MODULE-I

Elements of the form a_{ij} for which $i=j$ are called diagonal elements.

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ Elements below the diagonal elements are zero :

Upper triangular matrix.

$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ 1 & 2 & 6 \end{bmatrix}$ Elements above the diagonal elements are zero :

Lower triangular matrix

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ All elements other than diagonal elements are zero:

Diagonal matrix

* All diagonal elements same is diagonal matrix : Scalar

* All diagonal elements $= 1$ is a diagonal matrix : Unity.

$$c_{ik} = a_{ij} \times b_{jk}$$

$$= \sum_{j=1}^{n} a_{ij} b_{jk}$$

* If $A^2 = A \Rightarrow A$ is Idempotent

* If P is the least +ve integer such that $A^P = 0$, then A is called nilpotent index of P

A is $m \times n$

A^T is $n \times m$

$A^T B^T = AB$

$(A^T)^T = A$

$(\lambda A)^T = \lambda A^T$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

$a_{ij} = a_{ji}$

A is SYMMETRIC if $A^T = A$

A is SKEW SYMMETRIC if $A^T = -A$

$\begin{cases} a_{ij} = (-a_{ji}) \\ a_{ii} = 0 \end{cases}$

Properties.

* AA^T & A^TA are Symmetric

proof: $C = AA^T$

$C^T = (AA^T)^T = (A^T)^TA = AA^T = C$

proof $D = A^TA$

$D^T = (A^TA)^T = A^T(A^T)^T = A^TA = D$

* If A and B are symmetric, then AB is also symmetric if $AB = BA$

proof: given $A^T = A$ & $B^T = B$

let $AB = BA$

$(AB)^T = B^TA^T = BA = AB$

ie, AB is symmetric

let AB is symmetric

ie, $(AB)^T = AB$

$B^TA^T = AB$

$BA = AB$

If A is a square matrix

$A + A^T$ is Symmetric

$A - A^T$ is skew symmetric

Conjugate

$$A = \begin{bmatrix} 1+i & 2i \\ -3i & 7-2i \end{bmatrix}$$

$$A' = \begin{bmatrix} 1-i & -2i \\ 3i & 7+2i \end{bmatrix}$$

properties

$\overline{A+B} = \bar{A} + \bar{B}$

$(\bar{A})^T = (\overline{A^T})$

→ Hermitian & Skew Hermitian matrices

A square matrix is Hermitian if $\boxed{(\bar{A})^T = A}$

$A = a_{ij}$

$\bar{A} = \bar{a}_{ij}$

$(\bar{A})^T = \bar{a}_{ji}$

$$\left.\begin{aligned} (\bar{A})^T &= A \\ \bar{a}_{ji} &= a_{ij} \end{aligned}\right\} \text{proof}$$

ie, ji^{th} element is conjugate of $(ij)^{th}$ element

when $i = j$

$a_{ii} = \bar{a}_{ii} \Rightarrow$ a_{ii} is real

Diagonal elements are real.

eg $\begin{bmatrix} 5 & 1-i \\ 1+i & 3 \end{bmatrix}$

$$\begin{bmatrix} 5 & 2-i & 1+3i \\ 2+i & 2 & -7i \\ 1-3i & 7i & -1 \end{bmatrix}$$

A square matrix is Skew Hermitian if $\boxed{(\bar{A})^T = -A}$

$a_{ji} = -a_{ij}$

$\bar{\bar{a}}_{ji} = -\bar{a}_{ij}$

$a_{ji} = -\bar{a}_{ij}$

ji^{th} element is negative of conjugate of ij^{th} element

when $i = j$

$a_{ii} = -\bar{a}_{ii}$

$a_{ii} + \bar{a}_{ii} = 0$

$2\,Re(a_{ii}) = 0$

ie for skew hermitian, diagonal elements are purely imaginary or real part is 0.

$$A = \begin{bmatrix} 3i & 1-i \\ -1+i & 5i \end{bmatrix}, \begin{bmatrix} 2i & 3 & -2+i \\ -3 & 0 & 1-i \\ 2-i & -1+i & -5i \end{bmatrix}$$

→ Minor & cofactor of an element

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & 7 \\ 0 & 4 & 6 \end{bmatrix}$$

minor of 1 $= \begin{vmatrix} 5 & 7 \\ 4 & 6 \end{vmatrix} = 2$

minor of 5 $= \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} = 6$

minor of 6 $= \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = 8$

cofactor of $a_{ij} = (-1)^{i+j}$ minor

cofactor of $1 = (-1)^{1+1} \times 2 = 2$

cofactor of $5 = (-1)^{2+2} \times 6 = 6$

cofactor of $4 = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix}$

$= -1$

Determinant of 3×3 matrix

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & -1 & 5 \\ 1 & 6 & 7 \end{vmatrix}$$

$$|A| = 1\begin{vmatrix} -1 & 5 \\ 6 & 7 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ 1 & 7 \end{vmatrix} + 3\begin{vmatrix} 4 & -1 \\ 1 & 6 \end{vmatrix}$$

Sum of product of elements of any row or column with their cofactor

Properties

$|A| = |A^T|$

$|AB| = |A||B|$

If every element of row or column is 0, determinant is 0

If 2 rows or columns are identical, determinant is 0

If 2 rows or columns are interchanged, determinant value changes only in sign.

→ Adjoint of a square matrix

adj A is transpose of matrix obtained by replacing every element by its cofactor

$\text{adj } A = (A_{ij})^T$

cofactor of $a_{ij} = A_{ij}$

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

$$\text{cofactor matrix} = \begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\text{transpose (Adjoint } A) = \begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix}$$

$A \cdot \text{adj}(A) = |A|\, I$

$|A \text{ adj } A| = \big||A| I\big|$

$|A|\,|\text{adj } A| = |A|^n$

$|\text{adj } A| = |A|^{n-1}$

If there exists a matrix B such that $AB = BA = I$ then B is inverse of A

$A A^{-1} = A^{-1} A = I$

$A \cdot \dfrac{\text{adj } A}{|A|} = I$

* $\boxed{A^{-1} = \dfrac{\text{adj } A}{|A|}}$

* A^{-1} exists if $|A| \neq 0$

$|A^{-1}| = \dfrac{1}{|A|}$

* Inverse are unique

proof, If A has 2 inverses, B & C

$AB = BA = I$ — ①

$AC = CA = I$ — ②

$(AB)C = I(C) = C$ — ③

$= BA(C)$

② → $B(AC) = B(CA) = BI = B$ — ④

$(BA)C = B(AC)$

$C = B$

Inverse is unique

$(AB)^{-1} = B^{-1}A^{-1}$ ✓

$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A I A^{-1}$

$= AA^{-1} = I$

ie, $B^{-1}A^{-1}$ is inverse of AB

ie $(AB)^{-1} = B^{-1}A^{-1}$

For orthogonal matrix ✓

$\boxed{A^{-1} = A^T}$

eg $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$AA^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= I$

Q If A is orthogonal, then

a) $|A| = \pm 1$

b) If A & B are orthogonal so is AB

c) Inverse of orthogonal matrix is orthogonal

If A is orthogonal then $|A| = \pm 1$

$$AA^T = I$$
$$|AA^T| = 1$$
$$|A||A^T| = 1$$
$$|A||A| = 1$$
$$|A|^2 = 1$$
$$|A| = \pm 1$$

If A & B are orthogonal, so is AB.

$$AA^T = A^TA = I$$
$$BB^T = B^TB = I$$
$$(AB)(AB)^T = (AB)(B^T)(A^T)$$
$$= A(BB^T)A^T$$
$$= (AI)A^T$$
$$= AA^T$$
$$= I$$

