FUZZY LOGIC-I
MODULE 7

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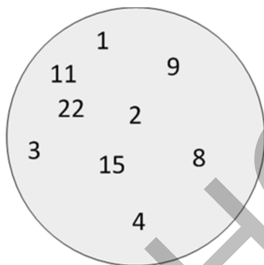
Agenda

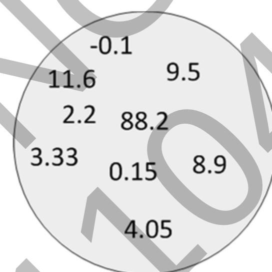
- Brief review of conventional sets
- Introduction to fuzzy sets
- Membership functions
- Operations on fuzzy sets

REVIEW OF CONVENTIONAL SETS

- Set
  - A collection of objects having one or more common characteristics
- Members/Elements
  - Objects belonging to a set is represented as $x \in A$, where A is a set

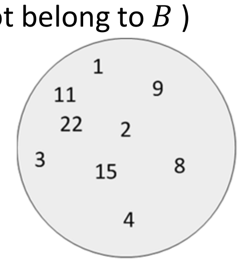


SET OF NATURAL NUMBERS

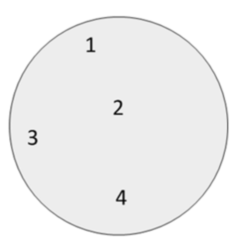


SET OF REAL NUMBERS

- Subset B
  - B is said to be a subset of set A, i.e., $B \subseteq A$ iff
    $$y \in B \Rightarrow y \in A \; \forall y$$
  (y belongs to B implies y belongs to A for all y)
- Proper subset
  - Set B is said to be proper subset of A iff
    $$B \subseteq A \text{ and } \exists x, x \in A \text{ but } x \notin B$$
  (B is a subset of A and there exists x that belongs to A but does not belong to B)

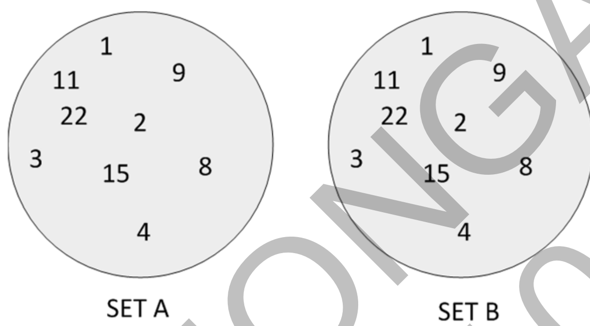


SET A



SET B

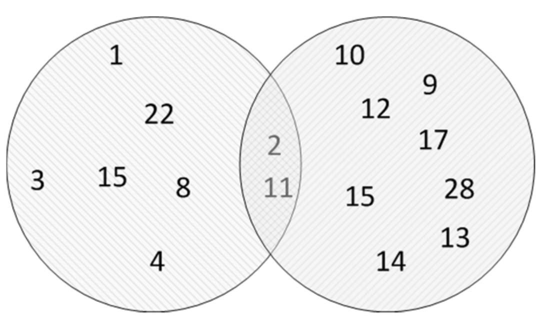
- Equal sets
  - Two sets A and B are said to be equal iff
    $\forall x \in A \; and \; \forall y \in B, x = y$



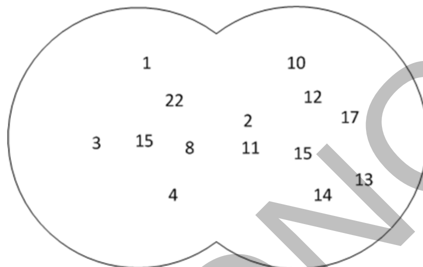
Set operations

- Intersection operation
  - For any two sets A and B if $\exists x$ common to both A and B, then
    $x \in (A \cap B)$

