FUZZY LOGIC-III
Fuzzy Inference System
MODULE 9

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Fuzzy Inference System

- Fuzzy inference system is a popular computing framework based on the concepts of fuzzy set theory, fuzzy if-then rules, and fuzzy reasoning
- Its applications include :
  - automatic control
  - data classification
  - decision analysis
  - expert systems
  - time series prediction
  - Robotics
  - pattern recognition
- Also called as
  - fuzzy-rule-based system
  - fuzzy expert system
  - fuzzy model
  - fuzzy associative memory
  - fuzzy logic controller
  - fuzzy system

Rule based Systems

- Worldly knowledge is very conveniently expressed in natural language
- The rule-base is one of the ways to represent knowledge using natural language
- A generic form of rule base is :

$\boldsymbol{IF\ premise[antecedent], THEN\ conclusion[consequence]}$

- This form is commonly referred to as IF-THEN rule-base form
- It typically expresses an ***inference*** such that if we know a fact we can infer or derive another fact
- Given a rule, we can derive another rule OR given a rule and the associated relation and another rule , we can predict what should be the consequence

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- Fuzzy information can be represented in the form of a RULE-BASE which consists of a set of rules in conventional antecedent-consequent form such as:

$$Rule\ 1: IF\ x\ is\ A\ , THEN\ y\ is\ B$$

Where A and B represent fuzzy propositions (sets)

- Now suppose we introduce a new antecedent A' and consider the rule:

$$Rule\ 2: IF\ x\ is\ A'\ , THEN\ y\ is\ B'$$

- From this information derived from Rule 1, is it possible to derive the consequent in Rule 2, i.e., B'?
- The answer is yes
- The consequent B' is found from the composition equation:

$$B' = A' \circ R$$

Where R is the relational matrix

Fuzzy Implication Relations

- A fuzzy implication relation for a given rule:

$$Rule\ 1: IF\ x\ is\ A\ , THEN\ y\ is\ B$$

is formally denoted by:

$$R_i(x,y) = \left\{\frac{\mu_{R_i}(x,y)}{(x,y)}\right\}$$

- $\mu_{R_i}(x,y)$ is computed using various implication rules, $if\ p\ then\ q\ (p \rightarrow q)$, where both p and q are fuzzy propositions

$$IF\ \underbrace{x\ is\ A}_{p}, THEN\ \underbrace{y\ is\ B}_{q}\ ,$$

Dienes-Rescher Implication

- If $p\ then\ q$states that p is true but q is false is **IMPOSSIBLE**, i.e.,$p \wedge -q$ is false
- Using DeMorgan's law

$$p \wedge -q = -p \vee q$$

- Thus the relational matrix can be computed as:

$$\boldsymbol{\mu_{R_i}(x,y) = \max[1 - \mu_{A_i}(x), \mu_{B_i}(y)]}$$

Mamdani Implications

- When fuzzy IF-THEN rules are locally true then using Mamdani implication $p \rightarrow q$ implies $p \wedge q$ is **TRUE**
- Thus the relational matrix can be computed using any of the following expressions:

$$\mu_{R_i}(x, y) = \min[\mu_{A_i}(x), \mu_{B_i}(y)]$$

or

$$\mu_{R_i}(x, y) = \mu_{A_i}(x) \cdot \mu_{B_i}(y)$$

- Remember that each rule is locally true
- Mamdani implication rule is widely used in fuzzy systems and fuzzy control engineering
- Example: If the temperature is HOT, then the fan should run FAST
- This rule does **NOT** imply , if temperature is COLD then the fan should run SLOW

Zadeh Implication

- If $p \rightarrow q$ may imply either p and q are TRUE or p is FALSE
- Thus $p \rightarrow q = (p \wedge q) \vee (-p)$
- The relational matrix can be computed as:

$$\mu_{R_i}(x, y) = \max[min(\mu_{A_i}(x), \mu_{B_i}(y)), 1 - \mu_{A_i}(x)]$$

- Given a set of rules we use various schemes by which we can construct a relational matrix between the antecedent and consequent
- The next step would be to utilize this relational matrix for inference
- This method is commonly known as **compositional rule of inference**
- Associated with each rule we have the relational matrix
- So given a rule means given a relational matrix, and another antecedent we can compute the consequent

Fuzzy Compositional Rules

Following are the different rules for fuzzy composition operation $B = A \circ R$

