# FUZZY LOGIC-II
# MODULE 8

Dr. R.B.Ghongade,
VIIT, Pune-411048



## Classical and Fuzzy Relations

- Relation is of fundamental importance in all-engineering, science, and mathematically based fields
- Associated with graph theory, a subject of wide impact in design and data manipulation
- Intimately involved in logic, approximate reasoning, classification, rule-based systems, pattern recognition, and control
- Relations represent the mapping of the sets
- Degrees of association can be represented by membership grades in a fuzzy relation by membership grades in a fuzzy relation in the same way as degrees of set membership are represented in the fuzzy set

# Cartesian Product of Relation

- An ordered sequence of *n* elements is called as ordered *n*-tuple
- The ordered sequence is in the form of $a_1, a_2, \ldots, a_n$
- For the crisp sets $A_1, A_2, \ldots, A_n$, the set of *n*-tuples $a_1, a_2, \ldots, a_n$, where $a_1 \in A_1$, $a_2 \in A_2 \ldots$, $a_n \in A_n$,is called the Cartesian product of $A_1, A_2, \ldots, A_n$
- Cartesian product is denoted by $A_1 \times A_2 \times \ldots \times A_n$
- The first element in each pair is a member of $x$ and the second element is a member of $y$ formally

$$x \times y = \{(x, y) | x \in X \ and \ y \in Y\},$$
$$If \ x \neq y \ then \ x \times y \neq y \times x$$

**Example:** The elements in two sets *A* and *B* are given as $A = \{0, 1\}$ and $B = \{e, f, g\}$, find the Cartesian product $A \times B, B \times A, A \times A, B \times B$

**Solution:**

- $A \times B = \{(0, e), (0, f), (0, g), (1, e), (1, f), (1, g)\}$
- $B \times A = \{(e, 0), (e, 1), (f, 0), (f, 1), (g, 0), (g1)\},$
- $A \times A = A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\},$
- $B \times B = B^2 = \{(e, e), (e, f), (e, g), (f, e), (f, f), (f, g), (g, e), (g, f), (g, g)\}.$

# Classical Relations

- A relation among classical sets $x_1, x_2, \dots, x_n \; and \; y_1, y_2, \dots, y_n$ is a subset of the Cartesian product
- Denoted either by $R$ or by the abbreviated form

$$X \times Y = \{(x, y)/x \in X, y \in Y\}$$

- The strength of the relationship between ordered pairs of elements in each universe is measured by the characteristic function denoted by $\chi$, where a value of unity is associated with complete relationship and a value of zero is associated with no relationship

$$\chi_{X\times Y}(x, y) = f(x) = \begin{cases} 1 (x, y) \in X \times Y \\ 0 (x, y) \notin X \times Y \end{cases}$$



- When the universe or the set are finite, a matrix called as relation matrix can conveniently represent the relation
- A two-dimensional matrix represents the binary relation
- **Example 1**: If $X = \{2, 4, 6\}$ and $Y = \{p, q, r\}$, if they both are related to each other entirely, then the relation between them can be given by:

| | p | q | r |
|---|---|---|---|
| 2 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 |

- **Example 2:** Let *R* be a relation among the three sets

$$X = \{Hindi, English\},$$
$$Y = \{Dollar, Euro, Pound, Rupees\}$$
$$Z = \{India, Nepal, United\ States, Canada\}$$
$$R\ (x, y, z) = \{Hindi, Rupees, India\}$$
$$\{Hindi, Rupees, Nepal\}$$
$$\{English, Dollar, Canada\}$$
$$\{English, Dollar, United\ States\}$$

The relation can be represented as follows:

| | India | Nepal | US | Canada |
|---|---|---|---|---|
| Dollar | 0 | 0 | 0 | 0 |
| Euro | 0 | 0 | 0 | 0 |
| Pound | 0 | 0 | 0 | 0 |
| Rupee | 1 | 1 | 0 | 0 |
| Hindi | | | | |

| | India | Nepal | US | Canada |
|---|---|---|---|---|
| Dollar | 0 | 0 | 1 | 1 |
| Euro | 0 | 0 | 0 | 0 |
| Pound | 0 | 0 | 0 | 0 |
| Rupee | 0 | 0 | 0 | 0 |
| English | | | | |



# Cardinality of Crisp Relation

- Suppose $n$ elements of the universe $X$ are related to $m$ elements of the universe $Y$
- If the cardinality of $X$ is $n_x$ and the cardinality of $Y$ is $n_y$, then the cardinality of the relation $R$, between these two universe

$$n_{x \times y} = n_x \times n_y$$

- The cardinality of the power set describing this relation, $P(X \times Y)$ is then

$$n_P(x \times y) = 2^{n_x n_y}$$

# Operations on Crisp Relation

- ***Union***
  $R \cup S = \chi_{R\cup S}(x,y): \chi_{R\cup S}(x,y) = max[\chi_R(x,y), \chi_S(x,y)]$
- ***Intersection***
  $R \cap S = \chi_{R\cap S}(x,y): \chi_{R\cap S}(x,y) = min[\chi_R(x,y), \chi_S(x,y)]$
- ***Complement***
  $$\bar{R} = \chi_{\bar{R}}(x,y) \; = 1 - \chi_R(x,y)$$
- ***Containment***
  $$R \subset S = \chi_R(x,y) \leq \chi_S(x,y)$$

# Properties of Crisp Relations

- The properties of commutativity, associativity, distributivity, involution(complementation) , and idempotency for the classical sets also hold good for crisp relation

# Composition

- Let $R$ be relation that relates elements from universe $X$ to universe $Y$
- Let $S$ be the relation that relates elements from universe $Y$ to universe $Z$
- Let $T$ relate the same element in universe that $R$ contains to the same elements in the universe $Z$ that $S$ contains
- The two methods of the composition operations are:
  - Max–min composition
  - Max–product composition

**Max–min composition**

- The max–min composition is defined by the set-theoretic and membership function-theoretic expressions:

$$T = R \circ S$$

$$\chi_T(x,z) = \underbrace{max}_{y \in Y}\ [\min(\chi_R(x,y), \chi_S(y,z))]$$

**Max–product composition**

- The max–product composition is defined by the set-theoretic and membership function-theoretic expressions:

$$T = R \circ S$$

$$\chi_T(x,z) = \underbrace{max}_{y \in Y}(\chi_R(x,y) \cdot \chi_S(y,z))$$



- **Example**: Using max–min composition find relation between $R$ and $S$:

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}$$

Solution:

$\mu_T(x_1, z_1)$ = max (min (1, 0), min (1, 1), min (0, 1))= max [0, 1, 0] = 1,

$\mu_T(x_1, z_2)$ = max (min (1, 1), min (1, 0), min (0, 1))= max [1, 0, 0] = 1,

$\mu_T(x_2, z_1)$ = max (min (0, 0), min (0, 1), min (1, 1))= max [0, 0, 1] = 1,

$\mu_T(x_2, z_2)$ = max (min (0, 1), min (0, 0), min (1, 1))= max [0, 0, 1] = 1,

$\mu_T(x_3, z_1)$ = max (min (0, 0), min (1, 1), min (0, 1))= max [0, 1, 0] = 1,

$\mu_T(x_3, z_2)$ = max (min (0, 1), min (1, 0), min (0, 1))= max [0, 0, 0] = 0,

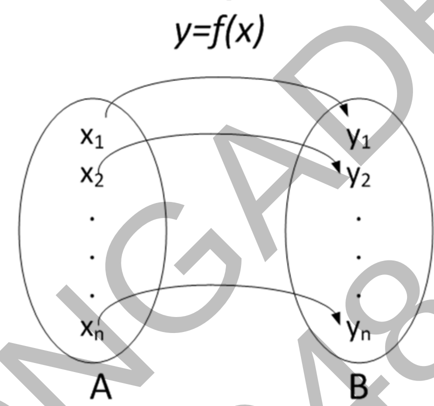
$$\therefore R \circ S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

## Extension principle of Fuzzy Sets

- Consider the **one to one** mapping from fuzzy set A to fuzzy set B, given by a function $y = f(x)$ such that

$$y_1 = f(x_1)$$
$$y_2 = f(x_2)$$
$$\vdots$$
$$y_n = f(x_n)$$

- In terms of membership grades:



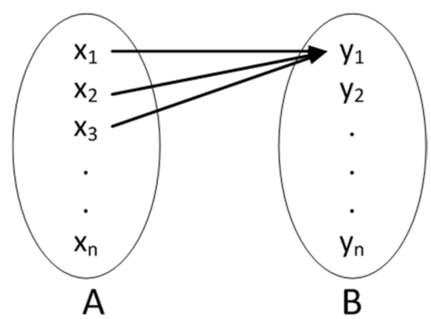

$$A = \left\{ \frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \dots, \frac{\mu_A(x_n)}{x_n} \right\}$$
$$B = \left\{ \frac{\mu_B(y_1)}{y_1}, \frac{\mu_B(y_2)}{y_2}, \dots, \frac{\mu_B(y_n)}{y_n} \right\}$$

where

$$\mu_B(y_i) = \mu_A(x_i)$$

## Extension principle of Fuzzy Sets

- Consider the **many to one** mapping from fuzzy set A to fuzzy set B, given by a function $y = f(x)$



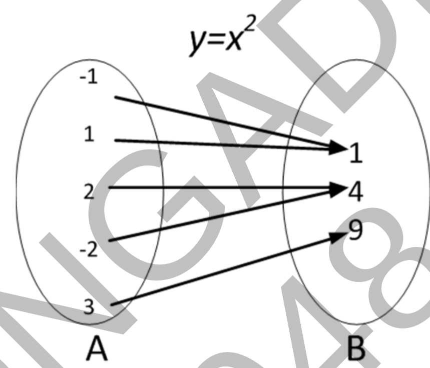

$$A = \left\{ \frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \dots, \frac{\mu_A(x_n)}{x_n} \right\}$$
$$B = \left\{ \frac{\mu_B(y_1)}{y_1}, \frac{\mu_B(y_2)}{y_2}, \dots, \frac{\mu_B(y_n)}{y_n} \right\}$$

where

$$\mu_B(y_i) = max\left[\mu_A(x_j): x_j \in f^{-1}(y_i)\right]$$

# Example: many to one mapping

- Consider two fuzzy sets A and B with the mapping A→B
- Given
  $A = \left\{\frac{0.2}{-1}, \frac{0.6}{1}, \frac{0.4}{-2}, \frac{0.8}{2}, \frac{0.9}{3}\right\}$
  determine fuzzy set B




Answer:

$$\mu_B(1) = \max(0.2, 0.6) = 0.6$$
$$\mu_B(4) = \max(0.4, 0.8) = 0.8$$
$$\mu_B(9) = \max(0.9) = 0.9$$
$$\therefore B = f(A) = \left\{\frac{0.6}{1}, \frac{0.8}{4}, \frac{0.9}{9}\right\}$$

# Fuzzy Relations

- $X$ and $Y$ are two universal sets. The fuzzy relation $R(x, y)$ is given as:

$$R(x,y) = \left\{\frac{\mu_R(x,y)}{(x,y)} \middle| (x,y) \in X \times Y\right\}$$

**Example:** Let $X = \{1,2,3\}$ $and$ $Y = \{1,2\}$. If membership function associated with each ordered pair $(x, y)$ is given by $\mu_R(x, y) = \exp(-(x - y)^2)$ then derive the fuzzy relation $R(x, y)$

**Solution:** The fuzzy relation $R(x, y)$ can be defined in two ways, one way using the standard nomenclature is :

$$R(x,y) = \left\{\frac{e^{-(1-1)^2}}{(1,1)}, \frac{e^{-(1-2)^2}}{(1,2)}, \frac{e^{-(2-1)^2}}{(2,1)}, \frac{e^{-(2-2)^2}}{(2,2)}, \frac{e^{-(3-1)^2}}{(3,1)}, \frac{e^{-(3-2)^2}}{(3,2)}\right\}$$

$$\therefore R(x,y) = \left\{\frac{1.0}{(1,1)}, \frac{0.43}{(1,2)}, \frac{0.43}{(2,1)}, \frac{1.0}{(2,2)}, \frac{0.16}{(3,1)}, \frac{0.43}{(3,2)}\right\}$$

- The second method of defining relation is through the relation matrix

$$\begin{array}{cc} & Y \rightarrow \\ & \begin{array}{cc} 1 & 2 \end{array} \\ X \downarrow \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1.0 & 0.43 \\ 0.43 & 1.0 \\ 0.16 & 0.43 \end{bmatrix} \end{array}$$

- Since the membership function describes the closeness between X and Y, it is obvious that a higher value implies stronger relation

# Fuzzy Relations: Formal Definition

- Definition: A fuzzy relation is a fuzzy set defined in the Cartesian product of crisp sets $X_1, X_2, \ldots, X_n$, a fuzzy relation $R(x_1, x_2, \ldots, x_n)$ thus is defined as

$$R(x_1, x_2, \ldots, x_n) = \left\{ \frac{\mu_R(x_1, x_2, \ldots, x_n)}{(x_1, x_2, \ldots, x_n)} \middle| (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n \right\}$$

Where

$$\mu_R : X_1 \times X_2 \times \cdots \times X_n \rightarrow [0,1]$$

## Operations on Fuzzy relations

- Fuzzy relations are very important because they can describe interactions between variables
- Let $R$ and $S$ be two binary fuzzy relations on $X \times Y$
- **Intersection:**
  - The intersection of R and S is defined as

$$(R \wedge S)(u, v) = \min\big(R(u, v), S(u, v)\big)$$

  Note that $R: X \times Y \to [0,1]$

Means that the domain of $R$is the whole Cartesian product $X \times Y$

- **Union:**
  - The union of R and S is defined as

$$(R \vee S)(u, v) = \max\big(R(u, v), S(u, v)\big)$$



- **Projection:**
  - A fuzzy relation $R$ is usually defined in the Cartesian space $X \times Y$
  - Often a projection of this relation on any of the sets $X$ or $Y$ may become useful for further information processing
  - The projection of $R(x, y)$ on $X$ , denoted by $R1$, is given by:

$$\mu_{R1}(x) = \underbrace{max}_{y \in Y} [\mu_R(x, y)]$$

  - The projection of $R(x, y)$ on $Y$ , denoted by $R2$, is given by:

$$\mu_{R2}(y) = \underbrace{max}_{x \in X} [\mu_R(x, y)]$$

## Example of projection

- Consider R(x,y) from previous example:

| | 1 | 2 |
|---|---|---|
| 1 | 1.0 | 0.43 |
| 2 | 0.43 | 1.0 |
| 3 | 0.16 | 0.43 |

