MODULE : 3

FUNCTIONS OF COMPLEX VARIABLE *

Let w be a complex variable (z also). Corresponding to each value of z, there is a value for variable w, we say w is a fn of z, and write it as $w = f(z)$.

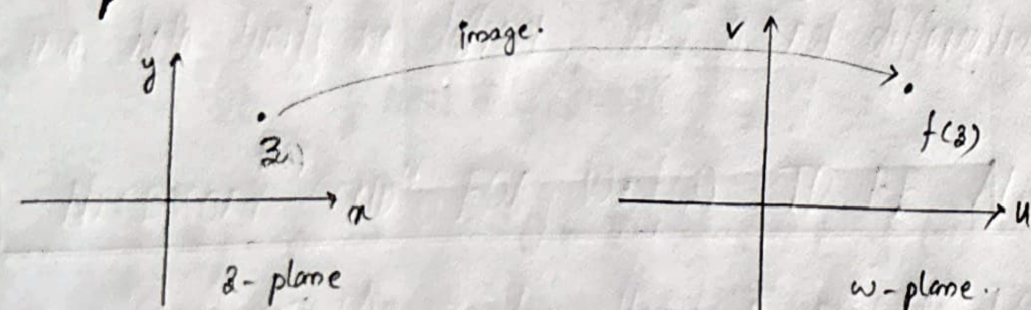
Note :- In general, $w = f(z)$ can be put in the form $u + iv$, where u & v are functions of x & y.

eg: * $w = z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy) = u + iv$

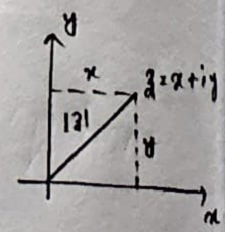
* $w = e^z = e^{(x+iy)} = e^x \cdot e^{iy} = e^x[\cos y + i \sin y] = e^x \cos y + i[e^x \sin y] = u + iv.$

* $w = \sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cdot \sinh x + [\cos x \sinh y] i$
$= u + iv.$

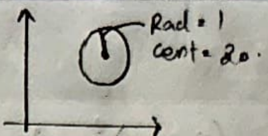
Note ① :-
Consider $w = f(z)$, to represent $z = x + iy = (x, y)$, we require a plane called z-plane / xy plane. Similarly to represent $w = f(z) = u + iv = (u, v)$, we require another plane called w-plane / (uv)-plane.



Note ② :- $|z| = |x+iy| = \sqrt{x^2+y^2}$ is the distance of z from origin.
ie $|z_1 - z_2|$ gives distance b/w z_1 & z_2.



eg: $|z - z_0| = 1$ $\Rightarrow$



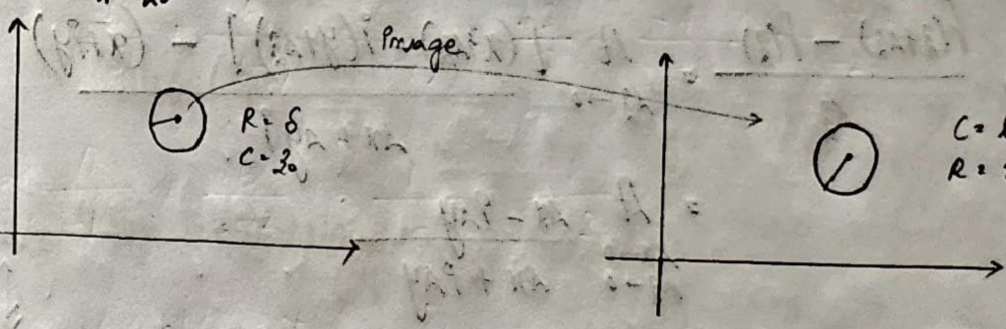
Also $z = x+iy$, $\bar{z} = x - iy$ $\Rightarrow$ $z + \bar{z} = 2x$ & $z - \bar{z} = 2iy$

$$\boxed{x = \frac{z+\bar{z}}{2}} \quad \boxed{y = \frac{z-\bar{z}}{2i}}$$

LIMIT OF A FUNCTION

Consider $w = f(z)$ tends to l, as $z \to z_0$. If $|f(z) - l| < \varepsilon$ whenever $|z - z_0| < \delta$. Then

$$\lim_{z \to z_0} f(z) = l \quad \left(|f(z) - l| < \varepsilon \text{ whenever } |z - z_0| < \delta \right).$$



Note: in the real case x can approach a in only 2 ways, by choosing values of $x > a$ or less than a. [LHL & RHL]. But in the complex case z can approach z_0 by choosing any value of z closed to z_0. ie z can approach z throug any curve passing through z_0. ie z approches z_0 in infinite ways.

Note: If $w = f(z)$ is differentiable, not only @ z_0, but in a neighbourt, then $f(z)$ is said to be analytic.

① $w = z^2 = f(z)$

$$\frac{dy}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \frac{z^2 + \Delta z^2 + 2\Delta z \cdot z - z^2}{\Delta z}$$

$$= \lim_{z \to 0} \frac{\Delta z[2z + \Delta z]}{\Delta z} = \lim_{\Delta z \to 0} 2z + \Delta z = \underline{\underline{2z}}$$

$w = z^2$ is analytic and $\frac{dw}{dz} = 2z$.

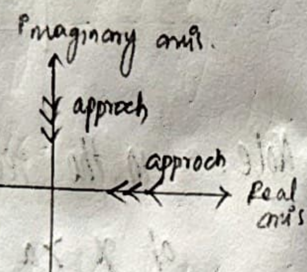
PYQ

② $w = \bar{z} = f(z) \Rightarrow f(z+\Delta z) = (\overline{z+\Delta z})$ Put $z = x+iy$
$\bar{z} = x - iy$.

ie $(\overline{z+\Delta z}) = (\overline{x+iy + \Delta x + i\Delta y}) = \overline{(x+\Delta x) + i(y+\Delta y)} = (x+\Delta x) - i(y+\Delta y)$

ie $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{[(x+\Delta z) - i(y+\Delta z)] - (\overline{x+iy})}{\Delta x + \Delta y i}$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$



Case (i) :- Let Δz is real : $\Delta y = 0 \Rightarrow \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta x} = 1$
(point in x axis)

Case (ii) :- Let Δz is imaginary : $\Delta x = 0 \Rightarrow \lim_{\Delta z \to 0} \frac{-\Delta y}{\Delta y} = -1$

Since both limits are diffrent, $w = \bar{z}$ is not differentiable.

