n\to\infty} \frac{2n\left(1-\frac{7}{2n}\right)}{3n\left(1+\frac{2}{3n}\right)} = \frac{2}{3}$$

$\Rightarrow$ The sequence $< s_n >$ tends to $2/3$.

***Example 39:*** ***Prove that sequence***

$$\sqrt{2}, \sqrt{(2\sqrt{2})}, \sqrt{[2\sqrt{2\sqrt{2}})]} .... \textit{ converges to } 2$$

***Or***

***If*** $s_1 = \sqrt{2}$ ***and*** $s_n = \sqrt{(2s_{n-1})}$ ***for*** $n > 1$, ***prove that*** $s_n \to 2$. **[B.C.A. (Meerut) 2005; B.C.A. (Delhi) 2008, 04, 01]**

***Solution:*** We have $$s_1 = \sqrt{2} = (2)^{1/2}$$

$$s_2 = \sqrt{2s_2} = \sqrt{2\sqrt{2}} > \sqrt{2}$$

$$\therefore \qquad s_2 > s_1$$

Now we shall use the principal of induction. Suppose

$$s_m < s_{m+1}$$

Then $\qquad 2s_m < 2s_{m+1} \quad \Rightarrow \sqrt{2s_m} < \sqrt{2s_{m+1}}$

$\Rightarrow \qquad s_{m+1} < s_{m+2}$

$\Rightarrow < s_n >$ is monotonically increasing. Now we have

$$s_1 = \sqrt{2} < 2$$

Suppose $\qquad s_{nq} < 2 \Rightarrow 2s_n < 2\,.2 \quad \Rightarrow \sqrt{2s_n} < \sqrt{2^2}$

$\Rightarrow \qquad s_{n+1} < 2$

$\Rightarrow \qquad s_n < 2, \forall n$

$\Rightarrow < s_n >$ is monotonically increasing and bounded and hence converges

Suppose, $\qquad \lim\limits_{n\to\infty} s_n = l$, then $s_n = \sqrt{2s_{n-1}} \quad \Rightarrow s_n^2 = 2s_{n-1}$

$\Rightarrow \qquad \lim\limits_{n\to\infty} s_n^2 = \lim\limits_{n\to\infty} 2s_{n-1}$

$\Rightarrow \qquad l^2 = 2l \;\Rightarrow l = 0$ or $l = 2$

But $\qquad \sqrt{2} \le s_n \;\forall n \;\Rightarrow l \ne 0$ and hence $l = 2$

$\Rightarrow \qquad < s_n > \to 2.$

***Example 40:*** ***Prove that the sequence*** $< a_n >$ ***where*** $a_n = \sqrt{n^2 + m} - n$ ***is convergent. Also find its limit.*** **[B.C.A. (Meerut) 2002]**

***Solution:*** Now, $\qquad a_n > 0 \;\forall\, n \in N$ since $\sqrt{n^2 + m} > n$

Further, $\qquad a_n = \dfrac{(\sqrt{n^2 + m} - n)(\sqrt{n^2 + m} + n)}{(\sqrt{n^2 + m} + n)} = \dfrac{m}{\sqrt{n^2 + m} + n}$

Also $\qquad a_{n+1} = \dfrac{m}{\sqrt{(n+1)^2 + m} + (n+1)}$

But $\qquad \sqrt{(n+1)^2 + m} + (n+1) > \sqrt{n^2 + m} + n$

$\Rightarrow \qquad \dfrac{m}{\sqrt{(n+1)^2 + m} + (n+1)} < \dfrac{m}{\sqrt{n^2 + m} + n}$

$\Rightarrow \qquad a_{n+1} < a_n \forall\, n \in N$

$\Rightarrow < a_n >$ is monotonic decreasing sequence and bounded below by 0

$\Rightarrow < a_n >$ is convergent

Further, $\qquad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{m}{\sqrt{n^2 + m} + n} = 0.$

**Example 41:** *If* $a_n = \sqrt{n}\,(\sqrt{n+1} - \sqrt{n})$, *then does* $< a_n >$ *converge?*

**[B.C.A. (Rohilkhand) 2010, 00; B.C.A. (Lucknow) 2009, 06]**

**Solution:** We have
$$a_n = \frac{\sqrt{n}(\sqrt{(n+1)} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{\sqrt{n}(n+1-n)}{(\sqrt{n+1} + \sqrt{n})} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}.$$

Also
$$a_{n+1} = \frac{\sqrt{n+1}}{\sqrt{n+2} + \sqrt{n+1}}$$

$\Rightarrow$
$$a_{n+1} - a_n = \frac{\sqrt{n+1}}{\sqrt{n+2} + \sqrt{n+1}} - \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{n+1 + \sqrt{n}\sqrt{n+1} - \sqrt{n}\sqrt{n+2} - \sqrt{n}\sqrt{n+1}}{(\sqrt{n+2} + \sqrt{n+1})(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{n+1 - \sqrt{n(n+2)}}{(\sqrt{n+2} + \sqrt{n+1})(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{\sqrt{(n+1)^2} - \sqrt{n^2 + 2n}}{(\sqrt{n+2} + \sqrt{n+1})(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{\sqrt{(n^2 + 2n + 1)} - \sqrt{n^2 + 2n}}{(\sqrt{n+2} + \sqrt{n+1})(\sqrt{n+1} + \sqrt{n})} > 0, \text{ for all } n, \text{ as}$$

$$n^2 + 2n + 1 > n^2 + 2n.$$

$\Rightarrow$ $a_{n+1} > a_n \Rightarrow < a_n >$ is monotonically increasing sequence.

Also
$$1 - a_n = 1 - \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{\sqrt{n+1}}{\sqrt{n+1} + \sqrt{n}} > 0$$

$\Rightarrow$ $a_n < 1$, for all $n \in N$.

$\Rightarrow$ $< a_n >$ is a monotonically increasing and bounded (above) sequence.

$\Rightarrow$ $< a_n >$ converges.

Further,
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}(\sqrt{1 + 1/n} + 1)}$$

$$= \lim_{n\to\infty} \frac{1}{(\sqrt{1 + 1/n} + 1)} = \frac{1}{2}$$

$\Rightarrow$ $< a_n >$ converges to $1/2$.

**Example 42:** ***Prove that the sequence*** $< s_n > = \left\langle \frac{e^n}{n} \right\rangle$ ***is monotonic increasing, bounded below but not bounded above.*** **[B.C.A. (Delhi) 2010, 07]**

**Solution:** We have $s_n = \frac{e^n}{n}$ and $s_{n+1} = \frac{e^{n+1}}{n+1}$

$\therefore$ $$\frac{s_n}{s_{n+1}} = \frac{e^n}{n} \times \frac{n+1}{e^{n+1}} = \left(\frac{n+1}{n}\right)\frac{1}{e} = \left(1+\frac{1}{n}\right)\frac{1}{e}$$

or $\frac{s_n}{s_{n+1}} < 1, \forall n \in N$ since $e > 2$ and $1 + \frac{1}{n}$ will have value at most 2

or $s_n < s_{n+1} \forall n \in N$

$\therefore$ The sequence $< s_n >$ is monotonic increasing. Also putting $n = 1, 2, 3, 4 \ldots\ldots..$ we find that

$$< s_n > = < e, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \ldots\ldots.. >, \text{ where } e > 1$$

$\therefore$ The sequence $< s_n >$ is monotonic increasing and we know $e^n = 1 + n + \frac{n^2}{(2)!} + \frac{n^3}{(3)!} + \ldots\ldots.$

or $$\frac{e^n}{n} = \frac{1}{n} + 1 + \frac{n}{(2)!} + \frac{n^2}{(3)!} + \ldots\ldots..$$

Thus shows that as $n$ increase $s_n = \frac{e^n}{n} \to \infty$, so $< s_n >$ is not bounded above.

**Example 43 :** ***Prove that the sequence*** $< x_n >$ ***defined by*** $x_1 = \sqrt{7}, x_{n+1} = \sqrt{7 + x_n}$ ***converges to the positive root of the equation*** $x^2 - x - 7 = 0$. **[B.C.A. (Lucknow) 2009, 05]**

**Solution:** We have $x_1 = \sqrt{7},\ x_{n+1} = \sqrt{7 + x_n}$ ...(1)

Putting $n = 1$ in (1) we have

$$x_2 = \sqrt{7 + x_1} = \sqrt{7 + \sqrt{7}}$$

or $x_2 > \sqrt{7},\ i.e.,\ x_2 > x_1$ ...(2)

Now suppose $x_{n+1} > x_n$ adding 7 to both the sides, we have

$$7 + x_{n+1} > 7 + x_n$$

Taking square root of both sides

$$\sqrt{7 + x_{n+1}} > \sqrt{7 + x_n} \Rightarrow x_{n+2} > x_{n+1} \quad \text{from (1)}$$

$\therefore$ We find that $x_{n+1} > x_n \quad \Rightarrow x_{n+2} > x_{n+1}$

Putting $n = 1$, we get $\quad x_2 > x_1 \quad \Rightarrow x_3 > x_2$

Putting $n = 2$, we get $\quad x_3 > x_2 \quad \Rightarrow x_4 > x_3$

With the help of principle of induction, we conclude that

$$x_1 < x_2 < x_3 < x_4 \qquad \text{...(3)}$$

*i.e.*, the sequence $< s_n >$ is monotonic increasing.

Also as $x_1 = \sqrt{7}$ given so from (3) the sequence $< x_n >$ is bounded below by $\sqrt{7}$.

Again $x_1 = \sqrt{7}$ (given) so $x_1 < 7$ and from (1)

$$x_2 = \sqrt{7 + x_1} = \sqrt{7 + \sqrt{7}}, \;\; i.e., \; x_2 < 7$$

Suppose $x_n < 7$ then $7 + x_n < 7 + 7$, additing 7 to both sides

or $\quad \sqrt{(7 + x_n)} = \sqrt{(14)}$ taking square roots of both sides

or $\quad x_{n+1} = \sqrt{14}, \quad$ from (1)

or $\quad x_{n+1} < \sqrt{49} \quad$ as $14 < 49 \Rightarrow x_{n+1} < 7$

$\therefore \quad x_n < 7 \Rightarrow x_{n+1} < 7$

$\therefore$ By mathematical induction, $x_n < 7, \forall n \in N$.

$\therefore$ The sequence $< x_n >$ is bounded above by 7.

Thus, we find that the sequence $< x_n >$ is monotonic increasing and bounded above and hence it is convergent.

Let $\lim\limits_{n \to \infty} x_n = l$. Then from (1), we have $x_{n+1}^2 = 7 + x_n$

So on taking limit as $n \to \infty$, we have

$$l^2 = 7 + l \quad \text{or} \quad l^2 - l - 7 = 0 \quad \text{or} \quad l = \frac{1}{2}[1 \pm \sqrt{29}]$$

But $\frac{1}{2}[1 - \sqrt{29}] < 0$ and cannot be taken as the limit of $x_n$ when $n \to \infty$ since the sequence $< x_n >$ is bounded below by $\sqrt{7}$ and above by 7.

$$\therefore \quad l = \frac{1}{2}[1 + \sqrt{29}]$$

which is the positive root of equation $l^2 - l - 7 = 0$.

Hence the sequence $< x_n >$ converges to the positive root of the equation

$$x^2 - x - 7 = 0$$

## 4.12 Cauchy's Sequences

A sequence $< s_n >$ is said to be a **Cauchy's sequence** if given $\in > 0$ there exists $m \in N$ such that

$$| s_n - s_m | < \in, \quad \forall n \geq m$$

or $$| s_{n+p} - s_n | < \in, \quad \forall n \geq m \text{ and every } p > 0$$

or $$| s_p - s_q | < \in, \quad \forall p, q \geq m.$$ **[B.C.A. (Meerut) 2007, 03]**

**Example 44:** ***The sequence*** $\left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots \frac{1}{n}, \ldots\ldots \right\rangle$ ***is a cauchy's sequence.***

**Solution:** Let the given sequence $< s_n >$, where $s_n = \frac{1}{n}$ and $\in > 0$. If $n \geq m$, then

$$| s_n - s_m | = \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{m-n}{mn} \right| = \frac{n-m}{mn} = \frac{n-m}{n} \cdot \frac{1}{m} < \frac{1}{m}$$

$$\left[ 0 \leq n - m < n \;\Rightarrow\; 0 \leq \frac{n-m}{n} < 1 \right]$$

If we consider $\frac{1}{m} < \in$, then

$$| s_n - s_m | = \left| \frac{1}{n} - \frac{1}{m} \right| < \in \quad \forall n \geq m$$

Hence, the given sequence is a Cauchy's sequence.

**Theorem 25: If $< s_n >$ is a Cauchy's sequence, then $< s_n >$ is bounded.**

**[B.C.A. (Rohilkhand) 2012, 08, 00; B.C.A. (Agra) 2012, 09; B.C.A. (Kanpur) 2010, 05; B.C.A. (Meerut) 2009, 03, 00]**

**Proof:** Let $< s_n >$ be Cauchy's sequence. Then for every $\in > 0$ *i.e.,* let $\in = 1$, there exists $m \in N$ such that $\quad | s_n - s_m | < 1, \quad \forall n \geq m.$

$\Rightarrow \quad (s_m - 1) < s_n < (s_m + 1), \quad \forall \; n \geq m$

Let $\quad k_1 = \min \{ s_1, s_2, s_3, \ldots\ldots\ldots s_{m-1}, s_m - 1 \}$

$\quad k_2 = \max \{ s_1, s_2, s_3, \ldots\ldots\ldots s_{m-1}, s_m + 1 \}$

$\therefore \quad k_1 \leq s_n \leq k_2, \quad \forall n \in N$

Hence, $< s_n >$ is bounded.

**NOTE:**

The converse of the above theorem need not be true. The sequences $< (-1)^n >$ is bounded but is not a Cauchy's sequence.

**Theorem 26:** **A sequence converges if and only if it is a Cauchy's sequence.**

**Or**

**State and prove Cauchy's convergence criterion.**

**[B.C.A. (Kurukshetra) 2012; B.C.A. (Avadh) 2010; B.C.A. (Meerut) 2010, 07, 06, 02; B.C.A. (Rohilkhand) 2003]**

**Proof :** **The necessary condition:** Let $< s_n >$ be a convergent sequence which converge to $l$. Then show it is Cauchy's sequence.

Since $< s_n >$ converge to $l$, then for every $\in\, > 0$ there exists $m \in N$ such that

$$| s_n - l | < \frac{\in}{2} \quad \forall n \geq m$$

In particular $\quad | s_m - l | < \dfrac{\in}{2}$

$\therefore$
$$| s_n - s_m | = | (s_n - l) - (s_m - l) | \leq | s_n - l | + | s_m - l |$$
$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

Thus $\quad | s_n - s_m | < \in, \quad \forall n \geq m$

$\Rightarrow \quad < s_n >$ is Cauchy's sequence.

**The sufficient condition** : Let $< s_n >$ Cauchy's sequence then show $< s_n >$ is convergent sequence. Since $< s_n >$ is Cauchy's sequence. Then $< s_n >$ is bounded.

$\Rightarrow \quad < s_n >$ has limit point $l$

Since $< s_n >$ is Cauchy's sequences then by definition of Cauchy's sequence

$$| s_n - s_m | < \in/3 \quad \forall n \geq m$$

Since $l$ is limit point of $< s_n >$, therefore every *nbd* of $l$ contains infinite terms of the sequence $< s_n >$. In particular the interval $]\ l - \dfrac{\in}{3},\ l + \dfrac{\in}{3}\ [$ contains infinite terms of $< s_n >$. Hence, there exists a positive integer $k \geq m$ such that

$$l - \frac{\in}{3} < s_k < l + \frac{\in}{3}$$

$\Rightarrow \quad | s_k - l | < \in/3$

$\therefore$
$$| s_n - l | = | (s_n - s_m) + (s_m - s_k) + (s_k - l) |$$
$$\leq | s_n - s_m | + | s_m - s_k | + | s_k - l |$$
$$< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3}, \quad \forall n \geq m.$$

Thus $\quad | s_n - l | < \in, \quad \forall n \geq m$

$\therefore \quad < s_n >$ converges to $l$.

**Example 45:** ***To show the sequence*** $< s_n > = < \dfrac{n+1}{n} >$ ***converges.***

**[B.C.A. (Rohilkhand) 2007, 00]**

**Solution:** Let $s_n = \dfrac{n+1}{n} = 1 + \dfrac{1}{n}$

We shall show that the sequence $<s_n>$ is Cauchy's sequence.

Let $\in > 0$ be given and if $n \geq m$ then

$$|s_n - s_m| = \left|\left(1+\frac{1}{n}\right)-\left(1+\frac{1}{m}\right)\right| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right| = \frac{n-m}{nm}$$

$$= \frac{n-m}{n}\,\frac{1}{m} < \frac{1}{m} \qquad \left[0 \leq n-m < n \Rightarrow 0 < \frac{n-m}{n} < 1\right]$$

If we take $m > \dfrac{1}{\in}$ or $\dfrac{1}{m} < \in$ then

$$|s_n - s_m| < \in, \ \forall n \geq m$$

Hence, the given series is Cauchy's sequence.

**Example 46:** ***Show, by applying Cauchy's convergence criterion that the sequence*** $< s_n >$ ***defined by*** $s_n = 1 + \dfrac{1}{3} + \dfrac{1}{5} + \ldots\ldots + \dfrac{1}{2n-1}$ ***does not convergent.***

**Solution:** We shall show the given sequences is not a Cauchy's sequence. For this we shall show that if we take $\in = \dfrac{1}{4} > 0$, then there exist no positive integer $m$ such that

$$s_n - s_m | < \in \ \ \forall\, n \geq m$$

Let $n = 2m+1 > m$

$$\therefore \qquad |s_n - s_m| = \left|\left(1+\frac{1}{3}+\frac{1}{5}+\ldots..+\frac{1}{2m-1}+\frac{1}{2m+1}+\ldots..+\frac{1}{4m+1}\right)\right.$$

$$\left. - \left(1+\frac{1}{3}+\frac{1}{5}+\ldots..+\frac{1}{2m-1}\right)\right|$$

$$= \frac{1}{2m+1} + \frac{1}{2m+3} + \ldots\ldots.. + \frac{1}{4m+1}$$

$$> \frac{1}{4m+1} + \frac{1}{4m+1} + \ldots\ldots\ldots\ldots + \text{upto } (m+1) \text{ terms}$$

$$= \frac{m+1}{4m+1} = \frac{\frac{1}{4}(4m+1) + \frac{3}{4}}{4m+1} = \frac{1}{4} + \frac{3}{4(4m+1)} > \frac{1}{4}$$

Thus, if we take $\in = \frac{1}{4}$, then whatever positive integer $m$ we take, we have $n = 2m+1 > m$

and $| s_n - s_m | = s_{2m+1} - s_m > \frac{1}{4}$

$\therefore \quad | s_n - s_m | > \in, \forall n \geq m$

$\therefore$ The given sequence is not Cauchy's sequence.

$\Rightarrow$ $< s_n >$ is not convergent by Cauchy's convergence criterion.

***Example 47:*** ***If $< s_n >$ be a sequence of positive number such that***

$$s_n = \frac{1}{2}(s_{n-1} + s_{n-2}), \text{ for all } n > 2.$$

***Then show that $< s_n >$ converges and find $\lim s_n$.***

***Solution:*** In case $s_1 = s_2 \Rightarrow s_n = s_1$, therefore $< s_n >$ converge to $s_1$.

Now we consider the case $s_1 \neq s_2$

$$\therefore \quad | s_n - s_{n-1} | = \left| \frac{1}{2}(s_{n-1} + s_{n-2}) - s_{n-1} \right| = \frac{1}{2} | s_{n-1} - s_{n-2} |$$

$$= \frac{1}{2} \cdot \frac{1}{2} | s_{n-2} - s_{n-3} | = \frac{1}{2^2} | s_{n-2} - s_{n-3} |$$

$$= \frac{1}{2^{n-2}} | s_2 - s_1 |, \text{ for } n \geq 2 \qquad ...(1)$$

Now for $n \geq m$, we have

$$| s_n - s_m | = | (s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \ldots\ldots + (s_{m+1} - s_m) |$$

$$\leq | s_n - s_{n-1} | + | s_{n-1} - s_{n-2} | + \ldots\ldots + | s_{m+1} - s_m |$$

$$= \left( \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \ldots\ldots\ldots + \frac{1}{2^{m-1}} \right) | s_2 - s_1 | \qquad \text{(using (1)}$$

$$= \frac{1}{2^{m-1}} \left[ 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-m-1}} \right] | s_2 - s_1 |$$

$$< \frac{1}{2^{m-2}} | s_2 - s_1 | \qquad ...(2)$$

$$\left[ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots\ldots + \left(\frac{1}{2}\right)^{n-m-1} < 2 \right]$$

Let $\in > 0$ be given, we can choose a positive integer $m$ such that

$$\frac{1}{2^{m-2}} | s_2 - s_1 | < \in$$

Then from (2)

$$|s_n - s_m| < \in \quad \forall n \geq m$$

Hence $< s_n >$ is a Cauchy's sequence and therefore by Cauchy's convergence criterian it converges.

Let lim $s_n = l$. Putting $n = 3, 4 \ldots . k$ in the relation.

$$\left.\begin{aligned} s_n &= \frac{1}{2}(s_{n-1} + s_{n-2}) \\ s_3 &= \frac{1}{2}(s_2 + s_1) \\ s_4 &= \frac{1}{2}(s_3 + s_2) \end{aligned}\right\} \qquad \ldots(3)$$

$$\left.\begin{aligned} s_{k-1} &= \frac{1}{2}(s_{k-2} + s_{k-3}) \\ s_k &= \frac{1}{2}(s_{k-1} + s_{k-2}) \end{aligned}\right\}$$

Adding the corresponding sides the relation in (3), we find

$$s_k + \frac{1}{2}\ s_{k-1} = \frac{1}{2}\ (s_1 + 2\ s_2)$$

Taking limit $k \to \infty$ on both sides

$$\lim_{k \to \infty} s_k + \frac{1}{2} \lim_{k \to \infty} s_{k-1} = \frac{1}{2}\ (s_1 + 2\ s_2)$$

$$\Rightarrow \qquad l + \frac{l}{2} = \frac{1}{2}\ (s_1 + 2\ s_2)$$

$$\Rightarrow \qquad l = \frac{1}{3} \quad (s_1 + 2\ s_2).$$

***Example 48:*** ***Let*** $< u_n >$ ***be a sequence and*** $s_n = u_1 + u_2 + \ldots\ldots + u_n$. ***If*** $t_n = | u_1 | + | u_2 | + \ldots\ldots\ldots + | u_n |$ ***for each*** $n \in N$ ***and*** $< t_n >$ ***is a Cauchy's sequence, then*** $< s_n >$ ***is also Cauchy's sequence.***

***Solution:*** Let $\in > 0$ be given, since $< t_n >$ is a Cauchy's sequence, therefore, for given $\in > 0$, there exists $m \in N$ such that

$$| t_n - t_m | < \in \ \forall n \geq m$$

$\Rightarrow$ $\qquad | u_{m+1} | + | u_{m+2} | + \ldots\ldots\ldots + | u_n | < \in \quad \forall n \geq m$

But $\qquad | u_{m+1} | + | u_{m+2} | + \ldots\ldots\ldots + | u_n | \geq | u_{m+1} + u_{m+2} + \ldots\ldots + u_m |$

$\therefore$ $\qquad | u_{m+1} + u_{m+2} + \ldots\ldots + u_n | < \in \ \forall n \geq m$

or $\qquad | s_n - s_m | < \in \forall n \geq m$

Hence, $< s_n >$ is a Cauchy's sequence.

**Example 49:** ***Show that the sequence $< s_n >$ defined by***

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+n}, \text{ converges.}$$

**Solution:** We have

$$s_{n+1} - s_n = \left(\frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}\right)$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0, \text{ for all } n$$

Hence, the sequence $< s_n >$ is monotonically increasing

Now $$|s_n| = s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+n} < \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}$$

$$= n \cdot \frac{1}{n} = 1$$

$\therefore \quad |s_n| < 1$, for all $n$

Hence, the sequence $< s_n >$ is bounded.

Since $< s_n >$ is bounded, monotonically increasing sequence, hence, it converges.

**Example 50:** ***Show that $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ exists and lies between 2 and 3.***

**[B.C.A. (Aligarh) 2012, 08, 06; B.C.A. (Kashi) 2012; B.C.A. (Meerut) 2001]**

**Solution:** Here $s_n = \left(1 + \frac{1}{n}\right)^n \Rightarrow s_1 = 2$

By the binomial theorem, we find

$$s_n = 1 + n.\frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots\ldots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots\ldots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{n-1}{n}\right) \quad \ldots(1)$$

Similarly

$$s_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \ldots\ldots$$

$$+ \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\ldots\ldots\left(1 - \frac{n}{n+1}\right)$$

$$> 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots\ldots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\ldots\left(1 - \frac{n-1}{n}\right)$$

$\therefore \qquad s_{n+1} > s_n, \ \forall n \in N$

Hence, the sequence $<s_n>$ is monotonically increasing

$\therefore \qquad s_n \geq s_1 = 2 \ \ \forall n \in N$

From (1) we see that

$$s_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} < 1 + \frac{1}{1 - \frac{1}{2}} = 3, \text{ for all } n$$

Thus, $2 \leq s_n < 3$, for all

Hence, the sequence $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence,

$\Rightarrow \quad \lim\limits_{n \to \infty} s_n$ exists and $\lim s_n = \sup < s_n >$

Now $2 \leq s_n < 3$ for all $n \Rightarrow 2 < \lim\limits_{n \to \infty} s_n < 3$

$\Rightarrow$ The limit lies between 2 and 3.

***Example 51:*** ***Discuss the convergence of the sequence*** $< \dfrac{1}{3n} >$. **[B.C.A. (Meerut) 2004]**

***Solution:*** For the convergence the given sequence, first we check that whether the given sequence is a Cauchy's sequence or not.

Given sequence $\qquad < \dfrac{1}{3n} > = \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{9} + \dfrac{1}{12} + \ldots\ldots + \dfrac{1}{3n}$

$$= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n}\right)$$

Let us choose a very small positive quantity $\in > 0$

Such that $| s_n - s_m | < \in \ \forall n \geq m$

where $m$ is a positive integer, if we take $n = 2m$

$$| s_n - s_m | = | s_{2m} - s_m |$$

$$= \left| \frac{1}{3.2m} - \frac{1}{3m} \right|$$

$$= \frac{1}{3} \left| \frac{1}{2m} - \frac{1}{m} \right|$$

$$= \frac{1}{3} \left| \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{2m} \right|, \qquad \begin{bmatrix} \frac{1}{m+1} < \frac{1}{m} \\ \frac{1}{m+2} < \frac{1}{m} \end{bmatrix}$$

$$> \frac{1}{3} \left| \frac{1}{2m} + \frac{1}{2m} + \ldots\ldots + m \text{ terms} \right|$$

$$= \frac{1}{3} \cdot m \cdot \frac{1}{2m} = \frac{1}{6}$$

If $\in = \frac{1}{6}$, then for any positive integer $m$, we have $n = 2m > m$ and

$$| s_n - s_m | = | s_{2m} - s_m | > \frac{1}{6}$$

$$| s_n - s_m | > \frac{1}{6} > \in$$

Thus there is no positive integer $m$ such that

$$| s_n - s_m | < \in \quad \forall n \geq m$$

So given sequence is not a Cauchy's sequence and that is why it is not convergent (By Cauchy's convergent criterion).

***Example 52:*** ***Show that the sequence $< s_n >$ defined by***

***$s_1 = \frac{1}{2}$, $s_{n+1} = \frac{2 s_n + 1}{3}$ $\forall n \in N$ is convergent. Also find its limit.*** **[B.C.A. (Delhi) 2007]**

***Solution:*** We have $\quad s_1 = \frac{1}{2}$ and $s_{n+1} = \frac{2s_n + 1}{3} \quad \forall n \in N \qquad \ldots(1)$

By mathematical induction we shall show that

$$s_{n+1} > s_n \quad \forall n \in N$$

When $n = 1$ in (1)

$\Rightarrow$ $$s_2 = \frac{2s_1 + 1}{3} = \frac{2\left(\frac{1}{2}\right) + 1}{3} = \frac{2}{3}$$

$\therefore$ $$s_2 > s_1$$

Suppose that $\quad s_{n+1} > s_n$

Then $\quad s_{n+1} > s_n \Rightarrow 2s_{n+1} > 2s_n$

$\Rightarrow \quad 2s_{n+1} + 1 > 2s_n + 1$

$$\Rightarrow \quad \frac{2s_{n+1}+1}{3} > \frac{2s_n+1}{3}$$

$$\Rightarrow \quad s_{n+2} > s_{n+1}$$

Thus $s_2 > s_1$ and $s_{n+1} > s_n$, then $s_{n+2} > s_{n+1}$

Hence $s_{n+1} > s_n \ \forall n \in N$

Thus, the sequence $< s_n >$ is monotonic increasing

Now, we shall show that the sequence $< s_n >$ is also bounded above

Now $\qquad s_{n+1} > s_n \quad \forall n \in N$

$$\Rightarrow \quad \frac{2s_n+1}{3} > s_n \quad \forall n \in N$$

$$\Rightarrow \quad 2s_n + 1 > 3s_n \quad \forall n \in N$$

$$\Rightarrow \quad s_n < 1 \quad \forall n \in N$$

$\therefore$ the sequence $< s_n >$ is bounded above by 1

Since the sequence $< s_n >$ is monotonic increasing and bounded above, therefore by monotone convergence theorem $< s_n >$ converges to its supremum.

Let $\qquad \lim_{n\to\infty} s_n = l$. Then $\lim_{n\to\infty} s_{n+1} = l$.

Now $\qquad s_{n+1} = \dfrac{2\,s_n+1}{3} \Rightarrow \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \dfrac{2s_n+1}{3}$

$$\Rightarrow \quad l = \frac{2\lim s_n + 1}{3}$$

$$\Rightarrow \quad l = \frac{2\,l+1}{3}$$

$$\Rightarrow \quad l = 1$$

Hence $s_n \to 1$, we have $\inf < s_n > \ = s_1 = \dfrac{1}{2}$

and $\sup < s_n > = \lim s_n = 1$.

***Example 53:** A sequence $< s_n >$ of positive terms is defined by*

$$\mathbf{s_1 = k > 0 : s_{n+1} = \frac{3+2s_n}{2+s_n}, \quad \forall n \in N}$$

***Show that the sequence converges to a limit independent of k and find the limit.***

**Solution:** We have $s_1 = k > 0$ and $s_{n+1} = \dfrac{3+2s_n}{2+s_n}, \quad \forall n \in N$

Then $s_2 > 0$, $s_3 > 0$ and so on

Therefore the terms of the sequence are all positive, then by mathematical induction we shall show that

$$s_{n+1} > s_n \quad \forall n \in N$$

we have $$s_2 - s_1 = \frac{3+2s_1}{2+s_1} - s_1 = \frac{3+2k}{2+k} - k = \frac{3-k^2}{2+k} > 0$$

if $$0 < k < \sqrt{3}$$

Thus $s_2 > s_1$ if $$0 < k < \sqrt{3}$$

Suppose $s_{n+1} > s_n$ ...(1)

Then $$s_{n+2} - s_{n+1} = \frac{3+2s_{n+1}}{2+s_{n+1}} - \frac{3+2s_n}{2+s_n}$$

$$= \frac{s_{n+1} - s_n}{(2+s_n)(2+s_{n+1})} > 0 \text{ (by (1))}$$

$\therefore$ $$s_{n+2} > s_{n+1}$$

Thus $s_2 > s_1$ and $s_{n+1} > s_n$, then we have also $s_{n+2} > s_{n+1}$

$\therefore$ by induction $s_{n+1} > s_n \quad \forall n \in N$

$$\Rightarrow \quad \frac{3+2\,s_n}{2+s_n} > s_n \;\Rightarrow\; \frac{3+2s_n}{2+s_n} - s_n > 0$$

$$\Rightarrow \quad \frac{3-s_n^2}{2+s_n} > 0 \;\Rightarrow\; 3 - s_n^2 > 0 \;\Rightarrow\; s_n^2 < 3$$

$$\Rightarrow \quad s_n < \sqrt{3} \quad \forall n \in N$$

$\therefore$ $< s_n >$ is bounded above by $\sqrt{3}$

Thus $< s_n >$ is bounded monotonically increasing sequence.

Hence it is convergent.

Let $\lim s_n = l$, Then $\lim s_{n+1} = l$

Now $$s_{n+1} = \frac{3+2s_n}{2+s_n} \Rightarrow \lim_{n\to\infty} (s_{n+1}) = \lim_{n\to\infty} \frac{3+2s_n}{2+s_n}$$

$$\Rightarrow \quad \lim_{n\to\infty} (s_{n+1}) = \frac{3 + 2\lim\limits_{n\to\infty} s_n}{2 + \lim\limits_{n\to\infty} s_n}$$

$$\Rightarrow \quad l = \frac{3+2l}{2+l} \Rightarrow l^2 = 3, \Rightarrow l = \pm\sqrt{3}$$

But $l$ cannot be negative because the terms of the sequence $< s_n >$ are all positive.

Hence $l = \sqrt{3}$ is independent of $k$.

***Example 54:*** ***Show that the sequence*** $<s_n>$ ***defined by*** $s_1 = 1$ ***and*** $s_{n+1} = \sqrt{2+s_n}\ \ \forall n \in N$ ***is monotonically increasing and bounded. Also find its limit.*** **[B.C.A. (Delhi) 2010, 08, 02]**

***Solution:*** We have $\quad s_1 = 1$ and $(s_{n+1})^2 = 2 + s_n \quad \forall n \in N$

$\therefore \quad s_2 = \sqrt{3},\ \ s_3 = \sqrt{(2+\sqrt{3})}\ldots..$

Now $\quad 1 < \sqrt{3} \ \Rightarrow\ s_1 < s_2$

Let us suppose that $s_m < s_{m+1}$, then $\sqrt{(2+s_m)} < \sqrt{(2+s_{m+1})}$

$\Rightarrow \quad s_{m+1} < s_{m+2}$

Hence by mathematical induction, we have

$$s_n < s_{n+1},\ \ \forall n \in N$$

*i.e.,* $< s_n >$ is monotonically increasing

Again, $\quad s_{n+1} > s_n \ \Rightarrow \sqrt{(2+s_n)} > s_n$

$$\Rightarrow\ 2 + s_n - s_n^2 > 0$$

$$\Rightarrow (2 - s_n)(1 + s_n) > 0$$

$$\Rightarrow (2 - s_n) > 0$$

$$\Rightarrow s_n < 2,\ \forall n \in N$$

Hence $< s_n >$ is bounded.

Thus $< s_n >$ is a monotonically increasing sequence bounded above by 2. $\Rightarrow$ it converges.

Let $\ \lim\ s_n = l$. Then $\lim\ (s_{n+1}) = l$

Now $\quad s_{n+1} = \sqrt{2+s_n} \ \Rightarrow\ \lim\,(s_{n+1}) = \lim \sqrt{(2+s_n)}$

$\Rightarrow \quad l = \sqrt{2+l} \ \ \Rightarrow\ l^2 - l - 2 = 0$

$\Rightarrow \quad (l+1)\ (l-2) = 0$

$\Rightarrow \quad l = -1,\ l = 2$

But $\ l \neq -1$, then $l = 2$

## 4.13 Limit Superior and Limit Inferior of a Sequence

Let $<s_n>$ be a sequence which is bounded above. Then, for each fixed $n \in N$, the set $\{s_n, s_{n+1} \ldots\ldots\}$ is bounded above and hence it must have a supremum. Let

$$\bar{s}_n = \sup\{s_n, s_{n+1} \ldots\ldots\ldots\}.$$

Since $\{s_{n+1}, s_{n+2} \ldots\ldots\}$ is a subset of $\{s_n, s_{n+1} \ldots\ldots\ldots\}$ therefore, it is obvious that $\bar{s}_n \geq \bar{s}_{n+1}$. Thus the sequence $<\bar{s}_n>$ is a monotonically decreasing sequence and consequently, it either converges or else it diverges to $-\infty$.

Similarly, if the sequence $<s_n>$ is bounded below, then the set $\{s_n, s_{n+1} \ldots\ldots\ldots\}$ has an infimum. Let $\underline{s_n} = \inf\{s_n, s_{n+1}, \ldots\ldots\ldots\}$ then the sequence $<s_n>$ is monotonically increasing and hence it either converges or diverges to $\infty$.

Keeping these notations in mind we now define **limit superior and limit inferior** these notations in mind we now define limit superior and limit inferior.

### 4.13.1 Definition 1

Let $<s_n>$ be a sequence of real number which is bounded above and let $\bar{s}_n = \sup\{s_n, s_{n+1}, \ldots\ldots\}$.

If $<\bar{s}_n>$ converges we define the limit superior of $<s_n>$ by

$$\lim_{n \to \infty} \sup s_n = \lim_{n \to \infty} \bar{s}_n$$

If $<\bar{s}_n>$ diverges to $-\infty$, we write $\lim\limits_{n \to \infty} \sup s_n = -\infty$ **[B.C.A. (Agra) 2007]**

If a sequence $<s_n>$ is not bounded above, we write

$$\lim_{n \to \infty} \sup s_n = \infty$$ **[B.C.A. (Delhi) 2008, 04, 02]**

### 4.13.2 Definition 2

Let $<s_n>$ be a sequence of real number which is bouned below and let $\underline{s_n} = \inf\{s_n, s_{n+1} \ldots\ldots\}$

If $<\underline{s_n}>$ converges we define the limit inferior of $<s_n>$ by

$$\lim_{n \to \infty} \inf s_n = \lim_{n \to \infty} \underline{s_n}$$

If $<\underline{s_n}>$ diverges to $\infty$, we write $\lim\limits_{n \to \infty} \inf s_n = \infty$. **[B.C.A. (Rohilkhand) 2003]**

If *a* sequence $<s_n>$ is not bounded below, we write

$$\lim_{n \to \infty} \inf s_n = -\infty$$ **[B.C.A. (Meerut) 2003]**

**NOTE:**

1. The notations $\overline{\lim}\ s_n$ and $\underline{\lim}\ s_n$ are also used for lim sup $s_n$ and lim inf $s_n$ respectively. In future, we shall use these notations.

2. The limit superior and the limit inferior are also called the **upper limit** and the **lower limit** of $< s_n >$ respectively.

3. We have $\overline{\lim}\ s_n = \inf\{\bar{s}_1, \bar{s}_2, \ldots\ldots\ldots, \bar{s}_n, \ldots\ldots\ldots.\}$

   and $\underline{\lim}\ s_n = \sup\{\underline{s_1}, \underline{s_2}, \ldots\ldots\ldots \underline{s_n}, \ldots\ldots\ldots\}$.

## 4.13.3 Illustrations

1. Let $< s_n >$ be the sequence defined by $s_n = (-1)^n \forall n \in N$.

   It is bounded above by 1 and bounded below by $-1$. For this sequence, $\bar{s}_n = 1$ and $\underline{s_n} = -1$ for all $n \in N$.

   Hence $\overline{\lim}\ s_n = 1$ and $\underline{\lim}\ s_n = -1$.

2. Let $< s_n >$ be sequence defined by $s_n = -n \ \forall\ n \in N$. It is bounded above by $-1$ but it is not bounded below.

$$\bar{s}_n = \sup\{-n, -n-1, \ldots\ldots\} = -n$$

   Since $\bar{s}_n \to -\infty$ as $n \to \infty$. Hence $\overline{\lim}\ s_n = -\infty$. Also since $< s_n >$ is not bounded below, by definition $\underline{\lim}\ s_n = -\infty$. Thus in this sequence both the limit superior and the limit inferior are $-\infty$.

3. Let $< s_n >$ be the sequence defined by $s_n = n \ \ \forall n \in N$. It is bounded below but not bounded above.

$$\underline{s_n} = \inf\{n, n+1, \ldots\ldots\} = n$$

   Since $\underline{s_n} \to \infty$ as $n \to \infty$, hence $\underline{\lim}\ s_n = \infty$.

   Also, since $< s_n >$ is not bounded above, by definition $\overline{\lim}\ s_n = \infty$. Thus in this sequence both the limit superior and the limit inferior are $\infty$.

4. Let $< s_n >$ be the sequence defined by $s_n = (-1)^n \left(1 + \frac{1}{n}\right)$.

   Then $< s_n > = < -2, \frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, -\frac{6}{5}, \frac{7}{6}, \ldots. >$

In this case $\bar{s}_1 = \frac{3}{2}$, $\bar{s}_2 = \frac{3}{2}$, $\bar{s}_3 = \frac{5}{4}$, $\bar{s}_4 = \frac{5}{4}$, $\bar{s}_5 = \frac{7}{6}$ etc.

and $\underline{s_1} = -2$, $\underline{s_2} = \frac{4}{3}$, $s_3 = -\frac{4}{3}$, $s_4 = -\frac{6}{5}$ etc.

Hence $\overline{\lim}\, s_n = \inf\left\{\frac{3}{2}, \frac{5}{4}, \frac{7}{6} \ldots\ldots\right\} = 1$

and, $\underline{\lim}\, s_n = \sup\left\{-2 - \frac{4}{3}, -\frac{6}{5}, \ldots\ldots\right\} = -1$

**Theorem 27: If $< s_n >$ is a convergent sequence of real numbers and if $\lim s_n = l$, then $\overline{\lim}\, s_n = \underline{\lim}\ s_n = l$. Conversely, if**

$$\overline{\lim}\, s_n = \underline{\lim}\, s_n = l \in R$$

**then $< s_n >$ is convergent and $\lim\limits_{n \to \infty} s_n = l$.**

**Proof:** First suppose that the sequence $< s_n >$ converges with $\lim s_n = l$. Let $\varepsilon > 0$ be given. Since $s_n \to l$, therefore, we can find a positive integer $m$, such that

$$| s_n - l | < \varepsilon \text{ for all } n \geq m$$

*i.e.,* $$l - \varepsilon < s_n < l + \varepsilon \text{ for all } n \geq m$$

This inequality shows that for all $n \geq m, l + \varepsilon$ is an upper bound of $\{s_n, s_{n+1} \ldots\}$ and $l - \varepsilon$ is not an upper bound of $\{s_n, s_{n+1} \ldots\ldots\}$.

Since $\overline{s_n} = \sup\{s_n, s_{n+1}, \ldots\ldots\}$ it follows that

$$l - \varepsilon < \bar{s}_n \leq l + \varepsilon, n \geq m$$

Taking limits as $n \to \infty$, we get

$$l - \varepsilon \leq \lim_{n \to \infty} \bar{s}_n \leq l + \varepsilon$$

Since $\varepsilon$ is arbitrary, it follows that $\overline{\lim}\, s_n = l$

Similarly, we can show that $\underline{\lim}\ s_n = l$

Thus $$\overline{\lim}\ s_n = \underline{\lim}\, s_n = l$$

**Conversely,** let $\overline{\lim}\ s_n = \underline{\lim}\ s_n = l$

Since $l = \lim\limits_{n\to\infty} \bar{s}_n$, given any $\varepsilon > 0$, there exists $m_1 \in N$ such that.

$$|\bar{s}_n - l| < \varepsilon \text{ for } n \geq m_1 \ i.e., \ l - \varepsilon < \bar{s}_n < 1 + \varepsilon \text{ for } n \geq m_1$$

The definition of $\bar{s}_n$ then gives that

$$s_n < l + \varepsilon \text{ for } n \geq m_1 \qquad ...(1)$$

Similarly, since $l = \lim\limits_{n\to\infty} \underline{s_n}$, there exists $m_2 \in N$ such that

$$\left| \underline{s_n} - l \right| < \varepsilon \text{ for } n \geq m_2$$

which implies as above that $s_n > 1 - \varepsilon$ for $n \geq m_2$ ...(2)

Let $m = \max, \{m_1, m_2\}$. Then from (1) and (2) we find that

$$\left| s_n - l \right| < \varepsilon \text{ for } n \geq m$$

This proves that the sequence $< s_n >$ converges and that

$$\lim_{n\to\infty} s_n = l$$

Similar results hold good for divergent sequences. Below we state them without proof.

***Illustration 1:*** A sequence $< s_n >$ diverges to $+\infty$ iff $\overline{\lim}\ s_n = \underline{\lim}\ s_n = \infty$.

***Illustration 2:*** A sequence $< s_n >$ diverges to $-\infty$ iff $\overline{\lim}\ s_n = \underline{\lim}\ s_n = -\infty$.

**Theorem 28:** **If $< s_n >$ and $< t_n >$ are bounded sequences of real numbers such that $s_n \leq t_n$ for all $n \in N$, then $\overline{\lim}\, s_n \leq \overline{\lim}\, t_n$ and $\underline{\lim}\, s_n \leq \underline{\lim}\ t_n$.**

**Proof:** Since $s_n \leq t_n$, therefore it is easy to see that

$$\bar{s}_n \leq \bar{t}_n \text{ and } \underline{s_n} \leq \underline{t_n}$$

Where $\overline{s_n}, \overline{t_n}, \underline{s_n}, \underline{t_n}$ have their usual meansings as deined earlier.

$$\lim \bar{s}_n \leq \lim \bar{t}_n \text{ and } \lim \underline{s_n} \leq \lim \underline{t_n}$$

or $$\overline{\lim}\ s_n \leq \overline{\lim}\ t_n \text{ and } \underline{\lim}\ s_n \leq \underline{\lim}\ t_n$$

**Theorem 29:** **If $< s_n >$ and $< t_n >$ are bounded sequences of real number, then**

**(i)** $\overline{\lim}\,(s_n + t_n) \le \overline{\lim}\, s_n + \overline{\lim}\, t_n$ ;

**(ii)** $\underline{\lim}\,(s_n + t_n) \ge \underline{\lim}\, s_n + \underline{\lim}\, t_n$.

**Proof:** Let $\bar{s}_n = \sup\{s_n, s_{n+1}, .......\}$, and $\bar{t}_n = \sup\{t_n, t_{n+1}, .....\}$

Then $$s_k \le \bar{s}_n, (k \ge n), t_k \le \bar{t}_n, (k \ge n)$$

$\therefore$ $$s_k + t_k \le \bar{s}_n + \bar{t}_n \text{ for } k \ge n$$

Thus $\bar{s}_n + \bar{t}_n$ is an upper bound for $\{s_n + t_n, s_{n+1} + t_{n+1} .......\}$.

Hence $$\overline{(s_n + t_n)} = \sup\{s_n + t_n, s_{n+1} + t_{n+1}, .....\} \le \bar{s}_n + \bar{t}_n$$

$\therefore$ $$\lim \overline{(s_n + t_n)} \le \lim(\bar{s}_n + \bar{t}_n) = \lim \bar{s}_n + \lim \bar{t}_n$$

*i.e.*, $$\overline{\lim}\,(s_n + t_n) \le \overline{\lim}\, s_n + \overline{\lim}\, t_n$$

Thus the result (i) has been proved. Similarly (ii) can be proved.

## Exercise 4.2

1. If $s_n = k\ (\in R)$ is constant sequence, then $\lim s_n = k$.

2. Show that the sequence $< s_n >$ where $s_n = \sin n\pi\theta$ and $\theta$ is a rational number such that $0 < \theta < 1$ is not convergent.

3. Prove that the sequence $< n^P >$ where $p >$ diverges to infinity.

4. Show that the sequence $< \log \frac{1}{n} >$ diverge to $(-\infty)$.

5. If $< t_n >$ diverges to $\infty$ and $s_n > t_n\ \forall n$, then $< s_n >$ diverge to $\infty$.

6. Show that the sequence $< s_n >$ defined by the relation $s_1 = 2$

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\ldots + \frac{1}{(n-1)!} \quad (n \geq 2) \text{ converges.}$$

7. Prove that $< s_n >$ is convergent where $s_n = 2 - \frac{1}{2^{n-1}}$.

8. Show that the sequence $< s_n >$ defined by $s_1 = \sqrt{2},\ \ s_{n+1} = \sqrt{2s_n}$ converge to 2.

9. Show that the sequence $< s_n >$ defined by $s_1 = 1,\ \ s_{n+1} = \frac{4 + 3s_n}{3 + 2s_n},\ n \in N$ is convergent and find its limit. **[B.C.A. (Lucknow) 2011]**

10. Show that the sequence $< s_n >$ defined by $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}$ does not converge.

11. If $< s_n >$ be a sequence of positive numbers such that $s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$, for all $n > 2$, then show that $< s_n >$ converges and find $\lim (s_n)$.

12. If $r > 0$, show that $(r)^{1/n} = 1$.

13. If $p > 0$ and $c$ is real, then find $\lim \frac{n^c}{(1+p)^n}$.

14. The usual definition of $e$ is given by $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. Show that $e$ is irrational.

15. If $s_n = \frac{n}{2^n}$, prove that $< s_n > \to 0$. **[B.C.A. (Meerut) 2004]**

16. Show that $\lim \left[\frac{1}{(n+1)^\lambda} + \frac{1}{(n+2)^\lambda} + \ldots\ldots + \frac{1}{(2n)^\lambda}\right] = 0,\ \ \lambda > 1.$

**17.** A sequence $< s_n >$ is defined as follows:

$$s_1 = a > 0,\ s_{n+1} = \sqrt{\{ab^2 + s_n^2\}}\,/\,(a+1),\ b > a,\ \ n \geq 1.$$

Show that $< s_n >$ is a bounded monotonically increasing sequence and $\lim\ s_n = b$.

**18.** If $s_n = \dfrac{2n-7}{3n+2}$, prove that $< s_n >$ is increasing and bounded. **[B.C.A. (Agra) 2006]**

**19.** If $s_n = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \ldots\ldots + \dfrac{1}{n(n+1)}$, prove that $< s_n >$ is increasing and convergent.

**20.** Prove that the sequence $< s_n >$ where $s_n = \dfrac{n+1}{n}$ converges. **[B.C.A. (Kashi) 2012]**

**21.** If $s_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots\ldots. + \dfrac{1}{n} - \log n$, prove that $< s_n >$ is decreasing and bounded.

**22.** Define with examples:

(i) Real sequence

(ii) Range of sequence **[B.C.A. (Meerut) 2010, 05]**

**23.** Define monotonic sequence with example. **[B.C.A. (Meerut) 2010, 07, 03, 02]**

**24.** Define monotonically increasing and decreasing sequence, give examples in your support. **[B.C.A. (Delhi) 2012, 04, 01]**

**25.** Define Cauchy's sequence with examples. **[B.C.A. (Lucknow) 2011]**

**26.** Define Cauchy's sequence, if $< s_n >$ is Cauchy's sequence, then $< s_n >$ is bounded. Is converse true then give example in support. **[B.C.A. (Delhi) 2007, 05, 03]**

## Answers 4.2

| | |
|---|---|
| **9.** | $l = \sqrt{2}$ |
| **11.** | $l = \dfrac{1}{3}(s_1 + 2s_2)$ |
| **13.** | 0 |

OOO

# Chapter 5

# Infinite Series

## 5.1 Definition

**Series:** *"An expression of the form $u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots.$ in which every term is followed by another term by some definite rule is called series".*

***For example:***

1. $1 + 3 + 5 + 7 + \ldots\ldots$ is a series.
2. $2 + 2^2 + 2^3 + \ldots\ldots.$ is a series.
3. $1 + 2 + 7 + 8 + 11 + 12 + \ldots\ldots\ldots$ is not a series.

Because every term is not followed by another term by a definite rule.

## 5.2 Finite Series

A series in which numbers of terms are finite is called **finite series.** It may be expressed as

$u_1 + u_2 + u_3 + \ldots\ldots + u_n$ or $\sum_{r=1}^{n} u_r$.

## 5.3 Infinite Series

A series in which numbers of terms are infinite is called **infinite series.** It may be expressed as $u_1 + u_2 + \ldots\ldots\ldots + u_n + \ldots\ldots\ldots$ or $\sum_{n=1}^{\infty} u_n$ or simply by $\sum u_n$, where $u_n$ is called **general terms** or $n$th term of the sequence.

In this chapter, we shall deal with the techniques of testing the infinite series as regards the given series is convergent, divergent or an oscillatory.

## 5.3.1 Behaviour of an Infinite Series

If $\Sigma u_n = u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots..$ be as infinite series such that $S_n$ is the sum of its first $n$ terms.

1. **Convergent Series:** A series $\sum u_n$ is said to be convergent if the sum of its first $n$ terms $S_n$ can not exceed numerically a finite quantity however $n$ large may be *i.e.,* $S_n$ tends to a finite and unique limit as $n$ tends to infinity. Therefore, $\sum u_n$ is convergent.

   If $\lim\limits_{n\to\infty} S_n =$ finite unique number.

2. **Divergent Series:** A series $\sum u_n$ is said to be divergent if the sum of its first $n$ terms tends to $+\infty$ or $-\infty$ as $n$ terms to infinity.

3. **Oscillatory Series:** The oscillatory series are of two types :

   (i) A series $\sum u_n$ is said to oscillates finitely if the sum of its first $n$ terms $S_n$ tends to a finite number but not unique limit as $n$ tends to infinity *i.e.,* $\sum u_n$ oscillates finitely.

   If $\lim\limits_{n\to\infty} S_n =$ finite number but not unique.

   (ii) A series $\sum u_n$ is said to oscillates infinitely if the sum of its first $n$ terms $S_n$ is $+\infty$ and $-\infty$ both as $n$ tends to infinity *i.e.,* $\sum u_n$ oscillates infinitely.

   If $\lim\limits_{n\to\infty} S_n = +\infty$ and $-\infty$ both.

## Solved Examples

***Example1 : Test the convergence of the following series:***

***(i)*** $\sum u_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$

***(ii)*** $\sum u_n = 1 + 2 + 3 + 4 + \ldots\ldots.$

***(iii)*** $\sum u_n = -1 - 2 - 3 - 4 - 5 - \ldots\ldots.$

***(iv)*** $\sum u_n = 1 - 1 + 1 - 1 + 1 - \ldots\ldots$

***(v)*** $\sum u_n = 1 - 2 + 3 - 4 + 5 - 6 + \ldots\ldots$

***Solution :*** (i) $\sum u_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$ is a geometric series with common ratio $\frac{1}{2} < 1$.

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots + \frac{1}{2^{n-1}}$$

$$= \frac{1[1-(1/2)^n]}{1/2} = 2\left(1 - \frac{1}{2^n}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right) = 2\left(1-\frac{1}{2^\infty}\right) = 2\left(1-\frac{1}{\infty}\right) = 2$$

which is finite unique quantity. Hence, the series is convergent.

(ii) $\sum u_n = 1+2+3+4+\ldots\ldots$

$S_n = 1+2+3+4+\ldots\ldots + n =$ Sum of first $n$ natural numbers $= \frac{1}{2}n(n+1)$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}(n)(n+1) = \frac{1}{2} \times \infty \times (\infty + 1) = \infty$$

Hence, the series is divergent and diverges to $\infty$.

(iii) $\sum u_n = -1-2-3-4-\ldots\ldots$

$$S_n = -1-2-3-4-\ldots\ldots-n = -(1+2+3+\ldots\ldots+n)$$

$$= -\text{ (Sum of first of natural numbers)}$$

$$= -\frac{1}{2}n(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left\{-\frac{1}{2}n(n+1)\right\} = -\frac{1}{2} \times \infty(\infty+1) = -\infty$$

Hence, the series is divergent and diverges to $-\infty$.

(iv) $\sum u_n = 1-1+1-1+1-\ldots$

$$S_n = \begin{cases} 0, \text{ if n is even} \\ 1, \text{ if n is odd} \end{cases}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \begin{cases} 0, \text{ if n is even} \\ 1, \text{ if n is odd} \end{cases}$$

$$= 0 \text{ if } n \text{ is even and } 1 \text{ if } n \text{ is odd.}$$

*i.e.,* $\lim_{n\to\infty} S_n$ is finite but no unique.

Hence, $\sum u_n$ oscillates finitely.

(v) $\sum u_n = 1-2+3-4+5-\ldots\ldots$

$$S_1 = 1, S_2 = -1, S_3 = 2, S_4 = -2, S_5 = 3, S_6 = -3$$

Thus, $$S_n = \begin{cases} -\dfrac{n}{2}, & \text{if } n \text{ is even} \\ \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(-\frac{n}{2}\right) = -\infty, \text{ if } n \text{ is even}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{n+1}{2}\right) = \infty, \text{ if } n \text{ is odd}$$

*i.e.,* $$\lim_{n\to\infty} S_n = -\infty \text{ and } \infty$$

Hence, $\sum u_n$ oscillates infinitely.

***Example 2:*** ***Discuss the convergence of a geometric series.***

**[B.C.A. (Kurukshetra) 2012, 10; B.C.A. (Meerut) 2010, 07]**

***Solution :*** Consider a geometric series with first terms $a$ and common ratio $x$ *i.e.,*

$$\sum u_n = a + ax + ax^2 + ax^3 + \ldots..$$

We shall discuss the convergence of the series for all real number $x$.

*i.e.,* $$-1 < x < 1, x = 1, \; x = -1, x > 1 \text{ and } x < -1$$

(i) When $-1 < x < 1$ or $|x| < 1$

We know that $\lim_{n\to\infty} x^n = 0$, when $x < 1$ or $|x| < 1$

Now $$S_n = a + ax + ax^2 + \ldots\ldots + ax^{n-1}$$

$$= \text{Sum of } n \text{ terms of geometric series when } x < 1 \text{ or } |x| < 1$$

$$= \frac{a(1-x^n)}{1-x}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left\{\frac{a(1-x^n)}{1-x}\right\} = \frac{a}{1-x}\left\{1 - \lim_{n\to\infty} x^n\right\} = \frac{a}{1-x}(1-0) = \frac{a}{1-x}$$

*i.e.,* $\lim_{n\to\infty} S_n$ is finite and unique number and therefore, $\sum u_n$ is convergent.

(ii) When $x = 1$, then $\sum u_n = a + a + a + \ldots\ldots$

$$S_n = na$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty \times a = \infty$$

*i.e.,* $\sum u_n$ is divergent and diverges to $+\infty$.

(iii) When $x = -1$, then $\sum u_n = a - a + a - a + \ldots..$

$$S_n = \begin{cases} 0, \text{ if } n \text{ is even} \\ a, \text{ if } n \text{ is odd} \end{cases}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 0 = 0, \text{ if } n \text{ is even}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 = 1, \text{ if } n \text{ is odd}$$

*i.e.*, $\lim_{n\to\infty} S_n$ is finite but not unique and therefore, $\sum u_n$ oscillates finitely.

(iv) When $x > 1$

We know that $\lim_{n\to\infty} x^n = \infty$ when $x > 1$

$$S_n = a + ax + ax^2 + \ldots\ldots + ax^{n-1}$$

$=$ Sum of $n$ terms of a geometric series when common ratio $x > 1$.

$$= \frac{a(x^n - 1)}{x - 1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(x^n - 1)}{x - 1} = \frac{a}{x-1}\left\{\lim_{n\to\infty} x^n - 1\right\}$$

$$= \frac{a}{x-1}(\infty - 1) = \infty \text{ if } a > 0$$

and $\lim_{n\to\infty} S_n = -\infty$ if $a < 0$

*i.e.*, $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$ and therefore, $\sum u_n$ is divergent.

(v) When $x < -1$ or $-x > 1$

Taking $r = -x$, so if $-x > 1 \Rightarrow r > 1$ and therefore $\lim_{n\to\infty} r^n = \infty$

Now, $S_n = a + ax + ax^2 + \ldots\ldots. + ax^{n-1}$

$=$ Sum of $n$ terms of the geometric series when $x < -1$

$$= \frac{a(1 - x^n)}{1 - x}$$

or $S_n = \dfrac{a\{1 - (-r)^n\}}{1 - (-r)}$, because $r = -x \Rightarrow x = -r$

$$= \begin{cases} \dfrac{a(1-r^n)}{1+r}, & \text{if } n \text{ is even} \\ \dfrac{a(1+r^n)}{1+r}, & \text{if } n \text{ is odd} \end{cases}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1+r} = \left(\frac{a}{1+r}\right)\left\{1 - \lim_{n\to\infty} r^n\right\}$$

$$= \frac{a}{1+r}(1-\infty) = -\infty, \text{ if } n \text{ is even}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1+r^n)}{1+r} = \left(\frac{a}{1+r}\right)\left\{1 + \lim_{n\to\infty} r^n\right\}$$

$$= \frac{a}{1+r}(1+\infty) = \infty, \text{ if } n \text{ is odd.}$$

*i.e.,* $\lim_{n\to\infty} S_n = -\infty$ and $\infty$ according as $n$ is even and odd (if $a > 0$).

Also $\lim_{n\to\infty} S_n = +\infty$ and $-\infty$ according as $n$ is even and odd (if $a < 0$).

*i.e.,* whether $a > 0$ or $a < 0$, we have $\lim_{n\to\infty} S_n$ is $-\infty$ and $\infty$ or $\infty$ and $-\infty$.

Hence, $\sum u_n$ oscillates infinitely.

## 5.3.2 Sequence of Partial Sums

If $\sum u_n = u_1 + u_2 + u_3 + u_4 + \ldots..$ is an infinite series. Now consider the sums

$$S_1 = u_1$$

$$S_2 = u_1 + u_2$$

$$S_3 = u_1 + u_2 + u_3$$

$$\vdots \qquad \vdots$$

$$S_n = u_1 + u_2 + \ldots\ldots + u_n$$

The sequence $< S_n > = < S_1, S_2, S_3, \ldots\ldots\ldots S_n \ldots\ldots >$ is called **sequence of partial sums.**

## 5.3.3 Behaviour of an Infinite Series in the Form of Partial Sums

1. **Convergent Series:** The series $\Sigma u_n$ is said to be convergent, if the sequence $< S_n >$ of partial sums of $\Sigma u_n$ is convergent.
2. **Divergent Series:** The series $\Sigma u_n$ is said to be divergent, if the sequence $< S_n >$ of partial sums of $\Sigma u_n$ is divergent.
3. **Oscillatory Series:** The series $\Sigma u_n$ is said to oscillate finitely or infinitely, if the sequence $< S_n >$ of partial sums of $\Sigma u_n$ oscillates finitely or infinitely.

**Example 3:** ***Show that the series*** $\boldsymbol{\Sigma(-1)^{n-1}2}$ ***oscillates.***

**Solution:** $\Sigma(-1)^{n-1}2 = 2-2+2-2+2-.....$

$$S_1 = 2, S_2 = 0, S_3 = 2, S_4 = 0...$$

Sequence of partial sums $<S_n> = <S_1, S_2, S_3, .....> = <2,0,2,0........>$ is an oscillatory sequence and oscillate finitely.

Hence, the series $\Sigma u_n$ oscillates finitely.

**Example 4:** ***Show that the series*** $\boldsymbol{\Sigma\ u_n = \Sigma(-1)^n n}$ ***is not convergent.***

**Solution:** $\Sigma(-1)^n n = -1+2-3+4-5+......$

$$S_1 = 1, S_2 = -1+2 = 1, S_3 = -1+2-3 = -2$$

$$S_4 = -1+2-3+4 = 2, S_5 = -1+2-3+4-5 = -3 ........$$

Sequence of partial sums $= <S_n> = <S_1, S_2, S_3, S_4, S_5, .....>$

$$= <-1, 1, -2, 2, -3, .....> \text{ is not convergent}$$

and therefore, $\Sigma u_n$ is not convergent.

**Example 5:** ***Show that the series*** $\boldsymbol{\Sigma u_n = \Sigma \sin \frac{n\pi}{3}}$ ***is not convergent.***

**Solution:**

$$\Sigma u_n = \Sigma \sin\frac{n\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0 - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 0 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + ...$$

Sequence of partial sums $= <S_n> = <S_1, S_2, S_3.......>$

$$= <\frac{\sqrt{3}}{2}, \sqrt{3}, \sqrt{3}, \frac{\sqrt{3}}{2}, 0, 0, \frac{\sqrt{3}}{2}, \sqrt{3}.....>$$

$$\overline{\lim_{n\to\infty}}\ S_n = \sqrt{3} \text{ and } \underline{\lim_{n\to\infty}}\ S_n = 0$$

*i.e.,*

$$\overline{\lim_{n\to\infty}}\ S_n \neq \underline{\lim_{n\to\infty}}\ S_n$$

Therefore, $<S_n>$ is not convergent and hence $\Sigma u_n$ is not convergent.

## 5.3.4 Necessary Condition for Convergence

**Theorem 1: If a series $\boldsymbol{\Sigma u_n}$ is convergent, the $\boldsymbol{\lim_{n\to\infty} u_n = 0}$ but the converse is not true.**

**Proof:** If $S_n$ denote the $n$th partial sum of the series $\Sigma u_n$ which is convergent and therefore, the sequence $<S_n>$ of partial sums is also convergent.

*i.e.,*

$$\lim_{n\to\infty} S_n = l \text{ (say), then } \lim_{n\to\infty} S_{n-1} = l$$

We have $$S_n = u_1 + u_2 + \ldots\ldots + u_{n-1} + u_n$$

and $$S_{n-1} = u_1 + u_2 + \ldots\ldots + u_{n-1}$$

Now, $$S_n - S_{n-1} = u_n$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} u_n$$

$$\Rightarrow \quad l - l = \lim_{n\to\infty} u_n$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n = 0$$

The converse of the above theorem is not always true. We shall give an example of a series $\Sigma\, u_n$ such that $\lim_{n\to\infty} u_n = 0$, but the series is not convergent.

$$\Sigma\, u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} + \ldots\ldots \text{ is a divergent series.}$$

But $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

**Remember:**

1. $\Sigma\, u_n$ convergent $\Rightarrow \lim_{n\to\infty} u_n = 0$.
2. $\lim_{n\to\infty} u_n = 0 \Rightarrow \Sigma\, u_n$ may or may not be convergent.
3. $\lim_{n\to\infty} u_n \neq 0 \Rightarrow \Sigma\, u_n$ is not convergent.

***Example 6:*** ***Discuss the convergence of the series***

$$\boldsymbol{\Sigma\, u_n = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots\ .}$$

***Solution:*** $\Sigma\, u_n = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots$

is a geometric series with common ratio $\frac{1}{3} < 1$, so $\Sigma u_n$ is convergent and $u_n = \frac{1}{3^n}$.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3^n} = \frac{1}{3^\infty} = \frac{1}{\infty} = 0$$

**NOTE:**

1. If $\lim_{n\to\infty} u_n = 0$, then we cannot say anything about the behaviour of the series.
2. If $\lim_{n\to\infty} u_n \neq 0$, then definitely $\Sigma\, u_n$ does not converse.

***Example 7:*** ***Prove that the series*** $\Sigma\, u_n = \Sigma \dfrac{n}{n+1}$ ***does not converse.***

***Solution:*** We have

$$\Sigma\, u_n = \Sigma \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + .....$$

$$u_n = \frac{n}{n+1} = \frac{1}{1+\dfrac{1}{n}}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\left(1+\dfrac{1}{n}\right)} = 1+0 = 1 \neq 0$$

So, $\Sigma\, u_n$ does not converse.

***Example 8:*** ***Prove that the series*** $\Sigma \sqrt{\dfrac{n}{2(n+1)}}$ ***is not convergent.***

**[B.C.A. (Avadh) 2008, 06]**

***Solution:*** Let $\Sigma\, u_n = \Sigma \sqrt{\dfrac{n}{2(n+1)}} = \sqrt{\dfrac{1}{4}} + \sqrt{\dfrac{2}{6}} + ..... + \sqrt{\dfrac{n}{2(n+1)}} + .....$

$$u_n = \sqrt{\frac{n}{2(n+1)}} = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{1}{1+1/n}}\right\}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{1}{1+1/n}}\right\} = \frac{1}{\sqrt{2}} \neq 0$$

Hence, the given series does not converse.

***Example 9:*** ***Show that the series*** $\Sigma \cos \dfrac{1}{n}$ ***does not converse.***

***Solution:*** Let
$$\Sigma\, u_n = \Sigma \cos \frac{1}{n}$$

$\Rightarrow$
$$u_n = \cos \frac{1}{n}$$

$\Rightarrow$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \cos \frac{1}{n} = \cos \frac{1}{\infty}$$

$$= \cos 0 = 1 \neq 0$$

Hence, $\Sigma \cos \dfrac{1}{n}$ is not convergent.

***Example 10:*** ***Show that the series*** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/n}$ ***does not converse.***

***Solution:*** Let
$$u_n = \left(\frac{1}{n}\right)^{1/n}$$

$$\Rightarrow \qquad \log u_n = \frac{\log 1 - \log n}{n} = -\frac{\log n}{n}$$

$$\lim_{n\to\infty} \log u_n = \lim_{n\to\infty}\left(-\frac{\log n}{n}\right) \text{ from } \frac{\infty}{\infty}$$

Applying **L-Hospitals rule**

$$\lim_{n\to\infty} \log u_n = -\lim_{n\to\infty}\frac{1/n}{1} = -\lim_{n\to\infty}\frac{1}{n} = -(0) = 0$$

*i.e.,* $$\lim_{n\to\infty} \log u_n = 0$$

$$\Rightarrow \qquad \log\left(\lim_{n\to\infty} u_n\right) = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty} u_n = e^0 = 1 \neq 0$$

Hence, $\Sigma u_n$ is not convergent.

***Example 11:*** ***Show that the series***

$\log_e 2 + \log_e \frac{3}{2} + \log_e \frac{4}{3} + \ldots\ldots = \Sigma \log_e\left(\frac{n+1}{n}\right)$ ***is divergent.*** **[B.C.A. (Purvanchal) 2011]**

***Solution:*** Let

$$\Sigma u_n = \Sigma \log_e\left(\frac{n+1}{n}\right) = \log_e\frac{2}{1} + \log_e\frac{3}{2} + \log_e\frac{4}{3} + \ldots\ldots + \log_e\frac{n+1}{n} + \ldots$$

$$S_n = \log_e\left\{\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\ldots\ldots\frac{n+1}{n}\right\} = \log_e(n+1)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \log_e(n+1) = \log\infty = \infty$$

Hence, $\Sigma u_n$ is divergent.

## 5.3.5 Cauchy's General Principle of Convergence for Series

**Theorem 2: The necessary and sufficient condition for the infinite series $\Sigma u_n$ to converse is that given $\in > 0$, however small, there exist a positive integer $m$ such that**

$$|u_{m+1} + u_{m+2} + \ldots\ldots + u_n| < \in \quad \forall n \geq m.$$

**Proof:** The series $\Sigma u_n$ is convergent iff the sequence of partial sums $< S_n >$ is convergent. By Cauchy's general principle of convergence for sequence, the sequence $< S_n >$ is convergent iff for each $\in > 0$, there exist a positive integer $m$ such that

$$| S_n - S_m| < \in \quad \forall n > m$$

$\Rightarrow$ $$| u_{m+1} + u_{m+2} + u_{m+3} + \ldots\ldots + u_n| < \in \quad \forall \; n > m$$

Hence the result.

***Example 12:*** ***Prove with the help of Cauchy's general principle of convergence that the series***

$$\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} + \ldots\ldots$$

***does not converge.***

***Solution:*** Suppose, the given series is convergent. Then, the sequence $< S_n >$ of partial sums of the given series is convergent. By Cauchy's general principle for sequences, for $\in = \frac{1}{2}$, there exist $m \in N$ such that

$$| S_n - S_m| < \frac{1}{2} \; \forall \; n \geq m$$

or $$\left| \left(1 + \frac{1}{2} + \ldots + \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n}\right) - \left(1 + \frac{1}{2} + \ldots\ldots + \frac{1}{m}\right) \right| < \frac{1}{2} \; \forall \, n \geq m \qquad \ldots(1)$$

or $$\left| \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n} \right| < \frac{1}{2} \qquad \forall \, n \geq m$$

or $$\frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{n} < \frac{1}{2} \qquad \forall \, n \geq m \text{ because } m \in N$$

Taking $n = 2m$, we get

$$\frac{1}{m+1} + \frac{1}{m+2} = \ldots\ldots + \frac{1}{m+m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \ldots\ldots + \frac{1}{2m} = m \frac{1}{2m} = \frac{1}{n} \quad \forall \, n = 2m > m$$

*i.e.,* $$\frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{n} > \frac{1}{2} \qquad \ldots(2)$$

Since (1) and (2) are contradictory statements, the given series does not converge.

## 5.3.6 Series of Positive Terms

An infinite series all of whose terms are positive is called **series of positive terms.**

*i.e.,* $\Sigma \, u_n$ is a series of positive terms, if $u_n > 0 \;\; \forall \; n \in N$.

We have $S_n = u_1 + u_2 + \ldots\ldots + u_n$ and

$$S_{n+1} = u_1 + u_2 + \ldots .. + u_n + u_{n+1}$$

$\Rightarrow$ $$S_{n+1} = S_n + u_{n+1}$$

$\Rightarrow$ $$S_{n+1} - S_n = u_{n+1} > 0 \quad \forall n \in N$$

*i.e.,* $$S_{n+1} - S_n > 0 \qquad \forall n \in N$$

So that sequence of partial sums $< S_n >$ of $\Sigma u_n$ is monotonically increasing *i.e.,* the sequence $< S_n >$ is monotonic.

We know that every monotonic sequence is either convergent or divergent but can not be an oscillatory sequence.

Some fundamental result for series of positive terms.

**Theorem 3: A series of positive terms $\Sigma u_n$ is convergent iff its sequence $< S_n >$ of partial sums is bounded above.**

**or**

**A series of positive terms $\Sigma u_n$ converges iff**

$$S_n = u_1 + u_2 + \ldots \ldots + u_n < K \; \forall n \in N, \text{ where } K \in R_+.$$

**Proof:** First suppose that $S_n = u_1 + u_2 + \ldots .. + u_n < K \; \forall n \in N$.

*i.e.,* $< S_n >$ is bounded above. Since the series $\Sigma u_n$ is of positive terms, then $< S_n >$ is monotonically increasing sequence which is bounded above, therefore by monotone convergence theorem for sequence $< S_n >$ is converge and hence $\Sigma u_n$ is convergent.

Conversely, let us suppose that $\Sigma u_n$is convergent. Then sequence of partial sums $< S_n >$ of $\Sigma u_n$ is also convergent. We know that every convergent sequence is bounded, therefore $< S_n >$ is bounded *i.e.,* $k \le S_n \le K \; \forall n \in N$. Now taking $S_n < K \; \forall n \in N$ *i.e.,*

$$S_n = u_1 + u_2 + \ldots \ldots + u_n < K \; \forall n \in N$$

**NOTE:**

In order to prove that the series of positive terms $\Sigma u_n$ is divergent, we have to show that the sequence of partial sums $< S_n >$ of $\Sigma u_n$ is not bounded.

**Theorem 4: An infinite series of positive terms is divergent, if each terms after a fixed stage is greater than some fixed positive quantity.**

**Proof:** Let $\Sigma u_n$ be a series of positive terms such that $u_n > k \; \forall n > m$, where $k$ is a fixed positive quantity.

$\Rightarrow$ $u_{m+1}, u_{m+2}, \ldots \ldots$ are all $> k$

Now, $u_1 + u_2 + \ldots\ldots + u_m$ = Sum of a finite number of positive terms of $\Sigma u_n$ = $a$ fixed, finite positive quantity = $M$ (say)

Let
$$S_n = (u_1 + u_2 + \ldots\ldots + u_m + \ldots\ldots + u_n)$$
$$= (u_1 + u_2 + \ldots\ldots + u_m) + (u_{m+1} + u_{m+2} + \ldots\ldots + u_n)$$
$$= M + (u_{m+1} + u_{m+2} + \ldots\ldots + u_n) > M + \underbrace{k + k + \ldots\ldots\ldots + k}_{(n-m)\text{ times}}$$

$\Rightarrow$ $$S_n > M + (n-m)k$$

$\Rightarrow$ $$\lim_{n\to\infty} S_n > \lim_{n\to\infty} [M + (n-m)k] = M + (\infty - m)k = \infty$$

$\Rightarrow$ $\Sigma u_n$ is divergent.

**Theorem 5: The nature of a series remain unaltered if the sign of all the terms of series are all together changed.**

**Proof:** Let $\Sigma u_n = u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ be an infinite series, then

$$S_n = u_1 + u_2 + \ldots\ldots + u_n$$

If the signs of all the terms of the given series are all together changed, then series will become

$$-u_1 - u_2 - u_3 - \ldots\ldots u_n - \ldots\ldots$$

Let $S_n'$ be sum of first $n$ terms of this series, then

$$S_n' = -u_1 - u_2 - \ldots\ldots u_n = -(u_1 + u_2 + \ldots\ldots + u_n)$$

$\Rightarrow$ $$S_n' = -S_n$$

$\Rightarrow$ $$\lim_{n\to\infty} S_n' = -\lim_{n\to\infty} S_n \qquad \ldots(1)$$

From (1) it is quite clear that if $\lim_{n\to\infty} S_n$ is finite and unique, then $\lim_{n\to\infty} S_n'$ will also be a finite and unique quantity *i.e.,* if the given series is convergent, then the new series is convergent.

Again, if $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$, then

$$\lim_{n\to\infty} S'_n = -\infty \quad \text{or} \quad +\infty$$

*i.e.,* if the given series is divergent, then the new series is also divergent.

Lastly, if $\lim_{n\to\infty} S_n$ oscillates then, $\lim_{n\to\infty} S_n'$ will also oscillate and therefore, if the given series is an oscillatory, then the new series will be oscillatory.

**Theorem 6: The nature of an infinite series remain unaltered by the addition or removal of a finite number of terms.**

**Proof:** Let $\Sigma u_n = u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ be a given series, and let $a + u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ be a series obtained by adding one more term to it. Let $S_n$ be the sum of first $n$ terms of the series $\Sigma u_n$ and $S_n'$ be the sum of first $n$ terms of the new series , then

$$S_n = u_1 + u_2 + \ldots\ldots + u_n \text{ and}$$

$$S_n' = a + u_1 + u_2 + \ldots\ldots + u_{n-1}$$

$$\Rightarrow \quad S_n' = a + S_{n-1}$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n' = a + \lim_{n\to\infty} S_{n-1} \quad \ldots(1)$$

If $\lim_{n\to\infty} S_n = S$ , then $\lim_{n\to\infty} S_{n-1} = S$, so from (1), we have $\lim_{n\to\infty} S_n' = a + S.$

If $S$ is finite and unique *i.e.,* $\Sigma\ u_n$ is convergent, then $\lim_{n\to\infty} S' = a + S$ is also finite and unique and therefore, the new series will also be convergent, if $\lim_{n\to\infty} S_n = \infty$ or $\lim_{n\to\infty} S_{n-1} = \infty$, then from (1), we have $\lim_{n\to\infty} S_n' = a + \infty = \infty$. This shows that if $\Sigma\ u_n$ is divergent, then the new series will also be divergent.

If $\Sigma u_n$ oscillates, then $\lim_{n\to\infty} S_{n-1}$ will also oscillate also $\lim_{n\to\infty} S_n' = a + S$ will also oscillate and therefore, the new series will also oscillate.

Similarly, we can show that for the removal of finite number of terms also.

**Theorem 7: If each term of a given series is multiplied or divided by a fixed non-zero number, then the new series so obtained will remain convergent or divergent according as it was originally convergent or divergent.**

**Proof:** Let the given series be

$$\Sigma u_n = u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$$

and

$$S_n = u_1 + u_2 + \ldots\ldots + u_n$$

Let the series be multiplied by a fixed non-zero number, say $\lambda$. Then, the series will become

$$\lambda u_1 + \lambda u_2 + \ldots\ldots + \lambda u_n + \ldots\ldots$$

If $S_n'$ be the sum of first $n$ terms of the new series. Then,

$$S_n' = \lambda u_1 + \lambda u_2 + \ldots\ldots + \lambda u_n = \lambda (u_1 + u_2 + \ldots\ldots + u_n)$$

$$\Rightarrow \quad S_n' = \lambda S_n$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n' = \lambda \lim_{n\to\infty} S_n \quad \ldots(1)$$

It is quite obvious from (1), that $\lim_{n\to\infty} S_n'$ is finite and unique, $+\infty$ and $-\infty$ and oscillates according as $\lim_{n\to\infty} S_n$ is finite and unique, $+\infty$ and $-\infty$ and oscillates. Hence, the nature of the new series will remain the same as that of the original series.

Again, let $\lambda = \frac{1}{m}$, where $m$ is finite and non-zero, then from (1), we have

$$\lim_{n\to\infty} S_n' = \lim_{n\to\infty} (\lambda S_n) = \frac{1}{m} \lim_{n\to\infty} S_n$$

With the same reasoning as discussed above we can prove that the nature of the series in this case even will remain the same as that of the original series.

**Theorem 8: If two infinite series are given, then the series formed by their sum will be**:

**(i) Convergent, if both the series are convergent**

**(ii) Divergent, if any one of them is divergent.**

**Proof:** Let $\Sigma u_n$ and $\Sigma v_n$ be the two given series. Then, the series formed by their sum

$$= \Sigma\, u_n + \Sigma\, v_n = \Sigma(u_n + v_n)$$
$$= (u_1 + v_1) + (u_2 + v_2) + \ldots\ldots + (u_n + v_n) + \ldots\ldots$$

Let $S_n, S_n'$ and $S_n''$ be the sum of first $n$ terms of the series $\Sigma u_n, \Sigma v_n$ and $\Sigma(u_n + v_n)$ respectively, then

$$S_n'' = (u_1 + v_1) + (u_2 + v_2) + \ldots\ldots + (u_n + v_n)$$
$$= (u_1 + u_2 + \ldots\ldots + u_n) + (v_1 + v_2 + \ldots\ldots + v_n) = S_n + S_n'$$

$$\lim_{n\to\infty} S_n'' = \lim_{n\to\infty} S_n + \lim_{n\to\infty} S_n' \qquad \ldots(1)$$

(i) If $\Sigma\, u_n$ and $\Sigma v_n$ both are convergent, then $\lim_{n\to\infty} S_n$ and $\lim_{n\to\infty} S_n'$ both are finite and definite numbers and sum of finite-definite number is also a finite-definite number therefore from (1), we have $\lim_{n\to\infty} S_n''$ is a finite definite-number. Hence, $\Sigma(u_n + v_n)$ is also convergent.

(ii) If any one of the given series (say first series) is divergent, then $\lim_{n\to\infty} S_n$ will be infinite which when added to finite-definite quantity $\left(\lim_{n\to\infty} S_n'\right)$ will result into an infinite quantity.

Hence, $\Sigma(u_n + v_n)$ is also divergent.

**Theorem 9: If each terms of a series $\Sigma u_n$ positive terms does not exceed the corresponding terms of a convergent series $\Sigma u_n$ of positive terms, then $\Sigma u_n$ is convergent. If on the contrary, each terms of $\Sigma u_n$ exceeds (or equals) the corresponding term of a divergent series of positive terms, then $\Sigma u_n$ is divergent.**

**Proof:** Let the given series are $\Sigma u_n$ and $\Sigma v_n$, then let

$$S_n = u_1 + u_2 + \ldots\ldots + u_n \text{ and } S_n' = v_1 + v_2 + \ldots\ldots + v_n$$

We have $\quad u_n \le v_n \quad \forall\, n \in N$, then

$$u_1 + u_2 + \ldots\ldots + u_n \le v_1 + v_2 + \ldots\ldots + v_n$$

$\Rightarrow \quad S_n \le S_n'$

But we have $\Sigma v_n$ is convergent series of positive terms,

therefore, $\quad \lim_{n\to\infty} S_n' =$ finite-definite number $= S'$ (say).

It follows that $\quad S_n \le S_n' \le S''$

$\Rightarrow \quad S_n \le S'' \quad \forall n \in N$

We have $\Sigma v_n$ is divergent *i.e.*, $\lim_{n\to\infty} S_n' = \infty$ or $-\infty$.

Hence, $\lim_{n\to\infty} S_n$ will be $\infty$ or $-\infty$. So $\Sigma v_n$ is divergent.

## 5.3.7 Hyper Harmonic Series or p-series $\Sigma \frac{1}{n^p}$

**The infinite series $\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots$ converges if $p > 1$ and diverges if $p \le 1$.**

**[B.C.A. (I.G.N.O.U.) 2012, 09; B.C.A. (Meerut) 2012, 07, 00; B.C.A. (Lucknow) 2011, 08, 06; B.C.A. (Agra) 2011, 06, 02]**

**Proof:** **Case I:** When $p > 1$

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}}, \left[\text{because } \frac{1}{3^p} < \frac{1}{2^p}\right]$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{(2^{p-1})^2}$$

$$\left[\text{because } \frac{1}{5^p} < \frac{1}{4^p}, \frac{1}{6^p} < \frac{1}{4^p} \text{ and } \frac{1}{7^p} < \frac{1}{4^p}\right]$$

Similarly, the sum of next eight terms

$$\frac{1}{8^p} + \frac{1}{9^p} + \ldots\ldots + \frac{1}{15^p} < \frac{1}{8^p} + \frac{1}{8^p} + \ldots\ldots + \frac{1}{8^p} = \frac{8}{8^p} = \frac{1}{8^{p-1}} = \frac{1}{(2^{p-1})^3}$$

and so on.

Now $$\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots\ldots \quad \ldots(1)$$

$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + \ldots\ldots + \frac{1}{15^p}\right) + \ldots\ldots$$

$$\le 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \frac{1}{(2^{p-1})^3} + \ldots\ldots \quad \ldots(2)$$

This is a geometric series whose common ratio is $\frac{1}{2^{p-1}} < 1$ and therefore (2) is convergent.

Hence, $\Sigma \frac{1}{n^p}$ is convergent.

**Case II:** When $p = 1$

$$\Sigma \frac{1}{n^p} = \Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

We have $$1 + \frac{1}{2} = 1 + \frac{1}{2}$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

and so on.

Now $$\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n} + \ldots..$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots\ldots$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots..$$

$$\Rightarrow \qquad \Sigma \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots\ldots \qquad \ldots(3)$$

$$S_n = 1 + (n-1)\frac{1}{2} = \text{Sum of first } n \text{ terms of the series } 1 + \frac{1}{2} + \frac{1}{2} + \ldots..$$

$$\Rightarrow \qquad \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left\{1 + (n-1)\frac{1}{2}\right\} = \infty$$

*i.e.,* $$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots.. \text{ is divergent.}$$

From (3), we have $\Sigma \frac{1}{n}$ is divergent because

$$\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots\ldots$$

is term by greater than a divergent series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots..$

**Case III:** When $p < 1$

If $$p < 1 \Rightarrow n^p < n \Rightarrow \frac{1}{n^p} > \frac{1}{n} \;\forall\, n = 2,3,4,\ldots\ldots$$

$$\Rightarrow \quad \Sigma \frac{1}{n^p} > \Sigma \frac{1}{n}$$

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$$

*i.e.,* $\Sigma \frac{1}{n^p}$ is term by term greater than the divergent series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots..$

Hence, $\Sigma \frac{1}{n^p}$ is divergent.

## 5.3.8 Comparison Tests

**Theorem 10: If $\Sigma u_n$ and $\Sigma v_n$ are series of positive terms and $\Sigma v_n$ is convergent and there is a positive constant $k$ such that $u_n \le k v_n \;\forall n \in N$, then $\Sigma\, u_n$ is also convergent.**

**Proof:** Let $S_n = u_1 + u_2 + \ldots\ldots + u_n$ and $S'_n = v_1 + v_2 + \ldots\ldots + v_n$

We have $u_n \le k v_n \;\forall n \in N$, therefore

$$S_n = u_1 + u_2 + \ldots\ldots + u_n \le k v_1 + k v_2 + \ldots\ldots + k v_n = k(v_1 + v_2 + \ldots.. + v_n) = kS_n'$$

*i.e.,* $$S_n \le kS'_n \qquad \ldots(1)$$

If $\Sigma v_n$ is convergent, then the sequence of partial series $< S_n' >$ of $\Sigma v_n$ is convergent and hence bounded.

$\therefore$ There exist a positive integer $k$ such that

$$S'_n < k_0 \qquad \forall n \in N$$

From (1) $$S_n \le kS'_n < kk_0 \qquad \forall n \in N$$

$$\Rightarrow \quad S_n < \lambda \;\; \forall n \in N, \text{ where } kk_0 = \lambda$$

This shows that $< S_n >$ is bounded and therefore bounded above where $< S_n >$ is the sequence of partial sums of $\Sigma\, u_n$ which is monotonically increasing.

Thus, we have $< S_n >$ is monotonically increasing and bounded above, so it is convergent.

If $< S_n >$ is convergent, then $\Sigma\, u_n$ is also convergent.

**Theorem 11:** **If $\Sigma\, u_n$ and $\Sigma\, v_n$ are series of positive terms and $\Sigma v_n$ is divergent and there is a positive constant $k$ such that $u_n \geq kv_n\ \forall n \in N$, then $\Sigma\, u_n$ is also divergent.**

**Proof:** Let $S_n = u_1 + u_2 + \ldots\ldots + u_n$ and $S'_n = v_1 + v_2 + \ldots\ldots + v_n$

We have $u_n \geq kv_n \quad \forall n \in N$

$\Rightarrow \quad u_1 + u_2 + \ldots\ldots + u_n \geq kv_1 + kv_2 + \ldots\ldots + kv_n = k(v_1 + v_2 + \ldots + v_n)$

$\Rightarrow \quad S_n \geq kS'_n \quad \forall n \in N$ ...(1)

We have $\Sigma\, v_n$ is divergent, therefore $< S'_n >$ the sequence of partial sums of $\Sigma\, v_n$ is also divergent.

$\Rightarrow$ For each positive number $k_0$ however large, there exist a positive interger $m$ such that

$$S'_n > k_0 \quad \forall n > m$$

$\Rightarrow \quad S_n > \lambda \quad \forall n > m$, where $kk_0 = \lambda$

This shows that $< S_n >$ the sequence of partial sums of $\Sigma\, u_n$ is divergent and hence, $\Sigma\, u_n$ is divergent.

**Theorem 12:** **If $\Sigma\, u_n$ and $\Sigma\, v_n$ are series of positive terms and $\Sigma\, v_n$ is convergent and there is a positive constant $k$ such that $u_n \leq kv_n\ \forall\, n > m$, then $\Sigma\, u_n$ is also convergent.**

**Proof:** Let $S_n = u_1 + u_2 + \ldots + u_n$ and $S'_n = v_1 + v_2 + \ldots + v_n$

We have $u_n \leq kv_n \quad \forall n > m$

$\Rightarrow$
$$u_{m+1} \leq kv_{m+1}$$
$$u_{m+2} \leq kv_{m+2}$$
$$\vdots \quad \vdots \quad \vdots$$
$$u_n \leq kv_n$$

Adding, $u_{m+1} + u_{m+2} + \ldots\ldots + u_n \leq k(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$

$\Rightarrow \quad (S_n - S_m) \leq k(S'_n - S'_m) \quad \forall n > m$

$\Rightarrow \quad S_n \leq k\, S'_n + (S_n - kS'_n) \quad \forall n > m$

$\Rightarrow \quad S_n \leq kS'_n + \mu \quad \forall n > m$ ...(1)

where $< S'_n >$ the sequence of partial sums of $\Sigma\, v_n$ is also convergent and hence bounded above.

From (1), we have $< S_n >$ is bounded above and also $< S_n >$ is monotonically increasing because $\Sigma\, u_n$ is the series of positive terms.

Now $< S_n >$ the sequence of partial sums of $\Sigma u_n$ is monotonically increasing and bounded above, therefore by monotone convergence theorem for sequence $< S_n >$ is convergent and hence, $\Sigma u_n$ is convergent.