- **max-min**

$$\mu_B(y) = max_{x \in X}\{min[\mu_A(x), \mu_R(x,y)]\}$$

- **max-product**

$$\mu_B(y) = max_{x \in X}\{\mu_A(x) \cdot \mu_R(x,y)\}$$

- **min-max**

$$\mu_B(y) = min_{x \in X}\{max[\mu_A(x), \mu_R(x,y)]\}$$

- **max-max**

$$\mu_B(y) = max_{x \in X}\{max[\mu_A(x), \mu_R(x,y)]\}$$

- **min-min**

$$\mu_B(y) = min_{x \in X}\{min[\mu_A(x), \mu_R(x,y)]\}$$

We have to choose these rules by looking at the behaviour of the specific data

Example

- Given a rule: $IF\ x\ is\ A, THEN\ y\ is\ B$, where $A = \left\{\frac{0.2}{1}, \frac{0.5}{2}, \frac{0.7}{3}\right\}$ $and\ B = \left\{\frac{0.6}{5}, \frac{0.8}{7}, \frac{0.4}{9}\right\}$

Infer B’ for another rule: $IF\ x\ is\ A', THEN\ y\ is\ B'$, where $A' = \left\{\frac{0.5}{1}, \frac{0.5}{2}, \frac{0.3}{3}\right\}$

- Solution: using Mamdani implication rule the relational matrix R can be found as:

$$R_{1,1} = min[A_1, B_1] = \min[0.2{,}0.6] = 0.2$$
$$R_{1,2} = min[A_1, B_2] = \min[0.2{,}0.8] = 0.2$$
$$R_{1,3} = min[A_1, B_3] = \min[0.2{,}0.4] = 0.2$$
$$R_{2,1} = min[A_2, B_1] = \min[0.5{,}0.6] = 0.5$$
$$R_{2,2} = min[A_2, B_2] = \min[0.5{,}0.8] = 0.5$$
$$R_{2,3} = min[A_2, B_3] = \min[0.5{,}0.4] = 0.4$$
$$R_{3,1} = min[A_3, B_1] = \min[0.7{,}0.6] = 0.6$$
$$R_{3,2} = min[A_3, B_2] = \min[0.7{,}0.8] = 0.7$$
$$R_{3,3} = min[A_3, B_3] = \min[0.7{,}0.4] = 0.4$$

R =

	5	7	9
1	0.2	0.2	0.2
2	0.5	0.5	0.4
3	0.6	0.7	0.4

- Using max-min composition relation

$B' = A' \circ R$ can be computed as:

$$A' = [0.5 \quad 0.5 \quad 0.3]\ and\ R = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.6 & 0.7 & 0.4 \end{bmatrix}$$

$\boldsymbol{B'_1}$ =

RC1	A’ T	min
0.2	0.5	0.2
0.5	0.5	0.5
0.6	0.3	0.3
max		0.5

$\boldsymbol{B'_2}$ =

RC2	A’ T	min
0.2	0.5	0.2
0.5	0.5	0.5
0.7	0.3	0.3
max		0.5

$\boldsymbol{B'_3}$ =

RC3	A’ T	min
0.2	0.5	0.2
0.4	0.5	0.4
0.4	0.3	0.3
max		0.4

$$\therefore B' = [0.5 \quad 0.5 \quad 0.4]$$

R =

	5	7	9
1	0.2	0.2	0.2
2	0.5	0.5	0.4
3	0.6	0.7	0.4

- Using max-product composition relation

$B' = A' \circ R$ can be computed as:

$$A' = [0.5 \quad 0.5 \quad 0.3] \; and \; R = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.6 & 0.7 & 0.4 \end{bmatrix}$$

$\boldsymbol{B'_1}$ =

RC1	A' T	prod
0.2	0.5	0.1
0.5	0.5	0.25
0.6	0.3	0.18
max		0.25

$\boldsymbol{B'_2}$ =

RC2	A' T	prod
0.2	0.5	0.1
0.5	0.5	0.25
0.7	0.3	0.21
max		0.25

$\boldsymbol{B'_3}$ =

RC3	A' T	prod
0.2	0.5	0.1
0.4	0.5	0.2
0.4	0.3	0.12
max		0.2

$$\therefore B' = [0.25 \quad 0.25 \quad 0.2]$$

R =

	5	7	9
1	0.2	0.2	0.2
2	0.5	0.5	0.4
3	0.6	0.7	0.4

- Using min-max composition relation

$B' = A' \circ R$ can be computed as:

$$A' = [0.5 \quad 0.5 \quad 0.3] \; and \; R = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.6 & 0.7 & 0.4 \end{bmatrix}$$

$\boldsymbol{B'_1}$ =

RC1	A' T	max
0.2	0.5	0.5
0.5	0.5	0.5
0.6	0.3	0.6
min		0.5

$\boldsymbol{B'_2}$ =

RC2	A' T	max
0.2	0.5	0.5
0.5	0.5	0.5
0.7	0.3	0.7
min		0.5

$\boldsymbol{B'_3}$ =

RC3	A' T	max
0.2	0.5	0.5
0.4	0.5	0.5
0.4	0.3	0.4
min		0.4

$$\therefore B' = [0.5 \quad 0.5 \quad 0.4]$$

R =

	5	7	9
1	0.2	0.2	0.2
2	0.5	0.5	0.4
3	0.6	0.7	0.4

- Using max-max composition relation

$B' = A' \circ R$ can be computed as:

$$A' = [0.5 \quad 0.5 \quad 0.3] \ and\ R = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.6 & 0.7 & 0.4 \end{bmatrix}$$