A NECESSORY CONDⁿ FOR $w = f(z)$ TO BE ANALYTIC

If $w = f(z) = u + iv$ is analytic, then 4 partial differentiations

$\frac{\partial u}{\partial x}$; $\frac{\partial u}{\partial y}$; $\frac{\partial v}{\partial x}$; $\frac{\partial v}{\partial y}$ should exist and satisfy the equation

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Proof :- Given $f(z) = w$ is analytic, $w = u + iv$, then

then $\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left(u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)\right) - \left(u(x,y) + i\,v(x,y)\right)}{\Delta z}$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left[u(x+\Delta x, y+\Delta y) - u(x,y)\right]}{\Delta x + i\Delta y} + i\,\frac{\left[v(x+\Delta x, y+\Delta y) - v(x,y)\right]}{\Delta x + i\Delta y} \longrightarrow ①$$

Case (i) : Let Δz is real $\Rightarrow \Delta y = 0$

$$\frac{dw}{dy} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \partial u / \partial x + i \frac{\partial u}{\partial y}.$$

Case (ii) :- Let Δz is imaginary : $\Delta x = 0$

$$\frac{dw}{dz} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\,\frac{v((x+\Delta z), y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since $f(z) = w$ is analytic, limits should be equal.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Hence : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

Note ① :- $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are denoted by U_x, U_y, V_x & V_y.

Note ② :- The necessory cond^n becomes $U_x = V_y$ & $U_y = -V_x$.

These are called Caushy - Riemann equations.

Note ③ :- If $w = f(z) = u + iv$ is analytic, then

$$f'(z) = u_x + iV_x \quad (\text{OR}) \quad f'(z) = V_y - iV_y$$

PYQ *

Note ④ :- A function $f = f(x,y)$ is said to be harmonic, if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad (\text{Laplace D.E}).$$

If $w = x+iy = u+iv$ is analytic, then u & v are harmonic.

To prove this :-

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

U is harmonic. $\therefore$ Sly V is harmonic.

Note ⑤ :- If $w = f(z) = u + iv$ is analytic, then $U(x,y) = C_1$ & $V = V(x,y) = C_2$ cut orthogonally.

Proof : $f(z) = u + iv$ is analytic : $U_x = V_y$ & $U_y = -V_x$

Consider $U = U(x,y) = C_1$

$m_1 = dy/dx = \frac{-\partial u/\partial x}{\partial u/\partial y} = \frac{-U_x}{U_y}$, Consider $V(x,y) = C_2$

$$m_2 = \frac{dy}{dx} = \frac{-V_x}{V_y}$$

$m_1 \cdot m_2 = \frac{-u_x}{u_y} \cdot \frac{-v_x}{v_y} = -1$, $u=c_1$ & $v=c_2$ are analytic.

Note ⑥ :- Useing If $f(z)=w=u+iv$ is analytic, then u & v are harmonic.

v is called harmonic conjugate.

If u is given, we can find v. Method :

Given $u \Rightarrow$ we can find u_x & u_y , But $-u_y = v_x \rightarrow \frac{\partial v}{\partial x} = -v_y$

$$\partial v = -v_y dx \quad \Rightarrow \quad v = -\int v_y dx + \psi(y) \longrightarrow ①$$

Simly : $v_y = u_x \Rightarrow \frac{\partial v}{\partial x} = u_x \rightsquigarrow v = \int v_x dy + \psi(x) \longrightarrow ②$

Combining ① & ② we get v.

Note ⑦ : <u>Milne - Thompson method of Constructing an analytical ~~method~~ function.</u>
<u>when $w = f(z) = u+iv$ & u and v are given</u>

Note : 8 :

Cauchy - Riemann eq^n's are not sufficient condition for $f(z)$ to be analytic.
ie there are fx^n which satisfy Cauchy-Riemann eq^n, but not differentiable.

PYQ

Eg. : Prove that $f(z) = \sqrt{|xy|}$ satisfy C-R eq's at the origin, but not differentiable at that point.

Ans : $f(z) = \sqrt{|xy|} = \sqrt{|xy|} + i0$ $\begin{cases} u = \sqrt{|xy|} \\ v = 0 \end{cases}$

General data.

But $u_x = \dfrac{\partial u}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

at $(0,0) \rightarrow$ $u_x = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x}$

Hence : $u_x = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x} = \dfrac{0-0}{x} = 0$

$u_y = \lim\limits_{y \to 0} \dfrac{f(0,y) - f(0,0)}{y} = \dfrac{0-0}{y} = 0$

$\Bigg\}$ $u_x = v_y$ and $u_y = -v_x$ (C-R eq^n satisfied).

Also $v = 0 \rightarrow v_x = 0 \quad v_y = 0$

$f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$, At $z=0 \rightarrow f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(\Delta z) - f(0)}{\Delta z}$

General data.

$= \lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z}$.

ie $f'(z) = \lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim\limits_{z \to 0} \dfrac{\sqrt{|xy|} - 0}{x+iy}$.

Let $z \to 0$ along the line $y = mx$, then $f'(0) = \lim\limits_{x \to 0} \dfrac{\sqrt{|x \cdot mx|} - 0}{x + i\,mx}$

ie $f'(0) = \frac{\sqrt{m}}{1+im}$ (since limit depends on m, it is not unique).

ie $f(z)$ is not differentiable at origin.

Qn$_2$:- Prove that $w = f(z) = \cancel{xy^2(x+iy)}\ \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, if $z \neq 0$

$= 0$, if $z = 0$

satisfy C-R equations, but not differentiable.

Ans: $f(z) = \frac{x^3[1+i] - y^3[1-i]}{x^2+y^2} = \frac{x^3 - y^3}{x^2+y^2} + i\frac{(x^3+y^3)}{(x^2+y^2)}$

$= U + iV$, if $z \neq 0$

$= 0$ if $z = 0$.

$U_x = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$

$U_y = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = \frac{-y^3}{y^3} = -1$

$V_x = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x}$ (at $0,0$)

$= \lim_{x \to 0} \frac{x - 0}{x} = 1$

$V_y = \lim_{x \to 0} \frac{V(0,y) - V(0,0)}{y} = \frac{y - 0}{y} = 1$

C·R Equations satisfied.

$f'(z) = f'(0) = \lim_{z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$.

$f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \frac{(x^3-y^3) + i(x^3+y^3)}{(x^2+y^2)\cdot(x+iy)}$

Let $z \to 0$ along line $y = mx$, then

$$f'(z) = \lim_{z\to 0} \frac{(x^2 - m^3x^3) + i(x^3 + m^3x^3)}{(x^2 + m^2x^2) * (x + imx)} = \frac{(1-m^3) + i(1+m^3)}{(1+m^2)(1+im)}.$$

ie $f'(z)$ depends on $m \Rightarrow$ It is not unique $\Rightarrow$ not differentiable.

Qn$_3$: Prove that $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$, if $z \neq 0$

$= 0$, if $z = 0$

Satisfy C-R equation at $(0,0)$, but not differentiable.

Ans: $f(z) = \frac{x^2y^2}{x^2+y^4} + i \cdot \frac{xy^3}{x^2+y^4} = U + iV$

$U_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \frac{0}{x} = 0$, $\quad U_y = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = 0$

$V_x = \lim_{x\to 0} \frac{V(x,0) - V(0,0)}{x} = 0$, $\quad V_y = \lim_{y\to 0} \frac{V(0,y) - f(0,0)}{y} = 0$

Hence C-R eq^n is satisfied.

$$f'(z) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{(x^2y^2) + i(xy^3)}{(x^2+y^4)(x+iy)} = \lim_{z\to 0} \frac{xy^2}{x^2+y^4}$$

Let $z \to 0$ along $y = mx$

the fx^n depends on x.