If A is orthogonal, A^{-1} is also

$$AA^T = A^TA = I$$
$$(AA^T)^{-1} = (A^TA)^{-1} = I^{-1}$$
$$(A^T)^{-1}A^{-1} = A^{-1}A^{T^{-1}} = I$$
$$(A^{-1})^T A^{-1} = A^{-1}(A^{-1})^T = I$$

Inverse of A^{-1} is $(A^{-1})^T$

Q Find A^{-1} if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$A_{11} = (-1)^{1+1} 4 = 4$$
$$A_{12} = (-1)^{1+2} 3 = -3$$
$$A_{21} = (-1)^{2+1} 2 = -2$$
$$A_{22} = (-1)^{2+2} 1 = 1$$

$$\text{adj } A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{-1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ find A^{-1}

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$
$$= 1(-3) - 2(-2) + 2(2)$$
$$= 5$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$
$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 2$$
$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2$$
$$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} = 2$$
$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$
$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 2$$
$$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2$$
$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2$$
$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

$$\text{adj } A = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}^T$$
$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj} A}{|A|} = \frac{1}{5} \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

→ Eigen values & Eigen vectors

Let A is an $n \times n$ matrix & λ is an unknown. Consider matrix $A - \lambda I$ where I is the unit matrix of order n then $|A - \lambda I| = 0$ is called the characteristic equation of A. On expanding we get the ~~th~~ polynomial of degree which is called characteristic polynomial

On solving $|A - \lambda I| = 0$ we get n values for λ, $\lambda_1, \lambda_2 \ldots \lambda_n$ which are called Eigen values of A

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$$

Ch. eqn is $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
$$= \begin{vmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} = 0$$
$$\lambda^2 - 5\lambda - 14 = 0$$

solving $\lambda = 7, -2$

These are Eign values of $7, -2$

Q Find Eigen values.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 1-\lambda \end{vmatrix}$$

$= (2-\lambda)(\lambda^2-5\lambda+6)-2(1-\lambda)$
$+1(\lambda-1)$

$= 2\lambda^2-10\lambda+12-\lambda^3+5\lambda^2-6\lambda$
$-2+2\lambda+\lambda-1$

$= \cancel{-\lambda^3+7\lambda^2-13\lambda+9}$

$= \lambda^3-7\lambda^2+11\lambda-5=0$

$$\begin{array}{r} \lambda^2-6\lambda+5 \\ \lambda-1 \overline{)\lambda^3-7\lambda^2+11\lambda-5} \end{array}$$

$= (\lambda-1)(\lambda-1)(\lambda-5)$

$\lambda = 1, 1 \& 5$ eigen value

Eigen vector

Consider a square matrix A
Let X be a column matrix of n unknowns.

$(A-\lambda I)X = 0$ will give rise to n homogeneous eqns in the n unknowns.

Let $\lambda = \lambda_1, \lambda_2, \ldots$ be the eigen values of A

put $\lambda = \lambda_1$, a system ① when we try to solve these equations we get a non-trivial soluation say

$x_1 = \mu_1, x_2 = \mu_2 \cdots x_n = \mu_n$
(sol other than $x_1 = x_2 = x_3 = 0$)

then
$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

called Eigen vector corresponding to $\lambda = \lambda_1$

similarly we can have, eigen vectors corresponding to eigen values.

Q Find Eigen value & Eigen vectors of

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix}$$

$|A-\lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0$$

$(-2-\lambda)(4-\lambda)+5=0$

$\lambda = -1, 3$

consider $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$(A-\lambda I)X = 0$

$$\begin{bmatrix} -2-\lambda & -1 \\ 5 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-2-\lambda)x - y = 0$ —— ①
$5x + (4-\lambda)y = 0$ —— ②

case i, $\lambda = -1$

$-x-y=0$
$5x+5y=0$ Identical

$y = -x$

$x = a, y = -a$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ -a \end{bmatrix}$$

case ii, $\lambda = 3$

$-5x-y=0$
$5x+y=0$

ie $5x+y=0$
$y = -5x$

if $x=a, y=-5a$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ -5a \end{bmatrix}$$

Note: $\lambda = 2$ (any value other than Eigen)

$4x-y=0$ —①
$5x+2y=0$ —②

①×2+② →

$-3x=0$
$x=0$
$y=0$
solution is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We get a non zero solution for $(A-\lambda I)X=0$ only for eigen value of λ

H.W

Q i) $\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

i

$|A-\lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} = 0$$

$(3-\lambda)(2-\lambda)-20=0$

$\lambda^2-5\lambda-14=0$

$\lambda = 7, -2$

consider $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$(A-\lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(3-\lambda)x + 4y = 0$ —— ①
$5x + (2-\lambda)y = 0$ —— ②

case 1, when $\lambda = 7$,

eq ① →

$-4x + 4y = 0$

$x = y$

eq ② →

$5x - 5y = 0$

$x = y$

if $x = a$, $y = a$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$$

case 2, when $\lambda = -2$

$5x + 4y = 0$

$5x + 4y = 0$

$y = \frac{-5}{4} x$

when $x = a$, $y = \frac{5}{4} a$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ \frac{-5}{4}a \end{bmatrix}$$

ii) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

$|A - I\lambda| = 0$

$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(4-\lambda) - 4 = 0$

$\lambda^2 - 5\lambda = 0$

$\lambda = 0, 5$

when $\lambda = 5$

eq ① →

$X = \begin{bmatrix} x \\ y \end{bmatrix}$

$|A - I\lambda| X = 0$

$$\begin{bmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x(1-\lambda) - 2y = 0$

$-2x + y(4-\lambda) = 0$

when $\lambda = 5$

$-4x - 2y = 0$

$-2x - y = 0$

$2x = y$

when $x = a$, $y = 2a$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ -2a \end{bmatrix}$$

Generally, A is a square matrix & X is a column matrix, then AX is also a column matrix.

ie, A acts as a function.

Usually change in magnitude & direction.

But, Eigen values [illegible], change only in magnitude. No change in direction

Eigen value is the scalar such that there is a non zero vector such that $AX = \lambda X$

When we multiply matrix A with X, we get a new vector that is just a scalar multiple of X.

Q. Find Eigen values & vectors QP

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

characteristic eqn → $|A - I\lambda| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$2-\lambda((2-\lambda)(3-\lambda) - 2) - 2(1-\lambda) + 1(-1+\lambda)$

$2-\lambda(\lambda^2 - 5\lambda + 4) + (1-\lambda)[-2-1] =$

$(2-\lambda)(\lambda-1)(\lambda-4) + -3(1-\lambda) = 0$

$(1-\lambda)[(2-\lambda)(\lambda-4) + 3] = 0$

$(1-\lambda)(\lambda^2 - 6\lambda + 5) = 0$

$(1-\lambda)(\lambda-1)(\lambda-5) = 0$

$\lambda = 1, 1, 5$

Take $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$(2-\lambda)x + 2y + z = 0$

$x + y(3-\lambda) + z = 0$

$x + 2y + z(2-\lambda) = 0$

case 1, $\lambda = 1$

Identical

① → $x + 2y + z = 0$

case 2, ~~$\lambda = 5$~~

when $x = a$, $y = b$

$z = -a - 2b$

$\therefore$ Eigen vector = $\begin{bmatrix} a \\ b \\ -a-2b \end{bmatrix}$

case 2, $\lambda = 5$

$-3x + 2y + z = 0$ — ①

$x + -2y + z = 0$ — ②

$x + 2y - 3z = 0$ — ③

①+③ → $2x - 2z = 0$

$x = z$

sub in ① →

$-3x + 2y + x = 0$

$x = y$

ie, $x = y = z = a$

$\therefore$ Eigen vector = $\begin{bmatrix} a \\ a \\ a \end{bmatrix}$

Q. $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$(3-\lambda)((3-\lambda)^2 - 1) - 1(3 - \lambda + \overset{+1}{\cancel{3}})$

$-1(1 + 3 - \lambda)$.