X-projection

| |
|---|
| 1.0 |
| 1.0 |
| 0.43 |

For X-projection the maximum value in each **row** is retained

Y-projection

| | |
|---|---|
| 1.0 | 1.0 |

For Y-projection the maximum value in each **column** is retained

## Projection: Formal Definition

- Projection of a fuzzy relation $R(x_1, x_2, \dots, x_n)$ on to $X_i \times X_j \times \cdots \times X_k$ for any $i, j$ and $k$ in $[1, n]$ is defined as a fuzzy relation $R_P$ where

$$R_P = \left\{ \underbrace{max}_{x_i \in X_i \dots\dots x_k \in X_k} \frac{\mu_{R_P}(x_i, x_j, \dots, x_k)}{x_i, x_j, \dots, x_k} \right\}$$

# FUZZY IF-THEN RULES

- Linguistic Variables
  - Conventional techniques for system analysis are intrinsically unsuited for dealing with humanistic systems, whose behavior is strongly influenced by human judgment, perception, and emotions
  - Principle of incompatibility:
    - "As the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance become almost mutually exclusive characteristics"

# Linguistic variables and other related terminology

- A linguistic variable is characterized by a quintuple ($\boldsymbol{x}$, $\boldsymbol{T}(\boldsymbol{x}), \boldsymbol{X}, \boldsymbol{G}, \boldsymbol{M}$) in which

  x is the name of the variable

  T(x) is the term set of x—that is, the set of its linguistic values or linguistic terms;

  X is the universe of discourse;

  G is a syntactic rule which generates the terms in T(x); and

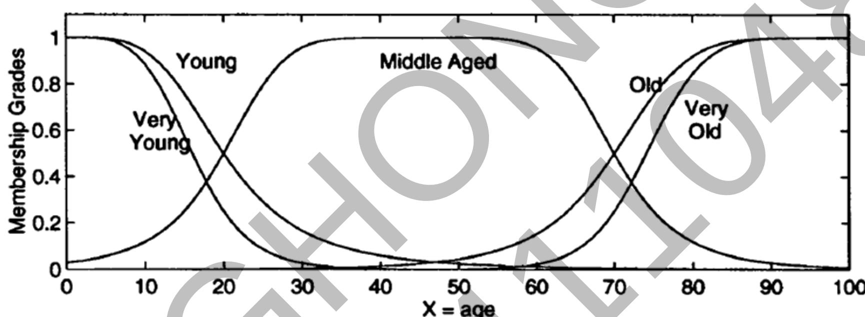
  M is a semantic rule which associates with each linguistic value A its meaning M(A), where M(A) denotes a fuzzy set in X.

# Example

- If age is interpreted as a linguistic variable, then its term set $T(age)$ could be

$$T(age) = \{young, not\ young, very\ young, .., \\ middle\ aged, not\ middle\ aged, \dots, \\ old, not\ old, very\ old, more\ or\ less\ old, not\ very\ old, \dots, \\ not\ very\ young, not\ very\ old\}$$

- Where each term in $T(age)$ is characterized by a fuzzy set of a universe of discourse $X = [0,100]$




- Usually we use "age is young" to denote the assignment of the linguistic value "young" to the linguistic variable age
- By contrast, when age is interpreted as a numerical variable, we use the expression "age = 20" instead to assign the numerical value "20" to the numerical variable age
- The syntactic rule refers to the way the linguistic values in the term set $T(age)$ are generated
- The semantic rule defines the membership function of each linguistic value of the term set
- The term set consists of several **primary terms** (young, middle aged, old) modified by the **negation** ("not") and/or the **hedges** (very, more or less, quite, extremely, and so forth), and then linked by **connectives** such as and, or, either, and neither.

## Concentration and dilation of linguistic values

- Let A be a linguistic value characterized by a fuzzy set with membership function $\mu_A(\cdot)$
- Then $A^k$ is interpreted as a modified version of the original linguistic value expressed as

$$A^k = \int_X \frac{[\mu_A(x)]^k}{x}$$

- The operation of concentration is defined as

$$CON(A) = A^2$$

- Dilation is expressed by

$$DIL(A) = A^{0.5}$$

- Conventionally, we take CON(A) and DIL(A) to be the results of applying the **hedges** "very" and "more or less", respectively, to the linguistic term A

- We can interpret the negation operator NOT and the connectives AND and OR as

$$NOT(A) = \bar{A} = \int_X \frac{1 - \mu_A(x)}{x}$$

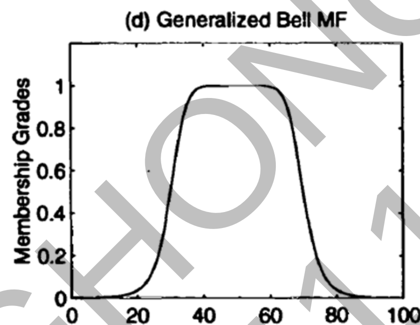
$$A\ AND\ B = A \cap B = \int_X \frac{\mu_A(x) \wedge \mu_B(x)}{x}$$

$$A\ OR\ B = A \cup B = \int_X \frac{\mu_A(x) \vee \mu_B(x)}{x}$$

where A and B are two linguistic values whose meanings are defined by $\mu_A(\cdot)$and $\mu_B(\cdot)$

## One more MF: Generalized bell

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}},$$




## Constructing MFs for composite linguistic terms

- Let the meanings of the linguistic terms young and old be defined by the following membership functions:

$$\mu_{young}(x) = bell(x, 20, 2, 0) = \frac{1}{1 + \left(\frac{x}{20}\right)^4}$$

$$\mu_{old}(x) = bell(x, 30, 3, 100) = \frac{1}{1 + \left(\frac{x-100}{30}\right)^6}$$

Where $x$ is the age of a given person, with the interval [0,100] as the universe of discourse

- We can construct MFs for the following composite linguistic terms:
- $more\ or\ less\ old = DIL(old) = old^{0.5}$

$$= \int_X \frac{\sqrt{\dfrac{1}{1+\left(\dfrac{x-100}{30}\right)^6}}}{x}$$

- $not\ young\ and\ not\ old = \overline{young} \cap \overline{old}$

$$= \int_X \frac{\left[1 - \dfrac{1}{1+\left(\dfrac{x}{20}\right)^4}\right] \wedge \left[1 - \dfrac{1}{1+\left(\dfrac{x-100}{30}\right)^6}\right]}{x}$$