**Theorem 13: If $\Sigma u_n$ and $\Sigma v_n$ are two series of positive terms and $\Sigma\, v_n$ is divergent and there is a positive constant $k$ such that $u_n > kv_n \;\forall n > m$, then $\Sigma\, u_n$ is also divergent.**

**Proof:** Let $\quad S_n = u_1 + u_2 + \ldots\ldots + u_n \quad$ and $\quad S'_n = v_1 + v_2 + \ldots\ldots + v_n$

It is give that $\quad u_n > kv_n \quad \forall n > m$

$$\Rightarrow \quad \begin{aligned} u_{m+1} &> kv_{m+1} \\ u_{m+2} &> kv_{m+2} \\ \vdots \quad & \quad \vdots \\ u_n &> kv_n \end{aligned}$$

Adding, $\quad u_{m+1} + u_{m+2} + \ldots\ldots + u_n > k(v_{m+1} + v_{m+2} + \ldots\ldots + v_n)$

$$\Rightarrow \quad S_n - S_m > k\,(S'_n - S'_m)$$

$$\Rightarrow \quad S_n > k\,S'_n + (S_m - kS'_m)$$

$$\Rightarrow \quad S_n > k\,S'_n + \mu \qquad \forall n > m \qquad \ldots(1)$$

where, $\quad S_m - k\,S'_m = \mu$

We have $\Sigma v_n$ is divergent, therefore $< S'_n >$, the sequence of partial sums of $\Sigma\, v_n$ is also divergent.

$\Rightarrow$ For each positive integer $k_1$, however large, there exist a positive integer $m'$ such that

$$S'_n > k_1 \qquad \forall n > m'$$

Let $\quad m_2 = \max\{m. m'\}$, then

$$S'_n > k_1 \qquad \forall n > m_2$$

From (1), we have $\quad S_n > kk_1 + \mu \quad \forall n > m_2$

$\Rightarrow \quad S_n > h \;\; \forall\, n > m_2$, where $kk_1 + \mu = h$

$\Rightarrow \quad < S_n >$ is divergent.

$\Rightarrow \Sigma\, u_n$ is divergent because $< S_n >$ be the sequence of partial sums of $\Sigma\, u_n$.

---

**Theorem 14:** **If $\Sigma\, u_n$ and $\Sigma v_n$ are two series of positive terms and there exist two positive constants $H$ and $k$ (independent of $n$) and a positive integer $m$ such that**

$$H < \frac{u_n}{v_n} < k \;\; \forall n > m,$$

**then the two series $\Sigma u_n$ and $\Sigma v_n$ converge or diverge together.**

**Proof:** We have $\Sigma v_n$ is a series of positive terms *i.e.,*

$$v_n > 0, \qquad \forall n \in N$$

$$\therefore \qquad H < \frac{u_n}{v_n} < k \quad \forall n > m$$

$$\Rightarrow \qquad Hv_n < u_n < kv_n \;\; \forall n > m \qquad \text{...(1)}$$

**Case I:** When $\Sigma\, v_n$ is convergent then from (1), we have

$$u_n < kv_n \qquad \forall n > m$$

From theorem (3), we have if $u_n < kv_n\; \forall n > m$ and $\Sigma v_n$ is convergent, then $\Sigma\, u_n$ is convergent.

**Case II:** When $\Sigma\, v_n$ is divergent, then from (1), we have $u_n > Hv_n \;\; \forall n > m$ and $\Sigma\, v_n$ is divergent, there from theorem (4) we have $\Sigma\, u_n$ is divergent.

**Case III:** When $\Sigma u_n$ is convergent.

From (1) , we have $\qquad Hv_n < u_n \qquad \forall n > m$

$$\Rightarrow \qquad v_n < \frac{1}{H} u_n \qquad \forall n > m$$

If $\Sigma\, u_n$ is convergent, then from theorem (3) we have $\Sigma\, v_n$ is convergent.

**Case IV:** When $\Sigma\, u_n$ is divergent.

From (1) , we have $kv_n > u_n \qquad \forall n > m$

$$\Rightarrow \qquad v_n > \frac{1}{k} u_n \qquad \forall n > m$$

If $\Sigma\, u_n$ is divergent, then from theorem (4), we have $\Sigma\, v_n$ is divergent.

**Theorem 15:** **If $\Sigma\, u_n$ and $\Sigma\, v_n$ be the two series of positive terms such that $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$ (finite and non-zero), then $\Sigma\, u_n$ and $\Sigma\, v_n$ both converge or diverge together.**

**Proof:** If $\Sigma\, u_n$ and $\Sigma\, v_n$ be the series of positive terms,

then $u_n > 0$ and $v_n > 0 \quad \forall n \in N$, and therefore $\dfrac{u_n}{v_n} > 0 \;\; \forall n \in N$.

If $\lim\limits_{n \to \infty} \dfrac{u_n}{v_n} = l$, then $l \geq 0$ but we have $l \neq 0$, so $l > 0$.

If $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = l, < \dfrac{u_n}{v_n} >$ converges to $l$ *i.e.,* for any $\in > 0$. There exist $m \in M$ such that

$$\left|\frac{u_n}{v_n} - l\right| < \in \qquad \forall n > m$$

$$\Rightarrow \qquad l - \in < \frac{u_n}{v_n} < l + \in \qquad \forall n > m$$

$$\Rightarrow \qquad (l - \in)(v_n) < u_n < (l + \in)(v_n) \;\; \forall n > m \qquad ...(1)$$

Let $\in > 0$ be so chosen that $l - \in > 0$.

Taking $l - \in = H$ and , $l + \in = k$, then from (1), we have

$$Hv_n < u_n < kv_n \quad \forall n > m \qquad ...(2)$$

Now, if $\Sigma\, v_n$ is convergent, then $\Sigma\, kv_n$ is also convergent. From (2), we have

$$\Rightarrow \qquad \Sigma\, u_n < \Sigma\, kv_n$$

*i.e.,* $\Sigma\, u_n$ is term by term less than a convergent series $\Sigma\, kv_n$ except possibly for a finite number of terms.

Therefore, $\Sigma\, u_n$ is also convergent.

Again if $\Sigma\, v_n$ is divergent, then $\Sigma\, Hv_n$ is also divergent.

From (2), we have $\Sigma\, Hv_n < u_n$.

$\Rightarrow \Sigma\, Hv_n < \Sigma u_n$ *i.e.,* $\Sigma\, u_n$ is term by term greater than a divergent series $\Sigma\, Hv_n$ except possibly for a finite number of terms. Therefore, $\Sigma\, u_n$ is also divergent. Hence, the series $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

> **Note:**
>
> This theorem is important from the point of view of application to the solution of problems.

**Theorem 16:** **If $\Sigma u_n$ and $\Sigma v_n$ be the series of positive terms.**

**(i)** **If $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$ $\forall n > m$ and $\Sigma v_n$ is convergent, then $\Sigma u_n$ is also convergent.**

**(ii)** **If $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ $\forall n > m$ and $\Sigma v_n$ is divergent, then $\Sigma u_n$ is also divergent.**

**Proof:** (i) $$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \quad \forall n > m, \text{ then}$$

$$\frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}}$$

$$\frac{u_{m+2}}{u_{m+3}} > \frac{v_{m+2}}{v_{m+3}}$$

$$\frac{u_{m+3}}{u_{m+4}} > \frac{v_{m+3}}{v_{m+4}}$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}$$

Multiplying the corresponding sides of the above inequalities, we have

$$\frac{u_{m+1}}{u_n} > \frac{v_{m+1}}{v_n} \qquad \forall n > m$$

$$\Rightarrow \qquad u_n < \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall n > m$$

$\Rightarrow \quad u_n < kv_n \quad \forall n > m$, where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed positive quantity.

$\Sigma v_n$ is convergent $\Rightarrow \Sigma kv_n$ is convergent and hence, $\Sigma u_n$ is convergent.

(ii) Using $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ $\forall n > m$

Now proceeding as in part (i), we have

$$\frac{u_{m+1}}{u_n} < \frac{v_{m+1}}{v_n} \qquad \forall n > m$$

$$\Rightarrow \qquad u_n > \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n \qquad \forall n > m$$

$$\Rightarrow \qquad u_n > kv_n \qquad \forall n > m \qquad \ldots(1)$$

where $k = \dfrac{u_{m+1}}{v_{m+1}}$ is a fixed positive quantity. We have $\Sigma v_n$ is divergent.

$\Rightarrow$ $\Sigma\, kv_n$ is also divergent, therefore from (1) we have $\Sigma\, u_n$ is divergent.

**Working rule for applying the comparison test given into the form of theorem 6.**

The theorem 6 states that if $\Sigma\, u_n$ and $\Sigma\, v_n$ be the two series of positive terms and if $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = l$, where $l$ is finite and non-zero, then the two series converge or diverge together.

If $\Sigma\, u_n = u_1 + u_2 + \ldots + u_n + \ldots$ be the series of positive terms, where $n$th term of the series $u_n$ contains the powers of $n$ only which may be positive or negative, integral or fractional. Let

$$u_n = \frac{an^q + bn^{q-1} + \ldots\ldots}{\alpha n^p + \beta n^{p-1} + \ldots\ldots}$$

$$= \frac{n^q}{n^p}\left(\frac{a + \frac{b}{n} + \ldots\ldots}{\alpha + \frac{\beta}{n} + \ldots\ldots}\right) = \frac{1}{n^{p-q}}\left(\frac{a + \frac{b}{n} + \ldots\ldots}{\alpha + \frac{\beta}{n} + \ldots\ldots}\right)$$

where $p$ and $q$ are the highest indices of $n$ in the denominator and numerator of $u_n$ respectively.

Now taking $v_n = \frac{1}{n^{p-q}}$, then $\Sigma\, v_n = \Sigma \frac{1}{n^{p-q}}$ is convergent if $p - q > 1$ and divergent if $p - q \le 1$.

Now $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{a + \frac{b}{n} + \ldots\ldots}{\alpha + \frac{\beta}{n} + \ldots\ldots}\right) = \frac{a}{\alpha}.$$

If $\frac{a}{\alpha}$ is finite and non-zero, then

1. $\Sigma\, u_n$ is convergent, if $\Sigma\, v_n$ is convergent and
2. $\Sigma\, u_n$ is divergent, if $\Sigma\, v_n$ is divergent.

**NOTE:**

If $$u_n = \frac{1}{\alpha n^p + \beta n^{p-1} + \ldots\ldots},$$ then $u_n$ can be written as

$$u_n = \frac{n^0}{\alpha n^p + \beta n^{p-1} + \ldots\ldots} = \frac{1}{n^{p-0}}\left[\frac{1}{\alpha + \frac{\beta}{n} + \ldots\ldots}\right]$$

Taking, $v_n = \frac{1}{n^p}$ and we have

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{1}{\alpha + \frac{\beta}{n} + \ldots\ldots}\right] = \frac{1}{\alpha}.$$

**Example 13:** ***Test each of the following series for convergence :*** **[B.C.A. (Meerut) 2006]**

*(i)* $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{3}{2\cdot 3\cdot 4}+\dfrac{5}{3\cdot 4\cdot 5}+\ldots..$

*(ii)* $\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\dfrac{1}{\sqrt{4}+\sqrt{5}}+\ldots..$ **[B.C.A. (Rohilkhand) 2012]**

*(iii)* $\dfrac{\sqrt{1}}{4\cdot 6}+\dfrac{\sqrt{3}}{6\cdot 8}+\dfrac{\sqrt{5}}{8.10}+\ldots..$

*(iv)* $\dfrac{1}{3\cdot 7}+\dfrac{1}{4\cdot 9}+\dfrac{1}{5\cdot 11}+\dfrac{1}{6\cdot 13}$ . **[B.C.A. (Rohilkhand) 2011, 07]**

**Solution:** (i) Let $\Sigma u_n=\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{3}{2\cdot 3\cdot 4}+\dfrac{5}{3\cdot 4\cdot 5}+\ldots...$

If $u_n$ be the $n$th term of the given series $\Sigma\, u_n$, then numerator of $u_n$ is the $n$th term of the progression 1,3,5..... by the formula $a+(n-1)\,d$, we get $n$th term is $(2n-1)$ of the numerator. The denominator of $u_n$ = ($n$th term of 1, 2, 3....) × ($n$th term of 2, 3,4......) × ($n$th term of 3, 4, 5,.....)= $n\ (n+1)(n+2)$.

Therefore, $$u_n=\frac{2n-1}{n(n+1)(n+2)}=\frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=\frac{\left(2-\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Taking, $v_n=\dfrac{1}{n^2}$ *i.e.,* $\Sigma\, v_n=\Sigma\dfrac{1}{n^2}$ is a convergent series and

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=2$$

which is finite and non-zero. Therefore, by comparison test $\Sigma\, u_n$ is convergent.

(ii) $$\Sigma\, u_n=\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\ldots...$$

$$u_n=\frac{(n\text{th term of } 1,1,1,\ldots..)}{(n\text{th term of } \sqrt{2},\sqrt{3},\sqrt{4},\ldots..)+(n\text{ th term of} \sqrt{3},\sqrt{4},\sqrt{5},\ldots...)}$$

$$=\frac{1}{\sqrt{n+1}+\sqrt{n+2}}=\frac{1}{\sqrt{n}\left\{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}\right\}}$$

Taking $v_n=\dfrac{1}{\sqrt{n}}$ *i.e.,* $\Sigma\, v_n=\Sigma\dfrac{1}{n^{1/2}}$ is divergent and

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{\left[\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}\right]}=\frac{1}{2}$$

which is finite and non-zero. Therefore, by comparison test $\Sigma\, u_n$ is divergent.

(iii) $$\Sigma\, u_n = \frac{\sqrt{1}}{4\cdot 6} + \frac{\sqrt{3}}{6\cdot 8} + \frac{\sqrt{5}}{8\cdot 10} + \ldots..$$

$$u_n = \frac{(n\text{th term of } \sqrt{1}, \sqrt{3}, \sqrt{5}, \ldots..)}{(n\text{th term of } 4,6,8,\ldots..) + (n\text{th term of } 6,8,10,\ldots...)}$$

$$= \frac{\sqrt{2n-1}}{(2n+2)(2n+4)} = \frac{\sqrt{n}}{n^2}\left[\frac{\sqrt{2-1/n}}{\left(2+\frac{2}{n}\right)\left(2+\frac{4}{n}\right)}\right]$$

$$= \frac{1}{n^{3/2}}\left[\frac{\sqrt{2-\frac{1}{n}}}{\left(2+\frac{2}{n}\right)\left(2+\frac{4}{n}\right)}\right]$$

Taking $v_n = \frac{1}{n^{3/2}}$, then we have $\Sigma\, v_n = \Sigma \frac{1}{n^{3/2}}$ is a convergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{\sqrt{2-1/n}}{\left(2+\frac{1}{n}\right)\left(2+\frac{4}{n}\right)}\right]$$

$$= \frac{\sqrt{2}}{4}$$

which is finite and non-zero. Therefore, by comparison test $\Sigma\, u_n$ is convergent.

(iv) $$\Sigma\, u_n = \frac{1}{3\cdot 7} + \frac{1}{4\cdot 9} + \frac{1}{5\cdot 11} + \frac{1}{6\cdot 13} + \ldots..$$

$$u_n = \frac{(n\text{th term of } 1,1,1,\ldots..)}{(n\text{th term of } 3,4,5,6,\ldots..) \times (n\text{th term of } 7,9,11,13,\ldots...)}$$

$$= \frac{1}{(n+2)(2n+5)} = \frac{1}{n^2\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)}$$

Taking $v_n = \frac{1}{n^2}$, then we have $\Sigma\, v_n = \Sigma \frac{1}{n^2}$ is a convergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)}$$

$= \frac{1}{2}$ which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent.

**Example 14:** ***Test the convergence of the series whose nth term is :***

*(i)* $\dfrac{\sqrt{n+1}-\sqrt{n-1}}{n}$ *(ii)* $\dfrac{1}{\sqrt{n}+\sqrt{n+1}}$ *(iii)* $\sqrt{n+1}-\sqrt{n}$ **[B.C.A. (Meerut) 2009, 04]**

*(iv)* $\sqrt{n^3+1}-\sqrt{n^3}$ *(v)* $\sqrt{n^4+1}-\sqrt{n^4}$ *(vi)* $[(n^3+1)^{1/3}-n]$ **[B.C.A. (Meerut) 2008]**

*(vii)* $\sum \dfrac{\sqrt{n+1}-\sqrt{n}}{n^p}$. **[B.C.A. (Meerut) 2009, 08]**

**Solution:** (i) We have

$$u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \times \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \frac{(n+1)-(n-1)}{n[\sqrt{n+1}+\sqrt{n-1}]} = \frac{1}{n^{3/2}}\left[\frac{2}{\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}\right]$$

Taking $v_n = \dfrac{1}{n^{3/2}}$, then $\Sigma\, v_n = \Sigma \dfrac{1}{n^{3/2}}$ is a convergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left[\frac{2}{\sqrt{\left(1+\frac{1}{n}\right)}+\sqrt{\left(1-\frac{1}{n}\right)}}\right] = \frac{2}{2} = 1$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent series.

(ii)
$$u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}} = \frac{1}{\sqrt{n}}\left\{\frac{1}{1+\sqrt{\left(1+\frac{1}{n}\right)}}\right\}$$

Taking $v_n = \dfrac{1}{\sqrt{n}}$, so that $\Sigma\, v_n = \Sigma\, \dfrac{1}{n^{1/2}}$ is divergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left\{\frac{1}{1+\sqrt{1+\frac{1}{n}}}\right\} = \frac{1}{2}$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is a divergent series.

(iii) $$u_n = \sqrt{n+1} - \sqrt{n} = \frac{\sqrt{n+1} - \sqrt{n}}{1} \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n}} \left\{ \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \right\}$$

Taking $v_n = \frac{1}{\sqrt{n}}$, so that $\Sigma\, v_n = \Sigma \frac{1}{n^{1/2}}$ is divergent series and

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \left\{ \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \right\} = \frac{1}{2}$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is divergent series.

(iv) $$\sqrt{n^3 + 1} - \sqrt{n^3} = \frac{\sqrt{n^3 + 1} - \sqrt{n^3}}{1} \times \frac{\sqrt{n^3 + 1} + \sqrt{n^3}}{\sqrt{n^3 + 1} + \sqrt{n^3}}$$

$$= \frac{(n^3 + 1) - n^3}{\sqrt{n^3 + 1} + \sqrt{n^3}} = \frac{1}{n^{3/2}\left[\sqrt{1 + \frac{1}{n^3}} + 1\right]}$$

Taking $v_n = \frac{1}{n^{3/2}}$, then $\Sigma\, v_n = \Sigma \frac{1}{n^{3/2}}$ is a convergent series and

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{1}{\sqrt{1 + \frac{1}{n^3}} + 1} = \frac{1}{2}$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent series.

(v) $$u_n = \sqrt{n^4 + 1} - \sqrt{n^4} = \frac{\sqrt{n^4 + 1} - \sqrt{n^4}}{1} \times \frac{\sqrt{n^4 + 1} + \sqrt{n^4}}{\sqrt{n^4 + 1} + \sqrt{n^4}}$$

$$= \frac{(n^4 + 1) - n^4}{\sqrt{n^4 + 1} + \sqrt{n^4}} = \frac{1}{n^2}\left[\frac{1}{\sqrt{1 + \frac{1}{n^4}} + 1}\right]$$

Taking $v_n = \frac{1}{n^2}$, then $\Sigma\, v_n = \Sigma \frac{1}{n^2}$ is a convergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^4}}+1} = \frac{1}{2}$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent series.

(vi) $$u_n = (n^3+1)^{1/3} - n = n\left\{\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right\}$$

$$= n\left\{\left(1+\frac{1}{3n^3}-\frac{1}{9n^6}+\ldots..\right)-1\right\} \text{ using binomial theorem}$$

$$= n\left(\frac{1}{3n^3}-\frac{1}{9n^6}+\ldots..\right) = \frac{1}{n^2}\left(\frac{1}{3}-\frac{1}{9n^3}+\ldots..\right)$$

Taking $v_n = \frac{1}{n^2}$, then $\Sigma\, v_n = \Sigma \frac{1}{n^2}$ is a convergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{3}-\frac{1}{9n^3}+\ldots...\right) = \frac{1}{3}$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent series.

(vii) We have $$u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$$

Multiplying numerator and a denominator by $(\sqrt{n+1}+\sqrt{n})$, we get

$$u_n = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})n^p} = \frac{n+1-n}{(\sqrt{n+1}+\sqrt{n})n^p}$$

$$= \frac{1}{n^p\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)}$$

Let $$v_n = \frac{1}{n^{p+1/2}}, \text{ then}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{p+(1/2)}}{n^p\sqrt{n}\left(1+\sqrt{1+\frac{1}{n}}\right)} = 1 \neq 0$$

Then by comparison test both series converge or divergent together, but we have to test

$$\Sigma\, v_n = \Sigma \frac{1}{n^{p+1/2}}$$

will be convergent if $p + \frac{1}{2} > 1$ or $p > \frac{1}{2}$ and divergent if $p + \frac{1}{2} \le 1$ or $p \le \frac{1}{2}$.

Hence, given series $\Sigma\, u_n$ will be convergent if $p > \frac{1}{2}$ and divergent if $p \le \frac{1}{2}$.

***Example 15:*** ***Test for the convergence the series***

***(i)*** $\boldsymbol{\Sigma \sin \frac{1}{n}}$ **[B.C.A. (Kanpur) 2010]**

***(ii)*** $\boldsymbol{\Sigma \frac{1}{\sqrt{n}} \tan \frac{1}{n}}$

***(iii)*** $\boldsymbol{\Sigma \frac{n+1}{n^p}}$

***(iv)*** $\boldsymbol{\Sigma \frac{1}{n^{\left(1+\frac{1}{n}\right)}}}$.

**[B.C.A. (Kanpur) 2010]**

***Solution:*** (i) Let $\Sigma\, u_n = \Sigma \sin \frac{1}{n}$

We have $u_n = \sin \frac{1}{n}$

Taking $v_n = \frac{1}{n}$, then $\Sigma\, v_n = \Sigma \frac{1}{n}$ is a convergent series and

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent series.

(ii) Let $\Sigma\, u_n = \Sigma \frac{1}{\sqrt{n}} \tan \frac{1}{n}$, then $u_n = \frac{1}{\sqrt{n}} \tan \frac{1}{n}$.

Taking $v_n = \frac{1}{n\sqrt{n}}$, then $\Sigma\, v_n = \Sigma \frac{1}{n^{3/2}}$ is a convergent series and

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{\frac{1}{\sqrt{n}} \tan \frac{1}{n}}{\frac{1}{n\sqrt{n}}} = \lim_{n \to \infty} \frac{\tan\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent series.

(iii) Let $\Sigma u_n = \Sigma \frac{n+1}{n^p}$, then $u_n = \frac{n+1}{n^p} = \frac{n}{n^p}\left(1+\frac{1}{n}\right) = \frac{1}{n^{p-1}}\left(1+\frac{1}{n}\right)$

Taking $v_n = \frac{1}{n^{p-1}}$, then $\Sigma v_n = \Sigma \frac{1}{n^{p-1}}$ is convergent if $p-1>1$ and divergent if $p-1 \le 1$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent series if $p-1>1$ and divergent $p-1 \le 1$.

(iv) Let $\Sigma\, u_n = \Sigma \frac{1}{n^{\left(1+\frac{1}{n}\right)}}$, then $u_n = \frac{1}{n^{\left(1+\frac{1}{n}\right)}} = \frac{1}{n.\, n^{\frac{1}{n}}}$

Taking $v_n = \frac{1}{n}$, then $\Sigma v_n = \Sigma \frac{1}{n}$ is a divergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{\lim\limits_{n\to\infty} n^{1/n}} = \frac{1}{1} = 1$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is divergent series.

***Example 16:*** ***Test the convergence of the series***

$$\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \ldots\ldots\ldots$$

**[B.C.A. (Agra) 2008]**

***Solution:***

$$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}}{n^3}\left\{\frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}}\right\}$$

Taking $v_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}$, then $\Sigma\, v_n = \Sigma \frac{1}{n^{5/2}}$ is a convergent series and

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left\{\frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}}\right\} = \frac{\sqrt{1+0}-0}{(1+0)^3-0} = 1$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is convergent series.

***Example 17:*** ***Show that the series***

$$\frac{1\cdot 2}{3^2\cdot 4^2}+\frac{3\cdot 4}{5^2\cdot 6^2}+\frac{5\cdot 6}{7^2\cdot 8^2}+\ldots\ldots \textit{ converges.}$$

***Solution:*** Let $\Sigma u_n = \frac{1\cdot 2}{3^2\cdot 4^2}+\frac{3\cdot 4}{5^2\cdot 6^2}+\frac{5\cdot 6}{7^2\cdot 8^2}+\ldots\ldots$ then

$$u_n = \frac{(n\text{th term of } 1,3,5,\ldots..)\times(n\text{th term of } 2,4,6\ldots\ldots..)}{(n\text{th term of } 3,5,7,\ldots..)^2\times(n\text{th term of } 4,\ 6,\ 8,\ldots\ldots)^2}$$

$$= \frac{(2n-1)2n}{(2n+1)^2(2n+2)^2} = \frac{n^2}{n^4}\left[\frac{\left(2-\frac{1}{n}\right).2}{\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}\right]$$

Taking $v_n = \frac{n^2}{n^4} = \frac{1}{n^2}$, then $\Sigma\, v_n = \Sigma\,\frac{1}{n^2}$ is a convergent series and

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n^2}{n^4}\left[\frac{\left(2-\frac{1}{n}\right).2}{\left(2+\frac{1}{n}\right)^2\left(2+\frac{2}{n}\right)^2}\right] = \frac{1}{4}$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is a convergent series.

***Example 18:*** ***Test for convergence the series*** $\sum_{n=1}^{\infty}\frac{1}{3^n+x}$, ***for all positive values of*** $x$. ***(i.e.,*** $x>0$***).***

***Solution:*** Clearly $3^n + x > 3^n$, because $x > 0$

$$\Rightarrow \qquad \frac{1}{3^n+x} < \frac{1}{3^n}\ \ \forall\, n\in N.$$

Now, $\Sigma\frac{1}{3^n}$ is a geometric series with common ratio $\frac{1}{3}<1$ is convergent. Hence by comparison test, the given series is convergent for all $x>0$.

***Example 19:*** ***Let*** $\Sigma\, a_n$ ***be a series of positive terms, prove that*** $\Sigma\,\frac{a_n}{n}$ ***is convergent.***

***Solution:*** We have $a_n > 0\ \forall\, n$ and let $\Sigma\, a_n$ is convergent.

Therefore, $\lim_{n\to\infty} a_n = 0$. It follows that for $\in = 1$, there exist $m\in N$

Such that $\quad |a_n - 0| < 1\ \forall n \geq m$

$\Rightarrow$ $$0 < a_n < 1 \,\forall n \geq m \Rightarrow a_n^2 < a_n \,\forall n \geq m$$

$\Rightarrow$ $$\Sigma a_n^2 < \Sigma a_n \forall n \geq m$$

By comparison test, we have $\Sigma a_n^2$ is convergent.

Now, $$\left(a_n - \frac{1}{n}\right)^2 \geq 0 \,\forall n \Rightarrow a_n^2 + \frac{1}{n^2} \geq 2\,\frac{a_n}{n} \,\forall n$$

or $$\frac{a_n}{n} \leq \frac{1}{2}\left(a_n^2 + \frac{1}{n^2}\right) \forall\, n \;\Rightarrow\; \Sigma \frac{a_n}{n} \leq \frac{1}{2}\Sigma\left(a_n^2 + \frac{1}{n^2}\right) \forall n$$

But $\Sigma a_n^2$ and $\Sigma \frac{2}{n^2}$ both are convergent, so $\frac{1}{2}\Sigma\left(a_n^2 + \frac{1}{n^2}\right)$ is convergent and therefore by comparison test, we have $\Sigma \frac{a_n}{n}$ is convergent.

**Example 20:** ***If $\Sigma a_n$ be the series of positive terms which is convergent, then $\Sigma \frac{a_n}{1 + a_n}$ is convergent.***

**Solution:** We have $a_n > 0 \Rightarrow 1 + a_n > 1 \;\; \forall n \in N$.

Now $\Sigma\, a_n$ is convergent, therefore by comparison test $\Sigma \frac{a_n}{1 + a_n}$ is convergent.

**Example 21:** ***Prove that the series $\frac{1}{\log 2} + \frac{1}{\log 3} + \ldots\ldots + \frac{1}{\log n} + \ldots\ldots$ is divergent.***

**Solution:** We have $\log n < n \,\forall n > 1$

$\Rightarrow$ $$\frac{1}{\log n} > \frac{1}{n} \,\forall n \geq 2$$

The series $\frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n} + \ldots\ldots\ldots$is divergent.

By comparison test $\Sigma \frac{1}{\log n}$ is divergent.

**Example 22:** ***Show that the series $1 + \frac{1}{2^2} + \frac{1}{3^3} + \ldots\ldots + \frac{1}{n^n} + \ldots\ldots\ldots$ is convergent.***

**Solution:** We know that $n^n > 2^n \forall\, n > 2$

$\Rightarrow$ $$\frac{1}{n^n} < \frac{1}{2^n} \,\forall n > 2$$

The series $\Sigma \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots$ is a geometric series with common ratio $\frac{1}{2} < 1$ so convergent. By comparison test $\Sigma \frac{1}{n^n}$ is convergent.

**Example 23:** ***Examine the following series for convergence.***

$$\Sigma \frac{1}{(\log n)^{\log n}}.$$

**Solution:** We have $\lim_{n \to \infty} \log(\log n) = \infty$

$\therefore$ We can find $n$ so large that $\log (\log n) > 2$

$\Rightarrow$ $$(\log n)\{\log(\log n)\} > 2 \log n$$

$\Rightarrow$ $$\log(\log n)^{\log n} > \log n^2$$

$\Rightarrow$ $$(\log n)^{\log n} > n^2$$

$\Rightarrow$ $$\frac{1}{(\log n)^{\log n}} < \frac{1}{n^2}$$

The series $\Sigma \frac{1}{n^2}$ is convergent, therefore by comparison test $\Sigma \frac{1}{(\log n)^{\log n}}$ is convergent.

**Example 24:** ***Test for convergence the series***

$$\Sigma \frac{\sqrt{n}}{n^2 + 1}.$$

**[B.C.A. (Kashi) 2011, 07; B.C.A. (Meerut) 2010, 06, 05]**

**Solution:** Here, we have $u_n = \frac{\sqrt{n}}{n^2 + 1}$

Let us take $$v_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

Now, $$\lim_{n \to \infty} \left(\frac{u_n}{v_n}\right) = \lim_{n \to \infty} \left\{\frac{\sqrt{n}}{n^2 + 1} . n^{3/2}\right\}$$

$$= \lim_{n \to \infty} \left\{\frac{n^2}{n^2 + 1}\right\}$$

$$= \lim_{n \to \infty} \left\{\frac{1}{1 + \left(\frac{1}{n}\right)^2}\right\} = 1$$

which is finite and non-zero.

Then by comparison test both series converge or diverge together.But $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ is convergent as $p = 3/2 > 1$, therefore by comparison test the given series $\Sigma u_n$ is also convergent.

## Exercise 5.1

**Test each of the following series for convergence:**

1. $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.

2. $\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 4}$.

3. $\sum_{n=1}^{\infty} \frac{1}{2^n + x}\ \forall x > 0.$

4. $\frac{1}{1+2} + \frac{1}{1+2^2} + \frac{1}{1+2^3} + \dots\dots$ . **[B.C.A. (Avadh) 2009]**

5. $\sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$.

6. $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \dots\dots\dots$ .

7. $\frac{1}{1\cdot3\cdot5} + \frac{1}{2\cdot4\cdot6} + \frac{1}{3\cdot5\cdot7} + \dots\dots$ .

8. $\frac{1}{\sqrt{3\cdot4}} + \frac{1}{\sqrt{4\cdot5}} + \frac{1}{\sqrt{5\cdot6}} + \frac{1}{\sqrt{6\cdot7}} + \dots$ .

9. $\frac{1}{1\cdot2^2} + \frac{1}{2\cdot3^2} + \frac{1}{3\cdot4^2} + \dots\dots$ .

10. $\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \dots$ .

11. $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \dots\dots$ .

12. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \dots\dots$ .

13. $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \dots\dots$ .

14. $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \dots\dots$ . **[B.C.A. (Agra) 2010]**

15. $1 + \frac{1}{2\cdot2^{1/100}} + \frac{1}{3\cdot3^{1/100}} + \frac{1}{4\cdot4^{1/100}} + \dots$ .

16. $\sum_{n=1}^{\infty} \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$.

17. $\sum_{n=1}^{\infty} \frac{n^p}{(1+n)^q}$. **[B.C.A. (Kashi) 2009]**

18. $\sum_{n=1}^{\infty} \frac{1}{1+\frac{1}{n}}$.

19. $\sum_{n=1}^{\infty} (\sqrt{n^4+1} - \sqrt{n^4-1})$.

20. $\sum_{n=1}^{\infty} (\sqrt{n^2+1} - n)$. **[B.C.A. (Rohtak) 2012, 09]**

21. $\sum \sin\frac{1}{n^2}$.

22. $\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{n}$. **[B.C.A. (Meerut) 2011]**

23. $\sum_{n=1}^{\infty} \frac{1}{n^{\left(a+\frac{b}{n}\right)}}$.

24. $\sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$.

**25.** $\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n$. **[B.C.A. (Kanpur) 2009]**

**26.** $\frac{1}{1+2^{-1}}+\frac{2}{1+2^{-2}}+\frac{3}{1+2^{-3}}+\ldots\ldots$ .

**27.** $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\sin\frac{1}{n}\right)$.

**28.** $\sum_{n=1}^{\infty} \cos\frac{1}{n}$. **[B.C.A. (Bundelkhand) 2009]**

**29.** If $a_n \geq 0 \;\forall n \in N$ and $\Sigma\, a_n$ converges, then show that $\Sigma\left(\frac{\sqrt{a_n}}{n}\right)$ converges.

**30.** Show that the series $\Sigma\, e^{-n^2}$ converges.

[ **Hint :** $e^{n^2} > n^2 \Rightarrow \frac{1}{e^{n^2}} < \frac{1}{n^2} \Rightarrow e^{-n^2} < \frac{1}{n^2} \;\; \forall n \in N$].

**31.** $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$.

**32.** $\sum_{n=1}^{\infty} \frac{1}{(n)!}$. **[B.C.A. (Lucknow) 2007]**

**33.** $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,!}$.

**34.** If $\Sigma\, u_n^2$ and $\Sigma\, v_n^2$ converge, prove that $\Sigma\, u_n\, v_n$ is convergent.

**35.** If $\Sigma\, u_n$ is convergent, prove that $\Sigma \frac{u_n}{1-u_n}$ $(u_n \neq 1)$ is convergent.

**36.** Examine the convergence of the series

(i) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\ldots\ldots.$ .

(ii) $a+b+a^2+b^2+a^3+b^3+\ldots..$

[**Hint:** Sum of two convergent series is convergent].

(iii) $\sum_{n=1}^{\infty} \tan^{-1}\frac{1}{n^2}$ **(iv)** $\sum_{n=1}^{\infty} \cot^{-1} n^2$ **[B.C.A. (Lucknow) 2012]**

(v) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\tan\frac{1}{n}$ **(vi)** $\sum_{n=1}^{\infty} \left(1+\frac{1}{\sqrt{n}}\right)^{-(n)^{3/2}}$. **[B.C.A. (Meerut) 2008, 03]**

## Answers 5.1

| | | | | | |
|---|---|---|---|---|---|
| **1.** | Divergent. | **2.** | Divergent. | **3.** | Convergent. |
| **4.** | Convergent. | **5.** | Convergent. | **6.** | Divergent. |
| **7.** | Convergent. | **8.** | Divergent. | **9.** | Convergent. |
| **10.** | Divergent. | **11.** | Convergent if $p > 2$ and divergent if $p \leq 2$. | **12.** | Divergent. |
| **13.** | Divergent. | **14.** | Convergent. | **15.** | Convergent. |
| **16.** | Divergent. | **17.** | Convergent if $p - q + 1 < 0$ and divergent if $p - q + 1 \geq 0$. | **18.** | Divergent. |
| **19.** | Convergent. | **20.** | Divergent. | **21.** | Convergent. |
| **22.** | Divergent. | **23.** | Convergent if $a > 1$ and divergent if $a \leq 1$. | **24.** | Convergent. |
| **25.** | Convergent. | **26.** | Divergent. | **27.** | Convergent. |
| **28.** | Divergent. | **31.** | Convergent. | **32.** | Convergent. |
| **33.** | Convergent. | | | | |
| **36.** | (i) Convergent | (ii) Convergent only when $\|a\|$ and $\|b\|$ are $< 1$. | | | |
| | (iii) Divergent | (iv) Convergent | (v) Convergent | (vi) Convergent. | |

### 5.3.9 Cauchy's Root Test (or Cauchy's Radical Test)

Let $\Sigma\, u_n$ be a series of positive terms such that

$$\lim_{n \to \infty} (u_n)^{1/n} = l. \text{ Then,}$$

1. $\Sigma\, u_n$ converges if $l < 1$
2. $\Sigma\, u_n$ diverges if $l > 1$
3. Test fails $l = 1$.

**Proof: Case I:** When $l < 1$

We can take $\in > 0$ such that $l + \in < 1$ or $\alpha < 1$, where $l + \in = \alpha$

If $\lim_{n\to\infty} (u_n)^{1/n} = l$, then

$$|(u_n)^{1/n} - l| < \in \qquad \forall n \geq m \qquad ...(1)$$

$$\Rightarrow \qquad l - \in < (u_n)^{1/n} < l + \in \qquad \forall n \geq m \qquad ...(2)$$

Consider $(u_n)^{1/n} < l + \in$ or $(u_n)^{1/n} < \alpha \quad \forall n \geq m$

$$\Rightarrow \qquad u_n < (\alpha)^n \qquad \forall n \geq m$$

Now $\Sigma(\alpha)^n = \alpha + \alpha^2 + \alpha^3 + ......$ is a geometric series with common ratio $\alpha < 1$, so convergent. By comparison test $\Sigma u_n$ is also convergent.

**Case II :** When $l > 1$

We can take $\in > 0$ such that $l - \in > 1$ or $\beta > 1$, where $\beta = l - \in$

If $\lim_{n\to\infty} (u_n)^{1/n} = l$, then

$$|(u_n)^{1/n} - l| < \in \qquad \forall n \geq m$$

$$\Rightarrow \qquad l - \in < (u_n)^{1/n} < l + \in \qquad \forall n \geq m$$

Consider $\quad \beta < (u_n)^{1/n} < \alpha \qquad \forall n \geq m$, where $\beta > 1$

$$\Rightarrow \qquad \beta < (u_n)^{1/n} \qquad \forall n \geq m$$

$$\Rightarrow \qquad \beta^n < u_n$$

Now $\Sigma \beta^n = \beta + \beta^2 + \beta^3 + ......$ being geometric series with common ratio $\beta > 1$, so divergent. By comparison test $\Sigma u_n$ is also divergent.

**Case III:** When $l = 1$

We shall give example of two series : One convergent and the other divergent, but both satisfying $\lim_{n\to\infty} (u_n)^{1/n} = 1$. So, Cauchy root test fails.

Consider the series $\Sigma\, u_n = \Sigma \frac{1}{n}$ is divergent and

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{(n)^{1/n}} = \frac{1}{1} = 1$$

Again, consider the series $\Sigma\, u_n = \Sigma \dfrac{1}{n^2}$ is convergent, but

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{(n^{1/n})^2} = \frac{1}{1} = 1.$$

***Example 25:*** ***Test the convergence of the following series :***

**[B.C.A. (Meerut) 2011, 04, 03, 02, 00]**

*(i)* $\Sigma \left(\dfrac{n}{n+1}\right)^{n^2}$ *or* $\Sigma \left(1 + \dfrac{1}{n}\right)^{-n^2}$

*(ii)* $\Sigma \dfrac{x^n}{n^n}$ **[B.C.A. (I.G.N.O.U.) 2010]**

*(iii)* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

*(iv)* $\Sigma\, x^n n^n (x > 0)$

*(v)* $\Sigma \left(\dfrac{n+1}{3n}\right)^n$

*(vi)* $\Sigma \dfrac{n^{n^2}}{(n+1)^{n^2}}$

**[B.C.A. (Bundelkhand) 2008]**

*(vii)* $\Sigma (n^{1/n} - 1)^n$

*(viii)* $\Sigma \left(1 + \dfrac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ **[B.C.A. (Kanpur) 2009]**

*(ix)* $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

*(x)* $\Sigma \dfrac{1}{n^n}$ **[B.C.A. (Purvanchal) 2011]**

*(xi)* $\Sigma\, 2^{-n-(-1)^n}$.

***Solution:*** (i) Let $u_n = \left(\dfrac{n}{n+1}\right)^{n^2}$, then $(u_n)^{1/n} = \left[\left(\dfrac{n}{n+1}\right)^{n^2}\right]^{1/n}$

$$= \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

By Cauchy's root test, the given series $\Sigma\, u_n$ is convergent .

(ii) **Hint:** $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{x}{n} = 0 < 1$ **(Ans.** Convergent**)**

(iii) **Hint:** $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = \frac{1}{\infty} = 0 < 1$ **(Ans.** Convergent**)**

(iv) **Hint:** $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} nx = \infty > 1$ **(Ans.** Divergent**)**

(v) **Hint:** $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{n+1}{3n} = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{n}\right) = \frac{1}{3} < 1$ **(Ans.** Convergent**)**

(vi) **Hint:** Similar to part (i)

$$u_n = (n^{1/n} - 1)^n \Rightarrow (u_n)^{1/n} = n^{1/n} - 1$$

(vii) $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} (n^{1/n} - 1) = \lim_{n\to\infty} n^{1/n} - 1$

$$= 1 - 1 = 0 < 1$$

Hence, by Cauchy's root test, the given series converges.

(viii) We have $u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}} = \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{n\sqrt{n}}}$

$$\Rightarrow \qquad (u_n)^{1/n} = \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{\lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$$

Hence, by Cauchy's root test, the given series converges.

(ix) We have $u_n = \frac{n^3}{3^n} \Rightarrow (u_n)^{1/n} = \frac{(n^{1/n})^3}{3}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \frac{\left(\lim_{n\to\infty} n^{1/n}\right)^3}{3} = \frac{1}{3} < 1$$

Hence, by Cauchy's root test, the given series converges.

(x) We have $u_n = \frac{1}{n^n} \Rightarrow (u_n)^{1/n} = \frac{1}{n}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0 < 1$$

So, the given series converges.

(xi) We have $u_n = 2^{-n-(-1)^n} = 2^{-\{n+(-1)^n\}} = \frac{1}{2^{n+(-1)^n}} = \frac{1}{2^n 2^{(-1)^n}}$

Now, $(u_n)^{1/n} = \frac{1}{2 \cdot 2^{(-1)^{n/n}}}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{2} \frac{1}{\lim_{n\to\infty} 2^{\frac{(-1)^n}{n}}} = \frac{1}{2} < 1 \qquad \left(\because \lim_{n\to\infty} \frac{(-1)^n}{n} = 0\right)$$

Hence, by Cauchy's root test, the given series converges.

***Example 26:*** ***Examine the convergence of the series***

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots\ldots .$$

**[B.C.A. (Delhi) 2012; B.C.A. (Bhopal) 2012; B.C.A. (Meerut) 2008]**

***Solution:***

$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

$$(u_n)^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1}$$

$$= \left[\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{1}{n}\right) - \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right]^{-1}$$

$$= (e.1-1)^{-1} = \frac{1}{e-1} < 1 \qquad \left(\because e > 2 \therefore e-1 > 1 \Rightarrow \frac{1}{e-1} < 1\right)$$

***Example 27:*** ***Test the convergence of series***

$$\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2 x^2+\left(\frac{4}{5}\right)^3 x^3+\ldots .$$

**[B.C.A. (Indore) 2011]**

***Solution:*** Neglecting the first term which do not effect the nature of the series, let the given series be denoted by $\Sigma\, u_n$. Then,

$$u_n=\left(\frac{n+1}{n+2}\right)^n x^n$$

$$(u_n)^{1/n}=\left(\frac{n+1}{n+2}\right)x=\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x$$

$$\lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)x=x$$

Hence, by Cauchy's root test, the given series will be convergent if $x<1$ and divergent if $x>1$.

Also the test fails if $x=1$.

Now, when $\quad x=1,\quad u_n=\left(\frac{n+1}{n+2}\right)^n=\left[\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{1}{n}\right)}\right]^n=\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}$

Taking $v_n=\frac{1}{n^0}$, so $\Sigma\, v_n=\Sigma\frac{1}{n^0}$ is divergent.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}=\frac{e}{e^2}=\frac{1}{e}$$

which is finite and non-zero.

By comparison test $\Sigma\, u_n$ is divergent, when $x=1$.

## Exercise 5.2

**Test the convergence of the following series:**

1. $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n}$.

2. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$.

3. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$.

**[B.C.A. (Bhopal) 2011]**

4. $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$.

5. $\sum_{n=1}^{\infty}\left[\log\left(1+\frac{1}{n}\right)\right]^{n}$.

6. $\sum_{n=1}^{\infty}\frac{n^{n^2}}{\left(n+\frac{1}{4}\right)^{n^2}}$.

7. $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\ldots$.

8. $\sum_{n=1}^{\infty}\frac{(n+1)^n}{n^{n+1}}$.

9. $\sum_{n=1}^{\infty}\left(\frac{1+nx}{n}\right)^{n}, x>1$.

10. $\sum_{n=1}^{\infty}\frac{n^n}{(n)!}$.

11. $\sum_{n=1}^{\infty}5^{-n-(-1)^n}$.

12. $\sum_{n=1}^{\infty}\frac{(n-\log n)^n}{2^n n^n}$.

**[B.C.A. (Meerut) 2007]**

13. $\sum_{n=1}^{\infty}\frac{x^n}{(n)!}$.

**[B.C.A. (Meerut) 2002]**

[**Hint:** $\lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\frac{x}{(n!)^{1/n}}=\lim_{n\to\infty}\left[\frac{n}{(n!)^{1/n}}\cdot\frac{x}{n}\right]$

$\lim_{n\to\infty}\frac{(n^n)^{1/n}}{(n!)^{1/n}}\cdot\frac{x}{n}=\lim_{n\to\infty}\frac{(n^n)^{1/n}}{(n!)^{1/n}}\lim_{n\to\infty}\frac{x}{n}=e\times 0=0<1$, By Cauchy root test, the given series converges.]

## Answers 5.2

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | Divergent. | 2. | Divergent. | 3. | Convergent. | 4. | Convergent. |
| 5. | Convergent. | 6. | Convergent. | 7. | Convergent. | 8. | Divergent. |
| 9. | Divergent. | 10. | Divergent. | 11. | Convergent. | 12. | Convergent. |
| 13. | Convergent. | | | | | | |

## 5.3.10 D' Alembert's Ratio Test

If $\Sigma u_n$ is a series of positive terms such that

$$\lim_{n\to\infty} \frac{u_{(n+1)}}{u_n} = l, \text{ then the series}$$

1. Converges, if $l < 1$
2. Diverges, if $l > 1$ and
3. The test fails, if $l = 1$ (*i.e.,* the series may converge, it may diverge if $l = 1$).

**Proof:** We have $\Sigma u_n = u_1 + u_2 + \ldots\ldots + u_{m-1} + u_m + u_{m+1} + \ldots\ldots$ and if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then for any $\in > 0$, however small, there exist $m \in N$ such that

$$\left| \frac{u_{n+1}}{u_n} - l \right| < \in \qquad \forall\, n \geq m$$

$$\Rightarrow \qquad l - \in < \frac{u_{n+1}}{u_n} < l + \in \qquad \forall\, n \geq m \qquad \ldots(1)$$

1. When $l < 1$

Let us choose $\in > 0$ such that $l < l + \in < 1$

If $l + \in = r$, then $r < 1$

From (1), we have $\frac{u_{n+1}}{u_n} < l + \in = r \qquad \forall\, n \geq m$

or $$\frac{u_{n+1}}{u_n} < r \qquad \forall\, n \geq m$$

$$\Rightarrow \qquad u_{n+1} < ru_n \qquad \forall\, n \geq m$$

Putting $n = m, m+1, m+2 \ldots\ldots$, we get

$$u_{m+1} < ru_m \qquad \ldots(1)$$

$$u_{m+2} < ru_{m+1} \Rightarrow u_{m+2} < r(ru_m) = r^2 u_m \qquad \ldots(2)$$

[using (1)]

Similarly, $$u_{m+3} < ru_{m+2} \Rightarrow u_{m+3} < r(r^2 u_m) = r^3 u_m \qquad \ldots(3)$$

[using (3)] and so on.

Adding the above inequalities, we get

$$u_{m+1} + u_{m+2} + u_{m+3} + \ldots\ldots < (r + r^2 + r^3 + \ldots\ldots) u_m$$

$\Rightarrow$ Each term of the given series $\Sigma\, u_n$ after leaving the first $n$ terms (*i.e.,* a finite number of terms) is less than the corresponding terms of a geometric series which is convergent ($\because$ its common ratio $r < 1$). Hence, the given series $\Sigma\, u_n$ is also convergent.

2. When $l > 1$

Let us choose $\in > 0$ such that $l - \in > 1$ and put $l - \in = R$. Therefore, $R > 1$.

Now from (1), we have $l - \in < \dfrac{u_{n+1}}{u_n}$ or $R < \dfrac{u_{n+1}}{u_n}$

$$\Rightarrow \qquad \frac{u_{n+1}}{u_n} > R \qquad \forall\, n \geq m$$

$$\Rightarrow \qquad u_{n+1} > Ru_n \qquad \forall n \geq m$$

Putting $n = m, m+1, m+2, \ldots\ldots$, we get

$$u_{m+1} > Ru_m \qquad \ldots(1)$$

$$u_{m+2} > Ru_{m+1} \;\Rightarrow\; u_{m+2} > R^2 u_m \qquad [\text{using } (1)]$$

Similarly, $u_{m+3} > R^3 u_m \ldots\ldots$ and so on.

Adding the above inequalities, we get

$$u_{m+1} + u_{m+2} + u_{m+3} + \ldots\ldots > (R + R^2 + R^3 + \ldots\ldots)u_m$$

$\Rightarrow$ Each term of the given series $\Sigma\, u_n$ after leaving the first $m$ terms (*i.e.*, a finite number of terms) is greater than the corresponding term of a geometric series which is divergent ($\because$ its common ratio $R > 1$). Hence, the given series $\Sigma u_n$ is also divergent.

3. When $l = 1$, the test fails to give any definite information about convergence or divergence of the series. We shall give example of two series : One convergent and the other divergent but both satisfying.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$$

The series $\Sigma u_n = \Sigma \dfrac{1}{n}$ is divergent, but

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \left( \frac{1}{1+\dfrac{1}{n}} \right) = 1$$

The series $\Sigma u_n = \frac{1}{n^2}$ is convergent, but

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \left( \frac{1}{1+\frac{1}{n^2}} \right) = 1$$

**Remark 1:** Another equivalent form of Ratio Test is as follows :

If $\Sigma\, u_n$ is the series of positive terms such that

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l, \text{ then}$$

(i) $\Sigma u_n$ is convergent if $l > 1$

(ii) $\Sigma u_n$ is divergent if $l < 1$ and

(iii) The test fails if $l = 1$.

**Remark 2:** If $\Sigma\, u_n$ be the series of positive terms such that

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty, \text{ then } \Sigma\, u_n \text{ is convergent.}$$

**Proof:** If $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \infty$, then the sequence $< \frac{u_n}{u_{n+1}} >$ diverges to $\infty$. So, there exist a positive integer $m$ such that

$$\frac{u_n}{u_{n+1}} > 2 \qquad \forall n \geq m$$

$$\Rightarrow \qquad \frac{u_n}{u_{n+1}} > \frac{2^{n+1}}{2^n} \text{ or } \frac{u_n}{u_{n+1}} > \frac{u_n}{u_{n+1}} \quad \forall\, n \geq m$$

where $\Sigma\, u_n = \Sigma \frac{1}{2^n}$ be a geometric series with common ratio $\frac{1}{2} < 1$ is convergent. Hence, by comparison test, $\Sigma\, u_n$ is convergent.

**Remark 3:** If $\Sigma\, u_n$ is a series of positive terms such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0, \text{ then } \Sigma\, u_n \text{ is convergent.}$$

**Remark 4:** If $\Sigma\, u_n$ is a series of positive terms such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \infty, \text{ then } \Sigma\, u_n \text{ is divergent.}$$

**Remark 5:** Another form of D' Alembert's Ratio Test. The ratio test can also be stated in the form given below.

If $\Sigma\, u_n$ is a series of positive terms such that:

(i) If from and after some fixed term $\frac{u_{n+1}}{u_n} < r < 1$,

where $r$ is fixed number, then the series $\Sigma\, u_n$ is convergent and

(ii) If from and after some fixed term $\frac{u_{n+1}}{u_n} > 1$, then the series $\Sigma\, u_n$ is divergent.

**Proof:** (i) We know that the convergence or divergence of a series is not affected by ignoring a finite number of terms. Thus by ignoring a finite number of terms (if needed), the series is $u_1 + u_2 + u_3 + \ldots\ldots$

and let $\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r \ldots\ldots$ where $r < 1$, then

$$u_1 + u_2 + u_3 + u_4 + \ldots\ldots = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \ldots\ldots\right)$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots\ldots\right)$$

$$= u_1\,(1 + r + r^2 + r^3 + \ldots\ldots) < \frac{u_1}{1-r} \qquad [\because r < 1]$$

which is a finite quantity.

Hence, the given series $\Sigma\, u_n$ is convergent.

(ii) Let the series after some particular fixed term be

$$u_1 + u_2 + u_3 + u_4 + \ldots\ldots \text{ and let } \frac{u_{n+1}}{u_n} > 1 \qquad \forall n \in N$$

Then $u_2 > u_1, u_3 > u_2 > u_1 \ldots\ldots$

Then $u_1 + u_2 + u_3 + u_4 + \ldots\ldots + u_n > nu_1$ *i.e.*, $S_n > nu_1$.

Now taking $n$ sufficiently large, $nu_1$ can be made greater than any finite number, however large.

Hence, the series $\Sigma\, u_n$ is divergent.

***Example 28:*** ***Test for convergence the series***

$$\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\ldots.$$

**[B.C.A. (I.G.N.O.U.) 2010, 07]**

***Solution:*** The series can be written as

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\ldots.$$

$$u_n=\frac{1}{(n\text{th term of } 1, 2, 3, ,\ldots\ldots)(n\text{th term of } 2, 2^2, 2^3,\ldots\ldots)}$$

$$=\frac{1}{n.2^n}$$

$$u_{n+1}=\frac{1}{(n+1)2^{n+1}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1}{n\cdot 2^n}\cdot\frac{(n+1)2^{n+1}}{1}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\cdot\frac{2^n\cdot 2}{2^n}=2\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=2>1$$

By ratio test, the given series converges.

***Example 29:*** ***Test for convergence the series***

$$\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\ldots\ldots.$$

**[B.C.A. (Kurukshetra) 2010]**

***Solution:*** We have

$$u_n=\left[\frac{1\cdot 2\cdot 3\ldots..n}{3\cdot 5\cdot 7\ldots\ldots(2n+1)}\right]^2$$

$$u_{n+1}=\left[\frac{1\cdot 2\cdot 3\ldots..n(n+1)}{3\cdot 5\cdot 7\ldots\ldots(2n+1)(2n+3)}\right]^2$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\left[\frac{1\cdot 2\cdot 3\ldots\ldots n}{3\cdot 5\cdot 7\ldots\ldots(2n+1)}\right]^2\times\left[\frac{3\cdot 5\cdot 7\ldots\ldots(2n+1)(2n+3)}{1\cdot 2\cdot 3\ldots..n(n+1)}\right]^2$$

$$=\lim_{n\to\infty}\left(\frac{2n+3}{n+1}\right)^2=\lim_{n\to\infty}\left(\frac{2+\frac{3}{n}}{1+\frac{1}{n}}\right)^2=4>1$$

By ratio test, the given series converges.

***Example 30:*** ***Test for convergence the series***

***(i)*** $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3\ldots\ldots n}{7\cdot 10\ldots\ldots(3n+4)}$ ***(ii)*** $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n+1}$.

***Solution:*** (i)

$$u_n = \frac{1\cdot 2\cdot 3\ldots\ldots n}{7\cdot 10\ldots\ldots(3n+4)}$$

$$u_{n+1} = \frac{1\cdot 2\cdot 3\ldots\ldots n(n+1)}{7\cdot 10\ldots\ldots(3n+4)(3n+7)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\ldots\ldots n}{7\cdot 10\ldots\ldots(3n+4)} \times \frac{7\cdot 10\ldots\ldots(3n+4)(3n+7)}{1\cdot 2\cdot 3\ldots\ldots n(n+1)}$$

$$= \lim_{n\to\infty} \frac{3n+7}{n+1} = \lim_{n\to\infty} \frac{3+\frac{7}{n}}{1+\frac{1}{n}} = 3 > 1$$

By ratio test, the given series converges.

(ii) $u_n = \dfrac{2^{n-1}}{3^n+1}, u_{n+1} = \dfrac{2^n}{3^{n+1}+1}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2^{n-1}}{3^n+1} \times \frac{3^{n+1}+1}{2^n}$$

$$\lim_{n\to\infty} \left(\frac{2^n}{2\cdot 2^n}\right)\left[\frac{3^n\left(3+\frac{1}{3^n}\right)}{3^n\left(1+\frac{1}{3^n}\right)}\right] = \frac{1}{2}\lim_{n\to\infty} \frac{3+\frac{1}{3^n}}{1+\frac{1}{3^n}} = \frac{3}{2} > 1$$

Because $\displaystyle\lim_{n\to\infty} \frac{1}{3^n} = 0$

By ratio test, the given series is convergent.

***Example 31:*** ***Test for convergence the series***

$$\frac{1^2\cdot 2^2}{1!} + \frac{2^2\cdot 3^2}{2!} + \frac{3^2\cdot 4^2}{3!} + \ldots\ldots .$$

**[B.C.A. (Meerut) 2008]**

***Solution:***

$$u_n = \frac{(T_n \text{ of } 1^2, 2^2, 3^2\ldots\ldots)(T_n \text{ of } 2^2, 3^2, 4^2\ldots\ldots)}{T_n \text{ of } 1!, 2!, 3!, \ldots\ldots.}$$

$$= \frac{n^2(n+1)^2}{n!}$$

$$u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2 (n+1)^2 (n+1)!}{n!\ (n+1)^2 (n+2)^2}$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+2)^2} \cdot \frac{(n+1)n!}{n!}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{2}{n}\right)^2} \lim_{n\to\infty} (n+1) = 1\times\infty = \infty > 1$$

Hence, the series converges.

**Example 32:** ***Test for convergence the series*** $\sum_{n=1}^{\infty} \frac{r^n}{n!}$, ***where r is any positive number.***

**Solution:** We have $u_n = \frac{r^n}{n!}, u_{n+1} = \frac{r^{n+1}}{((n+1)!}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{r^n}{n!} \times \frac{(n+1)!}{r^{n+1}} = \lim_{n\to\infty} \frac{1}{r} \frac{(n+1)n!}{n!}$$

$$= \frac{1}{r} \lim_{n\to\infty} (n+1) = \frac{1}{r} \times \infty = \infty > 1$$

By ratio test, the given series converges.

**Example 33:** ***Test for convergence the series whose nth term is*** $\frac{r^n}{n^n}, r > 0$.

**[B.C.A. (Delhi) 2012, 11, 07]**

**Solution:** We have $u_n = \frac{r^n}{n^n}, u_{n+1} = \frac{r^{n+1}}{(n+1)^{n+1}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{r^n}{n^n} \cdot \frac{(n+1)^{n+1}}{r^{n+1}} = \frac{1}{r} \lim_{n\to\infty} \frac{(n+1)^n (n+1)}{n^n}$$

$$= \frac{1}{r} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n (n+1) = \frac{1}{r} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \lim_{n\to\infty} (n+1)$$

$$= \frac{1}{r} \times e \times \infty = \infty > 1$$

By ratio test, the given series converges.

**Example 34:** ***Test for convergence the series*** $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

**Solution:** We have $u_n = \frac{n!}{n^n}, u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{(n+1)^n (n+1) n!}{n^n (n+1) n!}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

By ratio test, the given series converges.

**Example 35:** ***Test for convergence the series***

$$\frac{1}{5} + \frac{2!}{5^2} + \frac{3!}{5^3} + \frac{4!}{5^4} + \ldots\ldots .$$

**[B.C.A. (Garhwal) 2007]**

**Solution:** $u_n = \frac{n!}{5^n}$ and $u_{n+1} = \frac{(n+1)}{5^{n+1}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!}{5^n} \cdot \frac{5^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{n! 5^n . 5}{5^n (n+1) n!}$$

$$= 5 \lim_{n\to\infty} \frac{1}{n+1} = 5 \times \frac{1}{\infty} = 5 \times 0 = 0 < 1$$

By ratio test, the given series diverges.

**Example 36:** ***Test for convergence the series*** **[B.C.A. (Meerut) 2005]**

*(i)* $\frac{x}{1\cdot 3} + \frac{x^2}{2\cdot 4} + \frac{x^3}{3\cdot 5} + \frac{x^4}{4\cdot 6} + \ldots\ldots (x > 0)$

*(ii)* $\frac{x}{2\sqrt{3}} + \frac{x^2}{3\sqrt{4}} + \frac{x^3}{4\sqrt{5}} + \ldots\ldots (x > 0)$.

**[B.C.A. (Agra) 2007]**

**Solution:** (i) $u_n = \frac{x^n}{n(n+2)}$ and $u_{n+1} = \frac{x^{n+1}}{(n+1)(n+3)}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n}{n(n+2)} \cdot \frac{(n+1)(n+3)}{x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{x} = \frac{1}{x}$$

By ratio test, $\Sigma u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and $\Sigma u_n$ diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$.

The test fails for $x = 1$.

Now, for $x = 1$, we have $u_n = \frac{1}{n(n+2)} = \frac{1}{n^2}\left[\frac{1}{\left(1+\frac{2}{n}\right)}\right]$

Taking $v_n = \frac{1}{n^2}$ and $\Sigma v_n = \Sigma \frac{1}{n^2}$ converges.

$\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 1$ which is finite and non-zero. So by comparison test $\Sigma u_n$ is convergent for $x = 1$.

Hence, the given series converges for $x \le 1$ and diverges for $x > 1$.

(ii) $u_n = \frac{x^n}{(n+1)\sqrt{n+2}}$ and $u_{n+1} = \frac{x^{n+1}}{(n+2)\sqrt{n+3}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n}{(n+1)\sqrt{n+2}} \frac{(n+2)\sqrt{n+3}}{x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}}\left[\frac{\left(1+\frac{2}{n}\right)\sqrt{1+\frac{3}{n}}}{\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}}\right]\frac{1}{x} = \frac{1}{x}$$

By ratio test, the series converges if $\frac{1}{x} > 1$ or $x < 1$ and diverges if $\frac{1}{x} < 1$ or $x > 1$. The test fails for $x = 1$.

For $x = 1$, we have $$u_n = \frac{1}{(n+1)\sqrt{(n+2)}}$$

$$= \frac{1}{n^{3/2}}\left[\frac{1}{\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}}\right]$$

Taking $v_n = \frac{1}{n^{3/2}}$ and $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$ converges.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}} = 1$$

which is finite and non-zero. Therefore by comparison test, the given series converges for $x = 1$.

Hence, the series converges for $x \le 1$ and diverges for $x > 1$.

***Example 37:*** ***Test for the convergence the series***

***(i)*** $\dfrac{x}{\sqrt{5}} + \dfrac{x^3}{\sqrt{7}} + \dfrac{x^5}{\sqrt{9}} + \ldots\ldots (x > 0)$ **[B.C.A. (Rohilkhand) 2009]**

***(ii)*** $\dfrac{x^2}{2\sqrt{1}} + \dfrac{x^3}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \ldots\ldots (x > 0)$ **[B.C.A. (Agra) 2008]**

***(iii)*** $\dfrac{x}{2\sqrt{3}} + \dfrac{x^2}{3\sqrt{4}} + \dfrac{x^3}{4\sqrt{5}} + \ldots\ldots (x > 0)$. **[B.C.A. (Kanpur) 2010]**

***Solution:*** (i) We have $u_n = \dfrac{x^{2n-1}}{\sqrt{2n+3}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{2n-1}}{\sqrt{2n+3}} \times \frac{\sqrt{2n+5}}{x^{2n+1}} = \lim_{n\to\infty} \sqrt{\frac{2n+5}{2n+3}} \cdot \frac{1}{x^2}$$

$$= \lim_{n\to\infty} \sqrt{\frac{2+5/n}{2+3/n}}\, \frac{1}{x^2} = \frac{1}{x^2}$$

By ratio test, $\Sigma\, u_n$ is convergent if $\dfrac{1}{x^2} > 1$ *i.e.,* $x^2 < 1$ or $x < 1$ and divergent if $\dfrac{1}{x^2} < 1$

*i.e.,* $x^2 > 1$ or $x > 1$.

The test fails if $\dfrac{1}{x^2} = 1$ or $x = 1$.

For $x = 1$, we have $u_n = \dfrac{1}{\sqrt{2n+3}} = \dfrac{1}{\sqrt{n}\left(\sqrt{2 + \dfrac{3}{n}}\right)}$, $v_n = \dfrac{1}{\sqrt{n}}$

Taking $$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{2 + \dfrac{3}{n}}} = \frac{1}{\sqrt{2}}$$

which is finite and non-zero. So by ratio test $\Sigma\, u_n$ diverges if $x = 1$.

Hence , the given series is convergent if $x < 1$ and divergent if $x \ge 1$.

(ii) We have $u_n = \dfrac{x^{n+1}}{(n+1)\sqrt{n}}$, $u_{n+1} = \dfrac{x^{n+2}}{(n+2)\sqrt{n+1}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+2}}{x^{n+2}}$$

$$= \lim_{n\to\infty} \frac{(n+2)\sqrt{n+2}}{(n+1)\sqrt{n}} \cdot \frac{1}{x} = \frac{1}{x} \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)\sqrt{1+\frac{2}{n}}}{\left(1+\frac{1}{n}\right)} = \frac{1}{x}$$

By ratio test, the given series converges if $\frac{1}{x} > 1$ or $x < 1$ and divergent if $\frac{1}{x} < 1$ or $x > 1$. The test fails if $\frac{1}{x} = 1$ or $x = 1$.

For $x = 1$, we have $\quad u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}}\left[\frac{1}{\left(1+\frac{1}{n}\right)}\right]$

Taking $v_n = \frac{1}{n^{3/2}}$, then $\Sigma\, v_n = \Sigma \frac{1}{n^{3/2}}$ converges.

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = 1$ which is finite and non-zero.

So by comparison test, the given series converges.

Hence, the given series converges if $x \le 1$ and diverges if $x > 1$.

(iii) We have $\quad u_n = \frac{x^n}{(n+1)\sqrt{n+2}},\ u_{n+1} = \frac{x^{n+1}}{(n+2)\sqrt{n+3}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+2)\sqrt{n+3}}{(n+1)\sqrt{n+2}} \cdot \frac{1}{x} = \frac{1}{x} \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)\sqrt{1+\frac{3}{n}}}{\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}} = \frac{1}{x}$$

By ratio test, the given series is converge if $\frac{1}{x} > 1$ *i.e.,* $x < 1$ and divergent if $\frac{1}{x} < 1$ *i.e.,* $x > 1$. The test fails if $\frac{1}{x} = 1$ or $x = 1$.

For $x = 1$, we have $\quad u_n = \frac{1}{(n+1)\sqrt{n+2}} = \frac{1}{n^{3/2}}\left[\frac{1}{\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}}\right]$

Taking $v_n = \frac{1}{n^{3/2}}$, then $\Sigma\, v_n = \Sigma \frac{1}{n^{3/2}}$ converges.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left[\frac{1}{\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}}\right]=1 \text{ which is finite and non-zero.}$$

So by comparison test, the given series converges.

Hence, the given series converges if $x \le 1$ and diverges if $x > 1$.

***Example 38:* Test for convergence the series**

***(i)*** $x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\ldots\ldots(x>0)$

**[B.C.A. (Lucknow) 2004]**

***(ii)*** $\displaystyle\sum_{n=1}^{\infty}\frac{n^n x^n}{n!}\,(x>0).$

**[B.C.A. (Meerut) 2010, 08]**

***Solution:***

(i) We have $u_n=\dfrac{x^{2n-1}}{(2n-1)!},\ u_{n+1}=\dfrac{x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{x^{2n-1}}{(2n-1)!}\cdot\frac{(2n+1)!}{x^{2n+1}}$$

$$=\frac{1}{x^2}\lim_{n\to\infty}\frac{(2n+1)2n(2n-1)!}{(2n-1)!}$$

$$=\frac{1}{x^2}\lim_{n\to\infty}2n(2n+1)=\frac{1}{x^2}\times\infty=\infty>1$$

Hence, by ratio test, the given series converges.

(ii) We have $u_n=\dfrac{n^n x^n}{n!}$ and $u_{n+1}=\dfrac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{n^n x^n}{n!}\times\frac{(n+1)!}{(n+1)^{n+1}.x^{n+1}}$$

$$=\frac{1}{x}\lim_{n\to\infty}\frac{n^n(n+1)n!}{n!(n+1)(n+1)^n}=\frac{1}{x}\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{ex}$$

By ratio test, the given series converges if $\dfrac{1}{ex}>1$ or $x<\dfrac{1}{e}$ and diverges if $\dfrac{1}{ex}<1$ or $x>\dfrac{1}{e}$. The test fails if $\dfrac{1}{ex}=1$ or $x=\dfrac{1}{e}$.

For $x = \frac{1}{e}$, we have $u_n = \frac{n^n}{n!}\left(\frac{1}{e}\right)^n$, $u_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}\left(\frac{1}{e}\right)^{n+1}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^n}{n!}\frac{1}{e^n}\cdot\frac{(n+1)!}{(n+1)^{n+1}}\cdot e^{n+1}$$

$$= \lim_{n\to\infty} \frac{n^n (n+1) n!}{n!(n+1)(n+1)^n} \times e$$

$$= \lim_{n\to\infty} \frac{e}{\left(1+\frac{1}{n}\right)^n} = \frac{e}{e}.\frac{e}{e} = \frac{e}{e} = 1$$

So, the given series diverges if $x = 1$. Hence, the given series converges if $x < 1$ and diverges if $x \geq 1$.

***Example 39:*** ***Test the series***

$$\mathbf{1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots (x > 0)} \textit{ for convergence.}$$

***Solution:*** Ignoring the first term of the series, we have

$$\Sigma\, u_n = \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots$$

$$u_n = \frac{x^{2n}}{2n} \text{ and } u_{n+1} = \frac{x^{2n+2}}{2n+2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{2n}}{2n}\cdot\frac{(2n+2)}{x^{2n+2}} = \frac{1}{x^2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{x^2}$$

By ratio test, $\Sigma\, u_n$ converges if $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1$ or $x < 1$

and $\Sigma\, u_n$ diverges if $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1 \Rightarrow x > 1$.

The test fails when $\frac{1}{x^2} = 1$ or $x = 1$.

For $x = 1$, we have $u_n = \frac{1}{2n}$.

Taking $v_n = \frac{1}{n}$ or $\Sigma\, v_n = \Sigma \frac{1}{n}$ diverges.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{1}{2}=\frac{1}{2}$$

which is finite and non-zero. So, $\Sigma u_n$ diverges for $x = 1$.

Hence, the given series converges if $x < 1$ and diverges if $x \geq 1$.

***Example 40:*** ***Test for convergence the series***

$$\mathbf{1+\frac{x}{2}+\frac{x^2}{5}+\ldots\ldots+\frac{x^n}{n^2+1}+\ldots\ldots (x>0)}.$$

***Solution:*** We have, $u_n=\dfrac{x^n}{n^2+1}$ and $u_{n+1}=\dfrac{x^{n+1}}{(n+1)^2+1}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{x^n}{n^2+1}\cdot\frac{(n+1)^2+1}{x^{n+1}}=\frac{1}{x}\lim_{n\to\infty}\frac{n^2+2n+2}{n^2+1}$$

$$=\frac{1}{x}\lim_{n\to\infty}\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}=\frac{1}{x}$$

By ratio test, the given series converges if $\dfrac{1}{x}>1$ or $x<1$ and diverges if $\dfrac{1}{x}<1$ or $x>1$.

The test fails if $x = 1$.

For $x = 1$, we have $u_n=\dfrac{1}{n^2+1}=\dfrac{1}{n^2}\left(\dfrac{1}{1+\frac{1}{n^2}}\right)$

Taking $v_n=\dfrac{1}{n^2}$, so $\Sigma\, v_n=\Sigma\dfrac{1}{n^2}$ converges.

Now $\lim\limits_{n\to\infty}\dfrac{u_n}{v_n}=\lim\limits_{n\to\infty}\left(\dfrac{1}{1+1/n^2}\right)=1$ which is finite and non zero and so by comparison test, the given series converges if $x = 1$.

Hence, the given series converges if $x \leq 1$ and diverges if $x > 1$.

***Example 41:*** ***Test for convergence the series with nth term***

***(i)*** $\dfrac{\sqrt{n}x^n}{\sqrt{n+1}}$ ***(ii)*** $\sqrt{\dfrac{n-1}{n^3+1}}\cdot x^n\ (x>0)$.

***Solution:*** (i) $u_n=\dfrac{\sqrt{n}}{\sqrt{n^2+1}}x^n$ and $u_{n+1}=\dfrac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}}x^{n+1}$

$$\lim_{n\to\infty}\frac{u_n}{n_{n+1}}=\frac{1}{x}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n^2+1}}\frac{\sqrt{n^2+2n+2}}{\sqrt{n+1}}=\frac{1}{x}\lim_{n\to\infty}\frac{\sqrt{1+\frac{2}{n}+\frac{2}{n^2}}}{\sqrt{1+\frac{1}{n^2}}\sqrt{1+\frac{1}{n}}}=\frac{1}{x}$$

By ratio test, the given series converges if $\frac{1}{x}>1$ or $x<1$ and diverges if $\frac{1}{x}<1$ or $x>1$.

The test fails if $x=1$.

If $x=1$, then we have $u_n=\frac{\sqrt{n}}{\sqrt{n^2+1}}=\frac{\sqrt{n}}{n}\left[\frac{1}{\sqrt{1+\frac{1}{n^2}}}\right]$

Taking $v_n=\frac{\sqrt{n}}{n}=\frac{1}{\sqrt{n}}$, then $\Sigma v_n=\Sigma\frac{1}{n^{1/2}}$ diverges.

$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\left(\frac{1}{\sqrt{1+\frac{1}{n^2}}}\right)=1$ which is finite and non-zero. So by comparison test, the given series diverges.

Hence, the given series converges if $x<1$ and diverges if $x\geq 1$.

(ii) $u_n=\frac{\sqrt{n-1}}{\sqrt{n^3+1}}x^n$ and $u_{n+1}=\sqrt{\frac{n}{(n+1)^3+1}}x^{n+1}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1}{x}\sqrt{\frac{n-1}{n^3+1}}\cdot\sqrt{\frac{(n+1)^3+1}{n}}$$

$$=\frac{1}{x}\lim_{n\to\infty}\sqrt{\frac{1-1/n}{1+1/n^3}}\cdot\sqrt{\frac{(1+1/n)^3+1/n^3}{1}}=\frac{1}{x}$$

By ratio test, the given series converges if $\frac{1}{x}>1$ or $x<1$ and diverges if $\frac{1}{x}<1$ or $x>1$.

The test fails if $\frac{1}{x}=1$ or $x=1$.

For $x=1$, we have $u_n=\sqrt{\frac{n-1}{n^3+1}}=\frac{\sqrt{n}}{\sqrt{n^3}}\left(\frac{1-\frac{1}{n}}{1+\frac{1}{n^3}}\right)^{1/2}$

Taking $v_n=\frac{\sqrt{n}}{\sqrt{n^3}}=\frac{1}{n}$, then $\Sigma v_n=\Sigma\frac{1}{n}$ diverges.

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1-\frac{1}{n}}{1+\frac{1}{n^3}}\right)^{1/2} = 1$, which is finite and non-zero. Therefore by comparison test, the series diverges. Hence, the given series converges for $x < 1$ and diverges for $x \geq 1$.

***Example 42:*** ***Show that*** $\frac{1}{1^p} + \frac{2}{2^p} + \frac{3}{3^p} + \ldots + \frac{n}{n^p} + \ldots\ldots$ ***converges if*** $p > 2$ ***and diverges if*** $p \leq 2$.

**[B.C.A. (Meerut) 2004]**

***Solution:*** The given series is $\sum_{n=1}^{\infty} \frac{n}{n^p}$ or $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ which converges if $p-1 > 1$ *i.e.*, $p > 2$ and diverges if $p-1 \leq 1$ *i.e.*, $p \leq 2$.

***Example 43:*** ***Test for convergence the series***

$$1 + \frac{x^2}{2^p} + \frac{x^4}{4^p} + \frac{x^6}{6^p} + \ldots\ldots .$$

**[B.C.A. (Meerut) 2002]**

***Solution:*** Ignoring the first terms of the given series,

the $n$th term is $u_n = \frac{x^{2n}}{(2n)^p}$ and $u_{n+1} = \frac{x^{2n+2}}{(2n+2)^p}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2} \lim_{n\to\infty} \left(\frac{2n+2}{2n}\right)^p = \frac{1}{x^2} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p = \frac{1}{x^2}(1)^p = \frac{1}{x^2}$$

By ratio test, the given series converges if $\frac{1}{x^2} > 1$ or $x^2 < 1 \Rightarrow |x| < 1$ and diverges if $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1$ or $|x| > 1$. The test fails if $\frac{1}{x^2} = 1$ or $x = 1$.

For $x = 1$ or $x^2 = 1$, we have $u_n = \frac{1}{(2n)^p} = \frac{1}{2^p n^p}$.

Let $v_n = \frac{1}{n^p}$, then $\Sigma\, v_n = \Sigma \frac{1}{n^p}$ is convergent if $p > 1$ and diverges if $p \leq 1$.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{2^p} = \frac{1}{2^p} \text{ which is finite and non-zero.}$$

By comparison test, the given series converges for $p > 1$ and diverges if $p \leq 1$ for $x = 1$.

Hence, the given series for $x = 1$ converges if $p > 1$ and diverges if $p \leq 1$.

**Example 44:** ***Test for convergence the series***

$$1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots..(p>0).$$

**[B.C.A. (Meerut) 2006, 05]**

**Solution:** We have $u_n=\frac{n^p}{n!}\left(\text{because } 1=\frac{1^p}{1!}\right)$ and $u_{n+1}=\frac{(n+1)^p}{(n+1)!}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{n^p}{n!}\cdot\frac{(n+1)!}{(n+1)^p}=\lim_{n\to\infty}\frac{n^p(n+1)n!}{n!\,n^p\left(1+\frac{1}{n}\right)^p}$$

$$=\lim_{n\to\infty}\frac{n+1}{\left(1+\frac{1}{n}\right)^p}=\infty>1$$

By ratio test $\Sigma\, u_n$ is convergent.

**Example 45:** ***Test for convergence the series***

$$1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\frac{30}{33}x^4+\ldots\ldots.$$

**[B.C.A. (Lucknow) 2006]**

**Solution:** Ignoring the first term, we have

$$u_n=\frac{2^{n+1}-2}{2^{n+1}+1}\,x^n \text{ and } u_{n+1}=\frac{2^{n+2}-2}{2^{n+2}+1}\,x^{n+1}$$

$$\frac{u_n}{u_{n+1}}=\frac{(2^{n+1}-2)}{(2^{n+1}+1)}\cdot\frac{(2^{n+2}+1)}{(2^{n+2}-2)}\cdot\frac{x^n}{x^{n+1}}=\frac{(2\,.2^n-2)(4\,.2^n+1)}{(2\,.2^n+1)(4\,.2^n-2)}\cdot\frac{1}{x}$$

$$=\left(\frac{2-\frac{2}{2^n}}{2+\frac{1}{2^n}}\right)\left(\frac{4+\frac{1}{2^n}}{4-\frac{2}{2^n}}\right)\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\left(\frac{2-0}{2+0}\right)\left(\frac{4+0}{4-0}\right)\frac{1}{x}=\frac{1}{x}$$

By ratio test, the given series converges if $\frac{1}{x}>1$ or $x<1$ and diverges if

$\frac{1}{x}<1$ or $x>1$. The test fails if $\frac{1}{x}=1$ or $x=1$.

For $x=1$, we have $$u_n=\frac{2^{n+1}-2}{2^{n+1}+1}=\frac{2\cdot2^n-2}{2\cdot2^n+1}=\frac{2-\frac{2}{2^n}}{2+\frac{1}{2^n}}$$

$$\lim_{n\to\infty} u_n = \frac{2-0}{2+0} \neq 0, \text{ so } \Sigma\, u_n \text{ diverges for } x = 1$$

Hence, the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

**NOTE:**

Cauchy's root test so more general than *D*' Alembert's ratio test.

This is so because:

1. $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n}$ exists $\Rightarrow \lim\limits_{n\to\infty} (u_n)^{1/n}$ exists and $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} (u_n)^{1/n}$.

   Therefore, whenever ratio test applicable, so is the root test.

2. $\lim\limits_{n\to\infty} (u_n)^{1/n}$ exists need not imply $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n}$ exists.

   *i.e.*, $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n}$ may not exist. So when the ratio test fails, the root test succeeds.

   Hence, the root test is more general than ratio test.

***Example 46:*** ***Show that Cauchy's root test establishes the convergence of the series $\Sigma u_n$, where***

$$u_n = \begin{cases} 2^{-n} & \textit{if } n \textit{ is odd} \\ 2^{-n+2} & \textit{if } n \textit{ is even} \end{cases}$$

**[B.C.A. (I.G.N.O.U.) 2012]**

***While D' Alembert's ratio test fails to do so.***

***Solution:*** When $n$ is odd $\quad \lim\limits_{n\to\infty} (u_n)^{1/n} = \lim\limits_{n\to\infty} (2^{-n})^{1/n}$

$$= \lim_{n\to\infty} 2^{-1} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} < 1$$

When $n$ is even, $\quad \lim\limits_{n\to\infty} (u_n)^{1/n} = \lim\limits_{n\to\infty} (2^{-n+2})^{1/n}$

$$= \lim_{n\to\infty} 2^{-1+2/n} = \lim_{n\to\infty} \left(\frac{1}{2}.2^{2/n}\right) = \frac{1}{2}.2^0 = \frac{1}{2} < 1$$

*i.e.*, $\quad \lim\limits_{n\to\infty} (u_n)^{1/n} = \dfrac{1}{2} < 1 \ \forall\, n$ even or odd

Therefore by Cauchy's root test, $\Sigma\, u_n$ is convergent.

Now, when $n$ is odd (so that $n+1$) is even

$$\frac{u_{n+1}}{u_n} = \frac{2^{-(n+1)+2}}{2^{-n}} = 2$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 2 > 1$$

When $n$ is even (so that $n+1$ is odd)

$$\frac{u_{n+1}}{u_n} = \frac{2^{-(n+1)}}{2^{-n+2}} = 2^{-3} = \frac{1}{8}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{8} < 1$$

Therefore, $D$' Alembert's ratio test is inconclusive.

***Example 47:*** ***Test of convergence the series*** $2 + \frac{3}{2}x + \frac{4}{3}x^2 + \frac{5}{4}x^3 + \ldots\ldots\infty$, ***where*** $x > 0$. **[B.C.A. (Meerut) 2005]**

***Solution:*** Let $\Sigma u_n = 2 + \frac{3}{2}x + \frac{4}{3}x^2 + \frac{5}{4}x^3 + \ldots\ldots\infty$

Here, $u_n = \frac{(n+2)}{(n+1)} x^n$

and $u_{n+1} = \frac{(n+3)}{(n+2)} x^{n+1}$

Applying ratio test, we get $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+2)^2}{(n+1)(n+3)} \frac{1}{x} = \frac{1}{x}$.

If $\frac{1}{x} > 1$ or $x < 1$ the series is convergent, if $\frac{1}{x} < 1$ or $x > 1$, then series is divergent, if $\frac{1}{x} = 1$ or $x = 1$, then test fails.

Put $x = 1$ in $u_n = \frac{n+2}{n+1}$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+2}{n+1} = \lim_{n\to\infty} \frac{n\left(1+\frac{2}{n}\right)}{n\left(1+\frac{1}{n}\right)} = 1 \neq 0$$

Therefore, $\Sigma u_n$ is divergent if $x = 1$.

Thus, if $x > 1$ convergent and divergent if $x \le 1$.

## Exercise 5.3

**Test for convergence the series:**

1. $\frac{1}{3}+\frac{1\cdot 2}{3\cdot 5}+\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+\ldots\ldots.$ **[B.C.A. (Meerut) 2010 (R)]**

2. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots\ldots.$

3. $\frac{2\,!}{3}+\frac{3\,!}{3^2}+\frac{4\,!}{3^3}+\frac{(n+1)\,!}{3^n}+\ldots.$

4. $1+\frac{3}{2\,!}+\frac{5}{3\,!}+\frac{7}{4\,!}+\ldots\ldots.$

5. $\sum_{n=1}^{\infty}\frac{n+2}{2^n+5}.$

6. $\frac{1}{2}+\frac{2\,!}{2^3}+\frac{3\,!}{2^5}+\ldots..$

7. $1+\frac{2\,!}{2^2}x+\frac{3\,!}{2^3}x^2+\ldots\ldots(x>0).$

8. $\sum_{n=1}^{\infty}\frac{5^n}{n^2+5}.$

9. $\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\frac{x^4}{17}+\ldots..$

10. $1+3x+5x^2+7x^3+\ldots\ldots(x>0).$

11. $\sum_{n=1}^{\infty}\frac{1}{x^n+x^{-n}}.$

12. $\sum_{n=1}^{\infty}\frac{a^n}{x^n+a^n}.$ **[ B.C.A. (Meerut) 2005]**

13. $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots..$

14. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\ldots\ldots+\frac{n^2-1}{n^2+1}x^n+\ldots\ldots$ **[B.C.A. (Avadh) 2008]**

15. $2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\ldots\ldots+\frac{(n+1)x^n}{n^3}+\ldots\ldots(x>0).$

16. $\frac{x^2}{2\sqrt{1}}+\frac{x^3}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\ldots\ldots(x>0).$ **[B.C.A. (Rohtak) 2010]**

17. $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\ldots..(x>0).$

18. $\Sigma\left(\frac{3n-1}{2^n}\right).$ **[B.C.A. (Kurukshetra) 2010]**

19. $\Sigma\frac{x^n}{a+\sqrt{n}}.$

**20.** Show that the series

$$1+\frac{\alpha+1}{\beta+1}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\ldots..$$

converges if $\beta>\alpha>0$ and diverges if $\alpha\geq\beta>0$, where $\alpha>0, \beta>0$.

**[B.C.A. (Kanpur) 2009]**

## *Answers 5.3*

| | |
|---|---|
| **1.** | Convergent. |
| **2.** | Convergent. |
| **3.** | Divergent. |
| **4.** | Divergent. |
| **5.** | Convergent. |
| **6.** | Divergent. |
| **7.** | Divergent. |
| **8.** | Divergent. |
| **9.** | Convergent for $x\leq 1$ and Divergent for $x>1$. |
| **10.** | Convergent for $x<1$ and Divergent for $x\geq 1$. |
| **11.** | Convergent if $x>1$ or $x<1$ and Divergent if $x=1$. |
| **12.** | Convergent if $x>a$ and Divergent if $x\leq a$. |
| **13.** | Convergent if $x\leq 1$ and Divergent if $x>1$. |
| **14.** | Convergent if $x<1$ and Divergent $x\geq 1$. |
| **15.** | Convergent if $x\leq 1$ and Divergent $x>1$. |
| **16.** | Convergent if $x\leq 1$ and Divergent $x>1$. |
| **17.** | Convergent if $x\leq 1$ and Divergent if $x>1$. |
| **18.** | Convergent. |
| **19.** | Convergent if $x<1$ and Divergent $x\geq 1$. |

## 5.3.11 Raabe's Test

**[B.C.A. (Meerut) 2005]**

If $\Sigma\, u_n$ is a series of positive terms and $\lim\limits_{n\to\infty}\left\{n\left(\dfrac{u_n}{u_{n+1}}-1\right)\right\}=l$, then the series is convergent if $l>1$ and divergent if $l<1$. The test fails if $l=1$.

**Proof:** **Case I:** If $l>1$, we choose a number $p$ so that $1<p<l$. Now, compare the given series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n=\Sigma\dfrac{1}{n^p}$. This series is convergent if $p>1$. Therefore, by comparison test, $\Sigma\, u_n$ will be convergent if from and after some particular terms

$$\frac{v_{n+1}}{v_n}>\frac{u_{n+1}}{u_n} \text{ or } \frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}$$

*i.e.*,
$$\frac{u_n}{u_{n+1}}>\frac{\dfrac{1}{n^p}}{\dfrac{1}{(n+1)^p}}=\left(\frac{n+1}{n}\right)^p=\left(1+\frac{1}{n}\right)^p$$

Now using binomial theorem, we have

$$\frac{u_n}{u_{n+1}}>1+p\frac{1}{n}+\frac{p(p-1)}{(2)!}\frac{1}{n^2}+\ldots..$$

$$\Rightarrow \quad \left(\frac{u_n}{u_{n+1}}-1\right)>p\frac{1}{n}+\frac{p(p-1)}{(2)!}\frac{1}{n^2}+\ldots..$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}}-1\right)>p+\frac{p(p-1)}{(2)!}\frac{1}{n}+\ldots..$$

Taking limit of both sides, as $n\to\infty$, we get

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}>p \text{ and we have } l>p>1, \text{ therefore}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=l>1$$

*i.e.*, $\Sigma\, u_n$ is convergent if $\lim\limits_{n\to\infty}\left\{n\left(\dfrac{u_n}{u_{n+1}}-1\right)\right\}>1$

**Case II:** If $l<1$, we choose $p$ such that $l<p<1$. Therefore, the auxiliary series $\Sigma\, v_n=\Sigma\dfrac{1}{n^p}$ is divergent.

Now by comparison test, $\Sigma\, u_n$ will be divergent if from and after some particular terms

$$\frac{v_n}{v_{n+1}} > \frac{u_n}{u_{n+1}}$$

*i.e.*, $$\frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} > \frac{u_n}{u_{n+1}} \text{ or } \left(\frac{n+1}{n}\right)^p > \frac{u_n}{u_{n+1}}$$

or $$\left(1+\frac{1}{n}\right)^p > \frac{u_n}{u_{n+1}}$$

Using binomial theorem, we get

$$1 + p\frac{1}{n} + \frac{p(p-1)}{(2)!}\frac{1}{n^2} + \ldots\ldots > \frac{u_n}{u_{n+1}}$$

$$p\frac{1}{n} + \frac{p(p-1)}{2!}\frac{1}{n^2} + \ldots\ldots > \frac{u_n}{u_{n+1}} - 1$$

$\Rightarrow$ $$p + \frac{p(p-1)}{2!}\frac{1}{n} + \ldots\ldots > n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

Taking limit as $n \to \infty$, we get

$$p > \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\}$$

But we have $l < p < 1$, therefore

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = l < 1$$

*i.e.*, $\Sigma\, u_n$ is divergent if $$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} < 1$$

**Case III:** If $\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = 1$, then test fails. For example, consider the two series $\Sigma\,\frac{1}{n}$ and $\Sigma\,\frac{1}{n(\log n)^2}$. The series $\Sigma\,\frac{1}{n}$ is divergent and the series $\Sigma\,\frac{1}{n(\log n)^2}$ is convergent but $\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = 1$ for both the series.

**Remark:** Raabe's Test is stronger than Ratio Test.

When Ratio test fails *i.e.*, $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1$, Raabe's test may be applied.

