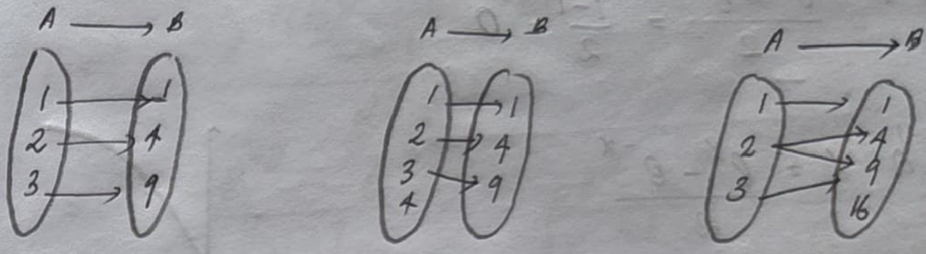
Module 1 *

Hyperbolic Functions



Fx^n Not Fx^n Not Fx^n

* Even fx^n : eg: $y = x^2 = (-x)^2$ odd Fx^n : eg: $y = x^3 \neq (-x)^3$

$y = \cos x$ $y = \sin x$.

- Any function can be written as the sum of an odd fx^n & even fx^n.

Fx^n = Even fx^n + odd fx^n

$$2f(x) = f(x) + f(-x) + f(x) - f(-x)$$

$$\Rightarrow f(x) = \underbrace{\frac{f(x)+f(-x)}{2}} + \underbrace{\frac{f(x)-f(-x)}{2}}$$

= even part + odd part

eg: $f(x) = e^x$ then $F(x) = \left[\frac{e^x + e^{-x}}{2}\right] + \left[\frac{e^x - e^{-x}}{2}\right]$

- Hyberbolic Cosine denoted as $\cosh x$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

- Hyberbolic sine denoted as $\sin h x$

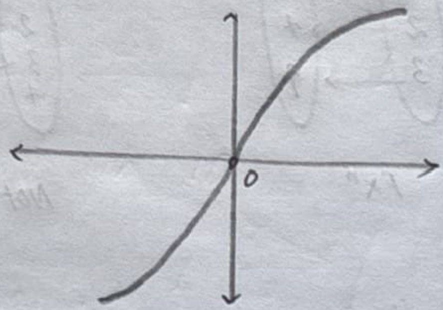
$$\sin h x = \frac{e^{x}-e^{-x}}{2}$$

⇒ Note : $\cos h\, 0 = \frac{e^{0}+e^{-0}}{2} = \frac{1+1}{2} = 1$

$\sin h\, 0 = \frac{e^{0}-e^{-0}}{2} = \frac{1-1}{2} = 0$

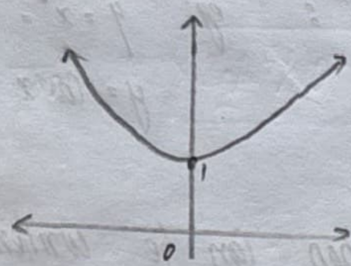
* For $y = \sin h\, x = \frac{e^{x}-e^{-x}}{2}$

Domain = Range = $\mathbb{R}$



* For $y = \cos h\, x = \frac{e^{x}+e^{-x}}{2}$

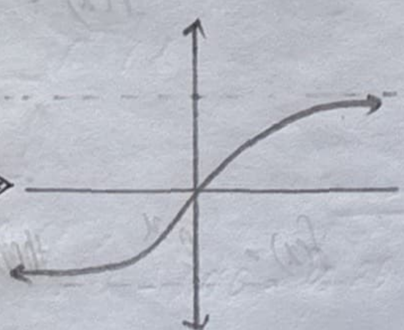
Domain = $\mathbb{R}$ range = $[1, \infty)$



⇒ Now we define : ① $\operatorname{Cosec} h x = \frac{1}{\sin h x} = \frac{2}{e^{x} \pm e^{-x}}$

② $\operatorname{Sec} h x = \frac{1}{\cos h x} = \frac{2}{e^{x}+e^{-x}}$

③ $\tan h x = \frac{\sin h x}{\cos h x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ ⇒



④ $\operatorname{Cot} h x = \frac{1}{\tan h x} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

