Module 3
Columns and Beam Columns

ANALYSIS OF STRUCTURES

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Revision of Euler's Theory

Failure of a Long Column

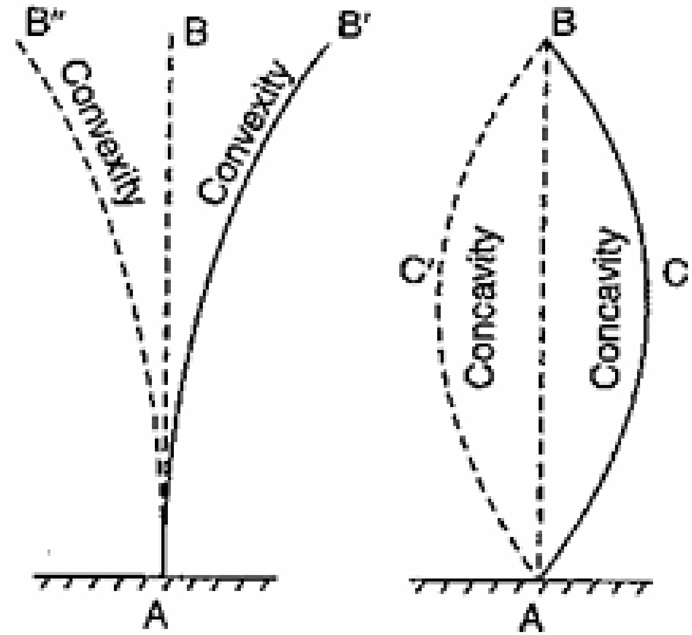
- A column is known as a **long column** if the length of the column in comparison to its lateral dimensions is very large.
- For long columns, the direct compressive stresses are negligible as compared to buckling stresses. Hence very long columns are subjected to buckling stresses only.
- The buckling load of a column is obtained through **Euler's theory.**

Assumptions in Euler's Theory

The following assumptions are made in the Euler's column theory:

1. The column is initially perfectly straight and the load is applied axially.
2. The cross-section of the column is uniform throughout its length.
3. The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke’s law.
4. The length of the column is very large as compared to its lateral dimensions.
5. The direct stress is very small as compared to the bending stress.
6. The column will fail by buckling alone.
7. The self-weight of column is negligible.

Sign Convention



(a) Positive (b) Negative

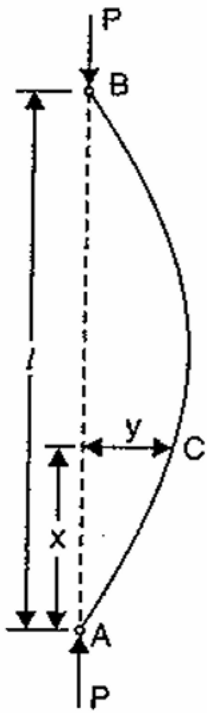
EXPRESSION FOR CRIPPLING LOAD WHEN BOTH THE ENDS OF THE COLUMN ARE HINGED

Consider any section at a distance x from the end A.

Let y = Deflection (lateral displacement) at the section.

The moment due to the crippling load at the section $= - P \cdot y$

(– ve sign is taken due to sign convention given in Art. 19.4.1)



But moment $$= EI \frac{d^2y}{dx^2}.$$

Equating the two moments, we have

$$EI \frac{d^2y}{dx^2} = - P \cdot y \quad \text{or} \quad EI \frac{d^2y}{dx^2} + P \cdot y = 0$$

or $$\frac{d^2y}{dx^2} + \frac{P}{EI} \cdot y = 0$$

The solution* of the above differential equation is

$$y = C_1 \cdot \cos\left(x\sqrt{\frac{P}{EI}}\right) + C_2 \cdot \sin\left(x\sqrt{\frac{P}{EI}}\right)$$

*The equation $\frac{d^2y}{dx^2} + \frac{P}{EI} \times y = 0$ can be written as $\frac{d^2y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \frac{P}{EI}$ or $\alpha = \sqrt{\frac{P}{EI}}$

(*i*) At A, $x = 0$ and $y = 0$

Substituting these values in equation (*i*), we get

$$\begin{aligned} 0 &= C_1 \, . \cos 0° + C_2 \sin 0 \\ &= C_1 \times 1 + C_2 \times 0 \\ &= C_1 \end{aligned}$$