$\boldsymbol{B'_1}$ =

RC1	A' T	max
0.2	0.5	0.5
0.5	0.5	0.5
0.6	0.3	0.6
max		0.6

$\boldsymbol{B'_2}$ =

RC2	A' T	max
0.2	0.5	0.5
0.5	0.5	0.5
0.7	0.3	0.7
max		0.7

$\boldsymbol{B'_3}$ =

RC3	A' T	max
0.2	0.5	0.5
0.4	0.5	0.5
0.4	0.3	0.4
max		0.5

$$\therefore B' = [0.6 \quad 0.7 \quad 0.5]$$

R =

	5	7	9
1	0.2	0.2	0.2
2	0.5	0.5	0.4
3	0.6	0.7	0.4

- Using min-min composition relation

$B' = A' \circ R$ can be computed as:

$$A' = [0.5 \quad 0.5 \quad 0.3] \ and\ R = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.4 \\ 0.6 & 0.7 & 0.4 \end{bmatrix}$$

$\boldsymbol{B'_1}$ =

RC1	A' T	min
0.2	0.5	0.2
0.5	0.5	0.5
0.6	0.3	0.3
min		0.2

$\boldsymbol{B'_2}$ =

RC2	A' T	min
0.2	0.5	0.2
0.5	0.5	0.5
0.7	0.3	0.3
min		0.2

$\boldsymbol{B'_3}$ =

RC3	A' T	min
0.2	0.5	0.2
0.4	0.5	0.4
0.4	0.3	0.3
min		0.2

$$\therefore B' = [0.2 \quad 0.2 \quad 0.2]$$

Approximate reasoning

- It means that given any logical system , it is very difficult (maybe unknown) to make exact reasoning
- Hence we are liberal from engineering perspective i.e., we do not want to be so precise as long as our system works
- We have a set of rules , so we use specific rule of inference and then we infer the consequent
- Given a rule R and given a condition A, the inference for B is done using composition rule $B = A \circ R$
- The fuzzy set associated with each rule base may be discrete or continuous
- A rule base may contain single rule or multiple rules
- Various inference mechanisms for a single rule are enumerated
- R , in case of continuous fuzzy sets is **NOT *defined*** (infinite values) hence the computation method is different

Example: Single Rule with Discrete Fuzzy Set

- RULE 1: If temperature is HOT, then fan should run FAST
- RULE 2: If temperature is MODERATELY HOT, then fan should run MODERATELY FAST

The temperature is expressed in °F and speed in 1000 rpm

Given:

$$H = HOT = \left\{\frac{0.4}{70}, \frac{0.6}{80}, \frac{0.8}{90}, \frac{0.9}{100}\right\}$$

$$\mathrm{F = FAST} = \left\{\frac{0.3}{1}, \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.9}{4}\right\}$$

$$H' = MODERATELY\ HOT = \left\{\frac{0.2}{70}, \frac{0.4}{80}, \frac{0.6}{90}, \frac{0.8}{100}\right\}$$

$Compute\ F' = MODERATELY\ FAST$

- Solution: Using Mamdani Implication Rule

Rule1, R =

	1	2	3	4
70	0.3	0.4	0.4	0.4
80	0.3	0.5	0.6	0.6
90	0.3	0.5	0.7	0.8
100	0.3	0.5	0.7	0.9

- Using max-min composition relation

$\because F' = H' \circ R$ can be computed as:

$$H' = [0.2 \quad 0.4 \quad 0.6 \quad 0.8] \; and \; R = \begin{bmatrix} 0.3 & 0.4 & 0.4 & 0.4 \\ 0.3 & 0.5 & 0.6 & 0.6 \\ 0.3 & 0.5 & 0.7 & 0.8 \\ 0.3 & 0.5 & 0.7 & 0.9 \end{bmatrix}$$

$F'_{1,1}$ =

RC1	H' T	min
0.3	0.2	0.2
0.3	0.4	0.3
0.3	0.6	0.3
0.3	0.8	0.3
max		0.3

$F'_{1,2}$ =

RC2	H' T	min
0.4	0.2	0.2
0.5	0.4	0.4
0.5	0.6	0.5
0.5	0.8	0.5
max		0.5

$F'_{1,3}$ =

RC3	H' T	min
0.4	0.2	0.2
0.6	0.4	0.4
0.7	0.6	0.6
0.7	0.8	0.7
max		0.7

$F'_{1,4}$ =

RC4	H' T	min
0.4	0.2	0.2
0.6	0.4	0.4
0.8	0.6	0.6
0.9	0.8	0.8
max		0.8

$$\therefore F' = \left\{ \frac{0.3}{1}, \frac{0.5}{2}, \frac{0.7}{3}, \frac{0.8}{4} \right\}$$