**Example 48:** ***Test for convergence the series***

$$\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6....2n} \cdot \frac{1}{n}.$$

**[B.C.A. (I.G.N.O.U.) 2007, 04]**

***Solution:*** We have

$$u_n = \frac{1 \cdot 3 \cdot 5.....(2n-1)}{2 \cdot 4 \cdot 6....2n} \cdot \frac{1}{n}$$

$$u_{n+1} = \frac{1 \cdot 3 \cdot 5....(2n-1)(2n+1)}{2 \cdot 4 \cdot 6....2n(2n+2)} \cdot \frac{1}{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(n+1)}{(2n+1)n} = \frac{\left(2+\frac{2}{n}\right)\left(1+\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)}$$

$\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} = 1$ means that ratio test fails.

Now, we apply Raabe's test. We have

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{(2n+2)(n+1)}{(2n+1)n} - 1\right] = \frac{3n+2}{2n+1} = \frac{3+\frac{2}{n}}{2+\frac{1}{n}}$$

$$\lim_{n \to \infty} \left\{ n\left(\frac{u_n}{u_{n+1}-1}\right) \right\} = \frac{3}{2} > 1$$

Hence, by Raabe's test, the given series converges.

**Example 49:** ***Test for convergence the series***

***(i)*** $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6.....2n}{1 \cdot 3 \cdot 5......(2n+1)}$ **[B.C.A. (Rohilkhand) 2009]**

***(ii)*** $\displaystyle\sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdot 9....3n}{7 \cdot 10 \cdot 13......(3n+4)} x^n, (x > 0)$ **[B.C.A. (Lucknow) 2007]**

***(iii)*** $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6.....(2n+2)}{3 \cdot 5 \cdot 7......(2n+3)} x^{n-1} (x > 0).$ **[B.C.A. (Rohilkhand) 2008]**

***Solution:*** (i) $u_n = \dfrac{2 \cdot 4 \cdot 6....2n}{1 \cdot 3 \cdot 5......(2n+1)}$ and

$$u_{n+1} = \frac{2 \cdot 4 \cdot 6 \ldots\ldots 2n(2n+2)}{1 \cdot 3 \cdot 5 \ldots\ldots (2n+1)(2n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+3}{2n+2} = \frac{2+\frac{3}{n}}{2+\frac{2}{n}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left( \frac{2+\frac{3}{n}}{2+\frac{2}{n}} \right) = \frac{2}{2} = 1$$

So ratio test fails.

Now, $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n+3}{2n+2} - 1\right) = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$$\lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} = \lim_{n\to\infty} \left( \frac{1}{2+\frac{1}{n}} \right) = \frac{1}{2} < 1$$

Therefore, by Raabe's test $\Sigma\, u_n$ diverges.

(ii) $$u_n = \frac{3 \cdot 6 \ldots\ldots 3n}{7 \cdot 10 \ldots\ldots (3n+4)} x^n \text{ and}$$

$$u_{n+1} = \frac{3 \cdot 6 \ldots\ldots (3n)(3n+3)}{7 \cdot 10 \ldots\ldots (3n+4)(3n+7)} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\left(\frac{3n+7}{3n+3}\right) = \frac{1}{x}\left( \frac{3+\frac{7}{n}}{3+\frac{3}{n}} \right)$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

By ratio test, $\Sigma\, u_n$ converges if $\frac{1}{x} > 1$ *i.e.*, $x < 1$ and diverges if $\frac{1}{x} < 1$ *i.e.*, $x > 1$. The ratio test fails if $x = 1$. We shall now apply Raabe's test.

For $x = 1$, $$\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$$

$\therefore$ $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{3n+7}{3n+3} - 1\right) = \frac{4n}{3n+3} = \frac{4}{3+\frac{3}{n}}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=\lim_{n\to\infty}\frac{4}{3+\frac{3}{n}}=\frac{4}{3}>1$$

By Raabe's test, $\Sigma\, u_n$ converges for $x=1$.

Hence, the given series converges for $x\le 1$ and diverges for $x>1$.

(iii)
$$u_n=\frac{2\cdot 4\ldots\ldots(2n+2)}{3\cdot 5\ldots\ldots(2n+3)}\,x^{n-1}$$

$$u_{n+1}=\frac{2\cdot 4\ldots..(2n+2)(2n+4)}{3\cdot 5\ldots\ldots(2n+3)(2n+5)}\,x^n$$

$$\frac{u_n}{u_{n+1}}=\frac{1}{x}\left(\frac{2n+5}{2n+4}\right)=\frac{1}{x}\left(\frac{2+\frac{5}{n}}{2+\frac{4}{n}}\right)$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x}\lim_{n\to\infty}\left(\frac{2+\frac{5}{n}}{2+\frac{4}{n}}\right)=\frac{1}{x}$$

By ratio test, $\Sigma\, u_n$ converges if $\frac{1}{x}>1$ *i.e.*, $x<1$ and diverges if $\frac{1}{x}<1$ *i.e.*, $x>1$. The ratio test fails if $x=1$.

For $x=1$, we have $\quad \dfrac{u_n}{u_{n+1}}=\dfrac{2n+5}{2n+4}$

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left(\frac{2n+5}{2n+4}-1\right)=\frac{n}{2n+4}=\frac{1}{2+\frac{4}{n}}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=\lim_{n\to\infty}\left(\frac{1}{2+\frac{4}{n}}\right)=\frac{1}{2}<1$$

By Raabe's test, $\Sigma\, u_n$ diverges for $x=1$.

Hence, the given series converges for $x<1$ diverges for $x\ge 1$.

***Example 50:* *Test for convergence the series***

$$\mathbf{1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\ldots\ldots(x>0)}.$$

***[Hint: Similar to example 49 (ii)].***

***Example 51:*** ***Examine the convergence of the following series:***

*(i)* $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n} \cdot \frac{x^{2n+1}}{2n+1}, \ (x > 0)$

**[B.C.A. (Rohilkhand) 2005]**

*(ii)* $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \ldots\ldots (4n-5)(4n-3)}{2 \cdot 4 \cdot 6 \ldots\ldots (4n-4)(4n-2)} \cdot \frac{x^{2n}}{4n}, \ (x > 0)$

**[B.C.A. (Kanpur) 2010]**

*(iii)* $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n} x^n, \ (x > 0)$

**[B.C.A. (Agra) 2006]**

*(iv)* $\sum_{n=1}^{\infty} \frac{2n!}{(n!)^2} x^n, \ (x > 0)$ *(v)* $\sum_{n=1}^{\infty} \frac{(n!)^2}{2n!} x^n$ .

***Solution:*** (i) We have , $u_n = \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n} \cdot \frac{x^{2n+1}}{2n+1}$

$$u_{n+1} = \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n(2n+2)} \cdot \frac{x^{2n+3}}{2n+3}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x^2} \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} = \frac{1}{x^2} \left\{ \frac{\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}{\left(2+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} \right\}$$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

By ratio test, $\Sigma\, u_n$ converges if $\frac{1}{x^2} > 1$ *i.e.,* $x^2 < 1$ or $x < 1$ (as $x > 0$ ) and diverges if $\frac{1}{x^2} < 1$ *i.e.,* $x^2 > 1$ or $x > 1$.

The test fails if $x = 1$.

For $x = 1$, we have $\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$

Now $$n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = n\left\{\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1\right\} = \frac{n(6n+5)}{(2n+1)^2}$$

$$\lim_{n \to \infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} = \lim_{n \to \infty} \frac{\left(6+\frac{5}{n}\right)}{\left(2+\frac{1}{n}\right)^2} = 3 > 1$$

By Raabe's test $\Sigma\, u_n$ converges for $x = 1$ .

Hence, the given series converges for $x \le 1$ and diverges for $x > 1$.

(ii) $$u_n = \frac{1\cdot 3\ldots\ldots(4n-3)}{2\cdot 4\ldots\ldots(4n-2)}\cdot\frac{x^{2n}}{4n}$$

$$u_{n+1} = \frac{1\cdot 3\ldots..(4n-3)(4n+1)}{2\cdot 4\ldots\ldots(4n-2)(4n+2)}\cdot\frac{x^{2n+2}}{(4n+4)}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x^2}\,\frac{(4n+2)(4n+4)}{4n(4n+1)} = \frac{1}{x^2}\,\frac{\left(4+\frac{2}{n}\right)\left(4+\frac{4}{n}\right)}{4\left(4+\frac{1}{n}\right)}$$

$\lim\limits_{n\to\infty}\dfrac{u_n}{u_{n+1}} = \dfrac{1}{x^2}$. By ratio test, $\Sigma\, u_n$ converges for $\dfrac{1}{x^2} > 1$ *i.e.*, $x^2 < 1$ or $x < 1$ (as $x > 0$) and diverges for $\dfrac{1}{x^2} < 1$ *i.e.*, $x^2 > 1$ or $x > 1$. The test fails for $x = 1$.

For $x = 1$, we have $$\frac{u_n}{u_{n+1}} = \frac{(4n+2)(4n+4)}{4n(4n+1)}$$

Now, $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\frac{(4n+2)(4n+4)}{4n(4n+1)}-1\right] = \frac{20n+8}{4(4n+1)}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} = \lim_{n\to\infty}\frac{\left(20+\frac{8}{n}\right)}{4\left(4+\frac{1}{n}\right)} = \frac{20}{16} > 1$$

By Raabe's test, $\Sigma\, u_n$ converges for $x = 1$.

Hence, the given series converges for $x \le 1$ and diverges for $x > 1$.

(iii) $$u_n = \frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)}{2\cdot 4\cdot 6..2n}\quad x^n$$

and $$u_{n+1} = \frac{1\cdot 3\cdot 5\ldots..(2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots.2n\ (2n+2)}\,x^{n+1}$$

$\therefore$ $$\frac{u_n}{u_{n+1}} = \frac{1}{x}\left(\frac{2n+2}{2n+1}\right) = \frac{1}{x}\left(\frac{2+\frac{2}{n}}{2+\frac{1}{n}}\right)$$

$\lim\limits_{n\to\infty}\dfrac{u_n}{u_{n+1}} = \dfrac{1}{x}$. By ratio test, $\Sigma\; u_n$ converges if $\dfrac{1}{x} > 1$ or $x < 1$ and diverges for $\dfrac{1}{x} < 1$ or $x > 1$. The ratio test fails if $x = 1$.

For $x = 1$, we have $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{2n+2}{2n+1}-1\right) = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} < 1$$

So by Raabe's test, $\Sigma\, u_n$ diverges for $x = 1$. Hence, $\Sigma\, u_n$ converges if $x < 1$ and diverges if $x \geq 1$.

(iv) $u_n = \dfrac{2n!}{(n!)^2} x^n$ and $u_{n+1} = \dfrac{(2n+2)!}{((n+1)!)^2} x^{n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x}\,\frac{2n!}{(n!)^2} \times \frac{((n+1)!)^2}{(2n+2)!} = \frac{1}{x}\,\frac{2n!}{(n!)^2} \times \frac{[(n+1)n!]^2}{(2n+2)(2n+1)2n!}$$

$$= \frac{1}{x}\,\frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{x}\,\frac{\left(1+\frac{1}{n}\right)}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{4x}$$

By ratio test, $\Sigma\, u_n$ converges for $\frac{1}{4x} > 1$ or $x < \frac{1}{4}$ and diverges for $\frac{1}{4x} < 1$ or $x > \frac{1}{4}$.

The ratio test fails for $x = \frac{1}{4}$.

For $x = \frac{1}{4}$, we have $n\left(\dfrac{u_n}{u_{n+1}}-1\right) = n\left[\dfrac{2(n+1)}{(2n+1)}-1\right]$

$$= \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} = \lim_{n\to\infty}\left(\frac{1}{2+\frac{1}{n}}\right) = \frac{1}{2} < 1$$

By Raabe's test $\Sigma\, u_n$ diverges for $x = \frac{1}{4}$.

Hence, the given series converges for $x < \frac{1}{4}$ and diverges for $x \geq \frac{1}{4}$.

(v) $u_n = \dfrac{(n!)^2}{(2n)!} x^n$ and $u_{n+1} = \dfrac{(n+1)!^2}{(2n+2)!} x^{n+1}$

$$\frac{u_n}{u_{n+1}} = \frac{(n!)^2}{(2n)!} \times \frac{(2n+2)!}{(n+1)!^2}\,\frac{1}{x} = \frac{(n!)^2}{2n!} \times \frac{(2n+1)(2n+2)2n!}{\{(n+1)n!\}^2\, x}$$

$$= \frac{(2n+1)(2n+2)}{(n+1)^2 x} = \frac{2(2n+1)(n+1)}{(n+1)^2 x} = \frac{2(2n+1)}{(n+1)x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2\left(2+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x} = \frac{4}{x}$$

By ratio test $\Sigma\, u_n$ converges if $\frac{4}{x} > 1$ *i.e.*, $x < 4$ and diverges if $\frac{4}{x} < 1$ *i.e.*, $x > 4$. The ratio test fails for $x = 4$.

For $x = 4$, we have

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left\{\frac{2n+1}{2(n+1)} - 1\right\} = \frac{-n}{2(n+1)} = -\frac{1}{2\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = -\frac{1}{2} < 1$$

By Raabe's test, $\Sigma\, u_n$ diverges. Hence, $\Sigma\, u_n$ convergent for $x < 4$ and divergent for $x \geq 4$.

***Example 52:*** ***Test for convergence the series***

$$\boldsymbol{x^2 + \frac{2^2}{3\cdot 4} x^4 + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6} x^6 + \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} x^8 + \ldots\ldots}$$

***Solution:*** Omitting the first term of the series because it will not effect the convergence or divergence of the series, we have

$$u_n = \frac{2^2\cdot 4^2\cdot 6^2 \ldots (2n)^2}{3\cdot 4\cdot 5\cdot 6 \ldots\ldots (2n+1)(2n+2)} x^{2n+2}$$

$$u_{n+1} = \frac{2^2\cdot 4^2\cdot 6^2 \ldots\ldots (2n)^2 (2n+2)^2}{3\cdot 4\cdot 5\cdot 6 \ldots\ldots (2n+1)(2n+2)(2n+3)(2n+4)} x^{2n+4}$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x^2} \frac{(2n+3)(2n+4)}{(2n+2)^2} = \frac{\left(2+\frac{3}{n}\right)\left(2+\frac{4}{n}\right)}{\left(2+\frac{2}{n}\right)^2} \frac{1}{x^2}$$

$\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$. By ratio test, $\Sigma\, u_n$ converges for $\frac{1}{x^2} > 1$ *i.e.*, $x^2 < 1$ and diverges for $\frac{1}{x^2} < 1$ *i.e.*, $x^2 > 1$ and the ratio test fails for $\frac{1}{x^2} = 1$ or $x^2 = 1$.

For $x^2 = 1$, we have

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left\{\frac{(2n+3)(2n+4)}{(2n+2)^2}-1\right\} = \frac{6n^2+8n}{4n^2+8n+4} = \frac{6+\frac{8}{n}}{4+\frac{8}{n}+\frac{4}{n^2}}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} = \frac{6}{4} > 1$$

By Raabe's test, $\Sigma\, u_n$ converges for $x^2 = 1$.

Hence, the given series converges for $x^2 \le 1$ and diverges for $x^2 > 1$.

***Example 53:*** ***Test the convergence of the series*** $\sum_{n=1}^{\infty} \frac{1}{1+\log n}$. **[B.C.A. (Rohilkhand) 2008]**

***Solution:*** $u_n = \frac{1}{1+\log n}$ and $u_{n+1} = \frac{1}{1+\log(n+1)}$

$$\frac{u_n}{u_{n+1}} = \frac{1+\log(n+1)}{1+\log n} = \frac{1+\log\left\{n\left(1+\frac{1}{n}\right)\right\}}{1+\log n}$$

$$= \frac{\log e+\log n+\log\left(1+\frac{1}{n}\right)}{\log e+\log n} = \frac{\log(en)+\log\left(1+\frac{1}{n}\right)}{\log(en)}$$

$$= 1+\frac{1}{\log(en)}\log\left(1+\frac{1}{n}\right) = 1+\frac{1}{\log(en)}\left\{\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\ldots\ldots\right\}$$

$$= 1+\frac{1}{n\log(en)}-\frac{1}{2n^2\log(en)}+\ldots\ldots$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1, \text{ so ratio test fails.}$$

Now, $$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left\{\frac{1}{n\log(en)}-\frac{1}{2n^2\log(en)}+\ldots\ldots\right\}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} = \lim_{n\to\infty}\left\{\frac{1}{\log(en)}-\frac{1}{2n\log(en)}+\ldots\ldots\right\}$$

$$= 0 < 1$$

Hence, by Raabe's test $\Sigma\, u_n$ is divergent.

**Example 54:** ***Test the convergence of the series***

*(i)* $\frac{1^2}{4^2}+\frac{1^2\cdot 5^2}{4^2\cdot 8^2}+\frac{1^2\cdot 5^2\cdot 9^2}{4^2\cdot 8^2\cdot 12^2}+\frac{1^2\cdot 5^2\cdot 9^2\cdot 13^2}{4^2\cdot 8^2\cdot 12^2\cdot 16^2}+\ldots..$ **[B.C.A. (Kanpur) 2006]**

*(ii)* $1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\frac{3\cdot 6\cdot 9\cdot 12}{7\cdot 10\cdot 13\cdot 16}x^4+\ldots..$

*(iii)* $1+a+\frac{a(a+1)}{1\cdot 2}+\frac{a(a+1)(a+2)}{1\cdot 2\cdot 3}+\ldots\ldots$ .

**Solution:** (i) We have $T_n$ of $1^2, 5^2, 9^2 \ldots..$ is $[1+(n-1)4]^2=(4n-3)^2$ and

$T_n$ of $4^2\cdot 8^2\cdot 12^2\ldots$ is $\{4+(n-1)4\}^2=(4n)^2$.

Therefore,
$$u_n=\frac{1^2\cdot 5^2\cdot 9^2\ldots..(4n-3)^2}{4^2\cdot 8^2\cdot 12^2\ldots.(4n)^2}$$

and
$$u_{n+1}=\frac{1^2\cdot 5^2\cdot 9^2\ldots\ldots(4n-3)^2(4n+1)^2}{4^2\cdot 8^2\cdot 12^2\ldots.(4n)^2(4n+4)^2}$$

Now
$$\frac{u_n}{u_{n+1}}=\frac{(4n+4)^2}{(4n+1)^2}=\frac{\left(4+\frac{4}{n}\right)^2}{\left(4+\frac{1}{n}\right)^2}=1, \text{ so ratio test fails.}$$

Now, applying Raabe's test. We have

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}=\lim_{n\to\infty}\left\{n\left(\frac{(4n+4)^2}{(4n+1)^2}-1\right)\right\}$$

$$=\lim_{n\to\infty}\left[\frac{n(24n+15)}{(4n+1)^2}\right]=\lim_{n\to\infty}\left[\frac{24+\frac{15}{n}}{\left(4+\frac{1}{n}\right)^2}\right]=\frac{24}{16}>1$$

Hence, by Raabe's test $\Sigma u_n$ converges.

(ii) Leaving the first terms of the series, we have

$$\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\ldots..$$

$T_n$ of $3, 6, 9\ldots$ is $3+(n-1)3=3n$

and $T_n$ of $7, 10, 13, \ldots..$ is $7+(n-1)3=3n+4$.

Therefore,
$$u_n=\frac{3\cdot 6\cdot 9\ldots.3n}{7\cdot 10\cdot 13\ldots..(3n+4)}x^n$$

and
$$u_{n+1}=\frac{3\cdot 6\cdot 9\ldots..3n(3n+3)}{7\cdot 10\cdot 13\ldots\ldots(3n+4)(3n+7)}x^{n+1}$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \left(\frac{3n+7}{3n+3}\right)\frac{1}{x} = \left(\frac{3+\frac{7}{n}}{3+\frac{3}{n}}\right)\frac{1}{x}$$

$\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{1}{x}$. By ratio test, $\Sigma\, u_n$ is convergent. For $\dfrac{1}{x} > 1$ or $x < 1$ and divergent for $\dfrac{1}{x} < 1$ or $x > 1$.

The ratio test fails for $\dfrac{1}{x} = 1$ or $x = 1$.

For $x = 1$, we have $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{3n+7}{3n+3}$

Now, applying Raabe's test

$$n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = n\left\{\frac{3n+7}{3n+3} - 1\right\} = \frac{4n}{3n+3} = \frac{4}{3+\frac{3}{n}}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \frac{4}{3} > 1, \text{ so by Raabe's test } \Sigma\, u_n \text{ is convergent for } x = 1.$$

Hence, the given series is convergent for $x \le 1$ and divergent for $x > 1$.

(iii) Leaving the first term, we have

$$\frac{a}{1} + \frac{a(a+1)}{1\cdot 2} + \frac{a(a+1)(a+2)}{1\cdot 2\cdot 3} + \ldots.$$

$$u_n = \frac{a(a+1)(a+2)\ldots..(a+n-1)}{1\cdot 2\cdot 3\ldots..n}$$

and
$$u_{n+1} = \frac{a(a+1)(a+2)\ldots..(a+n-1)(a+n)}{1\cdot 2\cdot 3\ldots..n(n+1)}$$

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{a+n} = \frac{1+\frac{1}{n}}{1+\frac{a}{n}}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1, \text{ so ratio test fails.}$$

Now, applying Raabe's test, we have

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{n+1}{a+n} - 1\right) = \frac{n(1-a)}{a+n} = \frac{1-a}{1+\frac{a}{n}}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = 1-a$$

Hence, by Raabe's test the given series is convergent if $1-a>1$ or $a<0$ and divergent if $1-a<1$ or $a>0$.

The test fails for $1-a=1$ or $a=0$.

When $a=0$, then $\Sigma\, u_n = 1+0+0......$ *i.e.*, $S_n = 1$.

$\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} 1 = 1$ which is a finite and unique number, so $\Sigma\, u_n$ is convergent.

Hence, the given series is convergent if $a \le 0$ and divergent if $a>0$.

## Exercise 5.4

**1.** Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5......(2n-1)}{2\cdot 4\cdot 6......2n}\cdot\frac{1}{2n+1}.$$

**[B.C.A. (Kurukshetra) 2008]**

**2.** Show that the series is divergent

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5......(2n-1)}{2\cdot 4\cdot 6......2n}\cdot\frac{x^{2n}}{2n}, (x>0).$$

**3.** Test the convergence of the series

$$\frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}\frac{x^2}{2} + \frac{1\cdot 4\cdot 5}{2\cdot 4\cdot 6}\frac{x^3}{3} + .....$$

**[B.C.A. (Bhopal) 2008]**

**4.** Discuss the convergence of the series

(i) $1 + \dfrac{1+\alpha}{1+\beta} + \dfrac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + .....$

(ii) $1 + \dfrac{\alpha}{\beta} + \dfrac{\alpha(\alpha+1)}{\beta(\beta+1)} + \dfrac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + ......(\alpha>0, \beta>0)$

(iii) $\dfrac{\alpha}{\beta} + \dfrac{1+\alpha}{(1+\beta)} + \dfrac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)} + ....$

**[B.C.A. (Meerut) 2009]**

(iv) $\dfrac{a}{b} + \dfrac{a(a+c)}{b(b+c)} + \dfrac{a(a+c)(a+2c)}{b(b+c)(b+2c)} + ......(a, b, c>0)$.

**[B.C.A. (Meerut) 2012, 07]**

**5.** Test for convergence the series

(i) $\dfrac{x}{1} + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1\cdot 3}{2\cdot 4}\dfrac{x^5}{5} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{x^7}{7} + ......(x>0)$

(ii) $1 + \dfrac{1}{2}\cdot\dfrac{1}{3} + \dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{1}{5} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{1}{7} + .....$ .

**[B.C.A. (Indore) 2009]**

**6.** Test for convergence the series

(i) $1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\ldots.$

(ii) $1+\frac{1}{2}\frac{x^2}{4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^4}{8}+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10}\frac{x^6}{12}+\ldots.$ **[B.C.A. (Lucknow) 2009]**

(iii) $1+x+\frac{1}{2}\frac{x^3}{2}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\ldots\ldots$

(iv) $x+\frac{1}{2}\frac{x^2}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^3}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^4}{7}+\ldots\ldots x>0$ .

**7.** Test the convergence of the series

(i) $\sum_{n=1}^{\infty}\frac{4\cdot 7\ldots\ldots(3n+1)}{1\cdot 2\ldots\ldots n}x^n$ **[B.C.A. (I.G.N.O.U.) 2009]**

(ii) $\sum_{n=1}^{\infty}\frac{n!x^n}{3\cdot 5\cdot 7\ldots\ldots(2n+1)}$

(iii) $\frac{a}{a+3}+\frac{a(a+2)}{(a+3)(a+5)}x+\frac{a(a+2)(a+4)}{(a+3)(a+5)(a+7)}x^2+\ldots\ldots$ .

## Answers 5.4

| | | |
|---|---|---|
| 1. | Convergent. | |
| 3. | Convergent if $x\le 1$ and Divergent if $x>1$. | |
| 4. | (i) | Convergent if $\beta>\alpha+1$. and Diverges if $\beta\le\alpha+1$. |
| | (ii) | Convergent if $\beta>\alpha+1$ and Diverges if $\beta\le\alpha+1$. |
| | (iii) | Convergent if $\beta>\alpha+1$ and Diverges if $\beta\le\alpha+1$. |
| | (iv) | Convergent if $b>a+c$ and Diverges if $b\le a+c$. |
| 5. | (i) | Convergent if $x^2\le 1$ and Diverges if $x^2>1$. |
| | (ii) | Convergent. |
| 6. | (i) | Convergent if $x<1$ and Divergent if $x\ge 1$. |
| | (ii) | Convergent if $x^2\le 1$ and Divergent if $x^2<1$. |
| | (iii) | Convergent if $x^2\le 1$ and Divergent if $x^2>1$. |
| | (iv) | Convergent if $x\le 1$ and Divergent if $x>1$. |
| 7. | (i) | Convergent if $x<\frac{1}{3}$ and Divergent if $x\ge\frac{1}{3}$. |
| | (ii) | Convergent if $x<2$ and Divergent if $x\ge 2$. |
| | (iii) | Convergent if $x\le 1$ and Divergent if $x>1$. |

## 5.3.12 Logarithmic Test

The series of positive terms $\Sigma u_n$ converges or diverges according as $\lim\limits_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) > 1$ or $< 1$.

**Proof:** If $\lim\limits_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = l$. Let us choose a number $p$ such that $l > p > 1$. Now compare the given series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n = \Sigma \frac{1}{n^p}$ which is convergent if $p > 1$ and divergent if $p \le 1$.

Now,
$$\frac{v_n}{v_{n+1}} = \frac{\frac{1}{n^p}}{\frac{1}{(n+1)^p}} = \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n}\right)^p = \left(1+\frac{1}{n}\right)^p$$

**Case I:** When $l > p > 1$, the series $\Sigma\, u_n$ will be convergent if from and after some term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} = \left(1+\frac{1}{n}\right)^p \quad \text{(By comparison test)}$$

or if
$$\log\left(\frac{u_n}{u_{n+1}}\right) > \log\left(1+\frac{1}{n}\right)^p = p\log\left(1+\frac{1}{n}\right)$$

or if
$$\log\frac{u_n}{u_{n+1}} > p\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\ldots\right] \quad \left[\because\ \log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}\ldots\ldots\right]$$

or if
$$n\log\frac{u_n}{u_{n+1}} > p\left[1-\frac{1}{2n}+\frac{1}{3n^2}\ldots\right]$$

or if
$$\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} > \lim_{n\to\infty}\left\{p\left(1-\frac{1}{2n}+\frac{1}{3n^2}\ldots\ldots\right)\right\}$$

or if
$$\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} > p$$

or if
$$\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} = l > p > 1$$

or if
$$\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} > 1$$

**Case II:** We have $\lim\limits_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = l$. Let us choose a number $p$ such that $l < p < 1$.

If $p < 1$ the $\Sigma\, v_n$ diverges.

In this case, $\Sigma\, u_n$ will also be divergent if from and after some particular term

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}} = \left(1+\frac{1}{n}\right)^p$$

or if $$\log\left(\frac{u_n}{u_{n+1}}\right) < \log\left(1+\frac{1}{n}\right)^p = p\log\left(1+\frac{1}{n}\right)$$

or if $$\log\left(\frac{u_n}{u_{n+1}}\right) < p\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\ldots.\right)$$

or if $$n\log\left(\frac{u_n}{u_{n+1}}\right) < p\left(1-\frac{1}{2n}+\frac{1}{3n^2}\ldots..\right)$$

or if $$\lim_{n\to\infty}\left\{n\log\left(\frac{u_n}{u_{n+1}}\right)\right\} < \lim_{n\to\infty}\left\{p\left(1-\frac{1}{2n}+\frac{1}{3n^2}\ldots.\right)\right\}$$

or if $$\lim_{n\to\infty}\left\{n\log\left(\frac{u_n}{u_{n+1}}\right)\right\} < p$$

or if $$\lim_{n\to\infty}\left\{n\log\left(\frac{u_n}{u_{n+1}}\right)\right\} = l < p < 1$$

or if $$\lim_{n\to\infty}\left\{n\log\left(\frac{u_n}{u_{n+1}}\right)\right\} < 1$$

Thus, $\Sigma u_n$ converges or diverges according as

$$\lim_{n\to\infty}\left\{n\log\left(\frac{u_n}{u_{n+1}}\right)\right\} > 1 \text{ or } < 1$$

**NOTE:**

1. The logarithmic test fails if, $\lim_{n\to\infty}\left\{n\log\left(\frac{u_n}{u_{n+1}}\right)\right\} = 1.$

2. Logarithmic test is applied after the failure of ratio test and generally, when in ratio test, $\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right)$ involved '$e$'.

***Example 55:*** ***Test the convergence or divergence of the following series***

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}\ldots.\ x>0.$$

**[B.C.A. (Purvanchal) 2009]**

***Solution:*** Let the given series be $\Sigma\, u_n$, then

$$u_n = \frac{n^{n-1}x^{n-1}}{n!} \text{ and } u_{n+1} = \frac{(n+1)^n x^n}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^{n-1}x^{n-1}}{n!}\cdot\frac{(n+1)!}{x^n(n+1)^n}$$

$$= \frac{n^{n-1}(n+1)n!}{n!\,n^n\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n}\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{ex}$$

By ratio test, $\Sigma\, u_n$ is convergent if $\frac{1}{ex} > 1 \Rightarrow ex < 1$ or $x < \frac{1}{e}$ and divergent if $\frac{1}{ex} < 1$ or $x > \frac{1}{e}$.

The test fails if $x = \frac{1}{e}$.

When $x = \frac{1}{e}$, we have $\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^n}.e$

$$\log\frac{u_n}{u_{n+1}} = \log\left(1+\frac{1}{n}\right) + \log e - n\log\left(1+\frac{1}{n}\right)$$

$$= \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\ldots.\right) + 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\ldots.\right)$$

or $$n\log\frac{u_n}{u_{n+1}} = \left(1 - \frac{1}{2n} + \frac{1}{3n^2}\ldots\right) + n - \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\ldots\right)n^2$$

$$= \left(\frac{3}{2} - \frac{5}{6n} + \ldots\right)$$

$\lim_{n\to\infty} n\,\log\frac{u_n}{u_{n+1}} = \frac{3}{2} > 1.$ By logarithmic test the given series is convergent if $x = \frac{1}{e}$.

Hence, the given series is convergent if $x \le \frac{1}{e}$ and divergent if $x < \frac{1}{e}$.

***Example 56:*** ***Test the convergence or divergence of the series***

$$\boldsymbol{x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots\ldots x > 0}\ .$$

**[B.C.A. (Meerut) 2007, 04]**

***Solution:*** Let the given series be $\Sigma\, u_n$, then

$$u_n = \frac{n^n x^n}{n!} \text{ and } u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n . x^n}{n!}\cdot\frac{(n+1)!}{(n+1)^{n+1} x^{n+1}} = \frac{1}{x}\ \frac{n^n(n+1)n!}{n!(n+1)(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}\frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} \frac{1}{x} = \frac{1}{ex}$$

By ratio test, $\Sigma\, u_n$ is convergent if $\frac{1}{ex} > 1$ or $x < \frac{1}{e}$ and divergent if $\frac{1}{ex} < 1$ or $x > \frac{1}{e}$.

The ratio test fails if $x = \frac{1}{e}$.

When $x = \frac{1}{e}$, we have $\frac{u_n}{u_{n+1}} = \frac{e}{\left(1+\frac{1}{n}\right)^n}$

$$\log \frac{u_n}{u_{n+1}} = \log e - n \log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \ldots\ldots\right)$$

$$= 1 - \left(1 - \frac{1}{2n} + \frac{1}{3n^2} \ldots\ldots\right) = \frac{1}{2n} - \frac{1}{3n^2} + \ldots\ldots$$

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\left(\frac{1}{2n} - \frac{1}{3n^2} + \ldots\ldots\right) = \frac{1}{2} < 1$$

By logarithmic test the given series is divergent.

Hence, $\Sigma\, u_n$ is convergent if $x < \frac{1}{e}$ and divergent if $x \geq \frac{1}{e}$.

***Example 57:*** ***Test for convergence the series***

$$\mathbf{1 + \frac{x}{2} + \frac{2!}{3^2} x^2 + \frac{3!}{4^3} x^3 + \ldots\ldots .}$$

**[B.C.A. (Meerut) 2008]**

***Solution:*** If the given series is $\Sigma\, u_n$, then

$$u_n = \frac{(n-1)!}{n^{n-1}} x^{n-1} \text{ and } u_{n+1} = \frac{n!\, x^n}{(n+1)^n}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n-1)!}{n^{n-1}} x^{n-1} \frac{(n+1)^n}{n!} \frac{1}{x^n}$$

$$= \frac{(n-1)!\, n^n \left(1+\frac{1}{n}\right)^n}{n^{n-1} n (n-1)!} \frac{1}{x} = \frac{\left(1+\frac{1}{n}\right)^n}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{e}{x}$$

By ratio test, $\Sigma\, u_n$ is converges if $\frac{e}{x} > 1$ or $x < e$ and diverges if $\frac{e}{x} < 1$ or $x > e$. The ratio test fails if $x = e$ .

Where $x = e$, we have $\quad \frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^n}{e}$

$$n \log\left(\frac{u_n}{u_{n+1}}\right) = n\{n \log\left(1+\frac{1}{n}\right) - \log e\}$$

$$= n\left\{n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \ldots\ldots\right) - 1\right\}$$

$$= n\left\{\left(1 - \frac{1}{2n^1} + \frac{1}{3n^2} \ldots\ldots\right) - 1\right\} = -\frac{1}{2} + \frac{1}{3n} \ldots\ldots$$

$$\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) = -\frac{1}{2} < 1$$

By logarithmic test $\Sigma\, u_n$ is divergent.

Hence, the given series is convergent if $x < e$ and divergent if $x \geq e$.

***Example 58:*** ***Test the convergence and divergence of the series***

$$\frac{a+x}{1} + \frac{(a+2x)^2}{2!} + \frac{(a+3x)^3}{3!} + \ldots. \quad .$$ **[B.C.A. (Kanpur) 2009]**

***Solution:*** If the given series be $\Sigma\, u_n$, then

$$u_n = \frac{(a+nx)^n}{n!} \text{ and } u_{n+1} = \frac{\{a+(n+1)x\}^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{(a+nx)^n}{n!} \times \frac{(n+1)!}{\{a+(n+1)x\}^{n+1}}$$

$$= \frac{(n+1)n!(nx)^n\left(1+\frac{a}{nx}\right)^n}{n!\{(n+1)x\}^{n+1}\left\{1+\frac{a}{(n+1)x}\right\}^{n+1}}$$

$$= \frac{(n+1)n^n x^n\left(1+\frac{a}{nx}\right)^n}{(n+1)^{n+1} x^{n+1}\left\{1+\frac{a}{(n+1)x}\right\}^{n+1}}$$

$$= \frac{n^n x^n\left(1+\frac{a}{nx}\right)^n}{(n+1)^n x^n . x\left\{1+\frac{a}{(n+1)x}\right\}^{n+1}}$$

$$= \frac{n^n\left(1+\frac{a}{nx}\right)^n}{n^n\left(1+\frac{1}{n}\right)^n\left\{1+\frac{a}{(n+1)x}\right\}^{n+1}}\frac{1}{x}$$

$$= \frac{\left[\left(1+\frac{a}{nx}\right)^{\frac{nx}{a}}\right]^{\frac{a}{x}}}{\left(1+\frac{1}{n}\right)^n\left[\left\{1+\frac{a}{(n+1)x}\right\}^{\frac{(n+1)x}{a}}\right]^{\frac{a}{x}}}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}\cdot\frac{1}{e}\cdot\frac{e^{a/x}}{e^{a/x}} = \frac{1}{ex}$$

By ratio test $\Sigma\, u_n$ is convergent if $\frac{1}{ex} > 1$ or $x < \frac{1}{e}$ and divergent if $\frac{1}{ex} < 1$ or $x > \frac{1}{e}$.

The ratio test fails if $x = \frac{1}{e}$ or $\frac{1}{ex} = 1.$

When $x = \frac{1}{e}$, we have
$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{ae}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{ae}{n+1}\right)^{n+1}}\times e$$

Taking log of both sides

$$\log\left(\frac{u_n}{u_{n+1}}\right) = n\log\left(1+\frac{ae}{n}\right)+\log e - n\log\left(1+\frac{1}{n}\right)-(n+1)\log\left(1+\frac{ae}{n+1}\right)$$

$$= n\left(\frac{ae}{n}-\frac{a^2e^2}{2n^2}+\ldots\ldots\right)+1-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\ldots.\right)$$

$$-(n+1)\left(\frac{ae}{n+1}-\frac{a^2e^2}{2(n+1)^2}+\frac{a^3e^3}{3(n+1)^3}\ldots\ldots\right)$$

$$= \left(ae-\frac{a^2e^2}{2n}+\frac{a^3e^3}{3n^2}\ldots\ldots\right)+1-\left(1-\frac{1}{2n}+\frac{1}{3n^2}\ldots..\right)$$

$$-\left(ae-\frac{a^2e^2}{2(n+1)}+\frac{a^3e^3}{3(n+1)^2}\ldots..\right)$$

$$= \frac{1}{2n}-\frac{e^2a^2}{2n}+\frac{e^2a^2}{2(n+1)}+\left(\frac{e^3a^3}{3}-\frac{1}{3}\right)\frac{1}{n^2}+\ldots.$$

$$n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{2}-\frac{e^2a^2}{2}+\frac{e^2a^2}{2\left(1+\frac{1}{n}\right)}+\frac{1}{n}\left(\frac{e^3a^3}{3}-\frac{1}{3}\right)$$

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = \frac{1}{2} - \frac{e^2a^2}{2} + \frac{e^2a^2}{2} = \frac{1}{2} < 1$$

By logarithmic test $\Sigma\, u_n$ is divergent if $x = \frac{1}{e}$.

Hence, the given series is convergent if $x < \frac{1}{e}$ and divergent if $x \geq \frac{1}{e}$.

## 5.3.13 De Morgan and Bertrand's Test

If $\Sigma\, u_n$ be the series of positive terms, such that $\lim\limits_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right\} \log n = l$, then the series

1. Converges if $l > 1$
2. Diverges if $l < 1$.

**Proof: Case I:** If $l > 1$, then we can choose a number $p$ so that $l > p > 1$ and compare the given series $\Sigma\, u_n$ with the auxiliary $\Sigma \frac{1}{n(\log n)^p}$ which is convergent because $p > 1$.

Now, $\Sigma\, u_n$ will be convergent if from and after some particular terms

$$\frac{v_{n+1}}{v_n} > \frac{u_{n+1}}{u_n} \text{ or } \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

where
$$\Sigma\, v_n = \Sigma \frac{1}{n(\log n)^p} \text{ and } n > 1$$

We have
$$\left(\frac{u_n}{u_{n+1}}\right) > \left(\frac{n+1}{n}\right)\left\{\frac{\log(n+1)}{\log n}\right\}^p \qquad ...(1)$$

For all sufficiently large values of $n$.

Now, for a large value of $n$, consider

$$\left(\frac{n+1}{n}\right)\left\{\frac{\log(n+1)}{\log n}\right\}^p = \left(1+\frac{1}{n}\right)\left\{\frac{\log n(1+1/n)}{\log n}\right\}^p$$

$$= \left(1+\frac{1}{n}\right)\left\{\frac{\log n + \log(1+1/n)}{\log n}\right\}^p$$

$$= \left(1+\frac{1}{n}\right)\left\{1+\frac{1}{\log n}\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right)\right\}^p$$

$$\left(\because \log(1+1/n) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right)$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n\log n}-\frac{1}{2n^2\log n}+\frac{1}{3n^3\log n}-\ldots\ldots\right)^p$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{p}{n\log n}+\ldots\ldots\right)$$

$$= 1+\frac{1}{n}+\frac{p}{n\log n}+\ldots\ldots+\text{terms containing higher powers of}$$

$n$ and $\log n$ in the denominator .

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n > p$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = l > p > 1$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n > 1$$

Hence, the series $\Sigma\, u_n$ is convergent if

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n > 1$$

**Case II:** If $l < 1$, then we can choose a number $p$ such that $l < p < 1$.

If $p < 1$, then the auxiliary series $\Sigma\, v_n = \Sigma \dfrac{1}{n(\log n)^p}$ is divergent. By comparison test $\Sigma\, u_n$ is divergent if $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$ .

or $\dfrac{u_n}{u_{n+1}} < 1+\dfrac{1}{n}+\dfrac{p}{n\log n}+\ldots\ldots$ terms containing higher powers of $n$ in the denominator.

$\therefore$
$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n < p < 1$$

Hence, the infinite series $\Sigma\, u_n$ is divergent, if

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n < 1$$

**NOTE:**

De Morgan and Bertrand's test is applied when Raabe's test fails.

## 5.3.14 Higher Logarithmic Test or An Alternative to Bertrand's Test

If $\Sigma u_n$ be the series of positive terms such that $\lim_{n\to\infty}\left\{\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n\right\}=l$, then the series

1. Converges if $l>1$
2. Diverges if $l<1$.

**Proof:** **Case I:** If $l>1$, then we choose $p$ such that $l>p>1$. Now compare the given series $\Sigma u_n$ with the auxiliary series $\Sigma\, v_n = \Sigma\,\frac{1}{n(\log n)^p}$ which is convergent when $p>1$.

Therefore, by comparison test, $\Sigma\, u_n$ will be convergent if from and after some particular terms.

$$\frac{v_{n+1}}{v_n}>\frac{u_{n+1}}{u_n} \text{ or } \frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}$$

Now proceding as in 5.3.13, we have

$$\frac{u_n}{u_{n+1}}>1+\frac{1}{n}+\frac{p}{n\log n}$$

or
$$\log\left(\frac{u_n}{u_{n+1}}\right)>\log\left\{1+\left(\frac{1}{n}+\frac{p}{n\log n}+.....\right)\right\}$$

or
$$n\log\left(\frac{u_n}{u_{n+1}}\right)>n\left\{\left(\frac{1}{n}+\frac{p}{n\log n}+....\right)-\frac{1}{2}\left(\frac{1}{n}+\frac{p}{n\log n}+.....\right)^2+....+....\right\}$$

or
$$n\log\frac{u_n}{u_{n+1}}>1+\frac{p}{\log n}-\frac{1}{2n}+....$$

or
$$\left(n\log\frac{u_n}{u_{n+1}}-1\right)>\frac{p}{\log n}-\frac{1}{2n}+......+\text{ terms of higher power of }\frac{1}{n}$$

or
$$\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n>p-\frac{\log n}{2n}+......\text{ terms of higher powers of }\frac{1}{n}$$

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n>p \text{ because } \lim_{n\to\infty}\frac{\log n}{n}=0$$

We have taken $p>1$, therefore

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log\ n=l>p>1$$

Hence, $\Sigma\, u_n$ is convergent if

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n > 1$$

**Case II:** If $l < 1$, then choose a number $p$ such that $l < p < 1$. Thus, the auxiliary series $\Sigma\, v_n$ is divergent because $p < 1$. By comparison test the given series $\Sigma\, u_n$ will be divergent if from and after some particular terms

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

*i.e.,*
$$\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n < p-\frac{1}{2}\frac{\log n}{n}+\ldots\text{ terms of higher power of }\frac{1}{n}$$

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{n_{n+1}}-1\right)\log n = l < p < 1$$

Hence, $\Sigma\, u_n$ is divergent if

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n = l < 1$$

**NOTE:**

Higher logarithmic test is applied when logarithmic test fails.

***Example 59:*** ***Test the convergence of the following series***

$$\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}x+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+\ldots\ .$$ **[B.C.A. (Kanpur) 2008]**

***Solution:*** We have $n$th term of the given series is

$$u_n = \frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2}{2^2.4^2.6^2\ldots\ldots(2n)^2}\,x^{n-1}$$

$$u_{n+1} = \frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2(2n+1)^2}{2^2.4^2.6^2\ldots\ldots(2n)^2(2n+2)^2}\,x^{n}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}\cdot\frac{1}{x} = \frac{4n^2+8n+4}{4n^2+4n+1}\cdot\frac{1}{x} = \frac{4+\frac{8}{n}+\frac{4}{n^2}}{4+\frac{4}{n}+\frac{1}{n^2}}\cdot\frac{1}{x}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}$$

By ratio test $\Sigma\, u_n$ is convergent if $\frac{1}{x} > 1$ or $x < 1$, divergent if $\frac{1}{x} < 1$ or $x > 1$ and the test fails if $\frac{1}{x} = 1$ or $x = 1$. If $\frac{1}{x} = 1$ or $x = 1$, then

$$\frac{u_n}{u_{n+1}} = \frac{4n^2 + 8n + 4}{4n^2 + 4n + 1}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{4n^2 + 8n + 4}{4n^2 + 4n + 1} - 1\right) = \frac{4n^2 + 3n}{4n^2 + 4n + 1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{4 + \frac{3}{n}}{4 + \frac{4}{n} + \frac{1}{n^2}} = \frac{4}{4} = 1$$

Hence, Raabe's test also fails.

Now $$\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} = \frac{4n^2 + 3n}{4n^2 + 4n + 1} - 1 = \frac{-n-1}{4n^2 + 4n + 1}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n = \lim_{n\to\infty}\left(\frac{-1-\frac{1}{n}}{4 + \frac{4}{n} + \frac{1}{n^2}}\right)\frac{\log n}{n}$$

$$= \lim_{n\to\infty}\left(\frac{-1-\frac{1}{n}}{4 + \frac{4}{n} + \frac{1}{n^2}}\right)\lim_{n\to\infty}\frac{\log n}{n}$$

$$= -\frac{1}{4}\times 0 = 0 < 1 \qquad \left(\text{because } \lim_{n\to\infty}\frac{\log n}{n} = 0\right)$$

Hence be De Morgan and Bertrand's test, the series $\Sigma\, u_n$ is divergent.

Therefore, the given series is convergent when $x < 1$ and divergent when $x \geq 1$.

***Example 60:*** ***Test the convergence of the following series***

$$\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots .$$

***Solution:*** The $n$th term of the given series is

$$u_n = \frac{a(a+1)(a+2)\ldots..(a+n-1)}{b(b+1)(b+2)\ldots\ldots.(b+n-1)}$$

$$u_{n+1} = \frac{a(a+1)(a+2)\ldots..(a+n-1)(a+n)}{b(b+1)(b+2)\ldots\ldots.(b+n-1)(b+n)}$$

Now, $$\frac{u_n}{u_{n+1}} = \frac{b+n}{a+n} = \frac{\frac{b}{n} + 1}{\frac{a}{n} + 1}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{b}{n}+1}{\frac{a}{n}+1} = 1$$

Hence, the ratio test fails.

Now, applying Raabe's test

$$\frac{u_n}{u_{n+1}} - 1 = \frac{b+n}{a+n} - 1 = \frac{b-a}{a+n}$$

$$\lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \lim_{n\to\infty} \frac{(b-1)n}{a+n} = \lim_{n\to\infty} \frac{b-a}{\frac{a}{n}+1} = b-a$$

By Raabe's test, $\Sigma u_n$ is convergent if $b-a>1$ or $b>a+1$, divergent if $b-a<1$ or $b<a+1$ and the test fails if $b-a=1$ or $b=a+1$.

If $b=a+1$, then

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \left(\frac{b-a}{a+n}\right)n = \left(\frac{a+1-a}{a+n}\right)n = \frac{n}{a+n}$$

Now, $$\left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} = \frac{n}{a+n} - 1 = \frac{n-a-n}{a+n} = \frac{-a}{a+n}$$

$$\lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n$$

$$= \lim_{n\to\infty} \left(\frac{-a}{a+n}\right) \log n$$

$$= \lim_{n\to\infty} \left(\frac{-a}{\frac{a}{n}+1}\right) \lim_{n\to\infty} \frac{\log n}{n} = -a \times 0 = 0 < 1 \quad \left[\because \lim_{n\to\infty} \frac{\log n}{n} = 0\right]$$

By De Morgan and Bertrand's test the given series $\Sigma u_n$ is divergent, when $b=a+1$. Hence, the given series is convergent if $b>a+1$ and divergent if $b \le a+1$.

***Example 61:*** ***Test the convergent of the following series***

$$1 + \frac{\alpha.\beta}{1.\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)} x^3 + \ldots..$$

***Solution:*** On leaving the first term of the series the $n$th term is

$$u_n = \frac{\alpha(\alpha+1)\ldots..(\alpha+n-1)\,\beta(\beta+1)\ldots\ldots(\beta+n-1)}{1.2.3\ldots\ldots.n\,\gamma(\gamma+1)\ldots.\,(\gamma+n-1)} x^n$$

$$u_{n+1} = \frac{\alpha(\alpha+1)\ldots.(\alpha+n-1)(\alpha+n)\ \beta\ (\beta+1)\ldots.(\beta+n-1)(\beta+n)}{1.2.3\ldots..n(n+1)\ \gamma(\gamma+1)\ldots.(\gamma+n-1)(\gamma+n)}\, x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\cdot\frac{1}{x}$$

$$= \frac{n^2+1(1+\gamma)n+\gamma}{n^2+(\alpha+\beta)n+\alpha\beta}\cdot\frac{1}{x} = \frac{1+\left(\frac{1+\gamma}{n}\right)+\frac{\gamma}{n^2}}{1+\frac{(\alpha+\beta)}{n}+\frac{\alpha\beta}{n^2}}\,\frac{1}{x} = \frac{1}{x}$$

By ratio test, $\Sigma\, u_n$ is convergent if $\frac{1}{x} > 1$ or $x < 1$, divergent if $\frac{1}{x} < 1$ or $x > 1$ and the test fails if $\frac{1}{x} = 1$ or $x = 1$.

If $\quad \frac{1}{x} = 1$ or $x = 1$, then $\frac{u_n}{u_{n+1}} = \frac{n^2+n(1+\gamma)+\gamma}{n^2+n(\alpha+\beta)n+\alpha\beta}$

$$\Rightarrow \quad \left(\frac{u_n}{u_{n+1}}-1\right) = \frac{n^2+n(1+\gamma)+\gamma}{n^2+n(\alpha+\beta)+\alpha\beta}-1 = \frac{n(\gamma-\alpha-\beta+1)+\gamma-\alpha\beta}{n^2+n(\alpha+\beta)+\alpha\beta}$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{n^2(\gamma-\alpha-\beta+1)+n\gamma-n\alpha\beta}{n^2+n(\alpha+\beta)+\alpha\beta}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\frac{(\gamma-\alpha-\beta+1)+\left(\frac{\gamma-\alpha\beta}{n}\right)}{1+\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}} = \gamma-\alpha-\beta+1$$

By Raabe's test, the series $\Sigma u_n$ is convergent if $\gamma-\alpha-\beta+1>1$ or $\gamma>\alpha+\beta$, divergent if $\gamma-\beta-\alpha+1<1$ or $\gamma<\alpha+\beta$ and the test fails if $\gamma=\alpha+\beta$. If $\gamma=\alpha+\beta$, then

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{n^2(1+\alpha+\beta-\alpha-\beta)+n(\alpha+\beta-\alpha\beta)}{n^2+n(\alpha+\beta)+\alpha\beta}$$

$$\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\} = \frac{n^2+n\ (\alpha+\beta-\alpha\beta)}{n^2+n(\alpha+\beta)+\alpha\beta}-1 = \frac{-n\alpha\beta-\alpha\beta}{n^2+n(\alpha+\beta)+\alpha\beta}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n = \lim_{n\to\infty}\left[\frac{-\alpha\beta-\frac{\alpha\beta}{n}}{1+\frac{\alpha+\beta}{n}+\frac{\alpha\beta}{n^2}}\right]\lim_{n\to\infty}\frac{\log n}{n}$$

$$= -\alpha\beta\times 0 = 0 < 1$$

By De Morgan and Bertrand's test, the series $\Sigma u_n$ is divergent if $\gamma=\alpha+\beta$ and $x=1$.

Hence, $\Sigma u_n$ is convergent, if $x<1$ or $x=1$ and $\gamma>\alpha+\beta$ also divergent if $x>1$ or $x=1$ and $\gamma\le\alpha+\beta$.

**Example 62:** ***Test the convergence of the following series***

$$x^1 + x^{1+1/2} + x^{1+1/2+1/3} + x^{1+1/2+1/3+1/4} + \ldots\ldots .$$

**Solution:** The $n$th term of the given series is

$$u_n = x^{1+1/2+1/3+\ldots\ldots+1/n}$$

$$u_{n+1} = x^{1+1/2+1/3+\ldots\ldots+1/n+1/(n+1)}$$

Now
$$\frac{u_n}{u_{n+1}} = \frac{x^{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right)}}{x^{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\frac{1}{n+1}\right)}} = \frac{1}{x^{(1/(n+1))}}$$

$\therefore$
$$\lim_{n\to\infty}\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\frac{1}{x^{(1/n+1)}} = \frac{1}{x^0} = 1$$

Thus, the ratio test fails.

$$\log\frac{u_n}{u_{n+1}} = \log\frac{1}{x^{(1/n+1)}} = \log 1 - \log x^{(1/n+1)} = -\left(\frac{1}{n+1}\right)\log x$$

$$= \frac{1}{n+1}\log\left(\frac{1}{x}\right)$$

$$n\log\frac{u_n}{u_{n+1}} = \frac{n}{n+1}\log\frac{1}{x} = \frac{1}{1+\frac{1}{n}}\log\frac{1}{x}$$

$\therefore$
$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}\log\frac{1}{x} = \log\frac{1}{x}$$

By logarithmic test, the given series $\Sigma\, u_n$ is convergent if

$$\log\frac{1}{x} > 1 \Rightarrow \log\frac{1}{x} > \log e \Rightarrow \frac{1}{x} > e$$

or
$$x < \frac{1}{e}$$

The given series is divergent if $\log\frac{1}{x} < 1 \Rightarrow \log\frac{1}{x} < \log e$

$\Rightarrow$
$$\frac{1}{x} < e \quad\text{or}\quad x > \frac{1}{e}$$

The test fails if
$$\log\frac{1}{x} = 1 \Rightarrow \log\frac{1}{x} = \log e \Rightarrow \frac{1}{x} = e \quad\text{or}\quad x = \frac{1}{e}$$

where $x = \frac{1}{e}$, we have

$$n \log \frac{u_n}{u_{n+1}} = \frac{n}{n+1} \log e = \frac{n}{n+1}$$

$$n \log \frac{u_n}{u_{n+1}} - 1 = \frac{n}{n+1} - 1 = \frac{-1}{n+1}$$

$$\lim_{n \to \infty} \left\{ n \log \frac{u_n}{u_{n+1}} - 1 \right\} \log n = \lim_{n \to \infty} \left( \frac{-1}{n+1} \right) \log n$$

$$= \lim_{n \to \infty} \left( \frac{-1}{1 + \frac{1}{n}} \right) \times \lim_{n \to \infty} \frac{\log n}{n} = -1 \times 0 = 0 < 1$$

By higher logarithmic test the given series is divergent for $x = \frac{1}{e}$.

Hence, the given series is convergent if $x < \frac{1}{e}$ and divergent if $x \geq \frac{1}{e}$.

## Exercise 5.5

**Test the convergent of the following series:**

1. $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \ldots .$

2. $1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \ldots .$ **[B.C.A. (Meerut) 2003, 00]**

3. $\frac{(1+a)(1+b)}{1.2.3} x + \frac{(2+a)(2+b)}{2.3.4} x^2 + \frac{(3+a)(3+b)}{3.4.5} x^3 + \ldots .$

4. $1 + \frac{a(1-a)}{1^2} + \frac{(1+a)a(1-a)(2-a)}{1^2.2^2} + \frac{(2+a)(1+a)a(1-a)(2-a)(3-a)}{1^2.2^2.3^2} + \ldots .$

5. $1 + \frac{\alpha}{1.\beta} x + \frac{\alpha(\alpha+1)^2}{1.2\beta(\beta+1)} x^2 + \frac{\alpha(\alpha+1)^2(\alpha+2)^2}{1.2.3.\beta(\beta+1)(\beta+2)} x^3 + \ldots .$ **[B.C.A. (Meerut) 2010]**

6. $(1)^p + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p = \left(\frac{1.3.5}{2.4.6}\right)^p + \ldots .$

7. Test the convergence of the series

   $x^2(\log 2)^q + x^3(\log 3)^q + x^4(\log 4)^q + \ldots .$ **[B.C.A. (Meerut) 2003]**

## Answers 5.5

| | |
|---|---|
| 1. | Divergent. |
| 2. | Divergent. |
| 3. | Convergent if $x < 1$; divergent if $\geq 1$. |
| 4. | Divergent. |
| 5. | Convergent if $x < 1$; if $x = 1$, then convergent if $\beta > 2\alpha$ and divergent if $\beta \leq 2\alpha$. |
| 6. | Convergent if $p > 2$ and divergent if $p \geq 2$. |
| 7. | Convergent if $x < 1$ and divergent $x \geq 1$. |

❍❍❍

Chapter 6

# Vector Differentiation

## 6.1 Vector Point Functions

The values of vector valued functions are vectors of the form

$$\mathbf{F(p)} = F_1(p)\ \hat{\mathbf{i}} + \mathbf{F}_2(p)\ \hat{\mathbf{j}} + \mathbf{F}_3\ (p)\ \hat{\mathbf{k}} \qquad ...(1)$$

The values depend on the point $P$ is space. A vector valued function defined a **vector field** in the region or on that surface or curve. This function may also depend on time $t$ or any other parameter.

1. Equation in cartesian co-ordinates $(x,y,z)$ can be written as

$$F(x, y,z) = F_1\ (x, y,z)\ \hat{\mathbf{i}} + F_2(x, y,z)\ \hat{\mathbf{j}} + F_3(x, y,z)\ \hat{\mathbf{k}}$$

***Illustration:***

Let $F(t) = \text{a}\cos t\ \hat{\mathbf{i}} + \text{b}\ \sin t\ \hat{\mathbf{j}} + t^2\ \hat{\mathbf{k}}$ is a vector function of the scalar variable $t$, where $F_1(t) = \text{a}\cos t,\ F_2\ (t) = \text{b}\sin t,\ F_3\ (t) = t^2$. This $\mathbf{F}$ is a vector function and $\mathbf{F}(t)$ is a vector quantity. $a\cos t$, $b\sin t$, $t$ are called components of the vector $\mathbf{F}(t)$ along the co-ordinate axes and $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be unit vectors along $x$, $y$ and $z$ axis.

## 6.2 Differentiation Coefficient (Derivative) of a Vector Function with Respect to Scalar

Let $\mathbf{r} = \mathbf{f(t)}$ be continuous and single valued function of a scalar variable $t$.Then, a small increment $\delta t$ in $t$ produces an increment $\delta \mathbf{r}$ in $\mathbf{r}$. Thus,

$$\mathbf{r} + \delta\mathbf{r} = f(t + \delta t)$$

$$\because \quad \delta r = (\mathrm{r} + \delta\mathbf{r}) - \mathbf{r}$$

$$\delta r = f(t + \delta t) - f(t)$$

Divide by $\delta t$ on both sides, we get

$$\frac{\delta r}{\delta t} = \frac{f(t+\delta t) - f(t)}{\delta t}$$

Taking limit $\delta t \to 0$ on both sides, we find

$$\lim_{\delta t \to 0} \left(\frac{\delta r}{\delta t}\right) = \lim_{\delta t \to 0} \left\{\frac{(\mathbf{r} + \delta r) - \mathbf{r}}{\delta t}\right\}$$

$$= \lim_{\delta t \to 0} \left\{\frac{f(t + \delta\, t) - f(t)}{\delta t}\right\}$$

When it exists, is called **derivative** or **differential coefficient** of the vector function $\mathbf{r}$ with respect to the scalar $t$ and is denoted by

$$\frac{dr}{dt}, \frac{d\vec{\mathrm{r}}}{dt}$$

$$\therefore \quad \frac{dr}{dt} = \lim_{\delta t \to 0} \frac{f(t + \delta t) - f(t)}{\delta t}$$

The vector $\mathbf{r}$ possessing a differential coefficient is called **derivable** or **differentiable**. The derivative $\frac{dr}{dt}$ of vector $\mathbf{r}$ is also a vector quantity.

## 6.3 Rules for Differentiation

Let $\mathbf{f(t)}$, $\mathbf{g(t)}$ and $\mathbf{h(t)}$ be three vector functions of scalar $t$. Then,

1. $\frac{d}{dt}[\mathbf{f}(\mathbf{t}) \pm \mathbf{g}(\mathbf{t})] = \frac{d\mathbf{f}(t)}{dt} \pm \frac{d\mathbf{g}(t)}{dt}$

2. $\frac{d}{dt}[\mathbf{c}\ \mathbf{f}(t)] = c\frac{df(t)}{dt}$, where c is scalar constant.

3. $\frac{d}{dt}[\mathbf{f(t)}.\mathbf{g(t)}] = \frac{d}{dt}[\mathbf{f}(t)].\mathbf{g}\,(\mathrm{t}) + \mathrm{f}(\mathrm{t}).\frac{d\,\mathbf{g(t)}}{dt}$

4. $\frac{d}{dt}[\mathbf{f(t)} \times \mathbf{g(t)}] = \frac{d}{dt}[\mathbf{f(t)}] \times \mathbf{g(t)} + \mathbf{f(t)} \times \frac{d}{dt}[\mathbf{g(t)}]$

5. $\frac{d}{dt}[\phi(t)\,\mathbf{f(t)}] = \mathbf{f(t)}\frac{d\,\phi(t)}{dt} + \phi\,(t)\ \frac{d\,\mathbf{f(t)}}{dt}$, where $\phi(t)$ is a scalar function of $t$.

6. $\frac{\boldsymbol{d}}{\boldsymbol{dt}}[\mathbf{f(t)g(t)\,h(t)}] = \frac{df(t)}{dt}\mathbf{g(t)\,h(t)} + \mathbf{f(t)}\frac{d}{dt}[\mathrm{g(t)}]\ \mathbf{h(t)} + \mathbf{f(t)\ g(t)}\frac{d}{dt}[\mathbf{h\ (t)}]$

7. $\frac{d}{dt}\left[\mathrm{f(t)} \times \mathrm{g(t)} \times \mathrm{h(t)}\right] = \frac{df(t)}{dt} \times \mathbf{g(t)} \times \mathbf{h(t)} + \mathbf{f(t)} \times \frac{d\mathbf{g(t)}}{dt} \times \mathbf{h(t)} + \mathbf{f(t)} \times \mathbf{g(t)} \times \frac{\mathbf{d\ (h\ (t))}}{\mathbf{dt}}$

## Solved Examples

***Example 1:*** ***Show that the derivative of a vector of constant length is either zero vector or is perpendicular to the vector.***

***Solution:*** Let $\mathbf{f}(t)$ be the vector of constant length

*i.e.*, $\qquad |\,\mathbf{f}(t)\,| = c$

Then, $\qquad |\,f(t)\,|^2 = \mathbf{f}(t)\,.\,\mathbf{f}(t) = c^2$

$$\therefore \qquad \frac{d}{dt}[\mathbf{f}(t)\,.\,\mathbf{f}(t)] = \frac{d}{dt}[\mathbf{f}(t)]\,.\,\mathbf{f}(t) + \mathbf{f}(t)\,.\,\frac{d}{dt}[f(t)] = 2\,\mathbf{f}(t)\,.\,\frac{d\,f(t)}{dt} = 0$$

$$\Rightarrow \qquad \mathbf{f}(t)\,.\,\frac{df(t)}{dt} = 0$$

$\Rightarrow$ $\qquad$ either $\mathbf{f}(t) = 0$ or $\frac{d}{dt}(\mathbf{f}(t)) = 0$ or $\frac{d}{dt}[\mathbf{f}(t)]$ is perpendicular to $\mathbf{f}(t)$.

## 6.4 Velocity and Acceleration

Let $\mathbf{r}$ be a vector function of the scalar variable $t$.

Then, $\qquad \mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \qquad$ ...(1)

be position vector of a point, where $x,y,z$ be components along $x$, $y$ and $z$ axis.

Differentiate w.r.t $(t)$ to (1), we get velocity $(v)$

$$v = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

$\therefore \qquad \frac{d\,\hat{\mathbf{i}}}{dt} = \frac{d\,\hat{\mathbf{j}}}{dt} = \frac{d\,\hat{\mathbf{k}}}{dt} = 0$ as $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be constant unit vector.

Where $\qquad \frac{dx}{dt}$ = component of velocity along $x$-axis

$\qquad \frac{dy}{dt}$ = component of velocity along $y$-axis and $\qquad \dfrac{dz}{dt} =$ component of velocity along $z$-axis.

Magnitude of velocity=speed $= \left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}$

$$\text{Acceleration } = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} + \frac{d^2z}{dt^2}\hat{\mathbf{k}}$$

Where $\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}$ and $\dfrac{d^2z}{dt^2}$ be components of accelerations. along $x, y$ and $z$ axis.

Magnitude of acceleration $= \sqrt{\left(\dfrac{d^2x}{dt^2}\right)^2 + \left(\dfrac{d^2y}{dt^2}\right) + \left(\dfrac{d^2z}{dt^2}\right)^2}$

$\therefore \qquad \text{Velocity } = \dfrac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$

$$\text{Acceleration } = \frac{d^2\mathbf{r}}{dt^2} = \ddot{r}$$

## Theorems

**Theorem 1: The necessary and sufficient condition for the vector F($t$) has a constant direction is that**

$$\mathbf{F} \times \frac{d\mathbf{F}}{dt} = \mathbf{0}.$$

**[B.C.A. (Bhopal), 2012, 09, 08; B.C.A. (Agra) 2008, 06, 04; B.C.A. (Kanpur) 2006; B.C.A. (Meerut) 2003]**

**Proof:** Let $|\mathbf{F}(t)| = f(t)$. Let $\mathbf{G}(t)$ be unit vector in the direction of **F(t)**. Then,

$$\frac{\mathbf{F}(t)}{|\mathbf{F}(t)|} = \mathbf{G}(t)$$

$$\Rightarrow \qquad \mathbf{F}(t) = f(t)\,\mathbf{G}(t) \qquad \ldots(1)$$

Differentiate w.r.t. ($t$)

$$\frac{d\mathbf{F}(t)}{dt} = f(t)\,\frac{d\,\mathbf{G}(t)}{dt} + \frac{d\,f(t)}{dt}\mathbf{G}(t)$$

Since $\mathbf{G}(t)$ has constant direction, then $\dfrac{d\,\mathbf{G}(t)}{dt} = 0$

$$\Rightarrow \qquad \frac{d\,\mathbf{F}(t)}{dt} = 0 + \frac{df(t)}{dt}\mathbf{G}(t)$$

Taking cross product with $\mathbf{F}(t)$ on both sides, we find

$$\mathbf{F} \times \frac{d\,\mathbf{F}(t)}{dt} = f(t)\,\mathbf{G}(t) \times \frac{df(t)}{dt}\mathbf{G}(t)$$

$$\Rightarrow \qquad \mathbf{F} \times \frac{d\,F(t)}{dt} = \mathbf{0} \qquad \text{But } [\mathbf{G(t)} \times \mathbf{G\ (t)}] = 0$$

## 6.5 Geometrical Interpretation of $\frac{d\mathbf{r}}{dt}$

**Theorem 2: A curve C is defined by the parametric equations**

$$x = x(s),\ y = y(s),\ z = z(s)$$

**Where s is the scalar representing the arc length C measures from a fixed point on C. If r is the position vector of any point on C, show that $\frac{dr}{dt}$ is a unit vector tangent to C.**

**[B.C.A. (Kanpur) 2011]**

**Proof:** Let $P$ be the point on the curve C whose position vector is $\mathbf{r}$

$\therefore$ $$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$\Rightarrow$ $$\frac{d\mathbf{r}}{ds} = \frac{d}{ds}[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}]$$

$$= \frac{dx}{ds}\hat{\mathbf{i}} + \frac{dy}{ds}\hat{\mathbf{j}} + \frac{dz}{ds}\hat{\mathbf{k}}$$ is the tangent to the curve C.

Magnitude of the vector $\frac{d\mathbf{r}}{ds} = \left|\frac{d\mathbf{r}}{ds}\right|$

$$= \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}$$

$$= \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(ds)^2}}$$

$$= \frac{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}}{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}} \qquad [\because (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2]$$

$$= 1$$

Hence, $\frac{dr}{ds}$ is the unit tangent vector to the curve or $\frac{dr}{ds}$ is a **unit vector** along the **tangent** to the curve.

> **NOTE:**
>
> 1. The component of velocity $\mathbf{v}$ in the direction of vector $\mathbf{a} = \mathbf{v}.\hat{\mathbf{a}}$.
> 2. The component of acceleration $\mathbf{a}$ in the direction $\mathbf{b}$ is $\mathbf{a}.\hat{\mathbf{b}}$.

***Example 2:*** ***A particle moves along the curve***

$$\mathbf{r} = (t^3 - 4t)\,\hat{\mathbf{i}} + (t^2 + 4t)\,\hat{\mathbf{j}} + (8t^2 - 3t^3)\,\hat{\mathbf{k}}$$

***where t is the time. Find the magnitude of the tangential components of its acceleration at $t = 2$.*** **[B.C.A. (Meerut) 2005]**

***Solution:*** We have

$$\mathbf{r} = (t^3 - 4t)\,\hat{\mathbf{i}} + (t^2 + 4t)\,\hat{\mathbf{j}} + (8t^2 - 3t^3)\,\hat{\mathbf{k}}$$

$$\therefore \quad \text{Velocity} = \frac{d\mathbf{r}}{dt} = (3t^2 - 4)\,\hat{\mathbf{i}} + (2t + 4)\,\hat{\mathbf{j}} + (16t - 9t^2)\,\hat{\mathbf{k}}$$

At $t = 2$, $\quad \text{Velocity} = 8\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$

$$\text{Acceleration} = \mathbf{a} = \frac{d^2 r}{dt^2} = 6t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (16 - 18t)\,\hat{\mathbf{k}}$$

At $t = 2$, $\quad \text{Acceleration} = \mathbf{a} = 12\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - 20\,\hat{\mathbf{k}}$

We know the velocity of particle on curve is the tangent to the curve. So, the tangent vector is velocity. Then, unit tangent vector

$$\hat{\mathbf{t}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{8\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{\sqrt{64 + 64 + 16}}$$

$$= \frac{8\,\hat{\mathbf{i}} + 8\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{12}$$

$$= \frac{2\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}}{3}$$

Tangential component of acceleration $= \mathbf{a} \,.\, \hat{\mathbf{t}}$

$$= (12\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - 20\,\hat{\mathbf{k}}) \,.\, \left(\frac{2\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}}{3}\right)$$

$$= \frac{24 + 4 + 20}{3} = \frac{48}{3} = 16$$

***Example 3:*** ***A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is time. Find the components of its velocity and acceleration at time $t = 1$ in the direction***

$$\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}.$$

**[B.C.A. (Rohilkhand) 2009; B.C.A. (Avadh) 2008]**

***Solution:*** Let $\quad \mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

$$\Rightarrow \quad \mathbf{r} = 2t^2\,\hat{\mathbf{i}} + (t^2 - 4t)\,\hat{\mathbf{j}} + (3t - 5)\,\hat{\mathbf{k}}$$

$$\therefore \quad \text{Velocity} = \frac{d\mathbf{r}}{dt} = 4t\,\hat{\mathbf{i}} + (2t - 4)\,\hat{\mathbf{j}} + (3)\,\hat{\mathbf{k}}$$

At $t=1$, Velocity $= 4\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}$

Let $\mathbf{a} = \hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$

Unit vector $\hat{\mathbf{a}} = \dfrac{\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}}{\sqrt{1+9+4}} = \dfrac{(\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})}{\sqrt{14}}$

The component of the velocity in the given direction

$$= \frac{d\mathbf{r}}{dt}.\hat{\mathbf{a}}$$

$$= (4\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}).\frac{(\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})}{\sqrt{14}}$$

$$= \frac{8\sqrt{14}}{7}$$

$$\text{Acceleration} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = 4\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}$$

Then, the component of the acceleration in the given direction

$$a = \frac{d^2\mathbf{r}}{dt^2}.\hat{\mathbf{a}}$$

$$= (4\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}}).\frac{(\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}})}{\sqrt{14}} = \frac{-\sqrt{14}}{7}.$$

***Example 4:*** ***Show that if*** $\mathbf{r} = \mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t$***. Where a, b,*** $\omega$ ***are constants, then***

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r} \;\; and \;\; \mathbf{r}\times\frac{d\mathbf{r}}{dt} = -\omega\,\mathbf{a}\times\mathbf{b}.$$

**[B.C.A. (Purvanchal) 2009]**

***Solution:*** We have $\mathbf{r} = \mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t$ ...(1)

$\Rightarrow$ $$\frac{d\mathbf{r}}{dt} = \mathbf{a}\omega\cos\omega t - \mathbf{b}\omega\sin\omega t$$

$\Rightarrow$ $$\frac{d^2\mathbf{r}}{dt^2} = -\mathbf{a}\omega^2\sin\omega t - \mathbf{b}\omega^2\cos\omega t$$

$\Rightarrow$ $$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2(\mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t)$$

$\Rightarrow$ $$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2\,\mathbf{r}$$ [by (1)]

Also $$\mathbf{r}\times\frac{d\mathbf{r}}{dt} = (\mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t)\times(\mathbf{a}\omega\cos\omega t - \mathbf{b}\omega\sin\omega t)$$

$$= \omega\,(-(\mathbf{a}\times\mathbf{b})\sin^2\omega t + (b\times a)\cos^2\omega t)$$ $[\because \mathbf{a}\times\mathbf{a} = \mathbf{b}\times\mathbf{b} = 0]$

$$= \omega\,(-(a\times b)\sin^2\omega t - (\mathbf{a}\times b)\cos^2\omega t)$$

$$= -\omega\,(\mathbf{a}\times\mathbf{b})\,[\sin^2\omega t + \cos^2\omega t]$$

$$= -\omega\,(\mathbf{a}\times\mathbf{b}).$$

**Example 5:** ***A particle P is moving on a circle of radius r with constant angular velocity $\omega = \frac{d\theta}{dt}$. Show that acceleration is $-\omega^2 r$.***

**Solution:** Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be the unit vectors along and perpendicular to radii *OA* and *OB* of circle and if *P* is any point on the circle such that $|OP| = \mathbf{r}$ make an angle $\theta$ with $\hat{\mathbf{i}}$.

Then, the position vector of *P* is given by

$$\mathbf{r} = r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}$$

Where $r$ is the radius of the circle and is constant.

Now, $$\frac{d\mathbf{r}}{dt} = -r\sin\theta\frac{d\theta}{dt}\hat{\mathbf{i}} + r\cos\theta\frac{d\theta}{dt}\hat{\mathbf{j}}$$

$$= -r\omega\sin\theta\,\hat{\mathbf{i}} + r\omega\cos\theta\,\hat{\mathbf{j}} \quad \left[\because \frac{d\theta}{dt} = \omega\right]$$

Hence, acceleration

$$= \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = (-r\omega\cos\theta\,\hat{\mathbf{i}} - r\omega\sin\theta\,\hat{\mathbf{j}})\frac{d\theta}{dt}$$

$$= -(r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}})\,\omega^2$$

$$\mathbf{a} = -\mathbf{r}\,\omega^2$$

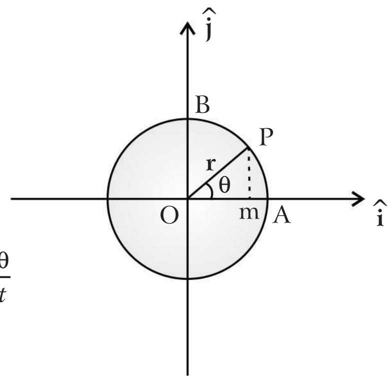


Fig. 6.1

**Example 6:** ***Find the unit tangent vector at any point on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$, where t is any variable. Also determine the unit tangent vector at the point $t = 2$.*** **[B.C.A. (Lucknow) 2008]**

**Solution:** If $\mathbf{r}$ is the position vector of any point $(x,y,z)$ on the given curve, then

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$

$$= (t^2+2)\,\hat{\mathbf{i}} + (4t-5)\,\hat{\mathbf{j}} + (2t^2-6t)\,\hat{\mathbf{k}}$$

We know $\frac{dr}{dt}$ is the tangent vector to the curve at point $(x,y,z)$, then

$$\frac{dr}{dt} = 2t\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}} + (4t-6)\,\hat{\mathbf{k}}$$

and $$\left(\frac{dr}{dt}\right) = \sqrt{(2t)^2 + (4)^2 + (4t-6)^2} = 2\sqrt{5t^2 - 12t + 13}$$

$\therefore$ The unit tangent vector $$\hat{\mathbf{t}} = \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|} = \frac{t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (2t-3)\,\hat{\mathbf{k}}}{\sqrt{5t^2 - 12t + 13}}$$

Also the unit tangent vector at the point $t = 2$

$$= \frac{2\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (2\times 2 - 3)\,\hat{\mathbf{k}}}{\sqrt{5\times 4 - 12\times 2 + 13}} = \frac{1}{3}(2\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

***Example 7:*** ***Show that radial and transverse accelerations of a particle moving in a plane are respectively.***

$$\frac{d^2\mathbf{r}}{dt^2} - \mathbf{r}\left(\frac{d\theta}{dt}\right)^2 \quad \textit{and} \quad \frac{1}{\mathbf{r}}\frac{d}{dt}\left(\mathbf{r}^2\frac{d\theta}{dt}\right)$$

**[B.C.A. (Lucknow) 2010; B.C.A. (Meerut) 2009]**

***Solution:*** Let $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ be two mutually perpendicular unit vectors in the plane of motion of the particle.

Let $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ be unit vectors along and perpendicular to $\mathbf{r}$

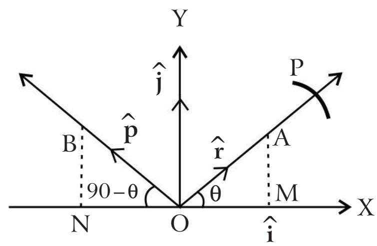

Fig. 6.2

Where $\qquad OP = r$

If $\qquad OA = OB = 1$

Then, $\qquad \hat{\mathbf{r}} = \mathbf{OA} = \mathbf{OM} + \mathbf{MA}$

$\Rightarrow \qquad \hat{\mathbf{r}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$

and $\qquad \hat{\mathbf{p}} = \mathbf{OB} = \mathbf{ON} + \mathbf{NB}$

$\Rightarrow \qquad \hat{\mathbf{p}} = \cos(90-\theta)(-\hat{\mathbf{i}}) + \sin(90-\theta)(\hat{\mathbf{j}})$

$\Rightarrow \qquad \hat{\mathbf{p}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$

But $\qquad \hat{\mathbf{r}} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$ or $\mathbf{r} = \hat{\mathbf{r}}\,r$

$\therefore \qquad \mathbf{r} = r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}$

If $\mathbf{v}$ is velocity, then we have

$$\mathbf{v} = \frac{dr}{dt} = \frac{d}{dt}(r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}})$$

$$= \frac{dr}{dt}\cos\theta\,\hat{\mathbf{i}} + \frac{dr}{dt}\sin\theta\,\hat{\mathbf{j}} - r\sin\theta\frac{d\theta}{dt}\,\hat{\mathbf{i}} + \mathbf{r}\cos\theta\frac{d\theta}{dt}\,\hat{\mathbf{j}}$$

$$= \frac{dr}{dt}(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + r\frac{d\theta}{dt}(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})$$

$$= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\mathbf{p}}$$

$\therefore$ Radial velocity = coefficient of $\hat{\mathbf{r}} = \dfrac{dr}{d\theta}$

and transverse velocity = coefficient of $\hat{\mathbf{p}} = r\dfrac{d\theta}{dt}$

Now, $$\frac{d\,\hat{\mathbf{r}}}{dt} = (-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})\frac{d\theta}{dt} = \hat{\mathbf{p}}\frac{d\theta}{dt}$$

and $$\frac{d\,\hat{\mathbf{p}}}{dt} = (-\cos\theta\,\hat{\mathbf{i}} - \sin\theta\,\hat{\mathbf{j}})\frac{d\theta}{dt} = -\hat{\mathbf{r}}\frac{d\theta}{dt}$$

Now if **a** be the acceleration, then we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\hat{\mathbf{r}}}{dt}\hat{\mathbf{p}} + \frac{d\mathbf{r}}{dt}\frac{d\theta}{dt}\hat{\mathbf{p}} + r\frac{d^2\theta}{dt^2}\hat{\mathbf{p}} + r\frac{d\theta}{dt}\frac{d\hat{\mathbf{p}}}{dt}$$

$$= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{2dr}{dt}\frac{d\theta}{dt}\hat{\mathbf{p}} + r\frac{d^2\theta}{dt^2}\hat{\mathbf{p}} - r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}}$$

$$= \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\hat{\mathbf{r}} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\mathbf{p}}$$

$$= \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\hat{\mathbf{p}}$$

$\therefore$ Radial acceleration $=$ coefficient of $\hat{\mathbf{r}} = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$

and transverse acceleration $=$ coefficient of $\hat{\mathbf{p}} = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

**Example 8:** ***If $\hat{\mathbf{r}}$ be the unit vector in the direction r, show that***

$$\hat{\mathbf{r}} \times d\,\hat{\mathbf{r}} = \frac{\mathbf{r}\times d\mathbf{r}}{r^2}.$$

**[B.C.A. (Meerut) 2007]**

**Solution:** $$d(\hat{\mathbf{r}}) = d\left(\frac{\mathbf{r}}{r}\right) = \frac{d\mathbf{r}}{r} - \mathbf{r}\frac{dr}{r^2}$$

$\therefore$ $$\hat{\mathbf{r}} \times d\,\hat{\mathbf{r}} = \hat{\mathbf{r}} \times \left(\frac{d\mathbf{r}}{r} - \mathbf{r}\frac{dr}{r^2}\right)$$

$$= \frac{\mathbf{r}}{r}\left(\frac{d\mathbf{r}}{r} - \mathbf{r}\frac{dr}{r^2}\right) \qquad \left[\because\ \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}\right]$$

$$= \frac{\mathbf{r}\times d\mathbf{r}}{r^2} - (\mathbf{r}\times\mathbf{r})\frac{dr}{r^3}$$

$$= \frac{\mathbf{r}\times d\mathbf{r}}{r^2} - 0 \qquad [\because\ \mathbf{r}\times\mathbf{r} = 0]$$

$$= \frac{\mathbf{r}\times d\mathbf{r}}{r^2}$$

**Example 9:** ***If $\dfrac{da}{dt} = c \times a, \dfrac{db}{dt} = c \times b$, show that $\dfrac{d}{dt}(a \times b) = c \times (a \times b)$.***

**[B.C.A. (Rohilkhand) 2008]**

**Solution:** We have $\dfrac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \dfrac{d\mathbf{a}}{dt}\times\mathbf{b} + \mathbf{a}\times\dfrac{d\mathbf{b}}{dt}$

$$= (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} + \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$$

$$= -\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{a} \times (\mathbf{c} \times \mathbf{b})$$

$$= -\{(\mathbf{b}.\mathbf{a})\,\mathbf{c} - (\mathbf{b}.\mathbf{c})\,\mathbf{a}\} + \{(\mathbf{a}.\mathbf{b})\,\mathbf{c} - (\mathbf{a}.\mathbf{c})\,\mathbf{b}\}$$

$$= (\mathbf{b}.\mathbf{c})\,\mathbf{a} - (\mathbf{a}.\mathbf{c})\mathbf{b}$$

$$= \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$$

***Example 10:*** ***If $\hat{\mathbf{r}}$ be the unit vector, then show that***

$$\left| \hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt} \right| = \left| \frac{d\hat{\mathbf{r}}}{dt} \right|.$$

**[B.C.A. (Purvanchal) 2012, 07]**

***Solution:*** Let $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$

$\Rightarrow$ $$\frac{d\hat{\mathbf{r}}}{dt} = (-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}})\frac{d\theta}{dt}$$

$\therefore$ $$\hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta\dfrac{d\theta}{dt} & \cos\theta\dfrac{d\theta}{dt} & 0 \end{vmatrix} = \hat{\mathbf{k}}\frac{d\theta}{dt}$$

Hence, $$\left| \hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt} \right| = \left| \hat{\mathbf{k}}\frac{d\theta}{dt} \right| = |\hat{\mathbf{k}}|\left| \frac{d\theta}{dt} \right| = \frac{d\theta}{dt} \qquad [\because |\hat{\mathbf{k}}| = 1] \qquad ...(1)$$

and $$\left| \frac{d\hat{\mathbf{r}}}{dt} \right| = \left| -\sin\theta\frac{d\theta}{dt}\,\hat{\mathbf{i}} + \cos\theta\frac{d\theta}{dt}\,\hat{\mathbf{j}} \right|$$

$$= \sqrt{\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + \cos^2\theta\left(\frac{d\theta}{dt}\right)^2}$$

$$= \frac{d\theta}{dt}\sqrt{\sin^2\theta + \cos^2\theta}$$

$$= \frac{d\theta}{dt} \qquad ...(2)$$

Hence, from (1) and (2) we find

$$\left| \hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt} \right| = \left| \frac{d\hat{\mathbf{r}}}{dt} \right|.$$

**Example 11: Evaluate**

*(i)* $\frac{d}{dt}\left[\mathbf{r}\frac{d\mathbf{r}}{dt}\frac{d^2\mathbf{r}}{dt^2}\right]$ *(ii)* $\frac{d^2}{dt^2}\left[\mathbf{r}\frac{d\mathbf{r}}{dt}\frac{d^2\mathbf{r}}{dt^2}\right]$ *(iii)* $\frac{d}{dt}\left[\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\right]$.

**[B.C.A. (Agra) 2010, 06; B.C.A. (Lucknow) 2007, 03; B.C.A. (Kanpur) 2005]**

**Solution:**

(i)
$$\frac{d}{dt}\left[\mathbf{r}\frac{d\mathbf{r}}{dt}\frac{d^2\mathbf{r}}{dt^2}\right]=\frac{d}{dt}[\mathbf{r}\,\dot{\mathbf{r}}\,\ddot{\mathbf{r}}]$$
$$=\frac{d}{dt}[\mathbf{r}\,.\,(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})]$$
$$=\dot{r}\,.\,(\dot{r}\times\ddot{r})+r\,.\,(\ddot{r}\times\ddot{r})+r\,.\,(\dot{r}\,.\,\dddot{r})$$
$$=[\dot{\mathbf{r}},\dot{\mathbf{r}},\ddot{\mathbf{r}}]+[\mathbf{r}\,\ddot{\mathbf{r}}\,\ddot{\mathbf{r}}]+[\mathbf{r}\,\dot{\mathbf{r}}\,\dddot{\mathbf{r}}]$$
$$=0+0+[\mathbf{r}\,\dot{\mathbf{r}}\,\ddot{\mathbf{r}}]\qquad\left[\because\ [\dot{\mathbf{r}},\ddot{\mathbf{r}},\dot{\mathbf{r}}]=0\ \ [\mathbf{r},\ddot{\mathbf{r}},\ddot{\mathbf{r}}]=0\right]$$
$$=[\mathbf{r}\,\dot{\mathbf{r}}\,\dddot{\mathbf{r}}].$$

(ii)
$$\frac{d^2}{dt^2}\left[\mathbf{r}\frac{d\mathbf{r}}{dt}\frac{d^2\mathbf{r}}{dt^2}\right]=\frac{d^2}{dt^2}[\mathbf{r}\,\dot{\mathbf{r}}\,\ddot{\mathbf{r}}]$$
$$=\frac{d}{dt}\frac{d}{dt}[\mathbf{r}\,\dot{\mathbf{r}}\,\ddot{\mathbf{r}}]$$
$$=\frac{d}{dt}\left\{[\dot{\mathbf{r}},\dot{\mathbf{r}},\ddot{\mathbf{r}}]+[\mathbf{r},\ddot{\mathbf{r}},\ddot{\mathbf{r}}]+[\mathbf{r},\dot{\mathbf{r}},\dddot{\mathbf{r}}]\right.$$
$$=\frac{d}{dt}\left[0+0+[\mathbf{r}\,\dot{\mathbf{r}}\,\dddot{\mathbf{r}}]\right.$$
$$=\frac{d}{dt}[\mathbf{r}\,\dot{\mathbf{r}}\,\dddot{\mathbf{r}}]$$
$$=[\dot{\mathbf{r}}\,\dot{\mathbf{r}}\,\dddot{\mathbf{r}}]+[\mathbf{r}\,\ddot{\mathbf{r}}\,\dddot{\mathbf{r}}]+[\mathbf{r}\,\dot{\mathbf{r}}\,\ddddot{\mathbf{r}}]$$
$$=0+[\mathbf{r}\,\ddot{\mathbf{r}}\,\dddot{\mathbf{r}}]+[\mathbf{r}\,\dot{\mathbf{r}}\,\ddddot{\mathbf{r}}]$$
$$=[\mathbf{r}\,\ddot{\mathbf{r}}\,\dddot{\mathbf{r}}]+[\mathbf{r}\,\dot{\mathbf{r}}\,\ddddot{\mathbf{r}}].$$

(iii)
$$\frac{d}{dt}\left[\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\right]=\frac{d}{dt}[\mathbf{r}\times(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})]$$
$$=\dot{\mathbf{r}}\times(\dot{r}\times\ddot{r})+r\times(\ddot{r}\times\ddot{r})+r\times(\dot{r}\times\dddot{r})$$
$$=[\dot{r}\,\dot{r}\,\ddot{r}]+[r\,\ddot{r}\,\ddot{r}]+[r\,\dot{r}\,\dddot{r}]$$
$$=0+0+[r\,\dot{r}\,\dddot{r}]$$
$$=[r\,\dot{r}\,\dddot{r}].$$

## Exercise 6.1

1. If $\mathbf{r}$ is a unit vector, then prove that

$$\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right|=\left|\frac{dr}{dt}\right|.$$

**[B.C.A. (Bhopal) 2007]**

2. If $\mathbf{r}=t^3\,\hat{\mathbf{i}}+\left(2t^3-\dfrac{1}{5t^2}\right)\hat{\mathbf{j}}$, show that $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}=\hat{\mathbf{k}}$.

**[B.C.A. (Meerut) 2001]**

3. If $\mathbf{a}=2t\,\hat{\mathbf{i}}+3t^2\,\hat{\mathbf{j}}-4t\,\hat{\mathbf{k}}$ and $\mathbf{b}=\hat{\mathbf{i}}+t^3\,\hat{\mathbf{j}}+t\,\hat{\mathbf{k}}$, then verify the formula

$$\frac{d}{dt}(\mathbf{a}\times\mathbf{b})=\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\mathbf{a}\times\frac{d\mathbf{b}}{dt}.$$

4. If $\mathbf{r}=\mathbf{a}\sin\omega t+\mathbf{b}\cos\omega t+\dfrac{\mathbf{c}t}{\omega^2}\sin\omega t$, prove that $\dfrac{d^2\mathbf{r}}{dt^2}+\omega^2\,\mathbf{r}=\dfrac{2\mathbf{c}}{\omega}\cos\omega t$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are constant vectors and $\omega$ is constant scalar.