$\therefore$ $$C_1 = 0.$$

(*ii*) At B, $x = l$ and $y = 0$

Substituting these values in equation (*i*), we get

$$\begin{aligned} 0 &= C_1 \, . \cos\left(l \times \sqrt{\frac{P}{EI}}\right) + C_2 \, . \sin\left(l \times \sqrt{\frac{P}{EI}}\right) \\ &= 0 + C_2 \, . \sin\left(l \times \sqrt{\frac{P}{EI}}\right) \\ &= C_2 \sin\left(l \sqrt{\frac{P}{EI}}\right) \end{aligned}$$

From equation (*iii*), it is clear that either $C_2 = 0$

or $$\sin\left(l \sqrt{\frac{P}{EI}}\right) = 0.$$

As $C_1 = 0$, then if C_2 is also equal to zero, then from equation (*i*) we will get $y = 0$. This means that the bending of the column will be zero or the column will not bend at all. Which is not true.

$$\therefore \qquad \sin\left(l\sqrt{\frac{P}{EI}}\right) = 0$$

$$= \sin 0 \text{ or } \sin \pi \text{ or } \sin 2\pi \text{ or } \sin 3\pi \text{ or } \ldots$$

or
$$l\sqrt{\frac{P}{EI}} = 0 \text{ or } \pi \text{ or } 2\pi \text{ or } 3\pi \text{ or } \ldots$$

Taking the least practical value,

$$l\sqrt{\frac{P}{EI}} = \pi$$

or
$$P = \frac{\pi^2 EI}{l^2}.$$

(…Refer Class notes…)

Buckling load formulae for columns with different end conditions

1.	Both ends hinged	$\frac{\pi^2 EI}{l^2}$
2.	One end is fixed and other is free	$\frac{\pi^2 EI}{4l^2}$
3.	Both ends fixed	$\frac{4\pi^2 EI}{l^2}$
4.	One end fixed and other is hinged	$\frac{4\pi^2 EI}{l^2}$

Effective length of column

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and cross-section with hinged ends, and having the value of the crippling load equal to that of the given column. Effective length is also called equivalent length.

S.No.	*End conditions of column*	*Crippling load in terms of*		*Relation between effective length and actual length*
		Actual length	*Effective length*	
1.	Both ends hinged	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = l$
2.	One end is fixed and other is free	$\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = 2l$
3.	Both ends fixed	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{2}$
4.	One end fixed and other is hinged	$\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{L_e^2}$	$L_e = \frac{l}{\sqrt{2}}$

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Beam Columns

- A member subjected to both axial and transverse forces is known as a **beam column**.
- In the elementary theory of bending, the influence of direct stress is usually ignored. But in beam columns, the direct stress has a considerable effect and must be taken into account.
- The deflection of a beam column depends on the magnitude and direction of the axial load as well as the transverse loads.
- Analysis of beam columns is important in the design of multi-storey frames, bridges, towers, and other structures where members are subjected to combined loading.

- In case of **Theory of pure bending**,

  σ, BM and SF $\propto$ P (applied load).

- But, these parameters (σ, BM and SF) are independent of the deflection produced.
- Furthermore, the principle of superposition is also valid for pure bending. i.e., total deflection at any point = sum of deflection due to individual set of loads.

- However, these conditions are not valid for the case when an axial load is also acting along with lateral loads **(for beam columns)**
- In case of beam columns, the deflection values also have an effect on the SF, BM, & stress values.

A beam column may be analyzed using:

1. **Equilibrium Criterion**

2. **Energy Criterion**

1. Equilibrium Criterion of beam columns

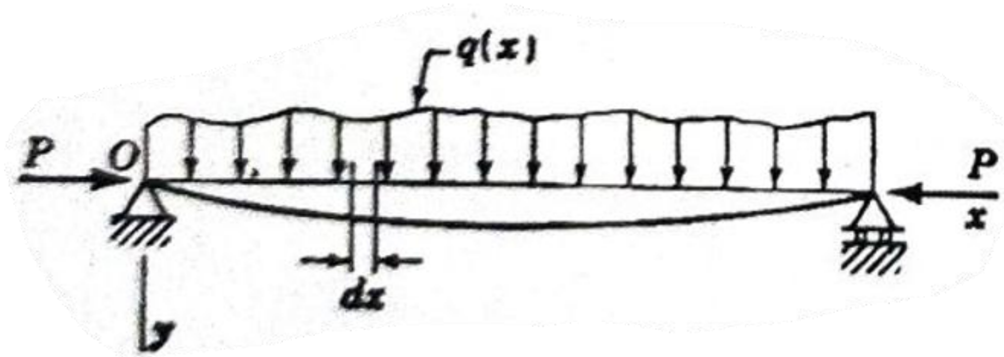
Equilibrium Criterion

- Based on static equilibrium equations.
- States that a structure is in equilibrium when the sum of forces and moments acting on it are zero.