- $young\ but\ not\ too\ young = young \cap \overline{(young)^2}$

$$= \int_X \frac{\left[\dfrac{1}{1+\left(\dfrac{x}{20}\right)^4}\right] \wedge \left[1 - \left(\dfrac{1}{1+\left(\dfrac{x}{20}\right)^4}\right)^2\right]}{x}$$

- $extremely\ old = CON(CON(CON(old))) = (((old)^2)^2)^2$

$$= \int_X \frac{\left(\dfrac{1}{1+\left(\dfrac{x-100}{30}\right)^6}\right)^8}{x}$$

MATLAB DEMO

## Fuzzy If-Then Rules

- A fuzzy if-then rule (also known as fuzzy rule, fuzzy implication, or fuzzy conditional statement) assumes the form:

$$\boldsymbol{if\ x}:\ \boldsymbol{is\ A\ then\ y\ is\ B}$$

- where A and B are linguistic values defined by fuzzy sets on universes of discourse X and Y, respectively
- $“x\ is\ A"$ is called the antecedent or premise, while "$y\ is\ B$" is called the consequence or conclusion
- Abbreviated as $\boldsymbol{A} \rightarrow \boldsymbol{B}$

## Fuzzy Intersection Re-visited

- The intersection of two fuzzy sets A and B is specified in general by a function

$$T : [0,1] \times [0,1] \rightarrow [0,1],$$

which aggregates two membership grades as follows:

$$\mu_{A \cap B}(x) = T\big(\mu_A(x), \mu_B(x)\big) = \mu_A(x) * \mu_B(x)$$

where * is a binary operator for the function T

- This class of fuzzy intersection operators, which are usually referred to as **T-norm (triangular norm) operators**
- Four of the most frequently used T-norm operators are:

1. *Minimum*: $T_{min}(a,b) = min(a,b) = a \wedge b$
2. *Algebraic product*: $T_{ap}(a,b) = ab$
3. *Bounded product*: $T_{bp}(a,b)| = 0 \vee (a + b - 1)$
4. *Drastic product*: $T_{dp}(a,b) = \begin{cases} a, & if\ b = 1 \\ b, & if\ a = 1 \\ 0, & if\ a,b < 1 \end{cases}$

## Fuzzy Union Re-visited

- The fuzzy union operator is specified in general by a function $S : [0,1] \times [0,1] \rightarrow [0,1]$

$$\mu_{A \cup B}(x) = S\big(\mu_A(x), \mu_B(x)\big) = \mu_A(x) + \mu_B(x)$$

where + is a binary operator for the function S.

- This class of fuzzy union operators, which are often referred to as **T-conorm (or S-norm) operators**
- Four of the most frequently used T-conorm operators are:

1. *Maximum*: $S_{max}(a,b) = max(a,b) = a \vee b$
2. *Algebraic sum*: $S_{as}(a,b) = a + b - ab$
3. *Bounded sum*: $S_{bs}(a,b)| = 1 \wedge (a + b)$
4. *Drastic sum*: $S_{ds}(a,b) = \begin{cases} a, & if\ b = 0 \\ b, & if\ a = 0 \\ 1, & if\ a,b > 1 \end{cases}$

## Formalizing $\boldsymbol{A} \rightarrow \boldsymbol{B}$ approach 1

- $\boldsymbol{A} \rightarrow \boldsymbol{B}$ means "A is coupled with B"
- Interpreted as:

$$R = A \rightarrow B = A \times B = \int_{X \times Y} \frac{\mu_A(x) * \mu_B(y)}{(x, y)}$$

where * is a T-norm operator and $A \rightarrow B$ is used again to represent the fuzzy relation $R$

## Formalizing $\boldsymbol{A} \rightarrow \boldsymbol{B}$ approach 2

- $\boldsymbol{A} \rightarrow \boldsymbol{B}$ means "A entails B"
- Entails: "To have, impose, or require as a necessary accompaniment or consequence"
- Can be written as four different formulas:
  1. Material implication: $R = A \rightarrow B = \bar{A} \cup B$
  2. Propositional calculus: $R = A \rightarrow B = \bar{A} \cup (A \cap B)$
  3. Extended propositional calculus: $R = A \rightarrow B = (\bar{A} \cap \bar{B}) \cup B)$
  4. Generalization of modus ponens:

  $\mu_R(x, y) = sup\{c | \mu_A(x) * \mu_B(y) and\ 0 \le c \le 1\}$

- Based on these two interpretations and various T-norm and T-conorm operators, a number of qualified methods can be formulated to calculate the fuzzy relation $R = A \rightarrow B$
- $R$ can be viewed as a fuzzy set with a two-dimensional MF

$$\mu_R(x,y) = f(\mu_A(x), \mu_B(y)) = f(a,b)$$

with a $= \mu_A(x)$, b $= \mu_B(y)$, where the function $f$, called the **fuzzy implication function**, performs the task of transforming the membership grades of $x \; in \; A$ and $y \; in \; B$ into those of $(x,y) \; in \; A \rightarrow B$.

# Using approach 1

- Four different fuzzy relations $A \rightarrow B$ result from employing four of the most commonly used T-norm

1. $R_m = A \times B = \int_{X \times Y} \frac{\mu_A(x) \wedge \mu_B(y)}{(x,y)}$ or $f_m(a,b) = a \wedge b$
2. $R_p = A \times B = \int_{X \times Y} \frac{\mu_A(x) \; \mu_B(y)}{(x,y)}$ or $f_p(a,b) = ab$
3. $R_{bp} = A \times B = \int_{X \times Y} \frac{\mu_A(x) \odot \mu_B(y)}{(x,y)}$ or $f_{bp}(a,b) = 0 \wedge (a + b - 1)$
4. $R_{dp} = A \times B = \int_{X \times Y} \frac{\mu_A(x) \hat{\cdot} \mu_B(y)}{(x,y)}$ or

$$f_{dp}(a,b) = a \hat{\cdot} b = \begin{cases} a, & if \; b = 1 \\ b, & if \; a = 1 \\ 0, & if \; a, b < 1 \end{cases}$$

## Using approach 2

- Four of the most commonly used fuzzy implication functions:

1. $R_a = \bar{A} \cup B = \int_{X\times Y} \frac{1 \wedge (\ 1-\mu_A(x)+\mu_B(y))}{(x,y)}$ or $f_m(a,b) = 1 \wedge (1 - a + b)$
2. $R_{mm} = \bar{A} \cup (A \cap B) = \int_{X\times Y} \frac{(1-\mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))}{(x,y)}$ or $f_m(a,b) = (1-a) \vee (a \wedge b)$
3. $R_s = \bar{A} \cup B = \int_{X\times Y} \frac{(1-\mu_A(x)) \vee \mu_B(y)}{(x,y)}$ or $f_{bp}(a,b) = (1-a) \vee b$
4. $R_\Delta = \int_{X\times Y} \frac{\mu_A(x) \hat{<} \mu_B(y)}{(x,y)}$ where

$$a \hat{<} b = \begin{cases} 1, if\ a \leq b \\ \frac{b}{a}, , if\ a > b \end{cases}$$

