MODULE : 2

LAPLACE TRANSFORMS

Definition :- Let $F(t)$ be a function defined for $t>0$, then Laplace transform of $F(t)$, defined by $L\{F(t)\}$ is defined as

$$\boxed{L\{F(t)\} = \int_0^\infty e^{-st} \cdot F(t)\, dt}$$

where s is a parameter. The RHS is a function of s, call it $\bar{f}(s)$. ie

$$\boxed{L\{f(t)\} = \bar{F}(s)}$$

$$\xrightarrow[\text{input}]{f(t)} \boxed{\text{Laplace transforms}} \xrightarrow[\text{output}]{\bar{F}(s)}$$

$f(t)$ is called inverse laplace transform of $\bar{f}(s)$.

LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

① $L\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{-1}{s} \cdot \left[\frac{1}{e^{st}}\right]_0^\infty = \frac{-1}{s}\left[\frac{1}{e^\infty} - \frac{1}{1}\right] = 1/s$$

$$\boxed{L\{1\} = \frac{1}{s}}$$

② $$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}\cdot e^{at}\,dt = \int_0^\infty e^{(a-s)t}\cdot dt = \left[\frac{e^{(a-s)t}}{(a-s)}\right]_0^\infty$$

$$= \frac{1}{(a-s)}\left[e^{(a-s)t}\right] = \frac{-1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^\infty = \frac{-1}{s-a}[0-1] = \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}} \quad \text{OR} \quad \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Similarly $$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}} \quad \text{OR} \quad \mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}.$$

$$\boxed{\begin{aligned}\int e^{ax}\cdot\sin bx\,dx &= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\\ \int e^{ax}\cos bx\,dx &= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\end{aligned}}$$

③ $$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\cdot\sin at\,dt$$

$$= \left(\frac{e^{-st}}{s^2+a^2}\left[(-s)\sin at - a\cos at\right]\right)_0^\infty$$

$$= \left(\frac{-e^{-st}}{s^2+a^2}\left[s\sin at + a\cos at\right]\right)_0^\infty = \frac{-1}{a^2+s^2}\left[0-\left(\frac{0+a\times 1}{1}\right)\right] = \frac{a}{a^2+s^2}$$

$$\boxed{\mathcal{L}\{\sin at\} = \frac{a}{a^2+s^2}} \quad \text{OR} \quad \mathcal{L}^{-1}\left\{\frac{a}{a^2+s^2}\right\} = \sin at.$$

④ $$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cdot\cos at\,dt$$

$$= \left[\frac{e^{-st}}{a^2+s^2}\left((-s)\cos at + a\sin at\right)\right]_0^\infty = \frac{1}{a^2+s^2}\left[\frac{-s\cos at + a\sin at}{e^{st}}\right]_0^\infty$$

$$= \frac{1}{a^2+s^2}\left[0-\left(\frac{0+s\times 1}{1}\right)\right] = \frac{+s}{a^2+s^2}$$

$$\boxed{\mathcal{L}\{\cos at\} = \frac{s}{a^2+s^2}}$$

(5) $L\{\sinh at\} = L\{\frac{e^{at}-e^{-at}}{2}\}$

$$= \int_0^\infty e^{-st}\left[\frac{e^{at}-e^{-at}}{2}\right] dt = \frac{1}{2}\int_0^\infty e^{(a-s)t} dt - \frac{1}{2}\int_0^\infty e^{-(a+s)t} \cdot dt$$

$$= \frac{1}{2} L\{e^{at}\} - \frac{1}{2} L\{e^{-at}\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

$$\boxed{L\{\sinh at\} = \frac{a}{s^2-a^2}}$$

(6) $L\{\cosh at\} = L\{\frac{e^{at}+e^{-at}}{2}\} = \frac{1}{2}\int_0^\infty e^{-st}(e^{at}+e^{-at})\, dt.$

$$= \frac{1}{2}\left[\int_0^\infty e^{-st}\cdot e^{at} dt + \int_0^\infty e^{-st}\cdot e^{-at} dt\right] = \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}$$

$$\boxed{L\{\cosh at\} = \frac{s}{s^2-a^2}}$$

LINEARITY PROPERTY. *

(i) $\boxed{L\{a f(t) + b g(t)\} = a \cdot L\{f(t)\} + b\, L\{g(t)\}}$

Proof :- $L\{a f(t) + b\, g(t)\} = \int_0^\infty e^{-st}(a\, f(t) dt + b\, g(t) dt)$

$$= a \int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt = a \cdot L\{f(t)\} + b\, L\{g(t)\}$$

(ii) SHIFTING PROPERTY

VIP

If $L\{f(t)\} = \bar{F}(s)$, then $\boxed{L\{e^{at}\cdot f(t)\} = \bar{F}(s-a)}$

Proof :- * Given $L\{f(t)\} = \bar{F}(s) \Rightarrow \int_0^\infty e^{-st}\cdot f(t)\,dt = \bar{F}(s) \longrightarrow (1)$

* $L\{e^{at}\cdot f(t)\} = \int_0^\infty e^{-st}\cdot e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t}\cdot f(t)\,dt = \bar{F}(s-a)$

Note :- If $L\{f(t)\}$ is $\bar{F}(s)$, then

$$L\{e^{-at}\cdot f(t)\} = \bar{F}(s-a)$$

Meaning :- 1) $L\{\sin t\} = \frac{1}{s^2+1} \Rightarrow L\{e^{2t}\cdot \sin t\} = \frac{1}{(s-2)^2+1}$

$L\{e^{-3t}\sin t\} = \frac{1}{(s+3)^2+1}$

ii) $L\{\cosh bt\} = \frac{s}{s^2-a^2} \Rightarrow L\{\cosh 2t\} = \frac{s}{s^2-4}$

$L\{e^{t}\cdot \cosh 2t\} = \frac{s-1}{(s-1)^2-4}$

$L\{e^{-2t}\cosh 3t\} = \frac{s+2}{(s-2)^2-9}.$

Note :- If $L\{f(t)\} = \bar{F}(s)$, then

$$\boxed{L\{\sinh at\cdot f(t)\} = \frac{1}{2}\left[\bar{F}(s-a) - \bar{F}(s+a)\right]}$$

Proof :- $L\{\sinh at\cdot f(t)\} = L\left\{\left[\frac{e^{at}-e^{-at}}{2}\right] f(t)\right\} = \frac{1}{2}\left[L\{e^{at}\cdot f(t)\} - L\{e^{-at}\cdot f(t)\}\right]$

$= \frac{1}{2}\left[\bar{F}(s-a) - \bar{F}(s+a)\right]$

(1) Similarly :- $L\{\cosh at\cdot f(t)\} = \frac{1}{2}\left[\bar{F}(s-a) + \bar{F}(s+a)\right]$

(7) $L\{t^n\} = n!/s^{(n+1)}$

* $L\{t\} = \int_0^\infty e^{-st}\cdot t\,dt = t\cdot\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 1\cdot\frac{e^{-st}}{-s}\,dt = \frac{-1}{s}\left[t/e^{st}\right]_0^\infty + \frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty$

$= \frac{-1}{s}\left[\lim_{t\to\infty}\frac{t}{e^{st}}\right] + \frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty$ → L. Hospital rule.