$f'(z) = \frac{m^2}{(1+m^4x^2)} \cdot x$

$\cdot\ 0/1 = 0$

Let $z \to 0$ along $y = \sqrt{x}$ / $y^2 = x$.

then $f'(z) = \frac{x \cdot x}{x^2 + x^2} = 1/2$. Since limit is not unique, not differentiable.

Qn ④: Prove that following fn's are harmonic and also find the harmonic Conjugate.

(i) $\frac{1}{2}\log(x^2+y^2)$ (ii) $e^x \sin y$ (iii) $\sin x \cosh y$ (iv) x^2-y^2.

Ans: (i) $u = \frac{1}{2}\log(x^2+y^2)$: $\partial u/\partial x = \frac{1}{2}\cdot\frac{2x}{(x^2+y^2)} = \frac{x}{x^2+y^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

simly: $\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$, ie $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Harmonic)

Condition for harmo-

$u_x = \frac{x}{x^2+y^2}$ $u_y = \frac{y}{x^2+y^2}$, from C-R eqn $v_x = -u_y$

ie $v_x = -1\times\frac{y}{x^2+y^2} \Rightarrow v = -\int\frac{y}{x^2+y^2}dx = -y\cdot\frac{1}{y}\tan^{-1}(x/y) + f(y)$ a integ. const.

also $v_y = u_x \rightarrow v_y = \frac{x}{x^2+y^2} \Rightarrow v = x\int\frac{1}{x^2+y^2}dy = \frac{x}{x}\tan^{-1}\left(\frac{y}{x}\right) + f(x)$ a integ. const.

ie we can say $v = \tan^{-1}(y/x) - \tan^{-1}(x/y) = \tan^{-1}(y/x) + c$

(ii) $u = e^x \sin y \Rightarrow u_x = \sin y\cdot e^x$ $u_y = e^x \cos y$

$u_{xx} = \frac{\partial^2 u}{\partial x^2} = e^x \sin y$ $u_{yy} = -e^x \cos y$ $\Rightarrow u_{xx} + u_{yy} = 0$ (Harmonic)

$v_x = -u_y = -e^x \cos y \Rightarrow v = -\int e^x \cos y\, dx = -e^x \cos y + f(y)$ —— ①

$v_y = u_x = e^x \sin y \Rightarrow v = \int e^x \sin y\, dy = -e^x \cos y + f(x)$ —— ②

Combining ① & ② : $V = -e^{x}\cos y + C$

Qn: Using Millie-Thompson method, find $f(z) = u + iv$, an analytic fn of

① $u = x$ ② $u = x^3 - 3xy^2 + y + 1$ ③ $e^x \sin y$ ④ $V = 3x^2y - y^3$

⑤ $V = 2xy$ ⑥ $V = x^4 - 6x^2y^2 + y^4$ ⑦ $u + v = \frac{x}{x^2+y^2}$. ⑨ $2u + v = e^x(\cos y - \sin y)$

Ans: ① $u = x$, $u_x = \frac{\partial u}{\partial x} = 1$ $u_y = 0$ $V_x = -u_y = 0$

$f'(z) = u_x + iV_x = 1 + i(0) = 1$

put $x = z$, $y = 0$, then $f'(z) = 1 \longrightarrow$ $f(z) = \int f'(z)dz = \int 1\,dz = z + c$

② $u = x^3 - 3xy^2 + y + 1 \Rightarrow u_x = 3x^2 - 3y^2$ $u_y = -6xy + 1$

$V_x = -u_y = 6xy - 1$

$f'(z) = u_x + iV_x = 3(x^2 - y^2) + i(6xy - 1)$

put $x = z$, $y = 0 \Rightarrow$ $f'(z) = 3z^2 - i$

the $f(z) = \int f'(z)\,dz = \int (3z^2 - i)\,dz = z^3 - iz + c$

③ $u = e^x \sin y \rightarrow u_x = e^x \sin y$ $u_y = e^x \cos y$ $V_x = -u_y = -e^x \cos y$

$f'(z) = e^x \sin y - [e^x \cos y]i = e^x[\sin y - i\cos y]$

the if $x = z$, $y = 0 \Rightarrow f'(z) = -ie^z$

$f(z) = \int f'(z)\,dz = -ie^z + c$

(iv) $v = 3x^2y - y^3 \rightarrow v_x = 6xy \quad v_y = 3x^2 - 3y^2 = 3(x^2 - y^2)$

$u_x = v_y = 3(x^2 - y^2)$

ie $f'(z) = u_x + i v_x = 3(x^2-y^2) + i \cancel{(x^2-y^2)}\, 6xy.$

put $x = z,\ y = 0 \Rightarrow f'(z) = 3z^2 \Rightarrow f(z) = \underline{z^3 + c}$

(v) $v = 2xy \Rightarrow v_x = 2y \quad v_y = 2x \Rightarrow u_x = v_y = 2x \Rightarrow f'(z) = 2(x + iy)$

Now putting $z = x$ & $y = 0 \Rightarrow f'(z) = 2z.$

ie $f(z) = \int f'(z)\,dz = \underline{z^2 + c}$

(vi) $v = x^4 - 6x^2y^2 + y^4 \rightarrow u_x = v_y = -12x^2y + 4y^3 \quad v_x = 4x^3 - 12xy^2$

ie $f'(z) = u_x + i v_x = 4y^3 - 12x^2y + i(4x^3 - 12xy^2).$

now $x = z$ & $y = 0 \rightarrow f'(z) = i4z^3 \Rightarrow f(z) = \int f'(z)\,dz = \underline{iz^4 + c}$

$\underline{f(z) = iz^4 + c}$

(vii) $u + v = \dfrac{x}{x^2+y^2} \Rightarrow u_x + v_x = \dfrac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \dfrac{y^2 - x^2}{(x^2+y^2)^2} \longrightarrow (1)$

$u_y + v_y = \dfrac{-2xy}{(x^2+y^2)^2} = -v_x + u_x \longrightarrow (2)$

Solve (1) & (2) $\Rightarrow$ (1) + (2) $\rightarrow \quad 2u_x = \dfrac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}$

$u_x = \dfrac{1}{2}\left[\dfrac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}\right]$

① - ② $\Rightarrow$ $2V_x = \frac{y^2-x^2+2xy}{(x^2+y^2)^2}$ $\Rightarrow$ $V_x = \frac{1}{2}\left[\frac{-x^2+y^2+2xy}{(x^2+y^2)^2}\right]$

$f'(z) = U_x + iV_x = \frac{1}{2}\left[\frac{(y^2-x^2-2xy)+i(y^2-x^2+2xy)}{(x^2+y^2)}\right]$

puting $z = x$ & $y = 0$ $\Rightarrow$ $f'(z) = \frac{(1+i)}{2}\cdot\left(\frac{-1}{z^2}\right)$.