5. If $\alpha=t^2\,\hat{\mathbf{i}}-t\,\hat{\mathbf{j}}+(2t+1)\,\hat{\mathbf{k}}$ and $\beta=(2t-3)\,\hat{\mathbf{i}}+\hat{\mathbf{j}}-t\hat{\mathbf{k}}$, find

$$\frac{d}{dt}\left(\alpha\times\frac{d\beta}{dt}\right)\text{ at } t=2.$$

6. Determine the velocity and acceleration at any time when the particle moves along the following curved paths:

(i) $x=a\cos t,\ y=a\sin t, z=at\tan\alpha$

(ii) $x=4\cos t, y=4\sin t, z=6t$

(iii) $x=e^{-t}, y=2\cos 3t, z=2\sin 3t$.

7. A particle moves so that its position vector is given by $\mathbf{r}=\cos\omega t\,\hat{\mathbf{i}}+\sin\omega t\,\hat{\mathbf{j}}$, where $\omega$ is constant, show that:

(i) The velocity of the particle is perpendicular to $\mathbf{r}$.

(ii) Acceleration is directed towards the origin and has magnitude proportional to distance from the origin.

(iii) $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$ is a constant vector.

**[B.C.A. (Meerut) 2009]**

8. A particle moves along the curve $x=t^3+1, y=t^2, z=2t+5$, where $t$ is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\,\hat{\mathbf{k}}$.

## Answers 6.1

5. $\hat{\mathbf{i}}+8\hat{\mathbf{j}}+2\hat{\mathbf{k}}$

6. (i) $v=-a\sin t\,\hat{\mathbf{i}}+a\cos t\,\hat{\mathbf{j}}+a\tan\alpha\,\hat{\mathbf{k}},\ \mathbf{a}=-a\cos t\,\hat{\mathbf{i}}-a\sin t\,\hat{\mathbf{j}}$

(ii) $v=-4\sin t\,\hat{\mathbf{i}}+4\cos t\,\hat{\mathbf{j}}+6\,\hat{\mathbf{k}},\ \mathbf{a}=-4\cos t\,\hat{\mathbf{i}}-a\sin t\,\hat{\mathbf{j}}$

(iii) $v=-e^{-6}\,\hat{\mathbf{i}}-6\sin 3t\,\hat{\mathbf{j}}+6\cos 3t\,\hat{\mathbf{k}},\ \ \mathbf{a}=e^{-t}\,i-18\cos 3t\,\hat{\mathbf{j}}-18\cos 3t\,\hat{\mathbf{k}}$

8. Component of velocity in given direction $=\sqrt{11}$ the component of acceleration in the given direction $=\dfrac{8}{\sqrt{11}}$

## Exercise 6.2

1. $\dfrac{d}{dt}\left[\mathbf{r},\dfrac{d\mathbf{r}}{dt},\dfrac{d^2\mathbf{r}}{dt^2}\right].$  2. $\dfrac{d^2}{dt^2}\left[\mathbf{r},\dfrac{d\mathbf{r}}{dt},\dfrac{d^2\mathbf{r}}{dt^2}\right].$

3. $\dfrac{d}{dt}\left[\mathbf{r}\times\left(\dfrac{d\mathbf{r}}{dt}\times\dfrac{d^2\mathbf{r}}{dt^2}\right)\right].$

4. If $\mathbf{r}=a\cos t\,\hat{\mathbf{i}}+a\sin t\,\hat{\mathbf{j}}+at\tan\alpha\,\hat{\mathbf{k}}$, find

$$\left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right| \text{ and } \left|\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right|.$$

5. Find the angle between the tangents to the curve $\mathbf{r}=t^3\,\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}}+t^3\,\hat{\mathbf{k}}$ at the point $t=\pm 1$.

6. Find the tangential and normal accelerations of a point moving in a plane curve.

7. Find the unit vector perpendicular to each of the vectors

$\mathbf{a}=\hat{\mathbf{i}}-2\,\hat{\mathbf{j}}$ and $\mathbf{b}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$.

8. If $\mathbf{r}=t^2\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}+3t\,\hat{\mathbf{k}}$, find value of $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$.

9. If $\mathbf{r}=\hat{\mathbf{i}}\cos\theta+\hat{\mathbf{j}}\sin\theta+\theta\,\hat{\mathbf{k}}$, find the value of $\dfrac{d\mathbf{r}}{dt}.\mathbf{r}$.

10. If $\mathbf{r}=t^2\,\hat{\mathbf{i}}-t^3\hat{\mathbf{j}}+t^4\hat{\mathbf{k}}$ represents the position vector of a moving particle, find its velocity at time $t=1$. **[B.C.A. (Rohilkhand) 2010]**

11. If $\mathbf{r} = 3t\,\hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}} + 2t^3\,\hat{\mathbf{k}}$, find the value of $\left[\dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}, \dfrac{d^3\mathbf{r}}{dt^3}\right]$.

12. A particle moves along the curve $x = \bar{e}^t$, $y = 2\cos 3t$, $z = 2\sin 3t$, where $t$ is time. Determine the velocity at $t = \pi$.

13. Prove that $\mathbf{a} \cdot \dfrac{d\mathbf{a}}{dt} = a\dfrac{da}{dt}$, where $|\mathbf{a}| = a$.

14. Find the tangents vector $r'(t)$ and the corresponding unit tangent vectors of the following curves:

(i) $\mathbf{r} = t\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}}$ at $(1, 1, 0)$

(ii) $\mathbf{r} = 2\cos t\,\hat{\mathbf{i}} + 2\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ at $(2, 0, 0)$.

15. If $\mathbf{a} = t^2\,\hat{\mathbf{i}} - t\,\hat{\mathbf{j}} + t^3\,\hat{\mathbf{k}}$ and $\mathbf{b} = \sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$, find:

(i) $\dfrac{d}{dt}(\mathbf{a} \cdot \mathbf{b})$ (ii) $\dfrac{d}{dt}(\mathbf{a} \times \mathbf{b})$.

16. If $\mathbf{r} = (\cos nt)\,\hat{\mathbf{i}} + (\sin nt)\,\hat{\mathbf{j}}$, then show that:

(i) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = n\hat{\mathbf{k}}$ (ii) $\mathbf{r} \cdot \dfrac{d\mathbf{r}}{dt} = 0$.

## Answers 6.2

1. $\left[\mathbf{r}, \frac{d\mathbf{r}}{dt}, \frac{d^3\mathbf{r}}{dt^3}\right]$

2. $\left[\mathbf{r}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right] + \left[\mathbf{r}, \frac{d\mathbf{r}}{dt}, \frac{d^4\mathbf{r}}{dt^4}\right]$

3. $\frac{d\mathbf{r}}{dt} \times \left(\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right) + \mathbf{r} \times \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3}\right)$

4. $a^2 \sec\alpha,\ a^3 \tan\alpha$

5. $\theta = \cos^{-1}\left(\frac{9}{11}\right)$

6. $\frac{d^2s}{dt^2}, \frac{v^2}{e}$

7. $\pm \frac{1}{\sqrt{6}}\left(-2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}\right)$

8. $6\,\hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}$

9. $\theta$

10. $2\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}$

11. $\frac{2}{6}$

12. $-e^{-\pi}\,\hat{\mathbf{i}} + 6\hat{\mathbf{k}}$

14. (i) $\hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}}$ and $\frac{\hat{\mathbf{i}} + 3t^2\,\hat{\mathbf{j}}}{(9t^4+1)^{1/2}}$

(ii) $-2\sin t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\frac{-2\sin t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{4\sin^2 t + 5}}$

15. (i) $(1+t^2)\sin t - t\cos t$

(ii) $(t^3 \sin t - 3t^2 \cos t)\,\hat{\mathbf{i}} + (t^3 \cos t + 3t^3 \sin t)\,\hat{\mathbf{j}} + (\cos t + t\sin t + t^2 \cos t)\,\hat{\mathbf{k}}$

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# Chapter 7

# Gradient, Divergence & Curl

## 7.1 Scalar and Vector Point Function

If a quantity assume one or more than one definite value at each point of a region, then that quantity is called **point function** in the given region.

## 7.2 Single Valued Function

A point function is said to be **single valued** or **uniform function** if it has only one definite value at each point of the region otherwise it is called **multiple valued function** or **multivalued function** in the given region.

### 7.2.1 Scalar Point Function

If corresponding to each point $P$ of a region $R$, there is associated a scalar $\phi(P)$ or $\phi(x, y, z)$. Then the function $\phi(P)$ is called **scalar point function** or **scalar function of position** design in the region $R$.

***Illustration:***

1. The mass $m(P)$ at the point $P$ of a body occupying a certain region is a scalar point function.

2. The density $\rho(P)$ at any point $P$ of a body occupying a certain region is a scalar point function.

3. The distance $d(P)$ any point $P(x, y, z)$ in space $R^3$ from a fixed point $A(x_0, y_0, z_0)$ is also a **scalar point function** where $d(P) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$.

## 7.3 Scalar Field

The set of points of the region $R$ together with corresponding scalar function values $\phi(P)$ is said to be a **scalar field** over $R$.

***Illustration:***

1. $\phi(x, y, z) = x^2 y - z^3$
2. Gravitational potential of system of masses
3. The temperature distribution in a medium

## 7.4 Vector Point Function

If corresponding to each point $P$ of a region $R$, there is associated a vector $\mathbf{f}(P)$ than $f$ is called vector point function or vector function of position defined in the region $R$.

***Illustration:***

1. The velocity $v(P)$ of a particle in a moving fluid at any time $t$ occupying the position $P$ in a certain region is **vector point functions.**

2. The gravitational force $G(P)$ or $F(x, y, z)$ exerted by a given point mass $m$ at the origin on a unit point mass located at a point $P(x, y, z)$ other than the origin is

$$G(P) = \frac{Gm}{x^2 + y^2 + z^2}\, u(x, y, z)$$

where $G =$ the universal gravitational constant.

$u(x, y, z) =$ unit vector emanating from $P$ and directed toward the origin.

Here, $G(P)$ is a vector point function.

## 7.5 Vector Field

The set of points of the region $R$ together with the corresponding vector function values $\mathbf{f}(P)$ is said to be a **vector field** over $R$.

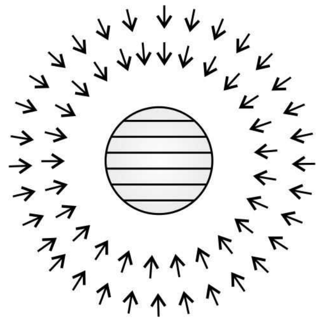

Fig. 7.1

***Illustration:***

1. We live on the earth in gravitational force field which may be considered as earth ward directed vector through each point of the space $\mathbf{R}^3$.

2. Electric intensity of force.

3. $F(x, y, z) = x^2 y\,\hat{\mathbf{i}} - 3yz^2\,\hat{\mathbf{j}} - 7x^2 z\,\hat{\mathbf{k}}$.

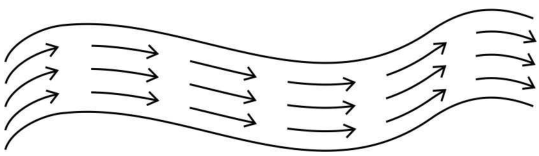

Fig. 7.2

## 7.6 Equipotential of Level Surfaces

If $\phi(x, y, z)$ be a single valued continuous scalar point function, then through any point $P(x, y, z)$ of the region considered, we can draw a surface such that at each point on it the function has the same value as at $P$. Such a surface is called a **level surface** of the function.

Thus, $$\phi(x, y, z) = C$$

where $C$ is on arbitrary constant, represents a level surface. If the scalar function $\phi$ is a potential function, the level surfaces are called the **equipotential surfaces.**

## 7.7 The Operator $\nabla$

The vector operator $\nabla$ is defined by the equation

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \qquad ...(1)$$

where $\dfrac{\partial}{\partial x}$ = partial differentiation w.r.t. '$x$' when $y$ and $z$ consider as constant.

$\dfrac{\partial}{\partial y}$ = partial differentiation w.r.t. '$y$' when $x$ and $z$ consider as constant.

$\dfrac{\partial}{\partial z}$ = partial differentiation w.r.t. '$z$' when $x$ and $y$ consider as constant.

Thus, $\nabla$ is called **"del"** or **"nabla"** operator, becomes popular for engineers and physicists due to its large applications.

> **NOTE:**
> It serves as a vector differential operator, when it operates upon a function.

## 7.8 Gradient

Let $\phi(x, y, z)$, a scalar function, be defined and differentiable at each point $(x, y, z)$ in the region of space (*i.e.*, $\phi$ is a differentiable scalar field). Then the **gradient** of $\phi$, written as $\nabla\phi$ or grad $\phi$, is defined by

$$\nabla\phi = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z}$$

Where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be unit vector along $x, y, z$ axis.

### 7.8.1 Geometrical Interpretation or Gradient as Surface Normal Vector

Let $f(x, y, z)$ be differentiable scalar field, $f(x, y, z) = C$ be the level surface and $P(a, b, c)$ be any point on it.

Consider a curve $C$ on the surface passing through a point $P(a, b, c)$. Let $x = x(t)$, $y = y(t)$, $z = z(t)$ is the parametric representation $C$. Then any $P$ on C has the position vector $\mathbf{r} = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$ and since the curve $C$ lies on the surface.

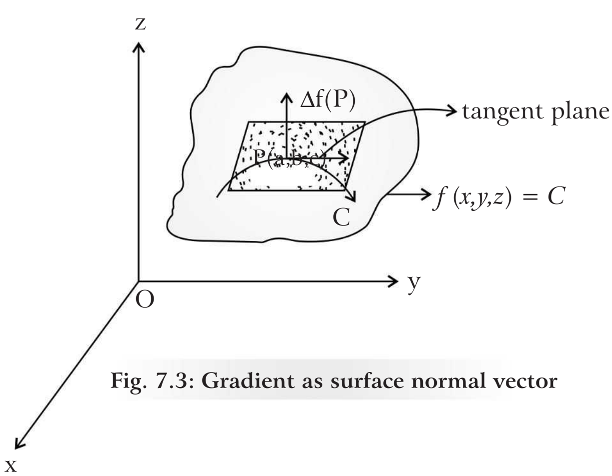


Fig. 7.3: Gradient as surface normal vector

$$f(x(t), y(t), z(t)) = C$$

$$\Rightarrow \quad \frac{d}{dt}[f(x(t), y(t), z(t))] = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0 \qquad \text{[using chain rule]}$$

$$\Rightarrow \quad \left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right) \cdot \left(\hat{\mathbf{i}}\frac{dx}{dt} + \hat{\mathbf{j}}\frac{dy}{dt} + \hat{\mathbf{k}}\frac{dz}{dt}\right) = 0$$

$$\Rightarrow \quad (\nabla f) . r'(t) = 0 \quad \text{or} \quad \nabla f \cdot dr = 0$$

Suppose $\nabla f \neq 0$, $r'(t) \neq 0$, $r'(t)$ is the tangent vector to the curve $C$ and it lies in the tangent plane to the surface $P$. Therefore, **$\nabla f(p)$ is orthogonal to every tangent vector at $p$**. Then, we can say $\nabla f(p)$ is a vector normal to the surface $f(x, y, z) = C$ at $P$.

## 7.9 Tangent Plane and Normal Plane to a Surface Level

1. The equation of tangent plane at $P(a,b,c)$ on surface $f(x, y, z) = C$ is

$$\nabla f(P)[(x-a)\hat{\mathbf{i}} + (y-b)\hat{\mathbf{j}} + (z-c)\hat{\mathbf{k}}] = 0$$

or $$(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y} + (z-c)\frac{\partial f}{\partial z} = 0$$

2. The equation of normal plane to the surface $f(x, y, z) = C$ at $P(a,b,c)$ is

$$\frac{(x-a)}{\dfrac{\partial f}{\partial x}} = \frac{(y-b)}{\dfrac{\partial f}{\partial y}} = \frac{(z-c)}{\dfrac{\partial f}{\partial z}}$$

3. Vector normal to surface $f(x, y, z) = C$ is $\nabla f$.

4. The unit vector normal to surface $f(x, y, z) = C$ is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{\nabla f}{|grad\ f|}$$

5. Angle between two surface $\phi_1(x, y, z) = C_1$ and $\phi_2(x, y, z) = C_2$ is given by

$$\cos\phi = \frac{(\nabla\phi_1).(\nabla\phi_2)}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

where $\phi$ is the angle between $\phi_1(x, y, z) = C_1$ and $\phi_2(x, y, z) = C_2$ surfaces

$$\nabla\phi_1 = \text{grad } \phi_1 = \frac{\partial\phi_1}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi_1}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi_1}{\partial z}\hat{\mathbf{k}}$$

and $$\nabla\phi_2 = \text{grad } \phi_2 = \frac{\partial\phi_2}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi_2}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi_2}{\partial z}\hat{\mathbf{k}}$$

## 7.10 Some Results Connected to Gradient

**Theorem 1: Prove the following relations:**

**(i)** $\nabla(f \pm g) = \nabla f \pm \nabla g$

**(ii)** $\nabla(fg) = f\,\nabla g + g\,\nabla f$ **[B.C.A. (Meerut) 2004]**

**(iii)** $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\,\nabla f - f\,\nabla g}{g^2},\ g \neq 0$

**where $f$, $g$ be two differentiable scalar fields.**

**Proof:** (i) We have $\nabla(f \pm g) = \left(\hat{\mathbf{i}}\dfrac{\partial}{\partial x} + \hat{\mathbf{j}}\dfrac{\partial}{\partial y} + \hat{\mathbf{k}}\dfrac{\partial}{\partial z}\right)(f \pm g)$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x}(f \pm g) + \hat{\mathbf{j}}\frac{\partial}{\partial y}(f \pm g) + \hat{\mathbf{k}}\frac{\partial}{\partial z}(f \pm g)$$

$$= \hat{\mathbf{i}}\left[\frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}\right] + \hat{\mathbf{j}}\left[\frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y}\right] + \hat{\mathbf{k}}\left[\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right]$$

$$= \left(\hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}\right) \pm \left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)$$

$$= \nabla f \pm \nabla g$$

$\therefore \qquad \nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) Similar proof as proof (1).

(iii) We have $\nabla\left(\dfrac{f}{g}\right) = \left(\hat{\mathbf{i}}\dfrac{\partial}{\partial x} + \hat{\mathbf{j}}\dfrac{\partial}{\partial y} + \hat{\mathbf{k}}\dfrac{\partial}{\partial z}\right)\left(\dfrac{f}{g}\right)$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \hat{\mathbf{j}}\frac{\partial}{\partial y}\left(\frac{f}{g}\right) + \hat{\mathbf{k}}\frac{\partial}{\partial z}\left(\frac{f}{g}\right)$$

$$= \hat{\mathbf{i}}\left\{\frac{g\dfrac{\partial f}{\partial x} - f\dfrac{\partial g}{\partial x}}{g^2}\right\} + \hat{\mathbf{j}}\left\{\frac{g\dfrac{\partial f}{\partial y} - f\dfrac{\partial g}{\partial y}}{g^2}\right\} + \hat{\mathbf{k}}\left\{\frac{g\dfrac{\partial f}{\partial z} - f\dfrac{\partial g}{\partial z}}{g^2}\right\}$$

$$= \frac{1}{g^2}\left\{g\left(\hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z}\right) - f\left(\hat{\mathbf{i}}\frac{\partial g}{\partial x} + \hat{\mathbf{j}}\frac{\partial g}{\partial y} + \hat{\mathbf{k}}\frac{\partial g}{\partial z}\right)\right\}$$

$$= \frac{g\,\nabla f - f\,\nabla g}{g^2}$$

## Solved Examples

**Example 1:** *If* $\phi = 3x^2 y - y^3 z^2$, *then find grad* $\phi$ *at the point* $(1, -2, -1)$.

**[B.C.A. (Meerut) 2007,06]**

**Solution:** Here,

$$\phi = 3x^2 y - y^3 z^2$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = 6xy, \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2 z^2, \frac{\partial \phi}{\partial z} = -2y^3 z$$

$$\therefore \quad \text{grad } \phi = \nabla \phi = \hat{i} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$= 6xy \,\hat{\mathbf{i}} + (3x^2 - 3y^2 z^2)\,\hat{\mathbf{j}} + (-2y^3 z)\,\hat{\mathbf{k}} \quad \text{at } (1, -2, -1)$$

$$= -12\,\hat{\mathbf{i}} - 9\,\hat{\mathbf{j}} - 16\,\hat{\mathbf{k}}$$

**Example 2:** *If* $r = |\mathbf{r}|$, *where* $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, *prove that*

*(i)* $\nabla f(r) = f'(r) \nabla r$ **[B.C.A. (Meerut) 2012,05,01]**

*(ii)* $\nabla r = \frac{\mathbf{r}}{r}$ **[B.C.A. (Agra) 2008,05]**

*(iii)* $\nabla f(r) \times \mathbf{r} = \mathbf{0}$ **[B.C.A. (Meerut) 2003]**

*(iv)* $\nabla \left(\frac{1}{r}\right) = \frac{-\mathbf{r}}{r^3}$ **[B.C.A. (Meerut) 2004]**

*(v)* $\nabla \log |r| = \frac{\mathbf{r}}{r^2}$ **[B.C.A. (Meerut) 2011]**

*(vi)* $\nabla r^n = n r^{n-2} \mathbf{r}$ **[B.C.A. (Lucknow) 2011,06]**

*(vii)* $\nabla |\mathbf{r}^2| = 2\mathbf{r}$ **[B.C.A. (Avadh) 2009]**

*(viii)* $\nabla e^{(x^2 + y^2 + z^2)} = 2e^{r^2} \mathbf{r}$.

**Solution:** If $\quad \mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

Then $\quad r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \quad \text{or} \quad r^2 = x^2 + y^2 + z^2$

Differentiate partially w.r.t. $x$, $y$ and $z$, we get

$$2r \frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r} \qquad \ldots(1)$$

(i) We have $$\nabla f(r) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) f(r)$$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x} f(r) + \hat{\mathbf{j}}\frac{\partial}{\partial y} f(r) + \hat{\mathbf{k}}\frac{\partial}{\partial z} f(r)$$

$$= \hat{\mathbf{i}}\, f'(r)\frac{\partial r}{\partial x} + \hat{\mathbf{j}}\, f'(r)\frac{\partial r}{\partial y} + \hat{\mathbf{k}}\, f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left[\hat{\mathbf{i}}\frac{\partial r}{\partial x} + \hat{\mathbf{j}}\frac{\partial r}{\partial y} + \hat{\mathbf{k}}\frac{\partial r}{\partial z}\right]$$

$$= f'(r)\,\nabla r$$

(ii) We have $$\nabla r = \hat{\mathbf{i}}\frac{\partial r}{\partial x} + \hat{\mathbf{j}}\frac{\partial r}{\partial y} + \hat{\mathbf{k}}\frac{\partial r}{\partial z} \qquad \text{[from (1)]}$$

$$= \hat{\mathrm{i}}\frac{x}{r} + \hat{\mathrm{j}}\frac{y}{r} + \hat{\mathrm{k}}\frac{z}{r} = \frac{x\,\hat{\mathrm{i}} + y\,\hat{\mathrm{j}} + z\,\hat{\mathrm{k}}}{r} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$$

(iii) We have $$\nabla f(r) \times \mathbf{r} = f'(r)\,\nabla r \times \mathbf{r} \qquad \text{[from (i)]}$$

$$= f'(r)\,\frac{\mathbf{r}}{r} \times \mathbf{r} \qquad \text{[from (ii)]}$$

$$= \frac{f'(r)}{r}\{\mathbf{r} \times \mathbf{r}\} \qquad [\because \mathbf{r} \times \mathbf{r} = 0]$$

$$= 0$$

(iv) We have $$\nabla\left(\frac{1}{r}\right) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{\mathbf{j}}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{\mathbf{k}}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{\mathbf{i}}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{\mathbf{j}}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{\mathbf{k}}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left(\frac{\partial r}{\partial x}\hat{\mathbf{i}} + \frac{\partial r}{\partial y}\hat{\mathbf{j}} + \frac{\partial r}{\partial z}\hat{\mathbf{k}}\right)$$

$$= \frac{-1}{r^2}\left(\frac{x}{r}\hat{\mathbf{i}} + \frac{y}{r}\hat{\mathbf{j}} + \frac{z}{r}\hat{\mathbf{k}}\right) \qquad \text{[from (1)]}$$

$$= \frac{-1}{r^3}\left(x\,\hat{\mathrm{i}} + y\,\hat{\mathrm{j}} + z\,\hat{\mathrm{k}}\right) = \frac{-\mathbf{r}}{r^3}$$

(v) We have $$\nabla \log|\mathbf{r}| = \nabla \log r$$

$$= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\log r$$

$$= \hat{\mathbf{i}} \frac{\partial}{\partial x}(\log r) + \hat{\mathbf{j}} \frac{\partial}{\partial y}(\log r) + \hat{\mathbf{k}} \frac{\partial}{\partial z}(\log r)$$

$$= \hat{\mathbf{i}} \frac{1}{r} \frac{\partial r}{\partial x} + \hat{\mathbf{j}} \frac{1}{r} \frac{\partial r}{\partial y} + \hat{\mathbf{k}} \frac{1}{r} \frac{\partial r}{\partial z}$$

$$= \frac{1}{r}\left(\hat{\mathbf{i}} \frac{x}{r} + \hat{\mathbf{j}} \frac{y}{r} + \hat{\mathbf{k}} \frac{z}{r}\right) \qquad \text{[from (1)]}$$

$$= \frac{1}{r^2}(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$$

$$= \frac{\mathbf{r}}{r^2}$$

(vi) We have

$$\nabla r^n = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}\right) r^n$$

$$= \hat{\mathbf{i}} \frac{\partial}{\partial x} r^n + \hat{\mathbf{j}} \frac{\partial}{\partial y} r^n + \hat{\mathbf{k}} \frac{\partial}{\partial z} r^n$$

$$= \hat{\mathbf{i}}\, n r^{n-1} \frac{\partial r}{\partial x} + \hat{\mathbf{j}}\, n r^{n-1} \frac{\partial r}{\partial y} + \hat{\mathbf{k}}\, n r^{n-1} \frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left(\hat{\mathbf{i}} \frac{\partial r}{\partial x} + \hat{\mathbf{j}} \frac{\partial r}{\partial y} + \hat{\mathbf{k}} \frac{\partial r}{\partial z}\right)$$

$$= n r^{n-1}\left(\hat{\mathbf{i}} \frac{x}{r} + \hat{\mathbf{j}} \frac{y}{r} + \hat{\mathbf{k}} \frac{z}{r}\right) \qquad \text{[from (1)]}$$

$$= n r^{n-1}\left(\frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{r}\right)$$

$$= n r^{n-1}\left(\frac{\mathbf{r}}{r}\right) = n r^{n-2}\,\mathbf{r}$$

(vii) We have

$$\nabla |\mathbf{r}|^2 = \nabla r^2 \quad (r^2 = x^2 + y^2 + z^2)$$

$$= \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}\right) r^2$$

$$= \hat{\mathbf{i}} \frac{\partial}{\partial x}(r^2) + \hat{\mathbf{j}} \frac{\partial}{\partial y}(r^2) + \hat{\mathbf{k}} \frac{\partial}{\partial z}(r^2)$$

$$= 2\,\hat{\mathbf{i}}\, r \frac{\partial r}{\partial x} + 2\,\hat{\mathbf{j}}\, r \frac{\partial r}{\partial y} + 2\,\hat{\mathbf{k}}\, r \frac{\partial r}{\partial z}$$

$$= 2r\left[\hat{\mathbf{i}} \frac{x}{r} + \hat{\mathbf{j}} \frac{y}{r} + \hat{\mathbf{k}} \frac{z}{r}\right]$$

$$= 2r \frac{(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})}{r} = 2\mathbf{r}$$

(viii) We have grad $e^{(x^2+y^2+z^2)} = \nabla e^{r^2}$

$$= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) e^{r^2}$$

$$= \hat{\mathbf{i}}\frac{\partial}{\partial x}(e^{r^2}) + \hat{\mathbf{j}}\frac{\partial}{\partial y}(e^{r^2}) + \hat{\mathbf{k}}\frac{\partial}{\partial z}(e^{r^2})$$

$$= \hat{\mathbf{i}}\, e^{r^2}\, 2r\frac{\partial r}{\partial x} + \hat{\mathbf{j}}\, e^{r^2}\, 2r\frac{\partial r}{\partial y} + \hat{\mathbf{k}}\, e^{r^2}\, 2r\frac{\partial r}{\partial z}$$

$$= 2\, r\, e^{r^2}\left[\hat{\mathbf{i}}\frac{\partial r}{\partial x} + \hat{\mathbf{j}}\frac{\partial r}{\partial y} + \hat{\mathbf{k}}\frac{\partial r}{\partial z}\right]$$

$$= 2\, r\, e^{r^2}\left[\hat{\mathbf{i}}\frac{x}{r} + \hat{\mathbf{j}}\frac{y}{r} + \hat{\mathbf{k}}\frac{z}{r}\right] \qquad \text{[from (1)]}$$

$$= 2\, r\, e^{r^2}\frac{(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})}{r} = 2\, e^{r^2}\,\mathbf{r}$$

***Example 3:*** ***If*** $\nabla u = 2r^4\,\mathbf{r}$***, find u.***

***Solution:*** Here, $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

$\Rightarrow$ $|\mathbf{r}| = r = \sqrt{x^2+y^2+z^2}$ or $r^2 = x^2+y^2+z^2$

$$\nabla u = \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}}$$

Then, from $\nabla u = 2r^4\mathbf{r}$

$\Rightarrow$ $$\frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}} = 2\,(x^2+y^2+z^2)^2\,(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$$

Comparing the coefficient of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ on both sides, we get

$$\frac{\partial u}{\partial x} = 2x\,(x^2+y^2+z^2)^2,\ \frac{\partial u}{\partial y} = 2\,y\,(x^2+y^2+z^2)^2,\ \frac{\partial u}{\partial z} = 2z\,(x^2+y^2+z^2)^2$$

From total differentiation

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$= 2\,(x^2+y^2+z^2)^2\,(x\,dx + y\,dy + z\,dz)$$

Let $x^2+y^2+z^2 = t \;\Rightarrow\; 2x\,dx + 2\,y\,dy + 2z\,dz = dt$

$\Rightarrow$ $du = t^2\,dt$

Integrating both sides, we get

$$u = \frac{t^3}{3} + C = \frac{1}{3}(x^2+y^2+z^2)^3 + C$$

**Example 4:** *If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$, prove that $(\text{grad } u).[(\text{grad } v)\times(\text{grad } w)] = \mathbf{0}$.*

*or*

*If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$, prove that grad u, grad v and grad w are coplanar.* **[B.C.A. (Rohilkhand) 2006]**

**Solution:** Here,

$$\text{grad } u = \nabla u = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x + y + z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\text{grad } v = \nabla v = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$$

$$= 2x\,\hat{\mathrm{i}} + 2y\,\hat{\mathrm{j}} + 2z\,\hat{\mathrm{k}}$$

$$\text{grad } w = \nabla w = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(xy + yz + zx)$$

$$= \hat{\mathbf{i}}\,(z + y) + \hat{\mathbf{j}}\,(z + x) + \hat{\mathbf{k}}\,(x + y)$$

$\therefore$ $(\text{grad } u).[\text{grad } v \times \text{grad } w] = [\text{grad } u, \text{grad } v, \text{grad } w]$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} \quad \text{Applying } R_2 = R_2 + R_3$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix} \text{taking common } (x+y+z) \text{ from } R_2$$

$$= 2\,(x + y + z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 0$$

**Example 5:** *Show that $\nabla(a\,.\,r) = a$, where $r = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and a is a constant vector.*

**Solution:** Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$

$$\therefore \qquad \mathbf{a}\,.\,\nabla = (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}).\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)$$

$$= \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)$$

Therefore, $(\mathbf{a}\,.\,\nabla)\,\mathbf{r} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}})$

$$= a_1\hat{\mathrm{i}} + a_2\hat{\mathrm{j}} + a_3\hat{\mathrm{k}} = \mathbf{a}$$

**Example 6:** ***If*** $\phi(x, y) = \log[(x^2 + y^2)]^{1/2}$***, show that*** **[B.C.A. (Kanpur) 2007]**

$$grad\ \phi = \frac{r - (\hat{\mathbf{k}}.\hat{\mathbf{r}})\hat{\mathbf{k}}}{\{r - (\hat{\mathbf{k}}.r)\hat{\mathbf{k}}\}\{r - (\hat{\mathbf{k}}.r)\hat{\mathbf{k}}\}}$$

***where*** r ***and*** $\hat{k}$ ***have usual meanings***.

**Solution:** grad $\phi = \nabla\phi$

$$= \nabla \log(x^2 + y^2)^{1/2} = \frac{1}{2}\nabla \log(x^2 + y^2)$$

$$= \frac{1}{2}\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\log(x^2 + y^2)$$

$$= \frac{1}{2(x^2 + y^2)}(2x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}) = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}}{x^2 + y^2}$$

$$= \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} - z\hat{\mathbf{k}}}{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}).(x\hat{\mathbf{i}} + y\hat{\mathbf{j}})} = \frac{\mathbf{r} - z\hat{\mathbf{k}}}{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} - z\hat{k}).(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} - z\hat{\mathbf{k}})}$$

$$(\because x\,i + y\,j + z\,k = r \Rightarrow k.r = z)$$

$$= \frac{\mathbf{r} - (\hat{\mathbf{k}}.\hat{\mathbf{r}})\hat{\mathbf{k}}}{\{\mathbf{r} - (\hat{\mathbf{k}}.\mathbf{r})\hat{\mathbf{k}}\}.\{\mathbf{r} - (\hat{\mathbf{k}}.\mathbf{r})\hat{\mathbf{k}}\}}$$

## 7.11 Directional Derivative

The rate of change of a scalar function $\phi$ at any point $P$ in any fixed direction **a** is called the **directional derivative** of $\phi$ at $P$ in the direction **a** and is denoted by $\frac{d\phi}{ds}$.

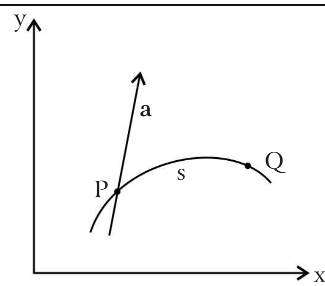


**Fig. 7.4: Directional derivative**

From advance calculus, we have

$$\nabla\phi = \frac{\partial\phi}{\partial x}\Delta x + \frac{\partial\phi}{\partial y}\Delta y + \frac{\partial\phi}{\partial z}\Delta z + \text{terms of higher}$$

powers of $\Delta x, \Delta y, \Delta z$

Then, $$\lim_{\Delta s\to 0}\frac{\Delta\phi}{\Delta s} = \lim_{\Delta s\to 0}\left(\frac{\partial\phi}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial\phi}{\partial y}\frac{\Delta y}{\Delta s} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial s}\right)$$

or $$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}$$

$$= \left(\hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z}\right).\left(\hat{\mathbf{i}}\frac{dx}{ds} + \hat{\mathbf{j}}\frac{dy}{ds} + \hat{\mathbf{k}}\frac{dz}{ds}\right)$$

$$= \nabla\phi \cdot \frac{dr}{ds} \qquad [\because r = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \Rightarrow \frac{dr}{ds} = \frac{dx}{ds}\hat{\mathbf{i}} + \frac{dy}{ds}\hat{\mathbf{j}} + \frac{dz}{ds}\hat{\mathbf{k}}]$$

The term $\nabla\phi$ represents the rate of change of $\phi$. The quantity $\frac{dr}{ds}$ is a unit vector in the direction $d\mathbf{r}$.

Hence, $\frac{d\phi}{ds}$ is the rate of change of $\phi$ in the direction of $\frac{d\mathbf{r}}{ds}$.

Thus, the directional derivative of $\phi$ in the direction **a**

$$= \nabla\phi \cdot \frac{a}{|\mathbf{a}|} \qquad \text{or} \qquad \nabla\phi \cdot \hat{\mathbf{a}}$$

**Theorem 2: The maximum value of the directional derivative of a scalar field $\phi$ is $|\nabla\phi|$, in the direction of $\nabla\phi$.**

**Proof:** We know the directional derivative of $\phi$ along the unit vector **a** is

$$\nabla\phi \cdot \mathbf{a} = |\nabla\phi||\mathbf{a}|\cos\theta \qquad \text{...(1)}$$

where $\theta$ is the angle between $\nabla\phi$ and **a**.

The maximum value of $\cos\theta$ is 1, when $\theta = 0$. Therefore, (1) will be maximum, when $\theta = 0$ *i.e.*, $\nabla\phi$ and **a** have same direction.

$$\therefore \qquad \nabla\phi \cdot \mathbf{a} = |\nabla\phi| \qquad [\text{from (1) } \cos 0 = 1, |\mathbf{a}| = 1]$$

The maximum value of $\nabla\phi \cdot \mathbf{a}$ is $|\nabla\phi|$.

## 7.11.1 Conservative Vector Field

A vector field $F$ is said to be conservative if $F$ can be expressed as the gradient of a scalar field $\phi$ *i.e.*, $F = \nabla\phi$ and $\phi$ is called the **scalar potential.**

## 7.11.2 Gradient in Co-ordinates

Let $r = f(\theta)$ be the curve. $P(r, \theta)$ be point on curve, where $\mathbf{r} = OP$ be the radius vector $\theta = \angle POX$. $O$ is pole, let $\hat{e}_r, \hat{e}_\theta$ be unit vectors along and perpendicular to radius vector **r**. If $s$ be the distance in the direction of **r**, then $ds = dr$.

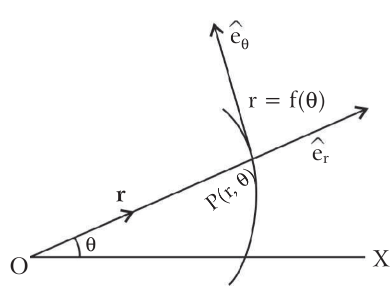


Fig. 7.5

The directional derivative of surface $\phi$ along $\hat{e}_r$

Thus, $$\frac{\partial\phi}{\partial r} = \nabla\phi \cdot \hat{e}_r \qquad \text{...(1)}$$

If we take distances in the direction perpendicular to **r**, then $ds = r\,d\theta$.

Also directional derivative of surface $\phi(r,\theta)$ along $\hat{\mathbf{e}}_\theta$

$$= \nabla\phi \, . \hat{\mathbf{e}}_\theta$$

Thus, $$\frac{\partial \phi}{r\,\partial \theta} = \nabla \phi \, . \hat{\mathbf{e}}_\theta \qquad ...(2)$$

Again, components of $\nabla\phi$ along and perpendicular to $\mathbf{r}$ are $\nabla\phi \, . \, \hat{\mathrm{e}}_r$ and $\nabla\phi \, . \, \hat{\mathrm{e}}_\theta$ respectively

Therefore, $$\nabla\phi = (\hat{\mathrm{e}}_r \, . \nabla\phi)\, \hat{\mathrm{e}}_r + (\hat{\mathrm{e}}_\theta \, . \nabla\phi)\, \hat{\mathrm{e}}_\theta$$

or $$\nabla\phi = \frac{\partial \phi}{\partial r}\hat{\mathrm{e}}_r + \frac{\partial \phi}{r\,\partial \theta}\hat{\mathrm{e}}_\theta \qquad ...(3)$$

[from (1) and (2)]

This is the expression of gradient in polar co-ordinates.

## 7.12 Gradient of Constant Vector

If $\phi$ is constant, then $$\frac{\partial \phi}{\partial x} = 0, \frac{\partial \phi}{\partial y} = 0, \frac{\partial \phi}{\partial z} = 0$$

$$\text{grad } \phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}}$$

$$= 0 \, . \, \hat{\mathbf{i}} + 0 \, . \, \hat{\mathbf{j}} + 0 \, . \, \hat{\mathbf{k}}$$

$$= 0$$

Thus, $\nabla\phi = 0 \Leftrightarrow$ function is constant.

***Example 7:*** ***Find the unit normal*** $\hat{\mathrm{n}}$ ***of the cone of revolution*** $z^2 = 4(x^2 + y^2)$ ***at the point*** $\boldsymbol{P(1,0,2)}$

***Solution:*** Let $$f = z^2 - 4x^2 - 4y^2 = 0 \qquad ...(1)$$

From (1) $$\frac{\partial f}{\partial x} = -8x, \ \frac{\partial f}{\partial y} = -8y, \ \frac{\partial f}{\partial z} = 2z$$

$\therefore$ $$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

$$= -8x\,\hat{\mathbf{i}} - 8y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}} \text{ at } (1,0,2)$$

$$\nabla f = -8\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}$$

$\Rightarrow$ $$|\nabla f| = \sqrt{(-8)^2 + (0)^2 + (4)^2} = \sqrt{64 + 0 + 16} = \sqrt{80}$$

The vector normal to surface $= \nabla f$

$$= -8\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{k}}$$

The unit vector normal to surface $= \pm \dfrac{\nabla f}{|\nabla f|}$

$$= \pm \frac{-8\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{k}}}{\sqrt{80}} = \pm \frac{-8\,\hat{\mathrm{i}} + 4\,\hat{\mathrm{k}}}{4\sqrt{5}} = \pm \frac{(-2\,\hat{\mathrm{i}} + \hat{\mathrm{k}})}{\sqrt{5}}$$

$$= \frac{2\,\hat{\mathrm{i}} - \hat{\mathrm{k}}}{\sqrt{5}}$$

***Example 8:*** ***Find the equations of the tangent plane and normal plane to the surface*** $2xz^2 - 3xy - 4x = 7$ ***at the point*** $(1, -1, 2)$. **[B.C.A. (Bundelkhand) 2011,07]**

***Solution:*** Let $$f = 2xz^2 - 3xy - 4x - 7 = 0 \quad \ldots(1)$$

Be given surface

Find $$\frac{\partial f}{\partial x} = 2z^2 - 3y - 4, \; \frac{\partial f}{\partial y} = -3x, \; \frac{\partial f}{\partial z} = 4xz$$

At $(1, -1, 2)$

$$\frac{\partial f}{\partial x} = 7, \; \frac{\partial f}{\partial y} = -3, \; \frac{\partial f}{\partial z} = 8$$

The equation of tangent plane to the surface $f(1)$ at which $(x_1, y_1, z_1)$ is

$$(x - x_1)\frac{\partial f}{\partial x} + (y - y_1)\frac{\partial f}{\partial y} + (z - z_1)\frac{\partial f}{\partial z} = 0$$

Then at $(1, -1, 2)$

$$(x - 1)(7) + (y + 1)(-3) + (z - 2)(8) = 0$$

$$7x - 3y + 8z - 7 - 3 - 16 = 0$$

or $$7x - 3y + 8z = 26$$

The equation of normal plane at $(x_1, y_1, z_1)$ is

$$\frac{x - x_1}{\frac{\partial f}{\partial x}} = \frac{y - y_1}{\frac{\partial f}{\partial y}} = \frac{z - z_1}{\frac{\partial f}{\partial z}}$$

$\therefore$ $$\frac{x - 1}{7} = \frac{y + 1}{-3} = \frac{z - 2}{8}$$

***Example 9:*** ***Find the angle between the surface*** $x^2 + y^2 + z^2 = 9$ ***and*** $z = x^2 + y^2 - 3$ ***at the point*** $(2, -1, 2)$.

***Solution:*** Let $$f_1 = x^2 + y^2 + z^2 - 9 = 0 \quad \ldots(1)$$

and $$f_2 = x^2 + y^2 - z - 3 = 0 \qquad \text{...(2)}$$

We know the angle between two surfaces at a point is the angle between their normals at that point.

Let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be normals to the surface (1) and (2), then $\boldsymbol{n}_1 = \nabla f_1$ and $\boldsymbol{n}_2 = \nabla f_2$

where $$\nabla f_1 = \frac{\partial f_1}{\partial x}\hat{\mathbf{i}} + \frac{\partial f_1}{\partial y}\hat{\mathbf{j}} + \frac{\partial f_1}{\partial z}\hat{\mathbf{k}}$$

$$= 2x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}} \qquad \text{[from (1)]}$$

and $$\nabla f_2 = \frac{\partial f_2}{\partial x}\hat{\mathbf{i}} + \frac{\partial f_2}{\partial y}\hat{\mathbf{j}} + \frac{\partial f_2}{\partial z}\hat{\mathbf{k}}$$

$$= 2x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} - \hat{\mathbf{k}} \qquad \text{[from (2)]}$$

At the point $(2, -1, 2)$

$$\mathbf{n}_1 = (\nabla f_1)_{(2,-1,2)} = 4\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}$$

and $$\mathbf{n}_2 = (\nabla f_2)_{(2,-1,2)} = 4\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

Let $\theta$ be the angle between $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, then

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\boldsymbol{n}_1||\boldsymbol{n}_2|\cos\theta$$

or $$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{4\times 4 + (-2)(-2) + 4(-1)}{\sqrt{16+4+16}\,\sqrt{16+4+1}}$$

$$= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

***Example 10:*** ***Find the directional derivative of*** $f(x, y, z) = x^2yz + 4xz^2$ ***at the point*** $(1, -2, -1)$ ***in the direction of the vector*** $2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}$. **[B.C.A. (Kurukshetra) 2012,10]**

***Solution:*** Here, $$f(x, y, z) = x^2yz + 4xz^2 \qquad \text{...(1)}$$

Find $$\frac{\partial f}{\partial x} = 2yzx + 4z^2,\ \frac{\partial f}{\partial y} = x^2z,\ \frac{\partial f}{\partial z} = x^2y + 8xz$$

$\therefore$ $$\text{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

$\Rightarrow$ $$\nabla f = (2yzx + 4z^2)\,\hat{\mathrm{i}} + (x^2z)\,\hat{\mathrm{j}} + (x^2y + 8xz)\,\hat{\mathrm{k}}$$

At $(1,-2,-1) \Rightarrow$ $\nabla f = 8\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 10\,\hat{\mathbf{k}}$

Let $\mathbf{a} = 2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}$, Then $\hat{\mathbf{a}} = \dfrac{a}{|\mathbf{a}|}$

$$\Rightarrow \quad \hat{\mathbf{a}} = \frac{2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}}{\sqrt{9}}$$

$$\Rightarrow \quad \hat{\mathbf{a}} = \frac{2\,\hat{\mathrm{i}} - \hat{\mathrm{j}} - 2\,\hat{\mathrm{k}}}{3}$$

The directional derivative of $f$ in the direction of $\mathbf{a}$ is

$$\frac{df}{ds} = \nabla f \,.\, \hat{\mathrm{a}} = \nabla f \,.\, \frac{\vec{\mathrm{a}}}{|\mathbf{a}|}$$

$$= (8\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - 10\,\hat{\mathbf{k}}) \,.\, \frac{(2\,\hat{\mathrm{i}} - \hat{\mathrm{j}} - 2\,\hat{\mathrm{k}})}{3}$$

$$= \frac{16 + 1 + 20}{3} = \frac{37}{3}$$

**Example 11:** ***Find the greatest rate of increase of*** $\phi = xyz^2$ ***at the point*** $(1, 0, 3)$.

**Solution:** We have $\phi = xyz^2$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = yz^2, \; \frac{\partial \phi}{\partial y} = xz^2, \; \frac{\partial \phi}{\partial z} = 2xyz$$

Now $$\nabla \phi = \hat{\mathbf{i}} \frac{\partial \phi}{\partial x} + \hat{\mathbf{j}} \frac{\partial \phi}{\partial y} + \hat{\mathbf{k}} \frac{\partial \phi}{\partial z}$$

$$= \hat{\mathbf{i}}\, y z^2 + \hat{\mathbf{j}}\, x z^2 + \hat{\mathbf{k}}\,(2xyz)$$

$\Rightarrow$ At $(1,0,3)$ $\nabla \phi = 0\,\hat{\mathbf{i}} + 9\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$

We know the greatest rate of increase of $\phi$

$$= |\nabla \phi| \text{ at } (1,0,3)$$

$$= \sqrt{(0)^2 + (9)^2 + (0)^2} = 9.$$

**Example 12:** ***Find the value of*** $a$ **and** $b$ ***such that the surface*** $ax^2 - byz = (a+2)x$ ***and*** $4x^2 y + z^3 = 4$ ***cut orthogonally at*** $(1, -1, 2)$. **[B.C.A. (Rohtak) 2009; B.C.A. (Avadh) 2003]**

**Solution:** Let $f = ax^2 - b\,z\,y - (a+2)x = 0$ ...(1)

and $g = 4x^2\,y + z^3 - 4 = 0$ ...(2)

Then, vector normal to surface $f$ and $g$ is

$$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} = \{2ax - (a+2)\}\hat{\mathbf{i}} - bz\hat{\mathbf{j}} - by\hat{\mathbf{k}}$$

and $$\nabla g = \frac{\partial g}{\partial x}\hat{\mathbf{i}} + \frac{\partial g}{\partial y}\hat{\mathbf{j}} + \frac{\partial g}{\partial z}\hat{\mathbf{k}} = 8xy\,i + 4x^2 j + 3z^2 k$$

At $(1,-1,2)$ $$\nabla f = (a-2)\hat{\mathbf{i}} - 2b\hat{\mathbf{j}} + b\hat{\mathbf{k}}$$

and $$\nabla g = -8\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$$

Since (1) and (2) cut orthogonally, then $\nabla f . \nabla g = 0$

$\Rightarrow$ $$-8(a-2) - 8b + 12b = 0$$

or $$-2a + b + 4 = 0 \qquad ...(3)$$

Also, the point $(1,-1,2)$ lies on (1) and (2), then

$$a + 2b - (a+2) = 0 \quad \text{or} \quad b = 1$$

From (3) $$-2a + 5 = 0 \quad \text{or} \quad a = \frac{5}{2}$$

Therefore, $a = \frac{5}{2},\ b = 1$

***Example 13:*** ***Evaluate the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q has co-ordinates $(5,0,4)$.***

***Solution:*** Here, position vector of $P = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Position vector of $$Q = 5\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

$\therefore$ $$\mathbf{PQ} = \text{p.v. of } Q - \text{ p.v. of } P$$

$$= (5\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

$$= 4\hat{\mathrm{i}} - 2\hat{\mathrm{j}} + \hat{\mathrm{k}} = \mathbf{a} \text{ (let)}$$

If $\hat{\mathrm{a}}$ is the unit vector along $PQ$, then

$$\hat{\mathrm{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{4\hat{\mathrm{i}} - 2\hat{\mathrm{j}} + \hat{\mathrm{k}}}{\sqrt{16+4+1}} = \frac{4\hat{\mathrm{i}} - 2\hat{\mathrm{j}} + \hat{\mathrm{k}}}{\sqrt{21}}$$

Now $$\text{grad } \phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}} = 2x\hat{\mathbf{i}} - 2y\hat{\mathbf{j}} + 4z\hat{\mathbf{k}}$$

At $(1,2,3)$ $$\Rightarrow \nabla\phi = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$$

Then, directional derivative of $\phi$ in the direction of $\mathbf{a}$ is

$$\frac{d\phi}{ds} = \nabla\phi\,.\,\hat{\mathbf{a}} = (2\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 12\,\hat{\mathbf{k}})\,.\,\frac{(4\,\hat{i} - 2\,\hat{j} + \hat{k})}{\sqrt{21}}$$

$$= \frac{2\times 4 + 4\times 2 + 12\times 1}{\sqrt{21}}$$

$$= \frac{28}{\sqrt{21}} = 4\sqrt{\frac{7}{3}}$$

***Example 14:*** ***For $\phi = x^2\ y^3\ z^4$, find the directional derivative of $\phi$ at $(2,3,-1)$ in the direction making equal angles with positive $x$, $y$ and $z$ axes.***

***Solution:*** Let $\hat{\mathbf{a}}$ be the unit vector in the direction making equal angles with positive $x$, $y$ and $z$ axes, then direction cosines of this vector are $\left(\underset{=l}{\cos\alpha}, \underset{=l}{\cos\alpha}, \underset{=l}{\cos\alpha}\right)$.

So, $\mathbf{a} = \cos\alpha\ \hat{\mathbf{i}} + \cos\alpha\ \hat{\mathbf{j}} + \cos\alpha\ \hat{\mathbf{k}}$ and $l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow \quad \cos\alpha = \dfrac{1}{\sqrt{3}}$

Then
$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\cos\alpha\ \hat{\mathbf{i}} + \cos\alpha\ \hat{\mathbf{j}} + \cos\alpha\ \hat{\mathbf{k}}}{\sqrt{\cos^2\alpha + \cos^2\alpha + \cos^2\alpha}}$$

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{3}}\,\hat{\mathbf{i}} + \frac{1}{\sqrt{3}}\,\hat{\mathbf{j}} + \frac{1}{\sqrt{3}}\,\hat{\mathbf{k}} = \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}}$$

Also,
$$\text{grad}\ \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\,\hat{\mathbf{k}}$$

$$= \frac{\partial}{\partial x}(x^2y^3z^4)\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}(x^2y^3z^4)\,\hat{\mathbf{j}} + \frac{\partial}{\partial z}(x^2y^3z^4)\,\hat{\mathbf{k}}$$

$$= 2xy^3z^4\,\hat{i} + 3x^2y^2z^4\,\hat{j} + 4x^2y^3z^3\,\hat{k}$$

At $(2,3,-1)$
$$\nabla\phi = 108\,\hat{\mathbf{i}} + 108\,\hat{\mathbf{j}} - 432\,\hat{\mathbf{k}}$$

Then, directional derivative of $\phi$ in the direction of $\mathbf{a}$ is

$$\frac{d\phi}{ds} = \nabla\phi\,.\,\hat{a}$$

$$= (108\,\hat{i} + 108\,\hat{j} - 432\,\hat{k})\,.\left(\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}\right)$$

$$= \frac{108 + 108 - 432}{\sqrt{3}} = -\frac{216}{\sqrt{3}}$$

***Example 15:*** ***Find the magnitude of directional derivative for the function*** $\phi = \dfrac{y}{x^2 + y^2}$ ***which makes an angle of*** $30°$ ***with positive direction of x-axis at the point*** $(1, 0)$.

***Solution:*** We have $\phi = \dfrac{y}{x^2 + y^2}$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = \frac{-2xy}{(x^2 + y^2)^2}, \frac{\partial \phi}{\partial y} = \frac{x^2 - y^2}{(x^2 + y^2)^2}, \frac{\partial \phi}{\partial z} = 0$$

$$\therefore \quad \text{grad } \phi = \nabla\phi = \hat{\mathbf{i}} \frac{\partial \phi}{\partial x} + \hat{\mathbf{j}} \frac{\partial \phi}{\partial y} + \hat{\mathbf{k}} \frac{\partial \phi}{\partial z}$$

$$= \frac{-2xy}{(x^2 + y^2)^2} \hat{\mathbf{i}} + \frac{x^2 - y^2}{(x^2 + y^2)^2} \hat{\mathbf{j}} + 0 \hat{\mathbf{k}}$$

At $(1,0) \Rightarrow \quad \nabla\phi = 0 \hat{\mathbf{i}} + \hat{\mathbf{j}}$

Also, $\hat{\mathbf{a}}$ = unit vector along the line making an angle of $30°$ with positive direction of $x$-axis at the point $(0, 1)$

$$= \cos 30° \hat{\mathbf{i}} + \sin 30° \hat{\mathbf{j}}$$

Then, directional derivative of $\phi$ in the direction of $\mathbf{a}$ is

$$\frac{d\phi}{ds} = \nabla\phi . \hat{\mathbf{a}} = (0 \hat{\mathbf{i}} + \hat{\mathbf{j}}).(\cos 30° \hat{\mathbf{i}} + \sin 30° \hat{\mathbf{j}})$$

$$= \sin 30° = \frac{1}{2}$$

***Example 16:*** ***Find the directional derivative of*** $f(x, y, z) = x^2 y^2 z^2$ ***at the point*** $(1, 1, -1)$ ***in the direction of the tangent to the curve*** $x = e^t,\ y = 2 \sin t,\ z = t - \cos t$, ***at,*** $t = 0$.

***Solution:*** Here, $f = x^2\ y^2\ z^2$

Then $\quad \dfrac{\partial f}{\partial x} = 2x y^2 z^2,\ \dfrac{\partial f}{\partial y} = 2x^2 yz^2,\ \dfrac{\partial f}{\partial z} = 2x^2 y^2 z$

$$\therefore \quad \text{grad } f = \nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} + \frac{\partial f}{\partial z} \hat{\mathbf{k}}$$

$$= 2xy^2z^2 \hat{\mathbf{i}} + 2x^2 yz^2 \hat{\mathbf{j}} + 2x^2 y^2 z \hat{\mathbf{k}}$$

At $(1, 1, -1)$, $\quad \nabla f = 2 \hat{\mathbf{i}} + 2 \hat{\mathbf{j}} - 2 \hat{\mathbf{k}}$

Now, $x = e^t,\ y = 2 \sin t,\ z = t - \cos t \ \Rightarrow\ \dfrac{dx}{dt} = e^t,\ \dfrac{dy}{dt} = 2 \cos t,\ \dfrac{dz}{dt} = 1 + \sin t$

Position vector $\quad \mathbf{r} = x \hat{\mathbf{i}} + y \hat{\mathbf{j}} + z \hat{\mathbf{k}}$

The tangent vector **t** is given by

$$\mathbf{t} = \frac{dr}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

$$\mathbf{t} = e^t\,\hat{\mathbf{i}} + 2\cos t\,\hat{\mathbf{j}} + (1+\sin t)\,\hat{\mathbf{k}}$$

At, $t = 0$ $\quad \mathbf{t} = \hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$

The unit tangent vector $\quad \hat{t} = \dfrac{\mathbf{t}}{|t|} = \dfrac{\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{1+4+1}} = \dfrac{\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{6}}$

Then, the directional derivative of $f$ at the point $(1, 1, -1)$ in the direction of $t$ is

$$\frac{df}{ds} = \nabla f \,.\, \hat{t} = (2\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} - 2\,\hat{\mathbf{k}})\,.\,\frac{(\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}})}{\sqrt{6}} = \frac{2+4-2}{\sqrt{6}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

***Example 17:*** ***Find the directional derivative of*** $\phi = 5x^2 y - 5y^2 z + \dfrac{5}{2} z^2 x$ ***at the point*** $P(1, 1, 1)$ ***in the direction of the line*** $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1}$.

***Solution:*** Here, $\quad \phi = 5x^2 y - 5y^2 z + \dfrac{5}{2} z^2 x$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x} = 10xy + \frac{5}{2}z^2, \quad \frac{\partial\phi}{\partial y} = 5x^2 - 10\,yz, \quad \frac{\partial\phi}{\partial z} = -5\,y^2 + 5xz$$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}}$$

$$= \left(10xy + \frac{5}{2}z^2\right)\hat{\mathbf{i}} + (5x^2 - 10\,yz)\,\hat{\mathbf{j}} + (-5\,y^2 + 5xz)\,\hat{\mathbf{k}}$$

$$\nabla\phi = \frac{25}{2}\,\hat{i} - 5\,\hat{j} + 0\,\hat{k} \;\text{ at }\; (1, 1, 1)$$

The direction ratios of the line $\dfrac{x-1}{2} = \dfrac{y-3}{-2} = \dfrac{z}{1}$ are $2, -2, 1$.

$\therefore$ Vector **a** in the direction of the given line is

$$\mathbf{a} = 2\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

The unit vector $\quad \hat{\mathbf{a}} = \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{2\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{4+4+1}} = \dfrac{2\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}}{3}$

$\therefore$ Required directional derivative of $\phi$ at $(1, 1, 1)$ in the direction of **a** is

$$\frac{d\phi}{ds} = \nabla\phi\,.\,\hat{\mathbf{a}} = \left(\frac{25}{2}\,\hat{\mathbf{i}} - 5\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}\right).\left(\frac{2\,\hat{\mathbf{i}} - 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}}{3}\right)$$

$$= \frac{25}{3} + \frac{10}{3} + 0 = \frac{35}{3}$$

***Example 18:*** *If the directional derivatives of* $\phi = ax^2y + by^2z + cz^2x$ *at the point* $(1,1,1)$ *has maximum* 15 *in the direction parallel to the line* $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, *find the values of* $a$, $b$ **and** $c$.

***Solution:*** Here, $\phi = ax^2y + by^2z + cz^2x$

Then $\frac{\partial \phi}{\partial x} = 2axy + cz^2,\ \frac{\partial \phi}{\partial y} = 2byz + ax^2,\ \frac{\partial \phi}{\partial z} = 2czx + by^2$

$\therefore$ $$\text{grad}\ \phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}}$$

$$= (2axy + cz^2)\,\hat{\mathbf{i}} + (2byz + ax^2)\,\hat{\mathbf{j}} + (by^2 + 2czx)\,\hat{\mathbf{k}}$$

$$= (2a + c)\,\hat{\mathbf{i}} + (a + 2b)\,\hat{\mathbf{j}} + (2c + b)\,\hat{\mathbf{k}} \qquad ...(1)$$

At $(1,1,1)$

Since the maximum value $= |\nabla\phi| = 15$

$\Rightarrow$ $$\sqrt{(2a+c)^2 + (a+2b)^2 + (2c+b)^2} = 15$$

or $$(2a+c)^2 + (2b+a)^2 + (2c+b)^2 = 225 \qquad ...(2)$$

But, the directional derivative is given to be maximum parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z-0}{1} \qquad ...(3)$$

Then, from (1) and (3)

$$\underset{\text{(i)}}{\frac{2a+c}{2}} = \underset{\text{(ii)}}{\frac{2b+a}{-2}} = \underset{\text{(iii)}}{\frac{2c+b}{1}}$$

From (i) and (ii)

$$2a + c = -2b - a \quad \text{or} \quad 3a + 2b + c = 0 \qquad ...(4)$$

From (ii) and (iii) $2b + a = -2(2c + b)$ or $a + 4b + 4c = 0$ ...(5)

Solve (4) and (5) for $a,b,c$ by cross multiplication

$$\frac{a}{2\times4 - 1\times4} = \frac{b}{1\times1 - 3\times4} = \frac{c}{3\times4 - 1\times2} = k$$

or $$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k$$

or $$a = 4k,\quad b = -11k,\quad c = 10k$$

Put in (2), we find

$$(18k)^2 + (18k)^2 + (9k)^2 = 225 \qquad \text{or} \qquad k = \pm\frac{5}{9}$$

Therefore, $$a = \pm\frac{20}{9},\ b = \pm\frac{55}{9},\ c = \pm\frac{50}{9}$$

***Example 19:*** ***Find the values of constant a, b and c so that the directional derivative of*** $\phi = axy^2 + byz + cz^3x^3$ ***at*** $(1, 2, -1)$ ***has a maximum magnitude 64 in the direction parallel to z-axis.***

***Solution:*** Here, $\phi = axy^2 + byz + cz^3x^3$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = ay^2 + 3cz^3x^2, \frac{\partial \phi}{\partial y} = 2axy + bz, \frac{\partial \phi}{\partial z} = by + 3cz^2x^3$$

Then,
$$\text{grad } \phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{\text{i}} + \frac{\partial \phi}{\partial y}\hat{\text{j}} + \frac{\partial \phi}{\partial z}\hat{\text{k}}$$

$$= (ay^2 + 3cz^3x^2)\,\hat{\text{i}} + (2axy + bz)\,\hat{\text{j}} + (by + 3cz^2x^3)\,\hat{\text{k}}$$

$$= (4a - 3c)\,\hat{\text{i}} + (4a - b)\,\hat{\text{j}} + (2b + 3c)\,\hat{\text{k}} \text{ at } (1, 2, -1)$$

Now, we know that directional derivative is maximum along the normal to the surface *i.e.*, along $\nabla\phi$. But here directional derivative is maximum in the direction parallel to $z$-axis *i.e.*, parallel to $\hat{\mathbf{k}}$.

Hence, the coefficient of $\hat{\text{i}}$ and $\hat{\text{j}}$ in $\nabla\phi$ should be zero and the coefficient of $\hat{\mathbf{k}}$ positive

Therefore, we have

$$4a - 3c = 0 \qquad \ldots(1)$$

$$4a - b = 0 \qquad \ldots(2)$$

and
$$2b + 3c > 0 \quad \text{or} \quad b > \frac{-3}{2}c \qquad \ldots(3)$$

Then,
$$\text{grad } \phi = \nabla\phi = (2b + 3c)\,\hat{\text{k}}$$

Now, max. directional derivative

$$= |\text{grad } \phi| = |\nabla\phi|$$

$$64 = 2b + 3c \qquad \ldots(4)$$

Solving (1), (2), (3) and (4)

$$a = \frac{16}{3}, b = \frac{64}{3}, C = +\frac{64}{9}$$

***Example 20:*** ***Find the directional derivative of*** $\nabla . (\nabla f)$ ***at the point*** $(1, -2, 1)$ ***in the direction of the normal to the surface*** $xy^2z = 3x + z^2$, ***where*** $f = 2x^3y^2z^4$.

**[B.C.A. (Meerut) 2009]**

***Solution:*** We have
$$f = 2x^3y^2z^4 \Rightarrow \frac{\partial f}{\partial x} = 6x^2y^2z^4, \frac{\partial f}{\partial y} = 4x^3yz^4, \frac{\partial f}{\partial z} = 8x^3y^2z^3$$

$$\nabla f = \left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}\right)$$

$\therefore \qquad \nabla f = 6x^2 y^2 z^4\ \hat{\mathbf{i}} + 4x^3 yz^4\ \hat{\mathbf{j}} + 8x^3 y^2 z^3\ \hat{\mathbf{k}}$

$$\Rightarrow \qquad \nabla . (\nabla f) = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right).(6x^2 y^2 z^4\ \hat{\mathbf{i}} + 4x^3 yz^4\ \hat{\mathbf{j}} + 8x^3 y^2 z^3\ \hat{\mathbf{k}})$$

$$= 12xy^2 z^4 + 4x^3 z^4 + 24x^3 y^2 z^2$$

Directional derivative of $\nabla . (\nabla f)$

$$= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(12xy^2 z^4 + 4x^3 z^4 + 24x^3 y^2 z^2)$$

$$= (12y^2 z^4 + 12x^2 z^4 + 72x^2 y^2 z^2)\ \hat{\mathbf{i}} + (24xyz^4 + 48x^3 yz^2)\ \hat{\mathbf{j}}$$

$$+ (48xy^2 z^3 + 16x^3 z^3 + 48x^3 y^2 z)\ \hat{\mathbf{k}}$$

Directional derivative at $(1, -2, 1)$

$$= (48 + 12 + 288)\ \hat{\mathbf{i}} + (-48 - 96)\ \hat{\mathbf{j}} + (192 + 16 + 192)\ \hat{\mathbf{k}}$$

$$= 348\ \hat{\mathrm{i}} - 144\ \hat{\mathrm{j}} + 400\ \hat{\mathrm{k}}$$

Normal to $(xy^2 z - 3x - z^2) = \nabla(xy^2 z - 3x - z^2) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(xy^2 z - 3x - z^2)$

$$= (y^2 z - 3)\ \hat{\mathbf{i}} + 2xyz\ \hat{\mathbf{j}} + (xy^2 - 2z)\ \hat{\mathbf{k}}$$

Normal at $(1, -2, 1)$ $\qquad = \hat{\mathbf{i}} - 4\ \hat{\mathbf{j}} + 2\ \hat{\mathbf{k}}$

Unit normal vector $\qquad = \dfrac{\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{1 + 16 + 4}} = \dfrac{(\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{21}}$

The directional derivative in the direction of normal

$$= (348\ \hat{\mathbf{i}} - 144\ \hat{\mathbf{j}} + 400\ \hat{\mathbf{k}}) . \frac{(\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{21}}$$

$$= \frac{348 + 576 + 800}{\sqrt{21}} = \frac{1724}{\sqrt{21}}$$

***Example 21:*** ***Find the directional derivative of*** $\mathrm{V}^2$***, where*** $\mathrm{V} = xy^2\ \hat{\mathbf{i}} + zy^2\ \hat{\mathbf{j}} + xz^2\ \hat{\mathbf{k}}$ ***at the point*** $(2, 0, 3)$ ***in the direction of the outward normal to sphere*** $x^2 + y^2 + z^2 = 14$ ***at the point*** $(3, 2, 1)$. **[B.C.A. (Agra) 2004]**

***Solution:*** We have $\mathbf{V}^2 = \mathbf{V} . \mathbf{V}$

$$= (xy^2\ \hat{\mathbf{i}} + y^2 z\ \hat{\mathbf{j}} + xz^2\ \hat{\mathbf{k}}) . (xy^2\ \hat{\mathbf{i}} + y^2 z\ \hat{\mathbf{j}} + xz^2\ \hat{\mathbf{k}})$$

$$= x^2 y^4 + z^2 y^4 + x^2 z^4$$

Directional derivative $= \nabla\, \mathbf{V}^2 = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2 y^4 + z^2 y^4 + x^2 z^4)$

$$=(2xy^4+2xz^4)\,\hat{\mathbf{i}}+(4x^2y^3+4y^3z^2)\,\hat{\mathbf{j}}+(2zy^4+4z^3x^2)\,\hat{\mathbf{k}}$$

$$=324\,\hat{\mathbf{i}}+432\,\hat{\mathbf{k}}$$

At (2,0,3)

The vector normal to surface $\phi=\nabla\phi$, where $\phi=x^2+y^2+z^2-14=0$

$$=\left(\frac{\partial\phi}{\partial x}\hat{\mathbf{i}}+\frac{\partial\phi}{\partial y}\hat{\mathbf{j}}+\frac{\partial\phi}{\partial z}\hat{\mathbf{k}}\right)$$

$$=2x\,\hat{\mathbf{i}}+2y\,\hat{\mathbf{j}}+2z\,\hat{\mathbf{k}}$$

$$=6\,\hat{\mathbf{i}}+4\,\hat{\mathbf{j}}+2\,\hat{\mathbf{k}}$$

At (3,2,1) $\quad =\mathbf{a}$ (let)

Unit vector normal to surface $(\hat{a})=\dfrac{\mathbf{a}}{|a|}=\dfrac{6\hat{\mathbf{i}}+4\hat{\mathbf{j}}+2\hat{\mathbf{k}}}{\sqrt{36+16+4}}$

$$=\frac{2\,(3\hat{i}+2\hat{j}+\hat{k})}{2\sqrt{14}}=\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$

∴ Required directional derivative along the normal to surface $\phi=\nabla(\mathbf{V}^2).\hat{a}$

$$=108\,(3\hat{i}+4\hat{k}).\frac{(3\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})}{\sqrt{14}}$$

$$=\frac{108\,(9+4)}{\sqrt{14}}=\frac{1404}{\sqrt{14}}$$

***Example 22:*** ***Find the directional derivative of*** $\dfrac{\mathbf{1}}{r}$ ***in the direction*** r, ***where*** $\mathbf{r}=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}$.

**[B.C.A. (Agra) 2010]**

***Solution:*** Let $\phi=\dfrac{1}{r}$

∴ $$\text{grad}\,\phi=\nabla\phi=\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$=\hat{\mathbf{i}}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\hat{\mathbf{j}}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\hat{\mathbf{k}}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$=-\hat{\mathbf{i}}\frac{1}{r^2}\frac{\partial r}{\partial x}-\frac{\hat{\mathbf{j}}}{r^2}\frac{\partial r}{\partial y}-\frac{\hat{\mathbf{k}}}{r^2}\frac{\partial r}{\partial z}$$

$$=-\frac{1}{r^2}\frac{x}{r}\hat{\mathbf{i}}-\frac{1}{r^2}\frac{y}{r}\hat{\mathbf{j}}-\frac{1}{r^2}\frac{z}{r}\hat{\mathbf{k}}$$

$$=\frac{-(x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}})}{r^3}=\frac{-\mathbf{r}}{r^3}\quad\left[\frac{\partial r}{\partial x}=\frac{x}{r},\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r},r^2=x^2+y^2+z^2\right]$$

Let $\hat{a}$ be the unit vector in the direction of **r**, then

$$\hat{a} = \frac{\mathbf{r}}{|\mathbf{r}|}$$

The directional derivative of $\phi$ in the direction of **r**

$$= \nabla\phi . \hat{a} = -\frac{\mathbf{r}}{r^3} . \frac{\mathbf{r}}{r} = -\frac{r^2}{r^4} = -1/r^2$$

**Example 23:** ***(i) Find the directional derivative of $\frac{1}{r^2}$ in the direction of* r*, where* $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.**

***(ii) Find the directional derivative of $\frac{1}{r^n}$ in the direction of* r *, where* $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.**

**Solution:**

(i) Let $\phi = \frac{1}{r^2} = r^{-2}$

By $\nabla f(r) = f'(r)\frac{\mathbf{r}}{r}$ put $f(r) = r^{-2}, f'(r) = -2r^{-3}$

$$\Rightarrow \quad \nabla\left(\frac{1}{r^2}\right) = \frac{-2}{r^3}\frac{\mathbf{r}}{r} = \frac{-2}{r^4}\mathbf{r}$$

Let $\hat{a}$ be the unit vector in the direction of **r**, then

$$\hat{a} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{r}$$

Therefore, directional derivative of $\phi$ in the direction of **a** is

$$\frac{d\phi}{ds} = \nabla\phi . \hat{\mathbf{a}} = \nabla\left(\frac{1}{r^2}\right) . \frac{\mathbf{r}}{r} = \frac{-2}{r^4}\mathbf{r} . \frac{\mathbf{r}}{r} = \frac{-2r^2}{r^5} = \frac{-2}{r^3}$$

(ii) Let $\phi = \frac{1}{r^n}$, by $\nabla\phi(r) = \phi'(r)\frac{\mathbf{r}}{r}$

$$\Rightarrow \quad \nabla\left(\frac{1}{r^n}\right) = \frac{-n}{r^{n+1}}\frac{\mathbf{r}}{r} = \frac{-n\,\mathbf{r}}{r^{n+2}}$$

Let $\hat{\mathbf{a}}$ be the unit vector in the direction of **r**, then

$$\hat{\mathbf{a}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{r}$$

Therefore, directional derivative of $\phi$ in the direction of **a** is

$$\frac{d\phi}{ds} = \nabla\phi . \hat{\mathbf{a}} = \nabla\left(\frac{1}{r^n}\right) . \frac{\mathbf{r}}{r} = \frac{-n}{r^{n+2}}\mathbf{r} . \frac{\mathbf{r}}{r} = \frac{-n}{r^{n+1}}$$

**Example 24:** ***If* $f = r\cos\theta + \tan\theta$*, find grad* $f$ *in polar co-ordinates.***

**Solution:** We have grad $f$ in polar form

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial f}{r\partial \theta}\hat{\mathbf{e}}_\theta$$

Where $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta$ be unit vector along and perpendicular to radius vector

$\therefore$ $$\nabla f = \cos\theta\ \hat{\mathbf{e}}_r + \frac{1}{r}[-r\sin\theta + \sec^2\theta]\hat{\mathbf{e}}_\theta.$$

## 7.13 Divergence of a Vector Point Function

Let **V** $(x, y, z)$ be a differentiable vector function, where $x$, $y$ and $z$ cartesian co-ordinates. The divergence of **V** is denoted by div (**V**) and is defined as

$$\begin{aligned} \text{div}(\mathbf{V}) &= \nabla.\mathbf{V} \\ &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).V \\ &= \hat{\mathbf{i}}\frac{\partial V}{\partial x} + \hat{\mathbf{j}}\frac{\partial V}{\partial y} + \hat{\mathbf{k}}\frac{\partial V}{\partial z} \end{aligned}$$

Clearly, **the divergence of a vector point function is a scalar point function.**

If $\mathbf{V} = V_1\ \hat{\mathbf{i}} + V_2\ \hat{\mathbf{j}} + V_3\ \hat{\mathbf{k}}$, then

$$\text{div}(\mathbf{V}) = \nabla.\mathbf{V} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).(V_1\ \hat{\mathbf{i}} + V_2\ \hat{\mathbf{j}} + V_3\ \hat{\mathbf{k}})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \qquad \left[\begin{aligned} \because\ \hat{\mathbf{i}}.\hat{\mathbf{i}} = \hat{\mathbf{j}}.\hat{\mathbf{j}} = \hat{\mathbf{k}}.\hat{\mathbf{k}} = 1 \\ \hat{\mathbf{i}}.\hat{\mathbf{j}} = \hat{\mathbf{j}}.\hat{\mathbf{k}} = \hat{\mathbf{k}}.\hat{\mathbf{i}} = 0 \end{aligned}\right.$$

### 7.13.1 Physical Interpretation of Divergence

We consider an example to interpret the meaning of divergence in fluid dynamics. If the region of space is filled with a compressible fluid then, the function $V(x, y, z)$ represents the velocity of a fluid particle at $(x, y, z)$ at any time $(t)$, then divergence of **V** is the rate of out-flow of fluid per unit volume with respect to time.

For incompressible fluid, however div (**V**) $= 0$

**NOTE:**

We remember that

$$\nabla.V \neq V.\nabla$$

Since $V.\nabla$ = scalar differential operator

and $\nabla.V$ = scalar function.

### 7.13.2 Solenoidal Vector

A vector field defined by the vector function $\mathbf{F}(x, y, z)$ is called solenoidal, if div $(\mathbf{F}) = 0$.

### 7.13.3 Divergence of a Constant Vector

**Theorem 3: If A is constant vector, then div (A) = 0.**

**Proof:** Since, **A** is constant vector

$$\Rightarrow \quad \frac{\partial A}{\partial x} = 0, \frac{\partial A}{\partial y} = 0, \frac{\partial A}{\partial z} = 0$$

Then,

$$\begin{aligned}
\text{div}(\mathbf{A}) &= \nabla . \mathbf{A} \\
&= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).(A) \\
&= \hat{\mathbf{i}}\frac{\partial A}{\partial x} + \hat{\mathbf{j}}\frac{\partial A}{\partial y} + \hat{\mathbf{k}}\frac{\partial A}{\partial z} \\
&= \hat{\mathbf{i}}\,0 + \hat{\mathbf{j}}\,0 + \hat{\mathbf{k}}\,0 \\
&= 0
\end{aligned}$$

## 7.14 Curl of a Vector Point Function

Let $\mathbf{V}(x, y, z)$ be a differentiable vector function, where $x, y$ and $z$ cartesian coordinates. **The curl (or rotation)** of **V** is denoted by curl (**V**) and defined as

$$\begin{aligned}
\text{curl } \mathbf{V} &= \nabla \times \mathbf{V} \\
&= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \mathbf{V} \\
&= \hat{\mathbf{i}} \times \frac{\partial \mathbf{V}}{\partial x} + \hat{\mathbf{j}} \times \frac{\partial \mathbf{V}}{\partial y} + \hat{\mathbf{k}} \times \frac{\partial \mathbf{V}}{\partial z}
\end{aligned}$$

Clearly, **the curl of a vector function is a vector point function.**

If $V = V_1\,\hat{\mathbf{i}} + V_2\,\hat{\mathbf{j}} + V_3\,\hat{\mathbf{k}}$, $\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$, then

$$\begin{aligned}
\text{curl } \mathbf{V} &= \nabla \times \mathbf{V} \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} \\
&= \hat{\mathbf{i}}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)
\end{aligned}$$

## 7.14.1 Irrotational Vector Field

**If curl V $= \mathbf{0}$ then,** vector field **V** is called irrotational vector field.

A field which is not irrotational is called a **vortex field.**

**NOTE:**
A vector **F** is conservative if curl $\mathbf{F} = 0$.

## 7.14.2 Physical Interpretation of Curl

Here, we shall interpret curl in the context of a uniform rotating rigid body about an axis.

Let $\omega = \omega_1 \hat{\mathbf{i}} + \omega_2 \hat{\mathbf{j}} + \omega_3 \hat{\mathbf{k}}$ be an angular velocity of a rigid body rotating about fixed point $O$.

The velocity **V** of any point $P(x, y, z)$ on the body is given by $\mathbf{V} = \omega \times \mathbf{r}$, where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ is position vector of $P$.

$$\therefore \qquad V = \omega \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= (\omega_2 z - \omega_3 y)\hat{\mathbf{i}} + (\omega_3 x - \omega_1 z)\hat{\mathbf{j}} + (\omega_1 y - \omega_2 x)\hat{\mathbf{k}}$$

$$\text{and} \qquad \text{curl}(\mathbf{V}) = \nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{\mathbf{i}} + (\omega_2 + \omega_2)\hat{\mathbf{j}} + (\omega_3 + \omega_3)\hat{\mathbf{k}}$$

[$\because \omega_1, \omega_2, \omega_3$ are constants.]

$$= 2(\omega_1 \hat{\mathbf{i}} + \omega_2 \hat{\mathbf{j}} + \omega_3 \hat{\mathbf{k}})$$

$$\Rightarrow \qquad \omega = \frac{1}{2}\,\text{curl}\,\mathbf{V}$$

Thus, **the angular velocity at any point is equal to half the curl of the linear velocity at that point of the body**.

## 7.15 Laplacian Operator

The differential operator $\nabla . \nabla$ or $\nabla^2$ is called **Laplacian Operator** and defined as

$$\nabla^2 = \nabla . \nabla = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) . \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$\nabla . \nabla \phi$ is usually denoted by $\nabla^2 \phi$, a scalar function is called the **Laplacian of $\phi$.** If it is zero *i.e.*, $\nabla^2 \phi = 0$, the equation is called the **Laplace equation.** A scalar function $\phi$ is called **harmonic**, if it satisfies the Laplace equation *i.e.*, is $\nabla^2 \phi = 0$.

## 7.16 Vector Identities

**Theorem 4: Prove the following results in connection with divergence.**

**(i)** $\mathbf{div\ (A + B) = div\ (A) + div\ (B)}$ **[B.C.A. (Meerut) 2001]**

**or**

$\nabla . (\mathbf{A} + \mathbf{B}) = \nabla . \mathbf{A} + \nabla . \mathbf{B}$

**(ii)** $\mathbf{div\ (\phi A) = grad\ \phi . A + \phi\, div\ (A)}$ **[B.C.A. (Lucknow) 2007]**

**or**

$\nabla . (\phi A) = \nabla\phi . \mathbf{A} + \phi\, \nabla . \mathbf{A}$

**(iii)** $\mathbf{div\ (A \times B) = B . curl\, A - A . curl\, B}$ **[B.C.A. (Meerut) 2002]**

**or**

$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) . \mathbf{B} - (\nabla \times \mathbf{B}) . \mathbf{A}$ **[B.C.A. (Kashi) 2010]**

**Proof:** (i) We have

$$\nabla . (\mathbf{A} + \mathbf{B}) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) . (\mathbf{A} + \mathbf{B})$$

$$= \hat{\mathbf{i}} . \frac{\partial}{\partial x}(\mathbf{A} + \mathbf{B}) + \hat{\mathbf{j}} . \frac{\partial}{\partial y}(\mathbf{A} + \mathbf{B}) + \hat{\mathbf{k}} . \frac{\partial}{\partial z}(\mathbf{A} + \mathbf{B})$$

$$= \hat{\mathbf{i}} . \left(\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x}\right) + \hat{\mathbf{j}} . \left(\frac{\partial \mathbf{A}}{\partial y} + \frac{\partial \mathbf{B}}{\partial y}\right) + \hat{\mathbf{k}} . \left(\frac{\partial \mathbf{A}}{\partial z} + \frac{\partial \mathbf{B}}{\partial z}\right)$$

$$= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} . \mathbf{A} + \hat{\mathbf{j}}\frac{\partial}{\partial y} . \mathbf{A} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\mathbf{A}\right) + \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} . \mathbf{B} + \hat{\mathbf{j}}\frac{\partial}{\partial y} . \mathbf{B} + \hat{\mathbf{k}}\frac{\partial}{\partial z} . \mathbf{B}\right)$$

$$= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) . \mathbf{A} + \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) . \mathbf{B}$$

$$= \nabla . \mathbf{A} + \nabla . \mathbf{B}$$

$$\therefore \qquad \nabla . (\mathbf{A} + \mathbf{B}) = \nabla . \mathbf{A} + \nabla . \mathbf{B}$$

(ii) We have

$$\nabla . (\phi \mathbf{A}) = \nabla . (\hat{\mathbf{i}} \phi A_1 + \hat{\mathbf{j}} \phi A_2 + \hat{\mathbf{k}} \phi A_3) \quad (\text{where } \mathbf{A} = A_1 \hat{\mathbf{i}} + A_2 \hat{\mathbf{j}} + A_3 \hat{\mathbf{k}})$$

$$= \left( \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) . (\hat{\mathbf{i}} \phi A_1 + \hat{\mathbf{j}} \phi A_2 + \hat{\mathbf{k}} \phi A_3)$$

$$= \frac{\partial}{\partial x} (\phi A_1) + \frac{\partial}{\partial y} (\phi A_2) + \frac{\partial}{\partial z} (\phi A_3)$$

$$= \left( \frac{\partial \phi}{\partial x} A_1 + \phi \frac{\partial A_1}{\partial x} \right) + \left( \frac{\partial \phi}{\partial y} A_2 + \phi \frac{\partial A_2}{\partial y} \right) + \left( \frac{\partial \phi}{\partial z} A_3 + \phi \frac{\partial A_3}{\partial z} \right)$$

$$= \left( \frac{\partial \phi}{\partial x} A_1 + \frac{\partial \phi}{\partial y} A_2 + \frac{\partial \phi}{\partial z} A_3 \right) + \phi \left( \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right)$$

$$= \left( \hat{\mathbf{i}} \frac{\partial \phi}{\partial x} + \hat{\mathbf{j}} \frac{\partial \phi}{\partial y} + \hat{\mathbf{k}} \frac{\partial \phi}{\partial z} \right) . (A_1 \hat{\mathbf{i}} + A_2 \hat{\mathbf{j}} + A_3 \hat{\mathbf{k}})$$

$$+ \phi \left( \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) . (\hat{\mathbf{i}} A_1 + \hat{\mathbf{j}} A_2 + \hat{\mathbf{k}} A_3)$$

$$= \text{grad } \phi . \mathbf{A} + \phi (\text{div } \mathbf{A})$$

(iii) We have

$$\nabla . (A \times B) = \left( \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) . (A \times B)$$

$$= \hat{\mathbf{i}} . \frac{\partial}{\partial x} (A \times B) + \hat{\mathbf{j}} . \frac{\partial}{\partial y} (A \times B) + \hat{\mathbf{k}} . \frac{\partial}{\partial z} (A \times B)$$

$$= \hat{\mathbf{i}} . \left( \frac{\partial A}{\partial x} \times B \right) + \hat{\mathbf{i}} . \left( A \times \frac{\partial B}{\partial x} \right) + \hat{\mathbf{j}} . \left( \frac{\partial A}{\partial y} \times B \right) + \hat{\mathbf{j}} . \left( A \times \frac{\partial B}{\partial y} \right) + \hat{\mathbf{k}} . \left( \frac{\partial A}{\partial z} \times B + A \times \frac{\partial B}{\partial z} \right)$$

$$\left[ \therefore \sum \hat{\mathbf{i}} \times \frac{\partial \mathbf{A}}{\partial x} = \hat{\mathbf{i}} \times \frac{\partial A}{\partial x} + \hat{\mathbf{j}} \times \frac{\partial A}{\partial y} + \hat{\mathbf{k}} \times \frac{\partial A}{\partial z} \right]$$

$$= \sum \hat{\mathbf{i}} . \left( \frac{\partial A}{\partial x} \times B \right) + \sum \hat{\mathbf{i}} . \left( A \times \frac{\partial B}{\partial x} \right) = \sum \hat{\mathbf{i}} \times \left( \frac{\partial A}{\partial x} \right) . B - \sum \hat{\mathbf{i}} . \left( \frac{\partial B}{\partial x} \times A \right)$$

$$= \sum \left( \hat{\mathbf{i}} \times \frac{\partial A}{\partial x} \right) . B - \sum \left( \hat{\mathbf{i}} \times \frac{\partial B}{\partial x} \right) . A \quad (\text{dot and cross interchange})$$

$$= \mathbf{B} . \text{curl } \mathbf{A} - \mathbf{A} \text{ curl } \mathbf{B}$$

**Theorem 5: Prove that the following results connected with curl.**

**(i)** $\mathbf{curl\ (A+B) = curl\,A + curl\,B}$

**or**

$\nabla \times (\mathbf{A}+\mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$ **[B.C.A. (Kashi) 2012]**

**(ii)** $\mathbf{curl\ (\phi A) = (grad\ \phi) \times A + \phi\, curl\, A}$

**or**

$\nabla \times (\phi \mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi\, \nabla \times \mathbf{A}$

**(iii)** $\mathbf{curl\ (A \times B) = (div\, B)A - (div\ A)B + (B\,.\,\nabla)A - (A\,.\,\nabla)B}$

**or**

$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla\,.\,\mathbf{B})\mathbf{A} - (\nabla\,.\,\mathbf{A})\mathbf{B} + (\mathbf{B}\,.\,\nabla)\mathbf{A} - (\mathbf{A}\,.\,\nabla)\mathbf{B}$ **[B.C.A. (Rohilkhand) 2011]**

**Proof:** (i) We have

$$\text{curl}\,(\mathbf{A}+\mathbf{B}) = \sum \left(\hat{\mathbf{i}} \times \frac{\partial}{\partial x}\right)(\mathbf{A}+\mathbf{B}) \qquad \left[\because \text{curl}\, A = \sum \left(\hat{\mathbf{i}} \times \frac{\partial}{\partial x}\right) A\right]$$

$$= \sum \hat{\mathbf{i}} \times \left(\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x}\right)$$

$$= \sum \hat{\mathbf{i}} \times \frac{\partial \mathbf{A}}{\partial x} + \sum \hat{\mathbf{i}} \times \frac{\partial \mathbf{B}}{\partial x}$$

$$= \text{curl}\,\mathbf{A} + \text{curl}\,\mathbf{B}$$

(ii) We have

$$\text{curl}\,(\phi\mathbf{A}) = \sum \hat{\mathbf{i}} \times \frac{\partial}{\partial x}(\phi\mathbf{A})$$

$$= \sum \hat{\mathbf{i}} \times \left(\frac{\partial \phi}{\partial x}\mathbf{A} + \phi\frac{\partial \mathbf{A}}{\partial x}\right)$$

$$= \sum \left(\hat{\mathbf{i}}\frac{\partial \phi}{\partial x}\right) \times \mathbf{A} + \phi \sum \hat{\mathbf{i}} \times \frac{\partial \mathbf{A}}{\partial x}$$

$$= \text{grad}\,\phi \times \mathbf{A} + \phi\ \text{curl}\,\mathbf{A}$$

(iii) We have

$$\text{curl}\,(A \times B) = \sum \hat{\mathbf{i}} \times \frac{\partial}{\partial x}(\mathbf{A} \times \mathbf{B}) = \sum \hat{\mathbf{i}} \times \left(\frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B} + \mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x}\right)$$

$$= \sum (\hat{\mathbf{i}}\,.\mathbf{B})\frac{\partial \mathbf{A}}{\partial x} - \sum \left(\hat{\mathbf{i}}\,.\frac{\partial \mathbf{A}}{\partial x}\right)\mathbf{B} + \sum \left(\hat{\mathbf{i}}\,.\frac{\partial \mathbf{B}}{\partial x}\right)\mathbf{A} - \sum (\hat{\mathbf{i}}\,.\mathbf{A})\frac{\partial \mathbf{B}}{\partial x}$$

$$= \sum \left(\mathbf{B}\,.\,\hat{\mathbf{i}}\frac{\partial}{\partial x}\right)\mathbf{A} - \mathbf{B}\sum \hat{\mathbf{i}}\,.\frac{\partial \mathbf{A}}{\partial x} + \mathbf{A}\sum \hat{\mathbf{i}}\,.\frac{\partial \mathbf{B}}{\partial x} - \sum \left(\mathbf{A}\,.\,\hat{\mathbf{i}}\frac{\partial}{\partial x}\right)\mathbf{B}$$

$$= (\mathbf{B}\,.\,\nabla)\,\mathbf{A} - \mathbf{B}\ \text{div}\,(\mathbf{A}) + \mathbf{A}\ \text{div}\,(\mathbf{B}) - (\mathbf{A}\,.\,\nabla)\,\mathbf{B}$$

**Theorem 6: Prove the following:**

**(i)** **curl (curl A) = grad (div A) − Laplacian A** **[B.C.A. (Indore) 2011]**

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla . \mathbf{A}) - \nabla^2 \mathbf{A}$$

**(ii)** **grad (A . B) = A × curl B + B × curl A + (A . ∇)B + (B. ∇)A.**

**or**

$$\nabla(\mathbf{A} . \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} . \nabla)\mathbf{B} + (\mathbf{B} . \nabla)\mathbf{A}.$$

**[B.C.A. (Bhopal) 2012, 08, 06]**

**Proof:** (i) We have

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \times \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \text{ where } \mathbf{A} = A_1 \hat{\mathbf{i}} + A_2 \hat{\mathbf{j}} + A_3 \hat{\mathbf{k}}$$

$$= \nabla \times \left[ \left( \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{\mathbf{i}} + \left( \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \hat{\mathbf{j}} + \left( \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{\mathbf{k}} \right]$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial A_3}{\partial y} - \dfrac{\partial A_2}{\partial z} & \dfrac{\partial A_1}{\partial z} - \dfrac{\partial A_3}{\partial x} & \dfrac{\partial A_2}{\partial x} - \dfrac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \sum \hat{\mathbf{i}} \left[ \frac{\partial}{\partial y} \left( \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) - \frac{\partial}{\partial z} \left( \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \right]$$

$$= \sum \hat{\mathbf{i}} \left[ \left( -\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right) + \left( \frac{\partial^2 A_2}{\partial y \, \partial x} + \frac{\partial^2 A_3}{\partial z \, \partial x} \right) \right]$$

$$= \sum \hat{\mathbf{i}} \left[ \left\{ -\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right\} + \left\{ \frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y \, \partial x} + \frac{\partial^2 A_3}{\partial z \, \partial x} \right\} \right]$$

$$= -\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (A_1 \hat{\mathbf{i}} + A_2 \hat{\mathbf{j}} + A_3 \hat{\mathbf{k}}) + \sum \hat{\mathbf{i}} \frac{\partial}{\partial x} \left( \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right)$$

$$= -\nabla^2 \mathbf{A} + \nabla \left( \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right)$$

$$= -\nabla^2 \mathbf{A} + \nabla(\nabla . \mathbf{A})$$

(ii) We have

$$\text{grad}(\mathbf{A}\,.\,\mathbf{B}) = \nabla(\mathbf{A}\,.\,\mathbf{B})$$

$$= \sum \hat{\mathbf{i}}\,\frac{\partial}{\partial x}(\mathbf{A}\,.\,\mathbf{B})$$

$$= \sum \hat{\mathbf{i}}\left[\frac{\partial \mathbf{A}}{\partial x}.\mathbf{B} + \mathbf{A}\,.\,\frac{\partial \mathbf{B}}{\partial x}\right] \qquad ...(1)$$

But $\mathbf{A} \times \left(\hat{\mathbf{i}} \times \frac{\partial \mathbf{B}}{\partial x}\right) = \mathbf{A}\,.\,\frac{\partial \mathbf{B}}{\partial x}\,\hat{\mathbf{i}} - (\mathbf{A}\,.\,\hat{\mathbf{i}})\,\frac{\partial \mathbf{B}}{\partial x}$

or $\left(\mathbf{A}\,.\,\frac{\partial \mathbf{B}}{\partial x}\right)\hat{\mathbf{i}} = \mathbf{A} \times \left(\hat{\mathbf{i}} \times \frac{\partial \mathbf{B}}{\partial x}\right) + (\mathbf{A}\,.\,\hat{\mathbf{i}})\,\frac{\partial \mathbf{B}}{\partial x}$

$\therefore$

$$\nabla(\mathbf{A}\,.\,\mathbf{B}) = \sum \left[\mathbf{A} \times \left(\hat{\mathbf{i}} \times \frac{\partial \mathbf{B}}{\partial x}\right) + (\mathbf{A}\,.\,\hat{\mathbf{i}})\,\frac{\partial \mathbf{B}}{\partial x} + \mathbf{B} \times \left(\hat{\mathbf{i}} \times \frac{\partial \mathbf{A}}{\partial x}\right) + (\mathbf{B}\,.\,\hat{\mathbf{i}})\,\frac{\partial \mathbf{A}}{\partial x}\right]$$

$$= \sum \left[\mathbf{A} \times \left(\hat{\mathbf{i}}\,\frac{\partial}{\partial x} \times \mathbf{B}\right) + \mathbf{B} \times \left(\hat{\mathbf{i}}\,\frac{\partial}{\partial x}\right)\mathbf{A} + \mathbf{A}\,.\left(\hat{\mathbf{i}}\,\frac{\partial}{\partial x}\right)\mathbf{B} + \mathbf{B}.\left(\hat{\mathbf{i}}\,\frac{\partial}{\partial x}\right)\mathbf{A}\right]$$

$$= \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A}\,.\,\nabla)\,\mathbf{B} + (\mathbf{B}\,.\,\nabla)\,\mathbf{A}$$

$$= \mathbf{A} \times \text{curl}\,\mathbf{B} + \mathbf{B} \times \text{curl}\,\mathbf{A} + (\mathbf{A}\,.\,\nabla)\,\mathbf{B} + (\mathbf{B}\,.\,\nabla)\,\mathbf{A}$$

**Theorem 7: For any scalar function $\phi(x, y, z)$, prove that curl grad $\phi = 0$.**

**[B.C.A. (Meerut) 2010]**

**Proof:** We have

$$\text{curl grad}\,\phi = \nabla \times (\nabla\phi)$$

$$= \nabla \times \left(\hat{\mathbf{i}}\,\frac{\partial \phi}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial \phi}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial \phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y}\right) + \hat{\mathbf{j}}\left(\frac{\partial^2\phi}{\partial z\,\partial x} - \frac{\partial^2\phi}{\partial x\,\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial y\,\partial x}\right)$$

$$= \hat{\mathbf{i}}\,(0) + \hat{\mathbf{j}}\,(0) + \hat{\mathbf{k}}\,(0)$$

$$= 0 + 0 + 0$$

$$= 0$$

***Example 25:*** *Show that if* $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

*(i) div* $(\mathbf{r}) = 3$ **[B.C.A. (Meerut) 2008]**

*(ii) curl* $(\mathbf{r}) = 0$ **[B.C.A. (Meerut) 2008]**

*(iii) div* $(\hat{\mathbf{r}}) = \dfrac{2}{r}$ *(iv) curl* $(\hat{\mathbf{r}}) = 0$.