  where ∩ denotes the logical intersection operation



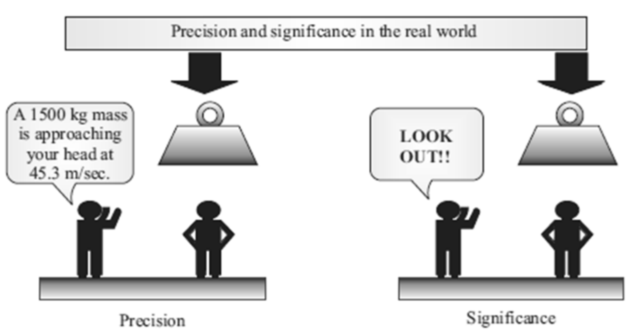
- Union Operation
  - For any two sets A and B if $\exists x$, which is a member of either A or B then $x \in (A \cup B)$, where $\cup$ denotes the union operation



- Universal Set
  - A universal set X is a set that has all possible members of a particular domain

Concept of Fuzziness

- When we talk about the real world , the way we describe or quantify the real world , is not precise
- E.g., Our description of a person's height- we use the terms short, medium, tall , which is imprecise
- Hence NOT to be PRECISE is FUZZY!
- Fuzzy Logic
  - The computation that involves the logic of imprecision is much powerful than computation that is being carried out in a precise manner



- Fuzzy logic is all about the relative importance of precision: How important is it to be exactly right when a rough answer will do?
- Fuzzy logic is a convenient way to map an input space to an output space
- ***As complexity rises, precise statements lose meaning and meaningful statements lose precision*** - L. A Zadeh (Father of Fuzzy Logic)
- ***So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality*** -Albert Einstein
- Fuzzy logic is a fascinating area of research because it does a good job of trading off between significance and precision - something that humans have been managing for a very long time

HISTORICAL BACKGROUND

- In viewing the evolution of fuzzy logic, three principal phases may be discerned
- The first phase, from 1965 to 1973, was concerned in the main with fuzzification, that is, with generalization of the concept of a set, with two-valued characteristic function generalized to a membership function taking values in the unit interval or, more generally, in a lattice
- The basic issues and applications which were addressed were, for the most part, set-theoretic in nature, and logic and reasoning were not at the center of the stage
- The second phase, 1973-1999, two key concepts were introduced in this paper
  - (a) the concept of a linguistic variable;
  - (b) the concept of a fuzzy if-then rule
- Today, almost all applications of fuzzy set theory and fuzzy logic involve the use of these concepts

- The term fuzzy logic was used for the first time in 1974. Today, fuzzy logic is used in two different contexts:
  (a) a narrow sense, in which fuzzy logic, abbreviated as FLn, is a logical system which is a generalization of multivalued logic;
  (b) a wide sense, in which fuzzy logic, abbreviated as FL, is a union of FLn, fuzzy set theory, possibility theory, calculus of fuzzy if-then rules, fuzzy arithmetic and calculus of fuzzy quantifiers
- An important development in the evolution of fuzzy logic, marking the beginning of the third phase, 1996 - is the genesis of computing with words and the computational theory of perceptions
- Basically, development of computing with words and perceptions brings together earlier strands of fuzzy logic and suggests that scientific theories should be based on fuzzy logic rather than on Aristotelian, bivalent logic, as they are at present
- A key component of computing with words is the concept of Precisiated Natural Language (PNL)

CHARACTERISTICS OF FUZZY LOGIC

Some of the essential characteristics of fuzzy logic relate to the following:

- In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning
- In fuzzy logic, everything is a matter of degree
- In fuzzy logic, knowledge is interpreted a collection of elastic or, equivalently, fuzzy constraint on a collection of variables
- Inference is viewed as a process of propagation of elastic constraints
- Any logical system can be fuzzified

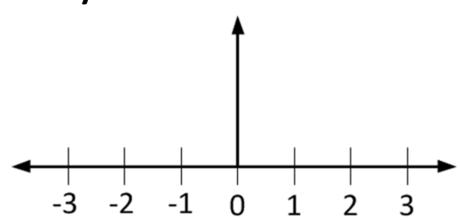
CHARACTERISTICS OF FUZZY SYSTEMS

There are two main characteristics of fuzzy systems that give them better performance for specific applications:

- Fuzzy systems are suitable for uncertain or approximate reasoning, especially for the system where a mathematical model that is difficult to derive.
- Fuzzy logic allows decision making with estimated values under incomplete or uncertain information

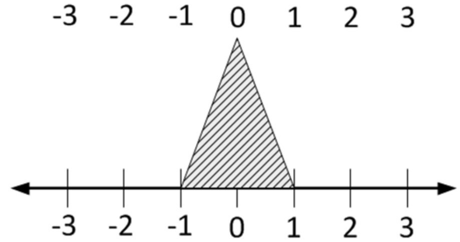
Concept of Fuzzy Number

- Precise zero



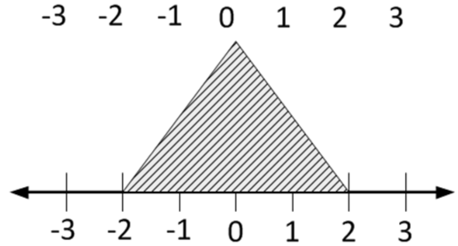
- Almost zero

(Can tolerate a band from -1 to +1)



- Near zero

(As we go away from zero towards -2 or +2 , the confidence level of how near to zero , reduces)



Fuzzy Sets

- Grade of membership
  - Every member x of a fuzzy set A is assigned a fuzzy index $\mu_A(x)$ in the interval $[0\,,1]$, which is called as the grade of membership of x in A
- In a conventional /crisp set membership grade $\mu_A(x)$ is either 0 or 1
- Fuzzy Sets
  - A set of ordered pairs , given by

$$A = \{(x, \mu_A(x)) : x \in X\}$$

Where X is a universal set and $\mu_A(x)$ is the grade of membership of the object x in A, usually $\mu_A(x)$ lies in $[0\,,1]$

- Membership Functions
  - A membership function$\mu_A(x)$ is characterized by:

$$\mu_A(x) : x \to [0,1], x \in X$$

Where x is a real number describing the object or its attribute , X is the universe of discourse and $A \subset X$

- Membership grade shows how likely the object belongs to the set

Comparing Classical and Fuzzy Approach

- Classical Approach
  - Consider a universal set T which stands for temperature
  - Cold, Normal and Hot are subsets of the universal set T

$Cold = \{temperature \in T: 5℃ < temperature < 15℃\}$

$Normal$

$= \{temperature \in T: 15℃ < temperature < 25℃\}$

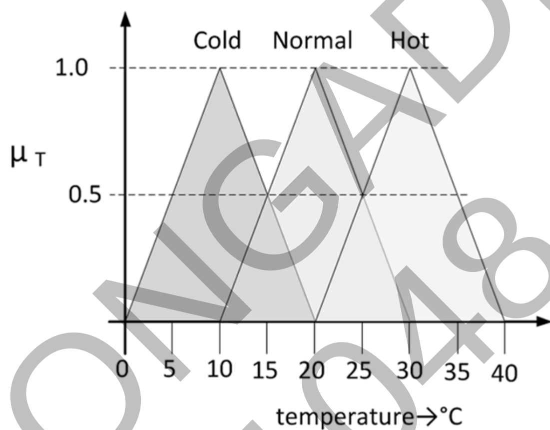
$Hot = \{temperature \in T: 25℃ < temperature < 35℃\}$

  - Notice here that 24.9°C is Normal while 25.1°C is Hot!
  - This implies that the classical sets have ***rigid*** boundaries
  - Because of this rigidity the expression of the data becomes difficult and may even feel absurd to a human being!

- Fuzzy Approach
  - In contrast the fuzzy sets have soft boundaries
  - One approach to define fuzzy subsets for the universal set T would be

- Means that
  - If temperature is around 10°C it is cold
  - If temperature is around 20°C it is normal
  - If temperature is around 30°C it is hot
- The temperature 25°C is a member of two fuzzy subsets "Cold" and "Normal" with a membership grade

$\boldsymbol{\mu_T(Normal) = \mu_T(Hot) = 0.5}$



- But if temperature is 28°C then it is more likely a temperature in the category of "Hot" , where as temperature of 22°C, belongs more to the category of "Normal"
- This is much better way of describing the temperature!