ie $f(z) = \int f'(z)\,dz = \left(\frac{1+i}{2}\right)\int \frac{-1}{z^2}\,dz = \frac{1+i}{2z}$

(IX) $2U+V = e^x(\sin y - \cos y)$ $\Rightarrow$ $2U_x + V_x = e^x[\sin y - \cos y]$ ⟶ ①

$2U_y + V_y = -e^x\sin y - e^x\cos y. = -2V_x + U_x$ → ②

$2\times$① + ② $\Rightarrow$ $5U_x = 2e^x[\cos y - \sin y] + e^x[\sin y - \cos y]$

$U_x = \frac{e^x}{5}[\cos y - 3\sin y]$

② × ② + ① $\Rightarrow$ $5V_x = e^x(3\cos y + \sin y]$

$f'(z) = U_x + iV_x = \frac{e^x}{5}[\cos y - 3\sin y] + \frac{ie^x}{5}[3\cos y + \sin y]$.

puting $x = z$ & $y = 0$ $\Rightarrow$ $f'(z) = \frac{1}{5}[e^z][1+3i]$.

$f(z) = \int f'(z)\,dz = \int \frac{e^z}{5}\,dz\cdot(1+3i) = \frac{(1+3i)}{5}\cdot e^z$ //.

PYQ ★ Qn: If $f(z) = U+iV$ is analytic, then with constant modulus, P.T the $f(z)$ = a const.

Ans: Given $f(z) = U+iV$ analytic

then $U_x = V_y$ $-V_x = U_y$

then $|f(z)| = \sqrt{u^2+v^2} = a$ const $= K$; 17 mark qn

to PT, $f(z) = a$ const $\rightarrow f'(z) = 0$ $\left\langle \begin{array}{l} u_x = 0 \\ v_x = 0 \end{array} \right.$

we have $u^2 + v^2 = k \Rightarrow$ d w r t x $\rightsquigarrow 2u \cdot u_x + 2v \cdot v_x = 0$

ie $uu_x + vv_x = 0 \longrightarrow ①$

d.w.r.t y $\Rightarrow uu_y + vv_y = 0 \Rightarrow$ from C-R eqn : $u_x = v_y$ $u_y = -v_x$ $\longrightarrow ②$

ie $u(-v_x) + v(u_x) = 0 \longrightarrow ③$

Solving $\rightarrow$ $① \cdot u + ③ \cdot v = u^2 u_x + v^2 u_x = 0 \Rightarrow u_x[u^2+v^2] = K u_x = 0$

$\Rightarrow u_x = 0$ (generally).

$u_x = 0$ in ③ $\Rightarrow v_x = 0 \Rightarrow f'(z) = u_x + i v_x = 0$ /

ie $f(z)$ is a constant.

pya

Qn: If $w = f(z) = u + iv$ is analytic, P T it is independent on $\bar{z}$.

Ans: $f(z) = u + iv =$ Analytic $\left\langle \begin{array}{l} u_x = v_y \\ v_y = -v_x \end{array} \right.$ (CR eqn).

So to prove $\frac{\partial w}{\partial \bar{z}} = 0$ (independent on $\bar{z}$).

since $u = u(x,y)$; $v = v(x,y)$ $\Rightarrow$

$$\begin{array}{l} z = u + iv \\ \bar{z} = u - iv \\ \hline z + \bar{z} = 2u \longrightarrow u = \frac{1}{2}(z + \bar{z}) \\ z - \bar{z} = 2iv \longrightarrow v = \frac{1}{2i}[z - \bar{z}] \end{array}$$

$\frac{\partial x}{\partial z} = \frac{1}{2}$, $\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$, $\frac{\partial y}{\partial z} = \frac{1}{2i}$, $\frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$, Then $w = u + iv$

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left(\frac{u_x}{2} + \frac{-v_y}{2i}\right) + i\left(\frac{v_x}{2} + \frac{v_y}{-2i}\right) \Rightarrow \text{from C.R eq}^n$$

$$= \frac{u_x}{2} - \frac{u_y}{2i} + \frac{v_x i}{2} + \frac{v_y}{2} = \frac{2u_x}{2} + \frac{i}{2}v_y + \frac{i}{2}v_y - \frac{i}{2}v_y$$

$$= \underline{\underline{0}}$$

$$u_x = 0 \quad \& \quad v_x = 0 \longrightarrow \partial w / \partial \bar{z} = 0 \text{ (independ.)}.$$

Qn8- If $f(z) = u + iv$ is analytic, P T $\left(\partial^2/\partial x^2 + \partial^2/\partial y^2\right)(R\, f(z))^2 = 2\left[f'(z)\right]^2$

Ans: Given $u + iv = f(z) =$ analytic $\left\langle \begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix} \right.$ and

u & v are harmonic $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$.

So $\quad R\, f(z) = u \rightarrow [R\, f(z)]^2 = u^2$

$\bullet\ \frac{\partial u^2}{\partial x} = \frac{\partial}{\partial x}(R\, f(z))^2 = 2uu_x \longrightarrow \frac{\partial^2}{\partial x^2}(R\, f(z))^2 = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left[\frac{\partial u}{\partial x}\right]^2\right]$

Simly : $\frac{\partial^2}{\partial y^2}(R f(z))^2 = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$$\left[\partial^2/\partial x^2 + \partial^2/\partial y^2\right](u^2) = 2u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + 2\left[u_x^2 + u_y^2\right] = 0 + 2\left[u_x^2 + u_y^2\right]$$

$$= 0 + 2\left[u_x^2 + (-v_x)^2\right] = 2\left[u_x^2 + v_x^2\right]$$

$$= \underline{\underline{2|f'(z)|^2}}.$$

$$|f'(z)| = \sqrt{u_x^2 + v_x^2}$$

Qn: If $w = f(z) = u + iv$ is analytic, P T $\left(\partial/\partial x\, |f(z)|\right)^2 + \left(\partial/\partial y\, |f(z)|\right)^2$

$= |f'(z)|^2$.