*or*

*Find div* $(\mathbf{v})$ *and curl* $(\mathbf{v})$ *if* $\mathbf{v} = \dfrac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + z^2}}$.

***Solution:*** (i) We have div $(\mathbf{r}) = \nabla . \mathbf{r} = \left(\hat{\mathbf{i}}\dfrac{\partial}{\partial x} + \hat{\mathbf{j}}\dfrac{\partial}{\partial y} + \hat{\mathbf{k}}\dfrac{\partial}{\partial z}\right) . (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \qquad [\because \hat{\mathbf{i}}.\hat{\mathbf{i}} = \hat{\mathbf{j}}.\hat{\mathbf{j}} = \hat{\mathbf{k}}.\hat{\mathbf{k}} = 1]$$

$$= 1 + 1 + 1 = 3$$

(ii) curl $(\mathbf{r}) = \nabla \times \mathbf{r}$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\mathbf{i}}\left\{\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right\} + \hat{\mathbf{j}}\left\{\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right\} + \hat{\mathbf{k}}\left\{\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right\}$$

$$= \hat{\mathbf{i}}.0 + \hat{\mathbf{j}}.0 + \hat{\mathbf{k}}.0 = 0$$

(iii) div $(\hat{\mathbf{r}}) = \nabla . \hat{\mathbf{r}} = \nabla . \left(\dfrac{\mathbf{r}}{r}\right) = \left(\hat{\mathbf{i}}\dfrac{\partial}{\partial x} + \hat{\mathbf{j}}\dfrac{\partial}{\partial y} + \hat{\mathbf{k}}\dfrac{\partial}{\partial z}\right) . \left(\dfrac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \left(\frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}\right) + \left(\frac{1}{r} - \frac{y}{r^2}\frac{\partial r}{\partial y}\right) + \left(\frac{1}{r} - \frac{z}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{3}{r} - \frac{x}{r^2}\left(\frac{x}{r}\right) - \frac{y}{r^2}\left(\frac{y}{r}\right) - \frac{z}{r^2}\left(\frac{z}{r}\right), \quad \left[r^2 = x^2 + y^2 + z^2 \text{ and } \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}\right]$$

$$= \frac{3}{r} - \frac{x^2 + y^2 + z^2}{r^3} = \frac{3}{r} - \frac{r^2}{r^3}$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

(iv) curl $\hat{\mathbf{r}} = \nabla \times \hat{\mathbf{r}} = \nabla \times \frac{\mathbf{r}}{r} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{r}$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{vmatrix} = \sum \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}\left(\frac{z}{r}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r}\right)\right]$$

$$= \sum \hat{\mathbf{i}}\left[z\frac{\partial}{\partial y}\left(\frac{1}{r}\right) - y\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right] = \sum \hat{\mathbf{i}}\left[\frac{-z}{r^2}\frac{\partial r}{\partial y} + \frac{y}{r^2}\frac{\partial r}{\partial z}\right]$$

$$= \sum \hat{\mathbf{i}}\left[\frac{-yz}{r^3} + \frac{yz}{r^3}\right] = 0$$

**Example 26:** ***(i) If*** $\mathrm{F}(x, y, z) = xz^3\,\hat{\mathrm{i}} - 2x^2yz\,\hat{\mathrm{j}} + 2yz^4\,\hat{\mathrm{k}}$, ***find divergence and curl of*** $\mathrm{F}(x, y, z)$.

**[B.C.A. (Agra) 2011, 08]**

***(ii) Find the divergence and curl of vector field***

$$\mathrm{V}(x, y, z) = x^2y^2\,\hat{\mathrm{i}} + 2xy\,\hat{\mathrm{j}} + (y^2 - xy)\,\hat{\mathrm{k}}.$$

**[B.C.A. (Meerut) 2011, 06]**

***(iii) Find the divergence and curl of the vector function***

$$\mathrm{F}(x, y, z) = e^{xyz}(xy^2\,\hat{\mathrm{i}} + yz^2\,\hat{\mathrm{j}} + zx^2\,\hat{\mathrm{k}})\ \textit{at the point}\ (1, 2, 3).$$

**[B.C.A. (Lucknow) 2008]**

***Solution:*** (i) div $(\mathbf{F}) = \nabla.\mathbf{F} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).(xz^3\,\hat{\mathrm{i}} - 2x^2yz\,\hat{\mathrm{j}} + 2yz^4\,\hat{\mathrm{k}})$

$$= \frac{\partial}{\partial x}(xz^3) - \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(2yz^4)$$

$$= z^3 - 2x^2z + 8yz^3$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{\mathbf{i}}[2z^4 + 2x^2y] - \hat{\mathbf{j}}[-3xz^2] + \hat{\mathbf{k}}[-4xyz]$$

$$= 2(x^2y + z^4)\,\hat{\mathbf{i}} + 3xz^2\,\hat{\mathbf{j}} - 4xyz\,\hat{\mathbf{k}}$$

(ii) div $(\mathbf{V}) = \nabla.\mathbf{V} = \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(y^2 - xy)$

$$= 2xy^2 + 2x$$

$$\text{curl } \mathbf{V} = \nabla \times V = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y^2 & 2xy & y^2 - xy \end{vmatrix}$$

$$= \hat{\mathbf{i}}\,(2y - x) - \hat{\mathbf{j}}\,(-y) + \hat{\mathbf{k}}\,(2y - 2x^2 y)$$

$$= (2y - x)\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + 2y\,(1 - x^2)\,\hat{\mathbf{k}}$$

(iii) div $(\mathbf{F}) = \nabla . F$

$$= \frac{\partial}{\partial x}(e^{xyz}\, x y^2) + \frac{\partial}{\partial y}(e^{xyz}\, yz^2) + \frac{\partial}{\partial z}(e^{xyz}\, z x^2)$$

$$= y^2\,[e^{xyz} + xyz\, e^{xyz}] + z^2\,[e^{xyz} + xyz\, e^{xyz}] + x^2\,[e^{xyz} + xyz\, e^{xyz}]$$

$$= (1 + xyz)\,(x^2 + y^2 + z^2)\, e^{xyz}$$

$$= 98\, e^6 \text{ at } (1,2,3)$$

$$\text{curl } \mathbf{F} = \nabla \times F = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2\, e^{xyz} & yz^2\, e^{xyz} & zx^2\, e^{xyz} \end{vmatrix}$$

$$= \hat{\mathbf{i}}\,\{x^3 z^2\, e^{xyz} - y\,(2e^{xyz}\, z + xyz^2 e^{xyz})\}$$

$$+ \hat{\mathbf{j}}\,\{x^2 y^3\, e^{xyz} - z\,(2x\, e^{xyz} + x^2\, yz\, e^{xyz})\}$$

$$+ \hat{\mathbf{k}}\,\{y^2 z^3\, e^{xyz} - x\,(2y\, e^{xyz} + y^2 z x\, e^{xyz})\}$$

$$= (x^3 z^2 - xy^2 z^2 - 2yz)\, e^{xyz}\, \hat{\mathbf{i}} - (2xz + x^2 yz^2 - x^2 y^3)\, e^{xyz}\, \hat{\mathbf{j}}$$

$$+ (y^2 z^3 - 2xy - x^2 y^2 z)\, e^{xyz}\, \hat{\mathbf{k}}$$

$$= -39\, e^6\, \hat{\mathbf{i}} - 16\, e^6\, \hat{\mathbf{j}} + 92\, e^6\, \hat{\mathbf{k}} \text{ at } (1,2,3)$$

***Example 27:*** ***Show that*** $\nabla^2 f(r) = f''(r) + \dfrac{2}{r} f'(r)$.

**[B.C.A. (Agra) 2010; B.C.A. (Kanpur) 2009; B.C.A. (Meerut) 2005; B.C.A. (Rohilkhand) 2010]**

***Solution:*** We have

$$\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r)$$

$$= \sum \frac{\partial^2}{\partial x^2} f(r) = \sum \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\right\} f(r)$$

$$= \sum \frac{\partial}{\partial x}\left\{f'(r)\frac{\partial r}{\partial x}\right\} = \sum \frac{\partial}{\partial x}\left\{f'(r)\frac{x}{r}\right\}$$

$$= \sum\left[f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r}\frac{\partial}{\partial x}f'(r)\right]$$

$$= \sum\left[f'(r)\left(\frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}\right) + \frac{x}{r}f''(r)\frac{\partial r}{\partial x}\right]$$

$$= \sum\left[f'(r)\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \frac{x^2}{r^2}f''(r)\right]$$

$$= \frac{3f'(r)}{r} - \frac{f'(r)}{r^3}(x^2+y^2+z^2) + \frac{f''(r)}{r^2}(x^2+y^2+z^2)$$

$$= \frac{3f'(r)}{r} - \frac{f'(r)}{r} + f''(r) = f''(r) + \frac{2}{r}f''(r)$$

***Example 28:*** ***If $\nabla^2 f(r) = 0$, show that $f(r) = (c_1/r) + c_2$, where $r^2 = x^2 + y^2 + z^2$ and $c_1, c_2$ are arbitrary constants.***

***Solution:*** We know that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r); \text{ where } r^2 = x^2 + y^2 + z^2$$

Now, we have

$$f''(r) + \frac{2}{r}f'(r) = 0 \quad \text{or} \quad \frac{f''(r)}{f'(r)} = -\frac{2}{r}$$

Integrating with respect to $r$, we get

$$\log f'(r) = -2\log r + \log c, \text{where } c \text{ is a constant}$$

$$= \log\frac{c}{r^2} \quad \Rightarrow \quad f'(r) = \frac{c}{r^2}$$

Again integrating, we get

$$f(r) = -\frac{c}{r} + c_2, \text{ where } c_2 \text{ is a constant}$$

$$= \frac{c_1}{r} + c_2, \text{ replacing } -c \text{ by } c_1.$$

***Example 29:*** ***Show that $\nabla^2(r^n \mathbf{r}) = n(n+3)r^{n-2}\mathbf{r}$.***

***Solution:*** $\nabla^2(r^n \mathbf{r}) = \sum \frac{\partial^2}{\partial x^2}(r^n \mathbf{r}) = \sum \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(r^n \mathbf{r})\right\}$

$$= \sum \frac{\partial}{\partial x}\left\{r^n\frac{\partial \mathbf{r}}{\partial x} + nr^{n-1}\frac{\partial r}{\partial x}\mathbf{r}\right\}$$

$$= \sum \frac{\partial}{\partial x}\left\{r^n\,\hat{\mathbf{i}} + nr^{n-1}\,\frac{x}{r}\,\mathbf{r}\right\} \qquad \left[\because \mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \;\because \frac{\partial\,\mathbf{r}}{\partial x} = \hat{\mathbf{i}}\right]$$

$$= \sum \left[n\,r^{n-1}\,\frac{\partial\,r}{\partial x}\,\hat{\mathbf{i}} + n\,r^{n-2}\;x\,\frac{\partial\,\mathbf{r}}{\partial\,x} + n\left\{r^{n-2} + x\,(n-2)\,r^{n-3}\,\frac{\partial\,r}{\partial\,x}\right\}\mathbf{r}\right]$$

$$= \sum\,[nr^{n-2}\;x\,\hat{\mathbf{i}} + nr^{n-2}\;x\,\hat{\mathbf{i}} + nr^{n-2}\;\mathbf{r} + n\,(n-2)\,r^{n-4}\;x^2\;\mathbf{r}]$$

$$= 2nr^{n-2}\;(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}) + 3nr^{n-2}\mathbf{r} + n\,(n-2)\,r^{n-2}\;\mathbf{r} \quad (\because x^2 + y^2 + z^2 = r^2)$$

$$= 2nr^{n-2}\;\mathbf{r} + 3nr^{n-2}\;\mathbf{r} + n(n-2)\,r^{n-2}\;\mathbf{r}$$

$$= nr^{n-2}\;\mathbf{r}\,(2 + 3 + n - 2) = n\,(n+3)\,r^{n-2}\;\mathbf{r}.$$

**Example 30:** ***Prove that div grad*** $\boldsymbol{r^n = n\,(n+1)\,r^{n-2}}$ ***i.e.,*** $\boldsymbol{\nabla^2 r^n = n\,(n+1)\,r^{n-2}}$.

**[B.C.A. (Meerut) 2008, 06]**

***Solution:*** We have

$$\nabla^2 r^n = \nabla\,.\,(\nabla r^n) = \text{div}\,(\text{grad}\,r^n) = \text{div}\,(nr^{n-1}\;\text{grad}\;r)$$

$$= \text{div}\left(n\,r^{n-1}\,\frac{1}{r}\,\mathbf{r}\right) = \text{div}\,(nr^{n-2}\mathbf{r})$$

$$= (nr^{n-2})\;\text{div}\;\mathbf{r} + \mathbf{r}\,.\,(\text{grad}\,nr^{n-2})$$

$$= 3nr^{n-2} + \mathbf{r}\,.\,[n\,(n-2)\,r^{n-3}\;\text{grad}\,\mathbf{r}]$$

$$= 3nr^{n-2} + r\,.\left[n\,(n-2)\,r^{n-3}\,\frac{1}{r}\,\mathbf{r}\right]$$

$$= 3nr^{n-2} + n\,(n-2)\,r^{n-4}\;r^2$$

$$= nr^{n-2}\;(3 + n - 2) = n\,(n+1)\,r^{n-2}.$$

**Example 31:** ***Prove that*** $\boldsymbol{\nabla^2\left(\frac{1}{r}\right) = 0}$ ***or div*** $\boldsymbol{\left(\text{grad}\,\frac{1}{r}\right) = 0}$.

**[B.C.A. (Lucknow) 2010]**

***Solution:*** Here,

$$\nabla^2\left(\frac{1}{r}\right) = \nabla\,.\left(\nabla\,\frac{1}{r}\right) = \text{div}\left(\text{grad}\,\frac{1}{r}\right)$$

$$= \text{div}\left(-\frac{1}{r^2}\,\text{grad}\;r\right) = \text{div}\left(-\frac{1}{r^2}\,\frac{1}{r}\,\mathbf{r}\right) = \text{div}\left(-\frac{1}{r^3}\,\mathbf{r}\right)$$

$$= \left(-\frac{1}{r^3}\right)\text{div}\;\mathbf{r} + \mathbf{r}\,.\,\text{grad}\left(-\frac{1}{r^3}\right) = -\frac{3}{r^3} + \mathbf{r}\,.\left[\frac{d}{dr}\left(-\frac{1}{r^3}\right)\text{grad}\,r\right]$$

$$= -\frac{3}{r^3} + \mathbf{r}\,.\left(\frac{3}{r^4}\,\frac{1}{r}\,\mathbf{r}\right) = -\frac{3}{r^3} + \frac{3}{r^5}\,(\mathbf{r}\,.\,\mathbf{r}) = -\frac{3}{r^3} + \frac{3}{r^5}\,r^2 = 0.$$

$1/r$ is a solution of Laplace's equation.

**Example 32:** ***Prove that*** $div\left\{\frac{f(r)\mathbf{r}}{r}\right\}=\frac{1}{r^2}\frac{d}{dr}(r^2 f)$.

**Solution:** Here, $\text{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\text{div}\left\{\frac{f(r)}{r}(x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}})\right\}$

$$=\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\}+\frac{\partial}{\partial y}\left\{\frac{f(r)}{r}y\right\}+\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}z\right\}$$

Now $\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\}=\frac{f(r)}{r}+x\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x}$

$$=\frac{f(r)}{r}+x\left\{\frac{f'(r)}{r}-\frac{1}{r^2}f(r)\right\}\frac{x}{r}=\frac{f(r)}{r}+\frac{x^2}{r^2}f'(r)-\frac{x^2}{r^3}f(r).$$

Similarly, $\frac{\partial}{\partial y}\left\{\frac{f(r)}{r}y\right\}=\frac{f(r)}{r}+\frac{y^2}{r^2}f'(r)-\frac{y^2}{r^3}f(r).$

and $\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}z\right\}=\frac{f(r)}{r}+\frac{z^2}{r^2}f'(r)-\frac{z^2}{r^3}f(r).$

Putting these values in (1), we find

$$\text{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\frac{3}{r}f(r)+\frac{r^2}{r^2}f'(r)-\frac{r^2}{r^3}f(r)$$

$$=\frac{2}{r}f(r)+f'(r)=\frac{1}{r^2}[2rf(r)+r^2 f'(r)]=\frac{1}{r^2}\frac{d}{dr}[r^2 f(r)].$$

**Example 33:** ***If r is the distance of a point $(x, y, z)$ from the origin, prove that***

$$curl\ (\mathbf{k}\times\mathbf{grad}\ 1/\mathbf{r}) + grad\ (\mathbf{k}.\mathbf{grad}\ 1/\mathbf{r})=0$$

***where k is the unit vector in the direction of z.***

**Solution:** The distance $r$ from the origin is given by

$$r^2=(x-0)^2+(y-0)^2+(z-0)^2=x^2+y^2+z^2$$

$$\Rightarrow\qquad \frac{1}{r}=(x^2+y^2+z^2)^{-1/2}$$

$$\because\qquad \text{grad}\left(\frac{1}{r}\right)=\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}$$

$$=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x\,\hat{\mathbf{i}}+2y\,\hat{\mathbf{j}}+2z\,\hat{\mathbf{k}})$$

$$=-\frac{x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}$$

Further $\quad \mathbf{k}\times\text{grad}\,\dfrac{1}{r}=\mathbf{k}\times\left[\dfrac{-(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})}{(x^2+y^2+z^2)^{3/2}}\right]=\dfrac{-(x\hat{\mathbf{j}}-y\hat{\mathbf{i}})}{(x^2+y^2+z^2)^{3/2}} \qquad [\because \hat{\mathbf{k}}\times\hat{\mathbf{k}}=\mathbf{0}]$

and curl $(\mathbf{k}\times\text{grad }1/r)=\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{y}{(x^2+y^2+z^2)^{3/2}} & \dfrac{-x}{(x^2+y^2+z^2)^{3/2}} & 0 \end{vmatrix}$

$$=\frac{3}{2}\frac{(-x)(2z)\,\hat{\mathbf{i}}}{(x^2+y^2+z^2)^{5/2}}-\frac{3}{2}\frac{y(2z)\,\hat{\mathbf{j}}}{(x^2+y^2+z^2)^{5/2}}+\left[-\frac{3}{2}\frac{(-x)(2x)}{(x^2+y^2+z^2)^{5/2}}\right.$$

$$\left.-\frac{1}{(x^2+y^2+z^2)^{3/2}}-\frac{3}{2}\frac{y(2y)}{(x^2+y^2+z^2)^{5/2}}-\frac{1}{(x^2+y^2+z^2)^{3/2}}\right]\hat{\mathbf{k}}$$

$$=\frac{-3xz\,\hat{\mathbf{i}}-3yz\,\hat{\mathbf{j}}+(x^2+y^2-2z^2)\,\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{5/2}} \qquad \ldots(1)$$

Now $\quad \mathbf{k}\,.\,\text{grad}\,\dfrac{1}{r}=\mathbf{k}.\left[-\dfrac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}\right]=\dfrac{-z}{(x^2+y^2+z^2)^{3/2}}$

$\therefore\quad$ grad $(\mathbf{k}.\text{grad}1/r)=\left(\hat{\mathbf{i}}\dfrac{\partial}{\partial x}+\hat{\mathbf{j}}\dfrac{\partial}{\partial y}+\hat{\mathbf{k}}\dfrac{\partial}{\partial z}\right)\dfrac{-z}{(x^2+y^2+z^2)^{3/2}}$

$$=\hat{\mathbf{i}}\,\frac{(-z)(-3/2)(2x)}{(x^2+y^2+z^2)^{5/2}}+\hat{\mathbf{j}}\,\frac{(-z)(-3/2)(2y)}{(x^2+y^2+z^2)^{5/2}}$$

$$+\hat{\mathbf{k}}\left[\frac{(-z)(-3/2)(2z)}{(x^2+y^2+z^2)^{5/2}}-\frac{1}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$=\frac{3xz\,\hat{\mathbf{i}}+3yz\,\hat{\mathbf{j}}-(x^2+y^2-2z^2)\,\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{5/2}}$$

***Example 34:*** ***Show that the vector field*** $\mathbf{F}=\dfrac{\boldsymbol{r}}{\boldsymbol{r}^3}$ ***is irrotational as well as solenoidal. Find the scalar potential.***

***Solution:*** Since **F** is irrotational then curl $\mathbf{F}=0$, where $F=\dfrac{\mathbf{r}}{r^3}=\dfrac{x\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}$

then $\quad \text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{(x^2+y^2+z^2)^{3/2}} & \dfrac{y}{(x^2+y^2+z^2)^{3/2}} & \dfrac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$

$$= \hat{\mathbf{i}}\left[-\frac{3}{2}\frac{2yz}{(x^2+y^2+z^2)^{5/2}} - \left(-\frac{3}{2}\right)\frac{2yz}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$+ \hat{\mathbf{j}}\left[-\frac{3}{2}\frac{2xz}{(x^2+y^2+z^2)^{5/2}} + \frac{3}{2}\frac{2xz}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$- \hat{\mathbf{k}}\left[-\frac{3}{2}\frac{2xy}{(x^2+y^2+z^2)^{3/2}} + \frac{3}{2}\frac{2xy}{(x^2+y^2+z^2)^{3/2}}\right] = 0\,\hat{\mathrm{i}} + 0\,\hat{\mathrm{j}} + 0\,\hat{\mathrm{k}} = 0$$

Therefore, $F$ is irrotational.

Now, if $\phi$ is scalar potential, then we have $F = \nabla\phi$

$\Rightarrow \qquad \mathbf{F}\,.\,d\mathbf{r} = \nabla\,\phi\,.\,d\mathbf{r}$ taking dot product with $d\mathbf{r}$

or $\qquad d\phi = F\,.\,dr$

or $\qquad d\phi = \dfrac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}}\,.\,(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}) = \dfrac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$

$$= d\{-(x^2+y^2+z^2)^{-1/2}\}$$

Integrating, we get

$$\phi = \frac{-1}{(x^2+y^2+z^2)^{1/2}}$$

or $\qquad \phi = -\dfrac{1}{r}$ is the required scalar potential.

For vector field $\mathbf{F}$ to be solenoidal div $\mathbf{F} = 0$.

We know $\quad \text{div}\,(\alpha\,\mathbf{a}) = \alpha\,\text{div}\,\mathbf{a} + \mathbf{a}\,\text{grad}\,\alpha$, where $\alpha$ is a scalar.

$\therefore \qquad \text{div}\left(\dfrac{\mathbf{r}}{r^3}\right) = \dfrac{1}{r^3}\,\text{div}\,\mathbf{r} + \mathbf{r}.\text{grad}\left(\dfrac{1}{r^3}\right) = \dfrac{3}{r^3} + \mathbf{r}.\left(-\dfrac{3}{r^4}\dfrac{\mathbf{r}}{r}\right)$

$$= \frac{3}{r^3} - \frac{3}{r^5}.r^2 = 0$$

Hence, vector field $\mathbf{F}$ is solenoidal.

***Example 35: Show that*:**

***(i)*** $\quad$ ***div*** $(r^n\,\mathbf{r}) = (n+3)\,r^n$ **[B.C.A. (Meerut) 2004]**

***(ii)*** $\quad$ ***Curl*** $(r^n\,\mathbf{r}) = \mathbf{0}$ **[B.C.A. (Meerut) 2004]**

***(iii)*** ***If*** $r^n\,\mathbf{r}$ ***is solenoidal, then*** $n + 3 = 0$. **[B.C.A. (Bhopal) 2008, 04, 02]**

**Solution:** We have $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\Rightarrow$ $$|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2} \quad \text{or} \quad r^2 = x^2 + y^2 + z^2$$

Then, $$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) div $(r^n \mathbf{r})$ $$= \nabla . (r^n \mathbf{r})$$

$$= \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}\right) . [r^n (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})]$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) \qquad ...(1)$$

Now, $$\frac{\partial}{\partial x}(r^n x) = r^n + x\, n r^{n-1} \frac{\partial r}{\partial x}$$

$$= r^n + nxr^{r-1} \frac{x}{r}$$

$$= r^n + nx^2\, r^{n-2}$$

Similarly, $$\frac{\partial}{\partial y}(r^n y) = r^n + n . y^2\, r^{n-2}, \frac{\partial}{\partial z}(r^n z) = r^n + nz^2 r^{n-2}$$

Put these values in (1), we get

$$\text{div } (r^n \mathbf{r}) = 3r^n + nr^{n-2}\,(x^2 + y^2 + z^2)$$

$$= 3r^n + nr^{n-2}\,(r^2)$$

$$= 3r^n + nr^n = (n+3)\, r^n$$

(ii) curl $(r^n \mathbf{r})$ $$= \nabla \times (r^n \mathbf{r})$$

$$= \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}\right) \times (r^n x\,\hat{\mathbf{i}} + r^n y\,\hat{\mathbf{j}} + r^n z\,\hat{\mathbf{k}})$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \sum \hat{\mathbf{i}} \left(\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right)$$

$$= \sum \hat{\mathbf{i}} \left(z \frac{\partial}{\partial y}(r^n) - y \frac{\partial}{\partial z}(r^n)\right)$$

$$= \sum \hat{\mathbf{i}} \left(z\, nr^{n-1} \frac{\partial r}{\partial y} - y\, nr^{n-1} \frac{\partial r}{\partial z}\right)$$

$$= \sum nr^{n-1}\, \hat{\mathbf{i}} \left(\frac{yz}{r} - \frac{yz}{r}\right) = 0$$

(iii) If $r^n\,\mathbf{r}$ is solenoidal, then we have

$$\text{div}\,(r^n\,\mathbf{r}) = 0$$

$$\Rightarrow \quad (n+3)\,r^n = 0 \qquad \text{[from (i) part]}$$

$$\Rightarrow \quad n+3 = 0 \text{ as } r \neq 0$$

**Example 36:** *If* $\mathbf{a}$ *be constant vector and* $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$*, show that*

***(i)*** ***div*** $(\mathbf{a} \times \mathbf{r}) = 0$ **[B.C.A. (Meerut) 2003]**

***(ii)*** ***Curl*** $(\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$ **[B.C.A. (Meerut) 2004]**

***(iii)*** $\mathbf{a} \times (\nabla \times \mathbf{r}) = \nabla(\mathbf{a}\,.\,\mathbf{r}) - (\mathbf{a}\,.\,\nabla)\,\mathbf{r}$.

***Solution:*** Let

$$\mathbf{a} = a_1\,\hat{\mathbf{i}} + a_2\,\hat{\mathbf{j}} + a_3\,\hat{\mathbf{k}}$$

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$

Then

$$\mathbf{a} \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{\mathbf{i}}\,(a_2 z - a_3 y) + \hat{\mathbf{j}}\,(a_3 x - a_1 z) + \hat{\mathbf{k}}\,(a_1 y - a_2 x)$$

and

$$\mathbf{a}\,.\,\mathbf{r} = a_1 x + a_2 y + a_3 z$$

(i) div $(\mathbf{a} \times \mathbf{r})$

$$= \nabla\,.\,(\mathbf{a} \times \mathbf{r})$$

$$= \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$= 0 + 0 + 0 = 0$$

(ii) curl $(\mathbf{a} \times \mathbf{r})$

$$= \nabla \times (\mathbf{a} \times \mathbf{r})$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \sum \hat{\mathbf{i}} \left\{ \frac{\partial}{\partial y}(a_1 y - a_2 x) - \frac{\partial}{\partial z}(a_3 x - a_1 z) \right\}$$

$$= \sum \hat{\mathbf{i}} \left\{ a_1 + a_1 \right\}$$

$$= \hat{\mathbf{i}}\,(a_1 + a_1) + \hat{\mathbf{j}}\,(a_2 + a_2) + \hat{\mathbf{k}}\,(a_3 + a_3)$$

$$= 2\,(\hat{\mathbf{i}}\,a_1 + \hat{\mathbf{j}}\,a_2 + \hat{\mathbf{k}}\,a_3) = 2\,\mathbf{a}$$

(iii) L.H.S. $= \mathbf{a} \times (\nabla \times \mathbf{r}) = \mathbf{a} \times \mathbf{0} = \mathbf{0}$ $(\because \text{curl } \mathbf{r} = \nabla \times \mathbf{r} = 0)$

R.H.S. $= \nabla(\mathbf{a} \,.\, \mathbf{r}) - (\mathbf{a} \,.\, \nabla)\,\mathbf{r}$

$$= \nabla(a_1 x + a_2 y + a_3 z) - \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right)\mathbf{r}$$

$$= a_1 i + a_2 j + a_3 k - a_1 i - a_2 j - a_3 k$$

$$= 0$$

L.H.S. = R.H.S.

**Example 37:** ***(i) If*** $\mathbf{F} = (x + 3y)\,\hat{\mathbf{i}} + (y - 2z)\,\hat{\mathbf{j}} + (x + az)\,\hat{\mathbf{k}}$ ***is solenoidal vector, then find the value of*** $a$.

***(ii) Determine the value of constant*** $a, b, c$ ***if***

$\mathbf{f} = (x + 2y + az)\,\hat{\mathbf{i}} + (bx - 3y - z)\,\hat{\mathbf{j}} + (4x + cy + 2z)\,\hat{\mathbf{k}}$ ***is irrotational.***

**Solution:** (i) If $\mathbf{F}$ is solenoidal vector then, div $(\mathbf{F}) = 0$

or $$\nabla \,.\, \mathbf{F} = 0$$

or $$\left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}\right).[(x + 3y)\,\hat{\mathrm{i}} + (y - 2z)\,\hat{\mathrm{j}} + (x + az)\,\hat{\mathrm{k}}] = 0$$

or $$\frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az) = 0$$

or $$1 + 1 + a = 0$$

or $$a = -2$$

(ii) If $\mathbf{f}$ is irrotational, then

$$\nabla \times \mathbf{f} = 0$$

or $$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

or $$\hat{\mathbf{i}}\,(c + 1) + \hat{\mathbf{j}}\,(a - 4) + \hat{\mathbf{k}}\,(b - 2) = 0$$

or $$(c + 1)\,\hat{\mathbf{i}} + (a - 4)\,\hat{\mathbf{j}} + (b - 2)\,\hat{\mathbf{k}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$$

or $$c + 1 = 0,\ a - 4 = 0,\ b - 2 = 0$$

or $$c = -1,\ a = 4,\ b = 2$$

**Example 38:** ***A fluid motion is given by $\mathbf{V} = (y+z)\,\hat{\mathbf{i}} + (z+x)\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}}$. Show that the motion is irrotational and hence find the velocity potential also show that motion is possible for an incompressible fluid.***

**Solution:** The motion of fluid will be irrotational, if curl $\mathbf{V} = 0$ or $\nabla \times \mathbf{V} = 0$

Now
$$\text{curl } \mathbf{V} = \nabla \times V = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] + \hat{\mathbf{j}}\left[\frac{\partial}{\partial z}(y+z) - \frac{\partial}{\partial x}(x+y)\right] + \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{\mathbf{i}}\,[1-1] + \hat{\mathbf{j}}\,[1-1] + \hat{\mathbf{k}}\,[1-1]$$

$$= 0$$

Hence, the motion is irrotational.

**Now to find velocity potential.**

Let $\phi$ be the velocity potential, then the velocity vector $V$ is given by

$$V = \text{grad}\,\phi \text{ or } V = \nabla\phi$$

But
$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}}\right).(dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}})$$

$$= \nabla\phi \,.\, d\mathbf{r} \qquad \text{(But } \nabla\phi = V\text{)}$$

$$= \mathbf{V}\,.\,(d\mathbf{r})$$

$$= [(y+z)\,\hat{\mathbf{i}} + (z+x)\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}}].[dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}]$$

$$= (y+z)dx + (z+x)\,dy + (x+y)\,dz$$

$$= (x\,dy + y\,dx) + (z\,dx + x\,dz) + (z\,dy + y\,dz)$$

$$d\phi = d(xy) + d(zx) + d(zy)$$

Integrating, we get

$$\phi = xy + yz + zx + C$$

which is the required velocity potential.

We know the incompressible is possible, if div $(\mathbf{V}) = 0$ or $\nabla . \mathbf{V} = 0$

Now
$$\nabla . \mathbf{V} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right).((y+z)\,\hat{\mathbf{i}} + (x+z)\,\hat{\mathbf{j}} + (x+y)\,\hat{\mathbf{k}})$$

$$= \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(x+z) + \frac{\partial}{\partial z}(x+y)$$

$$= 0 + 0 + 0 = 0$$

Hence, motion of the incompressible fluid is possible.

***Example 39:*** ***If the vector functions $F$ and $G$ are irrotational, show that $F \times G$ is solenoidal.***

***Solution:*** Since $F$ and $G$ are irrotational, then $\nabla \times \mathbf{F} = \nabla \times \mathbf{G} = 0$

Also, we know $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}).\mathbf{G} - (\nabla \times \mathbf{G}).\mathbf{F}$

$$= 0.\mathbf{G} - 0.\mathbf{F}$$

$$= 0$$

Thus, $\mathbf{F} \times \mathbf{G}$ is solenoidal.

***Example 40:*** ***Prove that $F = (y^2 \cos x + z^3)\hat{\mathbf{i}} + (2y \sin x - 4)\hat{\mathbf{j}} + (3xz^2 + 2)\hat{\mathbf{k}}$ is a conservative force field. Find the scalar potential for F. Also, find the work done in moving a particle in the field from $(0, 1, -1)$ to $(\pi/2, -1, 2)$.***

***Solution:*** If $\mathbf{F}$ is conservative, then curl $\mathbf{F} = 0$ or $\nabla \times \mathbf{F} = 0$ and there exists a scalar function $\phi$ such that

$$F = \text{grad}\,\phi$$

Now, $$\nabla \times F = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(0-0) - \hat{\mathbf{j}}(3z^2 - 3z^2) + \hat{\mathbf{k}}(2y \cos x - 2y \cos x) = 0$$

Hence, $F$ is conservative

Let $$F = \nabla\phi, \text{ where } \nabla\phi = \frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}}$$

Put $F$ and $\nabla\phi$ in (1)

$$(y^2 \cos x + z^3)\hat{\mathbf{i}} + (2y \sin x - 4)\hat{\mathbf{j}} + (3xz^2 + 2)\hat{\mathbf{k}} = \frac{\partial \phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial \phi}{\partial z}\hat{\mathbf{k}}$$

Equating the coefficient of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ on both sides, we get

$$\frac{\partial \phi}{\partial x} = y^2 \cos x + z^3, \frac{\partial \phi}{\partial y} = 2y \sin x - 4, \frac{\partial \phi}{\partial z} = 2xz^2 + 2$$

Integrating, we get

$$\phi = y^2 \sin x + xz^3 + f_1(y, z)$$

$$\phi = y^2 \sin x - 4y + f_2(x, z)$$

and $\phi = \frac{2}{3}xz^3 + 2z + f_3(x, y)$

We choose $f_1(y,z) = -4y + 2z,\ f_2(x,z) = xz^3 + 2z,\ f_3(x, y) = y^2 \sin x - 4y$

Thus $\phi = y^2 \sin x + xz^3 - 4y + 2z + \text{constant}$

$$\text{Work done} = \phi\left(\frac{\pi}{2}, -1, 2\right) - \phi(0, 1, -1)$$

$$= \left(1 + \frac{\pi}{2} \times 8 + 4 \times 4\right) - (-4 - 2) = 4\pi + 15$$

***Example 41:*** ***If*** $\nabla . E = 0,\ \nabla . H = 0,\ \nabla \times E = -\frac{\partial H}{\partial t}$

$$\nabla \times H = \frac{\partial E}{\partial t}, \textit{ show that } \nabla^2 H = \frac{\partial^2 H}{\partial t^2} \textit{ and } \nabla^2 E = \frac{\partial^2 E}{\partial t^2}.$$

***Solution:*** We know (curl E) = grad div $(\mathbf{E}) - \nabla^2\mathbf{E}$

But curl (curl E) $= \nabla \times (\nabla \times \mathbf{E})$

$$= \nabla \times \left(-\frac{\partial \mathbf{H}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}(\nabla \times H)$$

$$= -\frac{\partial}{\partial t}\left(\frac{\partial E}{\partial t}\right)$$

$\therefore$ $$-\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla(\nabla . \mathbf{E}) - \nabla^2 \mathbf{E} = 0 - \nabla^2 E \qquad (\because \nabla . E = 0)$$

or $$\nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Similarly, $$\nabla^2 \mathbf{H} = \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

***Example 42:*** ***Show that the function*** $\phi(x, y, z) = x^2 - y^2 + 4z$ ***is harmonic.***

***Solution:*** Here, $\phi = x^2 - y^2 + 4z$

Then, $$\frac{\partial \phi}{\partial x} = 2x, \frac{\partial \phi}{\partial y} = -2y, \frac{\partial \phi}{\partial z} = 4$$

and $$\frac{\partial^2 \phi}{\partial x^2} = 2, \frac{\partial^2 \phi}{\partial y^2} = -2, \frac{\partial^2 \phi}{\partial z^2} = 0$$

Thus, $$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 2 - 2 + 0 = 0$$

Hence, $\phi$ **is harmonic function.**

## Exercise 7.1

1. Find the directional derivative of $\phi = (x^2 + y^2 + z^2)^{-1/2}$ at the point $P(3,1,2)$ in the direction of the vector $yz\,\hat{\mathbf{i}} + zx\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}}$. **[B.C.A. (Avadh) 2006]**

2. Find the directional derivative of $f(x, y, z) = e^{2x}\cos(yz)$ at $(0,0,0)$ in the direction of the tangent to the curve $x = a\sin t$, $y = a\cos t$, $z = at$ at $t = \pi/4$. **[B.C.A. (Agra) 2007]**

3. Find the unit normal vector to the surface $x^2 y^2 z^2 = 4$ at the point $(-1,1,2)$.

4. Find the unit normal vector $\hat{n}$ of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point $P(1,0,2)$.

5. Find the directional derivative of the function:

   (i) $f = xy + yz + zx$ at point $(3,1,2)$ in the direction $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

   (ii) $f = 4e^{2x-y+z}$ at the point $(1,1,-1)$ in the direction towards the point $(-3,5,6)$

   (iii) $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at the point $P(2,1,3)$ in the direction of the vector $\mathbf{a} = \hat{\mathrm{i}} - 2\hat{\mathrm{k}}$. **[B.C.A. (Lucknow) 2008]**

6. Find the tangent plane and normal plane to the surface $2x^2 + y^2 + 2z = 3$ at the point $(2,1,-3)$.

7. If $(xyz)^m (x^n\,\hat{\mathbf{i}} + y^n\,\hat{\mathbf{j}} + z^n\,\hat{\mathbf{k}})$ is irrotational, show that either $m = 0$ or $n = -1$. **[B.C.A. (Kanpur) 2009]**

8. If $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, $r = |\mathbf{r}|$, show that

   (i) $\nabla r^2 = 2r$ (ii) $\nabla\left[\nabla . \dfrac{\mathbf{r}}{r}\right] = \dfrac{2\mathbf{r}}{r^3}$

   (iii) $\nabla \times \dfrac{(\mathbf{a}\times\mathbf{r})}{r^n} = \dfrac{2-n}{r^n}\mathbf{a} + \dfrac{n}{r^{n+2}}(\mathbf{a}\,.\,\mathbf{r})\,\mathbf{r}$, where $\mathbf{a}$ is constant vector.

   (iv) Curl $\dfrac{(\mathbf{A}\times\mathbf{r})}{r^3} = \dfrac{-a^3}{r^3} + 3\mathbf{r}\left(\dfrac{\mathbf{a}\,.\,\mathbf{r}}{r^3}\right)$ (v) $\nabla \times \dfrac{(\mathbf{a}\times\mathbf{r})}{r} = \dfrac{\mathbf{a}}{r} + \dfrac{\mathbf{a}\,.\,\mathbf{r}}{r^2}\mathbf{r}$

   (vi) div $[(a\,.\,\mathbf{r})\,\mathbf{r}] = 4\,(\mathbf{a}\,.\,\mathbf{r})$.

9. If $\nabla f = xy\,(2yz\,\hat{\mathbf{i}} + 2xz\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}})$, then find the scalar $f$.

**10.** Show that $\mathbf{F} = 2xyz^3\,\hat{\mathbf{i}} + x^2z^3\,\hat{\mathbf{j}} + 3x^2yz^2\,\hat{\mathbf{k}}$ is irrotational. Find the scalar function $f$ such that $F = \text{grad}\, f$. **[B.C.A. (Meerut) 2012, 06]**

**11.** Show that the following vector fields are irrotational:

(i) $F(x, y, z) = (y + z)\,\hat{\mathbf{i}} + (z + x)\,\hat{\mathbf{j}} + (x + y)\,\hat{\mathbf{k}}$

(ii) $F(x, y, z) = \cos(x^2 + y^2 + z^2)(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$. **[B.C.A. (Delhi) 2012, 06]**

**12.** Find div (**F**) and Curl (**F**) of the following vector field:

(i) $\mathbf{F} = (x^2 + yz)\,\hat{\mathbf{i}} + (y^2 + zx)\,\hat{\mathbf{j}} + (z^2 + xy)\,\hat{\mathbf{k}}$

(ii) $\mathbf{F} = xe^{-y}\,\hat{\mathbf{i}} + 2ze^{-y}\,\hat{\mathbf{j}} + xy^2\,\hat{\mathbf{k}}$

(iii) $F(x, y, z) = (x^2 + y^2 + z^2)^{3/2}(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}})$. **[B.C.A. (Rohilkhand) 2007]**

**13.** Find gradient of the scalar functions and evaluate at the given points.

(i) $f(x, y, z) = \sin(x\,y\,z), \quad (1, -1, \pi)$

(ii) $f(x, y, z) = \log(x + y + z), (1, 2, -1)$ **[B.C.A. (Meerut) 2008]**

(iii) $f(x, y, z) = e^{xyz}(x + y + z), \quad (2, 1, 1)$.

**14.** Find the angle between the surface $z = x^2 + y^2$ and $z = 3x^2 - 3y^2$ at the point $(2, 1, 5)$.

**15.** If a vector is given by

$\mathbf{F} = (x^2 - y^2 + x)\,\hat{\mathbf{i}} - (2xy + y)\,\hat{\mathbf{j}}$. Is this field irrotational? If so find its scalar potential.

## Answers 7.1

| | |
|---|---|
| **1.** | $\dfrac{-9}{49\sqrt{14}}$. |
| **2.** | 1. |
| **3.** | $\dfrac{-(\hat{\mathbf{i}}+3\,\hat{\mathbf{j}}-\hat{\mathbf{k}})}{\sqrt{11}}$. |
| **4.** | $\dfrac{2\,\hat{\mathbf{i}}-\hat{\mathbf{k}}}{\sqrt{5}}$. |
| **5.** | (i) $45/7$, (ii) $\dfrac{-20}{9}$, (iii) $\dfrac{-4}{\sqrt{5}}$. |
| **6.** | $x + y + z = 6$, $\dfrac{x-2}{4} = \dfrac{y-1}{1} = \dfrac{z+3}{1}$. |
| **9.** | $f(x, y, z) = x^2 y^2 z + c$, $c$ is an arbitrary constant. |
| **10.** | $x^2 yz^3 + c$. |
| **12.** | (i) $2(x+y+z), 0$.<br>(ii) $6x, -(x\,\hat{\mathbf{i}}+(2x-y)\,\hat{\mathbf{j}}-6y\,\hat{\mathbf{k}})$.<br>(iii) $6(x^2+y^2+z^2)^{3/2}, 0$. |
| **13.** | (i) $\pi(\hat{\mathbf{i}}-\hat{\mathbf{j}})+\hat{\mathbf{k}}$, (ii) $(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})/2$<br>(iii) $e^2(5\,\hat{\mathbf{i}}+9\,\hat{\mathbf{j}}+\hat{\mathbf{k}})$. |
| **14.** | $\theta = \cos^{-1}\left(\dfrac{\sqrt{21}}{101}\right)$. |
| **15.** | yes, $\dfrac{x^3}{3}+\dfrac{x^2}{2}-\dfrac{y^2}{2}-xy^2+C$. |

## Exercise 7.2

1. Show that:

   (i) grad $(\mathbf{r}.\mathbf{a}) = \mathbf{a}$ (ii) grad $[\mathbf{r}, \mathbf{a}, \mathbf{b}] = \mathbf{a} \times \mathbf{b}$.

2. If $\phi(x, y, z) = x^2 y + y^2 x + z^2$, find $\nabla\phi$ at the point (1, 1, 1).

3. Show that $\nabla\phi \,.\, d\mathbf{r} = d\phi$. **[B.C.A. (Purvanchal) 2010, 07]**

4. Prove that $\mathbf{A} \,.\, \left(\nabla \frac{1}{r}\right) = -\frac{\mathbf{A}\,.\,\mathbf{r}}{r^3}$.

5. Find the unit normal to surface $x^2 y + 2xz = 4$ at $(2, -2, 3)$.

6. In what direction from the point $(1, -1, -1)$ is the directional derivative of $f = x^2 - 2y^2 + 4z^2$ is maximum? Also find the value of this maximum directional derivative.

7. Find the equations of the tangent plane and normal plane to the surface $xyz = 4$ at the point $(1, 2, 2)$.

8. Find the gradient of each function $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$ be used if required:

   (i) $\phi = x^2 + y^2$ (ii) $\phi = e^x \sin 2y$.

9. Find the divergence of each of the following vector field at the point $(2, 1, -1)$:

   (i) $x^2 y\,\hat{\mathbf{i}} + y^2 x\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}$ (ii) $y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$.

10. If $\mathbf{a}$ is a constant vector and $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, show that:

    (i) $\nabla(\mathbf{a}\,.\,\mathbf{r}) = \mathbf{a}$

    (ii) $\nabla\,.\,(\mathbf{a} \times \mathbf{r}) = 0$

    (iii) $\nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$.

11. Find the curl of $f = xy\,\hat{i} + y^2\,\hat{j} + xz\,\hat{k}$ at $(-2, 4, 1)$.

12. If $f = xy - yz$, $g = 2y\,\hat{i} + 2z\,\hat{j} + (4x + z)\,\hat{k}$ find the following:

    (i) Curl (grad $f^2$)

    (ii) div (Curl $g$)

    (iii) div (grad $x^2 f$).

13. Show that the vector $V = 3y^4z^2\,\hat{\mathbf{i}} + 4x^3z^2\,\hat{\mathbf{j}} - 3x^2y^2\,\hat{\mathbf{k}}$ is solenoidal.

14. If $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $r = |\mathbf{r}|$, show that $\nabla(r^3\,\mathbf{r}) = 6r^3$.

15. Find the value of $\mu$ such that the function $f = x^2 - \mu y^2 + 4z$ is harmonic.

16. Show that $\nabla^2(\log r) = \dfrac{1}{r^2}$ where $r = \sqrt{x^2 + y^2 + z^2}$.

17. Find the unit normal to surface $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$.

18. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of vector $\boldsymbol{a} = \mathbf{2}\,\hat{\mathbf{i}} - \hat{\mathbf{j}} - \mathbf{2}\,\hat{\mathbf{k}}$.

19. Find the angle between the tangent planes to the surface $x\log z = y^2 - 1, x^2y = 2 - z$ at the point $(1, 1, 1)$.

20. Evaluate div ($\mathbf{F}$) and Curl ($\mathbf{F}$) at the point $(1, 2, 3)$:

(i) $\mathbf{F} = x^2yz\,\hat{\mathbf{i}} + xy^2z\,\hat{\mathbf{j}} + xyz^2\,\hat{\mathbf{k}}$

(ii) $\mathbf{F} = 3x^2\,\hat{\mathbf{i}} + 5xy^2\,\hat{\mathbf{j}} + 5xyz^3\,\hat{\mathbf{k}}$

(iii) $\mathbf{F} = \nabla[x^3y + y^3z + z^3x - x^2y^2z^2]$.

21. If $u = x^2 + y^2 + z^2$ and $V = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, show that div $(u\mathbf{V}) = 5u$.

22. Find $\nabla\phi$, if $\phi = \log(x^2 + y^2 + z^2)$. **[B.C.A. (Meerut) 2008]**

## Answers 7.2

| | |
|---|---|
| **2.** | $3\hat{\mathbf{i}}+3\hat{\mathbf{j}}+3\hat{\mathbf{k}}$. |
| **5.** | $\dfrac{\hat{\mathbf{i}}-2\hat{\mathbf{j}}-2\hat{\mathbf{k}}}{3}$. |
| **6.** | $2\hat{\mathbf{i}}-4\hat{\mathbf{j}}-8\hat{\mathbf{k}}, 2\sqrt{21}$. |
| **7.** | $2x+y+z=6$ (tangent plane), $\dfrac{x-1}{2}=\dfrac{y-2}{1}=\dfrac{z-2}{1}$ (normal plane). |
| **8.** | (i) $2x\hat{\mathbf{i}}+2y\hat{\mathbf{j}}$, (ii) $e^x \sin 2y\hat{\mathbf{i}}+2e^x \cos 2y\hat{\mathbf{j}}$. |
| **9.** | (i) 6, (ii) 0. |
| **11.** | $-\hat{\mathbf{j}}+2\hat{\mathbf{k}}$. |
| **12.** | (i) 0, (ii) 0, (iii) $6xy-2yz$. |
| **15.** | 1. |
| **17.** | $\dfrac{(-\hat{\mathbf{i}}+3\hat{\mathbf{j}}+2\hat{\mathbf{k}})}{\sqrt{14}}$. |
| **18.** | $\left(\dfrac{-5}{3}\right)$. |
| **19.** | $\cos^{-1}\left(\dfrac{-1}{\sqrt{30}}\right)$. |
| **20.** | (i) $12, 5\hat{\mathbf{i}}-16\hat{\mathbf{j}}+9\hat{\mathbf{k}}$, (ii) $278, 5(27\hat{\mathbf{i}}-54\hat{\mathbf{j}}+8\hat{\mathbf{k}})$, (iii) $-32, 0$. |
| **22.** | $\dfrac{2\vec{r}}{r}$. |

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Chapter 8

# Fourier Series

## 8.1 Periodic Functions

A function $f(x)$ is said to be periodic if it is defined for all real $x$ and there exist some positive number $T$ such that

$$f(x+T) = f(x) \qquad \forall x \in R$$

Positive number $T$ is called the period of the function $f(x)$. If $f(x)$ is a periodic function with period $T$, then $2T, 3T, \ldots.. nT$ are also periods of $f(x)$. Thus,

$$f(x) = f(x+T) = f(x+2T) = \ldots. = f(x+nT)$$

where $n$ is any integer.

***Illustrations:*** 1. The function $f(x) = \sin x$ and $g(x) = \cos x$ are periodic function. The period of each function is $2\pi$.

Since

$$f(x) = \sin x = \sin(x+2\pi) = \sin(x+4\pi) = \ldots..$$

and

$$g(x) = \cos x = \cos(x+2\pi) = \cos(x+4\pi) = \ldots...$$

The function $f(x) = \sin x$ is called sinusoidal periodic function.

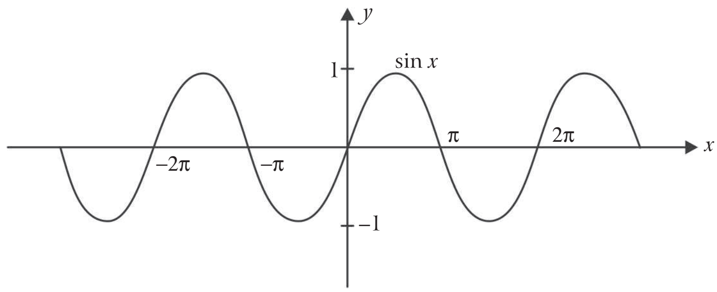


Fig. 8.1

2. The function sin $nx$ and cos $nx$ is periodic function whose period is $\frac{2\pi}{n}$.

3. Some of many periodic functions is also periodic function.

Let $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} b_n \sin nx$ be two periodic functions whose sum $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is also periodic and its period is $2\pi$.

## 8.2 Fourier Series

**[B.C.A. (Rohilkhand) 2010]**

If $f(x)$ be a periodic function with period $2\pi$ and $f(x)$ be represented by trigonometric series given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(1)$$

is called Fourier Series. Where $a_0$, $a_n$ and $b_n$ are called [Fourier coefficient of $f(x)$] whose values are obtained by following formulae:

1. **For Determination of $a_0$:** Integrating both sides of (1) w.r. to $x$ between the limits $-\pi$ and $\pi$, we find

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left[ a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx \right]$$

$$= a_0\,[2\pi] + 0$$

or $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx \qquad ...(2)$$

2. **For Determination of $a_n$:** Multiplying both sides of (1) by cos $nx$ where $n$ is fixed positive integer, and then integrating the limits $-\pi$ to $\pi$, we find

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = a_n \int_{-\pi}^{\pi} \cos^2 nx \, dx + 0 + 0 = \frac{a_n}{2} . 2\pi = \pi \, a_n$$

$$\Rightarrow \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \qquad ...(3)$$

**3.** **For Determination of $b_n$:** Multiplying both sides of (1) by $\sin nx$ where $n$ is a fixed positive integer and then integrating between the limits $-\pi$ to $\pi$, we find

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = b_n \int_{-\pi}^{\pi} \sin^2 nx \, dx$$

$$= \frac{b_n}{2} \int_{-\pi}^{\pi} (1 - \cos 2nx) \, dx = \frac{b_n}{2} \times 2\pi = \pi \, b_n$$

$$\Rightarrow \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x \, dx \qquad ...(4)$$

The formulae obtain from equation (2), (3) and (4) are known as Euler's formulae.

**Remark 1**

If we are given a periodic function $f(x)$ with period $2\pi$, then we can determine $a_0$, $a_n$ and $b_n$ by equation (2), (3) and (4) and form the trigonometric series (1). This is called Fourier Series corresponding to $f(x)$.

**Remark 2**

Since the function $f(x)$ is periodic with period $2\pi$, the interval of integration in the above formulae can be replaced by any other interval of length $2\pi$ for illustration, instead of interval $(-\pi, \pi)$ we can take the interval $(0, 2\pi)$ or $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

**Remark 3 (Generalized Rule Integration by Part)**

(i) Product of two functions = first function × integration of IInd function − $\int$ [differentiation of Ist × integration of IInd ] $dx$

(ii) Whenever a product of two function, one of which is a power of $x$, is to be integrated, it is better to apply the following rule:

$$\int u.v \, dx = u.v_1 - u' \, v_2 + u'' \, v_3 - u''' \, v_4 + .....$$

where $\qquad u' = \frac{dv}{dx}, u'' = \frac{d^2 u}{dx^2} ..., v_1 = \int v \, dx, v_2 = \int v_1 \, dx .....$ etc.

## 8.3 Some Important Results of the Definite Integral

1. $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ If $f(x)$, is even function of $x$ or $f(-x) = f(x)$.

2. $\int_{-a}^{a} f(x)\,dx = 0$ If $f(x)$, is odd function of $x$, $f(-x) = -f(x)$.

3. $\int_{-\pi}^{\pi} \sin nx\,dx = 0 = \int_{-\pi}^{\pi} \cos nx\,dx.$ $(n = 1, 2, 3....)$

4. $\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0.$ $(m \neq n \text{ or } m = n)$

5. $\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} (m, n = 0, 1, 2, ....).$

6. $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} (m, n = 0, 1, 2).$

7. $\int_{-\pi}^{\pi} \sin^2 mx\,dx = \pi\ (m = 1, 2, 3 ....).$

8. $\int_{-\pi}^{\pi} \cos^2 mx\,dx = \begin{cases} \pi & \text{when } m = 1, 2, 3 .... \\ 2\pi & \text{when } m = 0 \end{cases}$

9. $\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$

10. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$

    and $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx).$

11. $\int_{0}^{2\pi} \sin nx\,dx = 0.$

12. $\int_{0}^{2\pi} \cos nx\,dx = 0.$

13. $\int_{0}^{2\pi} \sin^2 nx\,dx = \pi.$

14. $\int_{0}^{2\pi} \cos^2 nx\,dx = \pi.$

15. $\int_{0}^{2\pi} \sin nx \sin mx\,dx = 0.$

16. $\int_{0}^{2\pi} \cos nx \cos mx\,dx = 0.$

17. $\int_{0}^{2\pi} \sin nx \cos mx\,dx = 0.$

18. $\int_{0}^{2\pi} \sin nx \cos nx\,dx = 0.$

## Solved Examples

***Example 1:*** ***Find the Fourier Series for expansion of $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.***

**[B.C.A. (Agra) 2009, 06; B.C.A.(Avadh) 2008]**

***Solution:*** Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ...(1)

where $f(x) = e^{-x}$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}dx = \frac{1}{2\pi}[-e^{-x}]_0^{2\pi} = \frac{1-e^{-2\pi}}{2\pi} \quad ....(2)$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}[e^{-x}(-\cos nx + n\sin nx)]_0^{2\pi}$$

$$= \frac{(1-e^{-2\pi})}{\pi}\cdot\frac{1}{n^2+1} \quad ...(3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi(n^2+1)}[e^{-x}(-\sin nx - n\cos nx)]_0^{2\pi} = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{n}{n^2+1}\,(n = 1, 2, \ldots) \quad ...(4)$$

Substitute the equation (2), (3), (4) in equation (1), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\ldots.\right)\right.$$

$$\left. + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\right)\right\}.$$

***Example 2:*** ***Find the Fourier Series for the function $f(x) = x,\ 0 < x < 2\pi$.***

**[B.C.A. (Kurukshetra) 2012, 08]**

***Solution:*** Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

or $$f(x) = a_0 + (a_1\cos x + a_2\cos 2x + a_3\cos 3x + \ldots..)$$

$$+ (b_1\sin x + b_2\sin 2x + b_3\sin 3x + \ldots..) \quad ...(1)$$

where $\quad f(x) = x$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \qquad (n = 1, 2, 3\,\ldots..\,\infty)$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos 2nx}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-x\frac{\cos nx}{n} - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2nx}{n}\right] = \frac{-2}{n}$$

Put the values of $a_0 = 2\pi, a_n = 0, b_n = \dfrac{-2}{n}$ in equation (1), we obtain

$$x = \pi + \sum_{n=1}^{\infty} 0.\cos nx + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\sin nx$$

or $$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots..\right].$$

***Example 3:*** ***Find the Fourier Series of the function*** $f(x) = x^2, -\pi < x < \pi$ ***and*** $f(x+2\pi) = f(x)$. ***Hence deduce that***

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots..$$

**[B.C.A. (Kanpur) 2006; B.C.A. (Meerut) 2004]**

***Solution:*** Let the Fourier of the function $f(x) = x^2$ be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \qquad \ldots(1)$$

Where the Fourier coefficients are given by

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$\left[\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{If } f(x) \text{ is even function} \\ 0 & \text{If } f(x) \text{ is odd function} \end{cases}\right.$$

$$a_n = \frac{2}{\pi}\left[x^2\,\frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 - 2\pi\left(\frac{-\cos n\pi}{n^2}\right) + 0 - 0\right] = \frac{4\pi\cos n\pi}{\pi n^2} = \frac{4(-1)^n}{n^2}$$

$$[\because \sin n\pi = 0, \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx = 0$$

Since, $\quad f(x) = x^2 \sin nx \Rightarrow f(-x) = (-x)^2 \sin(-nx) = -x^2 \sin nx = -f(x)$

$\therefore \quad f(x) = x^2 \sin nx$ is odd function

Put $\quad f(x) = x^2,\ a_0 = \dfrac{\pi^2}{3},\ a_n = \dfrac{4(-1)^n}{n^2}, b_n = 0$ in equation (1) we get

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty} 0.\sin nx$$

or
$$x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \ldots..\right] \qquad \ldots(2)$$

Put $x = 0$ in equation (2)

$$0 = \frac{\pi^2}{3} + 4\left[-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \ldots..\right] \quad \text{or} \quad \frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots\ldots..$$

***Example 4:*** ***Find the Fourier Series for the function*** $f(x) = x + x^2, -\pi < x < \pi$***. Hence deduce that***

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots .$$

**[B.C.A. (Meerut) 2007]**

***Solution:*** Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$

Then
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} (x + x^2)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]=\frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{\sin nx}{n}\right)-(1+2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2}-\frac{(-2\pi+1)\cos(-n\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right]=\frac{4(-1)^n}{n^2} \qquad [\because \cos n\pi = (-1)^n]$$

and
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right)+(1+2x)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{\cos nx}{n}+\frac{2\cos n\pi}{n^3}+(-\pi+\pi^2)\frac{\cos n\pi}{n}-2\frac{\cos n\pi}{n^3}\right]$$

$$= \frac{-1}{\pi}\left[\frac{2\pi}{n}\cos n\pi\right]=\frac{-2}{n}(-1)^n$$

Put the values of $a_0$, $a_n$ and $b_n$ in equation (1) we get the following Fourier Series:

$$x+x^2 = \frac{\pi^2}{3}+4\left[-\cos x+\frac{1}{2^2}\cos 2x-\frac{1}{3^2}\cos 3x+\ldots..\right]$$

$$-2\left[-\sin x+\frac{1}{2}\sin 2x-\frac{1}{3}\sin 3x+\ldots.\right] \qquad \ldots(2)$$

Now, put $x=\pi$ in equation (2), we obtain

$$\pi+\pi^2 = \frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.\right] \qquad \ldots(3)$$

Again, put $x=-\pi$ in equation (2), we obtain

$$-\pi+\pi^2 = \frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.\right] \qquad \ldots(4)$$

Adding equation (3) and (4), we get

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.\right]$$

$\Rightarrow$ $$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.\right]$$

or $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots. \quad \text{or} \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

***Example 5:*** ***Find the Fourier Series of the periodic function $f(x)$, where***

$$f(x)\begin{cases} -\pi, \text{ when } -\pi < x < 0 \\ x, \text{ when } \ 0 < x < \pi. \end{cases}$$

***Hence, deduce that*** $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$ . **[B.C.A. (Delhi) 2011]**

***Solution:*** The Fourier Series for given function is as follows:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

where $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} (-\pi)\, dx + \int_{0}^{\pi} x\, dx\right]$$

$$= \frac{1}{2\pi}\left[-\pi\,[0-(-\pi)] + \frac{1}{2}[\pi^2 - 0]\right] = \frac{1}{2\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{-\pi}{4}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(x) \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \cos nx\, dx + \int_{0}^{\pi} f(x) \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi) \cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos nx - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(\cos n\pi - 1) = \frac{1}{n^2\pi}[(-1)^n - 1]$$

and $$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\, dx + \int_{0}^{\pi} x\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0} + \left(\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}\cos n\pi\right]$$

$$= \frac{1}{n}(1 - 2\cos n\pi)\ (n = 1, 2, 3 \ldots.) = \frac{1}{n}(1 - 2(-1)^n)$$

Thus putting values of $a_0$, $a_n$ and $b_n$ in (1), the required Fourier Series of $f(x)$ is obtain as follows:

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots..\right)$$

$$+\left(3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4}\right) + \qquad \ldots(2)$$

To find the sum at $x = 0$ the series converges to

$$\frac{f(0+0) + f(0-0)}{2} = \frac{0 + (-\pi)}{2} = \frac{-\pi}{2}$$

and at $x = \pi$, the series converges to

$$\frac{f(-\pi+0) + f(\pi-0)}{2} = \frac{-\pi + \pi}{2} = 0$$

The L.H.S. of (2) is $\frac{-\pi}{2}$ at $x = 0$ and 0 at $x = \pi$. Therefore, putting $x = 0$ in R.H.S. of equation (2), we find

$$\left(\frac{-\pi}{2}\right) = \left(\frac{-\pi}{4}\right)\left(\frac{-2}{\pi}\right)\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots..\right]$$

or $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots. \qquad \ldots(3)$$

Similarly, if we put $x = \pi$ in equation (2), we get (3)

***Example 6:*** ***Find the Fourier Series for $f(x)$ in the interval $(-\pi, \pi)$, where***

$$f(x)=\begin{cases} x+\pi, & 0 \le x \le \pi \\ -x-\pi, & -\pi \le x < 0 \end{cases} \quad \textit{and} \quad f(x+2\pi)=f(x).$$

***Solution:*** Let the Fourier Series for function $f(x)$ is

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx=\dfrac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx+\int_{0}^{\pi} f(x)\,dx\right]$

$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0}(-x-\pi)\,dx+\int_{0}^{\pi}(x+\pi)\,dx\right]$$

$$=\frac{1}{2\pi}\left[\left(-\frac{x^2}{2}-\pi x\right)_{-\pi}^{0}+\left(\frac{x^2}{2}+\pi x\right)_{0}^{\pi}\right]=\frac{1}{2\pi}\left[\left(\frac{\pi^2}{2}-\pi^2\right)+\left(\frac{\pi^2}{2}+\pi^2\right)\right]=\frac{\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx+\int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-x-\pi)\cos nx\,dx+\int_{0}^{\pi}(x+\pi)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[(-x-\pi)\frac{\sin nx}{n}-(-1)\left\{\frac{-\cos nx}{n^2}\right\}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n}-(1)\left\{\frac{-\cos nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$=\frac{2}{\pi n^2}[(-1)^n-1] \qquad [\because \cos n\pi=(-1)^n, \sin n\pi=0]$$

and $b_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx+\int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$=\frac{1}{\pi}\left\{\int_{-\pi}^{0}(-x-\pi)\sin nx\,dx+\int_{0}^{\pi}(x+\pi)\sin nx\,dx\right\}$$

$$=\frac{1}{\pi}\left[(-x-\pi)\left(\frac{-\cos nx}{n}\right)-(-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}+\frac{1}{\pi}\left[(x+\pi)\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}\right]+\frac{1}{\pi}\left[\frac{-2\pi}{n}(-1)^n+\frac{\pi}{n}\right]=\frac{2}{n}\{1-(-1)^n\}$$

Putting above values in (1), we get the required Fourier Series of $f(x)$ as follows:

$$f(x)=\frac{\pi}{2}+\frac{4}{\pi}\left(\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\ldots\ldots\right).$$

***Example 7:*** ***Find the Fourier Series for the function $f(x)$, where***

$$f(x)=\begin{cases}-1 & \textit{when} \quad -\pi<x<-\dfrac{\pi}{2}\\ 0 & \textit{when} \quad \dfrac{-\pi}{2}<x<\dfrac{\pi}{2}.\\ 1 & \textit{when} \quad \dfrac{\pi}{2}<x<\pi.\end{cases}$$

**[B.C.A. (Bundelkhand) 2010, 04]**

***Solution:*** Let the Fourier Series for $f(x)$ is as follows:

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \qquad \ldots(1)$$

Then, we have

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2}f(x)\,dx+\int_{-\pi/2}^{\pi/2}f(x)\,dx+\int_{\pi/2}^{\pi}f(x)\,dx\right]$$

$$a_0=\frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2}(-1)\,dx+\int_{-\pi/2}^{\pi/2}0\,dx+\int_{\pi/2}^{\pi}1\,dx\right]$$

$$=\frac{1}{2\pi}[\{-x\}_{-\pi}^{-\pi/2}+[x]_{\pi/2}^{\pi}]=\frac{1}{2\pi}\left[\frac{\pi}{2}-\pi+\pi-\frac{\pi}{2}\right]=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}f(x)\,dx+\int_{-\pi/2}^{\pi/2}f(x)\,dx+\int_{\pi/2}^{\pi}f(x)\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)\cos nx\,dx+\int_{-\pi/2}^{\pi/2}(0)\cos nx\,dx+\int_{\pi/2}^{\pi}(1)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[-\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2}+\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{\sin\frac{n\pi}{2}}{n}\right]-\frac{1}{\pi}\left[\frac{\sin\frac{n\pi}{2}}{n}\right]=0$$

and $$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)\sin nx\,dx+\int_{-\pi/2}^{\pi/2}(0)\sin nx\,dx+\int_{\pi/2}^{\pi}(1)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] - \frac{1}{n\pi}\left[\cos n\pi - \cos\frac{n\pi}{2}\right] = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

Putting these values in equation (1), we get the required Fourier Series of $f(x)$ as follows:

$$f(x) = \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \ldots\right].$$

## Exercise 8.1

**Find the Fourier Series of the following function:**

**1.** $f(x) = e^{-ax}$ where $x = -\pi$ to $x = \pi$.

**2.** $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases}$ and $f(2\pi + x) = f(x)$.

**3.** $f(x) = \begin{cases} \pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi. \end{cases}$

**4.** $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$ and $f(x + 2\pi) = f(x)$.

**[B.C.A. (Rohtak) 2012]**

**5.** $f(x) = |\sin x|$, for $-\pi < x < \pi$.

$\left[\textbf{Hint}: f(x) = \begin{cases} -\sin x, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}\right]$

**6.** $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \frac{\pi x}{4}, & 0 < x < \pi \end{cases}$ Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$.

**[B.C.A. (Lucknow) 2011]**

**7.** $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

Hence deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots\ldots \infty = \frac{1}{4}(\pi - 2)$.

**[B.C.A. (Kanpur) 2010]**

**8.** $f(x) = \begin{cases} -x^2, & -\pi < x < 0 \\ x^2, & 0 < x < \pi. \end{cases}$

**9.** $f(x) = \begin{cases} 2k, & 0 < x < \pi \\ 0, & \pi < x < 2\pi. \end{cases}$

## Answers 8.1

1. $\frac{2\sinh(na\pi)}{\pi}\left[\left\{\left(\frac{1}{2a}-\frac{a\cos x}{1^2+a^2}+\frac{a\cos 2x}{2^2+a^2}+\ldots..\right)\right\}+\left\{\left(\frac{\sin x}{1^2}-\frac{2\sin x}{2^2+a^2}+\frac{3\sin 3x}{3^2+a^2}-\ldots.\right)\right\}\right].$

2. $\frac{\pi^2}{6}-2\left(\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots.\right)-\frac{1}{\pi}\left\{\left(\frac{2}{1^3}-\frac{\pi^2}{1}\right)\sin x-\left(\frac{2}{2^3}-\frac{\pi^2}{2}\right)\sin 2x+\ldots.\right\}.$

3. $\frac{2}{\pi}-\frac{4}{\pi}\sum\left[\frac{(-1)^n}{4n^2-1}\cos(2n\pi)\right]$ 4. $\frac{4}{\pi}\left[\frac{1}{1}\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\frac{1}{7}\sin 7x+\ldots..\right.$

5. $\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{3}\cos 2x+\frac{1}{15}\cos 4x+\frac{1}{35}\cos 6x+\ldots...\right].$

6. $\frac{\pi^2}{16}+\sum_{n=1}^{\infty}\left[\frac{\{(-1)^n-1\}}{4n^2}\cos nx-\frac{(-1)^n\pi\sin nx}{4n}\right].$

7. $\frac{1}{\pi}+\frac{\sin x}{2}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}.$

8. $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$ where $b_{2m}=\frac{-\pi}{m}, b_{2m+1}=\frac{2\pi}{2m+1}-\frac{8\pi}{(2m+1)^3}.$

9. $f(x)=k+\frac{4k}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\ldots..\right].$

## 8.4 Fourier Series of Even and Odd Functions

### 8.4.1 Even Function

A function $y=f(x)$ is said to be even function if $f(-x)=f(x)$.

Thus the graph of an even function $f(x)$ is symmetrical about $y$-axis.

### 8.4.2 Odd Function

A function $y=f(x)$ is said to be odd function if $f(-x)=-f(x)$.

Thus the graph of an odd function is symmetrical about the origin *i.e.*, it is symmetrical in opposite quadrant.

We shall now find the Fourier Series for even and odd function which are periodic in $(-\pi, \pi)$.

1. **When $f(x)$ is an Even Function:** The Fourier Series for $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad ...(1)$$

where $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$\because$ $f(x)$ is even and $\sin nx$ is odd function. Therefore, $f(x)\sin(mx)$ is and odd function.

$\therefore$ $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

Therefore, Fourier Series (1) is also called Fourier cosine series.

2. **When $f(x)$ is an Odd Function:** The Fourier Series for $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(1)$$

where $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = 0$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

$\because$ $f(x)\cos nx$ is odd function.

Again, $f(x)$ and $\sin nx$ both are odd function, therefore $f(x)\sin nx$ is even function

$\therefore$ $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Therefore, Fourier Series (1) is also called Fourier Sine Series.

***Example 8:*** ***Find the Fourier Series for $f(x) = x, -\pi < x < \pi$.*** **[B.C.A. (Lucknow) 2008]**

***Solution:*** The function $f(x) = x$ is odd function. Then

$\Rightarrow$ $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(1)$$

where $$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^\pi - \int_0^\pi 1.\left(\frac{-\cos nx}{n}\right)dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi}{n}\cos n\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi}{n}\cos n\pi + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^\pi\right\} = \frac{2}{\pi}\left\{\frac{-\pi}{n}\cos n\pi + 0\right\} = \frac{2}{n}(-1)^{(n+1)}$$

$\therefore$ Required Fourier Series is $x = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + .....\right]$.

***Example 9:*** ***Find the Fourier Series for the function $f(x) = x^3$, $-\pi < x < \pi$.***

***Solution:*** The function $f(x) = x^3$ is odd.

$$\Rightarrow \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(1)$$

Then, $a_0 = 0, a_n = 0$ and $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$

$$= \frac{2}{\pi}\int_0^\pi x^3 \sin nx\,dx$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right) - 3x^2\left(\frac{-\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{-\pi^3\cos n\pi}{n} + \frac{6\pi\cos n\pi}{n^3}\right]$$

$$= 2(-1)^n\left[\frac{-\pi^2}{n} + \frac{6}{n^3}\right] \qquad [\because \cos nx = (-1)^n,\ \sin n\pi = 0]$$

$\therefore$ The required Fourier Series is

$$x^3 = 2\left[-\left(\frac{-\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(\frac{-\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(\frac{-\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x + .....\right].$$

**Example 10:** ***Find the Fourier Series for the function*** $f(x) = |x|, -\pi < x < \pi$***. Hence deduce that***

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

**[B.C.A. (Meerut) 2005]**

***Solution:*** Here $f(x) = |x|$, is an even function

$\therefore$ The Fourier Series for $f(x) = |x|$

$$f(x) = |x| = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(1)$$

where $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} |x|\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2}$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} 1\cdot\left(\frac{\sin nx}{n}\right)dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{n}(\sin n\pi - \sin 0) - \frac{1}{n}\int_0^{\pi} \sin nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{0 - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi n^2}[\cos n\pi - 1] = \frac{2}{\pi n^2}[(-1)^n - 1] \qquad [\because \cos n\pi = (-1)^n]$$

Putting the above values in (1), we get required Fourier Series.

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} - \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \ldots\ldots\right] \qquad \ldots(2)$$

Now, put $x = 0$ in (2), we have

$$0 = |0| = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} - \frac{(-1)}{3^2} + \frac{1}{5^2} - \frac{(-1)}{7^2} + \ldots\ldots\right]$$

or

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots \frac{\pi^2}{8}.$$

**Example 11:** ***Obtain Fourier's Series of $f(x) = x\sin x$ in the interval $(-\pi, \pi)$. Hence deduce that***

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots$$

**[B.C.A. (Indore) 2012, 09, 06]**

**Solution:** Here, $f(x) = x\sin x$, $f(x)$ is an even function of $x$.

$\therefore$ The Fourier Series is given by

$$f(x) = \frac{1}{\pi}\int_0^\pi f(x)\,dx + \frac{2}{\pi}\sum_{n=1}^{\infty}\cos nx\int_0^\pi f(x)\cos nx\,dx$$

$$f(x) = x\sin x = \frac{1}{\pi}\int_0^\pi x\sin x\,dx + \frac{2}{\pi}\sum_{n=1}^{\infty}\cos nx\int_0^\pi x\sin x\cos nx\,dx$$

Now $\int_0^\pi x\sin x\,dx = [-x\cos x + \sin x]_0^\pi = \pi$

and $\int_0^\pi x\sin x\cos nx\,dx = \frac{1}{2}\int_0^\pi x[\sin(n+1)x - \sin(n-1)x]\,dx.$

$$= \frac{1}{2}\left[\frac{-[x\cos(n+1)x]}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^\pi - \frac{1}{2}\left[-\frac{x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^\pi$$

$$= \frac{\pi}{2}\left[\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right] = \frac{\pi}{2}\left[-\frac{\cos n\pi}{n-1} + \frac{\cos n\pi}{n+1}\right]$$

$$= \frac{\pi\cos n\pi}{1-n^2}, \text{ when } n > 1$$

Now, put $n = 1$, we have

$$\int_0^\pi x\sin x\cos x\,dx = \frac{1}{2}\int_0^\pi x\sin 2x\,dx = \frac{1}{2}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^\pi = -\frac{\pi}{4}$$

Therefore, $x\sin x = 1 + \frac{2}{\pi}\left(-\frac{\pi}{4}\cos x + \sum_{n=2}^{\infty}\frac{\pi\cos n\pi}{1-n^2}\cos nx\right)$

$$= 1 + 2\left[-\frac{1}{4}\cos x - \frac{1}{1.3}\cos 2x + \frac{1}{2.4}\cos 3x - \frac{1}{3.5}\cos 4x + \ldots\right]$$

Put $x = \frac{\pi}{2}$, we obtain

$$\frac{\pi}{2} = 1 + 2\left\{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\right\}$$

or $$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots.$$

***Example 12:*** ***Obtain the Fourier Series of the function $f(x) = x\cos x$ in the interval $(-\pi, \pi)$.*** **[B.C.A. (Bhopal) 2007]**

***Solution:*** Here, $x \cos x$ is odd function then, $a_0 = a_n = 0$

$$\therefore \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin nx\, dx,$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} x \cos x . \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\{\sin(n+1)\,x + \sin(n-1)\,x\}\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)\,x\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)\,x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)\,x}{n+1}\right) + \frac{\sin(n+1)\,x}{(n+1)^2}\right]_0^{\pi} + \frac{1}{\pi}\left[-x\frac{\cos(n-1)\,x}{n-1} + \frac{\sin(n-1)\,x}{(n-1)^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)\,x}{n+1} - \frac{\cos(n-1)\,x}{n-1}\right\}_0^{\pi} + 1\left\{\frac{\sin(n+1)\,x}{(n+1)^2} + \frac{\sin(n-1)\,x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\,\pi}{(n+1)} - \frac{\cos(n-1)\,\pi}{n-1}\right\}\right]$$

$$\Rightarrow \qquad b_n = \left\{-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right\}, n \neq 1$$

$$= (-1)^{n+1}\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} = \begin{cases} -\dfrac{2n}{n^2-1}, \text{ if } n \text{ is odd}; n \neq 1 \\ \dfrac{2n}{n^2-1}, \text{ if } n \text{ is even}; n \neq 1 \end{cases}$$

when
$$n = 1,\ b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\,.\,\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{-1}{2}\right)\right] = -\frac{1}{2}$$

Putting above values in equation (1), we get required Fourier Series.

$$x\cos x = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots$$

$$= -\frac{1}{2}\sin x + \frac{4\sin 2x}{2^2-1} - \frac{6\sin 3x}{3^2-1} + \ldots\ldots$$

***Example 13:*** ***If*** $f(x) = |\cos x|$ ***then find its expansion in*** $(-\pi, \pi)$.

***Solution:*** We observe that $f(-x) = |\cos(-x)| = |\cos x| = f(x)$ is even function. Therefore, Fourier Series for $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where $|\cos x| = \begin{cases} \cos x & \text{when } 0 < x < \dfrac{\pi}{2} \\ -\cos x & \text{when } \dfrac{\pi}{2} < x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} |\cos x|\,dx = \frac{1}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx\right]$$

$$= \frac{1}{\pi}[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}] = \frac{1}{\pi}[(1-0)-(0-1)] = \frac{2}{\pi}$$

and $$a_n = \frac{2}{\pi}\int_0^{x} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx\,dx + \int_{\pi/2}^{\pi} (-\cos x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} [\cos(n+1)x + \cos(n-1)x]\,dx - \int_{\pi/2}^{\pi} [\cos(n+1)x + \cos(n-1)x]\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_0^{\pi/2} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin\left\{(n+1)\dfrac{\pi}{2}\right\}}{n+1} + \frac{\sin\left\{(n-1)\dfrac{\pi}{2}\right\}}{n-1}\right\} - \left\{\frac{\sin\left\{(n+1)\dfrac{\pi}{2}\right\}}{n+1} + \frac{\sin\left\{(n-1)\dfrac{\pi}{2}\right\}}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left(\frac{\cos n(\pi/2)}{n+1} - \frac{\cos n(\pi/2)}{n-1}\right) = \frac{-4\cos n(\pi/2)}{\pi(n^2-1)} \qquad (n \neq 1)$$

Now, when $n = 1$ $a_1 = \dfrac{2}{\pi}\left[\displaystyle\int_0^{\pi/2} \cos^2 x\,dx - \int_{\pi/2}^{\pi} \cos^2 x\,dx\right] = 0$

$\therefore$ The required Fourier Series is

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left\{\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + .....\right\}.$$

## 8.5 Fourier Series of Functions with Period 2c

Let $f(x)$ be a function with period $2c$.

$\because \qquad f(x+2c) = f(x)\ \forall x$

Putting $x = \frac{ct}{\pi}$, we get $f\left(\frac{ct}{\pi} + 2c\right) = f\left(\frac{ct}{\pi}\right) \Rightarrow f\left(\frac{ct + 2c\pi}{\pi}\right) = f\left(\frac{ct}{\pi}\right)$

$\Rightarrow \qquad f\left[\frac{c}{\pi}(t + 2\pi)\right] = f\left(\frac{ct}{\pi}\right)$

$\therefore \quad f\left(\frac{ct}{\pi}\right)$ is a function with period $2\pi$.

