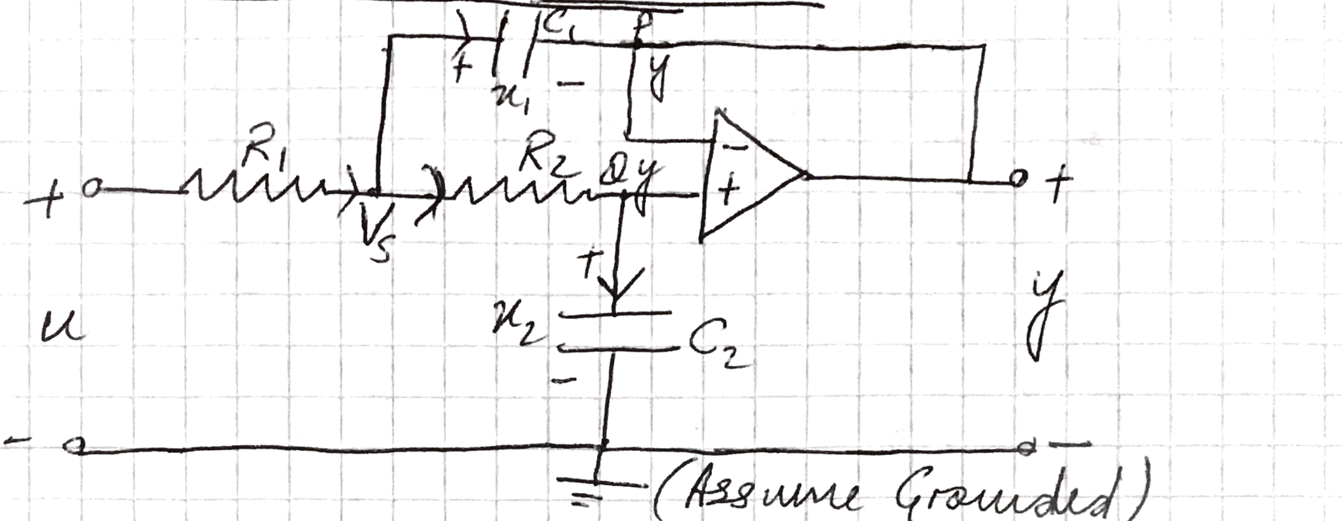
ECE 4550 - Problem Set #3 AADIT PANDEY

1.



a)

Applying Node Analysis at Point Q,

$$\frac{V_S - y}{R_2} = C_2 \dot{x}_2 \quad \text{—— (I)}, \quad \text{where } y = x_2$$

Node Analysis at Pt. V_S,

$$\frac{u - V_S}{R_1} = \frac{V_S - y}{R_2} + C_1 \dot{x}_1 \quad \text{—— (II)}$$

$$V_S - y = x_1 \Rightarrow V_S = x_1 + y = V_S = x_1 + x_2$$

Substituting this in (I) & (II):

$$\frac{x_1}{R_2} = C_2 \dot{x}_2 \Rightarrow \dot{x}_2 = \frac{1}{R_2 C_2} x_1$$

$$\frac{u - x_1 - x_2}{R_1} = \frac{x_1}{R_2} + C_1 \dot{x}_1$$

$$\dot{x}_1 = -x_1\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) - \frac{x_2}{R_1 C_1} + \frac{u}{R_1 C_1}$$

Therefore,

$$\dot{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & -\frac{1}{R_1 C_1} \\ \frac{1}{R_2 C_2} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1 C_1} \\ 0 \end{bmatrix} u(t)$$

b) First let us find the characteristic polynomial:

$$\det(sI - A) = \begin{vmatrix} s + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & \frac{1}{R_1 C_1} \\ -\frac{1}{R_2 C_2} & s \end{vmatrix}$$

$$= s^2 + s\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) + \frac{1}{R_1 R_2 C_1 C_2} \quad \text{———} \; \mathrm{I}$$

Note: If α, β are roots of $x^2 + bx + c$ then
$x^2 + bx + c = x^2 - (\alpha + \beta)x + \alpha\beta$.

Using this fact the roots of I are:

$\alpha = -\frac{1}{R_1 C_1}$ and $\beta = -\frac{1}{R_2 C_2}$. Thus,

$$\det(sI - A) = \left(s + \frac{1}{R_1 C_1}\right)\left(s + \frac{1}{R_2 C_2}\right)$$

So the system has <u>two</u> modes and they are located at: $s = -\frac{1}{R_1 C_1}$, $s = -\frac{1}{R_2 C_2}$

(c) Since we have <u>strictly positive components</u>, thus all the modes satisfy $Re\{s\} < 0$, hence the system is internally stable! (All roots lie inside the open ~~half~~ left-half plane!)

2. With $M_1 = 1$, $M_2 = 1$, $K_c = 1$ and $D_c = 0$ we have

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x$$

a) First, we find the characteristic polynomial of the system.

$$\det(sI - A) = \begin{vmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & -1 & s & 0 \\ -1 & 1 & 0 & s \end{vmatrix}$$

$$= s \begin{vmatrix} s & 0 & -1 \\ -1 & s & 0 \\ 1 & 0 & s \end{vmatrix} - \begin{vmatrix} 0 & s & -1 \\ 1 & -1 & 0 \\ -1 & 1 & s \end{vmatrix}$$

$$= s\left[s^3 - 1(-s)\right] - \left[-s(s) - (1-1)\right]$$

$$= s^4 + s^2 + s^2 = s^4 + 2s^2 = s^2(s^2+2)$$

$$= s^2(s+\sqrt{2}j)(s-\sqrt{2}j)$$

The system has 4 modes located at:

$s = 0$ (2), $s = -\sqrt{2}j$ and $s = \sqrt{2}j$

Since all the modes do not satisfy $Re\{s\} < 0$, the system is NOT internally stable.

b) $y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$

We know $H(s) = C(sI - A)^{-1}B + D$

$$H(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & -1 & s & 0 \\ -1 & 1 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$H(s) = \frac{s^2+1}{s^4+2s^2}$$ (Check Python Code Attached)

i) The system has 4 poles and they are located at:

$s = 0$ (2 repeated poles), $s = \sqrt{2}j$, $s = -\sqrt{2}j$.

ii) The system has 2 zeros and they are located at:

$s = j$, $s = -j$.

(C) In this case, the transfer function is $H(s) = \dfrac{1}{s^4+2s^2}$

i) The system has 4 poles and they are located at:

$s = 0$ (2 repeated poles), $s = \sqrt{2}j$, $s = -\sqrt{2}j$

ii) The system has 0 zeros.

ECE4550HW3 - Computations

September 12, 2019

[41]:
```
import sympy as sym
s = sym.symbols('s')
X = sym.Matrix([[s,0,-1,0],[0,s,0,-1],[1,-1,s,0],[-1,1,0,s]])
```

[42]:
```
Q = X.inv()
```

[53]:
```
C = sym.Matrix([1,0,0,0])
B = sym.Matrix([[0],[0],[1],[0]])
```

[55]:
```
sym.simplify(C.dot(Q*B))
```

[55]:
$$\frac{s^2+1}{s^2\left(s^2+2\right)}$$

[ ]:
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