## Fuzzy Reasoning

- The basic rule of inference in traditional two-valued logic is **modus ponens**, according to which we can infer the truth of a proposition *B* from the truth of *A* and the implication $A \to B$.
- For instance, if *A* is identified with "**the tomato is red**“ and *B* with "**the tomato is ripe**," then if it is true that "the tomato is red," it is also **true** that "the tomato is ripe."

premise 1 (fact): $x\ is\ A$
premise 2 (rule): $if\ \ x\ is\ A\ then\ y\ is\ B$
consequence (conclusion): $y\ is\ B$

- Much of human reasoning, modus ponens is employed in an ***approximate manner***
- For example, if we have the same implication rule "if the tomato is red, then it is ripe" and we know that "the tomato is more or less red," then we may infer that "the tomato -is more or less ripe."

premise 1 (fact): $x\ is\ A'$
premise 2 (rule): $if\ \ x\ is\ A\ then\ y\ is\ B$
consequence (conclusion): $y\ is\ B'$

- Where A' is close to A and B' is close to B
- When A, B, A', and B' are fuzzy sets of appropriate universes, this inference procedure is called approximate reasoning or fuzzy reasoning; it is also called generalized modus ponens(GMP)



## Approximate reasoning (fuzzy reasoning)

- Let *A, A'*, and *B* be fuzzy sets of *X, X,* and *Y,* respectively
- Assume that the fuzzy implication $A \ \rightarrow B$ is expressed as a fuzzy relation *R* on $X \ \times \ Y$
- Then the fuzzy set B induced by "x is A'" and the fuzzy rule "if x is A then y is B" is defined by

$$\mu_{B'}(y) = max_x \min[\mu_{A'}(x), \mu_R(x, y)]$$
$$= \vee_x[\mu_{A'}(x) \wedge \mu_R(x, y)]$$

Or

$$B' = A' \circ R$$

## Computational aspects of the fuzzy reasoning

- **Single Rule with Single Antecedent**

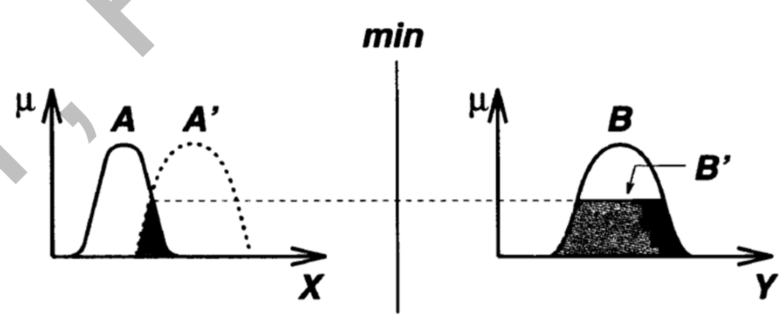
Given as

$$\mu_{B'}(y) = [\vee_x(\mu_{A'}(x) \wedge \mu_A(x))] \wedge \mu_B(y)$$
$$= w \wedge \mu_B(y)$$

- $w$ represents a measure of degree of belief for the antecedent part of a rule
- This measure gets propagated by the if-then rules and the resulting degree of belief or MF for the consequent part should not be greater than $w$



## *Graphic interpretation of GMP using Mamdani's fuzzy implication and the max-min composition*



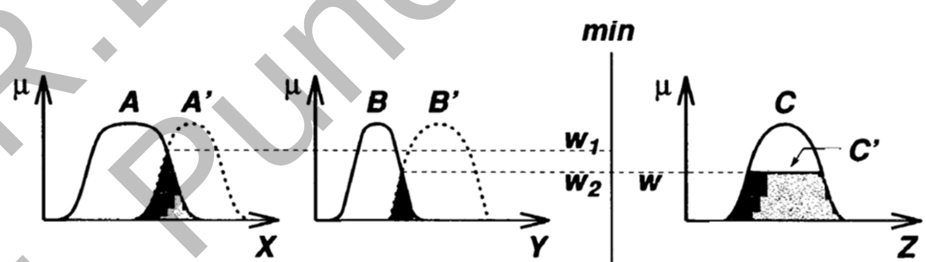
- **Single Rule with Multiple Antecedents**
- A fuzzy if-then rule with two antecedents is usually written as "if *x* is *A* and *y* is *B* then z is C"
- The corresponding problem for GMP is expressed as

premise 1 (fact): $x \text{ is } A' \text{ and } y \text{ is } B'$
premise 2 (rule): $if\ x \text{ is } A \text{ and } y \text{ is } B, then\ z \text{ is } C$
consequence (conclusion): $\mathbf{z}\ is\ C'$

- We then have

$$\mu_{C'}(z) = \underbrace{\{\vee_x [\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w1} \wedge \underbrace{\{\vee_y [\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w2} \wedge \mu_C(z)$$
$$= \underbrace{w1 \wedge w2}_{FIRING\ STRENGTH} \wedge \mu_C(z)$$

where $w1$ and $w2$ are the maxima of the MFs of $A \cap A'$ and $B \cap B'$, respectively




- $\boldsymbol{w1}$ denotes the **degrees of compatibility** between *A* and *A'*; similarly for $\boldsymbol{w2}$
- Since the antecedent part of the fuzzy rule is constructed by the connective "and," $\boldsymbol{w1} \wedge \boldsymbol{w2}$ is called the **firing strength or degree of fulfillment of the fuzzy rule**, which represents the degree to which the antecedent part of the rule is satisfied

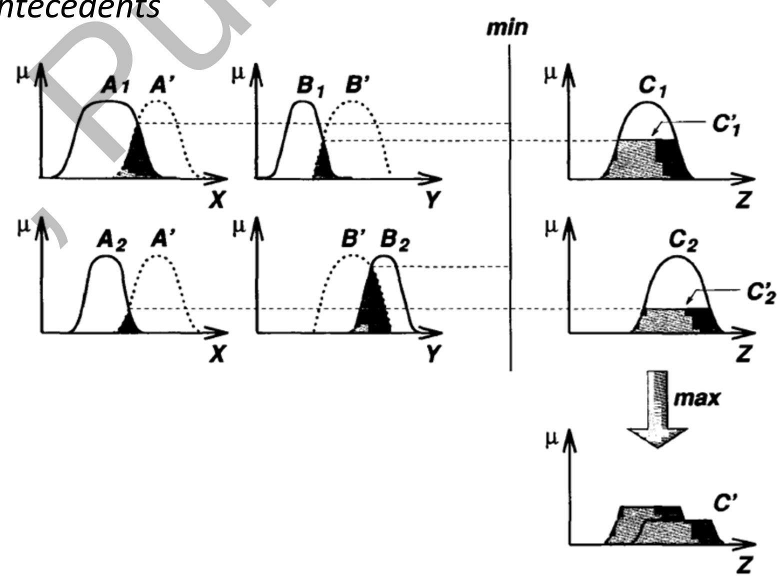
- **Multiple Rules with Multiple Antecedents**
- The interpretation of multiple rules is usually taken as the **union** of the fuzzy relations corresponding to the fuzzy rules
- Therefore, for a GMP problem written as

premise 1 (fact): $x\ is\ A'\ and\ y\ is\ B'$
premise 2 (rule 1): $if\ \ x\ is\ A1\ and\ y\ is\ B1, then\ z\ is\ C1$
premise 3 (rule 2): $if\ \ x\ is\ A2\ and\ y\ is\ B2, then\ z\ is\ C2$
consequence (conclusion): **z** $\boldsymbol{is\ C'}$