$3-\lambda((4-\lambda)(2-\lambda)) - (4-\lambda)$

$-(4-\lambda)$

$(4-\lambda)((3-\lambda)(2-\lambda) - 1 - 1)$

$(4-\lambda)(\lambda^2 - 5\lambda + 4) = 0$

$(4-\lambda)(\lambda - 4)(\lambda - 1) = 0$

$1, 4, 4 = \lambda$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$(A - I\lambda)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$x(3-\lambda) + y - z = 0$

$x + y(3-\lambda) + z = 0$

$-x + y + z(3-\lambda) = 0$

when $\lambda = 1$ case 1

$2x + y - z = 0$ — ①

$x + 2y + z = 0$ — ②

$-x + y + 2z = 0$ — ③

②+③ → $3x + 3y = 0$

$y = -x$

sub in ③

$-x - x + 2z = 0$

$z = x$

Eigen vector = $\begin{bmatrix} a \\ -a \\ a \end{bmatrix}$

$\lambda = 4$,

$-x + y - z = 0$ $x = a$

$x - y + z = 0$ $y = b$

$-x + y - z = 0$ $z = b - a$

Eigen vector = $\begin{bmatrix} a \\ b \\ b-a \end{bmatrix}$

Q.

3	10	5
-2	-3	-4
3	5	7.

$|A - I\lambda| = 0$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ 2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0.$$

$(3-\lambda)[(-3-\lambda)(7-\lambda) + 20]$

$-10((7-\lambda)2 + 12) + 5(10 - 3(-3-\lambda$

$\lambda = 2$ is a root

$$\begin{array}{r} \lambda^2 - 5\lambda + 6 \\ \lambda - 2 \,\big)\, \overline{\lambda^3 - 7\lambda^2 + 16\lambda - 12} \\ -\lambda^3 - 2\lambda^2 \\ \hline -5\lambda^2 + 16\lambda \\ -5\lambda^2 + 10\lambda \\ \hline 6\lambda - 12 \end{array}$$

$\lambda = 2, 2, 3$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 10 & 5 \\ 2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$(3-\lambda)x + 10y + 5z = 0$

$-2x + y(-3-\lambda) - 4z = 0$

$3x + 5y + z(7-\lambda) = 0$

$\lambda = 2$

$x + 10y + 5z = 0$ — ①

$2x - 5y - 4z = 0$ — ②

$3x + 5y + 5z = 0$ — ③

②+③ = $x + z = 0$

$z = -x$

① → $y = \frac{2}{5}x$

Eigen vector = $\begin{bmatrix} a \\ \frac{2}{5}a \\ -a \end{bmatrix}$

when $\lambda = 3$

① ⇒ $10y + 5z = 0$

② $-2x - 6y - 4z = 0$

3 $3x + 5y + 4z = 0$

2+3 $x = y$

$z = -2y$

Eigen vector = $\begin{bmatrix} a \\ a \\ -2a \end{bmatrix}$

H.W i) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Ans.

i) $\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$

$|A - I\lambda| =$

$(2-\lambda)((2-\lambda)^2 - 0)$

$= (2-\lambda)^3 \cancel{(3-\lambda)(1-\lambda)}$

$\lambda = 2, 2, 2$

$(A - I\lambda)X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$(2-\lambda)x + y = 0$ — ①

$(2-\lambda)y + z = 0$ — ②

$(2-\lambda)z = 0$ — ③

when $\lambda = 2$

$y = 0$, $z = 0$, $x = a$ eigen vector = $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

~~when $\lambda = 3$~~

~~$-x + y = 0$, $-y + z = 0$, $-z = 0$; $y = 0$, $x = 0$, $z = 0$~~ ~~$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$~~

~~when $\lambda = 1$~~

~~$x + y = 0$, $y + z = 0$, $z = 0$; $y = 0$, $x = 0$~~ ~~$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$~~

ii) $\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$

$(6-\lambda)((3-\lambda)^2 - 1) + 2(-6 + 2\lambda + 2) + 2(2 - 6 + 2\lambda)$

$= (6-\lambda)(4-\lambda)(2-\lambda) + 4(\lambda - 2) + 4(\lambda - 2)$

~~$= (2-\lambda)((6-\lambda)(4-\lambda) - 4) - 4(\lambda+2)$~~

~~$= (2-\lambda)(\lambda^2 - 10\lambda + 20) - 4(\lambda + 2)$~~

~~$= (2-\lambda)($~~

$= (\lambda - 2)((\lambda - 6)(4 - \lambda) + 8)$

$= (\lambda - 2)(\lambda^2 - 10\lambda + 16)$

$= (\lambda - 2)(\lambda - 2)(\lambda - 8)$

$\lambda = 2, 2, 8$

$(A - I\lambda)(X) = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$(6-\lambda)x - 2y + 2z = 0$

$-2x + y(3-\lambda) - z = 0$

$2x - y + z(3-\lambda) = 0$

case 1, $\lambda = 2$

$4x - 2y + 2z = 0$

$-2x + y - z = 0$

$2x - y + z = 0$

when $x = a$,

$y = b$

$z = b - 2a$

$\begin{bmatrix} a \\ b \\ b-2a \end{bmatrix}$

case 2, $\lambda = 8$

$-2x - 2y + 2z = 0$

$-2x - 5y - z = 0$

$2x - y - 5z = 0$

$x + y - z = 0$

$2x + 5y + z = 0$

$2x - y - 5z = 0$

$6y + 6z = 0$

$y = -z$

$x + y + y = 0$

$x + 2y = 0$

$y = -\frac{1}{2}x$

$z = \frac{1}{2}x$

$\begin{bmatrix} a \\ -\frac{1}{2}a \\ \frac{1}{2}a \end{bmatrix}$

→ Diagonalisation

Given a known diagonal matrix A we are transforming this matrix which is in the diagonal ~~mat~~ form

eigen vectors of A

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$$

eg. $P = \begin{bmatrix} 1 & 4 \\ 1 & -5 \end{bmatrix}$

cofactor matrix $= \begin{bmatrix} -5 & -1 \\ -4 & 1 \end{bmatrix}$

$|P| = -5-4 = -9$

$$P^{-1} = \frac{-1}{9}\begin{bmatrix} -5 & -4 \\ -1 & 1 \end{bmatrix}$$

$P^{-1}AP = D$

$$= \frac{-1}{9}\begin{bmatrix} -5 & -4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & -5 \end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix} -5 & -4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 7 & -8 \\ 7 & 10 \end{bmatrix}$$

$$= \frac{-1}{9}\begin{bmatrix} -63 & 0 \\ 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$$ eigen values

$\boxed{D = P^{-1}AP}$

$PD = PP^{-1}AP$

$PD = AP$

$\boxed{PDP^{-1} = A}$

$A^2 = A \cdot A$

$= (PDP^{-1})(PDP^{-1})$

$= PD^2P^{-1}$

$A^2 = PD^2P^{-1}$

$A^3 = PD^3P^{-1}$

$\boxed{A^n = PD^nP^{-1}}$

If a square matrix of power n has n linearly independent Eigen vectors. We form a new matrix P whose columns are the ~~ideo~~ Eigen vectors of A. The Diagonal form is given by $\boxed{D = P^{-1}AP}$

where D is in diagonal form

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{bmatrix}$$

$$D^n = \begin{bmatrix} d_1^n & 0 \\ 0 & d_2^n \end{bmatrix}$$

Q. Find the matrix P which diagonalises the following matrices. Also find A^3

i). $A = \begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix}$

$$\begin{vmatrix} -2-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} = 0$$

$(-2-\lambda)(4-\lambda) + 5 = 0$

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\lambda = 3, -1$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -2-\lambda & -1 \\ 5 & 4-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-2-\lambda)x - y = 0$