Ans:- $f(z) = u + iv$ is analytic & $u_x = v_y$; $u_y = -v_x$

$$|f(z)| = \sqrt{u^2+v^2} \Rightarrow \frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}\sqrt{u^2+v^2} = \frac{1}{2\sqrt{u^2+v^2}}(2uu_x + 2vv_x)$$

$$= \frac{(uu_x + vv_x)}{\sqrt{u^2+v^2}}$$

Simly ; $\frac{\partial}{\partial y}|f(z)| = \frac{uu_y + vv_y}{\sqrt{u^2+v^2}}$. So LHS becomes

$$\frac{(uu_x+vv_x)^2 + (uu_y+vv_y)^2}{(u^2+v^2)} = \frac{u^2u_x^2 + v^2v_x^2 + 2uvv_xu_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y}{(u^2+v^2)}$$

applying C.R equations now,

then $= \frac{u^2u_x^2 + v^2v_x^2 + \cancel{2uvu_xv_x} + u^2v_x^2 + v^2u_x^2 + \cancel{-2uvu_xv_x}}{u^2+v^2}$

$$= \frac{u_x^2\cancel{[u^2+v^2]} + v_x^2\cancel{[u^2+v^2]}}{\cancel{u^2+v^2}} = u_x^2 + v_x^2 = \underline{\underline{|f(z)|^2}}$$

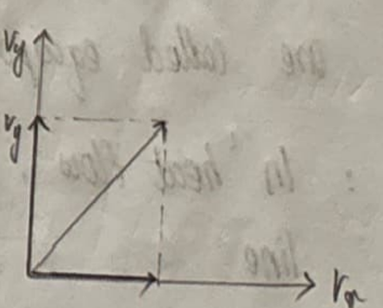
HW :- If $f(z) = u + iv$ is analytic, the $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(i\, f(z))^2 = 4[f'(z)]^2$

APPLICATION IN FLOW PROBLEM

(i) ~~Proportional~~ Irrotational motion of an incompressible fluid on 2-D

Let $\bar{v}$ be the velocity of the fluid particle in xy plane. Let v_x and v_y be the components of velocity along ox & oy.

$$\bar{V} = V_x i + V_y j \longrightarrow (1)$$



Motion is irrotational :- $\because$ Curl $\bar{V} = 0$

$$\bar{V} = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} \longrightarrow ③$$

ie from ① & ② $\quad V_x = \frac{\partial\phi}{\partial x}$, $\quad V_y = \partial\phi/\partial y$.

Again the motion is incompressible :- $\quad$ div $\bar{V} = 0$

$$\text{div } f = \sum i\frac{\partial f}{\partial x} \Rightarrow \text{div } \bar{V} = i\frac{\partial V_x}{\partial x} + j\frac{\partial V_y}{\partial y} = 0$$

Hence $\quad i\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = 0 \Rightarrow \boxed{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0}$

* Hence ϕ = Harmonic. So we can find $w = \phi + i\psi$ which is analytic where ψ is harmonic conjugate.
* Now we know $\phi = c$ & $\psi = c'$ cut orthogonally.

Consider $\psi = c' \Rightarrow \psi(x,y) = c'$ (from CR eq^n)

$$dy/dx = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} = \frac{\partial\phi/\partial y}{\partial\phi/\partial x} = V_y/V_x$$

* Velocity of fluid particle is along $\psi(x,y) = c'$. ie particle moves along this curve and $\psi(x,y)$ is called stream fn^n, & $\psi(x,y) = c'$ are called stream line.

* Note: in electrostatic and gravitational force, $\phi = c$ & $\psi = c'$ are called equipotential lines and lines of force.

Note$_2$: In heat flow, $\phi = c$ & $\psi = c'$ called irrotational & heat-flow line

Qn: If $w = \phi + i\psi$ represents a complex potential on electric field and $\psi = x^2 - y^2 + \frac{x}{(x^2+y^2)}$. Find ϕ

Ans:- $\psi = x^2 - y^2 + \frac{x}{(x^2+y^2)} \rightarrow \psi_x = 2x + \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2} \longrightarrow ①$

$\psi_y = -2y + \frac{-x}{(x^2+y^2)^2} \cdot 2y = -\left[2y + \frac{2xy}{(x^2+y^2)^2}\right] \longrightarrow ②$

From C.R eq^n's : $-\psi_x = \phi_y$ & $\psi_y = +\phi_x$

$$\partial\phi/\partial x = \phi_x = \psi_y = -\left[2y + \frac{2xy}{(x^2+y^2)^2}\right]$$

$$\phi = -\int\left(2y + \frac{2xy}{(x^2+y^2)^2}\right)dx = -2xy + -y\int\frac{2x}{(x^2+y^2)^2}dx = -2xy + y/(x^2+y^2) + f_1 \longrightarrow ③$$

Also $\phi_y = -\psi_x = -\left[2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right] = \frac{\partial\psi}{\partial y}$

$\Rightarrow \psi = -\int\left(2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right)dy = -2xy -$

Now from ③ : $\phi = -2xy + y/(x^2+y^2) + f_1(y)$, d.w.r.t. y gives

$$\frac{d\phi}{dy} = -2x + \frac{(x^2-y^2)}{(x^2+y^2)^2} + f_1'(y) \longrightarrow ④$$

② & ④ are identical, Hence, $\phi_y = -\psi_x$ and equal to $-2x + \frac{(x^2-y^2)}{(x^2+y^2)^2} \longrightarrow ⑤$.

From ④ & ⑤ $\rightarrow$ $f_1'(y) = 0$ and $f_1(y) = c$ (Not a fx^n of y.).

ie we can say $\phi = -2xy + \frac{(x^2-y^2)}{(x^2+y^2)} + c$ //

Qn: An electrostatic field in the xy plane is given by the potential fn
$\phi = 3x^2y - y^3$. Find the stream fn.

Ans:- Given $\phi = 3x^2y - y^3 \Rightarrow \phi_x = 6xy \quad \phi_y = 3x^2 - 3y^2$

From CR eqn: $\phi_x = \psi_y$ & $-\psi_x = \phi_y$, then

then $\psi = \int -(3x^2 - 3y^2)\,dx = -x^3 + 3xy^2 + f_1(y) \longrightarrow (1)$

$\psi = \int 6xy\,dy = 6x\frac{y^2}{2} + f_2(x) = 3xy^2 + f_2(x) \longrightarrow (2)$

then $\psi = 3xy^2 - x^3 + c$

Qn: In a 2-D flow, the stream fn ψ is given, find velocity potential.

(i) $\psi = \frac{-y}{x^2+y^2}$ (ii) $\psi = \tan^{-1}(y/x)$.

Ans: (1) $\psi = \frac{-y}{x^2+y^2} \Rightarrow \psi_x = \frac{+2xy}{(x^2+y^2)^2} \longrightarrow (1)$ $\psi_y = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = y^2 - x^2/(x^2+y^2)^2 \longrightarrow (2)$

then $\phi_x = \psi_y = \frac{y^2-x^2}{(y^2+x^2)^2} \Rightarrow \phi = \int \frac{y^2-x^2}{(x^2+y^2)^2}\,dx$

of the form $\frac{-1}{u^2} \rightarrow \frac{d}{dx}(1/u)$

$\phi_y = -\psi_x \Rightarrow \phi = -\int \frac{2xy}{(x^2+y^2)^2}\,dy = x\int \frac{-2y}{(x^2+y^2)^2}\,dy = \frac{x}{(x^2+y^2)} + f_1(x). \longrightarrow (3)$

From (3): $\phi_x = \frac{y^2-x^2}{(x^2+y^2)^2} + f_1'(x) \longrightarrow (4)$, also from CR eqn $\phi_x = \psi_y \Rightarrow \frac{y^2-x^2}{(x^2+y^2)^2} + 0 \longrightarrow (5)$

From (4) & (5) $f_1'(x) = 0$, ie $\phi = \frac{x}{(x^2+y^2)} + c$

(11) $\psi = \tan^{-1}(y/x) \Rightarrow \psi_x = \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2} = \frac{-y}{(x^2+y^2)}$

$\psi_y = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$

then $\phi_x = \psi_y = \frac{x}{x^2+y^2} \Rightarrow \phi = \int \frac{x}{x^2+y^2}\,dx = \frac{1}{2}\log(x^2+y^2) + f_1(y)$

$\phi_y = -\psi_x = \frac{y}{x^2+y^2} \Rightarrow \phi = \int \frac{y}{x^2+y^2}\,dy = \frac{1}{2}\log(x^2+y^2) + f_1(x).$

then $\phi = \log\sqrt{x^2+y^2}.$

SUMMATION SERIES: C + iS. method.