- Let $R_1 = A_1 \times B_1 \rightarrow C_1$ and $R_2 = A_2 \times B_2 \rightarrow C_2$
- We simply have

$$C' = (A' \times B') \circ (R_1 \cup R_2)$$
$$= [(A' \times B') \circ R_1] \cup [(A' \times B') \circ R_2]$$
$$\therefore C' = C_1{}' \cup C_2{}'$$

*Fuzzy reasoning for multiple rules with multiple antecedents*



## The process of fuzzy reasoning or approximate reasoning

- **Degrees of compatibility**
  - Compare the known facts with the antecedents of fuzzy rules to find the degrees of compatibility with respect to each antecedent MF.
- **Firing strength**
  - Combine degrees of compatibility with respect to antecedent MFs in a rule using fuzzy AND or OR operators to form a firing strength that indicates the degree to which the antecedent part of the rule is satisfied.
- **Qualified (induced) consequent MFs**
  - Apply the firing strength to the consequent MF of a rule to generate a qualified consequent MF. (The qualified consequent MFs represent how the firing strength gets propagated and used in a fuzzy implication statement.)
- **Overall output MF**
  - Aggregate all the qualified consequent MFs to obtain an overall output MF.



## Membership Functions-Revisited

- **Features of Membership Function**
  - The feature of the membership function is defined by three properties
  - They are:
    1. Core
    2. Support
    3. Boundary

## Core

- If the region of universe is characterized by full membership (1) in the set *A* then this gives the core of the membership function of fuzzy at *A*
- The elements, which have the membership function as 1, are the elements of the core, i.e., here $\mu_A(x) = 1$

## Support

- If the region of universe is characterized by nonzero membership in the set *A,* this defines the support of a membership function for fuzzy set *A.*
- The support has the elements whose membership is greater than 0 *i.e.,*$\mu_A(x) > 0$.

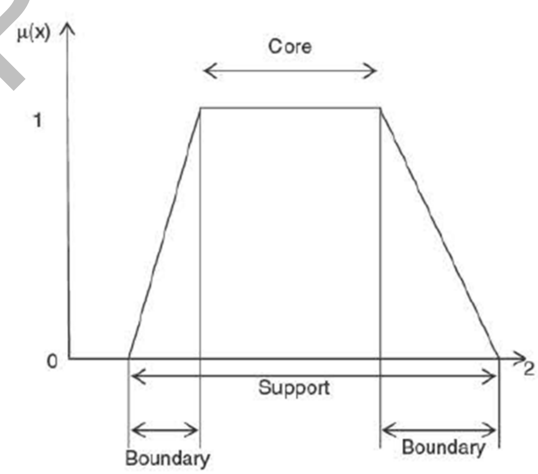
## Boundary

- If the region of universe has a nonzero membership but not full membership, this defines the boundary of a membership; this defines the boundary of a membership function for fuzzy set A
- The boundary has the elements whose membership is between 0 and 1,

  $0 < \mu_A\,(x) < 1$



## Features of membership function



## Important terms

- **Crossover Point**: The crossover point of a membership function is the elements in universe whose membership value is equal to 0.5

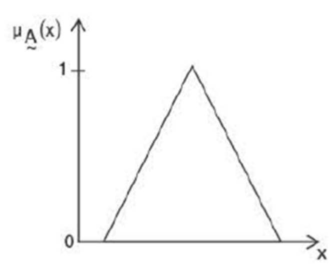
$$\mu_A(x) = 0.5$$

- **Height:** The height of the fuzzy set *A* is the maximum value of the membership function,

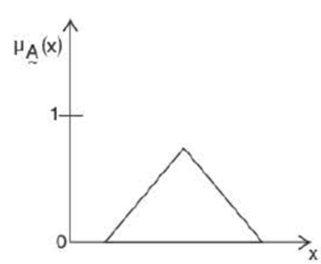
$$max(\mu_A(x))$$

## Classification of Fuzzy Sets

- The fuzzy sets can be classified based on the membership functions
  - *Normal fuzzy set:* If the membership function has at least one element in the universe whose value is equal to 1, then that set is called as normal fuzzy set
  - *Subnormal fuzzy set:* If the membership function has the membership values less than 1, then that set is called as subnormal fuzzy set

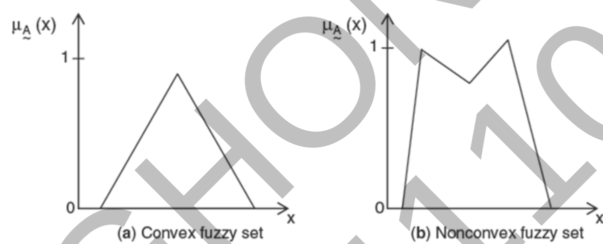



*Normal fuzzy set*

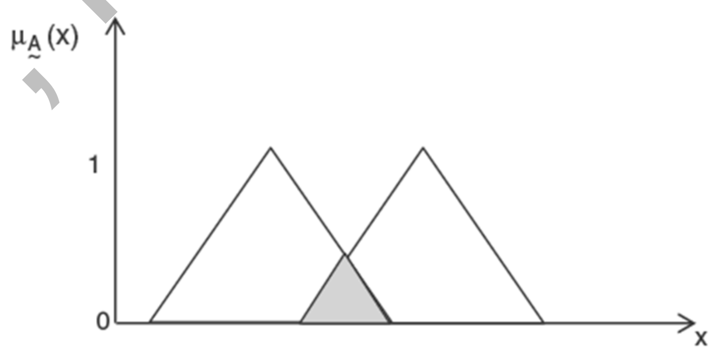



*Subnormal fuzzy set*

- *Convex fuzzy set:* If the membership function has membership values those are monotonically increasing, or, monotonically decreasing, or they are monotonically increasing and decreasing with the increasing values for elements in the universe ,the set is called convex fuzzy set
- *Nonconvex fuzzy set:* If the membership function has membership values which are not strictly monotonically increasing or monotonically decreasing or both monotonically increasing and decreasing with increasing values for elements in the universe, then this is called as nonconvex fuzzy set




- When intersection is performed on two convex fuzzy sets, the intersected portion is also a convex fuzzy set