Example: Multiple Rules with Discrete Fuzzy Set

- RULE 1: If height is TALL , then speed is HIGH
- RULE 2: If height is MEDIUM , then speed is MODERATE

The fuzzy sets for height (in feet) and speed (in m/s) are:

$$H_1 = TALL = \left\{\frac{0.5}{5}, \frac{0.8}{6}, \frac{1}{7}\right\}, S_1 = HIGH = \left\{\frac{0.4}{5}, \frac{0.7}{7}, \frac{0.9}{9}\right\}$$

$$H_2 = MEDIUM = \left\{\frac{0.6}{5}, \frac{0.7}{6}, \frac{0.6}{7}\right\}, S_2 = MODERATE = \left\{\frac{0.6}{5}, \frac{0.8}{7}, \frac{0.7}{9}\right\}$$

Given H $= ABOV\ EAV\ ERAGE$= $\left\{\frac{0.5}{5}, \frac{0.9}{6}, \frac{0.8}{7}\right\}$

Compute $S' = ABOV\ ENORMAL$

- Solution: Relational matrices for RULE 1 and RULE 2 are computed using Mamdani Implication Rule as:

R_1 =

	5	7	9
5	0.4	0.5	0.5
6	0.4	0.7	0.8
7	0.4	0.7	0.9

R_2 =

	5	7	9
5	0.6	0.6	0.6
6	0.6	0.7	0.7
7	0.6	0.6	0.6

$$S' = ABOV\ ENORMAL = \max[H' \circ R_1, H' \circ R_2]$$
$$= \mathrm{H}' \circ \max[R_1, R_2]$$

$$\max[R_1, R_2] = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.8 \\ 0.6 & 0.7 & 0.9 \end{bmatrix}$$

$$H' = [0.5 \quad 0.9 \quad 0.8]\ \ and\ R = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.8 \\ 0.6 & 0.7 & 0.9 \end{bmatrix}$$

S'_1 =

RC1	A' T	min
0.6	0.5	0.5
0.6	0.9	0.6
0.6	0.8	0.6
max		0.6

S'_2 =

RC2	A' T	min
0.6	0.5	0.5
0.7	0.9	0.7
0.7	0.8	0.7
max		0.7

S'_3 =

RC3	A' T	min
0.6	0.5	0.5
0.8	0.9	0.8
0.9	0.8	0.8
max		0.8

$$\therefore S' = \left\{\frac{0.6}{5}, \frac{0.7}{7}, \frac{0.8}{9}\right\}$$

Multiple rules with Continuous Fuzzy Sets

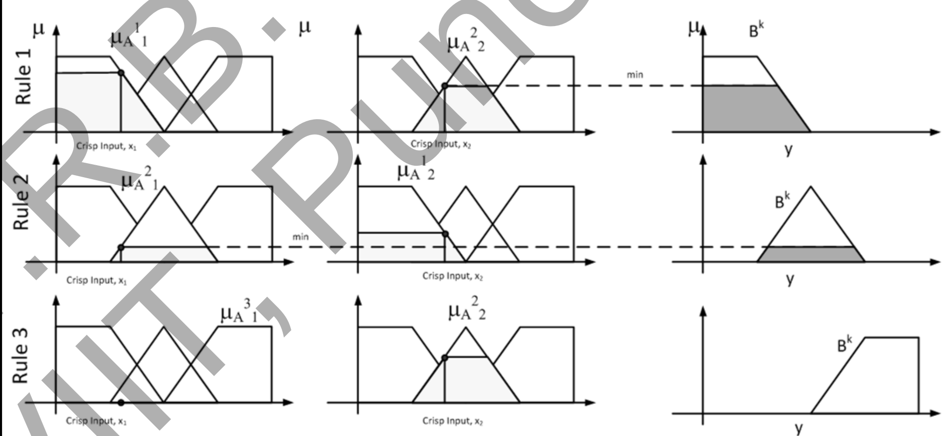
- A continuous fuzzy system with two non-interactive inputs x_1 $and x_2$(antecedents) and as ingle output y (consequent) is described by a collection of r linguistic IF-THEN rules