* Given a cosine series, to find sum we take C = given cosine series and S is the corresponding sine series. Now we find C+iS, which can be simplified using idea $\cos\theta + i\sin\theta = e^{i\theta}$, so that the sum C+iS reduces to one of the standard forms given below.
* Now equating real or imaginary parts, we can find sine series.

Std. formulae :- • $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

• $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

• $\sin x = x - x^3/3! + x^5/5! - \cdots$

• $\cos x = 1 - x^2/2! + x^4/4! - \cdots$

• $\sinh x = x/1! + x^3/3! + x^5/5! + \cdots = \frac{e^x - e^{-x}}{2}$

• $\cosh x = 1 + x^2/2! + x^4/4! + \cdots = \frac{e^x + e^{-x}}{2}.$

- $\log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$
- $\log(1-x) = -x - x^2/2 - x^3/3 - x^4/4 + \cdots$
- $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x + x^3/3 + x^5/5 + \cdots$
- $\tan^{-1}x = x - x^3/3 + x^5/5$
- $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots$
- $(1+x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$
- $a + ar + ar^2 + \cdots = a[1 + r + r^2 + \cdots] = a/(1-r)$
- $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \cdots$
- $(1+x)^{-1/2} = 1 - x/2 + \frac{1\cdot 3}{2\cdot 4}x^2 + \cdots$

Qn: Find the following sum:

① $1 + \frac{1}{2}\cos x + \frac{1}{4}\cos 2x + \frac{1}{6}\cos 3x + \cdots$

Ans:

$$C = 1 + \tfrac{1}{2}\cos x + \tfrac{1}{4}\cos 2x + \tfrac{1}{6}\cos 3x$$

$$S = 0 + \tfrac{1}{2}\sin x + \tfrac{1}{4}\sin 2x + \tfrac{1}{6}\sin 3x$$

$$C + iS = 1 + \tfrac{1}{2}(\cos x + i\sin x) + \tfrac{1}{4}(\cos 2x + i\sin 2x) + \cdots$$

$$= 1 + \tfrac{1}{2}e^{ix} + \tfrac{1}{4}e^{i2x} + \tfrac{1}{6}e^{i3x} + \cdots$$

$$= 1 + \tfrac{1}{2}y + \tfrac{1}{4}y^2 + \tfrac{1}{6}y^3 + \cdots \quad \text{If } y = e^{ix}$$

$$= 1 + \tfrac{1}{2}\left[y + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 + \cdots\right] = 1 + \tfrac{1}{2}(-\log(1-y))$$

$$= 1 + \log\frac{1}{\sqrt{1-e^{ix}}}$$

$\cos 2x - 1$, $1 - 2\sin^2 x$ $\sin^2 x + \cos^2 x = 1$, $\cos^2 x - \sin^2 x$, $\sin x$ $\frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$

ie $C + iS = 1 - \tfrac{1}{2}\log(1 - e^{ix}) = 1 - \tfrac{1}{2}[(1-\cos x) - i\sin\theta]\log$

$$= 1 - \tfrac{1}{2}\left[\log\left((1-\cos x)^2 + \sin^2 x\right)\right] + \frac{-i}{2}\tan^{-1}\left(\frac{\sin\theta}{1-\cos\theta}\right)$$

$$= 1 - \tfrac{1}{2}\log\left(2(1-\cos\theta)\right) + -\tfrac{i}{2}\tan^{-1}\left(\frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}\right)$$

$\log(a+ib) = \log(re^{i\theta})$
$= \log r + \log e^{i\theta}$
$= \log\sqrt{a^2+b^2} + i\theta$
where $\theta = \tan^{-1}(y/x)$.

$= 1 + -\frac{1}{2}\log(4\sin^2\theta/2) + -\frac{i}{2}\tan^{-1}(\cot\theta/2)$

ie $C = \log r = \frac{1}{2}\log(4\sin^2\theta/2)$.

(ii) $1 + C\cos\theta + \frac{c^2}{2!}\cos 2\theta + \frac{c^3}{3!}\cos 3\theta + \cdots\cdots$

Ans :- Let $C = 1 + c\cos\theta + \frac{c^2}{2!}\cos 2\theta + \cdots$

$S = c\sin\theta + \frac{c^2}{2!}\sin 2\theta$

$C + iS = 1 + c(\cos\theta + i\sin\theta) + \frac{c^2}{2!}(\cos\theta + i\sin 2\theta)$

$= 1 + ce^{i\theta} + \frac{c^2}{2!}e^{i2\theta}$

$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ if $x = ce^{i\theta}$

$= e^x = e^{ce^{i\theta}} = e^{c(\cos\theta + i\sin\theta)} = e^{c\cos\theta} \cdot e^{ic\sin\theta}$

(iii) $\sum_1^\infty \frac{\sin n\theta}{2^n}$; find the sum to infinity (∞).

Ans :- Given $S = \sum_1^\infty \frac{\sin n\theta}{2^n} = \frac{\sin\theta}{2} + \frac{\sin 2\theta}{4} + \frac{\sin 3\theta}{8} + \cdots\cdots + \frac{\sin n\theta}{2^n}\Big|_\infty$.