- Euler's Formula : $e^{i\theta} = \cos\theta + i \sin\theta$

$e^{-i\theta} = \cos\theta - i \sin\theta$

ie, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

Result ① $\Rightarrow$ $\sin(i\theta) = i\sin h\theta$ Relation of circular & hyperbolic f^n

Let $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$ Here replace θ with $i\theta$

then: $\sin(i\theta) = \dfrac{e^{i(i\theta)}-e^{-i(i\theta)}}{2i}$

$$= \frac{e^{-\theta}-e^{\theta}}{2i} = -\left(\frac{e^{\theta}-e^{-\theta}}{2i}\right) = \frac{i^2}{i}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)$$

$$= \underline{\underline{i\sin hx}}$$

Result ② $\Rightarrow$ $\cos(i\theta) = \cos h\theta$

Let $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ Replace θ with $i\theta$

$\Rightarrow$ $\cos(i\theta) = \dfrac{e^{i^2\theta}+e^{-i^2\theta}}{2} = \dfrac{e^{-\theta}+e^{\theta}}{2} = \cos h\theta$

$\Rightarrow$

$$\boxed{\begin{array}{ll} \sin(i\theta) = i\sin h\theta & \cos(i\theta) = \cos h\theta \\ \text{ie } \tan(i\theta) = i\tan\theta h & \end{array}}$$

- Relation with $\sin(i\theta)$ & $\cos(i\theta)$

we have $\sin^2\theta + \cos^2\theta = 1$ Replace θ with $i\theta$

$\sin^2(i\theta) + \cos^2(i\theta) = 1$

$(i\sin h\theta)^2 + (\cos h\theta)^2 = 1$

ie. $\boxed{\cosh\theta^2 - \sinh\theta^2 = 1}$

Smly :- ① $1 + \tan^2\theta = \sec^2\theta$ Replace θ with $i\theta$

$\Rightarrow 1 + \tan^2(i\theta) = \sec^2(i\theta)$ $\Rightarrow 1 + (i\tanh\theta) = \sec^2 h\theta$

ie $\boxed{1 - \tan^2 h\theta = \sec^2 h\theta}$

② $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ $\Rightarrow$ Rep. θ with $i\theta$

$\Rightarrow 1 + \frac{1}{i^2}\cot^2(i\theta) = \frac{1}{i^2}\operatorname{cosec}^2(i\theta)$

$\Rightarrow 1 - \coth^2\theta = -\operatorname{cosec}^2 h\theta$ or $\boxed{\cot^2 h\theta - 1 = \operatorname{cosec} h^2\theta}$

③ $\sin(2\theta) = 2\sin\theta\cos\theta$ Rep. θ with $i\theta$

$\Rightarrow$ ~~$2f\theta$~~ $\sin(i 2\theta) = 2\sin(i\theta)\cos(i\theta)$

$\cancel{i}\sinh(2\theta) = 2\cancel{i}\sinh\theta\cosh\theta$

ie $\boxed{\sinh(2\theta) = 2\sinh\theta\cosh\theta}$

④ $\cos 2\theta = \begin{cases} \cos^2\theta - \sin^2\theta \\ 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{cases}$ $* \Rightarrow$ $\cos^2\theta - \sin^2\theta = \cos 2\theta$, R θ with $i\theta$

ie $\cos(2i\theta) = \cos^2(i\theta) - \sin^2(i\theta)$

$= \cos h^2\theta - i^2\sin^2 h\theta$

$= \cos h^2\theta + \sin h^2\theta$

* $\cos 2\theta = 2\cos^2\theta - 1$

$x \to \theta \to w \to i\theta$

ie $\cos(2i\theta) = 2\cos^2(i\theta) - 1$

$\boxed{\cos h^2\theta = 2\cos h^2\theta - 1}$

* $\cos 2\theta = 1 - 2\sin^2\theta$

$x \to \theta \to w \to i\theta$

$\cos(2i\theta) = 1 - 2\sin^2(i\theta)$

$\boxed{\cos 2h\theta = 1 + 2\sin h^2\theta}$

(5) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ ⇒ Rep θ with $i\theta$

$\sin(3i\theta) = 3\sin(i\theta) - 4\sin^3(i\theta)$

$i\sin h3\theta = 3i\sin h\theta - 4i^3\sin^3 h\theta$

$\Rightarrow \boxed{\sin h3\theta = 3\sin h\theta + 4\sin^3 h\theta}$

(6) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2\tan(i\theta)}{1-\tan^2(i\theta)}$ ($\theta \Rightarrow$