$= 0/1$

$= -1/s \times 0 + 1/s \cdot L\{1\} = \frac{1}{s}\cdot\frac{1}{s} = 1/s^2$

ie $L\{t\} = 1/s^2$

* $L\{t^2\} = \int_0^\infty e^{-st}\cdot t^2\,dt = t^2\cdot\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 2t\cdot\frac{e^{-st}}{-s}\,dt = \frac{-1}{s}\left[\frac{t^2}{e^{st}}\right]_0^\infty + \frac{2}{s}L\{t\}$

$= -1/s\,[0-0] + \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$

* similarly $L\{t^3\} = 1\times2\times3/s^4$, $L\{t^5\} = 5!/s^5$ -----

STANDARD RESULTS

(I) $L\{1\} = 1/s$

(II) $L\{e^{at}\} = \frac{1}{s-a}$

(III) $L\{e^{-at}\} = \frac{1}{s+a}$

(IV) $L\{\sin at\} = \frac{a}{s^2+a^2}$

(V) $L\{\cos at\} = \frac{s}{s^2+a^2}$

(VI) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

(VII) $L\{\cosh at\} = \frac{s}{s^2-a^2}$

(VIII) $L\{t\} = 1/t$

(IX) $L\{t^n\} = \frac{n!}{t^{n+1}}$

◉ Shifting property :- If $L\{f(t)\} = \bar{F}(s)$, then $L\{e^{at}f(t)\} = \bar{F}(s-a)$.

Qn :- Find the Laplace transform of following fxⁿs.

ⓐ $L\{\sin^2 3t\} = \int_0^\infty e^{-st}\cdot \sin^2 3t \cdot dt = L\left\{\frac{1-\cos 6t}{2}\right\} = \frac{1}{2}\left[L(1) - L(\cos 6t)\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right] = \frac{1}{2}\frac{\left[36 \cancel{-36} -s^2\right.}{(s^2+36)}$

$= \underline{18/s^2+36}$

ⓑ $L\{\sin 4t \cdot \cos 2t\} = \frac{1}{2} L\{2 \sin 4t \cos 2t\}$

$= \frac{1}{2} L\{\sin 6t + \cos 2t\}$

$= \underline{\frac{1}{2}\left[\frac{6}{s^2+36} + \frac{s}{s^2+4}\right]}$

$$\boxed{\begin{aligned} 2\sin A \cos B &= \sin(A+B) + \sin(A-B) \\ 2\cos A \cdot \sin B &= \sin(A+B) - \sin(A-B) \\ 2\cos A \cos B &= \cos(A+B) + \cos(A-B) \\ 2\sin A \cdot \sin B &= \cos(A-B) - \cos(A+B) \end{aligned}}$$

ⓒ $L\{\cos^2 t \cdot \cos 4t\} = L\left\{\left(\frac{1+\cos 2t}{2}\right) \cdot \cos 4t\right\}$

$= \frac{1}{2} L\{\cos 4t + \cos 4t \cdot \cos 2t\} = \frac{1}{2} L\{\cos 4t + \frac{1}{2} \cdot 2\cos 4t \cdot \cos 2t\}$

$= \frac{1}{2} L\{\cos 4t\} + \frac{1}{4} L\{\cos 6t + \cos 2t\}$

$= \underline{\frac{1}{2}\left[\frac{s}{s^2+4^2}\right] + \frac{1}{4}\left[\frac{6}{s^2+36} + \frac{s}{s^2+4}\right]}$

(HW) :- i) $\cos^2 3t$ ii) $\sin^2 t \cdot \sin 3t$ iii) $\cos 6t + \cos 4t + \cos 2t$.

Ans: i) $L\{\cos^2 3t\} = L\left\{\frac{1+\cos 6x}{2}\right\} = \frac{1}{2}\left[L\{1\} + L\{\cos 6x\}\right]$

$= \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+36}\right\}$

$= \underline{\frac{s^2+18}{s\left[s^2+36\right]}}$

ii) $L\{\sin^2 t \cdot \sin 3t\} = L\left\{\left(\frac{1-\cos 2t}{2}\right)\sin 3t\right\}$

$= \frac{1}{2}\left[\left(L(1) - L\{\cos 2t\}\right) \sin 3t\right]$

$= \frac{1}{2}\left[L\{\sin 3t\} + -L\{\cos 2t \cdot \sin 3t\}\right]$

$= \frac{1}{2}\left[\frac{6-3}{s^2+9} - \frac{1}{2}L\{\sin 5t + \sin t\}\right]$

$= \frac{1}{2}\left[\frac{3}{s^2+9} - \frac{1}{2}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)\right]$

iii) $L\{\cos 6t + \cos 4t + \cos 2t\} = L\{\cos 6t\} + L\{\cos 4t\} + L\{\cos 2t\}$

$= \frac{s^2}{s^2+36} + \frac{s^2}{s^2+16} + \frac{s^2}{s^2+4}$

(d) Find $L\{e^{3t}\sin^2 t\}$

Ans : $L\left\{e^{3t}\left[\frac{1-\cos 2t}{2}\right]\right\} = \frac{1}{2}L\{e^{3t} - e^{3t}\cos 2t\}$

$= \frac{1}{2}\left[L\{e^{3t}\} - L\{e^{3t}\cos 2t\}\right]$

$= \frac{1}{2}\left[\frac{1}{s-3} - \frac{s-3}{(s-3)^2+2^2}\right]$ → shifting property.

(e) $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$?

Ans :- $L\{e^{-3t} 2\cos 5t - e^{-3t} \cdot 3\sin 5t\} = L\{e^{-3t} \cdot 2\cos 5t\} - L\{3 \cdot e^{-3t} \cdot \sin 5t\}$

$= 2 \cdot \frac{(s+3)}{(s+3)^2+5^2} - 3 \cdot \frac{5}{(s+3)^2+25}$

(f) $L\{e^{4t} \cdot \sin 2t \cos t\} = L\left\{\frac{1}{2}\left[e^{4t} \cdot 2\sin 2t \cdot \cos t\right]\right\} = \frac{1}{2}L\{e^{4t}(\sin 3t + \sin t)\}$

$= \frac{1}{2}L\{e^{4t}\sin 3t\} + \frac{1}{2}L\{e^{4t} \cdot \sin t\}$

$= \frac{1}{2}\left[\frac{3}{(s-4)^2+3^2} + \frac{1}{(s-4)^2+1^2}\right]$

(g) $\mathcal{L}\{t^2 \cdot e^{2t}\}$

First we find $\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$ $\Rightarrow$ $\mathcal{L}\{t^2 e^{2t}\} = \frac{2!}{(s-2)^3}$