$$\sum F_y = 0, \quad \sum M = 0$$

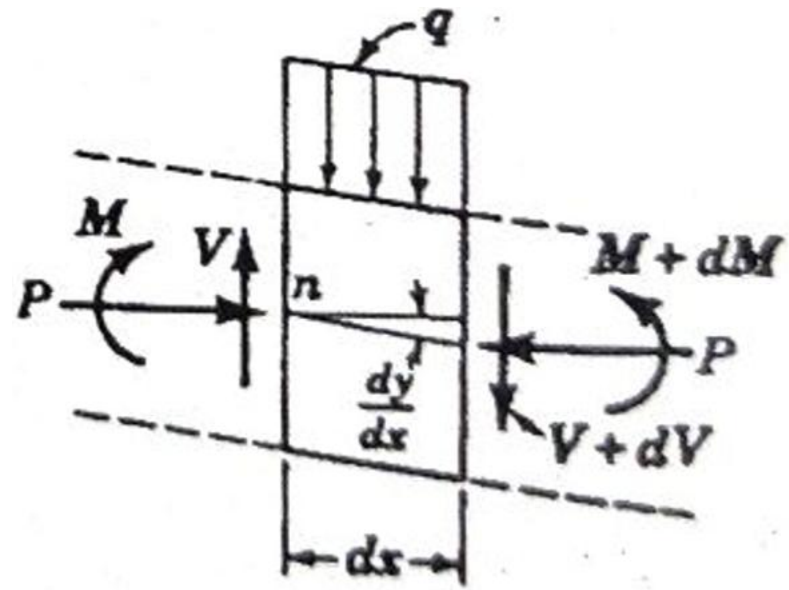
Differential equations for beam columns (using Equilibrium approach)

- Let take the case of a beam subjected to axial load P and lateral load of intensity q which varies along the length of the beam.



- Consider an element of length dx in the beam.
- Let the positive y direction be downwards.
- Considering the equilibrium of the element, Summing forces in the y direction gives,

$$-V + q\,dx + (V + dV) = 0$$



$$q = -\frac{dV}{dx} \qquad \text{..............(1)}$$

Taking moments about point n and assuming that the angle between the axis of the beam and the horizontal is small, we obtain

$$M + \left(q\,dx \cdot \frac{dx}{2}\right) + (V + dV)\,dx - (M + dM) + P\overbrace{\frac{dy}{dx}dx}^{\text{(of the form L = r}\theta\text{)}} = 0$$

- If terms of second order are neglected and dividing the equation throughout by dx, this equation becomes,

$$V = \frac{dM}{dx} - P\frac{dy}{dx} \quad \text{...............(2)}$$

- If the shearing and axial effects are neglected, the expression for the curvature of the axis of the beam is,

$$EI\frac{d^2y}{dx^2} = -M \quad \text{...............(3)}$$

- Substituting eqn (3) in eqn (2), we get,

$$EI\frac{d^3y}{dx^3} + P\frac{dy}{dx} = -V \quad \text{...............(4)}$$

Substituting eqn (4) in eqn (1),

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = q \qquad \text{……………(5)}$$

Equations 1 to 5 are the basic differential equations for bending of beam columns.

Note: If the axial force P equals zero, these equations reduce to the usual equations for bending by lateral loads only, i.e., beam bending case.

Energy Approach in Beam Columns

Theorem of stationary potential energy

- The energy method for analyzing elastic stability is based on the **theorem of stationary potential energy**.
- This theorem is based on the **principle of conservation of energy** for an ideal elastic body:
- **Work Done = Energy Stored**: When external forces are applied gradually to an elastic body, the work done by the external forces (W) is stored in the body as elastic strain energy (U).