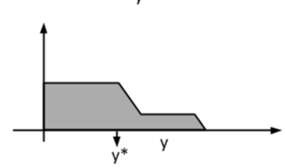
$$IF x_1 \ is \ A_1{}^k \ and \ x_2 \text{ is } A_2{}^k, then \ y^k \ is \ B^k$$

for $k = 1,2, \dots, r$

where $A_1{}^k$ and $A_2{}^k$ are the fuzzy sets representing k^{th} antecedent pairs and B^k are the fuzzy sets representing the k^{th} consequent

- Consider a two input system and three rule system
- The inputs to system are crisp values and we use a max-min inference method



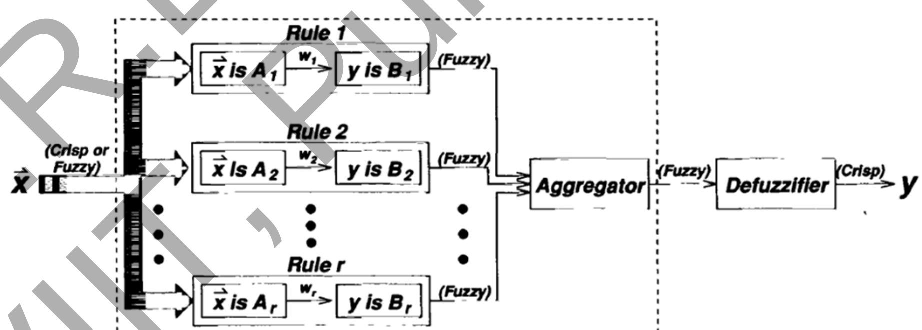


Result of overlapping (max operation)→
Find center of gravity to get crisp output y^*

Structure of FIS

- The basic structure of a fuzzy inference system consists of three conceptual components:
  - a rule base, which contains a selection of fuzzy rules;
  - a database(or dictionary), which defines the membership functions used in the fuzzy rules;
  - a reasoning mechanism, which performs the inference procedure (usually the fuzzy reasoning) upon the rules and given facts to derive a reasonable output or conclusion

Block Diagram of a typical FIS

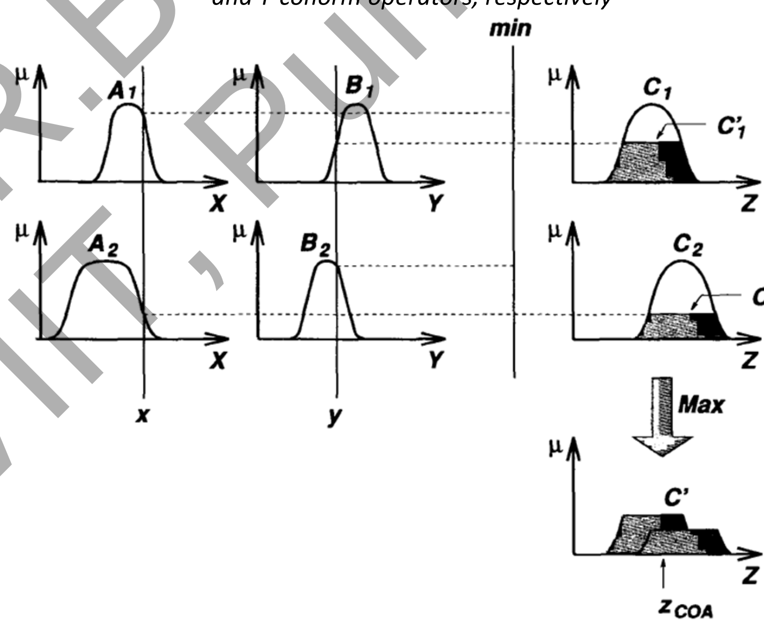


- With crisp inputs and outputs, a fuzzy inference system implements a nonlinear mapping from its input space to output space
- This mapping is accomplished by a number of fuzzy if-then rules, each of which describes the local behavior of the mapping
- The antecedent of a rule defines a fuzzy region in the input space, while the consequent specifies the output in the fuzzy region