$C = \sum_1^\infty \frac{\cos n\theta}{2^n} = \frac{\cos\theta}{2} + \frac{\cos 2\theta}{4} + \frac{\cos 3\theta}{8} + \cdots$

$C + iS = \frac{1}{2}[\cos\theta + \sin\theta i] + \frac{1}{4}[\cos 2\theta + i\sin 2\theta] + \frac{1}{8}[\cos 3\theta + i\sin 3\theta] + \cdots\cdots$

$= \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{i2\theta} + \frac{1}{8}e^{i3\theta} + \cdots\cdots = \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \cdots$ If $e^{i\theta} = x$

$= x/2\left[1 + x/2 + x/2^2 + x/2^3 + \cdots\right] = \frac{x/2}{1 - x/2} = \frac{x}{2 - x} = \frac{e^{i\theta}}{2 - e^{i\theta}}$

$= \frac{\cos\theta + i\sin\theta}{2 - [\cos\theta + i\sin\theta]} = \frac{[\cos\theta + i\sin\theta][2 + \cos\theta + i\sin\theta]}{[2 - \cos\theta + i\sin\theta][2 + (\cos\theta + i\sin\theta)]} \Rightarrow \frac{\sin\theta\cos\theta + \sin\theta[2 - \cos\theta]}{((2 - \cos\theta)^2 + (i\sin\theta)^2)}$

$$= \frac{\sin\theta\cos\theta + \sin\theta\left[2-\cos\theta\right]}{5 - 4\cos\theta} = S$$

Qn 8- $\cos\alpha\cos\alpha + \cos^2\alpha\cos 2\alpha + \cos^3\alpha\cos 3\alpha + \dots$, find the sum

Ans:
$$C = \sum_1^\infty \cos n\alpha \cdot \cos^n\alpha = \cos\alpha \cos\alpha + \cos^2\alpha\cos 2\alpha + \dots$$

$$S = \cos\alpha\sin\alpha + \cos^2\alpha\sin 2\alpha + \cos^3\alpha\sin 3\alpha + \dots$$

$$\Rightarrow C + iS = \cos\alpha\left[\cos\alpha + i\sin\alpha\right] + \cos^2\alpha\left[\cos 2\alpha + i\sin 2\alpha\right] + \dots$$

$$= \cos\alpha\, e^{i\alpha} + \cos^2\alpha\, e^{i2\alpha} + \cos^3\alpha\, e^{i3\alpha} + \dots$$

$$= \cos\alpha\, e^{i\alpha}\left[1 + \cos\alpha\, e^{i\alpha} + \cos^2\alpha\, e^{i2\alpha} + \dots\right] = \frac{a}{1-r} = \frac{\cos\alpha\, e^{i\alpha}}{1 - \cos\alpha\, e^{i\alpha}}$$

$$= \frac{\cos\alpha\left[i\sin\alpha + \cos\alpha\right]\cancel{(1+\cos\alpha\, e^{i\alpha})}}{(1-\cos\alpha\, e^{i\alpha})\cancel{(1+\cos\alpha\, e^{i\alpha})}} \Rightarrow$$

$$C + iS = \frac{\cos^2\alpha + i\sin\alpha\cos\alpha \cdot \left((1-\cos^2\alpha) + i\sin\alpha\right)}{(1-\cos^2\alpha - i\sin\alpha)\left(1-\cos^2\alpha + i(\sin\alpha)\right)} = \frac{\cos\alpha\left[(1-\cos^2\alpha)\cos\alpha - \sin^2\alpha\cos\alpha\right]}{(1-\cos^2\alpha)^2 + \sin^2\alpha\cos\alpha}$$

$$= \underline{\underline{0}}$$

Qn 8- $\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1\cdot 3}{2\cdot 4}\cos 3\theta + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 4\theta + \dots$

Ans:-
$$C = \cos\theta + \frac{1}{2}\cos 2\theta + \frac{1\cdot 3}{2\cdot 4}\cos 3\theta + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 4\theta + \dots$$

$$iS = i\sin\theta + \frac{i}{2}\sin 2\theta + \frac{i\cdot 3}{2+4}\cos 3\theta + \dots$$

$$C + iS = [\cos\theta + i\sin\theta] + \frac{1}{2}[\cos 2\theta + i\sin 2\theta] + \frac{1\cdot 3}{2\cdot 4}[\cos 3\theta + i\sin 3\theta] + \dots$$

$$= e^{i\theta} + \frac{1}{2}e^{i2\theta} + \frac{3}{2\cdot 4}\cdot e^{i3\theta} = x + \frac{x^2}{2} + \frac{1\cdot 3}{2\cdot 4}x^3 + \dots \quad \left\{ e^{i\theta} = x \right.$$

$$= x(1-x)^{-1/2} = e^{i\theta}\left[1 - e^{i\theta}\right]^{-1/2} = \frac{\cos\theta + i\sin\theta}{\sqrt{1 - [\cos\theta + i\sin\theta]}}$$

$= [\cos\theta + i\sin\theta]\left[(2\sin^2\theta/2) - (i\,2\sin\theta/2\cos\theta/2)\right]^{-1/2} \cdot (\cos\theta + i\sin\theta)\left[2\sin\theta/2\left[\sin\theta/2 - i\cos\theta/2\right]\right]^{-1/2}$

apply De-Moivre's theorem

$= \dfrac{\cos\theta + i\sin\theta}{\sqrt{2\sin\theta/2}} \cdot \left[\cos\theta/2 - i\sin\theta/2\right]^{-1/2} = \dfrac{\cos\theta + i\sin\theta}{\sqrt{2\sin\theta/2}}\left[\cos\left[\pi/2 - \frac{\theta}{2}\right] - i\sin\left(\pi/2 - \theta/2\right)\right]^{-1/2}$

$= \dfrac{\cos\theta + i\sin\theta}{\sqrt{2\sin\theta/2}}\left[\cos\left(\frac{\pi}{4} - \frac{\theta}{4}\right) + i\sin\left[\frac{\pi}{4} - \frac{\theta}{2}\right]\right]$

ie $C = \dfrac{\cos\theta \cdot \cos\left[\frac{\pi-\theta}{4}\right]}{\sqrt{2\sin\theta/2}} - \dfrac{\sin\theta\left(\sin\left(\frac{\pi-\theta}{4}\right)\right)}{\sqrt{2\sin\theta/2}} = \dfrac{1}{\sqrt{2\sin\theta/2}}\cos\left[\theta + \dfrac{(\pi-\theta)}{4}\right]$

$$\boxed{[\cos\theta \pm i\sin\theta]^n = \cos n\theta \pm i\sin n\theta}$$

decide wether s/c

Qn 8- $\cos x \cdot \sin x + \frac{1}{2!}\cos^2 x\,\sin 2x + \frac{1}{3!}\cos^3 x\,\sin 3x + \cdots$

Ans: Let $S = \cos x\sin x + \frac{1}{2!}\cos^2 x\sin 2x + \cdots$

$C = \cos x\cos x + \frac{1}{2}\cos^2 x\cos 2x + \cdots$

$C + iS = \cos x[\cos x + i\sin x] + \dfrac{\cos^2 x}{2!}(\cos 2x + i\sin 2x) + \cdots$

$= \cos x\, e^{ix} + \dfrac{\cos^2 x}{2!}e^{i2x} + \dfrac{\cos^3 x}{3!}e^{i3x} + \cdots$

$= \cos x\, e^{ix}\left[1 + \dfrac{\cos x\, e^{ix}}{2!} + \dfrac{\cos^2 x\, e^{i2x}}{3!} + \cdots\right]$ $\qquad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$y = \cos x\, e^{ix}$

$C + iS = e^y - 1$

$= e^{\cos x\, e^{ix}} - 1 = e^{\cos x[\cos x + i\sin x]} - 1 = e^{\cos^2 x} \cdot e^{i\cos x\sin x} - 1$