Hence, $f\left(\frac{ct}{\pi}\right)$ may be expanded in the Fourier Series in the interval $-\pi \le t \le \pi$ in the form.

$$f\left(\frac{ct}{\pi}\right) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \qquad ...(1)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{ct}{\pi}\right)dt,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{ct}{\pi}\right)\cos nt\, dt$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{ct}{\pi}\right)\sin nt\, dt$

$\because \qquad x = \frac{ct}{\pi}$ therefore $-\pi \le t \le \pi \Rightarrow -\pi \le \frac{\pi x}{c} \le \pi$

$\Rightarrow \qquad -\pi \times \frac{c}{\pi} \le x \le \pi \times \frac{c}{\pi}$ or $-c \le x \le c$

Now $\qquad t = \frac{\pi x}{c} \Rightarrow dt = \frac{\pi}{c}dx$

$$\therefore \qquad a_0 = \frac{1}{2\pi}\int_{-c}^{c} f(x)\frac{\pi}{c}dx = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)\cdot\frac{\pi}{c}dx = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

and $$b_n = \frac{1}{\pi}\int_{-c}^{c} f(x)\sin\left(\frac{\pi n x}{c}\right)\frac{\pi}{c}dx = \frac{1}{c}\int_{-c}^{c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

Putting $t = \dfrac{\pi x}{c}$ in (1), it becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)\right].$$

**NOTE:**

If given interval is $(0, 2c)$ and we want to transform in $(0, 2\pi)$ then we used.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{c}\right)$$

where $a_0 = \dfrac{1}{2c}\displaystyle\int_0^{2c} f(x)\,dx,\ a_n = \dfrac{1}{c}\int_0^{2c} f(x)\cos\left(\dfrac{n\pi x}{c}\right)dx,$

$$b_n = \frac{1}{c}\int_0^{2c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

***Example 14:*** ***Find the Fourier Series where*** $F(t) = \begin{cases} 0, & \textit{when } -2 < t < -1 \\ k, & \textit{when } -1 < t < 1. \\ 0, & \textit{when } 1 < t < 2 \end{cases}$

***Solution:*** We know that

$$f\left(\frac{\pi t}{c}\right) = F(t) = a_0 + \sum a_n \cos\frac{n\pi t}{c} + \sum b_n \sin\frac{n\pi t}{c} \text{ (Let)}$$

where $a_0 = \dfrac{1}{2c}\displaystyle\int_{-c}^{c} f\left(\dfrac{\pi t}{c}\right)dt = \dfrac{1}{2c}\int_{-c}^{c} F(t)\,dt$

$$a_n = \frac{1}{c}\int_{-c}^{c} F(t)\cos\frac{n\pi t}{c}\,dt \text{ and } b_n = \frac{1}{c}\int_{-c}^{c} F(t)\sin\frac{n\pi t}{c}\,dt$$

Here $c = 2$

$\therefore$ $a_0 = \dfrac{1}{4}\left[\displaystyle\int_{-2}^{-1} + \int_{-1}^{1} + \int_{1}^{2}\right]F(t)\,dt = \dfrac{1}{4}\int_{-1}^{1} k\,dt = \dfrac{k}{2}$

$$a_n = \frac{1}{2}\left[\int_{-2}^{-1} + \int_{-1}^{1} + \int_{1}^{2}\right]F(t)\cos\frac{n\pi t}{2}\,dt = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi t}{2}\,dt = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

or $b_n = \dfrac{1}{2}\left[\displaystyle\int_{-2}^{-1} + \int_{-1}^{1} + \int_{1}^{2}\right]F(t)\sin\left(\dfrac{n\pi t}{2}\right)dt = \dfrac{1}{2}\int_{-1}^{1} k\sin\left(\dfrac{n\pi t}{2}\right)dt = 0$

Therefore, $F(t) = \dfrac{k}{2} + \displaystyle\sum_{n=1}^{\infty}\left(\dfrac{2k}{n\pi}\right)\sin\left(\dfrac{n\pi}{2}\right)\cos\left(\dfrac{n\pi t}{2}\right).$

***Example 15:*** ***Find the Fourier Series for the function***

$$f(x) = x - x^2,\ -1 < x < 1.$$ **[B.C.A. (Meerut) 2003]**

***Solution:*** Here, $f(x)$ has (period), $2c = 2 \Rightarrow c = 1$

Let the Fourier Series for $f(x)$ be

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx\pi) + b_n \sin(n\pi x)] \qquad [\because c = 1] \qquad ...(1)$$

where $$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (x - x^2)\,dx = -\frac{1}{3}$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{1} (x - x^2)\cos n\pi x\,dx$$

$$= \int_{-1}^{1} x\cos n\pi x\,dx - \int_{-1}^{1} x^2 \cos n\pi x\,dx$$

I$^{st}$ integral is odd and II$^{nd}$ integral is even.

$$= -2\int_0^1 x^2 \cos n\pi x\,dx = -2\left[\frac{2\cos n\pi}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2} \qquad [\because \sin n\pi = 0]$$

and $$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{1}(x - x^2)\sin n\pi x\,dx = 2\int_0^1 x\sin n\pi x\,dx = 2\left[\frac{(-1)^{n+1}}{n\pi}\right]$$

$$\left[\because \int_{-1}^{1} x^2 \sin n\pi x\,dx = 0\ f(x) \text{ is odd function in this integral}\right]$$

Putting $a_0$, $a_n$ and $b_n$ in (1), we obtain

$$x - x^2 = -\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2}\cos n\pi x + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin n\pi x.$$

***Example 16:*** ***Prove that*** $\boldsymbol{\frac{l}{2} - x = \frac{l}{x}\sum_{1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l},\ 0 < x < l.}$ **[B.C.A. (Kanpur) 2009]**

***Solution:*** Here, $F(x) = (l/2) - x$

and $$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{l} \qquad ....(1)$$

where $$a_0 = \frac{1}{l}\int_0^l F(x)\,dx = \frac{1}{l}\int_0^l \left(\frac{l}{2} - x\right)dx = \frac{1}{l}\left[\frac{l\,x}{2} - \frac{x^2}{2}\right]_0^l = 0$$

$$a_n = \frac{2}{l}\int_0^l F(x)\cos\frac{2n\pi x}{l}\,dx = \frac{2}{l}\int_0^l \left(\frac{l}{2}-x\right)\cos\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left\{\left(\frac{l}{2}-x\right)\frac{l}{2n\pi}\sin\frac{2n\pi x}{l}\right\}_0^l - \frac{l}{2n\pi}\int_0^l (-1)\sin\frac{2n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\frac{l^2}{4n^2\pi^2}(\cos 2n\pi - 1) = \frac{l}{2n^2\pi^2}(1-1) = 0$$

and $$b_n = \frac{2}{l}\int_0^l F(x)\sin\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \left(\frac{l}{2}-x\right)\sin\frac{2n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left\{\left(\frac{l}{2}-x\right)\left(-\frac{l}{2n\pi}\cos\frac{2n\pi x}{l}\right)\right\}_0^l - \int_0^l (-1)\left(-\frac{l}{2n\pi}\cos\frac{2n\pi x}{l}\right)dx\right]$$

$$= \frac{2}{l}\left[\frac{l}{2}\frac{l}{2n\pi}\cos 2n\pi + \frac{l}{2}\frac{l}{2n\pi} - \left\{\left(\frac{l}{2n\pi}\right)\left(\frac{l}{2n\pi}\right)\sin\frac{2n\pi x}{l}\right\}_0^l\right] = \frac{l}{n\pi}$$

Putting these values in equation (1), we get required Fourier Series as follows:

$$\left(\frac{l}{2}-x\right) = \frac{l}{\pi}\sin\frac{2\pi x}{l} + \frac{l}{2\pi}\sin\frac{4\pi x}{l} + \frac{l}{3\pi}\sin\frac{6\pi x}{l} + \ldots\ldots = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\sin\frac{2n\pi x}{l}.$$

***Example 17:*** ***Find the Fourier Series for*** $F(x) = 1 + \frac{2x}{l}, -l < x < 0$ ***and*** $F(x) = 1 - \frac{2x}{l}$***, if*** $0 < x < l$.

**[B.C.A. (Lucknow) 2010, 06]**

***Solution:*** We know that $F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

where $$a_0 = \frac{1}{2l}\int_{-l}^{l} F(x)\,dx$$

$$= \frac{1}{2l}\int_{-l}^{0}\left(1+\frac{2x}{l}\right)dx + \frac{1}{2l}\int_0^l \left(1-\frac{2x}{l}\right)dx \qquad [\text{Here } c = l]$$

$$= \frac{1}{2l}\left(x+\frac{x^2}{l}\right)_{-l}^{0} + \frac{1}{2l}\left(x-\frac{x^2}{l}\right)_0^l = \frac{1}{2l}[l-l] + \frac{1}{2l}[l-l] = 0$$

$$a_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{0}\left(1+\frac{2x}{l}\right)\cos\frac{n\pi x}{l}\,dx + \frac{1}{l}\int_{0}^{l}\left(1-\frac{2x}{l}\right)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\left(1+\frac{2x}{l}\right)\frac{l}{n\pi}\sin\frac{n\pi x}{l} - \left(\frac{2}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{0}$$

$$+\frac{1}{l}\left[\left(1-\frac{2x}{l}\right)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - \left(-\frac{2}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\right)\cos\frac{n\pi x}{l}\right]_{0}^{1}$$

$$= \frac{2}{n^2\pi^2}[1-(-1)^n-(-1)^n+1] = \frac{4}{n^2\pi^2}[1-(-1)^n] = \begin{cases}\dfrac{8}{n^2\pi^2}, & \text{when } n \text{ is odd funciton}\\ 0, & \text{when } n \text{ is even funciton}\end{cases}$$

Now $b_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} F(x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{1}{l}\int_{-l}^{0}\left(1+\frac{2x}{l}\right)\sin\frac{n\pi x}{l}\,dx + \frac{1}{l}\int_{0}^{l}\left(1-\frac{2x}{l}\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\left(1+\frac{2x}{l}\right)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{2}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{0}$$

$$+\frac{1}{l}\left[\left(1-\frac{2x}{l}\right)\left(-\frac{l}{nx}\cos\frac{n\pi x}{l}\right) - \left(-\frac{2}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\right)\sin\frac{n\pi x}{l}\right]_{0}^{l}$$

$$= 0$$

$$F(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\ldots\right].$$

***Example 18:* Find the Fourier Series for the function $f(x)$ in interval $(-1, 1)$.**

***where*** $\quad f(x) = \begin{cases} x+1 & \textit{where } -1 < x < 0, \\ x-1 & \textit{where } 0 < x < 1, \end{cases}.$

**[B.C.A. (Kurukshetra) 2011, 08, 05]**

***Solution:*** We know that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{c} \qquad \ldots(1)$$

where $\quad a_0 = \dfrac{1}{2c}\displaystyle\int_{-c}^{c} f(x)\,dx,\quad$ Here $c = 1$

$$= \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left\{\int_{-1}^{0}(x+1)\,dx + \int_{0}^{1}(x-1)\,dx\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{x^2}{2}+x\right)_{-1}^{0} + \left(\frac{x^2}{2}-x\right)_{0}^{1}\right\} = 0$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx$$

$$= \int_{-1}^{0} (x+1)\cos n\pi x\,dx + \int_{0}^{1} (x-1)\cos n\pi x\,dx$$

$$= \left\{(x+1)\frac{\sin n\pi x}{n\pi}\right]_{-1}^{0} - \left\{-\frac{\cos n\pi x}{n^2\pi^2}\right]_{-1}^{0} + \left\{(x-1)\frac{\sin n\pi x}{n\pi}\right]_{0}^{1} - \left\{-\frac{\cos n\pi x}{n^2\pi^2}\right]_{0}^{1}$$

$$= 0 + \left\{\frac{1}{n^2\pi^2} - \frac{\cos n\pi}{n^2\pi^2}\right\} + 0 + \left\{\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right\} = 0$$

and $$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx$$

$$= \int_{-1}^{0} (x+1)\sin n\pi x\,dx + \int_{0}^{1} (x-1)\sin n\pi x\,dx$$

$$= \left\{(x+1)\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_{-1}^{0} + \left\{(x-1)\left(-\frac{\cos n\pi x}{n\pi}\right) - \left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{0}^{1}$$

$$= -\frac{2}{n\pi}$$

Hence, the required Fourier Series is

$$f(x) = -\frac{2}{\pi}\left\{\frac{1}{1}\sin \pi x + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \ldots\ldots\right\}.$$

***Example 19:*** ***Find the Fourier Series for function*** $f(x) = \begin{cases} 2, & -2 \le x < 0 \\ x, & 0 < x < 2 \end{cases}$ ***in interval*** $(-2, 2)$.

**[B.C.A. (Meerut) 2006]**

***Solution:*** Here, interval is $(-2, 2)$ *i.e.*, $c = 2$

$$f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{c} + \sum b_n \sin\frac{n\pi x}{c} \qquad \ldots(1)$$

Now $$a_0 = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2.2}\left[\int_{-2}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx\right]$$

$$= \frac{1}{4}\left[\int_{-2}^{0} 2.\,dx + \int_{0}^{2} x.\,dx\right] = \frac{1}{4}[4+2] = \frac{3}{2}$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} f(x)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} f(x)\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\left(\sin\frac{n\pi x}{2}\right)_{-2}^{0} + \left(x\frac{2}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)_{0}^{2}\right]$$

$$= \frac{2}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} -\dfrac{4}{n^2\pi^2}, & \text{when } n \text{ is odd,} \\ 0, & \text{when } n \text{ is even,} \end{cases}$$

Also $$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left(-2\,\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)_{-2}^{0} + \left(-x\frac{2}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)_{0}^{2}\right] = -\frac{2}{n\pi}$$

Putting all above values in (1), we get required Fourier Series

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots\ldots\right\}$$
$$-\frac{2}{\pi}\left\{\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \ldots\ldots\right\}$$

***Example 20:* Obtain the Fourier coefficients for the function**

$$f(t) = \begin{cases} 0, & \text{when } -5 < t < 0 \\ 3, & \text{when } 0 < t < 5 \end{cases}, \quad T = 10.$$

**[B.C.A. (Rohilkhand) 2008]**

***Solution:*** We know that $a_0 = \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} f(t)\,dt = \dfrac{1}{10}\int_{-5}^{5} f(t)\,dt$

$$= \frac{1}{10}\left[\int_{-5}^{0} f(t)\,dt + \int_{0}^{5} f(t)\,dt\right]$$

$$= \frac{1}{10}\left[\int_{-5}^{0} 0\,.\,dt + \int_{0}^{5} 3\,.\,dt\right] = \frac{1}{10}[3\,.5] = \frac{3}{2}$$

$$a_n = \frac{1}{5}\int_{-5}^{5} f(t)\cos\left(\frac{n\pi}{5}t\right)dt$$

$$= \frac{1}{5}\left[\int_{-5}^{0} 0 . \cos\left(\frac{n\pi}{5}t\right)dt + \int_{0}^{5} 3 . \cos\left(\frac{n\pi}{5}t\right)dt\right]$$

$$= \frac{1}{5} . 3 . \frac{5}{n\pi}\left[\sin\left(\frac{n\pi}{5}t\right)\right]_0^5 = \frac{3}{n\pi}\sin n\pi = 0$$

$$b_n = \frac{1}{5}\int_{-5}^{5} f(t)\sin\left(\frac{n\pi}{5}t\right)dt = \frac{1}{5}\left[\int_{-5}^{0} 0 . \sin\left(\frac{n\pi}{5}t\right)dt + \int_{0}^{5} 3 . \sin\left(\frac{n\pi}{5}t\right)dt\right]$$

$$= \frac{1}{5} . 3 . \frac{5}{n\pi}\left[-\cos\left(\frac{n\pi}{5}t\right)\right] = \frac{3}{n\pi}[1-\cos n\pi] = \begin{cases} 0, & \text{when } n \text{ is odd} \\ \frac{6}{n\pi}, & \text{when } n \text{ is even} \end{cases}$$

$\therefore$
$$f(t) = \frac{3}{2} + \frac{6}{\pi}\left[\sin\frac{\pi t}{5} + \frac{1}{3}\sin\frac{3\pi t}{5} + \frac{1}{5}\sin\frac{5\pi t}{5} + \ldots\ldots\right].$$

***Example 21:*** ***In interval*** $(-\pi, \pi)$ ***find the Fourier Series for function*** $f(x) = \dfrac{\pi}{2\sinh\pi}e^x$.

***Solution:*** Here, $$\frac{\pi}{2\sinh\pi}e^x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\pi e^x}{2\sinh\pi}dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos nx\int_{-\pi}^{\pi}\frac{\pi}{2\sinh\pi}e^x\cos nx\,dx$$
$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\sin nx\int_{-\pi}^{\pi}\frac{\pi}{2\sinh\pi}e^x\sin nx\,dx$$

Now
$$\int_{-\pi}^{\pi}e^x dx = [e^x]_{-\pi}^{\pi} = e^{\pi} - e^{-\pi} = 2\sinh\pi$$

$$\int_{-\pi}^{\pi}e^x\cos nx\,dx = \left(\frac{e^x\sin nx}{n}\right)_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}e^x\sin nx\,dx$$

$$= \frac{1}{n^2}[e^x\cos nx]_{-\pi}^{\pi} - \frac{1}{n^2}\int_{-\pi}^{\pi}e^x\cos nx\,dx$$

or
$$\left(1+\frac{1}{n^2}\right)\int_{-\pi}^{\pi}e^x\cos nx\,dx = \frac{1}{n^2}(e^{\pi}-e^{-\pi})\cos n\pi$$

or
$$\int_{-\pi}^{\pi}e^x\cos nx\,dx = \frac{2}{1+n^2}\sinh\pi\cos n\pi \qquad (\because e^{\pi}-e^{-\pi} = 2\sinh\pi)$$

Now
$$\int_{-\pi}^{\pi}e^x\sin nx\,dx = \left[-\frac{e^x\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}e^x\cos nx\,dx$$

$$= -\frac{1}{n}(e^{\pi}-e^{-\pi})\cos n\pi + \frac{1}{n}\left[\left\{\frac{e^x\sin nx}{n}\right\}_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^x\sin nx}{n}dx\right]$$

or $$\left(1+\frac{1}{n^2}\right)\int_{-\pi}^{\pi} e^x \sin nx\, dx = -\frac{1}{n}(e^{\pi}-e^{-\pi})\cos n\pi$$

or $$\int_{-\pi}^{\pi} e^x \sin nx\, dx = -\frac{2n}{1+n^2}\sinh\pi \cos n\pi$$

Therefore

$$\frac{\pi}{2\sinh\pi}e^x = \frac{1}{2}+\sum_{n=1}^{\infty}\frac{\cos n\pi}{1+n^2}\cos n\pi - \sum_{n=1}^{\infty}\left\{\frac{n}{1+n^2}\cos n\pi \sin n\pi\right\}$$

$$= \frac{1}{2}-\left(\frac{1}{2}\cos x - \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x - \frac{1}{17}\cos 4x + \ldots..\right)$$

$$+\left(\frac{1}{2}\sin x - \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x - \frac{4}{17}\sin 4x + \ldots.\right).$$

## Exercise 8.2

1. Find the sine series of $f(x) = x$ in interval $0 < x < \pi$.
2. For function $f(x) = x+1, 0 < x < \pi$ find a series:
   (i) Fourier sine series
   (ii) Fourier cosine series
3. Express the following functions as a half range sine series:
   (i) $f(x) = x, \quad 0 < x < 2$  (ii) $f(x) = x, \quad 0 < x < \pi$
   (iii) $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \frac{x}{2}, & \frac{\pi}{2} < x < \pi \end{cases}$  (iv) $f(x) = (x-1)^2, \; 0 < x < 1.$
4. Express the following functions as a half range cosine series:
   (i) $f(x) = x, \; 0 < x < 2$
   (ii) $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$ **[B.C.A. (Bhopal) 2007, 03]**
   (iii) $f(x) = (x-1)^2, \; 0 < x < 1.$
5. In interval $(-\pi, \pi)$ show that $\dfrac{x(\pi^2 - x^2)}{1^2} = \dfrac{\sin x}{1^3} - \dfrac{\sin 2x}{2^3} + \dfrac{\sin 3x}{3^3} + \ldots\ldots.$

**6.** Find the Fourier Series of the following function:

$$f(x)=\begin{cases}\left(\dfrac{\pi-x}{2}\right), & 0<x<\pi\\ -\left(\dfrac{\pi-x}{2}\right), & -\pi<x<0\\ 0, & x=0 \text{ or } x=\pm\pi\end{cases}$$

**[B.C.A. (Delhi) 2012, 08]**

## Answers 8.2

**1.** $2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\dots\right]$.

**2.** (i) $\frac{2}{\pi}\left[(\pi+2)\sin x-\frac{\pi}{2}\sin 2x-\frac{1}{3}(\pi+2)\sin 3x-\frac{\pi}{4}\sin 4x+\dots\right]$.

(ii) $\frac{\pi}{2}+1=4\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\dots\right)$.

**3.** (i) $f(x)=\frac{4}{\pi}\left[\frac{\sin \pi x}{2}-\frac{1}{2}\frac{\sin 2\pi x}{2}+\frac{1}{3}\frac{\sin 3\pi x}{2}-\dots\right]$.

(ii) $f(x)=2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\dots\right]$.

(iii) $f(x)=\left(1+\frac{2}{\pi}\right)\sin x-\frac{1}{2}\sin 2x+\left(\frac{1}{3}-\frac{2}{9\pi}\right)\sin 3x-\frac{1}{4}\sin 4x+\dots$.

(iv) $\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin n\pi x$.

**4.** (i) $f(x)=1-\frac{8}{\pi^2}\left[\frac{\cos(\pi x/2)}{1^2}+\frac{\cos(3\pi x/2)}{3^2}+\dots\right.$

(ii) $f(x)=\frac{\pi}{4}-\frac{2}{\pi}\left[\frac{\cos(2x)}{1^2}+\frac{\cos 6x}{3^2}+\frac{\cos 10x}{5^2}+\dots\right]$.

(iii) $\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$.

**6.** $f(x)=\sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\dots$.

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# Chapter 9

# Differential Equations of First Order and First Degree

## 9.1 Differential Equations

**Definition:** "*A **Differential Equation** is an equation that involves independent and dependent variables and the derivatives of the dependent variables .*"

### 9.1.1 Ordinary Differential Equation

A differential equation which involves only one independent variable is an **Ordinary Differential Equation.** Thus, the differential equations:

$$\sin y\, dy = \cos x\, dx \qquad \text{...(1)}$$

$$\frac{d^2 y}{dx^2} = a^2 y \qquad \text{...(2)}$$

$$x^3 \left(\frac{d^2 y}{dx^2}\right)^3 + y^2 \left(\frac{dy}{dx}\right)^4 + y^3 = 0 \qquad \text{...(3)}$$

$$\frac{d^2 y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0 \qquad \text{...(4)}$$

$$\frac{[1 + (dy/dx)^2]^{3/2}}{d^2 y/dx^2} = \rho \qquad \text{...(5)}$$

are all examples of ordinary differential equations. They involve single independent variable $x$.

### 9.1.2 Order of a Differential Equation [B.C.A. (Agra) 2009]

**Definition:** "*The* ***order of a differential equation*** *is the order of the highest differential coefficient which occurs in it.*"

Thus, if a differential equation contains $n$th and lower derivatives, it is said to be of **$n$th order.**

### 9.1.3 Degree of a Differential Equation

**Definition:** "*The* ***degree of a differential equation*** *is the degree of the highest differential coefficient which occurs in it, when the differential equation is independent of radicals and fractional powers.*"

Equations of degree higher than one are also called **non-linear.**

Thus the nature of the above differential equations are as follows:

**Table 9.1**

| Equation | Type | Order | Degree |
|---|---|---|---|
| 1. | Ordinary | 1 | 1 |
| 2. | Ordinary | 2 | 1 |
| 3. | Ordinary | 2 | 3 |
| 4. | Ordinary | 2 | 2 |
| 5. | Ordinary | 2 | 2 |

## 9.2 The Derivation of a Differential Equation or Formulation of Differential Equation

Let us consider the following examples:

1. Let $$y = ce^x \qquad ...(1)$$

or $$\frac{dy}{dx} = ce^x \qquad ...(2)$$

Eliminating the arbitrary constant $c$ between (1) and (2), we obtain

$$\frac{dy}{dx} = y.$$

Thus, we observe that eliminating one arbitrary constant $c$, we have a differential equation of the **first order.**

2. Let $$y = c_1 \cos x + c_2 \sin x \qquad ...(3)$$

$\therefore$ $$\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x \qquad ...(4)$$

and $$\frac{d^2y}{dx^2} = -c_1 \cos x - c_2 \sin x = -y \qquad \text{[by (3)]}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} + y = 0$$

Thus, by eliminating two arbitrary constants $c_1$ and $c_2$, we have a differential equation of **second order.**

Now, consider the general process. The equation

$$f(x, y, c_1, c_2, \ldots, c_n) = 0 \qquad \ldots(5)$$

contains, besides $x$ and $y, n$ arbitrary constants $c_1, c_2, \ldots, c_n$. Differentiating (5) $n$ times in succession with respect to $x$ gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y}\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dx^2} = 0$$

$$\frac{\partial^n f}{\partial x^n} + \ldots + \frac{\partial f}{\partial x}\frac{d^n y}{\partial x^n} = 0.$$

Between the original equation (5) and the $n$ equations thus obtained by differentiation, making $n+1$ equations in all, the $n$ constants $c_1, c_2, \ldots, c_n$ can be eliminated, and thus we obtain a differential equation of $n$th order, *i.e.,*

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^n y}{dx^n}\right) = 0. \qquad \ldots(6)$$

## 9.3 General and Particular Solutions

### 9.3.1 General Solution **[B.C.A. (Kanpur) 2012, 06]**

**Definition:** "*The solution which contains as many arbitrary constants as the order of the differential equation, is called the* ***general solution*** *or the* ***complete integral.***"

### 9.3.2 Particular Solution **[B.C.A. (Kanpur) 2009]**

**Definition:** "*Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a* ***particular solution*** *or* ***particular integral.***"

## Solved Examples

**Example 1:** ***Show that*** $y = c_1 \cos(\log x) + c_2 \sin(\log x)$ ***is a solution of the differential equation***

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

**[B.C. A. (Agra) 2010, 03, 02]**

**Solution:** We have $y = c_1 \cos(\log x) + c_2 \sin(\log x)$ ...(1)

Differential (1) w.r.t ($x$), we get

$$\frac{dy}{dx} = c_1 \{-\sin(\log x)\}\frac{1}{x} + c_2 \cos(\log x)\cdot\frac{1}{x}$$

or $$x\frac{dy}{dx} = -c_1 \sin x(\log x) + c_2 \cos(\log x) \quad ...(2)$$

Again differentiate (2) w.r.t ($x$), we get

$$\frac{dy}{dx} + x\frac{d^2 y}{dx^2} = -\frac{c_1 \cos(\log x)}{x} - \frac{c_2 \sin(\log x)}{x}$$

or $$x\frac{dy}{dx} + x^2\frac{d^2 y}{dx^2} = -\{c_1 \cos(\log x) + c_2 \sin(\log x)\}$$

or $$x\frac{dy}{dx} + x^2\frac{d^2 y}{dx^2} = -y \quad \text{[From (1)]}$$

or $$x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Hence, $y = c_1 \cos(\log x) + c_2 \sin(\log x)$

be a solution of the given differential equation.

**Example 2:** ***Form the differential equation from*** $y = Ae^{2x} + Be^{x} + C$, ***where*** $A$, $B$ ***and*** $C$ ***are constant.*** **[B.C.A. (Bundelkhand) 2011, 09, 06]**

**Solution:** The given equation is

$$y = Ae^{2x} + Be^{x} + C \quad ...(1)$$

Differentiating (1) w.r.t to $x$, we get

$$\frac{dy}{dx} = 2Ae^{2x} + Be^{x}$$

$$\Rightarrow \quad e^{-x}\frac{dy}{dx} = 2Ae^{x} + B \quad ...(2)$$

Differentiating (2) w.r.t. to $x$, we get

$$-e^{-x}\frac{dy}{dx}+e^{-x}\frac{d^2y}{dx^2}=2Ae^x$$

$$\Rightarrow \qquad -e^{-2x}\frac{dy}{dx}+e^{-2x}\frac{d^2y}{dx^2}=2A \qquad \ldots(3)$$

Differentiating once again w.r.t. to $x$, we get

$$2e^{-2x}\frac{dy}{dx}-e^{-2x}\frac{d^2y}{dx^2}-2e^{-2x}\frac{d^2y}{dx^2}+e^{-2x}\frac{d^3y}{dx^3}=0$$

$$\Rightarrow \qquad \frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

which is the required differential equation.

***Example 3:*** ***Find the order and the degree of the following differential equations:***

(*i*) $\dfrac{dy}{dx}=\cot x,$ **[B.C.A. (Bhopal) 2012]**

(*ii*) $\left(\dfrac{d^2y}{dx^2}\right)^3-xy\left(\dfrac{dy}{dx}\right)^4+y=0,$ **[B.C.A. (Bhopal) 2012]**

(*iii*) $\dfrac{d^2y}{dx^2}-k^2y=0,$ **[B.C.A. (Rohilkhand) 2012]**

(*iv*) $\dfrac{d^2y}{dx^2}+\sqrt{1+\left(\dfrac{dy}{dx}\right)^3}=0,$ **[B.C.A. (Rohilkhand) 2012 ]**

(*v*) $\left(\dfrac{d^3y}{dx^3}\right)^2-xy\left(\dfrac{dy}{dx}\right)^3+y=0,$ **[B.C.A. (Kanpur) 2008, 06]**

(*vi*) $\dfrac{d^2x}{dy^2}+\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}=0,$ **[B.C.A. (Meerut) 2004]**

(*vii*) $\dfrac{d^2y}{dx^2}+y=0,$

(*viii*) $\dfrac{dy}{dx}=\sin x.$ **[B.C. A. (Rohilkhand) 2008, 06]**

***Solution:***

(i) First order, first degree.

(ii) Second order, third degree.

(iii) Second order, first degree.

(iv) Second order, first degree.

(v) Third order, second degree.

(vi) Second order, first degree.

(vii) Second order, first degree.

(viii) First order, first degree.

***Example 4:*** ***Find the differential equation by the family of curves*** $y=c_1\cos ax+c_2\sin ax$, ***where*** $c_1$ ***and*** $c_2$ ***are arbitrary constants.*** **[B.C.A. (Rohtak) 2009]**

***Solution:*** The equation of given family of curves is

$$y=c_1\cos ax+c_2\sin ax \qquad \ldots(1)$$

Since this involves two arbitrary constants $c_1$ and $c_2$, we have to differentiate it twice to eliminate $c_1$ and $c_2$.

Now, $$\frac{dy}{dx} = -ac_1 \sin ax + ac_2 \cos ax$$

and $$\frac{d^2y}{dx^2} = -a^2 (c_1 \cos ax + c_2 \sin ax) = -a^2 y \qquad \text{[by (1)]}$$

or $$\frac{d^2y}{dx^2} + a^2 y = 0$$

which is the required differential equation.

***Example 5:*** ***Find the differential equation by family of curves $y = Ae^{3x} + Be^{5x}$, for different values of A and B.*** **[B.C.A. (I.G.N.O.U.) 2008]**

***Solution:*** The equation of given family of curves is

$$y = Ae^{3x} + Be^{5x}. \qquad ...(1)$$

Differentiating (1) with respect to $x$, we get

$$\frac{dy}{dx} = 3Ae^{3x} + 5Be^{5x} \qquad ...(2)$$

Differentiating (2) again with respect to $x$, we get

$$\frac{d^2y}{dx^2} = 9Ae^{3x} + 25Be^{5x} \qquad ...(3)$$

Now, $$\frac{dy}{dx} - 3y = 2Be^{5x} \quad \text{or} \quad Be^{5x} = \frac{1}{2}\left(\frac{dy}{dx} - 3y\right) \qquad ...(4)$$

From (1), we get

$$Ae^{3x} = y - Be^{5x} = y - \frac{1}{2}\left(\frac{dy}{dx} - 3y\right)$$

or $$Ae^{3x} = \frac{1}{2}\left(5y - \frac{dy}{dx}\right) \qquad ...(5)$$

Eliminating $A$ and $B$ from (3), (4) and (5), we get

$$\frac{d^2y}{dx^2} = 9\left[\frac{1}{2}\left(5y - \frac{dy}{dx}\right)\right] + 25\left[\frac{1}{2}\left(\frac{dy}{dx} - 3y\right)\right]$$

$$= \frac{45}{2}y - \frac{9}{2}\frac{dy}{dx} + \frac{25}{2}\frac{dy}{dx} - \frac{75}{2}y = -15y + 8\frac{dy}{dx}$$

or $$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0.$$

**Example 6:** ***Form the differential equation of*** $y = A\cos(x^2) + B\sin(x^2)$.

**[B.C.A. (Kashi) 2010, 06; B.C.A. (Agra) 2009, 06]**

**Solution:** The given equation is

$$y = A\cos(x^2) + B\sin(x^2) \quad \text{...(1)}$$

Differentiating (1) with respect to $x$, we get

$$\frac{dy}{dx} = A[-\sin(x^2)].2x + B[\cos(x^2)].2x$$

$$= 2x[-A\sin(x^2) + B\cos(x^2)]. \quad \text{...(2)}$$

Again differentiating with respect to $x$, we get

$$\frac{d^2y}{dx^2} = 2[-A\sin(x^2) + B\cos(x^2)] + 2x[-A\{\cos(x^2)\}(2x) + B\{-\sin(x^2)\}.(2x)]$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2[-A\sin(x^2) + B\cos(x^2)] - 4x^2[A\cos(x^2) + B\sin(x^2)] \qquad \text{[by (1) and (2)]}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x}\frac{dy}{dx} - 4x^2 y \;\Rightarrow\; x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3 y = 0.$$

**Example 7:** ***Show that*** $v = \frac{A}{r} + B$ ***is a solution of differential equation***

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0.$$

**[B.C.A. (Avadh) 2008, 04, 02]**

**Solution:** The given relation is

$$v = \frac{A}{r} + B. \quad \text{...(1)}$$

Differentiating (1) with respect to $r$, we get

$$\frac{dv}{dr} = -\frac{A}{r^2}. \quad \text{...(2)}$$

Again differentiating both sides with respect to $r$, we get

$$\frac{d^2v}{dr^2} = \frac{2A}{r^3}. \quad \text{...(3)}$$

From (2) and (3), we have

$$\frac{d^2v}{dr^2} \Big/ \frac{dv}{dr} = \frac{2A/r^3}{-A/r^2} \quad \Rightarrow \frac{d^2v}{dr^2} = -\frac{2}{r}\frac{dv}{dr} \quad \Rightarrow \frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0.$$

**Example 8:** ***Show that*** $Ax^2 + By^2 = 1$ ***is the solution of***

$$x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=y\frac{dy}{dx}.$$

***Or***

***Find the differential equation of all conics whose principal axes coincide with the co-ordinate axis.*** **[B.C.A. (Lucknow) 2006]**

**Solution:** The given relation is

$$Ax^2 + By^2 = 1. \quad \text{...(1)}$$

Differentiating both sides of (1) with respect to $x$, we get

$$2Ax + 2By\,.\frac{dy}{dx} = 0$$

or

$$Ax + By\frac{dy}{dx} = 0. \quad \text{...(2)}$$

Again differentiating with respect to $x$, we get

$$A + B\left[y\frac{d^2y}{dx^2}+\frac{dy}{dx}\,.\,\frac{dy}{dx}\right]=0$$

or

$$Ax + Bx\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=0 \quad \text{...(3)}$$

Subtracting (2) from (3), we get

$$Bx\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-By\frac{dy}{dx}=0 \quad \text{or} \quad x\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=y\frac{dy}{dx}.$$

**Example 9:** ***By the elimination of the constants h and k, find the differential equation of which*** $(x-h)^2 + (y-k)^2 = a^2$ ***is a solution.*** **[B.C. A. (Agra) 2011, 06]**

**Solution:** The given solution is

$$(x-h)^2 + (y-k)^2 = a^2. \quad \text{...(1)}$$

Three relations are necessary to eliminate two constants and thus differentiating the given relations twice successively, we have

$$(x-h) + (y-k)\frac{dy}{dx} = 0 \quad \text{...(2)}$$

$$1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0. \quad \text{...(3)}$$

From (2) and (3), we obtain

$$y-k=-\frac{1+(dy/dx)^2}{d^2y/dx^2},\quad x-h=\frac{[1+(dy/dx)^2]\,dy/dx}{d^2y/dx^2}.$$

Substituting these values in (1), we obtain

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

## Exercise 9.1

1. What is the order of differential equation whose solution is the circle $(x-\alpha)^2+y^2=\alpha^2$, where $\alpha$ is an arbitrary constant? **[B.C.A. (Bhopal) 2009, 07, 03]**
2. By the elimination of the constant $a$, obtain the differential equation of which $y^2=4a(x+a)$ is the solution. **[B.C.A. (Kanpur) 2009]**
3. Find the differential equation corresponding to the family of curves $y=c(x-c)^2$, where $c$ is an arbitrary constant. **[B.C.A. (Rohilkhand) 2011]**
4. Show that $y=A\cos x+\sin x$ is a solution of the differential equation $\cos x\frac{dy}{dx}+y\sin x=1.$ **[B.C.A. (I.G.N.O.U.) 2012]**
5. Find the differential equation of the family of curves $y=e^x(A\cos x+B\sin x)$, where $A$ and $B$ are arbitrary constants. **[B.C.A. (Avadh) 2010]**
6. Find the differential equation whose general solution is $y=Ae^{2x}+Be^{-2x}$.
7. Find the differential equation corresponding to $y=ae^{2x}+be^{-3x}+ce^x$, where $a, b, c$ are arbitrary constants. **[B.C.A. (Meerut) 2004]**
8. Discuss the general and a particular solution of a differential equation. **[B.C.A. (Lucknow) 2011, 04, 02]**

## Answers 9.1

| | | | |
|---|---|---|---|
| 1. | First order. | 2. | $y\left[1-\left(\frac{dy}{dx}\right)^2\right]=2x\frac{dy}{dx}.$ |
| 3. | $\left(\frac{dy}{dx}\right)^3-4xy\frac{dy}{dx}+8y^2=0.$ | 5. | $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0.$ |
| 6. | $\frac{d^2y}{dx^2}=4y.$ | 7. | $\frac{d^3y}{dx^3}-7\frac{dy}{dx}+6y=0.$ |

## 9.4 Separation of Variables

We start with procedures for solving first order differential equations whose variables can be separated. This method is called **separation of variables** and is outlined as follows:

Let the first order differential equation be

$$\phi\left(x, y \frac{dy}{dx}\right) = 0 \qquad ...(1)$$

Solving the differential equation (1) for $\frac{dy}{dx}$ we get, say,

$$\frac{dy}{dx} = f(x, y) \qquad ...(2)$$

Suppose that $f(x, y)$ can be written in the following form

$$f(x, y) = f_1(x) / f_2(y), \qquad ...(3)$$

where $f_1$ and $f_2$ are continuous, then (2) can be written with variables separated in the differential form:

$$f_2(y)\frac{dy}{dx} = f_1(x) \quad \text{or} \quad f_2(y)\, dy = f_1(x)\, dx \qquad ...(4)$$

Thus, a differential equation which can be represented in the form (3) is called **variables separable form.**

Now integrating both sides of (4), we get

$$\int f_2(y)\, f(y)\, dy = \int f_1(x)\, dx + c \qquad ...(5)$$

where $c$ is an arbitrary constant of integration. This constant, being arbitrary, can be put in any form and either side of (5) as we like. Since the integral involved in (5) can be evaluated hence the result will be free from any differential and there by giving the general solution of the differential equation (2).

***Example 10:*** ***Solve*** $\mathbf{(x + 1)\frac{dy}{dx} = x(y^2 + 1)}$**.** **[B.C.A. (Agra) 2012, 08, 06, 02]**

***Solution:*** We have $(x + 1)\frac{dy}{dx} = x(y^2 + 1)$

Separate the variables, we get

$$\frac{dy}{1 + y^2} = \left(\frac{x}{x + 1}\right) dx$$

Integrate both sides and add constant of integration

$$\int \frac{dy}{1 + y^2} = \int \left(1 - \frac{1}{x + 1}\right) dx + c$$

$$\Rightarrow \qquad \tan^{-1}(y) = x - \log(1+x) + c$$

or
$$\log(1+x) = c + x - \tan^{-1} y$$

or
$$(1+x) = e^{c + x - \tan^{-1} y}$$

or
$$(1+x) = e^{c}\, e^{x - \tan^{-1} y}$$

or
$$(1+x) = \lambda\, e^{x - \tan^{-1} y}. \qquad [\text{where } \lambda = e^{c} = \text{constant}]$$

**Example 11:** ***Solve*** $\boldsymbol{x\,(e^{y} + 4)\,dx + e^{x+y}\,dy = 0.}$ **[B.C.A. (Kurukshetra) 2010]**

***Solution:*** We have

$$x\,(e^{y} + 4)\,dx + e^{x+y}\,dy = 0$$

Separate the variables, we get

$$\left(\frac{x}{e^{x}}\right) dx + \left(\frac{e^{y}}{e^{y} + 4}\right) dy = 0$$

Integrate we obtain

$$\int x e^{-x}\,dx + \int \frac{e^{y}}{e^{y} + 4}\,dy = c$$

or
$$-\,x e^{-x} - e^{-x} + \log(e^{y} + 4) = c$$

or
$$\log(e^{y} + 4) - (x+1)\,e^{-x} = c.$$

**Example 12:** ***Solve*** $\boldsymbol{\dfrac{dy}{dx} = \sqrt{y - x}.}$

***Solution:*** We have $\dfrac{dy}{dx} = \sqrt{y - x}$ ...(1)

Put $y - x = t^{2} \Rightarrow \dfrac{dy}{dx} - 1 = 2t\,\dfrac{dt}{dx}$

$$\Rightarrow \qquad \frac{dy}{dx} = 2t\,\frac{dt}{dx} + 1$$

or
$$2t\,\frac{dt}{dx} = t - 1$$

or
$$\left(\frac{2t}{t-1}\right) dt = dx$$

or
$$\left(2 + \frac{2}{t-1}\right) dt = dx$$

Integrate we get

$$2t + 2\log(t-1) = x + c$$

or $$2\sqrt{y-x} + 2\log(\sqrt{y-x} - 1) = x + c$$

***Example 13:* Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y}$.**

***Solution:*** The given differential equation is

$$\frac{dy}{dx} = \frac{x}{y} \qquad \text{or} \qquad y\,dy = x\,dx$$

which is in variables separable form.

Hence integrating both sides, we get

$$\int y\,dy = \int x\,dx + C, \text{ where } C \text{ is arbitrary constant of integration.}$$

or $$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C \qquad \text{or} \qquad y^2 = x^2 + 2C$$

or $$y^2 = x^2 + c, \qquad \text{where} \qquad c = 2C$$

which is the required solution.

***Example 14:* Solve $(x^2 - yx^2)\,dy + (y^2 + xy^2)\,dx = 0$.** **[B.C.A. (Rohilkhand) 2009]**

***Solution:*** Rearranging the given differential equation, we have

$$x^2(1-y)\,dy + y^2(1+x)\,dx = 0$$

or $$\left(\frac{1-y}{y^2}\right)dy + \left(\frac{1+x}{x^2}\right)dx = 0$$

or $$\left(\frac{1}{y^2} - \frac{1}{y}\right)dy + \left(\frac{1}{x^2} + \frac{1}{x}\right)dx = 0$$

which is in variables separable form, hence on integration, we get

$$\int \left(\frac{1}{y^2} - \frac{1}{y}\right)dy + \int \left(\frac{1}{x^2} + \frac{1}{x}\right)dx = c,$$

$c$ being constant of integration.

or $$-\frac{1}{y} - \log y - \frac{1}{x} + \log x = c \qquad \text{or} \qquad \log\frac{x}{y} - \frac{x+y}{xy} = c$$

which given the general solution of the given differential equation.

**Example 15:** ***Solve*** $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$.

**[B.C.A. (Agra) 2008, 05]**

***Solution:*** Rearranging the given differential equation, we have

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

which is in variables separable form, hence integrating both sides, we get

$$\tan^{-1} y = \tan^{-1} x + C \qquad \text{or} \qquad \tan^{-1} y - \tan^{-1} x = C$$

or $$\tan^{-1}\left(\frac{y-x}{1+yx}\right) = \tan^{-1} c, \qquad \text{where} \qquad C = \tan^{-1} c$$

or $$y - x = c\,(1 + yx)$$

where $c$ is an arbitrary constant and gives the general solution of the given differential equation.

**Example 16:** ***Solve the differential equation*** $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$.

**[B.C. A. (Rohilkhand) 2010, 08, 03]**

***Solution:*** On separating the variables the given differential equation can be written as

$$\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

On integrating, we get

$$\sin^{-1} y + \sin^{-1} x = c,$$

where $c$ is arbitrary constant of integration which is the required solution.

**Example 17:** ***Solve the differential equation***

$$(1-x^2)(1-y)\,dx = xy\,(1+y)\,dy.$$

**[B.C. A. (Agra) 2007, 00]**

***Solution:*** On separating the variables the given differential equation can be written as

$$\frac{1-x^2}{x}\,dx = \frac{y\,(1+y)}{1-y}\,dy$$

*i.e.,* $$\left(\frac{1}{x} - x\right) dx = \left(-y - 2 + \frac{2}{1-y}\right) dy.$$

On integration, we get

$$\int \left(\frac{1}{x} - x\right) dx = \int \left(-y - 2 + \frac{2}{1-y}\right) dy + c$$

or $$\log x - \frac{1}{2}x^2 = -\frac{1}{2}y^2 - 2y - 2\log(1-y) + c$$

or $$\log\{x(1-y)^2\} = \frac{1}{2}(x^2 - y^2) - 2y + c$$

which is the required solution.

**Example 18:** ***Solve the differential equation*** $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$.

**[B.C.A. (Lucknow) 2006]**

***Solution:*** On separating the variables, the given differential equation can be written as

$$\frac{\cos x}{\sin x}dx + \frac{e^y}{1+e^y}dy = 0.$$

On integration, we get

$$\log \sin x + \log(1+e^y) = \log c, \text{ where } \log c \text{ is arbitrary constant of integration.}$$

$\Rightarrow$ $$\log\{\sin x(1+e^y)\} = \log c$$

$\Rightarrow$ $$\sin x(1+e^y) = c$$

which is the required solution.

**Example 19:** ***Solve the differential equation*** $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$.

**[B.C. A. (Agra) 2004]**

***Solution:*** On separating the variables, the given equation can be written as

$$\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0.$$

On integration, we get

$$\log \tan x + \log \tan y = \log c, \text{ where } \log c \text{ is arbitrary constant of integration.}$$

$\Rightarrow$ $$\tan x \tan y = c$$

which is the required solution.

**Example 20:** ***Solve the differential equation***

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}.$$ **[B.C.A. (Kashividhyapeeth) 2012, 06]**

***Solution:*** On separating the variables, the given differential equation can be written as

$$(\sin y + y\cos y)\,dy = x(2\log x + 1)\,dx.$$

On integration, we get

$$-\cos y + y\sin y - \int 1 . \sin y\, dy = 2\left[\log x . \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2}\, dx\right] + \frac{x^2}{2} + c,$$

where $c$ is arbitrary constant of integration.

$$\Rightarrow \qquad -\cos y + y\sin y + \cos y = 2\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right] + \frac{x^2}{2} + c$$

$$\Rightarrow \qquad y\sin y = x^2 \log x + c$$

which is the required solution.

**Example 21:** ***Solve*** $\left(y - x\dfrac{dy}{dx}\right) = a\left(y^2 + \dfrac{dy}{dx}\right).$ **[B.C. A. (Agra) 2002]**

***Solution:*** Solving the given differential equation for $\dfrac{dy}{dx}$, we get

$$(x + a)\frac{dy}{dx} = y - ay^2 \qquad \text{or} \qquad \frac{dy}{y - ay^2} = \frac{dx}{x + a}$$

which is in variables separable form, hence integrating both sides, we get

$$\int dy / [y(1 - ay)] = \int dx / (x + a) + C$$

or $$\int \left(\frac{1}{y} + \frac{a}{1 - ay}\right) dy = \int \frac{dx}{x + a} + C$$

or $$\log y - \log(1 - ay) = \log(x + a) + \log c, \quad \text{where } C = \log c$$

or $$\log \frac{y}{1 - ay} = \log[c . (x + a)]$$

or $$y / (1 - ay) = c(x + a) \qquad \text{or} \qquad y = c(x + a)(1 - ay)$$

where $c$ is an arbitrary constant and gives general solution of the given differential equation.

**Example 22:** ***Solve*** $(2ax + x^2)\dfrac{dy}{dx} = a^2 + 2ax.$ **[B.C.A. (Kanpur) 2006]**

***Solution:*** Solving the given differential equation for $\dfrac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{a^2 + 2ax}{2ax + x^2}$$

or $$\frac{dy}{dx} = \frac{a}{2}\left[\frac{1}{x} + \frac{3}{x+2a}\right],$$ by breaking into partial fractions

or $$dy = \frac{a}{2}\left[\frac{1}{x} + \frac{3}{x+2a}\right].$$

Thus, the variables has been separated. Now integrating both sides, we get

$$\int dy = \frac{a}{2}\left\{\int \frac{1}{x}\,dx + \int \frac{3}{x+2a}\,dx\right\} + c$$

or $$y = (a/2)\,[\log x + 3\log(x+2a)] + c$$

which gives the general solution of the given differential equation, where $c$ is an arbitrary constant.

***Example 23:*** ***Find the equation of the curve through the point*** $(1, 0)$ ***which satisfies the differential equation*** $(1 + y^2)\,dx - xy\,dy = 0$. **[B.C. A. (Kanpur) 2006]**

***Solution:*** The given differential equation is

$$(1 + y^2)\,dx - xy\,dy = 0 \qquad \text{or} \qquad \frac{dx}{x} - \frac{y\,dy}{1+y^2} = 0,$$

which is in variables separable form. Hence, on integration, we get

$$\int \frac{dx}{x} - \frac{1}{2}\int \frac{2y}{1+y^2}\,dy = C,$$ where $C$ is an arbitrary constant.

or $$\log x - \frac{1}{2}\log(1+y^2) = \log c, \qquad \text{where} \qquad C = \log c$$

or $$\log\left[x/\sqrt{1+y^2}\right] = \log c \qquad \text{or} \qquad x = c\sqrt{1+y^2}$$

or $$x^2 = c^2(1+y^2),$$ where $c$ is an arbitrary constant.

As this curve passes through $(1, 0)$, so we have

$$1 = c^2(1+0) \qquad i.e., \qquad c^2 = 1.$$

Hence, the equation of required curve is

$$x^2 = 1 + y^2 \qquad \text{or} \qquad x^2 - y^2 = 1$$

which is a rectangular hyperbola.

***Example 24:*** ***Solve the differential equation*** $ye^{x^2}\,dx + [(y^2 - 1)/x]\,dy = 0$.

**[B.C. A. (Garhwal) 2008]**

***Solution:*** Separate the variables,

$\Rightarrow$ $$xe^{x^2}\,dx + [(y^2 - 1)/y]\,dy = 0.$$

Thus, the variables have been separated. On integration, we get

$$\int xe^{x^2}\,dx + \int [(y^2 - 1)/y]\,dy = C, \quad \text{where } C \text{ is an arbitrary constant.}$$

or $$\frac{1}{2}\int e^{x^2}\,2x\,dx + \int [(y - 1/y)]\,dy = C$$

or $$\frac{1}{2}e^{x^2} + \frac{1}{2}y^2 - \log y = C$$

or $$e^{x^2} + y^2 - \log y^2 = c, \qquad \text{where} \qquad c = 2C$$

which is the required solution.

***Example 25:*** ***Solve*** $\boldsymbol{e^{2x-3y}\,dx + e^{2y-3x}\,dy = 0.}$

***Solution:*** On multiplying both sides of the given differential equation by $e^{3x+3y}$, we get

$$e^{5x}\,dx + e^{5y}\,dy = 0$$

which is in variables separable form, hence on integration, we get

$$\int e^{5x}\,dx + \int e^{5y}\,dy = C \qquad \text{or} \qquad \frac{1}{5}e^{5x} + \frac{1}{5}e^{5y} = C$$

or $$e^{5x} + e^{5y} = c, \text{ replacing } 5C \text{ by } c$$

which gives the general solution of the given differential equation, where $c$ is an arbitrary constant.

***Example 26:*** ***Solve*** $\boldsymbol{\frac{dy}{dx} = e^{x-y} + x^2e^{-y}}$.

**[B.C.A. (Agra) 2005]**

***Solution:*** On multiplying both sides of the given differential equation by $e^y$, we get

$$e^y\frac{dy}{dx} = e^x + x^2.$$

On separating variables, we have

$$e^y\,dy = e^x\,dx + x^2\,dx.$$

Integrating both sides, we get

$$\int e^y\,dy = \int e^x\,dx + \int x^2\,dx + c, \text{ where } c \text{ is an arbitrary constant.}$$

or $$e^y = e^x + \frac{1}{3}x^3 + c$$

which is the general solution of the given differential equation.

**Example 27:** ***Solve*** $(x+y)^2 \dfrac{dy}{dx} = a^2.$ **[B.C.A. (Meerut) 2007]**

***Solution:*** Let $x + y = v$. Differentiating both sides w.r. to $x$, we get

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dv}{dx} - 1.$$

Substituting these values in the given equation, we get

$$v^2 \left( \frac{dv}{dx} - 1 \right) = a^2 \qquad \text{or} \qquad v^2 \frac{dv}{dx} = a^2 + v^2$$

or $$\frac{v^2}{a^2 + v^2} dv = dx \qquad \text{or} \qquad \frac{a^2 + v^2 - a^2}{a^2 + v^2} dv = dx$$

or $$\left[ 1 - \frac{a^2}{a^2 + v^2} \right] dv = dx$$

which is a differential equation in variables separable form. Hence on integration, we get

$$v - a^2 \, [(1/a) \tan^{-1} (v/a)] = x + c, \text{ where } c \text{ is an arbitrary constant.}$$

or $$(x + y) - a \tan^{-1} \{(x + y)/a\} = x + c$$

or $$y - a \tan^{-1} \{(x + y)/a\} = c$$

which is the general solution of the given differential equation.

**Example 28:** ***Solve*** $(x + y)(dx - dy) = dx + dy.$ **[B.C. A. (Rohtak) 2009]**

***Solution:*** The given differential equation can be written as

$$dx - dy = \frac{dx + dy}{x + y} \qquad \text{or} \qquad d(x - y) = \frac{d(x + y)}{x + y}. \qquad ...(1)$$

Let $x - y = u$ and $x + y = v$, then the above equation reduces to

$$du = dv/v.$$

Integrating, we have

$$u = \log v - \log c, \text{ where } c \text{ is an arbitrary constant.}$$

or $$u = \log (v/c) \qquad \text{or} \qquad v/c = e^u \qquad \text{or} \qquad v = ce^u$$

or $$x + y = ce^{x - y} \qquad [\because \; u = x - y, \, v = x + y]$$

which is the general solution of the given differential equation.

## Exercise 9.2

**Solve the following Differential Equations:**

1. $y(1+x)\,dx + x(1+y)\,dy = 0.$ **[B.C.A. (Meerut) 2001]**
2. $2x(y+1)\,dx - y\,dy = 0.$
3. $x^2 \dfrac{dy}{dx} + y = 1.$ **[B.C.A. (Meerut) 2002]**
4. $y\,dx + (1+x^2)\tan^{-1} x\,dy = 0.$ **[B.C.A. (Kurukshetra) 2012]**
5. $(e^x + 1)\,y\,dy = (y+1)\,e^x\,dx.$
6. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0.$ **[B.C.A. (Meerut) 2004]**
7. $3e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0.$ **[B.C.A. (Indore) 2012]**
8. $y\sec^2 x + (y+7)\tan x \dfrac{dy}{dx} = 0.$
9. $\dfrac{dy}{dx} = \dfrac{\sin x + x\cos x}{y(2\log y + 1)}.$ **[B.C.A. (Bhopal) 2012, 07, 06]**
10. $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y} + xe^{-y}.$
11. $(x-y)^2 \dfrac{dy}{dx} = a^2.$ **[B.C.A. (Meerut) 2006 (B.P.)]**

## Answers 9.2

| | | | |
|---|---|---|---|
| 1. | $\log(xy) + (x+y) = c.$ | 2. | $x^2 = y - \log(y+1) + c.$ |
| 3. | $y - 1 = ce^{1/x}.$ | 4. | $y\tan^{-1} x = c.$ |
| 5. | $y - \log(y+1) = c + \log(e^x + 1).$ | 6. | $(x^2+1)(y^2+1) = c.$ |
| 7. | $\tan y = c(1 - e^x)^3.$ | 8. | $y^7 \tan x = ce^{-y}.$ |
| 9. | $y^2 \log y = x\sin x + c.$ | 10. | $e^y = e^x + \frac{1}{3}x^3 + \frac{1}{2}x^2 + c.$ |
| 11. | $2y + 2c = a\log\left(\dfrac{x-y-a}{x-y+a}\right).$ | | |

## 9.5 Homogeneous Function

**Definition:** "*The function given by $z = f(x, y)$ is said to be homogeneous function of degree n if $f(tx, ty) = t^n f(x, y)$.*"

### 9.5.1 Homogeneous Differential Equations

**Definition:** "*A differential equation of the form*

$$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)} \qquad ...(1)$$

*where $f_1(x, y)$, $f_2(x, y)$ are homogeneous functions of the same degree, n (say) in x and y, is called* ***homogeneous differential equation.***"

**Changing a Homogeneous Differential Equation to Variables Separable Form:** The differential equation (1) can be transformed into an equation whose variables are separable by letting $v = y/x$, then

$$y = vx \qquad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}. \qquad ...(2)$$

Put this value in (1), we get

$$v + x\frac{dv}{dx} = f(v) \qquad ...(3)$$

Because $f_1(x, y)$ and $f_2(x, y)$ both are homogeneous functions of the same degree (say $n$) So that

$$\frac{f_1(x, y)}{f_2(x, y)} = \frac{x^n f_1(y/x)}{x^n f_2(y/x)} = \frac{f_1(v)}{f_2(v)} = f(v).$$

on separating the variables, (3) can be written as

$$\frac{dv}{f(v) - v} = \frac{dx}{x} \qquad ...(4)$$

Integrating both sides of (4), we obtain a solution in $v$ and $x$ as follows:

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + c.$$

Replacing $v$ by $y/x$ in the above equation, we get the general solution of the differential equation (1).

***Example 29:*** ***Solve $x^2 dy + y(x + y)\,dx = 0$.*** **[B.C.A. (Rohilkhand) 2012, 07]**

***Solution:*** The given differential equation can be written as

$$\frac{dy}{dx} + \frac{y(x + y)}{x^2} = 0 \qquad ...(1)$$

which is a homogeneous differential equation.

Putting $\quad y = vx \quad$ and $\quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

(1) becomes $\quad v + x\dfrac{dv}{dx} + \dfrac{vx(x+vx)}{x^2} = 0,$

or $\quad v + x\dfrac{dv}{dx} + v(1+v) = 0 \quad$ or $\quad x\dfrac{dv}{dx} + v^2 + 2v = 0$

or $\quad \dfrac{dv}{(v^2+2v)} + \dfrac{dx}{x} = 0 \quad$ or $\quad \dfrac{1}{2}\left(\dfrac{1}{v} - \dfrac{1}{v+2}\right)dv + \dfrac{1}{x}dx = 0$

which is a differential equation in variables separable form. Hence on integration, we get

$$\frac{1}{2}[\log v - \log(v+2)] + \log x = C$$

or $\quad \log x\sqrt{\dfrac{v}{v+2}} = C \quad$ or $\quad \log x\sqrt{\dfrac{y/x}{(y/x)+2}} = C \quad [\because\ v = y/x]$

or $\quad x\sqrt{\dfrac{y}{y+2x}} = e^C = c',$ say

or $\quad x^2 y = c(y+2x),$

$c$ being an arbitrary constant $c = c'^2$ which is the required solution.

**Example 30:** ***Solve*** $(x^2 - y^2)\,dx + 3xy\,dy = 0.$ **[B.C.A. (Bundelkhand) 2007]**

***Solution:*** The given differential equation can be written as

$$\frac{dy}{dx} + \frac{x^2 - y^2}{3xy} = 0 \qquad ...(1)$$

which is a homogeneous differential equation. Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$;

(1) becomes

$$v + x\frac{dv}{dx} + \frac{x^2 - v^2x^2}{3x^2v} = 0 \quad \text{or} \quad v + x\frac{dv}{dx} + \frac{1-v^2}{3v} = 0$$

or $\quad x\dfrac{dv}{dx} + \dfrac{1+2v^2}{3v} = 0 \quad$ or $\quad \dfrac{dx}{x} = -\dfrac{3v}{1+2v^2}dv$

which is a differential equation in variables separable form. Hence on integration, we get

$$\int \frac{dx}{x} = -\frac{3}{4}\int \frac{4v}{1+2v^2}dv + C,\ C \text{ being an arbitrary constant.}$$

or $\log x = -(3/4)\log(1+2v^2) + (1/4)\log c$, where $(1/4)\log c = C$

or $\log x^4 = \log \dfrac{c}{(1+2v^2)^3}$ or $x^4 = \dfrac{c}{(1+2v^2)^3}$

or $x^4 = \dfrac{c}{[1+(2y^2/x^2)]^3}$ or $(x^2+2y^2)^3 = cx^2$

which is the general solution of the given differential equation.

***Example 31:*** ***Solve*** $y - x\dfrac{dy}{dx} = x + y\dfrac{dy}{dx}$.

***Solution:*** The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y-x}{y+x} \qquad \text{...(1)}$$

which is a homogeneous differential equation.

Putting $y = vx$ and $\dfrac{dy}{dx} = v\,.1 + x\dfrac{dv}{dx}$ in equation (1), then it becomes

$$v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{v-1}{v+1}$$

$\Rightarrow$ $x\dfrac{dv}{dx} = \dfrac{v-1}{v+1} - v = -\dfrac{1+v^2}{v+1}$ $\Rightarrow$ $\dfrac{v+1}{v^2+1}dv = -\dfrac{dx}{x}$

which is a differential equation in variables separable form. Hence on integration, we get

$$\int\left\{\frac{v}{v^2+1} + \frac{1}{v^2+1}\right\}dv = -\int\frac{dx}{x} + c$$

$\Rightarrow$ $\dfrac{1}{2}\log(v^2+1) + \tan^{-1}v = -\log x + c$

$\Rightarrow$ $\log\{\sqrt{(v^2+1)}\,.\,x\} + \tan^{-1}v = c$

$\Rightarrow$ $\log\left\{\sqrt{\left(\dfrac{y^2}{x^2}+1\right)}\,.\,x\right\} + \tan^{-1}\dfrac{y}{x} = c$

$\Rightarrow$ $\log\sqrt{x^2+y^2} + \tan^{-1}\dfrac{y}{x} = c.$

This is the required solution.

**Example 32:** ***Solve*** $x(x-y)\,dy + y^2\,dx = 0$. **[B.C.A. (Kurukshetra) 2009]**

**Solution:** The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y^2}{x(y-x)}.$$

Putting $y = vx$ and $\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$ in equation (1), it reduces to

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{x(vx-x)} = \frac{v^2}{v-1}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v}{v-1}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{v-1}{v}\,dv = \left(1 - \frac{1}{v}\right)dv$$

which is a differential equation in variables separable form. Hence on integration, we get

$$\log x = v - \log v + c_1 = \log e^v - \log v + \log c, \quad \text{where } c_1 = \log c$$

$$= \log \frac{ce^v}{v}$$

*i.e.,* $x = \frac{ce^v}{v} = \frac{ce^{y/x}}{y/x} = \frac{cxe^{y/x}}{y}$ *i.e.,* $y = ce^{y/x}$.

This is the required solution.

**Example 33:** ***Solve*** $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$. **[B.C.A. (Avadh) 2007]**

**Solution:** The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} \quad ...(1)$$

which is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

(1) becomes

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+v^2x^2}}{x} = v + \sqrt{1+v^2}$$

or $x\frac{dv}{dx} = \sqrt{1+v^2}$ or $\frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}}$

which is a differential equation in variables separable form. Hence on integration, we get

$$\int \frac{dx}{x} = \int \left[\frac{dv}{\sqrt{1+v^2}}\right] + C, \quad C \text{ being an arbitrary constant.}$$

or $$\log x = \log \{v + \sqrt{1+v^2}\} + \log c, \quad \text{where } \log c = C$$

or $$x = c\,\{v + \sqrt{1+v^2}\} \qquad \text{or} \qquad x = c\left\{\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right\}$$

or $$x^2 = c\,\{y + \sqrt{x^2+y^2}\}$$

which is the general solution of the given differential equation.

**Example 34:** ***Solve*** $x^2 y\, dx - (x^3 + y^3)\, dy = 0.$ **[B.C.A. (Agra) 2011]**

**Solution:** The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3} \qquad ...(1)$$

which is a homogeneous differential equation. Hence on putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in equation (1), it reduces to

$$v + x\frac{dv}{dx} = \frac{x^2 \,.\, vx}{x^3 + v^3 x^3} = \frac{v}{1+v^3}$$

$\Rightarrow$ $$x\frac{dv}{dx} = \frac{v}{1+v^3} - v = -\frac{v^4}{1+v^3}$$

$\Rightarrow$ $$-\left(\frac{1+v^3}{v^4}\right) dv = \frac{dx}{x}, \text{ on separating the variables.}$$

$\therefore$ On integration, we get

$$\frac{1}{3v^3} - \log v = \log x + c_1, \text{ where } c_1 \text{ is arbitrary constant of integration.}$$

$\Rightarrow$ $$\frac{1}{v^3} = 3\log vx + 3c_1$$

$\Rightarrow$ $$\frac{x^3}{y^3} = \log y^3 + \log c, \qquad \text{where } \log c = 3c_1$$

$\Rightarrow$ $$cy^3 = e^{x^3/y^3}.$$

This is the required solution.

**Example 35:** ***Solve*** $x \cos \frac{y}{x} (y\,dx + x\,dy) = y \sin \frac{y}{x} (x\,dy - y\,dx).$

**[B.C.A. (Rohtak) 2012]**

***Solution:*** The given differential equation can be written as

$$x \cos \frac{y}{x}\left(y + x\frac{dy}{dx}\right) = y \sin \frac{y}{x}\left(x\frac{dy}{dx} - y\right)$$

or $$x\frac{dy}{dx}\left(x \cos \frac{y}{x} - y \sin \frac{y}{x}\right) = -y\left(y \sin \frac{y}{x} + x \cos \frac{y}{x}\right)$$

or $$\frac{dy}{dx} = \frac{y\,[y \sin (y/x) + x \cos (y/x)]}{x\,[y \sin (y/x) - x \cos (y/x)]} \qquad \ldots(1)$$

which is a homogeneous differential equation. Hence, putting $y = vx$ and

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

(1) reduces in the following form

$$v + x\frac{dv}{dx} = \frac{vx\,(vx \sin v + x \cos v)}{x\,(vx \sin v - x \cos v)} \quad \text{or} \quad x\frac{dv}{dx} = \frac{2v \cos v}{v \sin v - \cos v}$$

or $$\left(\frac{v \sin v - \cos v}{v \cos v}\right) dv = 2\frac{dx}{x} \quad \text{or} \quad \left(\tan v - \frac{1}{v}\right) dv = 2\frac{dx}{x}$$

which is a differential equation in variables separable form. Hence on integration, we get

$$\log \sec v - \log v = 2 \log x + C$$

or $$\log \frac{\sec v}{vx^2} = \log c, \text{ replacing } C \text{ by } \log c$$

or $$\sec (y/x) = c\,xy. \qquad [\because\ y = vx]$$

which is the general solution of the given differential equation.

## Exercise 9.3

**Solve the following Differential Equations:**

1. $\frac{dy}{dx} = \frac{2x + y}{x - 3y}.$ **[B.C.A. (Agra) 2008, 00]**
2. $x^2\,dy + (xy + y^2) = 0.$
3. $\frac{dy}{dx} = \frac{x + y}{2x}.$ **[B.C.A. (Meerut) 2003]**
4. $\frac{dy}{dx} = \frac{x - y}{x + y}.$
5. $(x^2 + y^2)\,dx - 2xy\,dy = 0.$ **[B.C.A. (Rohtak) 2005, 03]**
6. $(x - y)^2\,dx + 2xy\,dy = 0.$

7. $\dfrac{dy}{dx} = \dfrac{x^2 + xy}{x^2 + y^2}$. **[B.C. A. (Agra) 2007]**

8. $\dfrac{dy}{dx} + \dfrac{x^2 + 3y^2}{3x^2 + y^2} = 0$. **[B.C.A. (Lucknow) 2006, 04]**

9. $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y$. **[B.C.A. (Purvanchal) 2006]**

10. $y^2\,dx + (xy + x^2)\,dy = 0$. **[B.C.A. (Avadh) 2004]**

11. $x^2\dfrac{dy}{dx} = \dfrac{y(x+y)}{2}$.

12. $x\,dy - y\,dx = 2\sqrt{y^2 - x^2}\,dx$. **[B.C.A. (Kurukshetra) 2005]**

13. $(x^2 - y^2)\dfrac{dy}{dx} = xy$. **[B.C.A. (Meerut) 2005]**

14. $x\dfrac{dy}{dx} = y + x\tan\dfrac{y}{x}$. **[B.C.A. (Bundelkhand) 2007]**

15. $(x^2 + 2xy)\,dy + (2xy + y^2 + 3x^2)\,dx = 0$. **[B.C.A. (Bhopal) 2009]**

16. $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$.

17. $\dfrac{dy}{dx} = \dfrac{y^3 + 3x^2y}{x^3 + 3xy^2}$. **[B.C.A. (I.G.N.O.U.) 2009, 07, 04]**

## Answers 9.3

1. $\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{\sqrt{3}\,y}{\sqrt{2}\,x} - \log\sqrt{2x^2 + 3y^2} = c$.

2. $yx^2 = c\,(y + 2x)$.

3. $x = c\,(x - y)^2$.

4. $y^2 + 2xy - x^2 = c$.

5. $y^2 = x\,(x + c)$.

6. $\log[x\,(3y^2 - 2xy + x^2)]\,\sqrt{2}\,\tan^{-1}[(3y - x)/x\sqrt{2}] = c$.

7. $(x - y)^{2/3}(x^2 + xy + y^2)^{1/6} = ce^{(1/\sqrt{3})\tan^{-}(x+2y)/x\sqrt{3})}$.

8. $2xy/(x + y)^2 + \log(x + y) = c$.

9. $cx = e^{x/y}$.

| | | | |
|---|---|---|---|
| **10.** | $xy^2 = c^2\,(x + 2y)$. | **11.** | $(y - x)^2 = cxy^2$. |
| **12.** | $y + \sqrt{y^2 - x^2} = cx^3$. | **13.** | $cy = e^{-x/2y}$. |
| **14.** | $\sin\left(\dfrac{y}{x}\right) = cx$ . | **15.** | $x\,(y^2 + xy + x^2) = c$. |
| **16.** | $y^2 - x^2 = c\,(y^2 + x^2)^2$. | **17.** | $xy = c^2\,(x^2 - y^2)^2$. |

## 9.6 Non-homogeneous Equations of the First Degree in x and y

**Equations Reducible to a Homogeneous Form**

**Definition:** "*Equations of the form*

$$\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C} \qquad ...(1)$$

*as known as* ***differential equations reducible to homogeneous form.***"

**Case I:** When$(a/A) \neq (b/B)$.

In this case, (1) can be reduced to homogeneous form by the following substitution which change variables $x$ and $y$ to new variables $X$ and $Y$,

$$x = X + h,\ y = Y + k \qquad ...(2)$$

where $h$ and $k$ are arbitrary constants and they can be so chosen that the new differential equation may be homogeneous. Now (2) gives

$$dx = dX \qquad \text{and} \qquad dy = dY \qquad ...(3)$$

Thus, (1) reduces to

$$\frac{dY}{dX} = \frac{a\,(X + h) + b\,(Y + k) + c}{A\,(X + h) + B\,(Y + k) + C}$$

or

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{AX + BY + (Ah + Bk + C)}. \qquad ...(4)$$

Now, choose $h$ and $k$ such that

$$ah + bk + c = 0 \qquad ...(5)$$

and

$$Ah + Bk + C = 0 \qquad ...(6)$$

Solving the above two equations for $h$ and $k$, we have

$$\frac{h}{bC - cB} = \frac{k}{cA - aC} = \frac{1}{aB - bA} \qquad \text{...(7)}$$

With these values of $h$ and $k$, (1) transform to the form

$$\frac{dY}{dX} = \frac{aX + bY}{AX + BY} \qquad \text{...(8)}$$

which is a homogeneous differential equation and can be solved by substitution $Y = vX$. Finally, replacing $X$ and $Y$ in the solution so obtained by $x - h$ and $y - k$ respectively, we can get the general solution of the given differential equation in terms of $x$ and $y$, the original variables.

**Case II:** When $(a / A) = (b / B)$.

In such case the values of $h$ and $k$ become infinite, that is meaningless and thus the above procedure cannot be adopted. In such a case, let

$$a / A = b / B = 1 / m \qquad \text{or} \qquad A = am,\ B = bm \qquad \text{...(9)}$$

Then equation (1) takes the form

$$\frac{dy}{dx} = \frac{ax + by + c}{m\,(ax + by) + c} \qquad \text{...(10)}$$

where $m$ is any number. In such cases, the substitution

$$ax + by = v, \quad a + b\,\frac{dy}{dx} = \frac{dv}{dx} \qquad \text{...(11)}$$

transform the differential equation to the form

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = \frac{v + c}{mv + c}$$

or

$$\frac{dv}{dx} = a + b\,\frac{v + c}{mv + c} \qquad \text{...(12)}$$

which is a differential equation in variables separable form and it can easily be solved.

***Example 36:* *Solve*** $\dfrac{dy}{dx} = \dfrac{x + 2y + 3}{2x + 3y + 4}$.

**[B.C. A. (Agra) 2009, 03]**

***Solution:*** The given differential equation belongs to reducible to homogeneous form. Here $(a / A) \neq (b / B)$, that is $\frac{1}{2} \neq \frac{2}{3}$. Hence, putting $x = X + h$ and $y = Y + k$, then the given differential equation reduces to the form

$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k + 3)}{2X + 3Y + (2h + 3k + 4)}. \qquad \text{...(1)}$$

Now, choose $h$ and $k$ such that

$$h + 2k + 3 = 0 \qquad \text{and} \qquad 2h + 3k + 4 = 0$$

Solving these equations for $h$ and $k$ gives

$$\frac{h}{8-9} = \frac{k}{6-4} = \frac{1}{3-4} \qquad \Rightarrow \qquad h = 1,\ k = -2.$$

Substituting these values in (1), we obtain

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + 3Y} \qquad \text{...(2)}$$

which is a differential equation in homogeneous form in $X$ and $Y$, $Y$ being dependent variable. Hence, putting $Y = vX$

and
$$\frac{dY}{dX} = v + X\frac{dv}{dX}.$$

(2) transforms in the form

$$v + X\frac{dv}{dX} = \frac{X + 2vX}{2X + 3vX} \qquad \text{or} \qquad \left(\frac{2 + 3v}{3v^2 - 1}\right) dv = -\frac{dX}{X}$$

or
$$\left(\frac{2}{3v^2 - 1} + \frac{3v}{3v^2 - 1}\right) dv = -\frac{dX}{X},$$

which belongs to variables separable form in $v$ and $X$. Hence on integration, we get

$$\frac{1}{\sqrt{3}} \log \frac{\sqrt{3}\, v - 1}{\sqrt{3}\, v + 1} + \frac{1}{2} \log (3v^2 - 1) = -\log X + \log c,$$

$c$ being as arbitrary constant.

Now, putting
$$X = x - h = x - 1,\ Y = y - k = y + 2,\ v = \frac{Y}{X} = \frac{y+2}{x-1}.$$

The general solution of the given differential equation becomes

$$\frac{1}{\sqrt{3}} \log \left[\frac{\{\sqrt{3}\,(y+2)/(x-1)\} - 1}{\{\sqrt{3}\,(y+2)/(x-1)\} + 1}\right] + \frac{1}{2} \log \left[3\left(\frac{y+2}{x-1}\right)^2 - 1\right] = \log\left(\frac{c}{x-1}\right)$$

or
$$\log \left[\left\{\frac{\sqrt{3}\; y - x + 2\sqrt{3} + 1}{\sqrt{3}\; y + x + 2\sqrt{3} - 1}\right\}^{1/\sqrt{3}} \cdot \left\{\frac{3y^2 - x^2 + 12y + 2x + 11}{(x-1)^2}\right\}^{1/2}\right] = \log \frac{c}{x-1}$$

or
$$\left\{\frac{\sqrt{3}\, y - x + 2\sqrt{3} + 1}{\sqrt{3}\; y + x + 2\sqrt{3} - 1}\right\}^{1/\sqrt{3}} \cdot \left\{\frac{(3y^2 - x^2 + 12y + 2x + 11)^{1/2}}{(x+1)}\right\} = \frac{c}{x-1}$$

or
$$(\sqrt{3}\; y - x + 2\sqrt{3} + 1)^{1/\sqrt{3}}\, (3y^2 - x^2 + 12y + 2x + 11)^{1/2} = c\,(\sqrt{3}\, y + x + 2\sqrt{3} - 1)^{1/\sqrt{3}}.$$

**Example 37:** ***Solve*** $(2x + y - 3)\, dy = (x + 2y - 3)\, dx$. **[B.C.A. (Lucknow) 2011, 09]**

***Solution:*** The given differential equation

$$\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3} \qquad \text{...(1)}$$

is reducible to homogeneous form. Here $(a / A) \neq (b / B)$, that is $\left(\frac{1}{2}\right) \neq \left(\frac{2}{1}\right)$. Hence, putting $x = X + h$ and $y = Y + k$ the given differential equation reduces to the form

$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)}. \qquad \text{...(2)}$$

Now, choose $h$ and $k$ such that

$$h + 2k - 3 = 0;\ \ 2h + k - 3 = 0.$$

Solving these equations, we get

$$\frac{h}{-6 + 3} = \frac{k}{-6 + 3} = \frac{1}{1 - 4}$$

$$h = 1,\ k = 1.$$

Substituting these values in (1), we obtain

$$\frac{dY}{dX} = \frac{X + 2Y}{2X + Y} \qquad \text{...(3)}$$

which is a differential equation in homogeneous form in $X$ and $Y$, $Y$ being dependent variable. Hence, putting

$$Y = vX \qquad \text{and} \qquad \frac{dY}{dX} = v + X\frac{dv}{dX}.$$

(3) transforms in the form

$$v + X\frac{dv}{dX} = \frac{X + 2vX}{2X + vX}$$

or $$X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 - v^2}{2 + v} \qquad \text{or} \qquad \frac{2 + v}{1 - v^2}\, dv = \frac{dX}{X}$$

which is in variables separable form in $v$ and $X$, hence on integration, we get

$$\int \frac{2 + v}{1 - v^2}\, dv = \int \frac{dX}{X} + C$$

or $$\frac{1}{2}\int \left[\frac{3}{1 - v} + \frac{1}{1 + v}\right] dv = \log X + C$$

or $$\frac{1+v}{(1-v)^3} = e^{2C} \,.\, X^2 \quad \text{or} \quad \frac{1+(Y/X)}{[1-Y/X]^3} = e^{2C} \,.\, X^2 \qquad [\because \; v = Y/X]$$

or $$(X+Y)/(X-Y)^3 = c, \qquad \text{where} \qquad c = e^{2C}$$

or $$x + y - 2 = c\,(x-y)^3 \qquad [\because \; X = x-1, Y = y-1]$$

which is the general solution of the given differential equation.

**Example 38:** ***Solve*** $\dfrac{dy}{dx} = \dfrac{x-y+3}{2x-2y+5}$.

**[B.C.A. (Kashi) 2010]**

***Solution:*** The given differential equation can be written as

$$\frac{dy}{dx} = \frac{(x-y)+3}{2\,(x-y)+5} \qquad \text{...(1)}$$

Here $\dfrac{a}{A} = \dfrac{b}{B}$, that is $\dfrac{1}{2} = \dfrac{-1}{-2}$. Therefore, putting $x - y = v$ and $1 - \dfrac{dy}{dx} = \dfrac{dv}{dx}$, (1) reduces in the form

$$1 - \frac{dv}{dx} = \frac{v+3}{2v+5}$$

or $$\frac{dv}{dx} = 1 - \frac{v+3}{2v+5} = \frac{v+2}{2v+5} \qquad \Rightarrow \qquad \frac{2v+5}{v+2}\,dv = dx$$

or $$\left(2 + \frac{1}{v+2}\right) dv = dx$$

which is in variables separable form. Hence on integration, we get

$$2v + \log(v+2) = x + c, \; c \text{ being an arbitrary constant.}$$

or $$x - 2y + \log(x-y+2) = c \qquad [\because \; v = x-y]$$

which is the general solution of the given differential equation.

**Example 39:** ***Solve*** $(x+y)(dx-dy) = dx + dy$.

***Solution:*** The given differential equation can be written as

$$(x+y-1)\,dx = (x+y+1)\,dy \quad \text{or} \quad \frac{dy}{dx} = \frac{x+y-1}{x+y+1} \qquad \text{...(1)}$$

which belongs to reducible homogeneous form. Hence, putting

$$x + y = v \qquad \text{and} \quad (dy/dx) = (dv/dx) - 1. \qquad \text{...(2)}$$

(1) transforms in the form

$$\left(\frac{dv}{dx}\right)-1=\frac{(v-1)}{(v+1)}$$

or $$\frac{dv}{dx}=\frac{v-1}{v+1}+1=\frac{2v}{v+1}$$

or $$\frac{v+1}{v}\,dv=2\,dx \qquad \text{or} \quad (1+1/v)\,dv=2\,dx$$

which is in variables separable form, hence on integration, we get

$$v+\log v=2x+c,\ c \text{ being an arbitrary constant.}$$

or $$x+y+\log(x+y)=2x+c \qquad [\because\ v=x+y]$$

or $$y-x+\log(x+y)=c$$

which is the general solution of the given differential equation.

***Example 40:*** ***Solve*** $\boldsymbol{\dfrac{dy}{dx}=\dfrac{3y+2x+4}{4x+6y+5}}$. **[B.C.A. (Purvanchal) 2008]**

***Solution:*** The given differential equation can be written as

$$\frac{dy}{dx}=\frac{2x+3y+4}{2(2x+3y)+5}. \qquad ...(1)$$

Putting $2x+3y=v$ and $2+3\dfrac{dy}{dx}=\dfrac{dv}{dx}$ in equation (1), we get

$$\frac{1}{3}\left(\frac{dv}{dx}-2\right)=\frac{v+4}{2v+5}$$

$\Rightarrow$ $$\frac{dv}{dx}=\frac{3v+12}{2v+5}+2=\frac{7v+22}{2v+5} \qquad \Rightarrow \qquad \frac{2v+5}{7v+22}\,dv=dx$$

$\Rightarrow$ $$\left\{\frac{2}{7}-\frac{9}{7}\cdot\frac{1}{7v+22}\right\}dv=dx$$

which is a differential equation in variables separable form. Hence on integration, we get

$$\frac{2}{7}v-\frac{9}{7}\cdot\frac{\log(7v+22)}{7}=x+c_1,\ \text{where } c_1 \text{ is arbitrary constant.}$$

$\Rightarrow$ $$14v-9\log(7v+22)=49x+49c_1$$

$\Rightarrow$ $$14(2x+3y)-9\log\{7(2x+3y)+22\}=49x+49c_1$$

$\Rightarrow$ $$42y-21x-9\log\{14x+21y+22\}=49c_1$$

$\Rightarrow$ $$7(2y-x)-3\log\{14x+21y+22\}=c, \qquad \text{where } c=\frac{49c_1}{3}.$$

This is the required solution.

**Example 41:** ***Solve*** $\dfrac{dy}{dx} = \dfrac{x + y + 1}{2x + 2y + 3}$. **[B.C. A. (Agra) 2002]**

***Solution:*** The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x + y + 1}{2\,(x + y) + 3}. \qquad \text{...(1)}$$

Putting $x + y = v$ and $1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$ in equation (1), we obtain

$$\frac{dv}{dx} - 1 = \frac{v + 1}{2v + 3}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{v + 1}{2v + 3} + 1 = \frac{3v + 4}{2v + 3} \quad \Rightarrow \quad \frac{2v + 3}{3v + 4}\, dv = dx$$

$$\Rightarrow \quad \frac{1}{3}\left\{2 + \frac{1}{3v + 4}\right\} dv = dx.$$

$\therefore$ On integration, we get

$$\frac{1}{3}\left\{2v + \frac{1}{3}\log\,(3v + 4)\right\} = x + c_1$$

$$\Rightarrow \quad 2\,(x + y) + \frac{1}{3}\log\,(3x + 3y + 4) = 3x + 3c_1$$

$$\Rightarrow \quad -3x + 6y + \log\,(3x + 3y + 4) = c, \quad \text{where } c = 9c_1.$$

This is the required solution.

## Exercise 9.4

**Solve the following Differential Equations:**

**1.** $\dfrac{dy}{dx} = \dfrac{y - x + 1}{y + x + 5}$. **[B.C.A. (Avadh) 2005, 03]**

**2.** $\dfrac{dy}{dx} = \dfrac{x - y + 3}{2x - 4y + 5}$. **[B.C.A. (Kashi) 2010, 06]**

**3.** $\dfrac{dy}{dx} = \dfrac{2x - y + 1}{x + 2y - 3}$.

**4.** $(x + 2y - 2)\, dx + (2x - y + 3)\, dy = 0$. **[B.C.A. (Aligarh) 2010, 04]**

**5.** $(2x - y + 1)\, dx + (2y - x - 1)\, dy = 0$.

**6.** $(3y - 7x + 7)\, dx + (7y - 3x + 3)\, dy = 0$. **[B.C.A. (Bhopal) 2006, 02]**

7. $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+y-1}$.

8. $\dfrac{dy}{dx} = \dfrac{2y+x-1}{2x+4y+3}$. **[B.C.A. (Kurukshetra) 2010, 04]**

9. $(4x+6y+3)\,dx = (6x+9y+2)\,dy$.

10. $(x-y-2)\,dx - (2x-2y-3)\,dy = 0$. **[B.C.A. (Rohtak) 2008]**

## Answers 9.4

| | |
|---|---|
| 1. | $\tan^{-1}\dfrac{y+3}{x+2} + \log c\sqrt{(y+3)^2+(x+2)^2} = 0.$ |
| 2. | $\log c\,\{(x+\frac{7}{2})^2 - 3\,(x\,\frac{7}{2}) + 4\,(y+\frac{1}{2})^2\}^{1/2} + \dfrac{1}{\sqrt{7}}\tan^{-1}\dfrac{8\,(y+\frac{1}{2}) - 3\,(x+\frac{7}{2})}{\sqrt{7}\,(x+\frac{7}{2})} = 0.$ |
| 3. | $(5y-7)^2 + (5x-1)(5y-7) - (5x-1)^2 = k.$ |
| 4. | $x^2 + 4xy - y^2 - 4x + 6y = c.$ |
| 5. | $3x^2 + 3y^2 - 3xy - 3y + 3x + 1 - 3c = 0.$ |
| 6. | $(y-x+1)^2\,(y+x-1)^5 = c.$ |
| 7. | $\log(x+y) = y - x - c.$ |
| 8. | $2y - x + \dfrac{1}{4}\log(8y+4x+5) = c.$ |
| 9. | $24x - 36y + 5\log(24x+36y+12) + c = 0.$ |
| 10. | $\log(x-y-1) = x - 2y + c.$ |

## 9.7 Linear Differential Equations

A differential equation is called **linear** if the dependent variable $y$ and its derivatives with respect to independent variable $x$ occur in the first degree only and there is no restriction of any kind on the occurrence of the independent variable $x$.

**Definition:** "*A differential equation of the form*

$$\frac{dy}{dx} + Py = Q \qquad \ldots(1)$$

*where P and Q are constants or any function of x, is called a* ***linear differential equation of first order.***" **[B.C. A. (Agra) 2009]**

Equation (1) is called **standard form** of the linear differential equation of the first order.

**NOTE:**

1. Since the multiplication by the factor $e^{\int P\,dx}$ to both sides of the given differential equation, reduces it in integrable form, hence it is called **integrating factor** of the differential equation.
2. Sometimes a given differential equation becomes linear if we take $y$ as the independent variable and $x$ as dependent one. Since as equation can be written as
$$\frac{dx}{dy} + Px = Q.$$
where $P$ and $Q$ are constants or functions of $y$ only. Then its integrating factor will be of the form
$$\text{I.F.} = e^{\int P\,dy}.$$
3. Students should remember that during the course of finding the solution of linear differential equation, for every $f$
$$e^{\log_e f} = f \qquad \text{and} \qquad e^{-\log_e f} = \frac{1}{f}.$$

**Working Rule**

To solve a linear differential equation of the first order, students should remember the following points:

1. Arrange the given differential equation in the standard form.
$$\frac{dy}{dx} + Py = Q \qquad \text{or} \qquad \frac{dx}{dy} + Px = Q$$
2. Write down its I.F. $= e^{\int P\,dx}$ or $e^{\int P\,dy}$ and evaluate it.
3. The general solution of the differential equation will be written as
$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}\,dx + c$$
or
$$xe^{\int P\,dy} = \int Qe^{\int P\,dy}\,dy + c$$
$c$ being arbitrary constant of integration.

***Example 42:*** ***Solve*** $(1 + x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0.$ **[B.C.A. (Garhwal) 2007, 03]**

***Solution:*** The given equation can be written as
$$\frac{dy}{dx} + \frac{2x}{(1+x^2)}\,y = \frac{4x^2}{(1+x^2)}$$

which is a linear differential equation and its of the form

$$\frac{dy}{dx} + Py = Q, \qquad \text{where } P = \frac{2x}{(1+x^2)} \qquad \text{and } Q = \frac{4x^2}{(1+x^2)}.$$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{(1+x^2)}dx} = e^{\log(1+x^2)} = (1+x^2).$$

Hence, its solution is

$$y \times (\text{I.F.}) = \int \{Q \times (\text{I.F.})\}\,dx + c$$

*i.e.,* $$y(1+x^2) = \int \left\{\frac{4x^2}{(1+x^2)} . (1+x^2)\right\} dx + c,$$

or $$y(1+x^2) = \left(\frac{4}{3}\right)x^3 + c \qquad \text{or} \qquad 3y(1+x^2) - 4x^3 = 3c$$

which is the required solution.

***Example 43:*** **Solve** $(1 + x^2)\dfrac{dy}{dx} + 2xy = \cos x$. **[B.C. A. (Agra) 2002]**

***Solution:*** The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cos x}{1+x^2}$$

which is a linear differential equation in $y$.

Here $$P = \frac{2x}{1+x^2} \qquad \text{and} \quad Q = \frac{\cos x}{1+x^2}, \text{ so that}$$

$$\int P\,dx = \int \frac{2x}{1+x^2}dx = \log(1+x^2).$$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{\log(1+x^2)} = 1 + x^2.$$

Multiplying (1) by the integrating factor, we obtain

$$(1+x^2)\left[\frac{dy}{dx} + \frac{2x}{1+x^2}y\right] = \cos x$$

or $$(1+x^2)(dy/dx) + 2xy = \cos x \qquad \text{or} \qquad d[y(\text{I.F.})] = \cos x.$$

Integrating this equation, we obtain $y . (1+x^2) = \int \cos x\,dx + c$

or $$y . (1+x^2) = \sin x + c$$

which is the required solution.

**Example 44:** ***Solve*** $(1-x^2)\dfrac{dy}{dx}+2xy = x\sqrt{1-x^2}$.

**Solution:** Writing the given linear differential equation in the standard form, we have

$$\frac{dy}{dx}+\frac{2x}{1-x^2}\,y=\frac{x}{\sqrt{1-x^2}}. \qquad ...(1)$$

$$\therefore \qquad P=\frac{2x}{1-x^2} \qquad \text{and} \qquad Q=\frac{x}{\sqrt{1-x^2}}.$$

$$\therefore \qquad \text{I.F.}=e^{\int P\,dx}=e^{\int \frac{2x}{1-x^2}dx}=e^{-\log(1-x^2)}=\frac{1}{1-x^2}.$$

Multiplying both sides of equation (1) by integrating factor (I.F.) and the integrating, we get

$$y\,.\frac{1}{1-x^2}=\int \frac{x}{\sqrt{1-x^2}}\,.\,\frac{1}{1-x^2}\,dx+c$$

or
$$\frac{y}{1-x^2}=-\frac{1}{2}\int(-2x)\,.\,(1-x^2)^{-3/2}\,dx+c$$

or
$$\frac{y}{1-x^2}=-\frac{1}{2}\,.\,\frac{(1-x^2)^{-1/2}}{\left(-\frac{1}{2}\right)}+c$$

or
$$\frac{y}{1-x^2}=\frac{1}{\sqrt{1-x^2}}+c \qquad \text{or} \qquad y=c\,(1-x^2)+\sqrt{1-x^2}$$

which is the required solution.

**Example 45:** ***Solve*** $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy$. **[B.C.A. (Agra) 2008]**

**Solution:** The given equation can be written as

*i.e.,*
$$\frac{dx}{dy}+\frac{1}{1+y^2}\,x=\frac{\tan^{-1}y}{1+y^2}, \qquad ...(1)$$

which is a linear equation in $x$.

Here
$$P=\frac{1}{(1+y^2)} \qquad \text{and} \qquad Q=\frac{(\tan^{-1}y)}{(1+y^2)}.$$

Thus
$$\int P\,dy=\int\left[\frac{1}{(1+y^2)}\right]dy=\tan^{-1}y.$$

$$\therefore \qquad \text{I.F.}=e^{\int P\,dx}=e^{\tan^{-1}y}.$$

Hence, the required solution is

$$x\,(\text{I.F.}) = \int \{Q \times (\text{I.E.})\}\, dy + c$$

*i.e.,* $$x \,.\, e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \,.\, e^{\tan^{-1} y}\, dy + c = \int t\, e^t\, dt + c, \quad \text{where } t = \tan^{-1} y$$

$$= te^t - e^t + c = \tan^{-1} y \,.\, e^{\tan^{-1} y} - e^{\tan^{-1} y} + c$$

*i.e.,* $$x = (\tan^{-1} y - 1) + c \,.\, e^{-\tan^{-1} y}.$$

**Example 46:** ***Solve*** $(1+y^2) + (x - e^{-\tan^{-1}}(y))\dfrac{dy}{dx} = 0.$ **[B.C. A. (Agra) 2000]**

**Solution:** Writing the given linear differential equation in the standard form, we have

$$\frac{dx}{dy} + \frac{1}{1+y^2}\, x = \frac{e^{-\tan^{-1} y}}{1+y^2}. \qquad \text{...(1)}$$

This equation is linear in $x$.