# Fuzzification

- Fuzzification is the process where the crisp quantities are converted to fuzzy (crisp to fuzzy)
- By identifying some of the uncertainties present in the crisp values, we form the fuzzy values
- The conversion of fuzzy values is represented by the membership functions
- In any practical applications, in industries, etc., measurement of voltage, current, temperature, etc., there might be a negligible error
- This causes imprecision in the data which can be represented by the membership functions
- Thus fuzzification process may involve assigning membership values for the given crisp quantities

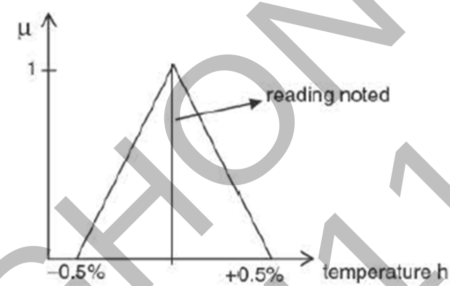


# Membership Value Assignments

- Various methods to assign the membership values or the membership functions to fuzzy variables are:
  - Intuition,
  - Inference,
  - Rank ordering,
  - Angular fuzzy sets,
  - Neural networks,
  - Genetic algorithms, and
  - Inductive reasoning

# Intuition

- Intuition is based on the human's own intelligence and understanding to develop the membership functions
- The thorough knowledge of the problem has to be known, the knowledge regarding the linguistic variable should also be known
- Example: Membership function for imprecision in crisp temperature reading




# Inference

- This method involves the knowledge to perform deductive reasoning
- The membership function is formed from the facts known and knowledge
- Let us use inference method for the identification of the triangle
- Let *U* be universe of triangles and *A*,*B*, and *C* be the inner angles of the triangles
- Also $A \geq B \geq C \geq 0$ therefore the universe is given by:

$$U = \{(A, B, C), A \geq B \geq C \geq 0, A + B + C = 180°\}$$

- There are various types of triangles, for identifying, we define three types of triangles:
  - *I:*Appropriate isosceles triangle
  - *R:*Appropriate right triangle
  - *O:*Other triangles

- The membership vales can be inferred to all of these triangle types through the method of inference, as we know the knowledge about the geometry of the triangles
- The membership for the approximate isosceles triangle, for the given conditions

  *A* ≥ *B* ≥ *C* ≥ 0 and *A* + *B* + *C* = 180°,

  is given as:

$$\mu_I(A,B,C) = 1 - \frac{1}{60°}\min(A-B, B-C)$$
$$\mu_R(A,B,C) = 1 - \frac{1}{90°}(A-90°)$$

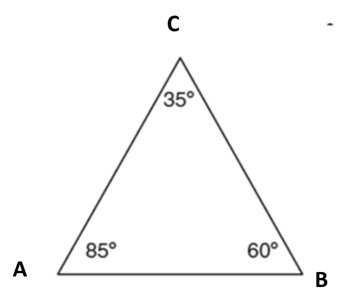
- The membership for the other triangles can be given as the complement of the logical union of the two already defined membership functions

$$\mu_O(A,B,C) = \overline{I \cup R}$$

i,.e.,

$$\mu_O(A,B,C) = \bar{I} \cap \bar{R} = min\{1-\mu_I(A,B,C), 1-\mu_R(A,B,C)\}$$

# Example




- Define the triangle for the figure shown in figure with the three given angles

- Solution: The condition is

$$A \geq B \geq C \geq 0$$

and

$$A + B + C = 180°$$

Here

$$\{U = A = 85° \geq B = 60° \geq C = 35° \geq 0, A + B + C = 180°\}$$

The membership for the triangle are

1. $\mu_I(x) = 1 - \frac{1}{60°}min(A-B, B-C)$

   $= 1 - \frac{1}{60°}min(25°, 25°) = 0.583$
2. $\mu_R(x) = 1 - \frac{1}{90°}(A-90°)$

   $= 1 - \frac{1}{90°}(5°) = 0.944$
3. $\mu_O(x) = min(1-\mu_I(x), 1-\mu_R(x)$

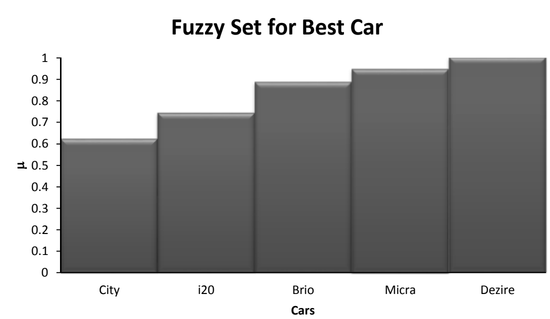
   $= min(1-0.583, 1-0.944) = 0.056$

## Rank Ordering

- The polling concept is used to assign membership values by rank ordering process
- Preferences are above the pairwise comparisons and from this the ordering of the membership is done
- **Example:** Suppose 1,000 people responds to a questionnaire about the pairwise preference among five cars, $x - \{Micra, Dezire, i20, Brio, City\}$.Define a fuzzy set as *A* on the universe of cars, “best cars”.

| Preferences | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Micra | Dezire | i20 | Brio | City | Total | Percentage | Rank order |
| Micra | – | 515 | 545 | 523 | 671 | 2,254 | 22.5 | 2 |
| Dezire | 481 | – | 475 | 845 | 580 | 2,381 | 23.8 | 1 |
| i20 | 469 | 624 | – | 141 | 536 | 1,770 | 17.7 | 4 |
| Brio | 457 | 530 | 470 | – | 649 | 2,114 | 21.1 | 3 |
| City | 265 | 425 | 402 | 389 | – | 1,481 | 14.8 | 5 |
| Total | | | | | | 10,000 | | |

- Hence there is highest membership for $\mu_R(x)$
- Thus inference method can be used to calculate the membership values
- From the table, it is clear that 515 preferred Micra compared to Dezire, 545 Astra to Dezire, etc
- The table forms an antisymmetric matrix
- There are about ten comparisons made which gives a ground total of 10,000.
- Based on preferences, the percentage is calculated
- The ordering is then performed
- It is found that Dezire is selected as the best car

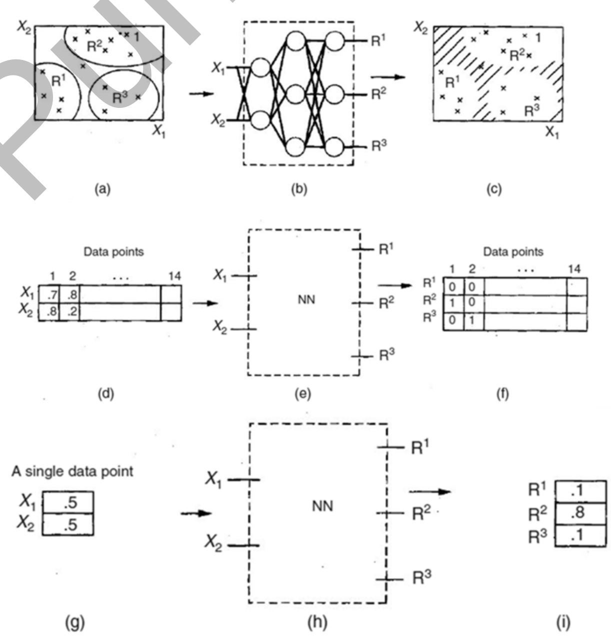


## Angular Fuzzy Sets

- The angular fuzzy sets are different from the standard fuzzy sets in their coordinate description
- These sets are defined on the universe of angles, hence are repeating shapes every $2\pi$ cycles
- Angular fuzzy sets are applied in quantitative description of linguistic variables known truth-values
- When membership of value 1 is true and that of 0 is false, then in between '0' and '1' is partially true or partially false
- The linguistic values are formed to vary with $\theta$, the angle defined on the unit circle and their membership values are on $\mu(\theta)$
- The membership of this linguistic term can be obtained from where $t$ is the horizontal projection of the radial vector and is given *as* cosθ, i.e., $t = \cos\theta$
- When the coordinates are in polar form, angular fuzzy sets can be used