(h) $\mathcal{L}\{t^2 \cdot e^{3t} \cdot \sinh t\} = \mathcal{L}\left\{t^2 e^{3t}\left[\frac{e^t - e^{-t}}{2}\right]\right\} = \frac{1}{2}\mathcal{L}\left[t^2 e^{4t} - t^2 e^{2t}\right]$

$$= \frac{1}{2}\left[\frac{2!}{(s-4)^3} - \frac{2!}{(s-2)^3}\right]$$

(i) $\mathcal{L}\{t^3 \cdot \cosh 2t\} = \mathcal{L}\left\{t^3\left(\frac{e^{2t}+e^{-2t}}{2}\right)\right\} = \frac{1}{2}\mathcal{L}\{t^3 e^{2t} + t^3 e^{-2t}\}$

$$= \frac{1}{2}\left[\frac{3!}{(s-2)^4} + \frac{3!}{(s+2)^{4}}\right]$$

(j) Find $\mathcal{L}\{f(t)\}$ if $f(t) = |t-1| + |t+1|$; $t \geq 0$

Ans:-
$$f(t) = \begin{cases} -(t-1)+(t+1) & ; \text{ if } 0<t<1 \\ (t-1)+(t+1) & ; \ t>1 \end{cases} = \begin{cases} 2, & \text{if } 0<t<1 \\ 2t, & \text{if } t>1 \end{cases}$$

ie $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt = \int_0^1 e^{-st} f(t)\, dt + \int_1^\infty e^{-st} f(t)\, dt$

$$= \int_0^1 e^{-st} \cdot 2\, dt + \int_1^\infty e^{-st} \cdot 2t\, dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^1 + 2\left[t \cdot \frac{e^{-st}}{-s}\Big|_1^\infty - \int_1^\infty 1 \times \frac{e^{-st}}{-s}\, dt\right]$$

$$= \frac{-2}{s}\left[\frac{1}{e^{st}}\right]_0^1 + 2\left[\frac{-1}{s}\left[t/e^{st}\right]_1^\infty + \frac{1}{s} \cdot \frac{-1}{s}\left[\frac{1}{e^{st}}\right]_1^\infty\right]$$

$$= \frac{-2}{s}\left[\frac{1}{e^s} - 1\right] + 2\left[\frac{-1}{s}\left[0 - 1/e^s\right] + \frac{-1}{s^2}\left[0 - 1/e^s\right]\right]$$

$$= \frac{2e^{-s}}{s^2} + \frac{2}{s}$$

(k) Find $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} t^2, & \text{if } 0<t<2 \\ t, & \text{if } 2<t<3 \\ 7, & \text{if } t>3 \end{cases}$

Ans: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt = \int_0^2 + \int_2^3 + \int_3^\infty$

$$= \int_0^2 e^{-st} \cdot t^2 dt + \int_2^3 e^{-st} \cdot t\, dt + 7\int_3^\infty e^{-st} dt$$

$$= \left| t^2 \cdot \frac{e^{-st}}{-s} - 2t \cdot \frac{-1}{s} \cdot \frac{e^{-st}}{-s} + 2 \cdot \frac{-1}{s^3} e^{-st} \right|_0^2 + \left| t \cdot \frac{e^{-st}}{-s} - 1 \times \frac{-1}{s} \cdot \frac{e^{-st}}{-s} \right|_2^3$$

$$+ 7 \left| \frac{e^{-st}}{-s} \right|_3^\infty$$

$$= \left[\left(\frac{-4}{s} \cdot e^{-2s} - \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} \cdot e^{-2s} \right) - \left(0 - \frac{2}{s^3}\right) \right] + \left[\left(\frac{-3}{s} \cdot e^{-3s} - \frac{1}{s^2} e^{-3s} \right) - \left(\frac{-2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s} \right) \right] + \frac{-7}{s}\left[0 - \frac{1}{e^{3s}}\right]$$

$$= \frac{-4}{s} \cdot e^{-2s} - \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} \cdot e^{-2s} + \frac{2}{s^3} + \frac{-3}{s} \cdot e^{-3s} - \frac{e^{-3s}}{s^2} + \frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2} + \frac{7e^{-3s}}{s}$$

$$= e^{-2s} \left[\frac{-4}{s} - \frac{4}{s^2} - \frac{2}{s^3} + \frac{2}{s} + \frac{1}{s^2} \right] + e^{-3s} \left[\frac{-3}{s} - \frac{1}{s^2} + \frac{7}{s} \right]$$

MULTIPLICATION WITH t^n

If $\mathcal{L}\{f(t)\} = \bar{F}(s)$, then $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\bar{F}(s))$.

Proof :- $\mathcal{L}\{f(t)\} = \bar{F}(s) = \int_0^\infty e^{-st} \cdot f(t)\, dt$

diff. w.r.t "s"

$$\frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\, dt\right] = \frac{d}{ds}(\bar{F}(s)) \Rightarrow \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) \cdot f(t)\, dt = \frac{d}{ds} \bar{F}(s)$$

now we got :- $\int_0^\infty e^{-st}[t]\, f(t)\, dt = -\frac{d}{ds}(\bar{F}(s))$ $\frac{d}{dx}\left(\frac{1}{x^2+s^2}\right)$

ie, $\int_0^\infty e^{-st} \cdot t\, f(t)\, dt = (-1)\frac{d}{ds}(\bar{F}(s))$

$= L\{t f(t)\} = (-1)\frac{d}{ds}(\bar{F}(s))$

similarly ; $L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}(\bar{F}(s))$ and in general

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\bar{F}(s))}$$

DIVISION BY t

If $L\{f(t)\} = F(s)$, then $\boxed{L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds}$

Proof :- $L\{f(t)\} = \bar{F}(s) = \int_0^\infty e^{-st} f(t)\, dt$.

Integrating both sides $\Rightarrow$ $\int_s^\infty \bar{F}(s)\, ds = \int_0^\infty \left(\int_s^\infty e^{-st} f(t)\, dt\right) ds$

$= \int_0^\infty \left(\int_s^\infty e^{-st}\, ds\right) f(t)\, dt$

$= \int_0^\infty \left|\frac{e^{-st}}{-t}\right|_s^\infty f(t)\, dt = \int_0^\infty \left(\frac{-1}{t}\left[0 - \frac{1}{e^{st}}\right]\right) f(t)\, dt = \int_0^\infty e^{-st}\left\{\frac{f(t)}{t}\right\} dt.$

$= L\left\{\frac{f(t)}{t}\right\}$

Qn :- Find the Laplace transforms of following

(a) $\frac{1-e^t}{t}$

Ans :- $L\{(1-e^t)\} = L\{1\} - L\{e^t\} = \frac{1}{s} - \frac{1}{s-1} = \frac{-1}{s(s-1)}$

$\therefore L\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = \log s - \log(s-1)\Big|_s^\infty$

$$= \log\left(\frac{s}{s-1}\right)_s^{\infty} = \log\left(\frac{1}{(1-1/s)}\right)_s^{\infty} = 0 - \log\left(\frac{1}{(1-1/s)}\right) = \underline{\underline{\log\left(\frac{s-1}{s}\right)}}$$

ⓑ $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Ans: $L\{e^{-at}-e^{-bt}\} = L\{e^{-at}\} - L\{e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$

$$\therefore L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^{\infty}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds = \Big[\log(s+a) - \log(s+b)\Big]_s^{\infty}$$

$$= \log\left[\frac{s+a}{s+b}\right]_s^{\infty} = 0 - \log\left(\frac{s+a}{s+b}\right) = \underline{\underline{\log\left(\frac{s+b}{s+a}\right)}}$$

apply L'Hospital rule for ∞/∞ limit

ⓒ $L\left\{\frac{e^{-t}.\sin t}{t}\right\}$

* Here we can consider function as $\left(\frac{e^{-t}}{t}\right).\sin t$. But it will be harder.