Nomenclature of Fuzzy Sets

- Let the elements of set X be $x_1, x_2,\ldots x_n$
- Then the fuzzy set $A \subseteq X$ is denoted by any of the following nomenclatures:

1. $\boldsymbol{A = \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), \ldots, (x_n, \mu_A(x_n))\}}$
2. $\boldsymbol{A = \left\{\frac{x_1}{\mu_A(x_1)}, \frac{x_2}{\mu_A(x_2)}, \ldots, \frac{x_n}{\mu_A(x_n)}\right\}}$
3. $\boldsymbol{A = \left\{\frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \ldots, \frac{\mu_A(x_n)}{x_n}\right\}}$

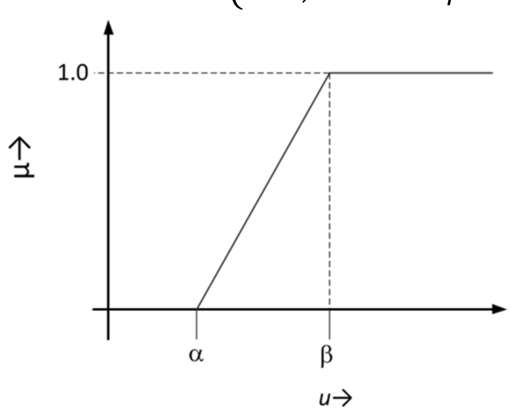
Typical Membership Functions

- Once we say that each member in a fuzzy set is associated with a membership function, we must know how to characterize this membership function
- Various membership functions have been employed, they are:
  1. γ-function
  2. S-function
  3. Triangular Function
  4. $\prod$-function
  5. Gaussian function

The γ-function

- Defined as :

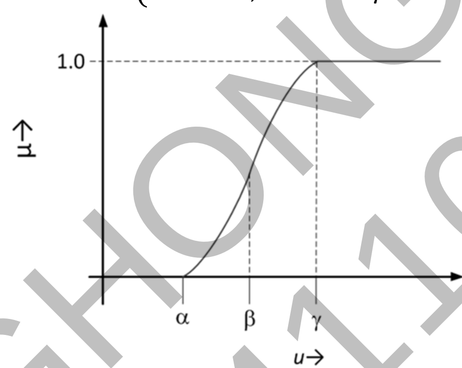
$$\gamma(u;\alpha,\beta)=\begin{cases}0, & u\le\alpha\\ \dfrac{(u-\alpha)}{(\beta-\alpha)}, & \alpha<u\le\beta\\ 1, & u>\beta\end{cases}$$



The S-function

- Defined as :

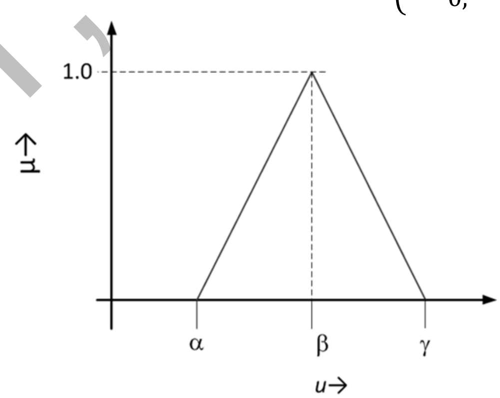
$$S(u;\alpha,\beta,\gamma)=\begin{cases}0, & u\le\alpha\\ 2\left[\dfrac{(u-\alpha)}{(\gamma-\alpha)}\right]^2, & \alpha<u\le\beta\\ 1-2\left[\dfrac{(u-\gamma)}{(\gamma-\alpha)}\right]^2, & \beta<u\le\gamma\\ 1, & u>\gamma\end{cases}$$