$\therefore$ $$P = \frac{1}{1+y^2} \quad \text{and} \quad Q = \frac{e^{-\tan^{-1} y}}{1+y^2}.$$

Now, $$\text{I.F.} = e^{\int P\, dx} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}.$$

Multiplying both sides of equation (1) by integrating factor (I.F.) and integrating, we get

$$x \,.\, e^{\tan^{-1} y} = \int \frac{e^{-\tan^{-1} y}}{1+y^2} \,.\, e^{\tan^{-1} y}\, dy + c$$

or $$x \,.\, e^{\tan^{-1} y} = \int \frac{dy}{1+y^2} + c$$

or $$x \,.\, e^{\tan^{-1} y} = \tan^{-1} y + c$$

which is the required solution.

**Example 47:** ***Solve*** $\sin 2x \dfrac{dy}{dx} = y + \tan x.$ **[B.C.A. (Lucknow) 2007]**

**Solution:** Writing the given linear differential equation in the standard form, we have

$$\frac{dy}{dx} - (\operatorname{cosec} 2x) \,.\, y = \frac{\tan x}{\sin 2x} = \frac{1}{2}\sec^2 x$$

$\therefore$ $$P = -\operatorname{cosec} 2x \quad \text{and} \quad Q = \frac{1}{2}\sec^2 x.$$

Now, integrating factor (I.F.) $= e^{\int P\, dx}$

$$= e^{-\int \operatorname{cosec} 2x\, dx} = e^{\log(\tan x)^{-1/2}} = \frac{1}{\sqrt{\tan x}}.$$

Multiplying both sides of equation (1) by integrating factor and integrating, we get

$$y \cdot \frac{1}{\sqrt{\tan x}} = \int \frac{\sec^2 x}{2\sqrt{\tan x}}\, dx + c$$

$$y \cdot \frac{1}{\sqrt{\tan x}} = \int \frac{dt}{2\sqrt{t}} + c, \qquad \text{where } t = \tan x = \sqrt{t} + c = \sqrt{\tan x} + c$$

or $$y = \tan x + c\sqrt{\tan x}$$

which is the required solution.

***Example 48:*** ***Solve*** $x(1-x^2)\,dy + (2x^2 y - y - ax^3)\,dx = 0.$ **[B.C.A. (Kashi) 2012]**

***Solution:*** Writing the given linear differential equation in the standard form, we have

$$\frac{dy}{dx} + \frac{2x^2 - 1}{x(1-x^2)}\, y = \frac{ax^2}{1-x^2} \qquad \text{...(1)}$$

$\therefore$ $$P = \frac{2x^2-1}{x(1-x^2)} \quad \text{and} \quad Q = \frac{ax^2}{1-x^2}.$$

Now, integrating factor (I.F.) $= e^{\int P\, dx}$

$$= e^{\int \frac{2x^2-1}{x(1-x^2)} dx} = e^{\int \frac{1-2x^2}{x(x-1)(x+1)} dx}$$

$$= e^{-\int \left\{\frac{1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}\right\} dx}$$

$$= e^{-[\log x + \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x+1)]} = e^{-\log\{x\sqrt{(x^2-1)}\}} = \frac{1}{x\sqrt{x^2-1}}.$$

Hence, the required solution of the linear differential equation (1) is

$$y \cdot \frac{1}{x\sqrt{x^2-1}} = a\int \frac{x^2}{1-x^2} \cdot \frac{1}{x\sqrt{x^2-1}}\, dx + c$$

$$= -a\int \frac{x\, dx}{(x^2-1)^{3/2}} + c = -\frac{a}{2}\int \frac{dt}{t^{3/2}} + c, \quad \text{on putting } x^2 - 1 = t$$

$$= \frac{a}{t} + c = \frac{a}{\sqrt{x^2-1}} + c$$

*i.e.,* $$y = ax + cx\sqrt{x^2-1}.$$

**Example 49:** ***Solve*** $(x+y+1)\dfrac{dy}{dx}=1.$ **[B.C.A. (Agra) 2000]**

***Solution:*** The given differential equation can be written as:

$$\frac{dx}{dy}=x+y+1 \qquad \text{or} \qquad \frac{dx}{dy}=x+y+1 \qquad ...(1)$$

This is the linear differential equation in $x$.

Now, $\int P\,dy=\int(-1)\,dy=-y.$

$\therefore$ Integrating factor (I.F.) $=e^{\int P\,dy}=e^{-y}.$

Hence, the required solution of the given differential equation (1) is

$$x\,.\,e^{-y}=\int(y+1)\,e^{-y}\,dy+c$$

$$=(y+1)\,.\,(-e^{-y})-(1)\,.\,(e^{-y})+c=(-y-2)\,e^{-y}+c$$

*i.e.,* $$x=ce^{y}-(y+2).$$

**Example 50:** ***Solve*** $(x+2y^3)\dfrac{dy}{dx}=y.$ **[B.C.A. (Aligarh) 2010]**

***Solution:*** The given equation can be written as

$$\frac{dx}{dy}=\frac{x+2y^3}{y} \qquad \text{or} \qquad \frac{dx}{dy}-\frac{1}{y}\,.\,x=2y^2 \qquad ...(1)$$

which is a linear equation in $x$ and is of the form

$$(dx/dy)+Px=Q, \qquad \text{where } P=-(1/y) \qquad \text{and} \quad Q=2y^2$$

$\therefore$ $$\text{I.F.}=e^{\int P\,dy}=e^{-\int(1/y)\,dy}=e^{-\log y}=e^{\log(1/y)}=1/y.$$

Hence, its solution is

$$x\,.\,(\text{I.F.})=\int\{Q\times(\text{I.F.})\}\,dy+c$$

*i.e.,* $$x\,.\,(1/y)=\int 2y^2\,.\,(1/y)\,dy+c$$

or $$x/y=y^2+c \qquad \text{or} \qquad x=y^3+cy$$

which is the required solution.

**Example 51:** ***Solve*** $x\cos x\dfrac{dy}{dx}+y\,(x\sin x+\cos x)=1.$

***Solution:*** Writing the given differential equation into the standard form, we have

$$\frac{dy}{dx}+\left(\frac{x\sin x+\cos x}{x\cos x}\right)y=\frac{1}{x\cos x}$$

or $$\frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right) y = \frac{1}{x \cos x} \quad ...(1)$$

which is linear in variable $y$. Now,

$$\int P\,dx = \int \left(\tan x + \frac{1}{x}\right) dx = \log \sec x + \log x = \log (x \sec x).$$

$\therefore$ Integrating factor (I.F.) $= e^{\int P\,dx} = e^{\log (x \sec x)} = x \sec x.$

Multiplying both sides of equation (1) by integrating factor (I.F.) and integrating, we get

$$y \,.\, (x \sec x) = \int \frac{1}{x \cos x} x \sec x\, dx + c = \int \sec^2 x\, dx + c$$

*i.e.,* $$yx \sec x = \tan x + c.$$

This is the required solution.

***Example 52:*** ***Solve*** $\boldsymbol{\dfrac{dy}{dx} + \dfrac{x}{1 + x^2}\, y = \dfrac{1}{2x\,(1 + x^2)}}$.

**[B.C. A. (Agra) 2002]**

***Solution:*** The given differential equation is linear in $y$ and is of the form

$$\frac{dy}{dx} + Py = Q, \qquad \text{where } P = \frac{x}{1 + x^2} \qquad \text{and} \quad Q = \frac{1}{2x\,(1 + x^2)}.$$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{\frac{1}{2}\int \frac{2x}{1 + x^2} dx} = e^{\frac{1}{2} \log (1 + x^2)} = e^{\log \sqrt{1 + x^2}} = \sqrt{1 + x^2}.$$

Hence the required solution is

$$y\,(\text{I.F.}) = \int \{Q \times (\text{I.F.})\}\, dx + c$$

*i.e.,* $$y \sqrt{1 + x^2} = \int \frac{1}{2x\,(1 + x^2)} \,.\, \sqrt{1 + x^2}\, dx + c = \int \frac{dx}{2x \sqrt{1 + x^2}} + c$$

$$= \int \frac{(-1/t^2)\, dt}{2\,.\,1/t \sqrt{1 + 1/t^2}} + c, \quad \text{putting, } x = \frac{1}{t}, \text{ so that } dx = -(1/t^2)\, dt$$

$$= -\frac{1}{2} \int \left[\frac{dt}{\sqrt{1 + t^2}}\right] + c = -\frac{1}{2} \sin h^{-1} t + c$$

or $$y \sqrt{1 + x^2} = c - \frac{1}{2} \sin h^{-1} \frac{1}{x}.$$

## Exercise 9.5

**Solve the following Equations:**

1. $\dfrac{dy}{dx} + ay = e^{mx}$.

2. $x\dfrac{dy}{dx} + y = x^2 + 3x + 2$. **[B.C.A. (Rohtak) 2009]**

3. $\dfrac{dy}{dx} = mx + ny + q$.

4. $(x-1)\dfrac{dy}{dx} + y = x^2 - 1$. **[B.C.A. (Purvanchal) 2010]**

5. $y - x\dfrac{dy}{dx} = b\,[1 + x^2]\,\dfrac{dy}{dx}$.

6. $\dfrac{dy}{dx}(x \log x) + y = 2 \log x$. **[B.C.A. (Kashi) 2011]**

7. $\dfrac{dy}{dx} + \dfrac{3x^2}{1+x^3}\,y = \dfrac{\sin^2 x}{1+x^3}$.

8. $x(x-1)\dfrac{dy}{dx} - (x-2)\,y = x^3(2x-1)$.

9. $x\dfrac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$. **[B.C.A. (Lucknow) 2008]**

10. $\sin x \dfrac{dy}{dx} + 3y = \cos x$. **[B.C. A. (Agra) 2001]**

11. $(x + \tan y)\,dy = \sin 2y\,dx$.

12. $\dfrac{dy}{dx} + y \cot x = \cos x$. **[B.C.A. (Meerut) 2001]**

13. $x^2\dfrac{dy}{dx} + y = 1$. **[B.C.A. (Rohilkhand) 2009]**

14. $\dfrac{dx}{dt} - ax = be^{at}$, $a$, $b$ are constants.

15. $(1-x^2)\dfrac{dy}{dx} - xy = 1\ (x > 1)$.

16. $(1 + y + x^2 y)\,dx + (x + x^3)\,dy = 0$. **[B.C.A. (Rohilkhand) 2006]**

17. $\dfrac{dy}{dx} + \dfrac{1-2x}{x^2}\, y = 1.$ [B.C.A. (Indraprastha) 2012]

18. $x\,(x^2+1)\dfrac{dy}{dx} - y\,(1-x^2) = x^3 \log x.$ [B.C.A. (Aligarh) 2007]

19. $\cos^2 x \dfrac{dy}{dx} + y = \tan x.$

20. $(1+x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1} x}.$ [B.C.A. (Meerut) 2002]

## Answers 9.5

| | |
|---|---|
| 1. | $y = e^{mx}/(m+a) + ce^{-ax}.$ |
| 2. | $xy = c + \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x.$ |
| 3. | $y = ce^{nx} - m\,(nx+1)/n^2 - q/n.$ |
| 4. | $3\,(x-1)\,y = x^3 - 3x + c.$ |
| 5. | $y\,(1+bx) = b + cx.$ |
| 6. | $y \log x = c + (\log x)^2.$ |
| 7. | $y\,(1+x^3) = \frac{1}{2}x - \frac{1}{4}\sin 2x + c.$ |
| 8. | $(x-1)\,y = x^2\,(x^2 - x + c).$ |
| 9. | $y = x \log \tan x + cx.$ |
| 10. | $y \tan^3 \frac{1}{2}x = c - \frac{1}{3}\tan^3 \frac{1}{2}x + 2\tan \frac{1}{2}x - x.$ |
| 11. | $x = c\sqrt{\tan y} + \tan y.$ |
| 12. | $y \sin x = \frac{1}{2}\sin^2 x + c.$ |
| 13. | $y = ce^{1/x} + 1.$ |

| | |
|---|---|
| 14. | $x = (bt + c)\, e^{at}$. |
| 15. | $y\sqrt{x^2 - 1} = c - \log\{x + \sqrt{x^2 - 1}\}$. |
| 16. | $xy = c - \tan^{-1} x$. |
| 17. | $y = x^2\,(ce^{1/x} + 1)$. |
| 18. | $y \cdot \dfrac{x^2 + 1}{x} = \dfrac{x^2}{2} \log x - \dfrac{x^2}{4} + c$. |
| 19. | $ye^{\tan x} = c + e^{\tan x}(\tan x - 1)$. |
| 20. | $ye^{\tan^{-1} x} = \dfrac{1}{2} e^{2\tan^{-1} x} + c$. |

## 9.8 Equations Reducible to Linear Form (Bernoulli's Equation)

**Definition:** *"An equation of the form*

$$\frac{dy}{dx} + Py = Q\,y^n \qquad ...(1)$$

*where P and Q are constants or function of x only and n is a constant, other than* 0 *and* 1, *is called a Bernoulli's differential equation."*

Equation (1) is called the **standard form of Bernoulli's equation.**

If we put $n = 0$ in (1), the equation becomes linear and $n = 1$ in (1) the variables are separable. We therefore concentrate on the case $n \neq 1$. Multiplying throughout by $y^{-n}$, the equation (1) becomes

$$y^{-n}\frac{dy}{dx} + Py^{-n+1} = Q \qquad ...(2)$$

Now put $y^{-n+1} = v$ and $(-n + 1)\, y^{-n}\dfrac{dy}{dx} = \dfrac{dv}{dx}$, the equation (2) transforms into

$$\frac{dv}{dx} + (1 - n)\,Pv = (1 - n)\,Q \qquad ...(3)$$

which is linear in $v$.

$\therefore$ $\quad$ I.F. $= e^{\int (1-n) P\,dx}$

Hence, the general solution of the Bernoulli's equation is

$$ve^{\int (1-n) P\,dx} = \int (1 - n)\,Qe^{\int (1-n) P\,dx}\,dx + c$$

or $$y^{(n-1)}\, e^{\int (1-n) P\,dx} = \int (1 - n)\,Qe^{\int (1-n) P\,dx}\,dx + c$$

**Remark:** If we have a differential equation of the form

$$f'(y)\,(dy/dx) + P\,.\,f(y) = Q$$

then by putting $f(y) = v$ and $f'(y)\,.\,(dy/dx) = dv/dx$, this equation is reduced to the linear form.

***Example 53:*** ***Solve*** $(1 - x^2)\dfrac{dy}{dx} + xy = xy^2.$ **[B.C.A. (Rohilkhand) 2003, 01]**

***Solution:*** Dividing throughout the given differential equation by $y^2\,(1 - x^2)$, we get

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{x}{1 - x^2}\,.\,\frac{1}{y} = x \qquad ...(1)$$

Putting $\dfrac{1}{y} = v$, so that $-\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$ in equation (1), we get

$$-\frac{dv}{dx} + \frac{x}{1 - x^2}\,v = x \quad \text{or} \quad \frac{dv}{dx} + \frac{x}{x^2 - 1}\,v = -x \qquad ...(2)$$

which is linear differential equation in $v$.

Here $P = \dfrac{x}{x^2 - 1}$ and $Q = -x$

$\therefore$ Integrating factor (I.F.) $= e^{\int P\,dx} = e^{\int \frac{x}{x^2 - 1}dx} = e^{\frac{1}{2}\log(x^2 - 1)} = \sqrt{x^2 - 1}.$

Multiplying both sides of equation (2) by integrating factor (I.F.) and integrating, we get

$$v\,.\,\sqrt{x^2 - 1} = c + \int (-x)\,.\,\sqrt{x^2 - 1}\,dx$$

$$= c - \int t^2\,dt, = c - \frac{1}{3}t^3 \quad (\text{on putting } x^2 - 1 = t^2)$$

or $$\frac{1}{y}\,.\,\sqrt{x^2 - 1} = c - \frac{1}{3}(x^2 - 1)^{3/2}$$

or $$\sqrt{x^2 - 1} = y\left[c - \frac{1}{3}(x^2 - 1)^{3/2}\right].$$

This is the required solution.

***Example 54:*** ***Solve*** $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2 \sin x.$ **[B.C.A. (Garhwal) 2008]**

***Solution:*** Clearly, the given differential equation is a Bernoulli's equation. Hence dividing the equation by $y^2$, we get

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\,.\,\frac{1}{y} = \sin x \qquad ...(1)$$

Putting $\frac{1}{y} = v$, so that $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$ in equation (1), we get

$$\frac{dv}{dx} - \frac{1}{x}\,.\,v = -\sin x \qquad \text{...(2)}$$

which is a linear equation in the variable $v$.

$\therefore$ Integrating factor (I.F.) $= e^{\int P\,dx} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$.

Hence, the solution of the linear equation in $v$ is:

$$v\,.\,\frac{1}{x} = c - \int \frac{\sin x}{x}\,dx$$

But $v = \frac{1}{v}$, thus the required solution of the given differential equation is

$$\frac{1}{xy} = c - \int \frac{\sin x}{x}\,dx.$$

***Example 55:*** ***Solve*** $\mathbf{3\frac{dy}{dx} + \frac{2}{x+1}\,y = \frac{x^3}{y^2}}$.

***Solution:*** Clearly, the given differential equation is a Bernoulli's equation. Hence multiplying the equation by $y^2$, we have

$$3\,y^2\frac{dy}{dx} + \frac{2}{x+1}\,y^3 = x^3 \qquad \text{...(1)}$$

On putting $y^3 = v$, so that $3\,y^2\frac{dy}{dx} = \frac{dv}{dx}$ in equation (1), we get

$$\frac{dv}{dx} + \frac{2}{x+1}\,v = x^3 \qquad \text{...(2)}$$

which is a linear differential equation in $v$.

$\therefore$ Integrating factor (I.F.) $= e^{\int P\,dx} = e^{\int \frac{2}{x+1}dx} = e^{2\log(x+1)} = (x+1)^2$.

Hence, the solution of the linear differential equation (2) in $v$ is given by

$$v\,(x+1)^2 = c + \int x^3\,(x+1)^2\,dx$$

$$= c + \int (x^5 + 2x^4 + x^3)\,dx = c + \frac{1}{6}x^6 + \frac{2}{5}x^5 + \frac{1}{4}x^4$$

But $v = y^3$, thus the required solution of the given differential equation is

$$y^2\,(x+1)^2 = \frac{1}{6}x^6 + \frac{2}{5}x^5 + \frac{1}{4}x^4 + c$$

**Example 56:** **Solve** $\dfrac{dy}{dx} - y \sec x = y^2 \cos x \sin x.$ **[B.C.A. (Kanpur) 2005]**

**Solution:** The given equation is a Bernoulli's equation. Multiplying throughout by $y^{-2}$, the given equation reduces to

$$y^{-2}\,(dy/dx) - \sec x \,.\, y^{-1} = \cos x \sin x \qquad ...(1)$$

Putting $\quad y^{-1} = v \quad$ and $\quad -y^{-2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$, the equation (1) reduces to

$$\frac{dv}{dx} + \sec x \,.\, v = -\cos x \sin x \qquad ...(2)$$

which is a linear equation in $v$.

Here $\quad P = \sec x \quad$ and $\quad Q = -\cos x \sin x$

$\therefore$ Its $\quad \text{I.F.} = e^{\int P\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$

Multiplying the integrating factor to equation (2) and integrating w.r.t to $x$, we obtain

$$v\,.\,(\sec x + \tan x) = \int -\cos x \sin x\,(\sec x + \tan x)\,dx + c$$

$$= -\int(\sin x + \sin^2 x)\,dx + c = -\int\left(\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x\right)dx + c$$

$$= \cos x - \frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

or $\quad y^{-1}(\sec x \tan x) = \cos x - \dfrac{1}{2}x + \dfrac{1}{4}\sin 2x + c$

which is the required solution of the given equation.

**Example 57:** **Solve** $\dfrac{dy}{dx} = e^{x-y}\,(e^x - e^y).$ **[B.C.A. (Agra) 2007]**

**Solution:** Multiplying both sides of the given differential equation by $e^y$, we get

$$e^y\frac{dy}{dx} = e^y\,e^{x-y}\,(e^x - e^y) \qquad \text{or} \qquad e^y\frac{dy}{dx} = e^x\,(e^x - e^y)$$

or $\quad e^y\dfrac{dy}{dx} + e^x\,e^y = e^{2x}$

Putting $\quad e^y = v \quad$ and $\quad e^y\dfrac{dy}{dx} = dv/dx$, we obtain

$$\frac{dv}{dx} + e^x\,.\,v = e^{2x}, \qquad ...(1)$$

which is clearly a linear equation in $v$.

Here $\qquad P = e^x \qquad$ and $\qquad Q = e^{2x}$

$\therefore$ Its $\qquad \text{I.F.} = e^{\int P\, dx} = e^{\int e^x dx} = e^{(e^x)}.$

Multiplying both sides of (1) by integrating factor and then integrating, we obtain

$$v \,.\, e^{(e^x)} = \int e^{2x} \,.\, e^{(e^x)}\, dx + c, \qquad \text{where } c \text{ is arbitrary constant.}$$

or $\qquad v \,.\, e^{(e^x)} = \int e^x \,.\, e^{(e^x)} \,.\, e^x\, dx + c = \int t \,.\, e^t\, dt + c = (t-1)\, e^t + c, \quad \text{where } t = e^x$

or $\qquad e^y \,.\, e^{(e^x)} = (e^x - 1)\, e^{(e^x)} + c \qquad [\because\ v = e^y,\ t = e^x]$

or $\qquad e^y = ce^{-(e^x)} + e^x - 1$

which is the required solution of the given equation.

***Example 58:*** ***Solve*** $x \dfrac{dy}{dx} + y = y^2 \log x$. **[B.C.A. (Meerut) 2002]**

***Solution:*** The given equation may be written as

$$\frac{dy}{dx} + \frac{1}{x} \,.\, y = \frac{\log x}{x}\, y^2.$$

which is a Bernoulli's equation. Multiplying throughout by $y^{-2}$, the above equation reduces to

$$y^{-2} \frac{dy}{dx} + \frac{1}{x} \,.\, y^{-1} = \frac{\log x}{x} \qquad \text{...(1)}$$

Putting $\qquad y^{-1} = v \qquad$ and $\quad -y^{-2} \,.\, \dfrac{dy}{dx} = \dfrac{dv}{dx}$ in (1), it becomes

$$\frac{dy}{dx} - \frac{1}{x} \,.\, v = -\frac{\log x}{x} \qquad \text{...(2)}$$

which is a linear equation in $v$.

Here $\qquad P = -\dfrac{1}{x} \qquad$ and $\qquad Q = -\dfrac{1}{x} \log x.$

$\therefore$ Its $\qquad \text{I.F.} = e^{\int P\, dx} = e^{-\int (1/x)\, dx} = e^{-\log x} = \dfrac{1}{x}.$

Multiplying (2) by integrating factor and integrating with respect to $x$, we obtain

$$v \,.\, \frac{1}{x} = \int \left(-\frac{1}{x} \log x\right) \frac{1}{x}\, dx + c, \qquad \text{where } c \text{ is arbitrary constant.}$$

$$= \int \left(-\frac{1}{x^2}\right) \log x \, dx + c = (\log x) \,.\, \frac{1}{x} - \int \left(\frac{1}{x} \,.\, \frac{1}{x}\right) dx + c$$

$$= \frac{\log x}{x} + \frac{1}{x} + c \qquad \text{or} \qquad \frac{1}{xy} = \frac{1}{x} (\log x + 1 + cx)$$

or $$y\,(1 + \log x + cx) = 1$$

***Example 59:*** ***Solve*** $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y.$

***Solution:*** The given equation can be written as

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \qquad ...(1)$$

Putting $\tan y = v$ and $\sec^2 y \dfrac{dy}{dx} = \dfrac{dv}{dx}$ in (1), we obtain

$$\frac{dv}{dx} + 2xv = x^3 \qquad ...(2)$$

which is a linear equation in $v$.

$\therefore$ Its $\text{I.F.} = e^{2\int x\,dx} = e^{x^2}.$

Multiplying both sides of (2) by integrating factor and integrating with respect to $x$, we obtain

$$v \,.\, e^{x^2} = \int x^3 e^{x^2} \, dx + c, \text{ where } c \text{ is arbitrary constant.}$$

$$= \frac{1}{2} \int t\, e^t \, dt + c, = \frac{1}{2} (t-1)\, e^t + c \qquad (\text{where } x^2 = t)$$

or $$e^{x^2} \tan y = \frac{1}{2} e^{x^2} (x^2 - 1) + c \qquad [\because \; v = \tan y]$$

or $$\tan y = \frac{1}{2} (x^2 - 1) + c e^{-x^2}$$

which is required solution.

***Example 60:*** ***Solve*** $2\dfrac{dy}{dx} - y \sec x = y^3 \tan x.$ **[B.C.A. (Meerut) 2002]**

***Solution:*** The given equation can be written as

$$\frac{2}{y^3} \frac{dy}{dx} - \frac{1}{y^2} \sec x = \tan x \qquad ...(1)$$

Putting $-\dfrac{1}{y^2} = v$ and $\dfrac{2}{y^3} \dfrac{dy}{dx} = \dfrac{dv}{dx}$, the equation (1) reduces to the form

$$\frac{dy}{dx} + \sec x \,.\, v = \tan x \qquad ...(2)$$

which is linear in $v$. Hence its integrating factor,

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

Hence. its solution is

$$v(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + c$$

$$= \int (\sec x \tan x + \sec^2 x - 1)\,dx + c = \sec x + \tan x - x + c$$

or $$-\frac{\sec x + \tan x}{x} = \sec x + \tan x - x + c \qquad \left[\because\ v = -\frac{1}{y^2}\right]$$

which is the required solution of the given equation.

## Exercise 9.6

**Solve the following Differential Equations:**

**1.** $y^2 \dfrac{dy}{dx} = x + y^3.$ **[B.C.A. (Avadh) 2004]**

**2.** $x \dfrac{dy}{dx} + y = x^2 y^2.$

**3.** $2 \dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}.$ **[B.C.A. (Purvanchal) 2006]**

**4.** $\dfrac{dy}{dx} = x^3 y^3 - xy.$

**5.** $(1 + x^2) \dfrac{dy}{dx} = xy - y^2.$ **[B.C.A. (Kashi) 2010]**

**6.** $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3 y^6.$

**7.** $\dfrac{dy}{dx} - y \tan x = y^2 \sec x.$ **[B.C.A. (Rohtak) 2007]**

**8.** $\dfrac{dy}{dx} + xy = y^2 e^{x^2/2} \sin x.$ **[B.C.A. (Rohilkhand) 2006]**

**9.** $\dfrac{dy}{dx} - \dfrac{\tan y}{1 + x} = (1 + x) e^x \sec y.$

**10.** $\dfrac{dy}{dx} + \dfrac{y}{x} \log y = \dfrac{y}{x^2} (\log y)^2.$ **[B.C.A. (Purvanchal) 2008]**

11. $\dfrac{dy}{dx}+\dfrac{y}{x}=y^2x.$ [B.C.A. (Aligarh) 2012]

12. $x\dfrac{dy}{dx}+y=xy^3.$ [B.C. A. (Agra) 2000]

13. $(x^2y^3+2xy)\,dy=dx.$

14. $y-x\dfrac{dy}{dx}=a\left[y^2+\dfrac{dy}{dx}\right].$ [B.C.A. (Bundelkhand) 2008]

15. $\dfrac{dy}{dx}+\dfrac{y}{x-1}=xy^{1/3}.$

16. $\dfrac{dy}{dx}(x^2y^3+xy)=1.$

17. $3x(1-x^2)\,y^2\dfrac{dy}{dx}+(2x^2-1)\,y^2=ax^3.$ [B.C.A. (Lucknow) 2009]

## Answers 9.6

| | |
|---|---|
| 1. | $y^3e^{-3x}=c-xe^{-3x}-\dfrac{1}{3}e^{-3x}.$ |
| 2. | $y(cx-x^2)=1.$ |
| 3. | $x=y(1+c\sqrt{x}).$ |
| 4. | $(1/y^2)=ce^x+x^2+1.$ |
| 5. | $\sqrt{1+x^2}=cy+y\ \sin h^{-1}\,x.$ |
| 6. | $cx^5y^5+5x^4y^5=1.$ |
| 7. | $\sec x=y(c-\tan x).$ |
| 8. | $e^{-x/2}=y(c+\cos x).$ |
| 9. | $\sin y=c(1+x)+e^x(1+x).$ |
| 10. | $\dfrac{1}{x(\log y)}=c+\dfrac{1}{2x^2}.$ |

| | |
|---|---|
| **11.** | $1 + x^2 y + cxy = 0.$ |
| **12.** | $(2 + cx)\, xy^2 = 1.$ |
| **13.** | $x\,(y^2 - 1) + 2 = Axe^{-y^2}.$ |
| **14.** | $(x + a) = ny\,(ax + c).$ |
| **15.** | $y^{2/3}\,(x-1)^{2/3} = \frac{2}{5}\, x\,(x-1)^{5/2} - \frac{3}{20}\,(x-1)^{8/3} + c.$ |
| **16.** | $1 + x\,(y^2 - 2 + ce^{-y/2}) = 0.$ |
| **17.** | $y = \cos x + \cos x \sqrt{1 - x^2}\ .$ |

***Example 61:*** ***Solve*** $\cos(x + y)\, dy = dx.$ **[B.C.A. (Meerut) 2005]**

***Solution:*** The given equation can be written as

$$\cos(x + y)\frac{dy}{dx} = 1 \qquad \ldots(1)$$

Put $x + y = v$; so that $1 + \frac{dy}{dx} = \frac{dv}{dx}$ or $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Substituting these in (1) the equation becomes,

$$\cos v\left(\frac{dv}{dx} - 1\right) = 1 \quad \text{or} \quad \cos v \frac{dv}{dx} = 1 + \cos v$$

or $$\left(\frac{\cos v}{1 + \cos v}\right) dv = dx \quad \text{or} \quad \left(1 - \frac{1}{1 + \cos v}\right) dv = dx$$

or $$\left(1 - \frac{1}{2\cos^2 \frac{1}{2} v}\right) dv = dx \qquad [\because\ 1 + \cos v = 2\cos^2 \tfrac{1}{2} v]$$

or $$[1 - \tfrac{1}{2}\sec^2 \tfrac{1}{2} v]\, dv = dx$$

which is an equation in which variables are separable. Hence on integration, we obtain the general solution

$$v - \tan \frac{1}{2} v = x + c, \quad c \text{ being an arbitrary constant.}$$

or $$x + y - \tan\frac{1}{2}(x + y) = x + c$$

or $$y = \tan\frac{1}{2}(x + y) + c$$

**Example 62:** ***Solve*** $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$.

**Solution:** On putting $x + y = v$, so that $1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$ in the given differential equation, we obtain

$$\frac{dv}{dx} = (1 + \cos v) + \sin v = 2\cos^2 \frac{1}{2} v + 2 \sin \frac{1}{2} v \cos \frac{1}{2} v$$

$$= 2\cos^2 \frac{1}{2} v \left(1 + \tan \frac{1}{2} v\right)$$

or
$$\frac{\frac{1}{2}\sec^2 \frac{v}{2}\, dv}{1 + \tan \frac{v}{2}} = dx$$

where variables are separated. Hence on integration, we get

$$\log\left(1 + \tan \frac{1}{2} v\right) = x + c$$

But $x + y = v$, thus the required solution of the given differential equation is

$$\log\left\{1 + \tan \frac{1}{2}(x + y)\right\} = x + c.$$

**Example 63:** ***Solve*** $\dfrac{dy}{dx} = (4x + y + 1)^2$. **[B.C.A. (Agra) 2010]**

**Solution:** On putting $4x + y + 1 = v$, so that $4 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$ in the given differential equation, we have

$$\frac{dv}{dx} - 4 = v^2 \qquad \Rightarrow \qquad \frac{dv}{v^2 + 4} = dx,$$

which is an equation in which variables are separated. Hence on integration, we get

$$\frac{1}{2}\tan^{-1}\frac{v}{2} = x + C$$

or
$$\frac{1}{2} v = \tan(2x + c), \qquad \text{where} \quad c = 2C$$

or
$$4x + y + 1 = 2\tan(2x + c)$$

This is the required solution.

**Example 64:** ***Solve*** $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\dfrac{a^2 - x^2 - y^2}{x^2 + y^2}}$. **[B.C.A. (Lucknow) 2009, 07, 04]**

**Solution:** Let $x = r\cos\theta$, $y = r\sin\theta$.

Then $x^2 + y^2 = r^2$ and $y/x = \tan\theta$.

Differentiating these relations, we obtain

$$x\,dy + y\,dx = r\,dr \qquad \text{and} \qquad \frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\, d\theta.$$

Substituting these in the given equation, we obtain

$$\frac{r\,dr}{x^2 \sec^2\theta\, d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}}$$

$$\Rightarrow \qquad \frac{r\,dr}{r^2\, d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}}, \qquad \text{since} \qquad x = r\cos\theta$$

$$\Rightarrow \qquad \frac{dr}{\sqrt{a^2 - r^2}} = d\theta$$

which is an equation in which variables are separable. Hence on integration, we obtain the general solution

$$\sin^{-1}\frac{r}{a} = \theta + c \qquad \text{or} \qquad r = a\sin(\theta + c)$$

or $$\sqrt{x^2 + y^2} = a\sin\left(\tan^{-1}\frac{y}{x} + c\right)$$

**Example 65:** ***Solve*** $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$.

***Solution:*** The given differential equation can be written as

$$2y\frac{dy}{dx} + y^2 = -(x^2 + 2x) \qquad \text{...(1)}$$

On putting $y^2 = v$, so that $2y\dfrac{dy}{dx} = \dfrac{dv}{dx}$ in equation (1), we get

$$\frac{dv}{dx} + v = -(x^2 + 2x) \qquad \text{...(2)}$$

which is a linear equation in $v$. Hence its integrating factor,

$$\text{I.F.} = e^{\int P\,dx} = e^{\int 1\,dx} = e^x$$

Hence, the solution of equation (2) is

$$v\,.\,e^x = c - \int (x^2 + 2x)\,e^x\,dx$$

$$= c - [(x^2 + 2x)\,e^x - (2x + 2)\,e^x + 2e^x] = c - x^2 e^x$$

But $v = y^2$, thus the required solution of the given differential equation is

$$y^2 e^x = c - x^2 e^x \qquad \text{or} \qquad y^2 = ce^{-x} - x^2$$

***Example 66:*** ***Solve*** $(x - y^2)\,dx + 2xy\,dy = 0$.

***Solution:*** The given differential equation can be written as

$$2xy\frac{dy}{dx} - y^2 + x = 0 \quad \text{or} \quad 2y\frac{dy}{dx} - \frac{1}{x}y^2 = -1 \qquad \ldots(1)$$

On putting $y^2 = v$, so that $2y\frac{dy}{dx} = \frac{dv}{dx}$ in equation (1), we get

$$\frac{dv}{dx} - \frac{1}{x}v = -1$$

which is a linear differential equation in $v$.

$\therefore$ Integrating factor (I.F.) $= e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$.

Hence, the solution of equation (2) is

$$v\,.\,\frac{1}{x} = c + \int(-1)\,.\,\frac{1}{x}\,dx = c - \log x$$

or $$y^2 = cx - x\log x \qquad [\because\ v = y^2]$$

This is the required solution.

***Example 67:*** ***Solve*** $(x^2 + y^2 + 2)\,dx + 2y\,dy = 0$. **[B.C.A. (Bhopal) 2012, 06]**

***Solution:*** The given differential equation can be written as

$$2y\frac{dy}{dx} + y^2 = -(x^2 + 2) \qquad \ldots(1)$$

On putting $y^2 = v$, so that $2y\frac{dy}{dx} = \frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dv}{dx} + v = -(x^2 + 2) \qquad \ldots(2)$$

which is linear differential equation in $v$.

$\therefore$ Integrating factor (I.F.) $= e^{\int P\,dx} = e^{\int 1\,dx} = e^x$.

$\therefore$ The solution of equation (2) is

$$v\,.\,e^x = c - \int(x^2 + 2)\,e^x\,dx = c - [(x^2 + 2)\,e^x - 2e^x\,(x - 1)]$$

or $$y^2 e^x = c - e^x\,(x^2 - 2x + 4),$$

which is the required solution.

## 9.9 Exact Differential Equations

The differential equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is called exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

where $\frac{\partial M}{\partial y}$ = partial differentiation of $M$ w.r.t. $y$ when $x$ as constant

and $\frac{\partial N}{\partial x}$ = partial differentiation of $N$ w.r.t. $x$ when $y$ as constant.

**Working Rule**

1. First integrate $M$ with respect to $x$ regarding $y$ as a constant.
2. Integrate $N$ with respect to $y$, keeping $x$ constant, and retaining only those terms which have not been already obtained by the integration of $M$.
3. The sum of the expressions, thus obtained equated to an arbitrary constant will be the required solution.

***Example 68:*** ***Solve the equation*** $(1 + 4xy + 2y^2)\,dx + (1 + 4xy + 2x^2)\,dy = 0$.

***Solution:*** Here $M = 1 + 4xy + 2y^2$ and $N = 1 + 4xy + 2x^2$

$\therefore$ $\frac{\partial M}{\partial y} = 4x + 4y$ and $\frac{\partial N}{\partial x} = 4y + 4x$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

it follows that the given differential equation is exact.

Now, $\int_{y-\text{constant}} M\,dx = \int (1 + 4xy + 2y^2)\,dx = x + 2x^2 y + 2xy^2$

and $\int_{x-\text{constant}} N\,dy = \int (1 + 4xy + 2x^2)\,dy = y + 2xy^2 + 2x^2 y$

Again, the only new term obtained on integrating $N$ with respect to $y$ is $y$, as the terms $2xy^2 + 2x^2 y$ are already present in the integrating of $M$.

Hence, the solution of the given differential equation is

$$(x + 2x^2 y + 2xy^2) + y = c \qquad \text{or} \qquad x + y + 2xy(x + y) = c$$

or $(x + y)(1 + 2xy) = c$.

***Example 69:*** ***Solve*** $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0$ ***and show that this differential equation represents a family of conics.*** **[B.C.A. (Indore) 2012]**

***Solution:*** The given differential equation can be written as

$$(ax + hy + g)\, dx + (hx + by + f)\, dy = 0$$

Here $\quad M(x, y) = ax + hy + g \quad$ and $\quad N(x, y) = hx + by + f.$

$\therefore \quad \dfrac{\partial M}{\partial y} = h \quad$ and $\quad \dfrac{\partial N}{\partial x} = h \quad \Rightarrow \quad \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Hence, the given differential equation is exact.

Now, $\quad \int_{y-\text{constant}} M\, dx = \int_{y-\text{constant}} (ax + hy + g)\, dx = \dfrac{ax^2}{2} + hxy + gx, \qquad ...(1)$

and $\quad \int_{x-\text{constant}} N\, dy = \int_{x-\text{constant}} (hx + by + f)\, dy = hxy + \dfrac{by^2}{2} + fy \qquad ...(2)$

The term $hxy$ is already present in the integration of $M$. Therefore, the only new terms obtained by integrating $N$ with respect to $y$ is $\dfrac{by^2}{2} + fy$.

Thus, adding $\dfrac{by^2}{2} + fy$ to the R.H.S. of (1) and then equated to an arbitrary constant $C$, the solution of the given exact equation is

$$\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = C$$

or $\quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \qquad$ where $c = -2C$.

This equation clearly represents a family or conics.

***Example 70:*** ***Solve*** $\boldsymbol{x\, dx + y\, dy + \dfrac{x\, dy - y\, dx}{x^2 + y^2} = 0.}$ **[B.C.A. (Rohilkhand) 2008]**

***Solution:*** The given equation can be written as

$$\left[x - \frac{y}{x^2 + y^2}\right] dx + \left[y + \frac{x}{x^2 + y^2}\right] dy = 0$$

or $\quad \left[\dfrac{x^3 + xy^2 - y}{(x^2 + y^2)}\right] dx + \left[\dfrac{x^2 y + y^3 + x}{(x^2 + y^2)}\right] dy = 0 \qquad ...(1)$

Here $\quad M(x, y) = \dfrac{x^3 + xy^2 - y}{(x^2 + y^2)}; \; N(x, y) = \dfrac{x^2 y + y^3 + x}{(x^2 + y^2)}.$

$$\therefore \qquad \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence, equation (1) is an exact differential equation.

Now, $$\int_{y-\text{constant}} M\,dx = \int \frac{x^3 + xy^2 - y}{x^2 + y^2}\,dx = \int \left(x - \frac{y}{x^2 + y^2}\right) dx$$

$$= \int x\,dx - y \int \frac{1}{x^2 + y^2}\,dx = \frac{1}{2}x^2 - y\,\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{1}{2}x^2 - \tan^{-1}\frac{x}{y}.$$

Also, $$\int_{x-\text{constant}} N\,dy = \int \frac{x^2 y + y^3 + x}{x^2 + y^2}\,dy = \int \left(y + \frac{x}{x^2 + y^2}\right) dy$$

$$= \frac{1}{2}y^2 + \tan^{-1}\frac{y}{x} = \frac{1}{2}y^2 + \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{y}\right).$$

The only new term obtained on integrating $N$ with respect to $y$ is $\frac{1}{2}y^2$.

Hence, the solution of the given differential equation is

$$\frac{1}{2}x^2 - \tan^{-1}\frac{x}{y} + \frac{1}{2}y^2 = C,$$

or $$x^2 + y^2 - 2\tan^{-1}\frac{x}{y} = c, \qquad \text{where } c = -2C.$$

***Example 71:*** **Solve $y \sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$.**

***Solution:*** Here $M = y \sin 2x;\ N = -(1 + y^2 + \cos^2 x)$.

Therefore, $$\frac{\partial M}{\partial y} = \sin 2x; \quad \frac{\partial N}{\partial x} = 2\cos x \sin x.$$

Since $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \sin 2x,$$

it follows that the given differential equation is exact.

Now, $$\int_{y-\text{constant}} M\,dx = \int y \sin 2x\,dx = -\frac{1}{2}y\cos 2x$$

Also, $$\int_{x-\text{constant}} N\,dy = -\int (1 + y^2 + \cos^2 x)\,dy = -y - \frac{1}{3}y^3 - y\cos^2 x$$

$$= -y - \frac{1}{3}y^3 - y\,.\,\frac{1}{2}(1 + \cos 2x) = -\frac{3}{2}y - \frac{1}{3}y^3 - \frac{1}{2}y\cos 2x.$$

The term $-\frac{1}{2}y\cos 2x$ is already present in the integration of $M$. Therefore, the only new terms obtained on integrating $N$ with respect to $y$ are $-\frac{3}{2}y - \frac{1}{3}y^3$.

Hence, the solution of the given differential equation is

$$-\frac{1}{2}y\cos 2x + \left(-\frac{3}{2}y - \frac{1}{3}y^3\right) = C$$

or $$3y\cos 2x + 9y + 2y^3 = c, \qquad \text{where} \quad c = -6C.$$

***Example 72:*** ***Solve*** $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$

***Solution:*** Here $M = (1 + e^{x/y})$; $N = e^{x/y}\left(1 - \frac{x}{y}\right).$

Therefore, $$\frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x}{y^2}e^{x/y}$$

and $$\frac{\partial N}{\partial x} = e^{x/y}\cdot\left(0 - \frac{1}{y}\right) + \left(1 - \frac{x}{y}\right)\cdot\left(e^{x/y}\cdot\frac{1}{y}\right) = -\frac{x}{y^2}e^{x/y}.$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, it follows that the given differential equation is exact.

Now, $$\int_{y\text{-constant}} M\,dx = \int (1 + e^{x/y})\,dx = x + ye^{x/y}$$

and $$\int_{y\text{-constant}} N\,dy = e^{x/y}\left(1 - \frac{x}{y}\right)dy;\ \text{put } y = \frac{1}{t}$$

$$= \int e^{xt}(1 - xt)\cdot\left(-\frac{dt}{t^2}\right) = -\int e^{xt}\frac{dt}{t^2} + \int xe^{xt}\cdot\frac{1}{t}\,dt$$

$$= -\int \frac{e^{xt}}{t^2}\,dt + \left\{e^{xt}\cdot\frac{1}{t} - \int e^{xt}\left(-\frac{1}{t^2}\right)dt\right\}, \text{ on integrating by parts}$$

$$= e^{xt/t}, \text{ as the two integrals cancel each other} = ye^{x/y}, \text{ as } t = 1/y.$$

The term $ye^{x/y}$ has already occurred in the integration of $M$. Therefore, no new term is obtained by integrating $N$ with respect to $y$.

Hence, the required solution is

$$x + ye^{x/y} = c, \ c \text{ being arbitrary constant.}$$

***Example 73:*** ***Solve*** $(e^y + 1)\cos x\,dx\ \ e^y \sin x\,dy = 0.$ **[B.C.A. (Kurukshetra) 2007]**

***Solution:*** Here $M = (e^y + 1)\cos x$ and $N = e^y \sin x$, so that

$$\frac{\partial M}{\partial y} = e^y\cos x \qquad \text{and} \qquad \frac{\partial N}{\partial x} = e^y\cos x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, it follows that the given differential equation is exact.

Now, $\int_{y-\text{constant}} M\,dx = \int_{y-\text{constant}} (e^y + 1)\cos x\,dx = (e^y + 1)\sin x$

and $\int_{x-\text{constant}} N\,dy = \int_{x-\text{constant}} e^y \sin x\,dy = e^y \sin x.$

But the term $e^y \sin x$ has already occurred in the integration of $M$. Therefore, no new term is obtained by integrating $N$ with respect to $y$.

Hence, the required solution of the given differential equation is $(e^y + 1)\sin x = c$.

**Example 74:** ***Solve*** $x\,dx + y\,dy = \dfrac{a^2 (x\,dy - y\,dx)}{x^2 + y^2}.$

***Solution:*** The given differential equation can written as

$$\left[x + \frac{a^2 y}{x^2 + y^2}\right]dx + \left[y - \frac{a^2 x}{x^2 + y^2}\right]dy = 0$$

or
$$\left[\frac{x^3 + xy^2 + a^2 y}{x^2 + y^2}\right]dx + \left[\frac{x^2 y + y^3 - a^2 x}{x^2 + y^2}\right]dy = 0$$

Clearly, here $M = \dfrac{x^3 + xy^2 + a^2 y}{x^2 + y^2}$ and $N = \dfrac{x^2 y + y^3 - a^2 x}{x^2 + y^2}$, so that

$$\frac{\partial M}{\partial y} = \frac{a^2 x^2 - a^2 y^2}{(x^2 + y^2)^3} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{a^2 x^2 - a^2 y^2}{(x^2 + y^2)^3}.$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, it follows that the given differential equation is exact.

Now, $\int_{y-\text{constant}} M\,dx = \int_{y-\text{constant}} \left[x + \dfrac{a^2 y}{x^2 + y^2}\right]dx = \dfrac{1}{2}x^2 + a^2 \tan^{-1}\dfrac{x}{y}$

$$= \frac{1}{2}x^2 + a^2\left(\frac{\pi}{2} - \tan^{-1}\frac{y}{x}\right)$$

and $\int_{x-\text{constant}} N\,dy = \int_{x-\text{constant}} \left[y - \dfrac{a^2 x}{x^2 + y^2}\right]dy = \dfrac{1}{2}y^2 - a^2 \tan^{-1}\dfrac{y}{x}$

But the term $-a^2 \tan^{-1}\dfrac{y}{x}$ has already occurred in the integration of $M$, therefore, the only new term obtained on integrating $N$ w.r.t. to $y$ is $\dfrac{1}{2}y^2$.

Hence, the required solution of the given differential equation is

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 - a^2 \tan^{-1}\frac{y}{x} + a^2\frac{\pi}{2} = C$$

*i.e.,* $x^2 + y^2 - 2a^2 \tan^{-1}\dfrac{y}{x} = c,$ **where** $c = 2C - \pi a^2.$

**Example 75:** ***Solve*** $y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$. **[B.C.A. (M.D.U. Rohtak) 2012]**

***Solution:*** Here $M = y \sin 2x$ and $N = -(y^2 + \cos^2 x)$, so that

$$\frac{\partial M}{\partial y} = \sin 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = -2 \cos x\,(-\sin x) = \sin 2x$$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$; hence the given differential equation is exact.

Now, $\int_{y-\text{constant}} M\, dx = \int_{y-\text{constant}} y \sin 2x\, dx = -\frac{1}{2} y \cos 2x$

and $\int_{x-\text{constant}} N\, dy = \int_{x-\text{constant}} -(y^2 + \cos^2 x)\, dy = -\left(\frac{1}{3} y^3 + y \cos^2 x\right)$

$$= -\frac{1}{3} y^3 - \frac{1}{2} y\,(1 + \cos 2x) = -\frac{1}{3} y^3 - \frac{1}{2} y^3 - \frac{1}{2} y - \frac{1}{2} y \cos 2x$$

But the term $-\frac{1}{2} y \cos 2x$, has already occurred in the integration of $M$, therefore, the only new terms obtained on integrating $N$ with respect to $y$ are $-\frac{1}{3} y^3 - \frac{1}{2} y$.

Hence, the required solution of the given differential equation is

$$-\frac{1}{2} y \cos 2x - \frac{1}{3} y^3 - \frac{1}{2} y = -\frac{1}{6} c, \text{ where } c \text{ is arbitrary constant.}$$

or $$3 y \cos 2x + 2 y^3 + 3 y = c.$$

## Exercise 9.7

**Show that the following Equations are Exact, and Solve Them:**

1. $x\, dx + y\, dy = 0$. **[B.C.A. (Meerut) 2006]**
2. $x\, dy + (x + y)\, dx = 0$.
3. $(x + y)\, dx + (x - y)\, dy = 0$. **[B.C.A. (Lucknow) 2007]**
4. $(4x + y)\, dx + (x + 2y)\, dy = 0$.
5. $2xy\, dx + (x^2 + 1)\, dy = 0$.
6. $(x + 2y - 3)\, dy - (2x - y + 1)\, dx = 0$. **[B.C.A. (Avadh) 2010]**
7. $(x^3 + 3xy^2)\, dx + (3x^2 y + y^3)\, dy = 0$.
8. $(x^2 - ay)\, dx = (ax - y^2)\, dy$.
9. $(x^2 - 2xy - y^2)\, dx - (x + y)^2\, dy = 0$. **[B.C.A. (Rohilkhand) 2002]**
10. $e^x \sin y\, dx + e^x \cos y\, dy = 0$.

## Answers 9.7

| | | | |
|---|---|---|---|
| 1. | $x^2 + y^2 = a^2$. | 2. | $xy + \frac{1}{2}x^2 = c$. |
| 3. | $x^2 + 2xy - y^2 = c$. | 4. | $2x^2 + xy + y^2 = c$. |
| 5. | $y = c/(1 + x^2)$. | 6. | $y^2 - x^2 + xy - x - 3y = c$. |
| 7. | $x^4 + y^4 + 6x^2y^2 = c$. | 8. | $x^3 - 3axy + y^3 = 3c$. |
| 9. | $\frac{1}{3}x^3 - x^2y - xy^2 - \frac{1}{3}x^3 = c$. | 10. | $e^x \sin y = c$. |

## 9.10 Integrating Factors

**Definition:** "*If the differential equation*

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad ...(1)$$

*is not exact, it may be possible to make it exact by multiplying by an appropriate function $F(x, y)$. Such a function is called an **integrating factor** for the differential equation. The number of such functions can be more than one.*"

## 9.11 Rules for Finding Integrating Factors

**Rule 1:** When $Mx + Ny \neq 0$, and the equation is homogeneous, then an integrating factor of the differential equation

$$M\,dx + N\,dy = 0 \text{ is } \frac{1}{Mx + Ny}.$$

**Rule 2:** If the equation $M\,dx + N\,dy = 0$ has the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0, \quad \text{and} \quad Mx - Ny \neq 0,$$

an integrating factor is $\frac{1}{Mx - Ny}$.

**Rule 3:** If in the equation $M\,dx + N\,dy = 0$,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$$

(a function of $x$ alone), then $e^{\int f(x)\,dx}$ is an integrating factor.

**Rule 4:** If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=f(y)$

be a function of $y$ alone, then $e^{\int f(y)\,dy}$ is an integrating factor of the differential equation

$$M\,dx+N\,dy=0$$

**Rule 5:** For the differential equation

$$x^a y^b\,(my\,dx+nx\,dy)+x^r y^s\,(py\,dx+qx\,dy)=0$$

where $a, b, m, n, s, p, q$ are constants, an integrating factor is $x^h y^k$ where $h, k$ are obtained by applying the condition that after multiplying by $x^h y^k$, the equation must become exact.

***Example 76:*** ***Solve*** $x^2 y\,dx-(x^3+y^3)\,dy=0.$ **[B.C.A. (Lucknow) 2008]**

***Solution:*** The given equation is homogeneous, and

$$Mx+Ny=(x^2 y)\,x-(x^3+y^3)\,y=-y^4\neq 0.$$

$\therefore$ An integrating factor $(I.F.)=\dfrac{1}{(Mx+Ny)}=-\dfrac{1}{y^4}.$

Therefore, multiplying the given equation by $-\frac{1}{y^4}$, we have

$$-\frac{x^2}{y^3}\,dx+\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=0. \qquad ...(1)$$

From the equation, we observe that

$$\frac{\partial}{\partial y}\left(-\frac{x^2}{y^3}\right)=\frac{3x^2}{y^2}=\frac{\partial}{\partial x}\left(\frac{x^3}{y^4}+\frac{1}{y}\right).$$

Therefore, equation (1) is exact.

Now, $$\int_{y-\text{constant}}\left(-\frac{x^2}{y^3}\right)dx=-\frac{x^3}{3y^3} \qquad ...(2)$$

and $$\int_{x-\text{constant}}\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=-\frac{x^3}{3y^3}+\log y \qquad ...(3)$$

But the term $-\frac{x^3}{3y^3}$ has already occurred in (2), therefore, the only new term in (3) is $\log y$. Hence, the required solution is $-\frac{x^3}{3y^3}+\log y=\log c,$ $c$ being arbitrary constant.

or $$\log\frac{y}{c}=\frac{x^3}{3y^3} \qquad \text{or} \qquad y=ce^{x^3/3y^3}.$$

**Example 77:** ***Solve*** $y(1+xy)\,dx + x(1-xy)\,dy = 0.$ ...(1) **[B.C.A. (Agra) 2011]**

***Solution:*** The given differential equation is of the form

$$M\,dx + N\,dy = 0,$$

where $M = y(1+xy) = yf_1(xy),\ N = x(1-xy) = xf_2(xy)$

and $Mx - Ny = y(1+xy)\,.\,x - x(1-xy)\,.\,y = 2x^2y^2 \neq 0$

$\therefore$ As integrating factor $= \dfrac{1}{Mx - Ny} = \dfrac{1}{2x^2y^2}$ or simply $\dfrac{1}{x^2y^2}$.

Hence, multiplying the given equation by integrating factor $1/x^2y^2$, we obtain

$$\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$$

From equation (1), we have

$$M = \frac{1}{x^2y} + \frac{1}{x};\ N = \frac{1}{xy^2} - \frac{1}{y}.$$

$$\therefore \qquad \frac{\partial M}{\partial y} = -\frac{1}{x^2y^2} = \frac{\partial N}{\partial x}.$$

Hence, equation (1) is exact. Therefore, the required solution is

$$U(x, y) = C$$

where $U(x, y) = \int M\,dx + \phi(y) = \displaystyle\int \left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \phi(y) = \frac{1}{xy} + \log x + \phi(y)$

and $\phi'(y) = N - \dfrac{\partial}{\partial y}[\int M\,dx] = \dfrac{1}{xy^2} - \dfrac{1}{y} - \dfrac{\partial}{\partial y}\left[-\dfrac{1}{xy} + \log x\right]$

$$= \frac{1}{xy^2} - \frac{1}{y} = -\frac{1}{xy^2} = -\frac{1}{y}.$$

Thus, on integration, we obtain

$$\phi(y) = -\log y + C_1.$$

Hence, the required solution is

$$-\frac{1}{xy} + \log x - \log y + C_1 = C$$

or $-\dfrac{1}{xy} + \log x - \log y = \log c,$ where $\log c = C - C_1$

or $$x = cy\,e^{1\,xy}.$$

**Example 78:** ***Solve*** $(y^2 + 2x^2 y)\, dx + (2x^3 - xy)\, dy = 0.$ **[B.C. A. (Rohilkhand) 2009]**

***Solution:*** The given equation is of the form

$$M\, dx + N\, dy = 0,$$

where $\quad M = y^2 + 2x^2 y \quad$ and $\quad N = 2x^3 - xy.$

$\therefore \quad \dfrac{\partial M}{\partial y} = 2\,(y + x^2) \quad$ and $\quad \dfrac{\partial N}{\partial x} = 6x^2 - y.$

Since $\quad \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x},$

Therefore, the given equation is not exact.

Now, the given equation may be written as

$$y\,(y\, dx - x\, dy) + x^2\,(2\,y dx + 2x\, dy) = 0$$

Hence, its integrating factor must be of the form $x^h y^k$ (refer rule 5), where $h$ and $k$ are constants to be determined.

Thus multiplying the given equation by $x^h y^k$, we get

$$(x^h y^{k+2} + 2h^{h+2} y^{k+1})\, dx + (2x^{h+3} y^k - x^{h+1} y^{k+1})\, dy = 0 \qquad ...(1)$$

Now, the equation (1) has been obtained by multiplying the given equation with its integrating factor, so it must be exact.

For the equation (1), we have

$$M = x^h y^{k+2} + 2x^{h+2} y^{k+1} \quad \text{and} \quad N = 2x^{h+3} y^k - x^{h+1} y^{k+1}.$$

$\therefore \quad \dfrac{\partial M}{\partial y} = (k+2)\, x^h y^{k+1} + 2\,(k+1)\, x^{h+2} y^k$

and $\quad \dfrac{\partial N}{\partial x} = 2\,(h+3)\, x^{h+2} y^k - (h+1)\, x^h x^{k+1}.$

Since the equation (1) is exact, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow \quad (k+2)\, x^h y^{k+1} + 2\,(k+1)\, x^{h+2} y^k = 2\,(h+3)\, x^{h+2} y^k - (h+1)\, x^h y^{k+1}$

Equating on both sides the coefficients of $x^h y^{k+1}$ and $x^{h+2} y^k$, we get

$$k + 2 = -(h+1) \quad \text{and} \quad 2\,(k+1) = 2\,(h+3)$$

$\Rightarrow \quad h + k = -3 \quad$ and $\quad h - k = -2$

$\Rightarrow \quad h = -\dfrac{5}{2};\ k = -\dfrac{1}{2}.$

$\therefore$ An integrating factor is $x^{-5/2} y^{-1/2}$.

Multiplying the given equation by $x^{-5/2}y^{-1/2}$, we get

$$(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})\,dx + (2x^{1/2}y^{-1/2} + x^{-3}y^{1/2})\,dy = 0.$$

which is an exact differential equation of the form

$$M\,dx + N\,dy = 0$$

Now, $$\int_{y-\text{constant}} M\,dx = \int_{y-\text{constant}} (x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})\,dx$$

$$= -\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} \qquad ...(2)$$

Also no new term is obtained by integrating $N$ with respect to $y$. Hence, the required solution is

$$-\frac{2}{3}x^{-3/2}y^{3/2} + hx^{1/2}y^{1/2} = C$$

or $$12x^{1/2}y^{1/2} - 2x^{-3/2}y^{3/2} = c, \qquad \text{where } c = 3C \text{ is an arbitrary constant.}$$

***Example 79:*** **Solve $(x^2 + y^2 + 1)\,dx - 2xy\,dy = 0$.**

***Solution:*** Here $M = x^2 + y^2 + 1$ and $N = -2xy$, so that

$$\frac{\partial M}{\partial y} = 2y \qquad \text{and} \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-2xy}(2y + 2y) = -\frac{2}{x} = f(x)$$

Hence, integrating factor (I.F.) $= e^{\int f(x)\,dx} = e^{-2\int \frac{dx}{x}} = e^{-2\log x} = \dfrac{1}{x^2}$

Multiplying the given differential equation by this integrating factor, we have

$$\left[1 + \left(\frac{y}{x}\right)^2 + \frac{1}{x^2}\right]dx - 2\left(\frac{y}{x}\right)dy = 0$$

or $$\left[1 + \frac{1}{x^2}\right]dx + \left[\frac{y^2}{x^2}\,dx - 2\left(\frac{y}{x}\right)dy\right] = 0$$

or $$\left[1 + \frac{1}{x^2}\right]dx + d\left(-\frac{y^2}{x}\right) = 0 \qquad \text{or} \qquad d\left[x - \frac{1}{x} - \frac{y^2}{x}\right] = 0$$

Hence on integration, the required solution is

$$x - \frac{1}{x} - \frac{y^2}{x} = c \qquad \text{or} \qquad x^2 - 1 - y^2 = cx.$$

**Example 80:** ***Solve*** $(x^2 y - 2xy^2)\, dx - (x^3 - 3x^2 y)\, dy = 0.$ **[B.C. A. (Kanpur) 2007, 04]**

***Solution:*** The given differential equation is homogeneous.

Also, $$Mx + Ny = (x^3 y - 2x^2 y^2) - (x^3 y - 3x^2 y^2) = x^2 y^2 \neq 0$$

$\therefore$ Integrating factor (I.F.) $= \dfrac{1}{Mx + Ny} = \dfrac{1}{x^2 y^2}.$

Hence, multiplying the given differential equation by this integrating factor, we get

$$\left(\frac{1}{y} - 2x\right) dx + \left(\frac{3}{y} - \frac{x}{y^2}\right) dy = 0 \qquad ...(1)$$

Clearly for equation (1), $\dfrac{\partial M}{\partial y} = -\dfrac{1}{y^2} = \dfrac{\partial N}{\partial x}.$

Therefore, equation (1) is an exact differential equation.

Now, $$\int_{y\text{-constant}} M\, dx = \int_{y\text{-constant}} \left(\frac{1}{y} - 2x\right) dx = \frac{x}{y} - x^2$$

and $$\int_{x\text{-constant}} N\, dy = \int_{x\text{-constant}} \left(\frac{3}{y} - \frac{x}{y^2}\right) dy = 3 \log y + \frac{x}{y}.$$

The only new term in the integration of $N$ is $3 \log y$, hence the required solution of the given differential equation is

$$\frac{x}{y} - x^2 + 3 \log y = c.$$

**Example 81:** ***Solve*** $(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0.$ **[B.C. A. (Agra) 2003]**

***Solution:*** Here $\dfrac{\partial M}{\partial y} = 2y$ and $\dfrac{\partial N}{\partial x} = 0$, so that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}.$

Hence, the given differential equation is not exact.

Now, $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2y}(2y - 0) = 1 = x^0 = f(x).$$

$\therefore$ Integration factor (I.F.) $= e^{\int f(x)\, dx} = e^{\int (1)\, dx} = e^x.$

Hence, multiplying the given differential equation by integrating factor, we obtain

$$(x^2 + y^2 + 2x)\, e^x\, dx + 2ye^x\, dy = 0 \qquad ...(1)$$

Now, from the equation (1), we have

$$\frac{\partial M}{\partial y} = 2ye^x = \frac{\partial N}{\partial x}.$$

Hence, the differential equation (1) is exact.

Now, $$\int_{y-\text{constant}} M\,dx = \int_{y-\text{constant}} (x^2 + y^2 + 2x)\,e^x\,dx = (x^2 + y^2)\,e^x$$

and $$\int_{x-\text{constant}} N\,dy = \int_{x-\text{constant}} 2\,ye^x\,dy = y^2 e^x$$

No new term is obtained by integrating $N$ with respect to $y$. Hence, the required solution is

$$(x^2 + y^2)\,e^x = c$$

**Example 82:** ***Solve*** $(x^2 y^2 + xy + 1)\,y\,dx + (x^2 y^2 - xy + 1)\,x\,dy = 0.$

**[B.C.A. (Purvanchal) 2004, 02]**

**Solution:** The given differential equation is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$.

Also, $$Mx - Ny = (x^2 y^2 + xy + 1)\,xy - (x^2 y^2 - xy + 1)\,xy = 2x^2 y^2 \neq 0.$$

$\therefore$ Integrating factor (I.F.) $= \dfrac{1}{Mx - Ny} = \dfrac{1}{2x^2 y^2}$ or only $\dfrac{1}{x^2 y^2}$.

Hence, multiplying the given differential equation by $\dfrac{1}{x^2 y^2}$, we get

$$\left(y + \frac{1}{x} + \frac{1}{x^2 y}\right) dx + \left(x - \frac{1}{y} + \frac{1}{xy^2}\right) dy = 0 \qquad \text{...(1)}$$

For the above equation, we have

$$\frac{\partial M}{\partial y} = 1 - \frac{1}{x^2 y^2} = \frac{\partial N}{\partial x}.$$

Hence, the differential equation (1) is exact.

Now, $$\int_{y-\text{constant}} M\,dx = \int_{y-\text{constant}} \left(y + \frac{1}{x} + \frac{1}{x^2 y}\right) dx = xy + \log x - \frac{1}{xy}$$

and $$\int_{x-\text{constant}} N\,dy = \int_{x-\text{constant}} \left(x - \frac{1}{y} + \frac{1}{xy^2}\right) dx = xy - \log y - \frac{1}{xy}.$$

The only new term obtained by integrating $N$ with respect to $y$ is $-\log y$. Hence, the required solution is

$$xy + \log x - \frac{1}{xy} - \log y = c \qquad \text{or} \qquad xy - \frac{1}{xy} + \log\left(\frac{x}{y}\right) = c.$$

***Example 83:*** **Solve** $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0.$

**[B.C. A. (Agra) 2001]**

***Solution:*** Here $\dfrac{\partial M}{\partial y} = 2xy$ and $\dfrac{\partial N}{\partial x} = 6xy^2 + 2xy - 6x^2$, so that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Hence, the given differential equation is not exact.

Now, $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^2 - x^2}(6xy^2 - 6x^2) = 6 = 6y^0 = f(y)$$

$\therefore$ Integrating factor (I.F.) $= e^{\int f(y)dy} = e^{\int 6\,dy} = e^{6y}$

Hence, multiplying the given equation by integrating factor $e^{6y}$, we obtain

$$(xy^2 - x^2)\,e^{6y}\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,e^{6y}\,dy = 0 \qquad ...(1)$$

Now, for differential equation (1),

$$\frac{\partial M}{\partial y} = (xy^2 - x^2)\,6e^{6y} + (2xy)\,e^{6y} = (6xy^2 + 2xy - 6x^2)\,e^{6y} = \frac{\partial N}{\partial x}$$

$\therefore$ Differential equation (1) is exact.

Now, $$\int_{y-\text{constant}} M\,dx = \int_{y-\text{constant}} (xy^2 - x^2)\,e^{6y}\,dx = \left(\frac{1}{2}x^2y^2 - \frac{1}{3}x^3\right)e^{6y}$$

and $$\int_{x-\text{constant}} N\,dy = \int_{x-\text{constant}} (3x^2y^2 + x^2y - 2x^3 + y^2)\,e^{6y}\,dy$$

$$= (3x^2y^2 + x^2y - 2x^3 + y^2).\left(\frac{e^{6y}}{6}\right) - (6x^2y + x^2 + 2y)\left(\frac{e^{6y}}{36}\right) + (6x^2 + 2)\left(\frac{e^{6y}}{216}\right)$$

$$= \left(\frac{x^2y^2}{2} - \frac{x^3}{3} + \frac{y^2}{6} - \frac{y}{18} + \frac{1}{108}\right)e^{6y}$$

Hence, the term $\left(\dfrac{1}{2}x^2y^2 - \dfrac{x^3}{3}\right)e^{6y}$ is common in integrals of $M$ and $N$. Hence, the required solution of the given differential equation is

$$\left(\frac{x^2y^2}{2} - \frac{x^3}{3} + \frac{y^2}{6} - \frac{y}{18} + \frac{1}{108}\right)e^{6y} = c.$$

***Example 84:*** **Solve** $(x^3y^3 + x^2y^2 + xy + 1)\,y\,dx + (x^3y^3 - x^2y^2 - xy + 1)\,x\,dy = 0.$

**[B.C. A. (Agra) 2006, 05]**

***Solution:*** Here $\dfrac{\partial M}{\partial y} = 4x^3y^3 + 3x^2y^2 + 2xy + 1$

and $$\frac{\partial N}{\partial x} = 4x^3y^3 - 3x^2y^2 - 2xy + 1, \text{ so that } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Hence, the given differential equation is not exact. Also, the given differential equation is of the form $f_1(xy)\ y\ dx + f_2(xy)\ x\ dy = 0$ and

$$\frac{1}{Mx - Ny} = \frac{1}{(x^3y^3 + x^2y^2 + xy + 1)\ xy - (x^3y^3 - x^2y^2 - xy + 1)\ xy}$$

$$= \frac{1}{2\ (x^3y^3 + x^2y^2)} = \frac{1}{2x^2y^2\ (xy+1)} \neq 0.$$

$\therefore$ Integrating factor (I.F.) $= \dfrac{1}{2x^2y^2\ (xy+1)}.$

Now, multiplying the given differential equation by this integrating factor, we have

$$\frac{1}{2x^2y^2\ (xy+1)}\left[\{x^2y^2\ (xy+1) + (xy+1)\}\ y\ dx + \{(x^2y^2+1) - xy\ (xy+1)\}\ x\ dy\right] = 0$$

or $$\left[\frac{(x^2y^2+1)}{x^2y^2}\right] y\ dx + \left[\frac{(x^2y^2 - xy + 1) - xy}{x^2y^2}\right] x\ dy = 0$$

or $$y\ dx + x\ dy + \frac{y\ dx + x\ dy}{x^2y^2} - \frac{2x^2y}{x^2y^2}\ dy = 0 \qquad \text{or} \quad d\,(xy) + \frac{d(xy)}{(xy)^2} - \frac{2}{y}\ dy = 0$$

or $$d\left[xy - \frac{1}{xy} - 2\log y\right] = 0$$

Hence on integration, the required solution is

$$xy - \frac{1}{xy} - 2\log y = c$$

***Example 85:*** **Solve $(3x + 2y^2)\ y\ dx + 2x\ (2x + 3y^2)\ dy = 0$.** **[B.C. A. (Agra) 2002]**

***Solution:*** The given equation is of the form

$$M\ dx + N\ dy = 0,$$

where $M = 3xy + 2y^3$ and $N = 4x^2 + 6xy^2$

$\therefore$ $\dfrac{\partial M}{\partial y} = 3x + 6y^2$ and $\dfrac{\partial N}{\partial x} = 8x + 6y^2.$

Thus, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

and so the given equation is not exact.

Now, the given equation can be written as

$$x\ (3y\ dx + 4x\ dy) + y^2\ (2y\ dx + 6x\ dy) = 0.$$

Let $x^h y^k$ be its integrating factor (refer rule 5), where $h$ and $k$ are constants to be determined.

Thus, multiplying the given equation by $x^h y^k$, we get

$$(3x^{h+1} y^{k+1} + 2x^h y^{k+3})\, dx + (4x^{h+2} y^k + 6x^{h+1} y^{k+2})\, dy = 0 \qquad ...(1)$$

Since this equation has been obtained by multiplying the given equation with its integrating factor and so it must be an exact equation.

Now comparing (1) with $M\, dx + N\, dy = 0$, we have

$$M = 3x^{h+1} y^{k+1} + 2x^h y^{k+3} \text{ and } N = 4x^{h+2} y^k + 6x^{h+1} y^{k+2}$$

$\therefore$ $$\frac{\partial M}{\partial y} = 3\,(k+1)\, x^{h+1} y^k + 2\,(k+3)\, x^h y^{k+2},$$

and $$\frac{\partial N}{\partial x} = 4\,(h+2)\, x^{h+1} y^k + 6\,(h+1)\, x^h y^{k+2}$$

Since (1) is exact, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore, $3\,(k+1)\, x^{h+1} y^k + 2\,(k+3)\, x^h y^{k+2} = 4\,(h+2)\, x^{h+1} y^k + 6\,(h+1)\, x^h y^{k+2}$

Equating on both sides the coefficients of $x^{h+1} y^k$ and $x^h y^{k+2}$, we get

$$3\,(k+1) = 4\,(h+2) \quad \text{and} \quad 2\,(k+3) = 6\,(h+1)$$

$\Rightarrow$ $$4h - 3k = -5 \quad \text{and} \quad 3h - k = 0 \quad \Rightarrow \quad h = 1 \quad \text{and} \quad k = 3.$$

Thus, the integrating factor is $xy^3$.

Now, multiplying the given equation by $xy^3$, we get

$$(3x^2 y^4 + 2xy^6)\, dx + (4x^3 y^2 + 6x^2 y^5)\, dy = 0$$

which is clearly an exact differential equation of the form

$$M\, dx + N\, dy = 0$$

Now, $$\int_{y-\text{constant}} M\, dx = \int_{y-\text{constant}} (3x^2 y^4 + 2xy^6)\, dx = x^3 y^4 + x^2 y^6 \qquad ...(2)$$

Also, no new term is obtained by integrating $N$ with respect to $y$.

Hence from (2), the required solution is

$$x^3 y^4 + x^2 y^6 = c,$$

where $c$ is an arbitrary constant.

## Exercise 9.8

**Solve the following Differential Equations:**

1. $y\,dx - (x + 6y^2)\,dy = 0.$

2. $(x + y)\,dx + \tan x\,dy = 0.$

3. $2y\,dx + (x - \sin\sqrt{y})\,dy = 0.$

4. $(4x^2 y + 2y^2)\,dx + (3x^3 + 4xy)\,dy = 0.$

5. $(2yx^2 + 2y^3 + x)\,dy - y\,dx = 0.$

## Answers 9.8

| | |
|---|---|
| 1. | Integrating factor : $1/y^2$; General solution : $(x/y) - 6y = c.$ |
| 2. | Integrating factor : $\cos x$; General solution : $y\sin x + x\sin x + \cos x = c.$ |
| 3. | Integrating factor : $1/\sqrt{y}$; General solution : $x\sqrt{y} + \cos\sqrt{y} = c.$ |
| 4. | Integrating factor : $xy^2$; General solution : $x^4y^3 + x^2y^4 = c.$ |
| 5. | $2y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0;\ y^2 + \tan^{-1}\dfrac{y}{x} = c.$ |

❍❍❍

# Chapter 10

# Differential Equations of Second Order with Constant Coefficients

## 10.1 Definition

**[B.C. A. (Agra) 2011, 09]**

We have already defined a linear differential equation of the first order in chapter 9. If a differential equation in variables $x$ and $y$ is such that the dependent variable $y$ and its derivatives occur only in the first degree and there occur no term containing product of derivatives of different order and the product of the derivative and the dependent variable, then such an equation is called a **linear differential equation** (of higher order). Also, the coefficients of the derivatives and the dependent variable $y$ are constants, then it is called a **linear differential equation** (of higher order) **with constant coefficients.** **[B.C.A. (Kanpur) 2008, 06]**

Thus, a linear differential equation of $n$th order of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = Q$$

or $$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_{n-1} y' + a_n y = Q \qquad \dots(1)$$

where $a_1, a_2, \dots, a_{n-1}, a_n$ are all constants and $Q$ is some function of $x$, is called a **linear differential equation with constant coefficients**. **[B.C.A. (Avadh) 2010, 08]**

If $Q = 0$, then the equation (1) is called **homogeneous differential equation**. If $Q \neq 0$, then the equation (1) is called **non-homogeneous differential equation**.

**[B.C.A. (Bundelkhand) 2008]**

Thus, the general solution of (1) consists of two parts, one of which contains $n$ arbitrary constants and is a solution of the equation obtained from (1) by putting the second member equal to zero, and the other contains no arbitrary constants.

The first part that contains $n$ arbitrary constants is called the **Complementary Functions (C.F.)** and the second part, $\phi(x)$ which does not involve any arbitrary constant is called the **Particular Integral (P.I.)**. We often denote these by $y_c$ and $y_p$ respectively.

Thus, the general solution of (1) is

$$y = \text{C.F.} + \text{P.I.} \quad \text{or} \quad y = y_c + y_p$$

## 10.2 Auxiliary Equation

Let us consider the differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = 0 \qquad \ldots(1)$$

Let $y = e^{mx}$ be a solution of equation (1), then by actual substitution, we have

$$e^{mx}\,[m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n] = 0$$

Hence, $e^{mx}$ will be a solution of (1) if $m$ is a root of the algebraic equation

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0 \qquad \ldots(2)$$

This equation is called the **auxiliary equation** or **characteristic equation** of the given differential equation (1).

In order to solve (1), we write the auxiliary equation (2) and solve it for $m$. Which gives different cases, according as the roots of the auxiliary equation (2) are real and distinct, real and repeated or complex.

**NOTE:**

It is worthwhile to note that the auxiliary equation can be written down from (1) by replacing $y$ by 1, $\frac{dy}{dx}$ by $m$, $\frac{d^2 y}{dx^2}$ by $m^2$, $\frac{d^3 y}{dx^3}$ by $m^3$ and so on.

## 10.3 Auxiliary Equation Having Distinct Roots

Let the roots of the auxiliary equation

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0 \qquad \ldots(1)$$

be $m_1, m_2, \ldots, m_n$ and they are all distinct. Then $e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x}$ are all distinct solutions of the differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = 0 \qquad \ldots(2)$$

and it can be shown that these are linearly independent on any interval $[a, b]$.

Thus, the general solution of (2) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x} \qquad \ldots(3)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

## Solved Examples

**Example 1:** ***Solve*** $\mathbf{\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} - 4y = 0}$. **[B.C.A. (Bhopal) 2012]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 3m - 4 = 0$$

*i.e.,* $(m-4)(m+1) = 0,$ $[\therefore\ m = -1, 4]$

Hence, the general solution of the given differential is

$$y = c_1 e^{-x} + c_2 e^{4x}$$

where $c_1$ and $c_2$ are arbitrary constants.

**Example 2:** ***Solve*** $\mathbf{\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + y = 0}$. **[B.C. A. (Agra) 2009]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 4m + 1 = 0 \Rightarrow m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

Hence, the required solution is

$$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x} \text{ or } y = e^{2x} \{c_1 e^{x\sqrt{3}} + c_2 e^{-x\sqrt{3}}\},$$

where $c_1$ and $c_2$ are arbitrary constants.

## Exercise 10.1

**Solve the following differential equation:**

1. $2 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - 2y = 0.$

   **[B.C.A. (Purvanchal) 2006]**

2. $\frac{d^2 y}{dx^2} - 7 \frac{dy}{dx} + 12y = 0.$

   **[B.C.A. (Meerut) 2003]**

3. $\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}+5y=0.$

4. $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}+4y=0.$

**[B.C.A. (Kashi) 2010, 06]**

5. $\dfrac{d^2y}{dx^2}-13\dfrac{dy}{dx}+12y=0.$

**[B.C.A. (Meerut) 2008, 04]**

6. $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=0.$

**[B.C.A. (Kanpur) 2009]**

7. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+4y=0.$

8. $\dfrac{d^2y}{dx^2}+(a^2+b^2)\dfrac{dy}{dx}+a^2b^2y=0.$

**[B.C.A. (Meerut) 2001]**

9. $\dfrac{d^3y}{dx^3}-9\dfrac{d^2y}{dx^2}+23\dfrac{dy}{dx}-15y=0.$

**[B.C.A. (Meerut) 2009]**

10. $\dfrac{d^3y}{dx^3}-13\dfrac{dy}{dx}+12y=0.$

**[B.C.A. (Purvanchal) 2008]**

## Answers 10.1

| | | | |
|---|---|---|---|
| 1. | $y=c_1e^{x/2}+c_2e^{-2x}.$ | 2. | $y=c_1e^{3x}+c_2e^{4x}.$ |
| 3. | $y=c_1e^{-x}+c_2e^{-5x}.$ | 4. | $y=c_1e^{-x}+c_2e^{-4x}.$ |
| 5. | $y=c_1e^{x}+c_2e^{12x}.$ | 6. | $y=e^{-\frac{1}{2}x}\left(c_1\cos\dfrac{\sqrt{3}}{2}x+c_2\sin\dfrac{\sqrt{3}}{2}x\right).$ |
| 7. | $y=e^x(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x).$ | 8. | $y=c_1e^{-a^2x}+c_2e^{-b^2x}.$ |
| 9. | $y=c_1e^{x}+c_2e^{3x}+c_3e^{5x}.$ | 10. | $y=c_1e^{x}+c_2e^{3x}+c_3e^{-4x}.$ |

## 10.4 The Symbol D

For the sake of convenience, the symbols $D$ and $D^n$ are used for $\dfrac{d}{dx}$ and $\dfrac{d^n}{dx^n}$ respectively in the treatment of linear differential equations with constant coefficients, because if an expression has a number of terms involving $y, Dy, D^2y, \ldots$, then $y$ may be written only ones by making use of brackets. Thus, a linear differential equation with constant coefficients

$$\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_{n-1}\frac{dy}{dx}+a_ny=0$$

may be written as $(D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_{n-1}D+a_n)\,y=0$ ...(1)

or $$f(D)\,y = 0. \qquad ...(2)$$

The expression $$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_{n-1} D + a_n \equiv f(D)$$

is called a differential operator of the ***n*th order.**

The symbolic coefficient of y in (1) is the same function of $D$ that the auxiliary equation (2). Since the roots of the auxiliary equation are $m_1, m_2, \ldots, m_n$, equation(1) may be written as

$$(D - m_1)(D - m_2) \ldots (D - m_n)\,y = 0. \qquad ...(3)$$

Hence, the integral of (1) can be found by putting its symbolic coefficient equal to zero, *i.e.*, $f(D) = 0$, and solving for $D$ it is thus apparent has the complete solution of (1) or (3) is made up of the solutions of

$$\left.\begin{array}{l} (D - m_1)\,y = 0; \\ (D - m_2)\,y = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (D - m_n)\,y = 0. \end{array}\right\} \qquad ...(4)$$

Hence, the general solution of given differential equation (4) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots. + c_n e^{m_n x}.$$

## 10.5 Auxiliary Equation Having Equal Roots

1. If the auxiliary equation has two equal roots, say $m_1 = m_2$, the solution of the equation $f(D)\ y = 0$ obtained is given by

$$y = (c_1 + c_2)\,e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

or $$y = A e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \text{ where } A = c_1 + c_2.$$

2. If $f(D) = 0$ has two equal roots equal to $m_1$, the general solution $f(D)\ y = 0$ is

$$y = (c_1 + c_2 x)\,e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}. \qquad ...(1)$$

3. If $f(D) = 0$ has three equal roots equal to $m_1$, then as before we can show that the general solution of $f(D)\ y = 0$ is

$$y = (c_1 + c_2 x + c_3 x^2)\,e^{m_1 x} + c_4\, e^{m_4 x} + \ldots + c_n\, e^{m_n x}. \qquad ...(2)$$

4. Proceeding in the similar manner, if $f(D) = 0$ has r equal roots equal to $m_1$, the general solution of $f(D)\ y = 0$ will be

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_{r-1} x^{r-2} + c_r x^{r-1})\, e^{m_1 x}$$

$$+ c_{r+1} e^{m_{r+1} x} + \ldots + c_n e^{m_n x}. \quad \ldots(3)$$

***Example 3:*** ***Solve*** $\boldsymbol{\dfrac{d^3 y}{dx^3} - 3\dfrac{d^2 y}{dx^2} + 4y = 0.}$ **[B.C. A. (Agra) 2005, 02]**

***Solution:*** Using the symbol $D = \dfrac{d}{dx}$, the given differential equation is

$$(D^3 - 3D^2 + 4)\, y = 0.$$

Thus, its auxiliary equation is

$$m^3 - 3m^2 + 4 = 0 \Rightarrow (m+1)(m-2)^2 = 0 \Rightarrow m = -1, 2, 2$$

*i.e.*, the roots of the auxiliary equation are $m = -1$, $m = 2, 2$ (two equal roots).
Hence, the required general solution is

$$y = c_1 e^{-x} + (c_2 + c_3 x)\, e^{2x}$$

where $c_1, c_2, c_3$ are arbitrary constants.

***Example 4:*** ***Solve*** $\boldsymbol{(D^4 - 4D^2 + 4)\, y = 0.}$ **[B.C.A. (Meerut) 2002]**

***Solution:*** Here, the auxiliary equation is

$$m^4 - 4m^2 + 4 = 0 \Rightarrow (m^2 - 2)^2 = 0 \Rightarrow m = \pm\sqrt{2}, \pm\sqrt{2}.$$

Hence, the required solution is

$$y = (c_1 + c_2 x)\, e^{\sqrt{2}x} + (c_3 + c_4 x)\, e^{-\sqrt{2}x}$$

***Example 5:*** ***Solve*** $\boldsymbol{(D^4 - 7D^3 + 18D^2 - 20D + 8)\, y = 0.}$

***Solution:*** Here, the auxiliary equation is

$\Rightarrow \quad m^4 - 7m^3 + 18m^2 - 20m + 8 = 0$

$\Rightarrow \quad m^3(m-1) - 6m^2(m-1) + 12m(m-1) - 8(m-1) = 0$

$\Rightarrow \quad (m-1)(m^3 - 6m^2 + 12m - 8) = 0$

$\Rightarrow \quad (m-1)(m-2)^3 = 0$

$\Rightarrow \quad m = 1$ and $m = 2, 2, 2$ (three equal roots).

Hence, the required solution is

$$y = c_1 e^x + (c_2 + c_3 x + c_4 x^2)\, e^{2x}.$$

## Exercise 10.2

**Solve the following differential equation:**

1. $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = 0.$ **[B.C. A. (Bhopal) 2009, 08]**

2. $16\dfrac{d^2 y}{dx^2} + 24\dfrac{dy}{dx} + 9y = 0.$ **[B.C. A. (Kanpur) 2007, 04]**

3. $(D^3 + 3D^2 - 4)y = 0.$

4. $[D^3 + (2\sqrt{3} - 1)D^2 + (3 - 2\sqrt{3})D - 3]\ y = 0.$ **[B.C. A. (Delhi) 2009, 04]**

5. $(D^4 + 4D^3 - 5D^2 - 36D - 36)\ y = 0.$ **[B.C.A. (Meerut) 2006]**

6. $(D^4 - 8D^2 + 16)\ y = 0.$ **[B.C. A. (Rohilkhand) 2007]**

7. $(D^4 - D^3 - 9D^2 - 11D - 4)\ y = 0.$ **[B.C. A. (Agra) 2010, 06]**

## Answers 10.2

| | | | |
|---|---|---|---|
| 1. | $y = (c_1 + c_2 x)\, e^x.$ | 2. | $y = (c_1 + c_2 x)\, e^{-\frac{3}{4}x}.$ |
| 3. | $y = c_1 e^x + (c_2 + c_3 x)\, e^{-2x}.$ | 4. | $y = c_1 e^x + (c_2 + c_3 x)\, e^{-\sqrt{3}x}.$ |
| 5. | $y = c_1 e^{3x} + c_2 e^{-3x} + (c_3 + c_4 x)\, e^{-2x}.$ | 6. | $y = (c_1 + c_2 x)\, e^{2x} + (c_3 + c_4 x)\, e^{-2x}.$ |
| 7. | $y = e^{-x}(c_1 + c_2 x + c_3 x^2) + c_4 e^{4x}.$ | | |

## 10.6 Auxiliary Equation Having Imaginary Roots

1. If the auxiliary equation $f(m) = 0$ has its roots imaginary, (say $a \pm ib$), then

$$\text{C.F.} = e^{ax}(c_1 \cos bx + c_2 \sin bx)$$

2. If the auxiliary equation $f(m) = 0$ has two pairs of imaginary roots $a \pm ib$ and $c \pm id$, then general solution is

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx) + e^{cx}(c_3 \cos dx + c_4 \sin dx)$$

3. If roots are repeated, then general solution is given by

$$y = e^{ax}\{(c_1 + c_2 x)\cos bx + (c_3 + c_4 x)\sin bx\}$$

**NOTE 1:**

If the roots of $f(m) = 0$ be $a \pm \sqrt{b}$ where b is positive, then general solution of differential equation is given by

$$y = e^{ax}\,[c_1 \cos h\sqrt{b}x + c_2 \sin h\sqrt{b}x].$$

**NOTE 2:**

The solution corresponding to the pair of imaginary roots $\alpha \pm \beta$ of $f(m) = 0$ can also be written as

$$y = c_1 e^{\alpha x}\sin(\beta x + c_2).$$

**NOTE 3:**

If a pair of roots of the auxiliary equation $f(m) = 0$ involves surds, say it is $\alpha \pm \sqrt{\beta}$, where $\beta$ is positive, then the corresponding term in the C.F. can similarly be written as

$$y = e^{\alpha x}(c_1 \cosh\sqrt{\beta}\,x + c_2 \sinh\sqrt{\beta}\,x)$$

or $$y = c_1 e^{\alpha x}\cosh(x\sqrt{\beta} + c_2)$$

or $$y = c_1 e^{\alpha x}\sinh(x\sqrt{\beta} + c_2).$$

***Example 6:*** ***Solve:*** $\mathbf{(D^2 \pm w^2)y = 0,\ w \neq 0.}$

***Solution:*** Hence, the auxiliary equation is

$$m^2 \pm w^2 = 0$$

$$\Rightarrow \quad m^2 + w^2 = 0 \quad \text{and} \quad m^2 - w^2 = 0$$

$$\Rightarrow \quad m = 0 \pm iw \text{ and } m = \pm w.$$

Hence, the required solution is

$$y = c_1 \cos wx + c_2 \sin wx \text{ and } y = c_3 e^{wx} + c_4 e^{-wx}$$

or $$y = c_1 \cos wx + c_2 \sin wx + c_3 e^{wx} + c_4 e^{-wx}.$$

***Example 7:*** ***Solve*** $\dfrac{d^3 y}{dx^3} - 8y = 0.$ **[B.C.A. (Purvanchal) 2010, 07, 03]**

***Solution:*** Here, the auxiliary equation is

$$m^3 - 8 = 0 \Rightarrow (m-2)(m^2 + 2m + 4) = 0$$

$$\Rightarrow \qquad m = 2, m = \frac{-2 \pm \sqrt{4-16}}{2} \Rightarrow m = 2, m = -1 \pm i\sqrt{3}.$$

Hence, the required general solution is

$$y = c_1 e^{2x} + e^{-x}(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x).$$

***Example 8:*** ***Solve*** $\dfrac{d^4 y}{dx^4} + m^4 y = 0.$ **[B.C.A. (Meerut) 2006 (B.P.) 05; B.C. A. (Agra) 2002]**

***Solution:*** The given differential equation can be written as

$$(D^4 + m^4) y = 0.$$

Thus, its auxiliary equation is

$$D^4 + m^4 = 0.$$

$$\Rightarrow \qquad D^4 + 2D^2 m^2 + m^4 - 2D^2 m^2 = 0$$

$$\Rightarrow \qquad (D^2 + m^2)^2 - (\sqrt{2}\, Dm)^2 = 0$$

$$\Rightarrow \qquad (D^2 - \sqrt{2}\, Dm + m^2)(D^2 + \sqrt{2}\, Dm + m^2) = 0$$

$$\Rightarrow \qquad D^2 - \sqrt{2}\, Dm + m^2 = 0, D^2 + \sqrt{2}\, Dm + m^2 = 0$$

$$\Rightarrow \qquad D = \frac{\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2}, D = \frac{-\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2}$$

$$\Rightarrow \qquad D = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}, D = -\frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}}.$$

Hence, the required general solution is

$$y = e^{mx/\sqrt{2}}\left(c_1 \cos\frac{m}{\sqrt{2}}x + c_2 \sin\frac{m}{\sqrt{2}}x\right) + e^{-mx/\sqrt{2}}\left(c_3 \cos\frac{m}{\sqrt{2}}x + c_4 \sin\frac{m}{\sqrt{2}}x\right),$$

where $c_1, c_2, c_3$ and $c_4$ are arbitrary constants.