$5x + y(4-\lambda) = 0$

$\lambda = 3$ case 1

$-5x - y = 0$ Identical

$5x + y = 0$

$$\begin{bmatrix} a \\ -5a \end{bmatrix}$$

$\lambda = -1$ case 2

$-x - y = 0$

$5x + 5y = 0$

$x = -y$

$$\begin{bmatrix} a \\ -a \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix}$$

$|P| = -5+1 = -4$

$A_{11} = (-1)^{1+1}-5 = -5$

$A_{12} = (-1)^{1+2}-1 = 1$

$A_{21} = (-1)^{2+1} 1 = -1$

$A_{22} = (-1)^{2+2} 1 = 1$

$$\begin{bmatrix} -5 & 1 \\ -1 & 1 \end{bmatrix}^T$$

$$P^{-1} = \frac{\text{adj} P}{|P|} = \frac{\begin{bmatrix} -5 & 1 \\ -1 & 1 \end{bmatrix}}{-4} = \frac{\begin{bmatrix} -5 & - \\ 1 & 1 \end{bmatrix}}{-4}$$

$$P^{-1}AP = \frac{-1}{4}\begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -5 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -15 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} 4 & 0 \\ 0 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = D.$$

$A = PDP^{-1}$

$A^3 = PD^3P^{-1}$

$= \frac{-1}{4}\begin{bmatrix}1 & 1\\ -1 & -5\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 27\end{bmatrix}\begin{bmatrix}-5 & -1\\ 1 & 1\end{bmatrix}$

$= \frac{-1}{4}\begin{bmatrix}1 & 1\\ -1 & -5\end{bmatrix}\begin{bmatrix}5 & 1\\ 27 & 27\end{bmatrix}$

$= \frac{-1}{4}\begin{bmatrix}32 & 28\\ -140 & -136\end{bmatrix}$

$= \begin{bmatrix}-8 & -7\\ 35 & 34\end{bmatrix}$

$\begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}$

$\begin{vmatrix}1-\lambda & 0 & -1\\ 1 & 2-\lambda & 1\\ 2 & 2 & 3-\lambda\end{vmatrix} = 0$

$(1-\lambda)((2-\lambda)(3-\lambda)-2)$

$-1(2-4+2\lambda)$

$(1-\lambda)(\lambda^2-5\lambda+4)-2(\lambda-1)$

$(1-\lambda)((\lambda-1)(\lambda-4)+2)$

$(1-\lambda)(\lambda^2-5\lambda+6)$

$(1-\lambda)(\lambda-2)(\lambda-3) = 0$

$\lambda = 1, 2, 3$

$\begin{bmatrix}1-\lambda & 0 & -1\\ 1 & 2-\lambda & 1\\ 2 & 2 & 3-\lambda\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = 0$

$(1-\lambda)x - z = 0$

$x + y(2-\lambda) + z = 0$

$2x + 2y + z(3-\lambda) = 0$

when $\lambda = 1$

$z = 0$

$x + y = 0$

$\begin{bmatrix}a\\ -a\\ 0\end{bmatrix}$

$\lambda = 2$

$-x - z = 0$

$x + z = 0$

$2x + 2y + z = 0$

$2x + 0 + 2z = 0$

$2y - z = 0$

$y = \frac{z}{2} = \frac{-x}{2}$

$\begin{bmatrix}a\\ \frac{-a}{2}\\ -a\end{bmatrix}$ $\quad \begin{matrix}2\\ -1\\ -2\end{matrix}$

$\lambda = 3$

$-2x - z = 0$

$x + -y + z = 0$

$2x + 2y = 0$

$x = -y$

$2x + z = 0$

$z = -2x$

$\begin{bmatrix}a\\ -a\\ -2a\end{bmatrix}$

$P = \begin{bmatrix}1 & 2 & 1\\ -1 & -1 & -1\\ 0 & -2 & -2\end{bmatrix}$

$|P| = 1(2+2) - 2(2) + 1(-2)$

$= 4 - 4 - 2 = -2$

$A_{11} = (-1)^{1+1}(2+2) = \cancel{4}\ \ 0$

$A_{12} = -(2) = \cancel{-2}\ \ -2$

$A_{13} = +(-2) = \cancel{-2}\ \ 2$

$A_{21} = -(-4-2) = \cancel{6}\ \ 2$

$A_{22} = (-2) = -2$

$A_{23} = -(2) = -2$

$A_{31} = +(-2+1) = -1$

$A_{32} = -(-1+1) = 0$

$A_{33} = +(-1+2) = 1$

$P^{-1} = \dfrac{\begin{bmatrix}0 & -2 & 2\\ 2 & -2 & -2\\ -1 & 0 & 1\end{bmatrix}^{T}}{-2}$

$= \frac{-1}{2}\begin{bmatrix}0 & 2 & -1\\ -2 & -2 & 0\\ 2 & -2 & 1\end{bmatrix}$

$D = P^{-1}AP$

$= \frac{-1}{2}\begin{bmatrix}0 & 2 & -1\\ -2 & -2 & 0\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ -1 & -1 & -1\\ 0 & -2 & -2\end{bmatrix}$

$= \frac{-1}{2}\begin{bmatrix}0 & 2 & -1\\ -2 & -2 & 0\\ 2 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 4 & 3\\ -1 & -2 & -3\\ 0 & -4 & -6\end{bmatrix}$

$= \frac{-1}{2}\begin{bmatrix}-2 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -6\end{bmatrix}$

$= \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$

$A^3 = PD^3P^{-1} = \begin{bmatrix}1 & 2 & 1\\ -1 & -1 & -1\\ 0 & -2 & -2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 27\end{bmatrix}\begin{bmatrix}0 & 2 & -1\\ -2 & -2 & 0\\ 2 & -2 & 1\end{bmatrix} \times \frac{-1}{2}$

$= \frac{-1}{2}\begin{bmatrix}1 & 2 & 1\\ -1 & -1 & -1\\ 0 & -2 & -2\end{bmatrix}\begin{bmatrix}0 & 2 & -1\\ -16 & -16 & 0\\ 54 & -54 & 27\end{bmatrix}$

$= \frac{-1}{2}\begin{bmatrix}22 & 24 & 26\\ -38 & -40 & -26\\ -76 & -76 & -54\end{bmatrix}$

Q. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

$|A - I\lambda| = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix}$

$(1-\lambda)(\lambda^2 - 5\lambda + 2)$

$-1(4) + 1(8 - 4\lambda)$

$\lambda = 1, 2, 3$

$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$(1-\lambda)x + y + z = 0$

$(2-\lambda)y + z = 0$

$-4x + 4y + z(3-\lambda) = 0$

$\lambda = 1$

$y + z = 0 \qquad y = -z$

$y + z = 0$

$-4x + 4y + 2z = 0$

$-4x + 4y - 2y = 0$

$-4x + 2y = 0$

$y = 2x$

$\begin{bmatrix} a \\ 2a \\ -2a \end{bmatrix} \quad \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$

$\lambda = 2$

$-x + y + z = 0$

$z = 0$

$-4x + 4y + z = 0$

$x = y$

$\begin{bmatrix} a \\ a \\ 0 \end{bmatrix}$

$\lambda = 3$

$-2x + y + z = 0$

$-y + z = 0 \qquad y = z$

$-4x + 4y = 0 \qquad x = y$

$\begin{bmatrix} a \\ a \\ a \end{bmatrix}$

$P = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

$A_{11} = 1$

$A_{12} = -(-2-2) = 4$

$A_{13} = -2$

$A_{21} = -1$

$A_{22} = -3$

$A_{23} = -2$

$A_{31} = 0$

$A_{32} = -1$

$A_{33} = 1$

$\cancel{\begin{bmatrix} 1 & 4 & -2 \\ -1 & -3 & -2 \\ 0 & -1 & 1 \end{bmatrix}}$

$\cancel{|P| = -1(1)}$

$-1(-2$

$+1(-2)$

$= -1 + 4 - 2$

$= 5$

$|P| = 1$

$P^{-1} = \dfrac{\begin{bmatrix} 1 & 4 & -2 \\ -1 & -3 & -2 \\ 0 & -1 & 1 \end{bmatrix}^T}{1}$

$= \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -1 \\ -2 & -2 & 1 \end{bmatrix}$

$D = P^{-1}AP$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A^3 = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -1 \\ -2 & -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -23 & 31 & 19 \\ -24 & 32 & 19 \\ -52 & -52 & 27 \end{bmatrix}$

→ Cayley - Hamilton Theorem

Every square matrix satisfies a characteristic equation.