Ans :- $L\left\{e^{-t}.\left(\frac{\sin t}{t}\right)\right\} \Rightarrow$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty}\frac{1}{1+s^2}ds = \tan^{-1}(s)\Big|_s^{\infty} = \underline{\underline{\pi/2 - \tan^{-1}s}}$$

ie $L\left\{e^{-t}\left(\frac{\sin t}{t}\right)\right\} = \underline{\underline{\pi/2 - \tan^{-1}(s+1)}}$

ⓓ $L\{t\cos 2t\}$

Ans:- we know; $L\{\cos 2t\} = \frac{s}{s^2+2^2}$

Now $L\{t^1\cdot\cos 2t\} = (-1)^1\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = (-1)\left[\frac{(s^2+4)\cdot 1 - s\,(2s)}{(s^2+4)^2}\right]$

$$= \underline{\underline{\frac{s^2-4}{(s^2+4)^2}}}$$

ⓔ $L\{t^3.e^{-3t}\}$

Ans :- $L\{e^{-3t}\} = \frac{1}{s+3}$ OR $L\{t^3\} = \frac{3!}{s^4}$

Then $\mathcal{L}\{e^{-3t}\cdot t^2\} = \dfrac{3!}{(s+3)^4}$ (easier way than first writing $\mathcal{L}\{e^{-3t}$

⑦ $\mathcal{L}\{te^{-t}\sin t\cdot 3\}$

Ans:- $\mathcal{L}\{\sin 3t\} = \dfrac{3}{s^2+9}$ and now $\mathcal{L}\{t\cdot\sin 3t\} = (-1)^1\dfrac{d}{ds}\left(\dfrac{3}{s^2+9}\right)$ — (1)

$$(1) \Rightarrow (-1)\cdot\frac{-3}{(s^2+9)^2}\cdot 2s = \frac{6s}{(s^2+9)^2}$$

ie $\mathcal{L}\{e^{-t}\cdot(t\sin 3t)\} = \dfrac{6(s+1)}{((s+1)^2+9)^2}$

⑨ $\mathcal{L}\{t^2e^{3t}\cdot\sinh t\} = \mathcal{L}\left\{t^2\cdot e^{3t}\left[\dfrac{e^t-e^{-t}}{2}\right]\right\} = \dfrac{1}{2}\mathcal{L}\{t^2(e^{4t}-e^{2t})\}$

now $\mathcal{L}\{t^2\} = \dfrac{2!}{s^3}$

$\Rightarrow \mathcal{L}\{t^2\cdot(e^{3t}\sinh t)\} = \dfrac{1}{2}\mathcal{L}\{t^2e^{4t} - t^2e^{2t}\} = \dfrac{2!}{(s-4)^2} - \dfrac{2!}{(s-2)^2}$.

HW → ① $\dfrac{1-e^t}{t}$ ⑪ $e^{-t}\cdot\sin t/t$ ⑪ $t^2[\sinh 3t - \cosh 3t]$ ⑩ $te^{2t}\cdot\cos 5t$.

Ans: ① $\mathcal{L}\{1-e^t\} = \left(1/s - \dfrac{1}{s-1}\right)$

$$\mathcal{L}\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \left(1/s - 1/s-1\right) ds = \log\left(\frac{s}{s-1}\right)_s^\infty = 0 - \log\left[\frac{s}{s-1}\right]$$

$$= \log\left[\frac{s-1}{s}\right]$$

② $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

$$\mathcal{L}\{\sin t/t\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \tan^{-1}(s)_s^\infty = \pi/2 - \tan^{-1}(s)$$

$$\mathcal{L}\{e^{-t}\sin t/t\} = \pi/2 - \tan^{-1}(s+1)$$

(3) $\sinh 3t - \cosh 3t = \frac{e^{3t}}{2} - \frac{e^{-3t}}{2} - \frac{e^{3t}}{2} - \frac{e^{-3t}}{2} = \underline{\underline{-e^{-3t}}}$

$\Rightarrow L\{\sinh 3t - \cosh 3t\} = L\{-e^{-3t}\} = \frac{-1}{s+3}$

$\Rightarrow L\{t^2(-e^{-3t})\} = (-1)^2 \frac{d}{ds^2}\left(\frac{-1}{s+3}\right) = \frac{-2}{(s+3)^3}$

(4)

EVALUATION OF INTEGRAL USING L.T

(1) $\int_0^\infty t \cdot e^{-2t} \sin t \, dt = \int_0^\infty e^{-2t}[t \sin t]\,dt = \int_0^\infty e^{-2t}(F(t))\,dt$ $\quad F(t) = t\sin t$, $s = 2$

$= L\{F(t)\}$

$= L\{t \sin t\}$

But $L\{\sin t\} = \frac{1}{s^2+1} \Rightarrow L\{t \sin t\} = (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -1 \times \frac{-1}{(s^2+1)^2} \cdot 2s$

$= \underline{\frac{2s}{(s^2+1)^2}}$

ie $L(t \sin t)_{s=2} = \frac{4}{(5)^2} = \underline{\underline{\frac{4}{25}}}$

② $\int_0^\infty t e^{-2t} \cos t \, dt = \int_0^\infty e^{-2t} (t \cos t) \, dt = \int_0^\infty e^{-2t} (F(t)) \, dt$ $F(t) = \cos t$, $s = 2$

$\Rightarrow$ Given Integral $= L\{t \cos t\}$

But $L\{\cos t\} = \frac{s}{s^2+1} \Rightarrow L\{t \cos t\} = (-1) \cdot \frac{d}{ds}\left(\frac{s}{s^2+1}\right)$

$= (-1)\left[\frac{(s^2+1) \cdot 1 - s \cdot 2s}{(s^2+1)^2}\right] = \frac{s^2-1}{(s^2+1)^2}$

ie $L\{t \cos t\}_{s=2} = \frac{4-1}{(5^2)} = 3/25$.