## Neural Networks



## Defuzzification

- Defuzzification means the fuzzy to crisp conversions
- The fuzzy results generated cannot be used as such to the applications, hence it is necessary to convert the fuzzy quantities into crisp quantities for further processing
- This can be achieved by using defuzzification process
- The defuzzification has the capability to reduce a fuzzy to a crisp single-valued quantity or as a set, or converting to the form in which fuzzy quantity is present
- Defuzzification can also be called as "rounding off" method
- Defuzzification reduces the collection of membership function values in to a single scaler quantity



## Lambda Cuts for Fuzzy Sets

- Consider a fuzzy set *A*, then the lambda cut set can be denoted by $A_\lambda$, where $\lambda$ ranges between 0 and 1 ($0 \le \lambda \le 1$)
- The set $A_\lambda$ is a crisp set
- This crisp set is called the lambda cut set of the fuzzy set *A*, where

$$A_\lambda = \left\{ \frac{x}{\mu_A(x)} \geq \lambda \right\}$$

- The value of lambda cut set is $x$, when the membership value corresponding to $x$ is greater that or equal to the specified $\lambda$
- This lambda cut set can also be called as alpha cut set

## Properties of Lambda Cut Sets

- There are four properties of the lambda cut sets, they are:
    1. $(A \cup B)_\lambda = A_\lambda \cup B_\lambda$
    2. $(A \cap B)_\lambda = A_\lambda \cap B_\lambda$
    3. $(\bar{A})_\lambda \neq \bar{A}_\lambda$ $except\ at\ \lambda = 0.5$
    4. *For any λ ≤ α, where α varies between 0 and 1, it is true that, $A_\alpha \subseteq A_\lambda$, where the value of $A_0$ will be the universe defined*
- From the properties it is understood that the standard set of operations or fuzzy sets is similar to the standard set operations on lambda cut sets

## Lambda Cuts for Fuzzy Relations

- The lambda cut procedure for relations is similar to that for the lambda cut sets
- Consider a fuzzy relation *R* , in which some of the relational matrix represents a fuzzy set
- A fuzzy relation can be converted into a crisp relation by depending the lambda cut relation of the fuzzy relation as:

$$R_\lambda = \left\{ \frac{x, y}{\mu_R(x, y)} \geq \lambda \right\}$$

## Properties of Lambda Cut Relations

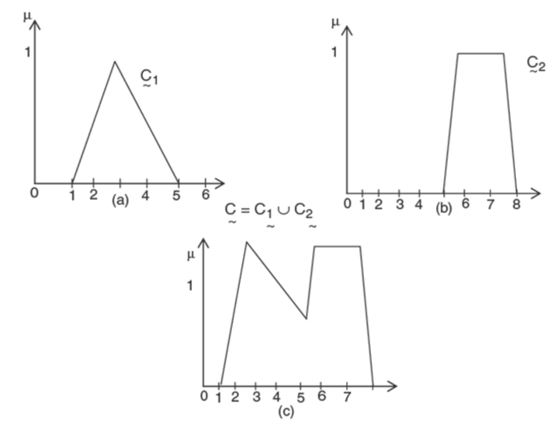
- Properties of the lambda cut relations, are:
  1. $(R \cup S)_\lambda = R_\lambda \cup S_\lambda$
  2. $(R \cap S)_\lambda = R_\lambda \cap S_\lambda$
  3. $(\bar{R})_\lambda \neq \bar{R}_\lambda$
  4. *For any $\lambda \le \alpha$, where $\alpha$ varies between 0 and 1, then,* $R_\alpha \subseteq R_\lambda$



## Defuzzification Methods

- There are other various defuzzification methods employed to convert the fuzzy quantities into crisp quantities
- The output of an entire fuzzy process can be union of two or more fuzzy membership functions
- Generally this can be given as:

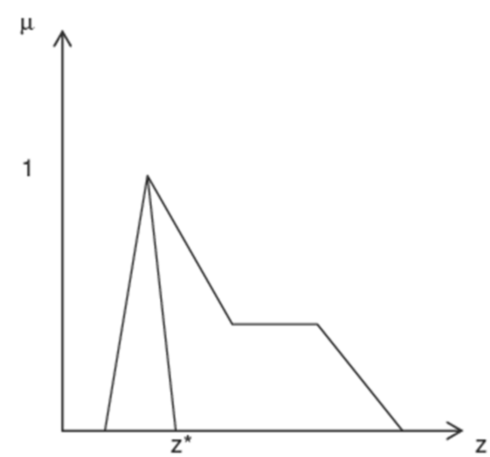
$$C_{\underset{r}{n}} = \sum_{i}^{n} C_{\underset{r}{i}} = C$$



- There are seven methods used for defuzzifying the fuzzy output functions:
  (1) Max-membership principle,
  (2) Centroid method,
  (3) Weighted average method,
  (4) Mean-max membership,
  (5) Centre of sums,
  (6) Centre of largest area,
  (7) First of maxima or last of maxima

## Max-membership-principle

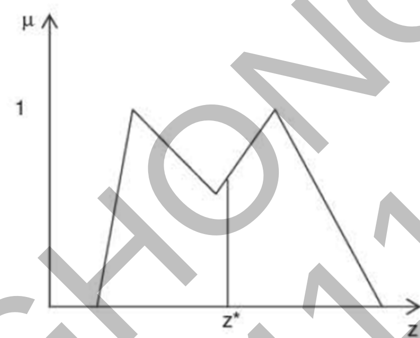
- This method is given by the expression
$$\mu_C(z^*) \geq \mu_C(z) \quad ,\forall z$$
- This method is also referred as height method



# Centroid method

- This is the most widely used method
- This can be called as center of gravity or center of area method
- It can be defined by the algebraic expression

$$z^* = \frac{\int \mu_C(z) \cdot z\, dz}{\int \mu_C(z) dz}$$




# Weighted average method

- This method cannot be used for asymmetrical output membership functions, can be used only for symmetrical output membership functions
- Weighting each membership function in the obtained output by its largest membership value forms this method
- The evaluation expression for this method is

$$z^* = \frac{\sum \mu_C(z_i) \cdot z_i}{\sum \mu_C(z_i)}$$