Verify CH for $A = \begin{bmatrix} -2 & 1 \\ 5 & 4 \end{bmatrix}$

also find A^3 & A^{-1}

$\begin{vmatrix} -2-\lambda & 1 \\ 5 & 4-\lambda \end{vmatrix}$

$= (-2-\lambda)(4-\lambda) - 5 = 0$

$= \lambda^2 - 2\lambda - 13 = 0$

To verify $A^2 - 2A - 13I = 0$

$A^2 = \begin{bmatrix} -2 & 1 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 5 & 4 \end{bmatrix}$

$= \begin{bmatrix} 9 & 2 \\ 10 & 21 \end{bmatrix}$

$A^2 - 2A = \begin{bmatrix} 9 & 2 \\ 10 & 21 \end{bmatrix} - \begin{bmatrix} -4 & 2 \\ 10 & 8 \end{bmatrix}$

$= \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$

$A^2 - 2A - 13I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^2 = 13I + 2A$

$A^3 = 13A + 2A^2$

$= \begin{bmatrix} -26 & 13 \\ 65 & 52 \end{bmatrix} + \begin{bmatrix} 18 & 4 \\ 20 & 42 \end{bmatrix}$

$= \begin{bmatrix} -8 & 17 \\ 85 & 94 \end{bmatrix}$

Q $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ find A^{-1} using CHT. Also find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix} \quad \left(\lambda^2 - 4\lambda - 5 = 0 \right. \text{ — ①}$$

consider

$$\lambda^2 - 4\lambda - 5 \,\big)\, A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I \,\big(\, \lambda^3 - 2\lambda + 3$$
$$\lambda^5 - 4\lambda^4 - 5\lambda^3$$
$$-2\lambda^3 + 11\lambda^2 - \lambda$$
$$-2\lambda^3 + 8\lambda^2 + 10\lambda$$
$$3\lambda^2 - 11\lambda - 10$$
$$3\lambda^2 - 12\lambda - 15$$
$$\lambda + 5$$

$$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I$$
$$\Rightarrow (\lambda^3 - 2\lambda + 3)(\lambda^2 - 4\lambda - 5) + \lambda + 5$$
$$= (\lambda^3 - 2\lambda + 3) \times 0 + (\lambda + 5)$$
$$= \lambda + 5$$
$$= A + 5I$$
$$= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$(1-\lambda)\left((1-\lambda)^2 - 1\right) + 3\left(-2 - (1-\lambda)\right)$$
$$= (1-\lambda)(\lambda^2 - 2\lambda) + 3(-3+\lambda) = 0$$
$$= \lambda^3 - 3\lambda^2 - \lambda + 9 = 0$$

by CHT

$$A^3 - 3A^2 - A + 9I = 0$$
$$A^2 - 3A - I + 9A^{-1} = 0$$
$$9A^{-1} = -A^2 + 3A + I$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

→ Properties of Eigen values

① A & A^T have same Eigen values.

proof:

Eigen values of A is obtained by $|A - \lambda I| = 0$

$$|A - \lambda I| = |(A - \lambda I)^T|$$
$$= |A^T - (\lambda I)^T|$$
$$|A - \lambda I| = |A^T - \lambda I|.$$

if $|A - \lambda I| = 0 \Rightarrow$
$|A^T - \lambda I| = 0$
same Eigen values

② Sum of Eigen values of matrix is the sum of diagonal elements

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= (a_{11} - \lambda)\left[(a_{22} - \lambda)(a_{33} - \lambda) - (a_{23} \times a_{32})\right]$$
$$- a_{12}\left[(a_{33} - \lambda)a_{21} - a_{23}a_{31}\right]$$
$$+ a_{13}\left[a_{21}a_{32} - (a_{22} - \lambda)a_{31}\right]$$

$$= -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2$$
$$+ (-a_{11}a_{22} - a_{11}a_{33} + a_{22}a_{32} + a_{13}a_{21} + a_{12}a_{21})\lambda$$
$$+ (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22})$$

$$|A - \lambda I| = -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$
$$= -(\lambda - \lambda_1)\left(\lambda^2 - (\lambda_2 + \lambda_3)\lambda + \lambda_2\lambda_3\right)$$
$$= -\left(\lambda^3 - (\lambda_2 + \lambda_3)\lambda^2 + \lambda_2\lambda_3\lambda - \lambda^2\lambda_1 + \lambda_1(\lambda_2 + \lambda_3)\lambda - \lambda_1\lambda_2\lambda_3\right)$$

② → $= -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_2\lambda_3 + \lambda_1\lambda_2 + \lambda_1\lambda_3)\lambda + \lambda_1\lambda_2\lambda_3$

$$\boxed{\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}}$$

Sum of Eigen values = Sum of diagonal elements
from ① & ②

③ Determinant of a matrix is the product of the Eigen values.

$\lambda = 0$ in ②

$|A| = \lambda_1 \lambda_2 \lambda_3$

$AA^T = A^T A = I$

$A^T = A^{-1}$

λ is Eigen value of A

$\Rightarrow \frac{1}{\lambda}$ is Eigen value of A^{-1}

$\Rightarrow \frac{1}{\lambda}$ is Eigen value of A^T

$\Rightarrow \frac{1}{\lambda}$ is Eigen value of A

④ If λ is Eigen value of A then $\frac{1}{\lambda}$ is an Eigen value of A^{-1}

proof.

Let λ be an Eigen value

Let λ be an Eigen value of A & X is the Eigen vector

$AX = \lambda X$

$A^{-1}AX = A^{-1}\lambda X$

$IX = \lambda A^{-1}X$

$X = \lambda A^{-1}X$

$A^{-1}X = \frac{1}{\lambda}X$

$\frac{1}{\lambda}$ is Eigen value of A^{-1}

⑤ If λ is Eigen value of orthogonal matrix, then $\frac{1}{\lambda}$ is also an Eigen value

⑥ If $\lambda_1, \lambda_2 \ldots \lambda_n$ are Eigen values of A then, $\lambda_1^m, \lambda_2^m \ldots \lambda_n^m$ are Eigen values of A^m

proof

λ is Eigen value & X is Eigen vector.

$AX = \lambda X$ ——①

$AAX = A\lambda X$

$A^2X = \lambda AX$

$A^2X = \lambda(\lambda X)$

$A^2X = \lambda^2 X$

$\therefore \lambda^2$ is Eigen value of A^2

⑦ Eigen values of Hermition matrix are real. Cannot be imaginary

proof. Given A is Hermition

$(\bar{A})^T = A$ ——①

Let λ is Eigen value & X is a Eigen vector.