PYQ

③ Prove that $\int_0^\infty \left[\frac{e^{-t} - e^{-3t}}{t}\right] dt = \log 3$

Ans:- LHS $= \int_0^\infty \frac{e^{-t} - e^{-3t}}{t} \cdot dt = \int_0^\infty \frac{e^{-t}}{t} \cdot dt - \int_0^\infty e^{-3t} \cdot \frac{1}{t} \, dt = L\{1/t\}_{s=1} - L\{1/t\}_{s=3}$

In this method, ∞ comes.

ie LHS $= \int_0^\infty e^{0t} \cdot \left(\frac{e^{-t} - e^{-3t}}{t}\right) dt = L\left\{\frac{e^{-t} - e^{-3t}}{t}\right\}$ with $s=0$

$L\{e^{-t} - e^{-3t}\} = \frac{1}{s+1} - \frac{1}{s+3} \Rightarrow L\left\{\frac{e^{-t} - e^{-3t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+1} - \frac{1}{s+3}\right) ds$

$= \log\left(\frac{s+1}{s+3}\right)\Big|_s^\infty = \log\left[\frac{1+1/s}{3+3/s}\right]_s^\infty$

$= 0 - \cancel{\log(1/3)} \log\left[\frac{1+1/s}{3+3/s}\right]$

Given, $s=0$, then ans $= -\log 1/3 = \log 3$

Qn :- Prove that $\int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t} \cdot dt = \frac{1}{4}\log 5$

Ans: $LHS = \int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t} \cdot dt = L\left\{\frac{\sin^2 t}{t}\right\}$ with $s = 1 \longrightarrow (1)$

$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds = \frac{1}{2}\cdot\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$

take s^2 outside

$= \frac{1}{2}\left|\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right|_s^\infty = \frac{1}{2}\left[0 - \log\left[\frac{s}{\sqrt{s^2+4}}\right]\right]$

$= \frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}$

$(1) \Rightarrow LHS = \frac{1}{2}\times\log\frac{\sqrt{1+5}}{1} = \frac{1}{2}\cdot\log 5^{1/2} = \underline{\frac{1}{4}\log 5}$

TRANSFORM OF DERIVATIVES

- If $L\{F(t)\}$ is $\bar{F}(s)$

(i) $L\{F'(t)\} = s\bar{F}(s) - F(0)$

(ii) $L\{F''(t)\} = s^2\bar{F}(s) - sF(0) - F'(0)$

(iii) $L\{F'''(t)\} = s^3\bar{F}(s) - s^2F(0) - sF'(0) - F'(0)$

Proof :- Given $L\{F(t)\} = \bar{F}(s)$

$L\{F'(t)\} = \int_0^\infty e^{-st}[F'(t)]\,dt = e^{-st}\cdot\int_0^\infty F'(t)\,dt - \int_0^\infty \frac{-e^{-st}}{s}F(t)\,dt$

$= [0 - F(0)] + s[L(F[t])] = sF(t) - F(0)$

$$\boxed{L\{f'(t)\} = s\bar{F}(s) - f(0)} \quad \text{if} \quad L\{F(t)\} = \bar{F}(s)$$

TRASFORM OF INTEGRAL

If $L\{f(t)\} = \bar{F}(s)$, then

then $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{F}(s)$

Proof :- Let $\int_0^t f(u)\,du = \phi(t)$ so that $\phi'(t) = f(t)$ & $\phi(0) = 0$

But $L\{\phi'(t)\} = sL(\phi(t)) - \phi(0) = s\bar{\phi}(s) - 0 = s\bar{\phi}(s)$

ie $\bar{\phi}(s) = \frac{1}{s}L(\phi'(t)) \Rightarrow \boxed{L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{F}(s)}$

INVERSE LAPLACE TRANSFORM

① $L^{-1}(s^{-1}) = 1$

② $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

③ $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

④ $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\cdot\sin at$.

⑤ $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

⑥ $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$.

⑦ $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$.

⑧ $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{1}{n!}\cdot t^n$

⑨ $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = e^{at}\cdot\frac{1}{b}\sin bt$.

⑩ $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cdot\cos bt$.

⑪ $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}\cdot t\sin at$.

⑫ $L^{-1}\left\{\frac{s^2-a^2}{(s^2+a^2)^2}\right\} = t\cdot\cos at$.

(13) $L^{-1}\left\{\frac{1}{[s^2+a^2]^2}\right\} = \frac{1}{2a^3}\left[\sin at - at\cos t\right]$

DERIVATION: $\frac{1}{[s^2+a^2]^2} = \frac{1}{a^2}\cdot\frac{[s^2-s^2+a^2]}{(s^2+a^2)^2} = \frac{1}{a^2}\left[\frac{1}{(s^2+a^2)} - \frac{s^2}{(s^2+a^2)^2}\right]$

$$= \frac{1}{a^3}\cdot\frac{a}{(s^2+a^2)} - \frac{1}{a^2}\left(\frac{s^2+a^2-a^2}{(s^2+a^2)^2}\right)$$

$$= \frac{1}{a^3}\frac{a}{(s^2+a^2)^2} - \frac{1}{a^2}\left[\frac{s^2-a^2}{(s^2+a^2)} - \frac{a^2}{(s^2+a^2)^2}\right]$$

$$\Rightarrow 2\times\frac{1}{[s^2+a^2]^2} = \frac{1}{a^3}\cdot\frac{a}{(s^2+a^2)} - \frac{1}{a^2}\cdot\frac{(s^2-a^2)}{(s^2+a^2)^2}$$

ie $2\times L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a^3}\cdot\sin at - \frac{1}{a^2}\cdot t\cos at\cdot$

$$\boxed{L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}\left[-\cos at\cdot at + \sin at\right]}$$

METHOD OF PARTIAL FRACTION ★

(I) $\frac{1}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$ Rule (1)

(II) $\frac{1}{(ax+b)^2(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \frac{C}{(cx+d)}$ Rule (2)

(III) can't further factorize

$\frac{1}{(ax^2+bx+c).(dx+e)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$ Rule (3)

Qn:- Find the inverse laplace transforms of following:

R1 (a) $\dfrac{s+2}{s^2-4s+3}$.