$= e^{\cos^2 x}\left[\cos(\sin x\cos x) + i\sin(\sin x\cos x)\right] - 1$

ie $S = e^{\cos^2 x}\sin(\sin x\cos x) - 0$ //

Qn 8- $\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{2!}\sin(\alpha+2\beta) + \cdots\cdots$

Ans 8- $S = \sin\alpha + x\sin(\alpha+\beta) + x^2/2!\,(\sin(\alpha+2\beta)) + \cdots\cdots$

$C = \cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{2!}\cos(\alpha+2\beta) + \cdots\cdots$

$C+iS = \cos\theta + i\sin\theta + x[\cos(\alpha+\beta) + i\sin(\alpha+\beta)] + \cdots = e^{i\theta} + xe^{i(\alpha+\beta)} + \frac{x^2}{2!}e^{i2(\alpha+\beta)}$

$= e^{i\alpha} + xe^{i\alpha}e^{i\beta} + \frac{x^2}{2!}e^{i\alpha}e^{i2\beta} + \cdots\cdots = e^{i\alpha}\left[1 + xe^{i\beta} + \frac{x^2}{2!}e^{i2\beta} + \cdots\right]$

$= e^{i\alpha}\left[1 + y + \frac{y^2}{2!} + \cdots\right] = e^{i\alpha}e^{y} = e^{i\alpha}e^{xe^{i\beta}}$ $\quad y = xe^{i\beta}.$

$= \cancel{e^{xi(\alpha+\beta)}} \cdot \cancel{e^{x}e^{i(\alpha+\beta)}}$

ie $C+iS = [\cos\alpha + i\sin\alpha]\,e^{x(\cos\beta + i\sin\beta)} = [\cos\alpha + i\sin\alpha]\,e^{x\cos\beta}\cdot e^{ix\sin\beta}$

$= [\cos\alpha + i\sin\alpha]\,e^{x\cos\beta}[\cos(x\sin\beta) + i\sin(x\sin\beta)] \Rightarrow e^{x\cos\beta}\left[\cos\alpha\sin(x\sin\beta) + \sin\alpha\cos(x\sin\beta)\right] = S$

$\sin A\cos B + \cos A\sin B.$

ie $S = e^{x\cos\beta}\sin[\alpha + x\sin\beta]$

Qn 8- $1 - \frac{\cos 2\theta}{2!} + \frac{\cos 4\theta}{4!} - \frac{\cos 6\theta}{6!} + \cdots$, find sum.

Ans:- $C = 1 - \frac{\cos\theta}{2!} + \frac{\cos 4\theta}{4!} + \cdots$

$iS = 0 - \frac{i\sin\theta}{2!} + \frac{i\cos 4\theta}{4!} + -\frac{i\sin 6\theta}{6!} + \cdots$ $\quad x = e^{i\theta}$

$C+iS = 1 - \frac{1}{2!}e^{i\theta} + \frac{1}{4!}e^{i4\theta} - \frac{1}{6!}e^{i6\theta} + \cdots = 1 - x^2/2! + x^4/4! + \cdots$

$= \cos x = \cos[e^{i\theta}] = \cos[i\sin\theta + \cos\theta] = \cos[\cos\theta]\cos(i\sin\theta) - \sin(\cos\theta)\sin(i\sin\theta)$

$= \cos(\cos\theta)\cosh(\sin\theta) - \sin(\cos\theta)\sinh(\sin\theta)$

$C = \cos(\cos\theta)\cosh(\sin\theta)$

Qn 8- If $C = \cos^2\theta - \frac{1}{3}\cos^3\theta\cos 3\theta + \frac{1}{5}\cos^5\theta\cos 5\theta + \cdots$, P T $\tan 2C = 2\cot^2\theta$.

Ans 8-

$$C = \cos\theta\cos\theta - \frac{1}{3}\cos^3\theta\cdot\cos 3\theta + \frac{1}{5}\cos^5\theta\cos 5\theta + \cdots$$

$$S = \cos\theta\sin\theta - \frac{1}{3}\cos^3\theta\sin 3\theta + \frac{1}{5}\cos^5\theta\sin 5\theta + \cdots$$

$$C+iS = \cos\theta\cdot e^{i\theta} - \frac{1}{3}\cos^3\theta\, e^{i3\theta} + \frac{1}{5}\cos^5\theta\, e^{i5\theta} - \cdots$$

gregories series $\leftarrow$ $= x - x^3/3 + x^5/5 - \cdots$ where $x = \cos\theta\, e^{i\theta}$

$$C+iS = \tan^{-1}(x) = \tan^{-1}(\cos\theta e^{i\theta}) = \tan^{-1}(\cos\theta[\cos\theta + i\sin\theta]) = \tan^{-1}(\cos^2\theta + i(\sin\theta\cos\theta))$$

ie $\tan(C+iS) = \cos^2\theta + i\sin\theta\cos\theta \longrightarrow (1)$ & $\tan(C-iS) = \cos^2\theta - i\sin\theta\cos\theta \longrightarrow (2)$

$$\tan 2C = \tan((C+iS)+(C-iS)) = \frac{\tan(C+iS)+\tan(C-iS)}{1-\tan(C+iS)\tan(C-iS)} = \frac{\cos^2\theta + \cancel{i\sin\theta\cos\theta} + \cos^2\theta - \cancel{i\sin\theta\cos\theta}}{1-[\cos^4\theta + \sin^2\theta\cos^2\theta]}$$

$$= \frac{2\cos^2\theta}{1-\cos^2\theta[\cos^2\theta+\sin^2\theta]} = \frac{2\cos^2\theta}{1-\cos^2\theta} = \underline{\underline{2\cot^2\theta}}.$$

Qn 8- $\sin\theta + \frac{\sin 3\theta}{3!} + \frac{\sin 5\theta}{5!} + \cdots$

Ans 8-

$$S = \sin\theta + \frac{\sin 3\theta}{3!} + \frac{\sin 5\theta}{5!}$$

$$C = \cos\theta + \frac{\cos 3\theta}{3!} + \frac{\cos 5\theta}{5!} + \cdots$$

$$C+iS = e^{i\theta} + \frac{1}{3!}e^{i3\theta} + \frac{1}{5!}e^{i5\theta} + \cdots \quad \Rightarrow e^{i\theta} = x$$

$$= x + x^3/3! + x^5/5! + \cdots$$

$$= \sinh x = \sinh[e^{i\theta}] = \sinh[\cos\theta + i\sin\theta] =$$

$$= -i\sin(i(\cos\theta + i\sin\theta)) = -i\sin(i\cos\theta - \sin\theta)$$

$$= -i[\sin(i\cos\theta)\cos(\sin\theta) + \cos(i\cos\theta)\sin(\sin\theta)] = -i[i\sinh(\cos\theta)\cos(\sin\theta) - \cosh(\cos\theta)\sin(\sin\theta)]$$

$$\boxed{\begin{array}{l}\sin(ix) = i\sinh x \\ \sinh x = \frac{1}{i}\sin x\end{array}}$$

Hence $S = \cosh(\cos\theta)\sin(\sin\theta)$. .