$AX = \lambda X$ ——②

$\overline{AX} = \overline{\lambda X}$

$\bar{A}\bar{X} = \bar{\lambda}\bar{X}$

$(\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T$ from ①

$\bar{X}^T . \bar{A}^T = \bar{\lambda} . \bar{X}^T$

$\bar{X}^T . A = \bar{\lambda} . \bar{X}^T$ ——③

$\bar{X}^T AX = \bar{\lambda}\bar{X}^T X$ ——③

② $\Rightarrow \bar{X}^T AX = \bar{X}^T \lambda X$ ——④

from ③ & ④

$\bar{\lambda}\bar{X}^T X = \lambda \bar{X}^T X$

$\bar{\lambda} = \lambda \Rightarrow \lambda$ REAL

⑧ Eigen values of Skew-Hermition matrix are purely imaginary

proof

$(\bar{A})^T = -A$ ——①

Let λ be Eigen value, X be Eigen vector

$AX = \lambda X$ ——②

$\overline{AX} = \overline{\lambda X}$

$\bar{A}\bar{X} = \bar{\lambda}\bar{X}$

$\bar{X}^T . \bar{A}^T = \bar{\lambda}(\bar{X})^T$ by ①

$-\bar{X}^T . A = \bar{\lambda}(\bar{X})^T$

$-\bar{X}^T . A = \bar{\lambda}\bar{X}^T$

$-\bar{X}^T AX = \bar{\lambda}\bar{X}^T X$ ——③

from ②

$-\bar{X}^T AX = \bar{X}^T \lambda X$

$-\bar{X}^T AX = -\lambda \bar{X}^T X$ ——④

from ③ & ④

$\bar{\lambda}\bar{X}^T X = -\lambda \bar{X}^T X$

$\bar{\lambda} = -\lambda$

$\lambda + \bar{\lambda} = 0$

λ is imaginary

⑨ The Eigen values of an orthogonal matrix are of magnitude 1

To prove $\lambda = \pm 1$

A is orthogonal

$$AA^T = A^TA = I \quad \text{---} ①$$

Let λ be an Eigen value & x be the Eigen vector

$$Ax = \lambda x$$

$$(Ax)^T = (\lambda x)^T$$

$$x^TA^T = \lambda x^T$$

② × ① ⇒

$$(x^TA^T)(Ax) = \lambda x^T \lambda x$$

$$x^T(A^TA)x = \lambda^2 x^T x$$

$$x^T I x = \lambda^2 x^T x$$

$$x^T x = \lambda^2 x^T x$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

⑩ The Eigen value of an Idempotent matrix are zero or unity

proof.

Given it is idempotent

$$A^2 = A \quad \text{---} ①$$

Let λ be an Eigen value & x be the Eigen value

$$Ax = \lambda x \quad \text{---} ②$$

$$A^2x = A\lambda x$$

$$A^2x = \lambda Ax$$

$$A^2x = \lambda^2 x$$

$$\Rightarrow Ax = \lambda^2 x \text{ by } ① \quad \text{---} ③$$

from ② & ③

$$\lambda x = \lambda^2 x$$

$$\lambda = \lambda^2$$

$$\lambda^2 - \lambda = 0$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = 0, 1$$

MODULE II

Laplace Transforms

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \qquad * \quad L(t^3) = \frac{3!}{s^4}$$

$$L(1) = \int_0^\infty e^{-st}\, \cancel{f(t)}\, 1\, dt$$

$$= -\frac{e^{-st}}{s}\Big]_0^\infty$$

$$= \frac{1}{s}(0-1) = \frac{1}{s}$$

Let $f(t)$ be a function defined ~~for~~ $t > 0$, then laplace transform of $f(t)$ denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

where s is a parameter

The RHS is a function of s call its $\bar{f}(s)$. $L\{f(t)\} = \bar{f}(s)$

$f(t)$ input → [LT] → $\bar{f}(s)$ output

$f(t)$ is called inverse Laplace transform of $\bar{f}(s)$

$$L^{-1}\{\bar{f}(s)\} = f(t)$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

proof:

$$L(t) = \int_0^\infty e^{-st}\, t\, dt$$

$$= \left[t \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{1 \times e^{-st}}{-s}\, dt$$

$$= \frac{-1}{s}\left[\frac{t}{e^{st}}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= \frac{-1}{s}\left[0 - \frac{0}{1}\right] + \frac{1}{s} L\{1\}$$

$$= \frac{1}{s} \times \frac{1}{s} = \frac{1}{s^2}$$

$$L\{t^2\} = \frac{2}{s^3}$$

similarly, $L(t^3) = \dfrac{6}{s^4} = \dfrac{3!}{s^4}$

→ Laplace transform of some elementary function

i) $L\{1\} = \int_0^\infty e^{-st}\, 1\, dt$

$= \left.\frac{e^{-st}}{-s}\right]_0^\infty$

$= -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty$

$= \frac{-1}{s}\left(0 - \frac{1}{0}\right) = \frac{1}{s}$

$L^{-1}\left\{\frac{1}{s}\right\} = 1$

ii) $L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$

$= \int_0^\infty e^{-(s-a)t}$

$= \left.\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$

$= \frac{-1}{s-a}\left(\frac{1}{e^{(s-a)t}}\right)_0^\infty$

$= \frac{-1}{s-a}\left(0 - \frac{1}{1}\right) = \frac{1}{s-a}$

$L\{e^{at}\} = \frac{1}{s-a}$

$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

similarly

$L\{e^{-at}\} = \frac{1}{s+a}$, $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

iii) $L\{\sin(at)\}$

$= \int_0^\infty e^{-st} \sin(at)\, dt$

$= \frac{e^{-st}}{s^2+a^2}\left(-s\sin at) - a\cos\right.$

$= \frac{-1}{s^2+a^2}\left[\frac{s\sin at + a\cos at}{e^{st}}\right]$

$= \frac{-1}{s^2+a^2}\left(0 - \left(\frac{0+a}{1}\right)\right)$

$= \frac{a}{s^2+a^2}$

$L\{\sin(at)\} = \frac{a}{s^2+a^2}$

$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at)$

* $\int e^{ax}\sin bx\, dx$

$= \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

* $\int e^{ax}\cos bx\, dx$

$= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

iv) $L\{\cos at\} = \int_0^\infty e^{-st}\cos at\, dt$

$= \frac{e^{-st}}{(-s)^2+a^2}\left(-s\cos at + a\sin at\right)_0^\infty$

$= \frac{1}{s^2+a^2}\left(\frac{-s\cos at + a\sin at}{e^{st}}\right)$

$= \frac{1}{s^2+a^2}\left(0 - \frac{(-s)}{1}\right)$

$L\{\cos(at)\} = \frac{s}{s^2+a^2}$

$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

v) $L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\}$

$= \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right) dt$

$= \frac{1}{2}\int_0^\infty e^{-st} e^{at}\, dt - \frac{1}{2}\int_0^\infty e^{-st}\cdot e^{-at}\, dt$

$= \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$

$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$

$= \frac{1}{2}\left(\frac{s+a-(s-a)}{(s-a)(s+a)}\right)$

$= \frac{1}{2}\left(\frac{2a}{s^2-a^2}\right) = \frac{a}{s^2-a^2}$

$L\{\sinh(at)\} = \frac{a}{s^2-a^2}$

vi) $L\{\cosh(at)\}$

$= L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$= \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt$

$= \frac{1}{2}\int_0^\infty e^{-st} e^{at}\, dt + \frac{1}{2}\int_0^\infty e^{-st} e^{-at}\, dt$

$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$= \frac{1}{2}\left(\frac{s+a+s-a}{s^2-a^2}\right) = \frac{s}{s^2+a^2}$