Ans:- $\dfrac{s+2}{(s^2-4s+3)} = \dfrac{s+2}{(s-1)(s-3)} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-3)}$ For A: put $s=1$, For B: put $s=3$

$= \dfrac{-3/2}{(s-1)} + \dfrac{5/2}{(s-3)}$

ie $L^{-1}\left(\dfrac{s+2}{s^2-4s+3}\right) = \dfrac{-3}{2} L^{-1}\left(\dfrac{1}{s-1}\right) + \dfrac{5}{2} L^{-1}\left(\dfrac{1}{s-3}\right) = \frac{1}{2}\left[e^{3t} - e^{t}\right]$

R2 (b) $\dfrac{4s+5}{(s-1)^2[s+1]}$

Ans:- $\dfrac{4s+5}{(s-1)^2[s+1]} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-1)^2} + \dfrac{C}{(s+1)} = \dfrac{A(s-1)(s+1) + B(s+1) + C(s-1)^2}{(s-1)^2(s+1)}$

Put $s=1 \Rightarrow 9 = 2B \Rightarrow B = 9/2$

$s=-1 \Rightarrow 1 = 4C \Rightarrow C = 1/4$

Equate co-eff. of $s^2 \Rightarrow A + C = 0 \Rightarrow A = -C = -1/4$

ie $L^{-1}\left[\dfrac{4s+5}{(s-1)^2[s+1]}\right] = L^{-1}\left[\dfrac{-1/4}{(s-1)} + \dfrac{9/2}{(s-1)^2} + \dfrac{1/4}{(s+1)}\right]$

$= \dfrac{-1}{4} \times e^{t} + \dfrac{9}{2} \cdot te^{t} + \dfrac{1}{4} e^{-t}$

R3 (c) $\dfrac{3s+1}{(s^2+1)(s-2)}$

Ans:- $\dfrac{3s+1}{(s^2+1)(s-2)} = \dfrac{As+B}{(s^2+1)} + \dfrac{C}{(s-2)} = \dfrac{(As+B)(s-2) + C(s^2+1)}{(s^2+1)(s-2)}$.

$$3s+1 = (As+B)(s-2) + C(s^2+1)$$

Put $s=2 \Rightarrow \quad 7 = 5C \Rightarrow C = 7/5$

Equating Coeff. of $s^2 \Rightarrow A + C = 0 \Rightarrow A = -C = 7/5$

Put $s=0 \Rightarrow \quad 1 = -2B + C \Rightarrow B = 1/5$.

ie $L^{-1}\left[\frac{3s+1}{(s^2+1)(s-2)}\right] = L^{-1}\left[\frac{-7/5\, s + 1/5}{(s^2+1)} + \frac{7/5}{(s-2)}\right]$

$= \frac{1}{5} L^{-1}\left[\frac{1-7s}{(s^2+1)} + \frac{7}{(s-2)}\right] = \frac{1}{5}\left[L^{-1}\left(\left(\frac{1}{s^2+1}\right) - \frac{7s}{(s^2+1)} + 7\frac{1}{(s-2)}\right)\right]$

$= 1/5\left[\sin t - 7.\cos t + 7e^{2t}\right]$

(8) $\frac{s^2}{(s^2+4)(s^2+9)}$ (every term is 2^{nd} degree).

Ans :- If $u = s^2 \Rightarrow \frac{u}{(u+4)(u+9)} = \frac{-4/5}{(u+4)} + \frac{-9/-5}{(u+9)} = \frac{-4/5}{s^2+4} + \frac{9/5}{s^2+9}$.

ie $L^{-1}\left(\frac{s^2}{(s^2+4)(s^2+9)}\right) = \frac{-4}{5} L^{-1}\left(\frac{1}{s^2+4}\right) + \frac{9}{5} L^{-1}\left(\frac{1}{s^2+9}\right)$

$= \frac{-4}{10}\sin 2t + \frac{9}{5}\sin 3t$

(HW) 8- (i) $\frac{2s-1}{(s^2-5s+6)(s+4)}$ (ii) $\frac{1}{(s+1)^2(s+2)}$ (iii) $\frac{2s+1}{(s^2+4)(s-2)}$ (iv) $\frac{s^2+1}{(s^2+4)(s^2+16)}$

Ans:- (i) $\frac{2s-1}{(s^2-5s+6)(s+4)} = \frac{2s-1}{(s-2)(s-3)(s+4)} = \frac{A}{(s-2)} + \frac{B}{(s-3)} + \frac{C}{(s+4)}$

for $A, B \& C \Rightarrow$ put $s=2 \Rightarrow A = -1/2$ and $s=-4 \Rightarrow C = -3/14$
$s=3 \Rightarrow B = 5/7$

ie $\frac{2s-1}{(s^2-5s+6)(s+4)} = \frac{-1/2}{(s-2)} + \frac{5/7}{(s-3)} + \frac{-3/14}{(s+4)} \longrightarrow$ ①

ie $L^{-1}\left\{\frac{2s-1}{(s^2-5s+6)(s+4)}\right\} = L^{-1}\{①\} = \frac{-1}{2} L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{7} L^{-1}\left\{\frac{1}{s-3}\right\}$
$+ \frac{-3}{14} L^{-1}\left\{\frac{1}{s+4}\right\}$

ie we got : $L^{-1}\{①\} = \frac{-1}{2} \cdot e^{2t} + \frac{5}{7} e^{3t} + \frac{-3}{14} e^{-4t}$

(ii) $\frac{1}{(s-1)^2(s+2)} = \frac{A}{(s+2)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2} = \frac{A(s-1)^2 + B(s-1)(s+2) + C(s+2)}{(s-1)^2(s+2)}$

① ie $s=1 \Rightarrow 1 = 3C \Rightarrow C = 1/3$
$s=-2 \Rightarrow 1 = 9A \Rightarrow A = 1/9$

equating coeffs. of $s^2 \Rightarrow 0 = A+B \Rightarrow B = -A = -1/9$

ie we get $\frac{1}{(s-1)^2(s+2)} = \frac{1/9}{(s+2)} + \frac{-1/9}{(s-1)} + \frac{1/3}{(s-1)^2} \longrightarrow$ eqⁿ ⓐ

ie $L^{-1}\left\{\frac{1}{(s-1)^2(s+2)}\right\} = L^{-1}\{eq^n ⓐ\} = L^{-1}\left(\frac{1}{9}\cdot\frac{1}{s+2}\right) + L^{-1}\left\{\frac{-1/9}{(s-1)}\right\} + L^{-1}\left\{\frac{1/3}{(s-1)^2}\right\}$

$= \frac{1}{9} \cdot e^{-2t} + \frac{-1}{9} e^{t} + \frac{1}{3} t e^{t}$

(iv) $\frac{s^2+1}{(s^2+4)(s^2+16)}$

If $u = s^2$, then given expression becomes $\frac{u+1}{(u+4)(u+16)} = \frac{-3/12}{(u+4)} + \frac{5/4}{(u+16)}$

ie $\frac{s^2+1}{(s^2+4)(s^2+16)} = \frac{-1/4}{(u+4)} + \frac{5/4}{(u+16)} = \frac{-1/4}{s^2+4} + \frac{5/4}{s^2+16}$

ie $L^{-1}\left\{\frac{s^2+1}{(s^2+4)(s^2+16)}\right\} = \frac{-1}{4} L^{-1}\left\{\frac{1}{s^2+2^2}\right\} + \frac{5}{4} L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$

$= \frac{-1}{4}\cdot\frac{1}{2}\sin 2t + \frac{5}{4}\times\frac{1}{4}\cdot\sin 4t$