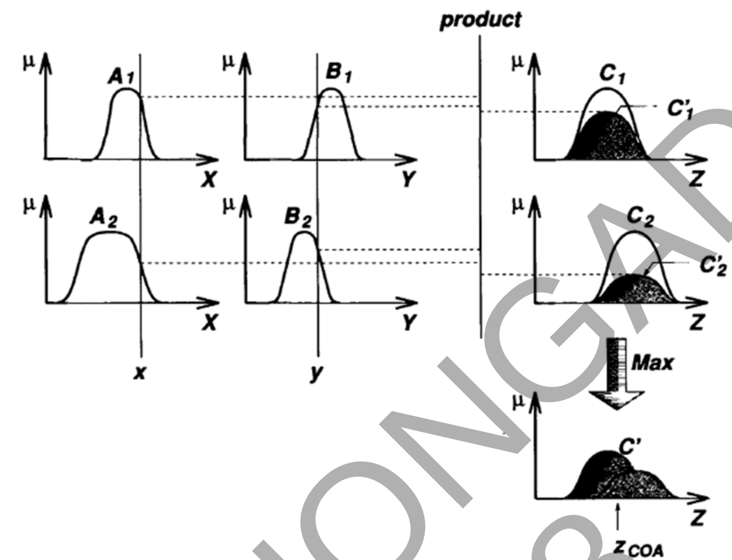
MAMDANI FUZZY MODELS

- The Mamdani fuzzy inference system was proposed as the first attempt to control a steam engine and boiler combination by a set of linguistic control rules obtained from experienced human operators
- It is a two-rule Mamdani fuzzy inference system which derives the overall output z when subjected to two crisp inputs x and y
- Two fuzzy inference systems were used as two controllers to generate the heat input to the boiler and throttle opening of the engine cylinder, respectively, to regulate the steam pressure in the boiler and the speed of the engine
- Since the plant takes only crisp values as inputs, we have to use a defuzzifier to convert a fuzzy set to a crisp value

The Mamdani fuzzy inference system using min *and* max *for T-norm and T-conorm operators, respectively*

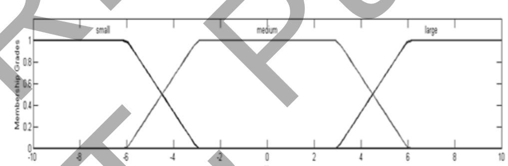


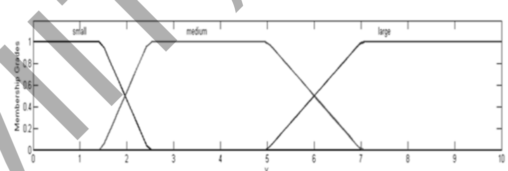
The Mamdani fuzzy inference system using product and max for Tnorm and T-conorm operators, respectively

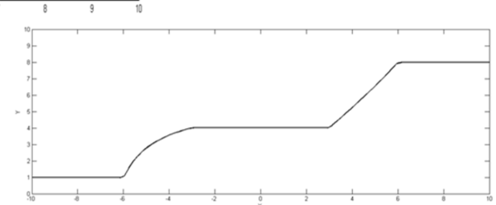


Example: Single-input single-output Mamdani fuzzy model

$$\begin{cases} If\ X\ is\ small\ then\ Y\ is\ small \\ If\ X\ is\ medium\ then\ Y\ is\ medium \\ If\ X\ is\ large\ then\ Y\ is\ large \end{cases}$$