The Triangular membership function

- Defined as :

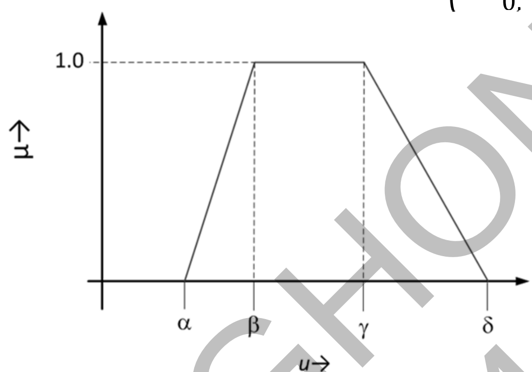
$$\Lambda(u;\alpha,\beta,\gamma)=\begin{cases}0, & u\le\alpha\\ \dfrac{(u-\alpha)}{(\beta-\alpha)}, & \alpha<u\le\beta\\ \dfrac{(\gamma-u)}{(\gamma-\beta)}, & \beta<u\le\gamma\\ 0, & u>\gamma\end{cases}$$



$\prod$-function

- Defined as :

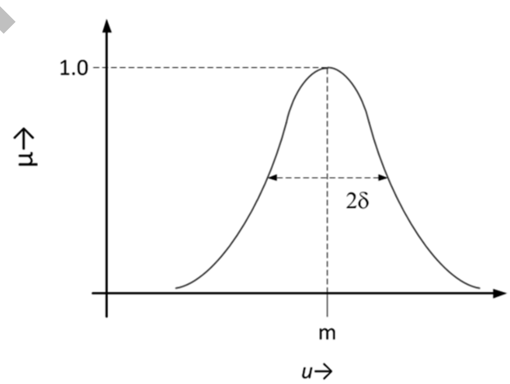
$$\Pi(u;\alpha,\beta,\gamma,\delta) = \begin{cases} 0, & u \le \alpha \\ \dfrac{(u-\alpha)}{(\beta-\alpha)}, & \alpha < u \le \beta \\ 1, & \beta < u \le \gamma \\ \dfrac{(\delta-u)}{(\delta-\gamma)}, & \gamma < u \le \delta \\ 0, & u > \delta \end{cases}$$



Gaussian function

- Defined as :

$$G(u;m,\delta) = \quad exp\left[-\frac{(u-m)^2}{2\delta^2}\right]$$

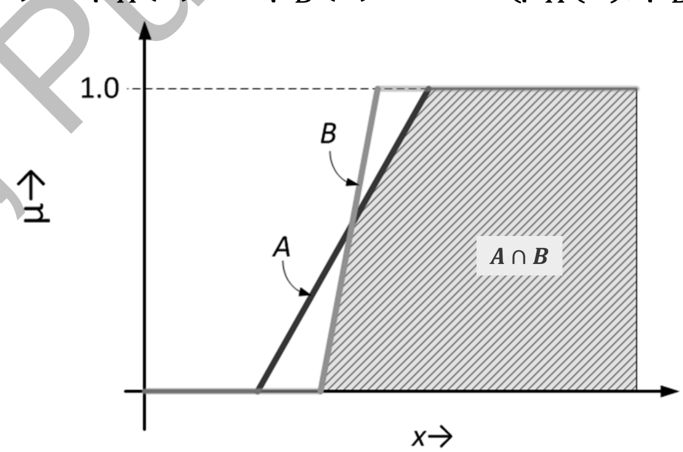


Operations on Fuzzy Sets

- The main feature of operations on fuzzy sets is that unlike conventional sets, operations on fuzzy sets are usually described with reference to membership function
- Thus operations manipulate the membership function
- Common operations defined on fuzzy sets are:
  1. Intersection or minimum function
  2. Union or maximum function
  3. Fuzzy complementation