$= \frac{1}{8}\left[\frac{5}{2}\sin 4t - \sin 2t\right]$

Qn:- Find $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}$

Ans:- $\frac{s^2-3s+4}{s^3} = \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3} \Rightarrow L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right\}$

ie $L^{-1}\{\sim\} = L^{-1}(1/s) - 3L^{-1}(1/s^2) + 4L^{-1}(1/s^3)$

$= 1 - 3t + \frac{4}{2!}t^2$

Qns- Find $L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\}$

Ans :- Now expressing s^2 in terms of $(s-2) \Rightarrow s^2 = (s-2)^2 - (4s-4)$

$= (s-2)^2 - 4(s-2) + 4$

ie $\frac{s^2}{(s-2)^2} = \frac{(s-2)^2 - 4(s-2) + 4}{(s-2)^3} = \frac{1}{(s-2)} - 4\cdot\frac{1}{(s-2)^2} + 4\cdot\frac{1}{(s-2)^3} \longrightarrow$ eqⁿ ⓐ

ie $L^{-1}\{\text{eq}^n\ ⓐ\} = L^{-1}\left\{\frac{1}{s-2}\right\} - 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4\,L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$

$= e^{2t} - 4\cdot te^{2t} + 4\cdot\frac{1}{2!}\cdot t^2e^{2t}$

ANOTER METHODS *

① $L\{f(t)\} = \bar{F}(s)$, then $L\{e^{at}f(t)\} = \bar{F}(s-a)$. from this

$L^{-1}\{\bar{F}(s-a)\} = e^{at}\cdot f(t)$

$L^{-1}\{\bar{F}(s+a)\} = e^{-at}f(t)$

(ii) If $L^{-1}\{\bar{F}(s)\} = f(t)$, then if $f(0)$ is also 0, then $L^{-1}\{s\bar{F}(s)\} = \frac{d}{dt} f(t)$

Proof :- we have $L\{f'(t)\} = s\bar{F}(s) - f(0) = s\bar{F}(s)$ (we have $f(0)=0$)

then $L^{-1}\{s\cdot\bar{F}(s)\} = f'(t) = \frac{d}{dt} f(t)$.

* eg : $L^{-1}\{\frac{s^2}{s^2+1}\} = L^{-1}\{s\cdot\frac{s}{s^2+1}\} = \frac{d}{dt}\cos t = \underline{\underline{-\sin t}}$

(iii) If $L^{-1}\{\bar{F}(s)\} = f(t)$, then $L^{-1}\{\frac{\bar{F}(s)}{s}\} = \int_0^{\infty} f(t)\, dt$.

Proof :- $L\{f(t)\} = \bar{F}(s)$, then $L^{-1}\{\int_0^t f(t)\,dt\} = \frac{1}{s}\cdot\bar{F}(s)$

Hence $L^{-1}\{\frac{\bar{F}(s)}{s}\} = \int_0^t f(t)\,dt$.

(iv) If $L^{-1}\{\bar{F}(s)\} = f(t)$, then $L^{-1}\{(-1)^n \frac{d}{ds}\bar{F}(s)\} = t\cdot f(t)$

Proof :- we have $L\{f(t)\} = \bar{F}(s)$, then

$L\{t\cdot f(t)\} = (-1)^1 \frac{d^1}{ds^1}\bar{F}(s)$

Examples :- *

(1) Find $L^{-1}\{\frac{s+2}{s^2-4s+13}\}$

Ans :- $\frac{s+2}{s^2-4s+13} = \frac{s+2}{(s-2)^2+3^2} = \frac{(s-2)+4}{(s-2)^2+3^2} = \frac{s-2}{(s-2)^2+3^2} + \frac{4\cdot 3}{3[(s-2)^2+3^2]}$ $\longrightarrow$ eq^n (a)

ie $L^{-1}\{\text{eq}^n\ (a)\} = e^{2t}\cos 3t + \frac{4}{3} e^{2t}\sin 3t$.

(2) Find $L^{-1}\{\frac{s+3}{s^2-4s+29}\}$

Ans :- $\frac{s+3}{s^2-4s+4+25} = \frac{(s+3)}{(s-2)^2+5^2} = \frac{(s-2)+5}{(s-2)^2+5^2} = \frac{(s-2)}{(s-2)^2+5^2} + \frac{5}{(s-2)^2+5^2}$ $\longrightarrow$ eq^n (b)

$L^{-1}\{eq^n\ (a)\} = e^{2t}\cdot\cos 5t + e^{2t}\sin 5t$

PYQ ③ Find $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$.

Ans :- Let $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = f(t) \Rightarrow L\{f(t)\} = \dfrac{s}{(s^2+a^2)^2} \Rightarrow L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_0^\infty \dfrac{s}{(s^2+a^2)^2}\,ds$

Hence we got :- $L\left\{\dfrac{f(t)}{t}\right\} = \dfrac{-1}{2}\times\displaystyle\int_0^\infty \dfrac{-2s}{(s^2+a^2)^2}\,ds = \dfrac{-1}{2}\left[\dfrac{1}{(s^2+a^2)}\right]_s^\infty$ $\quad u = \dfrac{1}{(s^2+a^2)}$, $du = \dfrac{-2s}{(s^2+a^2)^2}$

$= \dfrac{1}{2}\cdot\dfrac{1}{(s^2+a^2)}$

ie $\dfrac{f(t)}{t} = \dfrac{1}{2}L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{1}{2a}\sin at \Rightarrow f(t) = \dfrac{t\sin at}{2a}$.

Note :- $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{s\cdot\dfrac{s}{(s^2+a^2)^2}\right\} = \dfrac{1}{2a}\left[t\cos at + \sin at\right]$.

PYQ ④ Find $L^{-1}\left\{\log\left(\dfrac{s+1}{s-1}\right)\right\}$

Ans :- Let $L^{-1}\left\{\log\left(\dfrac{s+1}{s-1}\right)\right\} = f(t) \Rightarrow L\{f(t)\} = \log\left(\dfrac{s+1}{s-1}\right)$

ie $L\{t\,f(t)\} = (-1)\dfrac{d}{dt}\left(\log(s+1) - \log(s-1)\right) = \left[\dfrac{1}{s+1} - \dfrac{1}{s-1}\right](-1)$

$= \dfrac{1}{s-1} - \dfrac{1}{s+1}$

$\Rightarrow t\,f(t) = L^{-1}\left\{\dfrac{1}{s-1} - \dfrac{1}{s+1}\right\} = e^t - e^{-t} \Rightarrow f(t) = \dfrac{e^t - e^{-t}}{t} = \dfrac{2\sinh t}{t}$.

PYQ ⑤ Find $L^{-1}\{\cot^{-1} s/2\}$

Ans :- Let $L^{-1}\{\cot^{-1} s/2\} = f(t) \Rightarrow L\{f(t)\} = \cot^{-1}(s/2)$

$\Rightarrow L\{t\,f(t)\} = (-1)\dfrac{d}{ds}\cot^{-1}(s/2) = (-1)\cdot\dfrac{(-1)}{1+s^2/4}\cdot\dfrac{1}{2} = \dfrac{1}{2}\cdot\dfrac{1}{(1+s^2/4)}$

ie $t\,f(t) = L^{-1}\left\{\dfrac{1}{2}\dfrac{1}{(1+s^2/4)}\right\} = \sin 2t \Rightarrow f(t) = \dfrac{\sin 2t}{t}$.