- Consider the equilibrium of an elastic body subjected to external surface and body forces.
- During the application of these forces, the body deforms and consequently, these forces do a certain amount of work W.
- When external forces are applied gradually, and no dissipation of energy takes place due to friction etc., the work done by the external forces (W) should be equal to the internal elastic energy U, i.e.,

$$\mathbf{W = U \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)}$$

Concept of Virtual Displacements:

- These are **very small displacements** that are **consistent with the constraints (boundary conditions)** imposed on the body.
- For example, if a point of the body is fixed, then the virtual displacement there is zero.
- If a point of the body is constrained to lie on the surface of another body, then the virtual displacement there should be tangential to the surface of the contacting body (like sliding over).

- Since these virtual displacements being very small, the changes necessary in the external forces to bring about these virtual displacements will also be very small and will vanish in the limit.
- Thus, the work done by external surface and body forces P_i during these virtual displacements is:

$$\delta W = \sum P_i \delta \Delta_i$$

where $\delta\Delta_i$ are the virtual displacements.

Definition of Potential V of the external forces:

It is convenient to define potential *V* of the external forces in such a manner that the work done during virtual displacements is equal to a decrease in potential energy of the external forces (–δV). i.e.,

$$-\delta V = \sum P_i \delta \Delta_i = \delta W$$

$$-\delta V - \delta W = 0$$

Using Eq. (1), the above equation can be written as:

$$\delta(U + V) = 0 \quad \text{.................(2)}$$

- The expression *U + V* is known as the total potential of the system. Consequently, Eq. (2) can be stated as follows:

The first-order change in the total potential energy must vanish for every virtual displacement when an elastic body is in equilibrium.

- Equation (2) is **also stated as that in which the quantity *U + V* assumes a stationary value**

$$(U + V) = \text{stationary}$$

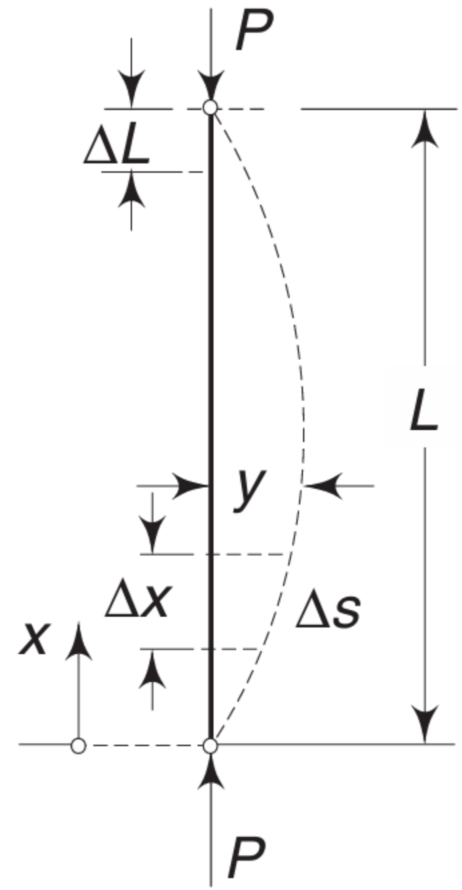
Application of energy approach in beam columns

- Consider the column shown in Fig. (a), carrying an axial load P.
- Let the buckled form be expressed by $y = f(x)$ .The elastic strain energy (due to bending) is given by:

$$U = \int_0^L \frac{M^2}{2EI}\, dx$$

where $$M = EI\frac{d^2y}{dx^2}$$

Thus, $$U = \int_0^L \frac{EI\left(\frac{d^2y}{dx^2}\right)^2}{2}\, dx$$



(a)

Let the moment of inertia $I_x$be variable.

$$U = \frac{1}{2}E\int_0^L I_x\left(\frac{d^2y}{dx^2}\right)^2 dx$$

- Taking the undeflected position as datum, the potential energy in the buckled form is