1.Fuzzy Intersection

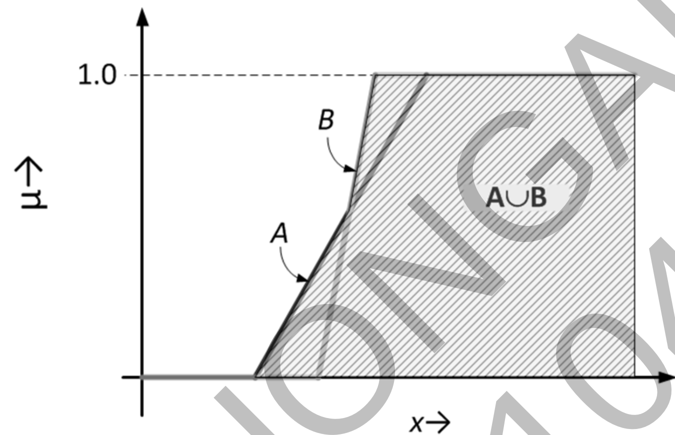
$$\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x) = \min(\mu_A(x), \mu_B(x))$$



- Overall membership function is the minimum of the two sets A and B

2.Fuzzy Union

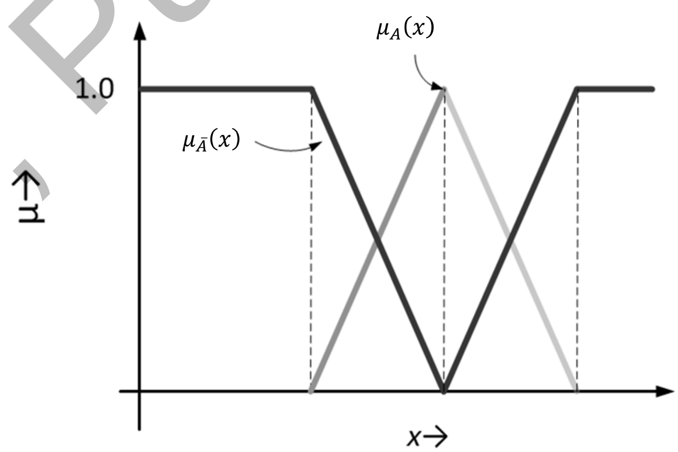
$$\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x) = \max(\mu_A(x), \mu_B(x))$$



- Overall membership function is the maximum of the two sets A and B

3.Fuzzy Complementation

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$



Other Fuzzy Operations

- De Morgan's Law:

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

For fuzzy sets:

$$\mu_{\overline{A \cap B}} = \mu_{\bar{A} \cup \bar{B}} = \max(1 - \mu_A, 1 - \mu_B)$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

For fuzzy sets:

$$\mu_{\overline{A \cup B}} = \mu_{\bar{A} \cap \bar{B}} = \min(1 - \mu_A, 1 - \mu_B)$$

- Difference:

$$A|B = A \cap \bar{B}$$

For fuzzy sets:

$$\mu_{A \cap \bar{B}} = \min(\mu_A, 1 - \mu_B)$$

$$B|A = B \cap \bar{A}$$

For fuzzy sets:

$$\mu_{B \cap \bar{A}} = \min(\mu_B, 1 - \mu_A)$$

Properties of Fuzzy Sets

1. Commutative

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. Distributive

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

3. Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Properties of Fuzzy Sets

4. Idempotency

$$A \cup A = A$$
$$A \cap A = A$$

5. Identity

$$A \cup \phi = A \quad and\ A \cap X = A$$
$$A \cap \phi = \phi\ and\ A \cup X = X$$

Where X is universal set and ϕ is a NULL set

Examples

- Given two discrete fuzzy sets:

$$A = \left\{\frac{0}{1}, \frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5}\right\}$$

$$B = \left\{\frac{0}{1}, \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5}\right\}$$

- Find $\bar{A}$, $\bar{B}$, $A \cup B$, $A \cap B$, $A|B$, $B|A$

Conclusion

- Fuzzy membership v/s Probability
  - Probability can be defined for an event that can be repeated again and again
  - Fuzzy membership can be defined for any event, the event need not be repeatable
- Example: Free-hand circle