***Example 9:*** ***Solve*** $(D^4 + 8D^2 + 16)y = 0.$ **[B.C. A. (Agra) 2003, 00]**

***Solution:*** The auxiliary equation of the given differential equation is

$$m^4 + 8m^2 + 16 = 0 \Rightarrow (m^2 + 4)^2 = 0$$

$$\Rightarrow \qquad m^2 + 4 = 0,\ m^2 + 4 = 0 \ \Rightarrow\ m = \pm 2i, \pm 2i.$$

Hence, the required general solution is

$$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x,$$

where $c_1, c_2, c_3$ and $c_4$ are arbitrary constant of integration.

***Example 10:*** ***Solve*** $(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0.$

***Solution:*** Here, the auxiliary equation is

$$(m^2 + 1)^3 (m^2 + m + 1)^2 = 0.$$

$$\Rightarrow \qquad m^2 = -1 \text{ (thrice)and } m^2 + m + 1 = 0 \text{ (twice)}$$

$$\Rightarrow \qquad m = \pm i \text{ (thrice) and } m = \{-1 \pm \sqrt{1-4}\}/2 \text{ (twice)}$$

$$\Rightarrow \qquad m = 0 \pm i \text{ (thrice) and } m = (-1 \pm i\sqrt{3})/2 \text{ (twice).}$$

Hence, the required general solution is

$$y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$$

$$+ e^{-x/2}\left\{(c_7 + c_8 x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10} x)\sin\frac{\sqrt{3}}{2}x\right\} \quad [\because e^{0x} = 1]$$

## Exercise 10.3

**Solve the following differential equations:**

1. $\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0.$

**[B.C.A. (Rohilkhand) 2009]**

2. $\dfrac{d^2 y}{dx^2} + \dfrac{dy}{dx} + 7y = 0.$

**[B.C.A. (Bundelkhand) 2006]**

3. $\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0.$

4. $\dfrac{d^2 y}{dx^2} + 2p\dfrac{dy}{dx} + (p^2 + q^2)y = 0.$

**[ B.C.A. (Rohilkhand) 2006]**

5. Solve $\dfrac{d^2 y}{dx^2} + y = 0$ given that $y = 0$ when $x = 0$ and $y = -2$, when $x = \pi/2$.

**[ B.C.A. (Rohilkhand) 2010; B.C. A. (Agra) 2002]**

6. $\dfrac{d^2 y}{dx^2} + 8\dfrac{dy}{dx} + 25y = 0.$

7. $(D^3 + 1)y = 0.$

**[B.C.A. (Agra) 2008]**

## Answers 10.3

| | | | |
|---|---|---|---|
| 1. | $y = e^{-x}(c_1 \cos x + c_2 \sin x)$. | 2. | $y = e^{-x/2}\left(c_1 \cos \frac{3\sqrt{3}}{2} x + c_2 \sin \frac{3\sqrt{3}}{2} x\right)$. |
| 3. | $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$. | 4. | $y = c_1 e^{-px} \cos(qx + c_2)$. |
| 5. | $y = 2 \cos x - 2 \sin x$ . | 6. | $y = e^{-4x}(c_1 \cos 3x + c_2 \sin 3x)$. |
| 7. | $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2}\right)$. | | |

## 10.7 To Find the Particular Integral

$$\frac{1}{D - \alpha} = Q$$

**[B.C.A. (Bhopal) 2012, 08, 04]**

Let $$\frac{1}{D - \alpha} Q = v. \qquad \text{...(1)}$$

We have, $$(D - \alpha) v = (D - \alpha)\left\{\frac{1}{D - \alpha} Q\right\}$$

$\Rightarrow$ $$\frac{dv}{dx} - \alpha \, . \, v = Q, \qquad \text{...(2)}$$

which is a linear differential equation of first order. Hence its solution (on leaving the constant of integration, since we need the particular integral ) is given by

$$v \, . \, (\text{I.F.}) = \int Q \, . \, (\text{I.F.}) \, dx$$

$\Rightarrow$ $$v \, . \, e^{-\alpha x} = \int Q \, . \, e^{-\alpha x} dx$$

$\Rightarrow$ $$v = e^{\alpha x} \int Q \, . \, e^{-\alpha x} dx$$

Hence, $$\frac{1}{D - \alpha} Q = e^{\alpha x} \int Q \, . \, e^{-\alpha x} dx. \qquad \text{...(3}$$

***Example 11:*** **Solve** $\frac{d^2 y}{dx^2} + a^2 y = \sec ax.$

**[B.C. A. (Agra) 2011, 08; B.C.A. (Lucknow) 2010, 06, 04]**

***Solution:*** Here, the auxiliary equation of the given differential equation is

$$m^2 + a^2 = 0 \qquad [\therefore \; m = \pm \, ai]$$

Therefore, the complementary function is

$$\text{C.F.} = c_1 \cos ax + c_2 \sin ax.$$

Also the particular integral is,

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{2ai}\left\{\frac{1}{D - ai} - \frac{1}{D + ai}\right\} \sec ax.$$

But
$$\frac{1}{D - ia} \sec ax = e^{iax} \int e^{-iax} \sec ax \, dx$$

$$= e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax} dx = e^{iax}\left\{x + \frac{i}{a} \log \cos ax\right\}.$$

Similarly,
$$\frac{1}{D + ia} \sec ax = e^{iax}\left\{x - \frac{i}{a} \log \cos ax\right\}.$$

Therefore, the particular integral is

$$\text{P.I.} = \frac{1}{2ai}\left\{e^{iax}\left(x + \frac{i}{a} \log \cos ax\right) - e^{-iax}\left(x - \frac{i}{a} \log \cos ax\right)\right\}$$

$$= \frac{1}{a}\left\{x \sin ax + \frac{1}{a} (\log \cos ax) \cos ax\right\}.$$

Hence, the general solution of the given equation is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.*,
$$y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a}\left\{x \sin ax + \frac{1}{a} (\log \cos ax) \cos ax\right\}.$$

## Exercise 10.4

**Solve the following differential equation:**

**1.** $\dfrac{d^2 y}{dx^2} + y = \sec x$

**2.** $\dfrac{d^2 y}{dx^2} + 9y = \sec 3x.$

**3.** $(D^2 + 4) y = \tan 2x.$

**[B.C.A. (Purvanchal) 2008]**

**4.** $(D + 1)^2 y = 2e^{2x}.$

**[B.C.A. (Bhopal) 2004]**

**5.** $\dfrac{d^2 y}{dx^2} + y = \operatorname{cosec} x.$

**[B.C.A. (Rohtak) 2011]**

**6.** $\dfrac{d^2 y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 5e^{3x}.$

**[B.C.A. (Agra) 2003]**

## Answers 10.4

| | |
|---|---|
| 1. | $c_1 \cos x + c_2 \sin x + x \sin x + \cos x \log(\cos x)$. |
| 2. | $c_1 \cos 3x + c_2 \sin 3x + \frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x \log(\cos 3x)$. |
| 3. | $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log \tan\left(\frac{\pi}{y} + x\right)$ |
| 4. | $y = (c_1 + c_2 x)e^{-x} + \frac{2e^{2x}}{9}$. |
| 5. | $y = c_1 \cos x + c_2 \sin x + \sin x \log(\sin x) - x \cos x$. |
| 6. | $y = (c_1 + c_2 x)e^{-3x} + \frac{5}{36} e^{3x}$. |

## 10.8 To Find $\frac{1}{f(D)} e^{ax}$, when $f(a) \neq 0$

By successive differential equation, we have

$$D(e^{ax}) = ae^{ax},\ D^2(e^{ax}) = a^2 e^{ax}, \ldots, D^n(e^{ax}) = a^n . e^{ax}.$$

In general, if $f(D)$ is a polynomial of $D$ *i.e.*,

$$f(D) = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n,$$

then $$f(D)e^{ax} = f(a)e^{ax}.$$

Now operating $\frac{1}{f(D)}$ on both sides, we get

$$\frac{1}{f(D)}[f(D)e^{ax}] = \frac{1}{f(D)} f(a)e^{ax}$$

$\Rightarrow$ $$e^{ax} = f(a).\frac{1}{f(D)} e^{ax} \quad [\because f(a) \text{ is merely an algebraic multiplier}]$$

Hence, $$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provided } f(a) \neq 0.$$

**NOTE:**
This method fails when a is a root of $f(D) = 0$, because in this case $f(a) = 0$ and so

$$\frac{1}{f(D)} e^{ax} = \infty . e^{ax}.$$

In this case, we write $f(D) = (D - a)^n \phi(D)$ and use the general method.

## 10.9 To Find $\dfrac{1}{f(D)}e^{ax}$

When $\quad f(a) \neq 0,$ where $f(D) = (D-a)^n \phi(D)$

Let $\quad f(D) = (D-a)^n\,\phi(D),$ where $\phi(a) \neq 0.$

Then
$$\frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)^n\phi(D)}e^{ax} = \frac{1}{(D-a)^n}\cdot\frac{1}{\phi(D)}e^{ax}$$

$$= \frac{1}{(D-a)^n}\frac{1}{\phi(a)}e^{ax} = \frac{1}{\phi(a)}\cdot\frac{1}{(D-a)^n}e^{ax} \qquad ...(1)$$

$[\because \phi(a) \neq 0]$

Now
$$\frac{1}{D-a}e^{ax} = e^{ax}\int e^{ax}e^{-ax}dx = xe^{ax},$$

$$\frac{1}{(D-a)^2}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{D-a}e^{ax}$$

$$= \frac{1}{D-a}(xe^{ax}) = e^{ax}\int xe^{ax}.e^{-ax}dx = e^{ax}\int x\,dx = \frac{x^2e^{ax}}{2}$$

$$\frac{1}{(D-a)^3}e^{ax} = \frac{1}{D-a}\cdot\frac{1}{(D-a)^2}e^{ax} = \frac{1}{D-a}\left(\frac{x^2e^{ax}}{2}\right)$$

$$= e^{ax}\int\frac{x^2e^{ax}}{2}.e^{-ax}dx = \frac{x^3e^{ax}}{3!}.$$

Proceeding in a similar manner, one can see that

$$\frac{1}{(D-a)^n}e^{ax} = \frac{x^ne^{ax}}{n!}. \qquad ...(2)$$

Three lines n/A.

***Example 12:*** ***Solve*** $\dfrac{d^2y}{dx^2} + 31\dfrac{dy}{dx} + 240y = 272\,e^{-x}.$

**[B.C. A. (Agra) 2010, 07, 04; B.C.A. (Purvanchal) 2008, 04]**

***Solution:*** Here, the auxiliary equation is

$$m^2 + 31m + 240 = 0$$

$\Rightarrow \qquad (m+15)(m+16) = 0 \qquad$ or $\qquad m = -15, -16.$

Therefore, the complementary function is

$$\text{C.F.} = c_1e^{-15x} + c_2e^{-16x}.$$

Now, the particular integral is,

$$\text{P.I.} = \frac{1}{D^2 + 31D + 240}(272e^{-x}) = 272\frac{1}{D^2 + 31D + 240}e^{-x}$$

$$= 272\frac{1}{(-1)^2 + 31(-1) + 240}e^{-x} \qquad [\because D = a = -1]$$

$$= \frac{136}{105}e^{-x}.$$

Hence, the general solution of the given equation is

$$y = c_1 e^{-15x} + c_2 e^{-16x} + \frac{136}{105}e^{-x}.$$

***Example 13:*** **Solve** $\boldsymbol{\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}}$.

**[B.C.A. (Bundelkhand) 2007; B.C.A. (Kanpur) 2007]**

***Solution:*** Here, the auxiliary equation is

$$m^2 + 4m + 3 = 0$$

$\Rightarrow$ $(m+1)(m+3) = 0$ or $m = -1, -3$.

$\therefore$ $\text{C.F.} = c_1 e^{-x} + c_2 e^{-3x}$

and $\text{P.I.} = \dfrac{1}{D^2 + 4D + 3}e^{-3x} = \dfrac{1}{(D+3)(D+1)}e^{-3x}$

$$= \frac{1}{D+3}\left\{\frac{1}{(-3)+1}e^{-3x}\right\} = -\frac{1}{2}\frac{1}{D+3}e^{-3x}$$

$$= -\frac{1}{2}e^{-3x}\int e^{-3x}e^{3x}dx = -\frac{1}{2}e^{-3x}\int 1\,dx = -\frac{1}{2}x\,e^{-3x}.$$

Hence, the required general solution is

$$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{2}xe^{-3x}.$$

***Example 14:*** **Solve** $\boldsymbol{\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + (p^2 + q^2)y = e^{ax}}$. **[B.C.A. (Kurukshetra) 2012]**

***Solution:*** Here, the auxiliary equation is

$$m^2 + 2pm + p^2 + q^2 = 0$$

$\Rightarrow$ $(m+p)^2 = -q^2 \Rightarrow m = -p \pm iq.$

Therefore, the complementary function is

$$\text{C.F.} = e^{-px}(c_1 \cos qx + c_2 \sin qx).$$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{D^2 + 2pD + p^2 + q^2} e^{ax}$$

$$= \frac{1}{a^2 + 2pa + p^2 + q^2} e^{ax} = \frac{1}{(a + p)^2 + q^2} e^{ax}.$$

Hence, the general solution of the given equation is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.*, $$y = e^{-px}(c_1 \cos qx + c_2 \sin qx) + e^{ax} / [(a + p)^2 + q^2].$$

**Example 15:** **Solve** $\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 7y = e^{x} + e^{-x}.$

**[B.C.A. (Kashi) 2010]**

**Solution:** The given differential equation can be written as $(D^2 - 6D + 7)y = e^x + e^{-x}$

$\therefore$ its auxiliary equation is

$$m^2 - 6m + 7 = 0$$

$\Rightarrow$ $$m = \frac{6 \pm \sqrt{36 - 28}}{2} = 3 \pm \sqrt{2}.$$

$\therefore$ $$\text{C.F.} = e^{3x}(c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x})$$

and $$\text{P.I.} = \frac{1}{D^2 - 6D + 7}(e^x + e^{-x}) = \frac{1}{D^2 - 6D + 7} e^x + \frac{1}{D^2 - 6D + 7} e^{-x}$$

$$= \frac{e^x}{1^2 - 6(1) + 7} + \frac{e^{-x}}{(-1)^2 - 6(-1) + 7} = \frac{e^x}{2} + \frac{e^{-x}}{14}.$$

Hence, the required general solution is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.*, $$y = e^{3x}(c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}) + \frac{e^x}{2} + \frac{e^{-x}}{14}.$$

**Example 16:** **Solve** $\dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^x.$

**Solution:** Here, the auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$\Rightarrow$ $$(m - 1)(m - 2) = 0 \Rightarrow m = 1, 2.$$

Therefore, the complementary function is

$$\text{C.F.} = c_1 e^x + c_2 e^{2x}.$$

Now, the particular integral is

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} e^x = \frac{1}{D-1} \cdot \frac{1}{(D-2)} e^x$$

$$= \frac{1}{D-1} \cdot \frac{e^x}{1-2} = -\frac{1}{D-1} e^x = -e^x \int e^x e^{-x} dx = -xe^x.$$

**Example 17:** ***Solve*** $\boldsymbol{\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = e^{-x}}$.

**Solution:** Here, the auxiliary equation is

$$m^3 + 3m^2 + 3m + 1 = 0$$

$\Rightarrow$ $$(m+1)^3 = 0 \qquad [\therefore\ m = -1, -1, -1]$$

Therefore, the complementary function is

$$\text{C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^{-x}.$$

Now, the particular integral is

$$\text{P.I.} = \frac{1}{(D+1)^3} e^{-x} = \frac{x^3 e^{-x}}{3!} = \frac{x^3 e^{-x}}{6} \qquad [\therefore\ a = -1, n = 3]$$

Hence, the required general solution is

$$y = (c_1 + c_2 x + c_3 x^2) e^{-x} + \frac{x^3 e^{-x}}{6}.$$

**Example 18:** ***Solve*** $\boldsymbol{(D+2)(D-1)^3\ y = e^x}$. **[B.C.A. (Agra) 2001]**

**Solution:** Here, the auxiliary equation is

$$(m+2)(m-1)^3 = 0; \qquad [\therefore\ m = -2, 1, 1, 1.]$$

Therefore, the complementary function is

$$\text{C.F.} = c_1 e^{-2x} + (c_2 + c_3 x + c_4 x^2)\, e^x.$$

$$\text{P.I.} = \frac{1}{(D-1)^3 (D+2)} e^x = \frac{x^3 e^x}{(1+2)\,.\,3!} = \frac{x^3 e^x}{18}.$$

Hence, the required general solution is

$$y = c_1 e^{-2x} + (c_2 + c_3 x + c_4 x^2) + \frac{x^3 e^x}{18}.$$

## Exercise 10.5

**Solve the following differential equation:**

1. $(D^2 - 2D + 1)\,y = e^{-x}$. **[B.C.A. (Meerut) 2002]**

2. $\dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{5x}$.

3. $3\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} - y = e^{x/2} + 2e^{3x}$.

4. $(3D^2 - 4D + 5)\,y = e^x - 2e^{2x} + 3e^{3x}$. **[B.C.A. (Rohilkhand) 2009]**

5. $(D^3 - 1)\,y = (e^x + 1)^2$. **[B.C.A. (Lucknow) 2008]**

## Answers 10.5

| | |
|---|---|
| 1. | $y = (c_1 + c_2 x)e^x + \dfrac{1}{4}e^{-x}$. |
| 2. | $y = c_1 e^x + c_2 e^{2x} + \dfrac{1}{12}x\,e^{5x}$. |
| 3. | $y = c_1 e^{x/3} + c_2 e^{-x} + \dfrac{4}{3}e^x + \dfrac{1}{16}e^{3x}$. |
| 4. | $y = e^{2x/3}\left(c_1 \cos\dfrac{\sqrt{11}}{3}x + c_2 \sin\dfrac{\sqrt{11}}{3}x\right) + \dfrac{1}{4}e^x - \dfrac{2}{9}e^{2x} + \dfrac{3}{20}e^{3x}$. |
| 5. | $y = e^{-x/2}\left(c_1 \cos\dfrac{\sqrt{3}}{2}x + c_2 \sin\dfrac{\sqrt{3}}{2}x\right) + e^x\left(\dfrac{2}{3}x\right) + \dfrac{1}{7}e^{2x} - 1$. |

## 10.10 To Find $\dfrac{1}{f(D^2)}\sin ax$ and $\dfrac{1}{f(D^2)}\cos ax$ when $f(-a^2) \neq 0$

By successive differentiation, we get

$$D \sin ax = a\cos ax$$

$$D^2 \sin ax = -a^2 \sin ax$$

$$D^3 \sin ax = -a^3 \cos ax$$

$$D^4 \sin ax = a^4 \sin ax = (-a^2)^2 \sin ax$$

$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In general, $(D^2)^n \sin ax = (-a^2)^n \sin ax$.

So that, $f(D^2)\sin ax = f(-a^2)\sin ax$.

Operating both sides of above by $\dfrac{1}{f(D^2)}$, we find

$$\frac{1}{f(D^2)}[f(D^2)\sin ax] = \frac{1}{f(D^2)}[f(-a^2)\sin ax]$$

$\Rightarrow$ $$\sin ax = f(-a^2)\frac{1}{f(D^2)}\sin ax.$$

Hence, $$\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax,\ f(-a^2) \neq 0.$$

Similarly,

$$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax,\ f(-a^2) \neq 0.$$

***Example 19:* Solve $(D^2+1)y = \cos 2x$.** **[B.C. A. (Agra) 2003]**

***Solution:*** The auxiliary equation of the differential equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i, \quad i.e.,\ 0 \pm i.$$

$\therefore$ $$(\text{C.F.}) = c_1 \cos x + c_2 \sin x.$$

and $$(\text{P.I.}) = \frac{1}{D^2+1}\cos 2x = \frac{1}{(-2^2)+1}\cos 2x = -\frac{\cos 2x}{3}.$$

Hence, the complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.,* $$y = c_1 \cos x + c_2 \sin x - \frac{1}{3}\cos 2x.$$

***Example 20:* Solve $\dfrac{d^2y}{dx^2} - 4y = e^x + \sin 2x$.** **[B.C. A (Agra) 2005, 02]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 4 = 0; \quad \therefore\ m = \pm 2$$

Therefore, the complementary function is

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{2x}.$$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{D^2-4}(e^x + \sin 2x) = \frac{1}{D^2-4}e^x + \frac{1}{D^2-4}\sin 2x$$

$$= \frac{1}{1^2 - 4} e^x + \frac{1}{(-2^2) - 4} \sin 2x = -\frac{e^x}{3} - \frac{\sin 2x}{8}.$$

Hence, the general solution is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.*,
$$y = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{3} e^x - \frac{1}{8} \sin 2x.$$

**Example 21:** ***Solve*** $\frac{d^2 y}{dx^2} - 8 \frac{dy}{dx} + 9y = 40 \sin 5x.$

**[B.C. A. (Lucknow) 2004; B.C.A. (Agra) 2000]**

**Solution:** Here, the auxiliary equation is

$$m^2 - 8m + 9 = 0, \text{ so that } m = 4 \pm \sqrt{7}.$$

Therefore,
$$\text{C.F.} = c_1 e^{4x} \cosh(\sqrt{7}x + c_2)$$

and
$$\text{P.I.} = \frac{1}{D^2 - 8D + 9} 40 \sin 5x = 40 \frac{1}{D^2 - 8D + 9} \sin 5x$$

$$= 40 \frac{1}{-5^2 - 8D + 9} \sin 5x = -5 \frac{1}{D + 2} \sin 5x$$

$$= -5 \frac{D - 2}{D^2 - 4} \sin 5x = -5(D - 2) \frac{1}{-5^2 - 4} \sin 5x$$

$$= \frac{5}{29} [D(\sin 5x) - 2 \sin 5x] = \frac{25}{29} \cos 5x - \frac{10}{29} \sin 5x.$$

Hence, the complete solution is

$$y = c_1 e^{4x} \cosh(\sqrt{7}x + c_2) + \frac{25}{29} \cos 5x - \frac{10}{29} \sin 5x.$$

**Example 22:** ***Solve*** $\frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} - \frac{dy}{dx} - y = \cos 2x.$ **[B.C.A. (Aligarh) 2010, 04]**

**Solution:** The auxiliary equation of the given differential equation is

$$m^3 + m^2 - m - 1 = 0$$

$$\Rightarrow \quad (m + 1)(m^2 - 1) = 0. \qquad [\therefore m = 1, -1, -1]$$

$$\therefore \quad \text{C.F.} = c_1 e^x + (c_2 + c_3 x) e^{-x}.$$

$$\text{P.I.} = \frac{1}{D^3 + D^2 - D - 1} \cos 2x = \frac{1}{D + 1} \left\{ \frac{1}{D^2 - 1} \cos 2x \right\}$$

$$= \frac{1}{D + 1} \left\{ \frac{\cos 2x}{-2^2 - 1} \right\} = -\frac{1}{5} \frac{1}{D + 1} \cos 2x$$

$$= -\frac{1}{5}\frac{D-1}{(D+1)(D-1)}\cos 2x$$

$$= -\frac{1}{5}\frac{D-1}{D^2-1}\cos 2x = -\frac{1}{5}\frac{D-1}{-5}\cos 2x$$

$$= \frac{1}{25}(D\cos 2x - \cos 2x) = \frac{1}{25}(-2\sin 2x - \cos 2x).$$

Hence, the required general solution of the differential equation is

$$y = c_1 e^x + (c_2 + c_3 x)e^{-x} - (2/25)\sin 2x - (1/25)\cos 2x.$$

**Example 23:** ***Solve*** $(D^2 - 3D + 2)y = 6e^{2x} + \sin 2x.$ **[B.C.A. (Avadh) 2004]**

***Solution:*** The auxiliary equation of the given differential equation is

$$m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0$$

$\Rightarrow$ $m = 1, 2.$

$\therefore$ $\text{C.F.} = c_1 e^x + c_2 e^{2x}$

and $\text{P.I.} = \frac{1}{D^2 - 3D + 2}(6e^{2x} + \sin 2x)$

$$= \frac{1}{(D-1)(D-2)}6e^{2x} + \frac{1}{D^2 - 3D + 2}\sin 2x$$

$$= 6\frac{e^{2x}}{2-1}\frac{1}{D+2-2}1 + \frac{1}{-2^2 - 3D + 2}\sin 2x$$

$$= 6e^{2x}\frac{1}{D}1 - \frac{1}{3D+2}\sin 2x = 6e^{2x}x - \frac{3D-2}{(3D+2)(3D-2)}\sin 2x$$

$$= 6xe^{2x} - \frac{3D-2}{9D^2-4}\sin 2x = 6xe^{2x} - (3D-2)\frac{1}{9(-2^2)-4}\sin 2x$$

$$= 6xe^{2x} + \frac{1}{40}(3D-2)\sin 2x = 6xe^{2x} + \frac{1}{40}(6\cos 2x - 2\sin 2x)$$

$$= 6xe^{2x} + \frac{1}{20}(3\cos 2x - \sin 2x).$$

Hence, the required general solution is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.,* $y = c_1 e^x + c_2 e^{2x} + 6xe^{2x} + \frac{1}{20}(3\cos 2x - \sin 2x).$

**Example 24:** ***Solve*** $\frac{d^2 y}{dx^2} - 4y = \cos^2 x.$ **[B.C.A. (Kurukshetra) 2008, 04]**

***Solution:*** The given differential equation can be written as

$$(D^2 - 4)y = \cos^2 x. \quad \text{...(1)}$$

$\therefore$ Its auxiliary equation is

$$m^2 - 4 = 0 \Rightarrow m = \pm 2.$$

$\therefore$ $$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

and $$\text{P.I.} = \frac{1}{D^2 - 4} \cos^2 x = \frac{1}{D^2 - 4} \frac{1}{2}(1 + \cos 2x)$$

$$= \frac{1}{D^2 - 4} \frac{1}{2} e^{0x} + \frac{1}{D^2 - 4} \frac{1}{2} \cos 2x$$

$$= \frac{1}{2} \cdot \frac{1}{0^2 - 4} + \frac{1}{2} \cdot \frac{1}{-2^2 - 4} \cos 2x = -\frac{1}{8} - \frac{\cos 2x}{16}.$$

Hence, the required general solution is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.,* $$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} - \frac{\cos 2x}{16}.$$

***Example 25:*** **Solve** $\dfrac{d^2 y}{dx^2} - 5\dfrac{dy}{dx} + 6y = \sin 3x.$ **[B.C.A. (Rohtak) 2012, 06]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$\Rightarrow$ $$(m-2)(m-3) = 0 \quad \text{or} \quad m = 2, 3.$$

$\therefore$ $$\text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$$

and $$\text{P.I.} = \frac{1}{D^2 - 5D + 6} \sin 3x = \frac{1}{-3^2 - 5D + 6} \sin 3x$$

$$= -\frac{5D - 3}{(5D+3)(5D-3)} \sin 3x$$

$$= -(5D - 3) \cdot \frac{1}{25D^2 - 9} \sin 3x$$

$$= -(5D - 3) \frac{1}{25(-3^2) - 9} \sin 3x = \frac{1}{234}(5D - 3) \sin 3x$$

$$= \frac{15 \cos 3x - 3 \sin 3x}{234} = \frac{5 \cos 3x - \sin 3x}{78}.$$

Hence, the required general solution is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.,* $$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{78}(5 \cos 3x - \sin 3x).$$

## Exercise 10.6

**Solve the following differential equations:**

1. $\dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx} + 2y = \cos 2x.$

2. $\dfrac{d^2 y}{dx^2} + a^2 y = \sin ax.$

3. $\dfrac{d^2 y}{dx^2} - 4y = 2\sin\dfrac{x}{2}.$

4. $\dfrac{d^2 y}{dx^2} + 4y = e^x + \sin 2x.$ **[B.C.A. (Agra) 2002]**

5. $\dfrac{d^2 y}{dx^2} + 4y = \cos 3x - \sin 3x.$

6. $\dfrac{d^2 y}{dx^2} - 4\dfrac{dy}{dx} + 4y = \sin 2x.$

7. $\dfrac{d^2 y}{dx^2} - 4\dfrac{dy}{dx} + y = a\cos 2x.$

8. $\dfrac{d^2 y}{dt^2} - 3\dfrac{dy}{dt} + 2y = 10\sin t.$

9. $2\dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx} + 4y = 3\cos\dfrac{x}{2}.$

10. $(D^2 - D - 8)y = -2\sin^2 x.$

11. $(D^2 + D + 1)y = (1 + \sin x)^2.$

12. $(D^4 - 1)y = \sin 2x.$

## Answers 10.6

1. $y = c_1 e^{2x} + c_2 e^x - (1/20)\cos 2x - (3/20)\sin 2x.$

2. $y = c_1 \cos ax + c\sin ax - \dfrac{x\cos ax}{2a}.$

3. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{8}{17}\sin\dfrac{1}{2}x.$

4. $y = c_1\cos 2x + c_2\sin 2x + \dfrac{1}{5}e^x - \dfrac{1}{4}x\cos 2x.$

| | |
|---|---|
| **5.** | $y = c_1 \cos 2x + c_2 \sin 2x - (1/5)(\cos 3x + \sin 3x)$. |
| **6.** | $y = (c_1 + c_2 x)e^{2x} + (1/8)\cos 2x$. |
| **7.** | $y = c_1 e^2 \cosh(\sqrt{3}x + c_2) - \frac{a}{73}(3\cos 2x + 8\sin 2x)$. |
| **8.** | $y = c_1 e^t + c_2 e^{2t} + \sin t + 3\cos t$. |
| **9.** | $y = \left(c_1 \cos \frac{\sqrt{23}}{4}x + c_2 \sin \frac{\sqrt{23}}{4}x\right)e^{\frac{3}{4}x} + \frac{3}{29}\left(7\cos\frac{x}{2} - 3\sin\frac{x}{2}\right)$. |
| **10.** | $y = e^{x/2}[c_1 e^{-\sqrt{33}x/2} + c_2 e^{\sqrt{33}x/2}] + \frac{1}{8} + \frac{1}{148}(-2\sin 2x - 12\cos 2x)$. |
| **11.** | $y = \left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)e^{-x/2} + \frac{3}{2} + (1/26)(3\cos 2x - 2\sin 2x) - 2\cos x$. |
| **12.** | $y = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x + (1/15)\sin 2x$. |

## 10.11 To Find $\frac{1}{f(D)}x^m$, where m is a Positive Integer

To evaluate $\frac{1}{f(D)}x^m$, we bring out common the lowest degree term of D from $f(D)$. Then the remaining factor in the denominator is of the form $[1 + F(D)]$ or $[1 - F(D)]$, which is taken in the numerator as a negative index.

Now, we expand $[1 + \pm F(D)]^{-1}$ in ascending powers of D by the binomial theorem upto $D^m$. Then operate upon $x^m$, using each term of the expansion thus obtained.

***Example 26:*** ***Solve*** $(D^2 - 5D + 6)y = x$. **[B.C.A. (Kanpur) 2009, 06]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow \quad (m-2)(m-3) = 0; \quad m = 2, 3.$$

$$\therefore \quad \text{C.F.} = c_1 e^{2x} + c_2 e^{3x}.$$

Also,
$$\text{P.I.} = \frac{1}{D^2 - 5D + 6}x = \frac{1}{6(1 - \frac{5}{6}D + \frac{1}{6}D^2)}x$$

$$= \frac{1}{6}[1 - \frac{1}{6}(5D - D^2)]^{-1}x$$

$$= \frac{1}{6}[1 + \frac{1}{6}(5D - D^2) + \ldots]x = \frac{1}{6}[1 + \frac{5}{6}D + \ldots]x$$

$$= \frac{1}{6}[x + \frac{5}{6}D(x)], \text{ as all other terms vanish } = \frac{1}{6}x + \frac{5}{36}.$$

Hence, the complete solution of the given equation is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}x + \frac{5}{36}.$$

***Example 27:*** ***Solve*** $(D^2 - 4)y = x^2$. **[B.C.A. (Agra) 2010]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 4 = 0; \ m = \pm 2.$$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$, where $c_1$ and $c_2$ are arbitrary constants.

Also
$$\text{P.I.} = \frac{1}{D^2 - 4}x^2 = \frac{1}{-4(1 - \frac{1}{4}D^2)}x^2$$

$$= -\frac{1}{4}(1 - \frac{1}{4}D^2)^{-1}x^2 = -\frac{1}{4}[1 + \frac{1}{4}D^2 + \ldots]x^2$$

$$= -\frac{1}{4}[x^2 + \frac{1}{4}D^2(x^2)], \text{ as all other terms vanish}$$

$$= -\frac{1}{4}(x^2 + \frac{1}{2}).$$

Hence, the complete solution of given equation is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}(x^2 + \frac{1}{2}).$$

***Example 28:*** ***Solve*** $(D^3 + 3D^2 + 2D)y = x^2$. **[B.C.A. (Agra) 2004, 03]**

***Solution:*** Here, the auxiliary equation is

$$m^3 + 3m^2 + 2m = 0$$

$\Rightarrow$
$$m(m + 2)(m + 1) = 0 \qquad [\therefore \ m = 0, -2, -1]$$

Therefore, the complementary function is

$$\text{C.F.} = c_1 + c_2 e^{-2x} + c_3 e^{-x}.$$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{2D + 3D^2 + D^3}x^2 = \frac{1}{2D}\left(1 + \frac{3}{2}D + \frac{D^2}{2}\right)^{-1}x^2$$

$$= \frac{1}{2D}(1 - (\frac{3}{2}D + \frac{7}{4}D^2) + \ldots)x^2$$

$$= \frac{1}{2D}[x^2 - (\frac{3}{2}D(x^2) + \frac{7}{4}D^2)(x^2)], \text{ as all other terms vanish}$$

$$= \frac{1}{2D}(x^2 - (3x + \frac{7}{2})) = \frac{1}{2}\left(\frac{x^3}{3} - (\frac{3}{2}x^2 + \frac{7}{2}x)\right); \text{ as } (1/D)x = \int x\,dx \text{ etc.}$$

Hence, the complete solution of the given equation is

$$y = c_1 + c_2 e^{-2x} + c_3 e^{-x} + \left(\frac{x}{12}\right)(2x^2 - 9x - 21).$$

**Example 29:** ***Solve*** $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x.$ **[B.C. A. (Agra) 2008]**

**Solution:** Here, the auxiliary equation is

$$m^3 + 2m^2 + m = 0 \Rightarrow m\,(m^2 + 2m + 1) = 0$$

$\Rightarrow$ $$m(m+1)^2 = 0 \Rightarrow m = 0, -1, -1.$$

$\therefore$ $$\text{C.F.} = c_1 + (c_2 + c_3 x)e^{-x}$$

and $$\text{P.I.} = \frac{1}{D(D+1)^2}(e^{2x} + x^2 + x)$$

$$= \frac{1}{D(D+1)^2}e^{2x} + \frac{1}{D(D+1)^2}(x^2 + x)$$

$$= \frac{1}{2(2+1)^2}e^{2x} + \frac{1}{D}(1+D)^{-2}(x^2 + x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\{1 - 2D + 3D^2 - \ldots\}(x^2 + x)$$

$$= \frac{e^{2x}}{18} + \frac{1}{D}\{(x^2 + x) - 2\,(2x + 1) + 6\}$$

$$= \frac{e^{2x}}{18} + \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x\right).$$

Hence, the required general solution is

$$y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{e^{2x}}{18} + \frac{1}{3}x^3 - \frac{3}{2}x^2 + 4x.$$

**Example 30:** ***Solve*** $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2.$ **[B.C. A. (Bhopal) 2009, 05, 03]**

**Solution:** Here, the auxiliary equations

$$m^3 - m^2 - 6m = 0$$

$\Rightarrow$ $$m(m+2)(m-3) = 0 \qquad [\therefore m = 0, -2, 3]$$

Therefore, the complementary function is

$$\text{C.F.} = c_1 + c_2 e^{-2x} + c_3 e^{3x}.$$

Also, $$\text{P.I.} = \frac{1}{-6D - D^2 + D^3}(1 + x^2)$$

$$= \frac{-1}{6D}\left(1 + \frac{1}{6}D - \frac{1}{6}D^2\right)^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{1}{6}D + \frac{7}{36}D^2 + \dots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[(1 + x^2) - \frac{1}{6}D(1 + x^2) + \frac{7}{36}D^2(1 + x^2)\right], \text{ as all other terms vanish}$$

$$= -\frac{1}{6D}\left(x^2 - \frac{1}{3}x + \frac{25}{18}\right)$$

$$= -\frac{1}{6}\left(\frac{x^3}{3} - \frac{1}{3}\cdot\frac{x^2}{2} + \frac{25}{18}x\right), \text{ as } \frac{1}{D}x = \int x\,dx \text{ etc.}$$

Hence, the complete solution of the given equation is

$$y = c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{18}x^3 + \frac{1}{36}x^2 - \frac{25}{108}x.$$

***Example 31:*** ***Solve*** $(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x.$ **[B.C.A. (Rohilkhand) 2009, 06]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 4m + 4 = 0 \Rightarrow (m - 2)^2 = 0 \Rightarrow m = 2, 2.$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 x)e^{2x}$$

and $$\text{P.I.} = \frac{1}{D^2 - 4D + 4}(x^2 + e^x + \cos 2x).$$

Now $$= \frac{1}{D^2 - 4D + 4}x^2 = \frac{1}{4}\left(1 - D + \frac{1}{4}D^2\right)^{-1}x^2$$

$$= \frac{1}{4}\left\{x^2 + 2x + \frac{3}{4}(2)\right\} = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

$$= \frac{1}{D^2 - 4D + 4}e^x = \frac{1}{1^2 - 4(1) + 4}e^x = e^x$$

$$\text{P.I. of } \cos 2x = \frac{\cos 2x}{(D^2 - 4D + 4)} \quad \text{put} \quad D^2 = -4$$

$$= \frac{\cos 2x}{-4D} = \frac{-1}{8}\sin 2x$$

Hence, the required general solution is

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{2}(x^2 + 2x + \frac{3}{2}) + e^x - \frac{1}{8}\sin 2x.$$

**Example 32:** ***Solve*** $(D^2 + D - 2)y = x + \sin x.$ **[B.C. A. (Agra) 2000]**

**Solution:** Here, the auxiliary equation is

$$\Rightarrow \quad (m-1)(m+2) = 0$$

$$\Rightarrow \quad m = 1, -2.$$

$$\therefore \quad \text{C.F.} = c_1 e^x + c_2 e^{-2x}$$

and
$$\text{P.I.} = \frac{1}{D^2 + D - 2}(x + \sin x)$$

$$= \frac{1}{D^2 + D - 2} x + \frac{1}{D^2 + D - 2} \sin x$$

$$= -\frac{1}{2}\left(1 - \frac{1}{2}D - \frac{1}{2}D^2\right)^{-1} x + \frac{1}{-1^2 + D - 2} \sin x$$

$$= -\frac{1}{2}\left(1 + \frac{1}{2}D + \ldots\right) x + \frac{D+3}{D^2 - 9} \sin x$$

$$= -\frac{1}{2}\left(x + \frac{1}{2}\right) + \frac{D+3}{-1^2 - 9} \sin x$$

$$= -\frac{1}{4}(2x + 1) - \frac{1}{10}(\cos x + 3\sin x).$$

Hence, the required general solution is

$$y = c_1 e^x + c_2 e^{-2x} - \frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x).$$

**Example 33:** ***Solve*** $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x + x^2.$ **[B.C.A. (Bundelkhand) 2008]**

**Solution:** Here, the auxiliary equation is

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$$

$$\Rightarrow \quad m = -1, -1.$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)e^{-x}$$

and
$$\text{P.I.} = \frac{1}{(D+1)^2}(2x + x^2) = (1 + D)^{-2}(2x + x^2)$$

$$= (1 - 2D + 3D^2 - \ldots)(2x + x^2)$$

$$= (2x + x^2) - 2(2 + 2x) + 3(0 + 2) = x^2 - 2x + 2.$$

Hence, the required general solution is

$$y = (c_1 + c_2 x)e^{-x} + x^2 - 2x + 2.$$

## Exercise 10.7

**Solve the following differential equation:**

1. $\dfrac{d^2 y}{dx^2} - y = 2 + 5x.$ **[B.C.A. (Rohilkhand) 2009]**

2. $(D^2 + 2D + 2)\, y = x^2.$ **[B.C.A. (Avadh) 2009]**

3. $(D^2 - 5D + 6)\, y = x + \sin 3x.$

4. $(D^3 - D^2 - 6D)\, y = x^2 + 1.$ **[B.C.A. (Kurukshetra) 2012]**

5. $(D^3 - 2D + 4)\, y = x^4 + 3x^2 - 5x + 2.$ **[B.C.A. (Agra) 2011]**

6. $\dfrac{d^3 y}{dx^3} + 2\dfrac{d^2 y}{dx^2} + 4\dfrac{dy}{dx} + 8y = x.$ **[B.C.A. (Kanpur) 2007]**

7. $\dfrac{d^3 y}{dx^3} - 3\dfrac{dy}{dx} - 2y = x^2.$ **[B.C.A. (Bhopal) 2009]**

8. $(D^4 + D^2 + 16)\, y = 16x^2 + 256.$ **[B.C.A. (Lucknow) 2008]**

9. $\dfrac{d^2 y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 8(x^2 + e^{2x} + \sin 2x).$ **[B.C. A. (Agra) 2009]**

## Answers 10.7

1. $y = c_1 e^x + c_2 e^{-x} - 2 - 5x.$
2. $y = e^{-x}(c_1 \cos x \;\; c_2 \sin x) + \frac{1}{2}(x^2 - 2x + 1).$
3. $y = c_1 e^{3x} + c_2 e^{2x} + \frac{1}{6}(x + \frac{5}{6}) + (1/234)\,(15 \cos 3x - 3 \sin 3x).$
4. $y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{6}(\frac{25}{18}x + \frac{1}{6}x^2 + \frac{1}{3}x^3).$
5. $y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x) + \frac{1}{4}(x^4 + 2x^3 + 6x^2 - 5x - \frac{7}{2}).$
6. $y = c_1 e^{-2x} + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{16}(2x - 1).$
7. $y = c_1 e^{2x} + (c_2 + c_3 x)e^{-x} - \frac{1}{4}(2x^2 - 6x + 9).$
8. $y = e^{-\frac{1}{2}x\sqrt{7}}[c_1 \cos(3/2)x + c_2 \sin(3/2)x]$
 $+ e^{\frac{1}{2}x\sqrt{7}}[c_3 \cos(3/2)x + c_4 \sin(3/2)x] + x^2 + \frac{127}{8}.$
9. $y = (c_1 + c_2 x)\, e^{2x} + 2(x^2 + 2x + \frac{3}{2}) + 4x^2 e^{2x} + \cos 2x.$

## 10.12 To Find $\frac{1}{f(D)} e^{ax} V$, where V is a Function if x

By successive differentiation, we get

$$D(e^{ax}V_1) = e^{ax}D(V_1) + ae^{ax}V_1,\ (V_1 \text{ is any function of x}) = e^{ax}(D+a)V_1$$

$$D^2(e^{ax}V_1) = e^{ax}D(D+a)V_1 + ae^{ax}(D+a)V_1 = e^{ax}(D+a)^2 V_1$$

$$\dots\dots\dots = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \dots\dots$$

$$\dots\dots\dots = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \dots\dots$$

$$\dots\dots\dots = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad \dots\dots$$

In general, $D^n(e^{ax}V_1) = e^{ax}(D+a)^n V_1$

So that $\quad f(D)\, e^{ax}V_1 = e^{ax} f(D+a)V_1.$ ...(1)

Now, suppose that $f(D+a)V_1 = V$, then $V$ will also be a function of $x$, since $V_1$ is a function of $x$, and $V_1 = \frac{1}{f(D+a)} V.$ ...(2)

Substituting this value of $V_1$ in (1), we get

$$f(D)e^{ax} \frac{1}{f(D+a)} V = e^{ax} V.$$

$$\therefore \qquad \frac{1}{f(D)} \left\{ f(D)e^{ax} \frac{1}{f(D+a)} V \right\} = \frac{1}{f(D)} e^{ax} V$$

*i.e.,*

$$e^{ax} \frac{1}{f(D+a)} V = \frac{1}{f(D)} e^{ax} V.$$

Hence,

$$\frac{1}{f(D)} e^{ax}.V = e^{ax} \frac{1}{f(D+a)} V$$

where $V$ is any function of $x$.

**NOTE:**

To find $\frac{1}{f(D)} e^{ax}$, when $f(a) = 0$.

In this case, we have

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{f(D)} (e^{ax}.1) = e^{ax} \frac{1}{f(D+a)} 1.$$

**Example 34:** ***Solve*** $(D^2 - 5D + 6)y = x^3 e^{2x}$. **[B.C.A. (Bundelkhand) 2005]**

**Solution:** Here, the auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$\Rightarrow$ $(m-2)(m-3) = 0$ $[\therefore\ m = 2, 3]$

Therefore, C.F. $= c_1 e^{2x} + c_2 e^{3x}$.

And
$$\text{P.I.} = \frac{1}{D^2 - 5D + 6} x^3 e^{2x} = e^{2x} \frac{1}{(D+2)^2 - 5(D+2) + 6} x^3$$
$$= e^{2x} \frac{1}{D^2 - D} x^3 = -e^{2x} \frac{1}{D(1-D)} x^3 = -e^{2x} \frac{1}{D} (1-D)^{-1} x^3$$
$$= -e^{2x} \left(\frac{1}{D}\right)(x^3 + 3x^2 + 6x + 6), \text{ all other terms vanish}$$
$$= -e^{3x} (\frac{1}{4} x^4 + x^3 + 3x^2 + 6x), \text{ as } \left(\frac{1}{D}\right) x = \int x\, dx \text{ etc.}$$

Hence, the required general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} - e^{2x} (\frac{1}{4} x^4 + x^3 + 3x^2 + 6x).$$

**Example 35:** ***Solve*** $(D^2 - 3D + 2)y = e^x$.

**Solution:** Here, the auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$\Rightarrow$ $(m-1)(m-2) = 0;$ $[\therefore\ m = 1, 2.]$

Therefore, C.F. $= c_1 e^x + c_2 e^{2x}$.

And
$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} e^x . 1 = e^x \frac{1}{(D+1)^2 - 3(D+1) + 2} 1$$
$$= e^x \frac{1}{D^2 - D} 1 = e^x . \frac{1}{-D} (1-D)^{-1} 1$$
$$= -e^x \frac{1}{D} [1 + D + \ldots] 1 = -e^x \frac{1}{D}(1) = -xe^x, \text{ as } \frac{1}{D} 1 = \int 1\, dx.$$

Hence, the required general solution is

$$y = c_1 e^x + c_2 e^{2x} - xe^x.$$

**Example 36:** ***Solve*** $(D^2 - 2D + 1)y = e^x x^2$. **[B.C.A. (Agra) 2001]**

**Solution:** Here, the auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$\Rightarrow \qquad (m-1)^2 = 0; \qquad [\therefore m = 1, 1]$

$\therefore \qquad \text{C.F.} = (c_1 + c_2 x)e^x$

and $\qquad \text{P.I.} = \dfrac{1}{D^2 - 2D + 1} e^x x^2 = \dfrac{1}{(D-1)^2} e^x x^2$

$$= e^x \frac{1}{(D+1-1)^2} x^2 = e^x \frac{1}{D^2} x^2$$

$$= e^x \frac{1}{D} \int x^2 dx = e^x \frac{1}{D}\left(\frac{x^3}{3}\right) = \frac{1}{3} e^x \int x^3 dx = \frac{1}{12} e^x x^4.$$

Hence, the required general solution is

$$y = \text{C.F.} + \text{P.I.} \;\; i.e., \;\; y = (c_1 + c_2 x)e^x + \frac{1}{12} e^x x^4.$$

***Example 37:* Solve $\dfrac{d^2 y}{dx^2} + 4y = e^x + \cos 2x.$** **[B.C.A. (Aligarh) 2011, 07]**

***Solution:*** The given differential equation can be written as

$$(D^2 + 4) y = e^x + \cos 2x.$$

$\therefore$ Its auxiliary equation is

$$m^2 + 4 = 0;$$

$\therefore \qquad m = \pm 2i.$

$\therefore \qquad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$

and $\qquad \text{P.I.} = \dfrac{1}{D^2 + 4}(e^x + \cos 2x) = \dfrac{1}{D^2 + 4} e^x + \dfrac{1}{D^2 + 4} \cos 2x$

$$= \frac{e^x}{1^2 + 4} + \text{real part of } \frac{1}{(D+2i)(D-2i)} e^{i2x}$$

$$= \frac{e^x}{5} + \text{real part of } \frac{e^{i2x}}{2i + 2i} \cdot \frac{1}{D + 2i - 2i} (1)$$

$$= \frac{e^x}{5} + \text{real part of } \frac{e^{i2x}}{4i} \cdot \frac{1}{D} 1 = \frac{e^x}{5} + \text{real part of } \frac{e^{i2x}}{4i} x$$

$$= \frac{e^x}{5} + \text{real part of } \frac{x}{4i}(\cos 2x + i \sin 2x) = \frac{e^x}{5} + \frac{x}{4} \sin 2x.$$

Hence, the required general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{5} + \frac{x}{4} \sin 2x.$$

**Example 38:** **Solve** $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = xe^x + e^x.$ **[B.C.A. (Kanpur) 2008]**

**Solution:** Here, the auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$\Rightarrow \quad (m-1)^3 = 0 \qquad [\therefore m = 1, 1, 1]$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x + c_3 x^2)e^x$$

and

$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 3D - 1}(xe^x + e^x) = \frac{1}{(D-1)^3} e^x (x+1)$$

$$= e^x \frac{1}{(D+1-1)^3}(x+1) = e^x \frac{1}{D^3}(x+1)$$

$$= e^x \iiint (x+1)(dx)^2 = e^x\left(\frac{x^4}{4!} + \frac{x^3}{3!}\right) = e^x\left(\frac{x^4}{24} + \frac{x^3}{6}\right).$$

Hence, the required general solution is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.,*

$$y = (c_1 + c_2 x + c_3 x^2)e^x + \frac{1}{24}e^x x^4 + \frac{1}{6}e^x x^3.$$

**Example 39:** **Solve** $(D^3 + 3D^2 + 3D + 1)y = x^2 e^{-x}.$

**Solution:** Here, the auxiliary equation is

$$m^3 + 3m^2 + 3m + 1 = 0 \Rightarrow (m+1)^3 = 0;$$

$$[\therefore m = -1, -1, -1.]$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^{-x}$$

and

$$\text{P.I.} = \frac{1}{D^3 + 3D^2 + 3D + 1} x^2 e^{-x} = \frac{1}{(D+1)^3} x^2 e^{-x}$$

$$= e^{-x} \frac{1}{(D-1+1)^3} x^2 = e^{-x}\frac{1}{D^3}x^2 = e^{-x}\iiint x^2\,(dx)^3 = e^{-x}\frac{x^5}{60}.$$

Hence, the required general solution is

$$y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + \frac{1}{60}x^5 e^{-x}.$$

**Example 40:** **Solve** $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2 e^{3x}.$ **[B.C.A. (Meerut) 2006, 04, 01]**

**Solution:** Here, the auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \quad (m-1)^2 = 0; \qquad [\therefore m = 1, 1]$$

Therefore, the complementary function is

$$\text{C.F.} = (c_1 + c_2 x)e^x.$$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} x^2 e^{3x} = \frac{1}{(D-1)^2} e^{3x} x^2$$

$$= e^{3x} \frac{1}{(D+3-1)^2} x^2 = e^{3x} \frac{1}{(D+2)^2} x^2$$

$$= e^{3x} \frac{1}{4\left(1+\frac{D}{2}\right)^2} x^2 = \frac{e^{3x}}{4}\left(1+\frac{D}{2}\right)^{-2} x^2$$

$$= \frac{e^{3x}}{4}\left[1 - 2.\frac{D}{2} + 3.\frac{D^2}{4} - .....\right] x^2$$

$$= \frac{e^{3x}}{4}\left[x^2 - D(x^2) + \frac{3}{4}D^2(x^2)\right],$$

all other terms vanish because they contain highest power of $D$

$$= \frac{e^{3x}}{4}\left(x^2 - 2x + \frac{3}{2}\right).$$

Hence, the required general solution is

$$y = (c_1 + c_2 x)\, e^x + \frac{e^{3x}}{4}\left(x^2 - 2x + \frac{3}{2}\right).$$

***Example 41:*** ***Solve*** $(D^2 - 2D + 5)y = e^{2x}\sin x.$ **[B.C.A. (Rohtak) 2010, 04]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 2m + 5 = 0$$

$$\Rightarrow \quad m = \frac{1}{2}[2 \pm \sqrt{4-20}] = 1 \pm 2i.$$

Therefore, the complementary function is

$$\text{C.F.} = e^x(c_1 \cos 2x + c_2 \sin 2x).$$

and

$$\text{P.I.} = \frac{1}{D^2 - 2D + 5} e^{2x} \sin x$$

$$= e^{2x} \frac{2}{(D+2)^2 - 2(D+2) + 5} \sin x$$

$$= e^{2x} \frac{1}{D^2 + 2D + 5} \sin x = e^{2x} \frac{1}{-1^2 + 2D + 5} \sin x$$

$$= e^{2x} \frac{1}{2D+4} \sin x = e^{2x} \frac{(2D-4)}{(2D+4)(2D-4)} \sin x$$

$$= e^{2x}(2D-4)\frac{1}{4D^2-16}\sin x = e^{2x}(2D-4)\frac{1}{\{4.(-1^2)-16\}}\sin x$$

$$= -\frac{e^{2x}}{20}[2D(\sin x) - 4\sin x] = -\frac{e^{2x}}{20}[2\cos x - 4\sin x]$$

$$= -\frac{e^{2x}}{10}(\cos x - 2\sin x).$$

Hence, the required general solution is

$$y = e^x(c_1\cos 2x + c_2\sin 2x) - \left(\frac{e^{2x}}{10}\right)(\cos x - 2\sin x).$$

***Example 42:*** ***Solve*** $(D^2-1)y = \cosh x \cos x.$ **[B.C. A. (Agra) 2005]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 1 = 0 \qquad [\therefore m = 1, -1]$$

Therefore, $\quad$ C.F. $= c_1 e^x + c_2 e^{-x}$.

Also, $\quad$ P.I. $= \dfrac{1}{D^2-1}\cosh x\cos x = \dfrac{1}{D^2-1}\dfrac{1}{2}(e^x + e^{-x})\cos x$

$$= \frac{1}{2}\frac{1}{D^2-1}e^x\cos x + \frac{1}{2}\frac{1}{D^2-1}e^{-x}\cos x$$

$$= \frac{e^x}{2}\cdot\frac{1}{(D+1)^2-1}\cos x + \frac{e^{-x}}{2}\frac{1}{(D-1)^2-1}\cos x$$

$$= \frac{e^x}{2}\frac{1}{D^2+2D}\cos x + \frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x$$

$$= \frac{e^x}{2}\frac{1}{-1^2+2D}\cos x + \frac{e^{-x}}{2}\frac{1}{(-1^2-2D)}\cos x$$

$$= \frac{e^x}{2}\frac{(2D+1)}{(2D-1)(2D+1)}\cos x - \frac{e^{-x}}{2}\frac{2D-1}{(2D+1)(2D-1)}\cos x$$

$$= \frac{e^x}{2}\frac{2D+1}{4D^2-1}\cos x - \frac{e^{-x}}{2}\frac{2D-1}{4D^2-1}\cos x$$

$$= \frac{e^x}{2}\frac{(2D+1)}{4(-1^2)-1}\cos x - \frac{e^{-x}}{2}\frac{(2D-1)}{4(-1^2)-1}\cos x$$

$$= -\frac{e^x}{10}(2D+1)\cos x + \frac{e^{-x}}{10}(2D-1)\cos x$$

$$= -\frac{e^x}{10}[2D(\cos x) + \cos x] + \frac{e^{-x}}{10}[2D(\cos x) - \cos x]$$

$$= -\frac{e^x}{10}[-2\sin x + \cos x] + \frac{e^{-x}}{10}[-2\sin x - \cos x]$$

$$= \frac{1}{5}.\left[\left(\frac{e^x - e^{-x}}{2}\right)2\sin x - \left(\frac{e^x + e^{-x}}{2}\right).\cos x\right]$$

$$= \frac{1}{5}[2\sinh x \sin x - \cosh x \cos x].$$

Hence, the required general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{5}(2\sinh x \sin x - \cosh x \cos x).$$

***Example 43:*** ***Solve*** $(D^2 + 4D - 12)y = e^{2x}(x-1).$ **[B.C.A. (Rohtak) 2008]**

***Solution:*** Here, the auxiliary equation is

$$m^2 + 4m - 12 = 0$$

$$\Rightarrow \quad (m-2)(m+6) = 0 \qquad [\therefore m = 2, -6]$$

$$\therefore \quad \text{C.F.} = c_1 e^{2x} + c_2 e^{-6x}$$

and $$\text{P.I.} = \frac{1}{D^2 + 4D - 12}e^{2x}(x-1) = \frac{1}{(D-2)(D+6)}e^{2x}(x-1)$$

$$= e^{2x}\frac{1}{(D+2-2)(D+2+6)}(x-1)$$

$$= e^{2x}\frac{1}{D(D+8)}(x-1) = e^{2x}\frac{1}{8D}\left(1+\frac{D}{8}\right)^{-1}(x-1)$$

$$= e^{2x}\frac{1}{8D}\left(1-\frac{D}{8}+\ldots\right)(x-1) = e^{2x}\frac{1}{8D}\left\{(x-1)-\frac{1}{8}\right\}$$

$$= e^{2x}\frac{1}{8}\left(\frac{1}{2}x^2 - \frac{9}{8}x\right) = \frac{e^{2x}}{16}\left(x^2 - \frac{9}{4}x\right).$$

Hence, the required general solution is

$$y = c_1 e^{2x} + c_2 e^{-6x} + \frac{e^{2x}}{16}\left(x^2 - \frac{9}{4}x\right).$$

***Example 44:*** ***Solve*** $(D^2 + 1)y = \cos x + e^x \sin x + xe^{2x}.$ **[B.C.A. (Bhopal) 2012, 07; B.C.A. (Rohilkhand) 2008, 06]**

***Solution:*** Here, the auxiliary equation is

$$m^2 + 1 = 0 \text{ or } m = \pm i.$$

$\therefore$ $\quad$ $\text{C.F.} = c_1 \cos x + c_2 \sin x$

and $\quad$ $\text{P.I.} = \dfrac{1}{D^2+1}(\cos x + e^x \sin x + xe^{2x})$

$$= \text{Real part of } \frac{1}{(D-i)(D+i)}e^{ix} + \frac{1}{D^2+1}e^x \sin x + \frac{1}{D^2+1}xe^{2x}$$

$$= \text{Real part of } \frac{1}{(D-i)(i+i)}e^{ix} + e^x \frac{1}{(D+1)^2+1}\sin x + e^{2x}\frac{1}{(D+2)^2+1}x$$

$$= \text{Real part of } \frac{e^{ix}}{2i}\frac{1}{D+i-i}1 + e^x \frac{1}{D^2+2D+2}\sin x + e^{2x}\frac{1}{D^2+4D+5}x$$

$$= \text{Real part of } \frac{e^{ix}}{2i}\frac{1}{D}(1) + e^x \frac{1}{-1^2+2D+2}\sin x + \frac{e^{2x}}{5}\left(1+\frac{4}{5}D+\frac{1}{5}D^2\right)^{-1} x$$

$$= \text{Real part of } \frac{e^{ix}}{2i}\int 1\,dx + e^x \frac{2D-1}{4D^2-1}\sin x + \frac{e^{2x}}{5}\left(1-\frac{4}{5}D-\dots\right)x$$

$$= \text{Real part of } \frac{xe^{ix}}{2i} + e^x \frac{(2\cos x - \sin x)}{4(-1^2)-1} + \frac{e^{2x}}{5}\left(x-\frac{4}{5}\right)$$

$$= \frac{1}{2}x\sin x - \frac{e^x}{5}(2\cos x - \sin x) + \frac{e^{2x}}{25}(5x-4).$$

Hence, the required solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.*, $\quad$ $y = c_1 \cos x + c_2 \sin x + \dfrac{x}{2}\sin x - \dfrac{e^x}{5}(2\cos x - \sin x) + \dfrac{e^{2x}}{25}(5x-4).$

***Example 45:*** ***Solve*** $\boldsymbol{\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x.}$ **[B.C.A. (Kashi) 2012, 08, 04]**

***Solution:*** Here, the auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$\Rightarrow$ $\quad$ $m^3 - m^2 - 2m^2 + 2m + 2m - 2 = 0$

$\Rightarrow$ $\quad$ $m^2(m-1) - 2m(m-1) + 2(m-1) = 0$

$\Rightarrow$ $\quad$ $(m-1)(m^2 - 2m + 2) = 0$

$\Rightarrow$ $\quad$ $m - 1 = 0$ and $m^2 - 2m + 2 = 0$

$\Rightarrow$ $\quad$ $m = 1, 1 \pm i.$

$\therefore$ $\quad$ $\text{C.F.} = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x)$

and $$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x)$$

$$= \frac{1}{(D-1)\{(D-1)^2+1\}}e^x + \frac{1}{D^3 - 3D^2 + 4D - 2}\cos x$$

$$= \frac{1}{(D-1)\{0^2+1\}}e^x + \frac{1}{D(-1^2) - 3(-1^2) + 4D - 2}\cos x$$

$$= \frac{1}{D-1}e^x + \frac{1}{3D+1}\cos x = e^x \frac{1}{D+1-1} 1 + \frac{3D-1}{9D^2-1}\cos x$$

$$= e^x \frac{1}{D}(1) + \frac{3D-1}{9(-1^2)-1}\cos x$$

$$= e^x \int 1\, dx - \frac{1}{10}(-3\sin x - \cos x)$$

$$= xe^x + \frac{1}{10}(3\sin x + \cos x).$$

Hence, the required general solution is

$$y = c_1 e^x + e^x(c_2 \cos x + c_3 \sin x) + xe^x + \frac{1}{10}(3\sin x + \cos x).$$

***Example 46:*** **Solve $(D^2 + 4)y = e^x + \sin 2x$.** **[B.C.A. (Indore) 2012, 07]**

***Solution:*** Here the auxiliary equation is

$$m^2 + 4 = 0.$$

Therefore, $\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$

and $$\text{P.I.} = \frac{1}{D^2+4}(e^x + \sin 2x) = \frac{1}{D^2+4}e^x + \frac{1}{D^2+4}\sin 2x$$

$$= \frac{1}{1+4}e^x + \left(-\frac{1}{2\times 2}x\cos 2x\right) = \frac{1}{5}e^x - \frac{1}{4}x\cos 2x.$$

Hence, the required general solution is

$$y = \text{C.F.} + \text{P.I.}\ i.e.,\ y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{1}{4}x\cos 2x.$$

## Exercise 10.8

**Solve the following differential equations:**

1. $(D^2 + D - 2)\, y = e^x$. **[B.C. A. (Agra) 2005]**

2. $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2 e^{3x}$. **[B.C.A. (Kanpur) 2010, 08, 06, 04]**

3. $\dfrac{d^2 y}{dx^2} - 4\dfrac{dy}{dx} + 4y = x^2 e^{3x}$.

4. $(D^2 + 2D + 4)\, y = e^x \sin 2x$. **[B.C. A. (Agra) 2004]**

5. $(D^2 - 1)\, y = e^x \cos x$. **[B.C.A. (Meerut) 2006]**

6. $(D^2 + 2D + 1)\, y = \dfrac{e^{-x}}{(x+2)}$.

7. $\dfrac{d^2 y}{dx^2} - 4\dfrac{dy}{dx} + 12y = (x-1)\, e^{2x}$.

8. $\dfrac{d^2 y}{dx^2} - y = \cosh x$. **[B.C.A. (Rohilkhand) 2010]**

9. $(D^2 + 4D + 4)\, y = e^{2x} - e^{-2x}$ or $2 \sinh 2x$. **[B.C.A. (Rohilkhand) 2009]**

10. $(D^2 - 3D - 2)\, y = 540 x^3 e^{-x}$.

11. $(D^2 + 2)\, y = x^2 e^{3x} + e^x \cos 2x$.

12. $\dfrac{d^3 y}{dx^3} + 3\dfrac{d^2 y}{dx^2} - 4y = x e^{-2x}$.

13. $(D^3 - 3D^2 + 3D - 1)\, y = x e^{-x} + e^x$. **[B.C.A. (Agra) 2005; B.C.A. (Meerut) 2003]**

14. $(D^3 - 7D - 6)\, y = e^{2x}(1 + x)$.

15. $\dfrac{d^4 y}{dx^4} - y = e^x \cos x$. **[B.C.A. (Lucknow) 2007; B.C.A. (Meerut) 2002]**

## Answers 10.8

1. $y = c_1 e^x + c_2 e^{-2x} + \dfrac{1}{3} x e^x$.
2. $y = (c_1 + c_2 x) e^{2x} + (1/8) e^{3x} (2x^2 - 4x + 3)$.
3. $y = (c_1 + c_2 x)\, e^{2x} + e^{3x} [x^2 - 4x + 6]$.
4. $y = e^{-x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + \dfrac{e^x}{73}(3 \sin 2x - 8 \cos 2x)$.
5. $y = c_1 e^x + c_2 e^{-x} + (1/5) e^x (2 \sin x - \cos x)$.
6. $y = (c_1 + c_2 x)\, e^{-x} + e^{-x} [x \log(x+2) - x + 2 \log(x+2)]$.

| | |
|---|---|
| 7. | $y = c_1 e^{2x} + c_2 e^{-6x} + \frac{1}{16} e^{2x}\{x^2 - (9/4/x)\}.$ |
| 8. | $y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} x \sinh x.$ |
| 9. | $y = e^{-2x}(c_1 + c_2 x) + \frac{1}{16} e^{2x} - \frac{1}{2} x^2 e^{-2x}.$ |
| 10. | $y = c_1 e^{2x} + (c_2 + c_3 x - 20x^2 - 20x^3 - 15x^4 - 9x^5)\, e^{-x}.$ |
| 11. | $y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x + \frac{1}{121} e^{3x}\left(11x^2 - 12x + \frac{50}{11}\right) + \frac{1}{17} e^{x}(4\sin 2x - \cos 2x).$ |
| 12. | $y = c_1 e^{x} + (c_2 + c_3 x)\, e^{-2x} - (1/18)(x^3 + x)e^{-2x}.$ |
| 13. | $y = e^{x}\left(c_1 + c_2 x + c_3 x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4\right).$ |
| 14. | $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{1}{12} e^{2x}\left(x + \frac{17}{12}\right).$ |
| 15. | $y = c_1 e^{x} + c_2 e^{-x} + c_3 \sin x + c_4 \cos x - 1/5\, e^{x} \cos x.$ |

## 10.13 To Find $\frac{1}{f(D^2)} \sin ax$ and $\frac{1}{f(D^2)} \cos ax$ when $f(-a^2) = 0$

If $D^2 + a^2$ is a factor of $f(D)$, the substitution of $-a^2$ for $D^2$ will make the denominator of the particular integral zero *i.e.*, $f(-a^2) = 0$.

Let $f(D) = (D^2 + a^2)\phi(D)$, where $\phi(-a^2) \neq 0$, then to evaluate

$\frac{1}{f(D)}(\sin ax \text{ or } \cos ax)$ we shall first find

$\frac{1}{D^2 + a^2}(\sin ax \text{ or } \cos ax)$ and

then apply the operator $\frac{1}{\phi(D^2)}$ to the result.

Now, $$\frac{1}{D^2 + a^2}\sin ax = \frac{1}{2i}\frac{1}{(D - ai)(D + ai)}(e^{iax} - e^{-iax}), \text{ as } \sin ax = (e^{iax} - e^{iax})/2i$$

$$= \frac{1}{2i}\left[\frac{xe^{iax}}{(1)!(2ai)} - \frac{xe^{iax}}{(1)!(-2ai)}\right],$$

$$= -\frac{x}{2a}\cdot\frac{e^{iax} + e^{-iax}}{2}.$$

Hence, $$\frac{1}{D^2 + a^2}\sin ax = -\frac{x}{2a}\cos ax.$$

Similarly, we can show that

$$\frac{1}{D^2 + a^2}\cos ax = \frac{x}{2a}\sin ax.$$

***Example 47:*** ***Solve*** $\dfrac{d^2y}{dx^2} + a^2y = \sin ay$.

***Solution:*** Here, the auxiliary equation is

$$m^2 + a^2 = 0; \qquad [\therefore\ m = \pm ia.]$$

$\therefore$ $$\text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

and $$\text{P.I.} = \frac{1}{D^2 + a^2} \sin ax = -\frac{x}{2a} \cos ax.$$

Hence, the required general solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{x}{2a} \cos ax.$$

***Example 48:*** ***Solve*** $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = \sin 3x.$

***Solution:*** Here, the auxiliary equation is

$$m^2 - 2m + 5 = 0; \qquad [\therefore\ m = 1 \pm 2i.]$$

$\therefore$ $$\text{C.F.} = e^x(c_1 \cos 2x + c_2 \sin 2x)$$

and $$\text{P.I.} = \frac{1}{D^2 - 2D + 5} \sin 3x = \frac{1}{-3^2 - 2D + 5} \sin 3x$$

$$= -\frac{1}{2} \frac{D-2}{D^2 - 4} \sin 3x = -\frac{1}{2}(D-2) \frac{1}{-3^2 - 4} \sin 3x$$

$$= \frac{1}{26}(D-2) \sin 3x = \frac{1}{26}(3 \cos 3x - 2 \sin 3x).$$

Hence, the required solution if the given differential equation is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.*, $$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{26}(3 \cos 3x - 2 \sin 3x).$$

***Example 49:*** ***Solve*** $(D^2 - 6D + 13)y = 8e^{3x} \sin 2x.$ **[B.C.A. (Aligarh) 2007]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 6m + 13 = 0$$

$\Rightarrow$ $$m = \frac{6 \pm \sqrt{36 - 52}}{2} = 3 \pm 2i.$$

$\therefore$ $$\text{C.F.} = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$$

and $$\text{P.I.} = \frac{1}{D^2 - 6D + 13} 8e^{3x} \sin 2x$$

$$= 8e^{3x} \frac{1}{(D+3)^2 - 6(D+3) + 13} \sin 2x$$

$$= 8e^{3x} \frac{1}{D^2 + 4} \sin 2x = 8e^{3x} \frac{-x}{2 \times 2} \cos 2x = -2xe^{3x} \cos 2x.$$

Hence, the required solution of the given differential equation is

$$y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - 2xe^{3x} \cos 2x.$$

***Example 50:*** ***Solve*** $(D^2 + 1)y = \sin x \ \ \sin 2x.$

***Solution:*** Here, the auxiliary equation is

$$m^2 + 1 = 0; \qquad [\therefore \ m = 0 \pm i]$$

Therefore, C.F. $= c_1 \cos x + c_2 \sin x.$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{D^2 + 1}(\sin x \sin 2x) = \frac{1}{D^2 + 1}\frac{1}{2}(2 \sin x \sin 2x)$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 + 1}(\cos x - \cos 3x)$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 + 1} \cos x - \frac{1}{2} \cdot \frac{1}{(D^2 + 1)} \cos 3x$$

$$= \frac{1}{2} \cdot \frac{x}{2(1)} \sin x - \frac{1}{2} \cdot \frac{1}{(-3^2 + 1)} \cos 3x = \frac{1}{4} x \sin x + \frac{1}{16} \cos 3x.$$

Hence, the complete solution of the given equation is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{4} x \sin x + \frac{1}{16} \cos 3x.$$

***Example 51:*** ***Solve*** $\dfrac{d^2 y}{dx^2} + a^2 y = \cos ax.$

***Solution:*** Here, the auxiliary equation is

$$m^2 + a^2 = 0, \qquad [\therefore \ m = \pm ai]$$

Therefore, C.F. $= c_1 \cos ax + c_2 \sin ax$

and $\text{P.I.} = \dfrac{1}{D^2 + a^2} \cos ax = \dfrac{x}{2a} \sin ax.$

Hence, the required general solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{2a} \sin ax.$$

**Example 52** ***Solve*** $\dfrac{d^3y}{dx^3} + a^2\dfrac{dy}{dx} = \sin ax.$ **[B.C.A. (I.G.N.O.U.) 2012, 06, 04]**

***Solution:*** Here, the auxiliary equation is

$$m^3 + a^2 m = 0$$

$$\Rightarrow \qquad m(m^2 + a^2) = 0 \qquad [\because m = 0, 0 \pm ai]$$

Therefore, the complementary function is

$$\text{C.F.} = c_1 + (c_2 \cos ax + c_3 \sin ax).$$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{D^3 + a^2 D}\sin ax = \frac{1}{D^2 + a^2}\cdot\frac{1}{D}\sin ax$$

$$= \frac{1}{D^2 + a^2}\left(\int \sin ax \, dx\right)$$

$$= \frac{1}{D^2 + a^2}\left(-\frac{\cos ax}{a}\right) = -\frac{1}{a}\cdot\frac{1}{D^2 + a^2}\cos ax$$

$$= -\frac{1}{a}\cdot\frac{x}{2a}\sin ax = -\frac{x}{2a^2}\sin ax.$$

Hence, the required general solution is

$$y = c_1 + (c_2 \cos ax + c_3 \sin ax) - \frac{x}{2a^2}\sin ax.$$

**Example 53:** ***Solve*** $(D^2 + 4)y = \sin^2 x.$ **[B.C.A. (Kurukshetra) 2012, 07]**

***Solution:*** Here, the auxiliary equation is

$$m^2 + 4 = 0; \qquad [\because \; m = \pm 2i.]$$

$$\therefore \qquad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

and

$$\text{P.I.} = \frac{1}{D^2 + 4}\sin^2 x = \frac{1}{D^2 + 4}\frac{1}{2}(1 - \cos 2x)$$

$$= \frac{1}{8}\left(1 + \frac{D^2}{4}\right)^{-1}.1 - \frac{1}{2}\frac{1}{D^2 + 4}\cos 2x$$

$$= \frac{1}{8}\left(1 - \frac{D^2}{4} + \ldots\right)1 - \frac{x}{8}\sin 2x = \frac{1}{8} - \frac{x}{8}\sin 2x.$$

Hence, the required general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8} - \frac{1}{8}x \sin 2x.$$

**Example 54:** ***Solve*** $(D^4 + D^2 + 1)y = e^{-x/2}\cos\left(\frac{1}{2}x\sqrt{3}\right)$.

**[B.C.A. (Lucknow) 2012, 07, 04, 02 ]**

**Solution:** Here, the auxiliary equation is

$$m^4 + m^2 + 1 = 0 \Rightarrow (m^2+1)^2 - m^2 = 0$$

$$\Rightarrow \quad (m^2 + 1 + m)(m^2 + 1 - m) = 0$$

$$\Rightarrow \quad m^2 + m + 1 = 0 \text{ and } m^2 - m + 1 = 0$$

$$\Rightarrow \quad m = \frac{1}{2}(-1 \pm i\sqrt{3}) \text{ and } m = \frac{1}{2}(1 \pm i\sqrt{3})$$

$$\therefore \quad \text{C.F.} = e^{-x/2}\left(c_1 \cos\frac{1}{2}x\sqrt{3} + c_2 \sin\frac{1}{2}x\sqrt{3}\right) + e^{x/2}\left(c_3\cos\frac{1}{2}x\sqrt{3} + c_4 \sin\frac{1}{2}x\sqrt{3}\right)$$

and
$$\text{P.I.} = \frac{1}{D^4 + D^2 + 1} e^{-x/2}\cos\left(\frac{1}{2}x\sqrt{3}\right)$$

$$= \text{Real part of } \frac{1}{D^4 + D^2 + 1} e^{-x/2} e^{ix\sqrt{3}/2}$$

$$= \text{Real part of } e^{ix\sqrt{3}/2} \frac{1}{\left(D + \frac{i\sqrt{3}}{2}\right)^4 + \left(D + \frac{i\sqrt{3}}{2}\right)^2 + 1} e^{-x/2}$$

$$= \text{Real part of } e^{ix\sqrt{3}/2} \frac{1}{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^4 + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2 + 1} e^{-x/2}$$

$$= \text{Real part of } e^{ix\sqrt{3}/2} \frac{-16e^{-x/2}}{(i\sqrt{3}-1)^4 + 4(i\sqrt{3}-1)^2 + 16}$$

$$= \text{Real part of } e^{ix\sqrt{3}/2} \frac{16e^{-x/2}}{-1} = -16\,e^{-x/2}\cos\left(\frac{1}{2}x\sqrt{3}\right).$$

Hence, the required solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.}$$

*i.e.,*
$$y = e^{-x/2}\left(c_1\cos\frac{1}{2}x\sqrt{3} + c_2\sin\frac{1}{2}x\sqrt{3}\right) + e^{x/2}\left(c_3\cos\frac{1}{2}x\sqrt{3} + c_4\sin\frac{1}{2}x\sqrt{3}\right) - 16e^{-x/2}\cos\left(\frac{1}{2}x\sqrt{3}\right).$$

**Example 55:** ***Solve*** $\frac{d^2y}{dx^2} - 4y = e^x \sin x$.

**Solution:** Here, the auxiliary equation is

$$m^2 - 4 = 0; \qquad [\therefore \ m = 2, -2.]$$

$\therefore$ $$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

and $$\text{P.I.} = \frac{1}{D^2 - 4} e^x \sin x = e^x \frac{1}{(D+1)^2 - 4} \sin x$$

$$= \frac{1}{2} e^x \frac{D+2}{D^2 - 4} \sin x = \frac{1}{2} e^x (D+2) \frac{1}{-1^2 - 4} \sin x$$

$$= -\frac{1}{10} e^x (D+2) \sin x = -\frac{1}{10} e^x (\cos x + 2 \sin x) .$$

Hence, the required solution of the given differential equation is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{10} e^x (\cos x + 2 \sin x) .$$

## Exercise 10.9

**Solve the following differential equations:**

1. $\dfrac{d^2 y}{dx^2} + 9y = \cos 3x.$ **[B.C.A. (Meerut) 2004, 02]**

2. $(D^2 + 1) y = e^x + \cos x.$ **[B.C.A. (Bhopal) 2010, 06, 04]**

3. $(D^2 + 1) y = e^{-x} + \cos x.$ **[B.C.A. (Kurukshetra) 2012, 07]**

4. $(D^2 + 4) y = \sin 2x.$ **[B.C.A. (Indore) 2012, 07]**

5. $(D^3 - 3D^2 + 9D - 27) y = \cos 3x.$ **[B.C.A. (Rohilkhand) 2011]**

6. $(D^4 - 1) y = \sin x.$ **[B.C.A. (Avadh) 2009, 05]**

## Answers 10.9

1. $y = c_1 \cos 3x + c_2 \sin 3x + \dfrac{1}{6} x \sin 3x.$
2. $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{2} (e^x + x \sin x) .$
3. $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{2} e^{-x} + \dfrac{1}{2} x \sin x.$
4. $y = c_1 \cos 2x + c_2 \sin 2x - \dfrac{1}{4} x \cos 2x.$
5. $y = c_1 e^{3x} + c_2 \cos 3x + c_3 \sin 3x - \dfrac{1}{36} x(\sin 3x + \cos 3x).$
6. $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \dfrac{1}{4} x \cos x.$

## 10.14 To Find $\frac{1}{f(D)}xV$, where V is any Function of x

By successive differentiation, we find

$$D(xV_1) = xDV_1 + V_1, \text{where } V_1 \text{ is any function if } x$$

$$D^2(xV_1) = xD^2V_1 + 2DV_1$$

$$\dots\dots\dots = \dots\dots\dots\dots\dots$$

$$D^n(xV_1) = x\,D^n\,V_1 + n\,D^{n-1}V_1, \text{by Leibnitz theorem}$$

$$= xD^nV_1 + \left(\frac{d}{dD}D^n\right)V_1$$

$\therefore$ $$f(D)xV_1 = xf(D)V_1 + f'(D)V_1.$$

Now, let $f(D)V_1 = V$, then obviously $V_1$ will also be a function of x, because $V_1$ is a function of x, and $V_1 = \frac{1}{f(D)}V$.

Substituting this value of $V_1$ in (1), we find

$$f(D)x\frac{1}{f()D}V_1 = xV + f'(D)\frac{1}{f(D)}V.$$

$\therefore$ $$\frac{1}{f(D)}\left\{f(D)x\frac{1}{f(D)}V\right\} = \frac{1}{f(D)}xV + \frac{1}{f(D)}\left\{f'(D)\frac{1}{f(D)}V\right\}$$

*i.e.*, $$x\frac{1}{f(D)}V = \frac{1}{f(D)}xV + \frac{1}{f(D)}, f'(D).\frac{1}{f(D)}V.$$

Hence, $$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V.$$

Similarly, we can find that

$$\frac{1}{f(D)}x^mV = x^m\frac{1}{f(D)}V + mx^{m-1}\left\{\frac{d}{dD}\frac{1}{f(D)}\right\}V + \frac{m(m-1)}{2!}x^{m-2}\left\{\frac{d}{dD^2}\frac{1}{f(D)}\right\}V + \dots$$

**Example 56:** ***Solve*** $(D^2 - 2D + 1)y = x \sin x$.

**[B.C.A. (Agra) 2010, 09, 03; B.C. A. (Kanpur) 2008, 04]**

**Solution:** Here, the auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \qquad (m-1)^2 = 0; \qquad [\therefore \ m = 1.]$$

Therefore, the complementary function is

$$\text{C.F.} = (c_1 + c_2 x)e^x.$$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} x \sin x$$

$$= x \frac{1}{D^2 - 2D + 1} \sin x - \frac{(2D - 2)}{(D^2 - 2D + 1)^2} \sin x$$

$$= x \frac{1}{-1 - 2D + 1} \sin x - \frac{(2D - 2)}{(-1 - 2D + 1)^2} \sin x$$

$$= -\frac{1}{2} x \frac{1}{D} \sin x - \frac{1}{4} \cdot \frac{1}{D^2} (2D - 2) \sin x$$

$$= -\frac{1}{2} x \int \sin x \, dx - \frac{1}{4} \cdot \frac{1}{D^2} [2D(\sin x) - 2 \sin x]$$

$$= \frac{1}{2} x \cos x - \frac{1}{2} \frac{1}{D^2} (\cos x - \sin x)$$

$$= \frac{1}{2} x \cos x - \frac{1}{2} \cdot \frac{1}{D} \int (\cos x - \sin x) \, dx$$

$$= \frac{1}{2} x \cos x - \frac{1}{2} \cdot \frac{1}{D} (\sin x + \cos x)$$

$$= \frac{1}{2} x \cos x - \frac{1}{2} \int (\sin x + \cos x) \, dx$$

$$= \frac{1}{2} x \cos x - \frac{1}{2} (-\cos x + \sin x) = \frac{1}{2} (x \cos x + \cos x - \sin x).$$

Hence, the required general solution is

$$y = (c_1 + c_2 x) \, e^x + \frac{1}{2} (x \cos x + \cos x - \sin x).$$

***Example 57:*** ***Solve*** $(D^4 + 2D^2 + 1)y = x^2 \cos x.$ **[B.C. A. (Bhopal) 2006, 03]**

***Solution:*** The auxiliary equation is

$$m^4 + 2m^2 + 1 = 0$$

$$\Rightarrow \qquad (m^2 + 1)^2 = 0; \qquad [\therefore\ m = \pm i, \pm i.]$$

Therefore, $\qquad$ C.F. $= (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x.$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x$$

$$= \text{Real part of} \frac{1}{(D^2+1)^2} x^2 e^{ix}, \qquad [\text{as } e^{ix} = \cos x + i\sin x]$$

$$= \text{Real part of } e^{ix} \frac{1}{\{(D+i)^2+1\}^2} x^2$$

$$= \text{Real part of } e^{ix} \frac{1}{(D^2+2iD)^2} x^2$$

$$= \text{Real part of } e^{ix} \frac{1}{-4D^2\left(1-\dfrac{iD}{2}\right)^2} x^2$$

$$= \text{Real part of} \frac{e^{ix}}{(-4)} \cdot \frac{1}{D^2}\left(1-\frac{iD}{2}\right)^{-2} x^2$$

$$= \text{Real part of} \frac{e^{ix}}{(-4)} \cdot \frac{1}{D^2}\left[1 + iD - \frac{3}{4}D^2 + \ldots\right] x^2$$

$$= \text{Real part of} \frac{e^{ix}}{(-4)} \cdot \frac{1}{D^2}\left[x^2 + 2ix - \frac{3}{2}\right], \text{ all other terms vanish}$$

$$= \text{Real part of} \frac{(\cos x + i\sin x)}{(-4)}\left[\frac{x^4}{12} - \frac{9x^2}{12} + \frac{ix^3}{3}\right]$$

$$= \frac{(9x^2 - x^4)}{48}\cos x + \frac{x^3}{12}\sin x.$$

Hence, the general solution of the given equation is

$$y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x - \frac{1}{12}x^3 \sin x + \frac{9x^2 - x^4}{48}\cos x.$$

***Example 58:*** ***Solve*** $\dfrac{d^4y}{dx^4}+\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-2y=x^2+e^x.$ **[B.C.A. (Rohtak) 2005]**

***Solution:*** Here, the auxiliary equation is

$$m^4+m^3+m^2-m-2=0$$

$$\Rightarrow \quad m^3(m+1)+m(m+1)-2(m+1)=0$$

$$\Rightarrow \quad (m+1)(m^3+m-2)=0$$

$$\Rightarrow \quad (m+1)\{m^2(m-1)+m(m-1)+2(m-1)\}=0$$

$$\Rightarrow \quad (m+1)(m-1)(m^2+m+2)=0$$

$$\Rightarrow \quad m=-1,1,-\frac{1}{2}\pm\frac{1}{2}\sqrt{7}i.$$

$$\therefore \quad \text{C.F.}=c_1e^{-x}+c_2e^{x}+e^{-x/2}(c_3\cos\sqrt{7}x/2+c_4\sin\sqrt{7}x/2).$$

Now, since $f(D)=D^4+D^3+D^2-D-2$, we therefore have

$$\frac{1}{f(D)}x^2=\frac{1}{D^4+D^3+D^2-D-2}x^2=\frac{-1}{2}\left(x^2-x+\frac{3}{2}\right)$$

$$\frac{1}{f(D)}e^x=\frac{1}{(D-1)(D+1)(D^2+D+2)}e^x$$

$$=\frac{1}{D-1}\left\{\frac{1}{(1+1)(1^2+1+2)}e^x\right\}$$

$$=\frac{1}{8}\cdot\frac{1}{D-1}e^x=\frac{1}{8}.e^x\frac{1}{(D+1)-1}(1)=\frac{1}{8}e^x\int dx=\frac{1}{8}e^x.x.$$

Thus, the particular integral is

$$\text{P.I.}=-\frac{1}{2}\left(x^2-x+\frac{3}{2}\right)+\frac{1}{8}xe^x.$$

Hence, the required general solution is

$$y=\text{C.F.}+\text{P.I.}$$

*i.e.,* $y=c_1e^{-x}+c_2e^{x}+e^{-x/2}(c_3\cos\sqrt{7}x/2+c_4\sin\sqrt{7}x/2)-\dfrac{1}{2}\left(x^2-x+\dfrac{3}{2}\right)+\dfrac{1}{8}xe^x.$

**Example 59:** ***Solve*** $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x.$ **[B.C. A. (Rohilkhand) 2004]**

***Solution:*** The auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \qquad (m-1)^2 = 0; \qquad [\therefore m = 1, 1.]$$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 x)e^x.$$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} xe^x \sin x = \frac{1}{(D-1)^2} e^x.(x \sin x)$$

$$= e^x . \frac{1}{(D+1-1)^2}(x \sin x) = e^x . \frac{1}{D^2}(x \sin x)$$

$$= e^x . \frac{1}{D}\int x \sin x \, dx = e^x . \frac{1}{D}(-x \cos x + \sin x)$$

$$= e^x \int (-x \cos x + \sin x) dx = e^x \{(-x \sin x - \cos x) - \cos x\}$$

$$= e^x (x \sin x + 2 \cos x).$$

**Example 60:** ***Solve*** $(D^2 - 1)y = x^2 \sin x.$ **[B.C. A. (Kurukshetra) 2006]**

***Solution:*** Here, the auxiliary equation is

$$m^2 - 1 = 0; \qquad m = \pm 1.$$

$$\therefore \qquad \text{C.F.} = c_1 e^x + c_2 e^{-x}.$$

Also, $$\text{P.I.} = \frac{1}{D^2 - 1} x^2 \sin x = \text{Imaginary part of } \frac{1}{D^2 - 1} x^2 e^{ix}$$

$$= \text{I.P. of } e^{ix} \frac{1}{\{(D+i)^2 - 1\}} x^2$$

$$= \text{I.P. of } \frac{1}{D^2 + 2iD - 2} x^2, \text{ as } i^2 = -1$$

$$= \text{I.P. of } -\frac{1}{2} e^{ix} [1 - iD - \frac{1}{2} D^2]^{-1} x^2$$

$$= \text{I.P. of } -\frac{1}{2} e^{ix} [1 + iD - \frac{1}{2} D^2 + \ldots] \, x^2, \text{ as } i^2 = -1$$

$$= \text{I.P. of } -\frac{1}{2}e^{ix}[x^2 + 2ix - 1]$$

$$= \text{I.P. of } [-\frac{1}{2}(\cos x + i\sin x)(x^2 + 2ix - 1)]$$

$$= \text{I.P. of } -x\cos x - \frac{1}{2}\times(x^2 - 1)\sin x.$$

Hence, the general solution of the given equation is

$$y = c_1e^x + c_2e^{-x} - x\cos x - \frac{1}{2}(x^2 - 1)\sin x.$$

***Example 61:*** **Solve** $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 8x^2e^{2x}\sin 2x.$ **[B.C.A. (Rohtak) 2007]**

***Solution:*** The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

*i.e.*, $(m-2)^2 = 0;$ $[\therefore m = 2, 2.]$

$\therefore$ $\text{C.F.} = (c_1 + c_2x)e^{2x}.$

Also, the particular integral is

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4}8x^2e^{2x}\sin 2x = \frac{1}{(D-2)}e^{2x}.(8x^2\sin 2x)$$

$$= e^{2x}.\frac{1}{(D+2-2)^2}(8x^2\sin 2x) = e^{2x}.\frac{1}{D^2}(8x^2\sin 2x)$$

$$= e^{2x}.\frac{1}{D}\int 8x^2\sin 2x\,dx$$

$$= e^{2x}\int(-4x^2\cos 2x + 4x\sin 2x + 2\cos 2x)\,dx$$

$$= e^{2x}(-2x^2\sin 2x - x - 4x\cos 2x + 3\sin 2x).$$

Hence, the required general solution is

$$y = (c_1 + c_2x)e^{2x} + (3\sin 2x - 4x\cos 2x - 2x^2\sin 2x)e^{2x}.$$

## Exercise 10.10

**Solve the following differential equations:**

**1.** $(D^2 + m^2)\, y = x \cos mx.$

**2.** $(D^2 + 3D + 2)\, y = x \cos 2x.$

**3.** $(D^2 - 1)\, y = x^2 \cos x.$ [**Hint:** Write $x^2 \cos x = x.(x \cos x)$].

**4.** $(D^4 - 1)\, y = x \sin x.$

## Answers 10.10

| | |
|---|---|
| 1. | $y = c_1 \cos mx + c_2 \sin mx + \dfrac{x^2}{4m} \sin mx + \dfrac{x^2}{4m}.$ |
| 2. | $y = c_1 e^{-x} + c_2 e^{-2x} + \dfrac{1}{20} x(\sin 3x - \cos 2x) + \dfrac{3}{25} \cos 2x - \dfrac{7}{200} \sin 2x.$ |
| 3. | $y = c_1 e^{x} + c_2 e^{-x} + x \sin x + \dfrac{1}{2}(1 - x^2) \cos x.$ |
| 4. | $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \dfrac{1}{8}(x^2 \cos x - 3 \sin x - x^2 - y^2).$ |

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