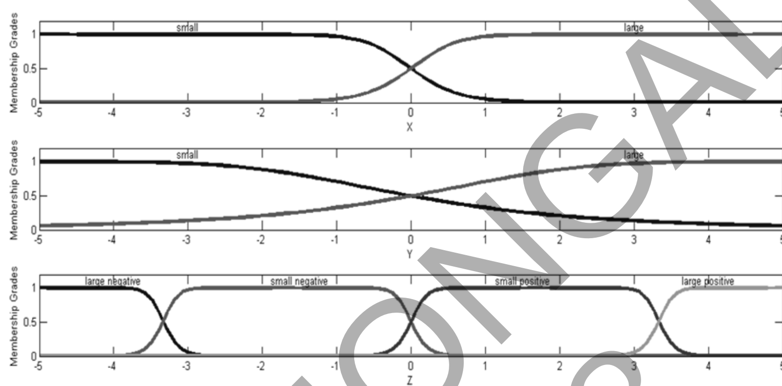




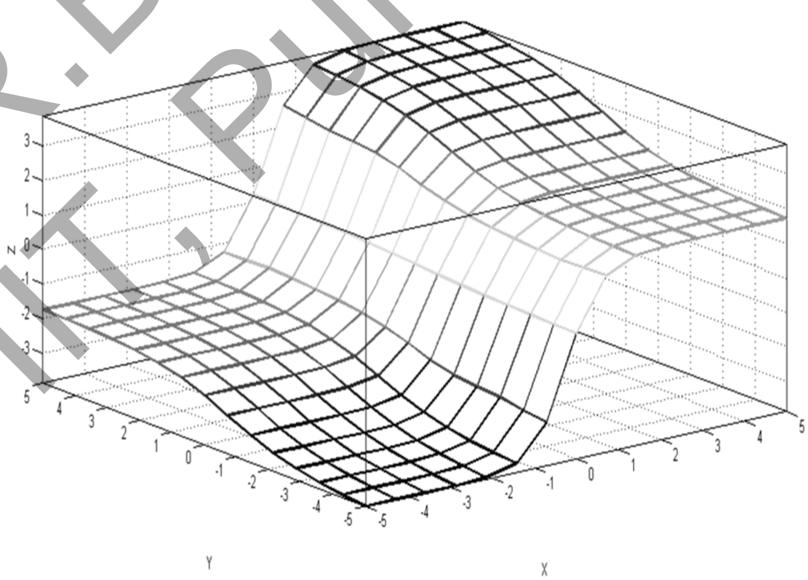


Example : Two-input single-output Mamdani fuzzy model

$$\begin{cases} If\ X\ is\ small\ and Y\ is\ small\ then\ Z\ is\ negative\ large \\ If\ X\ is\ small\ and Y\ is\ large\ then\ Z\ is\ negative\ small \\ If\ X\ is\ large\ and Y\ is\ small\ then\ Z\ is\ positive\ small \\ If\ X\ is\ large\ and Y\ is\ large\ then\ Z\ is\ positive\ large \end{cases}$$



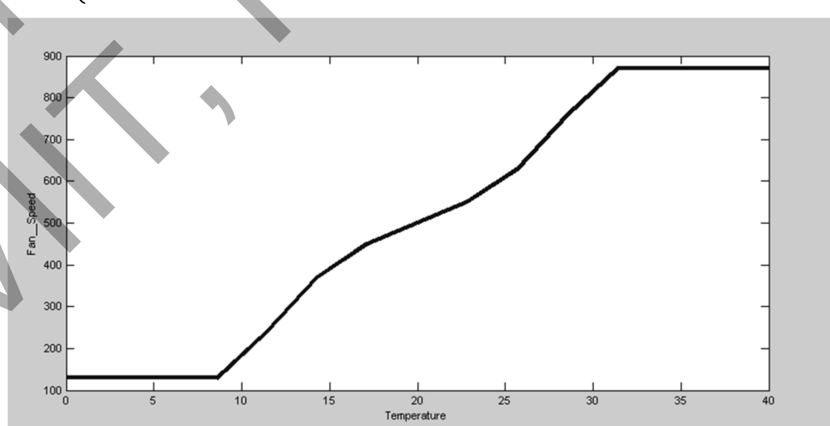
Input-Output mapping



MATLAB DEMO

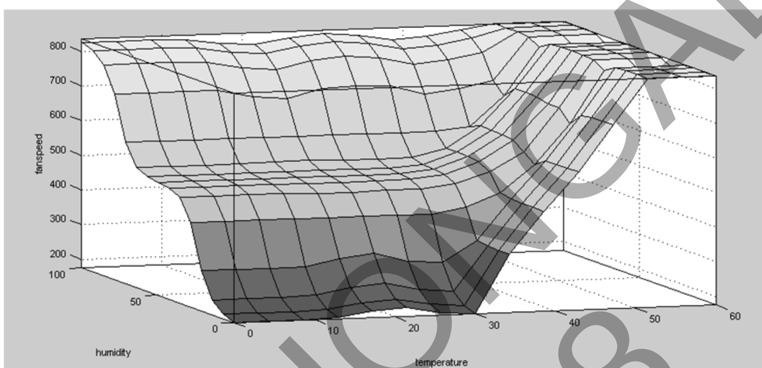
Ex 1: Fan Speed Control , single input as Temperature

$$\begin{cases} If\ Temperature\ is\ COLD\ then\ FanSpeed\ is\ SLOW \\ If\ Temperature\ is\ NORMAL\ then\ FanSpeed\ is\ MODERATE \\ If\ Temperature\ is\ HOT\ then\ FanSpeed\ is\ FAST \end{cases}$$



Ex 2: Fan Speed Control , two inputs as Temperature & Humidity

If (temperature is cold) and (humidity is low) then (fanspeed is slow)
If (temperature is cold) and (humidity is medium) then (fanspeed is moderate)
If (temperature is cold) and (humidity is high) then (fanspeed is fast)
If (temperature is normal) and (humidity is low) then (fanspeed is slow)
If (temperature is normal) and (humidity is medium) then (fanspeed is moderate)
If (temperature is normal) and (humidity is high) then (fanspeed is fast)
If (temperature is hot) and (humidity is low) then (fanspeed is fast)
If (temperature is hot) and (humidity is medium) then (fanspeed is fast)
If (temperature is hot) and (humidity is high) then (fanspeed is fast)



SUGENO FUZZY MODELS

- The Sugeno fuzzy model (also known as the TSK fuzzy model) was proposed by Takagi, Sugeno, and Kang in an effort to develop a systematic approach to generating fuzzy rules from a given input-output data set
- A typical fuzzy rule in a Sugeno fuzzy model has the form