PYQ ⑥ Find $L^{-1}\{\tan^{-1}(2/s^2)\}$

Ans :- Let $L^{-1}\{\tan^{-1}(2/s^2)\} = f(t) \Rightarrow L\{f(t)\} = \tan^{-1}(2/s^2) \Rightarrow L\{t f(t)\} = (-1)\frac{d}{ds}(\tan^{-1}(2/s^2))$

ie $L\{t f(t)\} = (-1)\cdot\frac{1}{1+(2/s^2)^2}\cdot\left[\frac{-4}{s^3}\right] = \frac{4s}{s^4+4} = \frac{4s}{(s^2+2)^2-(2s)^2}$

$$= \frac{4s}{(s^2+2+2s)(s^2+2-2s)} = \frac{4s}{[(s+1)^2+1][(s-1)^2+1]}$$

we use partial fraction $\Rightarrow L\{t f(t)\} = \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2-2s+2)}$

$$= \frac{[As+B][s^2-2s+2] + [Cs+D][s^2+2s+2]}{(s^2+2s+2)(s^2-2s+2)}$$

equating nrt^{rs} $\Rightarrow$ $4s = (As+B)(s^2-2s+2) + (Cs+D)(s^2+2s+2)$

put $s=0 \Rightarrow B+D=0$

equating coeffs. of $s^3 \Rightarrow 0 = A+C$

" " " $s^2 \Rightarrow B-2A+D+2C = 0$

" " " $s \Rightarrow 4 = 2A-2B+2C+2D \Rightarrow 2 = -B+D$

$A=0 \quad C=0 \quad D=1 \quad B=-1$

ie $L^{-1}\{t f(t)\} = \frac{-1}{s^2+2s+2} + \frac{1}{s^2-2s+2} = \frac{-1}{(s+1)^2+1^2} + \frac{1}{(s-1)^2+1^2}$

$t f(t) = L^{-1}\left\{(-1)\frac{1}{(s+1)^2+1^2} + \frac{1}{(s-1)^2+1^2}\right\} = (-1)\cdot e^{-t}\sin t + e^{t}\cdot\sin t$

$\Rightarrow f(t) = \frac{e^{t}\sin t - e^{-t}\sin t}{t}$.

CONVOLUTION THEOREM. [CT]

- If $L^{-1}\{\bar{F}(s)\} = f(t)$ and $L^{-1}\{\bar{g}(s)\} = g(t)$, then

$$L^{-1}\{\bar{F}(s)\cdot\bar{g}(s)\} = \int_0^t f(u)\, g(t-u)\, du$$

eg:- $L^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = L^{-1}\left\{\left(\frac{1}{s-1}\right)\left(\frac{1}{s-2}\right)\right\}$ But $L^{-1}\left\{\frac{1}{s-1}\right\} = e^t$, $L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$.

then $L^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = \int_0^t e^u e^{2(t-u)} du = \int_0^t e^{(2t-u)} du$

$$= e^{2t}\int_0^t e^{-u} du = e^{2t}\cdot\left(-e^{-u}\right)_0^t = -e^{2t}\left[e^{-t}-1\right]$$

② Using Convo. theorem (C·T), find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Ans:- $L^{-1}\left\{\frac{s}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}\right\} \Rightarrow$ But $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$ & $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$.

ie by C.T, $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au\cdot\frac{1}{a}\sin a(t-u)\, du = \frac{1}{2a}\int_0^t 2\cos au\, \sin a(t-u)\, du$

$(\sin(a+b) - \sin(a-b))$

$$= \frac{1}{2a}\int_0^t \left[\sin at - \sin(2au - at)\right] du$$

$$= \frac{1}{2a}\left[\sin at\,[u]_0^t + \left[\frac{\cos(2au-at)}{2a}\right]_0^t\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\left[\cos at - \cos at\right]\right] = \frac{1}{2a}\cdot t\sin at$$

③ Using C·T, Find $L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\}$

Ans:- $L^{-1}\left\{\frac{s}{s^2+1}\cdot\frac{s}{s^2+4}\right\} \Rightarrow$ But $L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$, $L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$

$$\Rightarrow L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\} = \int_0^t \cos u\cdot\cos 2(t-u)\, du = \frac{1}{2}\int_0^t 2\cos u\cdot\cos 2(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \left[\cos[u+2t-2u] + \cos[u-2t+2u] \right] du$$

$$= \frac{1}{2}\int_0^t \left[\cos(2t-u) + \cos[3u-2t] \right] du = \frac{1}{2}\left[-\sin(2t-u)\Big|_0^t + \frac{1}{3}\Big|\sin(3u-2t)\Big|_0^t \right]$$

$$= \frac{1}{2}\left[-(\sin t - \sin 2t) + \frac{1}{3}[\sin t \cdot (-\sin 2t)] \right]$$

$$= \frac{1}{2}\left[(\sin 2t - \sin t) + \frac{-1}{3}(\sin 2t \cdot \sin t) \right]$$

(8) Using C.T, Find $L^{-1}\left\{ \frac{1}{s^2(s+1)^2} \right\}$

Ans:- $L^{-1}\left\{ \frac{1}{s^2} \cdot \frac{1}{(s+1)^2} \right\} \Rightarrow$ But $L^{-1}\left\{ \frac{1}{s^2} \right\} = t$ $\quad L^{-1}\left\{ \frac{1}{(s+1)^2} \right\} = e^{-t} t$.

$$\Rightarrow L^{-1}\left\{ \frac{1}{s^2(s+1)^2} \right\} = \int_0^t e^{-u} u (t-u)\, du = \cancel{e^{-t} t \int_0^t (t-u)\, du = \frac{t}{e^t}\left[tu - \right.}$$

$$= \int_0^t (tue^{-u} - u^2 e^{-u})\, du = t\int_0^t (ue^{-u})\, du - \int_0^t u^2 e^{-u}\, du$$

$$= t\left[u\left(\frac{e^{-u}}{-1}\right)_0^t - 1 \times \left(\frac{e^{-u}}{-1}\right)_0^t \right] - \left[u^2\left(\frac{e^{-u}}{-1}\right)_0^t - 2u\left(\frac{e^{-u}}{-1}\right)_0^t + 2\left(\frac{e^{-u}}{-1}\right)_0^t \right]$$

$$= t\left((-te^{-t} - 0) - (e^{-t} - e^0) \right) - \left(-t^2 e^{-t} - 2te^{-t} - 2[e^{-t} - 1] \right)$$

$$= te^{-t} + 2e^{-t} + t - 2$$

HW → (i) $\frac{6}{(s^2+1)\cdot(s-1)}$ (ii) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$, find L^{-1} for these us C.T.