$$V = -P\Delta L \qquad \text{......(3)}$$

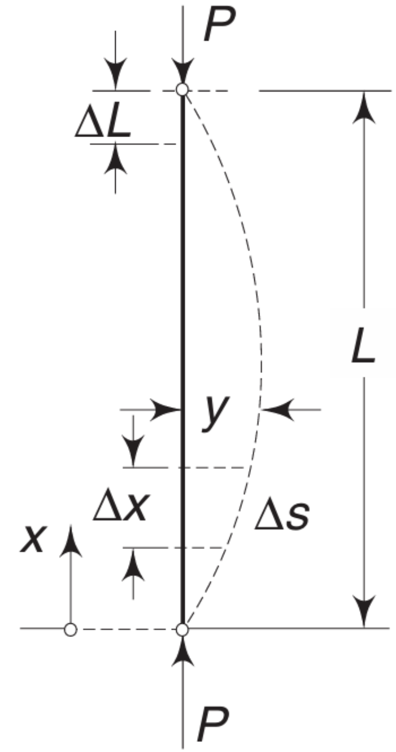
where ΔL is the reduction in length (axial compression) in the X-direction,

i.e. ΔL = Actual Length – Initial Deformed Length, i.e.,

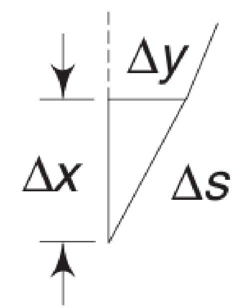
$$\Delta L = \int_0^L \Delta s - \int_0^L \Delta x$$

$$\Delta L = \int_0^L (\Delta s - \Delta x)$$

From Fig. (b), $\Delta s = (\Delta x^2 + \Delta y^2)^{1/2} \approx \Delta x + \frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2 \Delta x$



(a)



(b)

Substituting in equation (3), total potential V =

$$P\Delta L = \frac{1}{2} P \int_0^L \left(\frac{dy}{dx}\right)^2 dx$$

The total potential energy is, therefore, given by

$$U + V = \frac{1}{2} E \int_0^L I_x \left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{1}{2} P \left(\frac{dy}{dx}\right)^2 dx$$

For equilibrium, the variation of the above quantity should vanish, i.e.

$$\delta(U + V) = \delta \left[\frac{1}{2} E \int_0^L I_x \left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{1}{2} P \int_0^L \left(\frac{dy}{dx}\right)^2 dx\right] = 0$$

By assuming a suitable solution function for the deflection 'y' satisfying the boundary conditions, the critical load or buckling load (P_{cr}) can be calculated.

Numerical

Q) Consider a pin-ended column subjected to an axial compressive load P, as shown in Figure. Assume that the buckled shape is given by,

$$y = a \sin \frac{\pi x}{L}$$

where a is an unknown parameter. The coordinate function chosen satisfies the boundary conditions which are

$$y = 0 \quad \text{at } x = 0 \quad \text{and} \quad \text{at } x = L$$

$$\frac{d^2 y}{dx^2} = 0 \quad \text{at } x = 0 \quad \text{and} \quad \text{at } x = L$$

Solution:

$$U = \frac{1}{2} EI \int_0^L \left(\frac{d^2 y}{dx^2} \right)^2 dx$$

$$= \frac{1}{2} EI \int_0^L a^2 \left(\frac{\pi}{L} \right)^4 \sin^2 \frac{\pi x}{L} dx$$

$$= \frac{1}{4} \pi^4 a^2 \left(\frac{EI}{L^3} \right)$$

$$V = -\frac{1}{2} P \int_0^L \left(\frac{dy}{dx} \right)^2 dx$$

$$= -\frac{1}{2} P \int_0^L a^2 \left(\frac{\pi}{L} \right)^2 \cos^2 \frac{\pi x}{L} dx$$

$$= -\frac{1}{4} P\pi^2 \left(\frac{a^2}{L} \right)$$

Thus, the total potential energy is

$$U + V = \frac{1}{4} \pi^4 a^2 \frac{EI}{L^3} - \frac{1}{4} P\pi^2 \frac{a^2}{L}$$

For this to be an extremum, we must have

$$\frac{1}{2}\pi^4 a \frac{EI}{L^3} - \frac{1}{2} P\pi^2 \frac{a}{L} = 0$$

or
$$\frac{1}{2}\pi^2 \frac{a}{L}\left(\pi^2 \frac{EI}{L^2} - P\right) = 0$$

The non-trivial solution is obtained when

$$P = P_{cr} = \frac{\pi^2 EI}{L^2}$$

We have been able to obtain an exact solution since the coordinate function we used happens to give the exact deflected shape for the column.

References

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3) Srinath, L. S. (2009). Advanced mechanics of solids (3rd ed.). Tata McGraw-Hill Publishing Company.