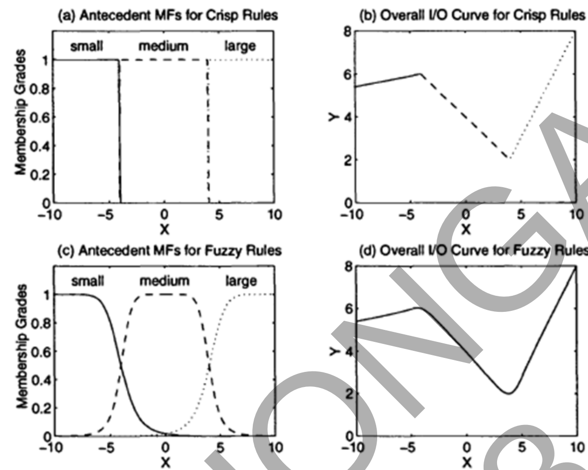
$$\boldsymbol{if\ x\ is\ A\ and\ y\ is\ B\ then\ z\ =\ f\ (\ x, y)},$$

- where A and B are fuzzy sets in the antecedent, $while\ z\ = f\ (\ x, y)$ is a crisp function in the consequent
- Usually $f(x, y)$ is a polynomial in the input variables x and y but it can be any function as long as it can appropriately describe the output of the model within the fuzzy region specified by the antecedent of the rule

Example : *Fuzzy and nonfuzzy rule set-a comparison*

- An example of a single-input Sugeno fuzzy model can be expressed as:

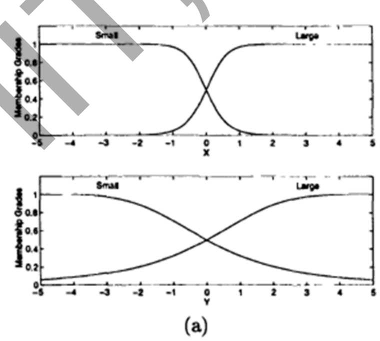
$$\begin{cases} If\ X\ is\ small\ then\ Y = 0.1X + 6.4 \\ If\ X\ is\ medium\ then\ Y = -0.5X + 4 \\ If\ X\ is\ large\ then\ Y = X - 2 \end{cases}$$



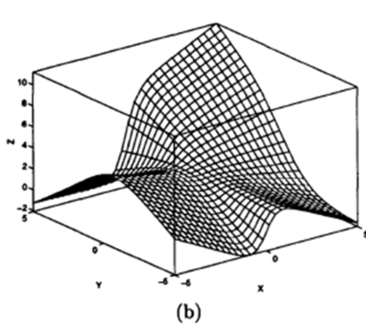
Example: *Two-input single-output Sugeno fuzzy model*

- An example of a two-input single-output Sugeno fuzzy model with four rules can be expressed as:

$$\begin{cases} If\ X\ is\ small\ andY\ is\ small\ then\ z = -x + y + 1 \\ If\ X\ is\ small\ andY\ is\ large\ then\ z = -y + 3 \\ If\ X\ is\ large\ andY\ is\ small\ then\ z = -x + 3 \\ If\ X\ is\ large\ andY\ is\ large\ then\ z = x + y + 2 \end{cases}$$



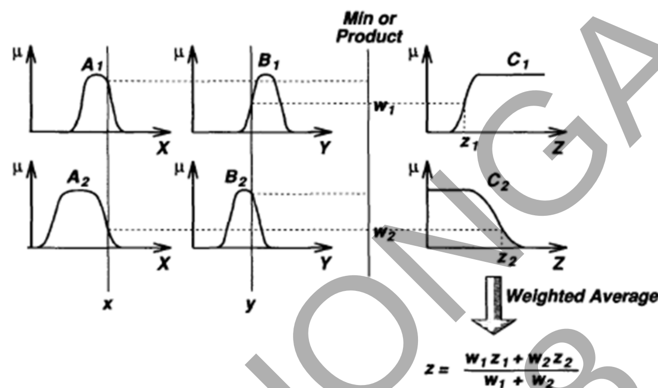
(a)



(b)

TSUKAMOTO FUZZY MODELS

- In the Tsukamoto fuzzy models , the consequent of each fuzzy if-then rule is represented by a fuzzy set with a monotonical MF
- The inferred output of each rule is defined as a crisp value induced by the rule's firing strength
- The overall output is taken as the weighted average of each rule's output
- Since each rule infers a crisp output, the Tsukamoto fuzzy model aggregates each rule's output by the method of weighted average and thus avoids the time-consuming process of defuzzification



Example : *Single-input Tsukamoto fuzzy model*

- An example of a single-input Tsukamoto fuzzy model can be expressed as:

$$\begin{cases} If\ X\ is\ small\ then\ Y\ is\ C1 \\ If\ X\ is\ medium\ then\ Y\ is\ C2 \\ If\ X\ is\ large\ then\ Y\ is\ C3 \end{cases}$$