$L\{\cosh at\} = \frac{s}{s^2-a^2}$

$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$

→ Linearity property

$L\{af(t) + bg(t)\}$

$= L\{af(t)\} + L\{bg(t)\}$

proof

$L\{af(t) + bg(t)\}$

$= \int_0^\infty e^{-st}(af(t) + bg(t))\, dt$

$= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt$

$= a.L\{f(t)\} + bL\{g(t)\}$

→ Shifting property

If $L\{f(t)\} = \bar{f}(s)$

then $L\{e^{at} f(t)\} = \bar{f}(s-a)$

proof

Given $L\{f(t)\} = \bar{f}(s)$

$\int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$ —— ①

$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt$

$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$= \bar{f}(s-a)$

Note: If $L\{f(t)\} = \bar{f}(s)$

then $L\{e^{-at} f(t)\} = \bar{f}(s+a)$

eg. $L\{\sin t\} = \frac{1}{s^2+1}$

$L\{e^{2t}\sin t\} = \frac{1}{(s-2)^2+1}$

$L\{e^{3t}\sin t\} = \frac{1}{(s-3)^2+1}$

eg $L\{\cosh 2t\} = \frac{s}{s^2-2^2}$

$L\{e^{t}\cosh 2t\} = \frac{s-1}{(s-1)^2-2^2}$

If $L\{f(t)\} = \bar{f}(s)$ then

$L\{\sinh at\, f(t)\}$

$: L\{\sinh(at) f(t)\}$

$= L\left\{\frac{e^{at}-e^{-at}}{2}\right\} f(t)$

$= \frac{1}{2}\left(L(e^{at} f(t)) - L(e^{-at} f(t))\right)$

$= \frac{1}{2}\left(\bar{f}(s-a) - \bar{f}(s+a)\right)$

Standard Results

$L\{1\} = \frac{1}{s}$

$L\{e^{at}\} = \frac{1}{s-a}$

$L\{e^{-at}\} = \frac{1}{s+a}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{\cos at\} = \frac{s}{s^2+a^2}$

$L\{\sinh at\} = \frac{a}{s^2-a^2}$

$L\{\cosh at\} = \frac{s}{s^2-a^2}$

$L\{t\} = \frac{1}{s^2}$

$L\{t^n\} = \frac{n!}{s^{n+1}}$

$L\{f(t)\} = \bar{f}(s)$ then

$L\{e^{at} f(t)\} = \bar{f}(s-a)$

$L\{f(t)\} = \bar{f}(s)$ then

$L\{e^{-at} f(t)\} = \bar{f}(s+a)$

$L\{f(t)\} = \bar{f}(s)$ then

$L\{\sinh at\, f(t)\} = \frac{1}{2}\left(\bar{f}(s-a) - \bar{f}(s+a)\right)$

Find Laplace transform of the following

i) $L\{\sin^2 3t\}$

$= L\left\{\frac{1-\cos 6t}{2}\right\}$

$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 6t\}$

$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+a^2}$

ii) $L\{\sin 4t\cos 2t\}$

$= \frac{1}{2}L\{2\sin 4t \cos 2t\}$

$= \frac{1}{2}L\{\sin 6t + \sin 2t\}$

$= \frac{1}{2}\left(\frac{6}{s^2+6^2} + \frac{2}{s^2+2^2}\right)$

iii) $L\{\cos^2 t \cos 4t\}$

$= L\left\{\frac{1+\cos 2t}{2}\cos 4t\right\}$

$= \frac{1}{2}L\{\cos 4t\} + \frac{1}{2}L\{\cos 4t \cos 2t\}$

$= \frac{1}{2}L\{\cos 4t\} + \frac{1}{4}L\{2\cos 4t \cos 2t\}$

$= \frac{1}{2}L\{\cos 4t\} + \frac{1}{4}L\{\cos 6t + \cos 2t\}$

$= \frac{1}{2}\left(\frac{s}{s^2+4^2}\right) + \frac{1}{4}\left(\frac{s}{s^2+6^2} + \frac{s}{s^2+2^2}\right)$

Q. $\cos^2 3t$

$\cos^2 3t = \frac{1+\cos 6t}{2}$

$L(\cos^2 3t) = \frac{1}{2}(L\{1\} + L\{\cos 6t\})$

$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2-6^2}\right)$

Q. $\sin^2 t \sin 3t$

$= \left(\frac{1-\cos 2t}{2}\right)\sin 3t$

$= \frac{1}{2}\sin 3t - \frac{1}{2}\sin 3t\cos 2t$

$= \frac{1}{2}\sin 3t - \frac{1}{4}(2\sin 3t\cos 2t)$

$= \cancel{\frac{1}{2} \times \frac{3}{s^2+3^2} - \frac{1}{4}}$

$= \frac{1}{2}\sin 3t - \frac{1}{4}(\sin 5t + \sin t)$

$= \frac{1}{2} \times \frac{3}{s^2+3^2} - \frac{1}{4}\left(\frac{5}{s^2+3^2} + \frac{1}{s^2+1}\right)$

Q. $\cos 6t \cos 4t \cos 2t$

$= \frac{1}{2}(2\cos 6t\cos 4t)\cos 2t$

$= \frac{1}{2}(\cos 10t + \cos 2t)\cos 2t$

$= \frac{1}{4}(2\cos 10t\cos 2t + 2\cos^2 2t)$

$= \frac{1}{4}(\cos 12t + \cos 8t + 1 + \cos 4t)$

$= \frac{1}{4}\left(\frac{s}{s^2+12^2} + \frac{s}{s^2+8^2} + \frac{1}{s} + \frac{s}{s^2+4^2}\right)$

Q. $e^{3t}\sin^2 t$

$= e^{3t}\left(\frac{1-\cos 2t}{2}\right)$

$= \frac{1}{2}(e^{3t} - e^{3t}\cos 2t)$ (SHIFT

$L = \frac{1}{2}\left(\frac{1}{s-3} - \frac{s-3}{((s-3)^2+2^2)}\right)$

Q. $e^{-3t}(2\cos 5t - 3\sin 5t)$

$L\{2\cos 5t - 3\sin 5t\}$

$= 2 \times \frac{s}{s^2+5^2} - \frac{3\times 5}{s^2+5^2}$

by SHIFT,

$L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

$= \frac{2(s+3)}{(s+3)^2+5^2} - \frac{15}{(s+3)^2+5^2}$

for e^{at} replace s with $(s-a)$

for e^{-at} replace s with $(s+a)$

Q. $e^{4t}\sin 2t\cos t$

$L\{\sin 2t\cos t\}$

$= \frac{1}{2}L\{2\sin 2t\cos t\}$

$= \frac{1}{2}L\{\sin 3t + \sin t\}$

$= \frac{1}{2} \times \left(\frac{3}{s^2+3^2} + \frac{1}{s^2+1}\right)$

$L\{e^{4t}\sin 2t\cos t\}$

$= \frac{1}{2}\left(\frac{3}{(s-4)^2+3^2} + \frac{1}{(s-4)^2+1}\right)$

Q. $t^2 e^{2t}$

$L\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

$L\{e^{2t}t^2\} = \frac{2}{(s-2)^3}$

Q. $t^2 e^{3t}\sinh t$

$= t^2 e^{3t}\frac{e^t - e^{-t}}{2}$

$= \frac{1}{2}t^2(e^{4t} - e^{2t})$

$= \frac{1}{2}(t^2e^{4t} - t^2e^{2t})$

$L\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

$= \frac{1}{2}\left(\frac{2}{(s-4)^3} - \frac{2}{(s-2)^3}\right)$

Q. $t^3\cosh 2t$

$= t^3\left(\frac{e^{2t}+e^{-2t}}{2}\right)$

$= \frac{t^3e^{2t} + e^{-2t}t^3}{2}$

$L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

$L\{t^3\cosh 2t\} = \frac{1}{2}\left(\frac{6}{(s-2)^4} + \frac{6}{(s+2)^4}\right)$

Q $f(t) = |t-1| + |t+1|$; $t \geq 0$

find $L\{f(t)\}$

ans

$$f(t) = \begin{cases} -(t-1)+(t+1) & \text{for } 0<t<1 \\ (t-1)+(t+1) & \text{for } t>1 \end{cases}$$

$$= \begin{cases} 2 ; & 0<t<1 \\ 2t ; & t>1 \end{cases}$$

$$= \int_0^1 e^{-st} 2\,dt + \int_1^\infty e^{-st} 2t\,dt$$

$$= 2 \frac{e^{-st}}{-s}\Big]_0^1 + 2\left(t \frac{e^{-st}}{-s}\Big]_1^\infty - \int_1^\infty \frac{e^{-st}}{-s} dt \right)$$

$$= \frac{-2}{s} \frac{1}{e^{st}}\Big]_0^1 - \frac{2}{s}\left(\frac{t}{e^{st}}\right)_1^\infty + \frac{2}{s}\left(\frac{e^{-st}}{s}\right)_1^\infty$$

$$\cancel{= \frac{-2}{s}\left(\frac{1}{e^{st}}\right)_0^1 - \frac{2}{s}\left(\frac{t}{e^{st}}\right)_1^\infty + \frac{2}{s}}$$

$$= \frac{-2}{s}\left(\frac{1}{e^{s}} - 1\right) - \frac{2}{s}\left(0 - \frac{1}{e^{s}}\right) + \frac{2}{s} - \frac{2}{s^2}\left(0 - \frac{1}{e^{s}}\right)$$

$$= e^{-s}\left(\frac{-2}{s} + \frac{2}{s} + \frac{2}{s^2}\right) + \frac{2}{s}$$

$$= \frac{2e^{-s}}{s^2} + \frac{2}{s}$$

Q $f(t) = \begin{cases} t^2, & 0<t<2 \\ t-1, & 2<t<3 \\ 7, & t>3 \end{cases}$

$$= \int_0^2 e^{-st} t^2\,dt + \int_2^3 e^{-st}(t-1)\,dt + \int_3^\infty 7\,dt$$

$$\cancel{= t^2\left(\frac{e^{-st}}{-s}\right)_0^2 - \int 2t\left(\frac{e^{-st}}{-s}\right)}$$

$\int uv = uv_1 - u'v_2 + u''v_3$

$$= t^2\left(\frac{e^{-st}}{-s}\right) - 2t\left(\frac{e^{-st}}{s^2}\right) + 2\left(-\frac{e^{-st}}{s^3}\right)\Big]_0^2$$

$$+ (t-1)\left(\frac{e^{-st}}{s}\right) - \left(\frac{e^{-st}}{s^2}\right)\Big]_2^3$$

$$+ 7\frac{e^{-st}}{-s}\Big]_3^\infty$$

$$= \frac{-1}{s}(4e^{-2s} - 0) - \frac{2}{s^2}(2e^{-2s} - 0) - \frac{2}{s^3}(e^{-2s} - 1)$$

$$+ \frac{-1}{s}(2e^{-3s} - e^{-2s}) - \frac{1}{s^2}(e^{-3s} - e^{-2s}) - \frac{7}{s}(0 - e^{-3s})$$

Multiplication by t^n

If $L\{f(t)\} = \bar{f}(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{\partial^n}{\partial s^n} \bar{f}(s)$ *

proof $L\{f(t)\} = \bar{f}(s)$

$$\int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$$

diff wrt s.

$$\frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right) = \frac{d}{ds}\bar{f}(s)$$

$$\int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt = \frac{d}{\partial s}\bar{f}(s)$$

$$\int_0^\infty e^{-st} \cdot -t\, f(t)\,dt = \frac{\partial}{\partial s}\bar{f}(s)$$

$$\int_0^\infty e^{-st}(t f(t))\,dt = (-1)\frac{\partial}{\partial s}\bar{f}(s)$$

$$L\{t f(t)\} = (-1)\frac{\partial}{\partial s}\bar{f}(s)$$

$$L\{t^2 f(t)\} = (-1)^2\frac{\partial^2}{\partial s^2}\bar{f}(s)$$

$$* \quad L\{t^n f(t)\} = (-1)^n\frac{\partial^n}{\partial s^n}\bar{f}(s)$$

Division by t

If $L\{f(t)\} = \bar{f}(s)$

then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$

proof

$$L\{f(t)\} = \bar{f}(s)$$

$$\int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$$

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left(\int_0^\infty e^{-st} f(t)\,dt\right) ds$$

$$= \int_0^\infty \left(\int_s^\infty e^{-st}\,ds\right) f(t)\,dt$$

$$= \int_0^\infty \frac{e^{-st}}{-t}\Big]_s^\infty f(t)\,dt$$

$$= \int_0^\infty \frac{-1}{t}\left(\frac{1}{e^{st}}\right) f(t)\,dt$$

$$= \int_0^\infty \frac{-1}{t}\left(0 - \frac{1}{e^{st}}\right) f(t)\, dt$$

$$= \int_0^\infty e^{-st}\left(\frac{f(t)}{t}\right) dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

$$= \int_s^\infty \bar{f}(s)\, ds$$

Q find Laplace transform of $\frac{1-e^{t}}{t}$

$$L\{1-e^{t}\} = \frac{1}{s} - \frac{1}{s-1}$$

$$L\left\{\frac{1-e^{t}}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$$

$$= \log s - \log(s-1)$$

$$= \log\left(\frac{s}{s-1}\right)\Big]_s^\infty$$

$$= \log\left(\frac{s}{s(1-\frac{1}{s})}\right)_s^\infty$$

$$= \cancel{\log(1)}$$

$$= 0 - \log\left(\frac{s}{s(1-\frac{1}{s})}\right)$$

$$= \log\left(1-\frac{1}{s}\right)$$

$$= \log \frac{s-1}{s}$$

Q $\frac{e^{-at} - e^{-bt}}{t}$

$$L\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$$

$$L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\} = \int_s^\infty \frac{1}{s+a} - \frac{1}{s+b}$$

$$= \log(s+a) - \log(s+b)$$

$$= \log \frac{s+a}{s+b}\Big]_s^\infty$$

$$= 0 - \log \frac{s+a}{s+b}$$

$$= \log \frac{s+b}{s+a}$$

Q $\frac{e^{-t}\sin t}{t} \approx e^{-t}\frac{\sin t}{t}$

$$L\{\sin t\} = \frac{1}{s^2+1^2}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \tan^{-1}(s)\Big]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \frac{\pi}{2} - \tan^{-1}(s+1)$$

Q $t\cos 2t$

$$L\{\cos 2t\} = \frac{s}{s^2+4}$$

$$L\{t\cos 2t\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= (-1)\times\frac{(s^2+4) - s(2s)}{(s^2+4)^2}$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

Q $t^3 e^{-3t}$

method 1

$$L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$$

$$L\{e^{-3t}t^3\} = \frac{6}{(s+3)^4}$$

method 2

$$L\{e^{-3t}\} = \frac{1}{s+3}$$

$$L\{t^3 e^{-3t}\} = (-1)^3 \frac{d^3}{ds^3}\frac{1}{s+3}$$

$$= -1 \times \frac{-6}{(s+3)^4} = \frac{6}{(s+3)^4}$$

Q $t e^{-t}\sin 3t$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{t\sin 3t\} = -1\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= -1 \times \frac{-3}{(s^2+9)^2} \times 2s$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\{e^{-t}t\sin 3t\} = \frac{6(s+1)}{((s+1)^2+9)^2}$$

Q $t^2 e^{3t}\sinh t$

$$t^2 e^{3t}\,\frac{e^{t}-e^{-t}}{2}$$

$$\frac{1}{2}t^2\left(e^{4t} - e^{2t}\right)$$

$$= \frac{1}{2}\left(e^{4t}t^2 - e^{2t}t^2\right)$$

$$L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

$$= \frac{1}{2}\left(\frac{2}{(s-4)^3} - \frac{2}{(s-2)^3}\right)$$