$i\tan h2\theta = \dfrac{2i\tan h\theta}{1-i^2\tan h^2\theta} = \dfrac{2\tan h\theta}{1+\tan h^2\theta}$

$\boxed{\tan h2\theta = \dfrac{2\tan h\theta}{1+\tan h^2\theta}}$

(7) $\sin(A+B) = \sin A\cos B + \cos A\sin B$ $[A = ix \quad B = iy]$

$\sin(ix+iy) = \sin(ix)\cos(iy) + \cos(ix)\sin(iy)$

$i\sinh(x+y) = i\sinh x\cos hy + \cos hx\, i\sin hy.$

$\boxed{\sinh(x+y) = \sin hx\cos hy + \cos hx\sin hy.}$

(8) $\cos(A-B) = \cos A \cos B + \sin A \sin B$ $\quad [A = ix \ \& \ B = iy]$

$\cos[ix - iy] = \cos(ix)\cos(iy) + \sin(ix)\sin(iy)$

$\Rightarrow \cos(x-y) = \cosh x \cosh y + i\sinh x \cdot i\sinh y$

$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y.$

(9) $\sin C + \sin D = 2\sin\left[\frac{C+D}{2}\right]\cos\left[\frac{C-D}{2}\right]$ $\quad [C = ix \ \& \ D = iy]$

ie $\sin(ix) + \sin(iy) = 2\sin\left[\frac{ix+iy}{2}\right]\cos\left[\frac{ix-iy}{2}\right]$

$i[\sinh x + \sinh y] = 2i^2 \sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

ie $\sinh x + \sinh y = 2\sinh\left(\frac{x+y}{2}\right)\cosh\left(\frac{x-y}{2}\right)$

Inverse trignometric functions

- If $x = \sinh\theta$, we define $\theta = \sinh^{-1} x$

* exam derivation

Result (1): Prove that $\sinh^{-1} x = \log[x + \sqrt{x^2+1}]$

Ans :- From definition : $\sinh y = x$ & $x = \frac{e^y - e^{-y}}{2}$

$\Rightarrow 2x = e^y - 1/e^y \Rightarrow 2x = \frac{(e^y)^2 - 1}{e^y}$

ie $(e^y)^2 - 2x e^y - 1 = 0$ (a quad. in e^y)

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = \frac{2[x \pm \sqrt{x^2+1}]}{2}$$

ie $e^y \to x + \sqrt{x^2+1}$

$\to x - \sqrt{x^2+1}$ (this term can not be since e^y defined only for +ve numbers).

therefore $\underline{y = \log(x + \sqrt{x^2+1})}$

Result ② : P.T $\cos^{-1}hx = \log|x + \sqrt{x^2-1}|$

let $y = \cos^{-1}hx \Rightarrow x = \cosh y = \frac{e^x + e^{-x}}{2}$

from this $2x e^x = (e^x)^2 + 1 \Rightarrow (e^x)^2 - 2x e^x + 1 = 0$

$$e^x = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2-1}$$

ie $e^x = x + \sqrt{x^2-1}$ or $e^x = x - \sqrt{x^2-1}$ × (when $x < 1$)

ie $e^x = |x + \sqrt{x^2-1}|$

ie $\underline{x = \log|x + \sqrt{x^2-1}|}$

⑥ HW : ① P.T $\sin h(2A) = \to \sin 2A = \frac{2\tan hA}{1+\tan^2 hA}$

② P T $\cos h(2A) = \to \frac{1 - \tan^2 hA}{1 + \tan^2 A}$

③ P T $\cos h(3A) = 4\cos^3 hA - 3\cos hA$

(v) P.T $\tanh(x-y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

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(2)

- P.T $\tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

Ans: $y = \tanh^{-1} x \Rightarrow x = \tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}} \Rightarrow x = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

ie $\dfrac{1}{x} = \dfrac{e^y + e^{-y}}{e^y - e^{-y}} \Rightarrow \dfrac{1+x}{1-x} = \dfrac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - (e^y - e^{-y})}$

$\left| \ \dfrac{a}{b} = \dfrac{c}{d} \Rightarrow \dfrac{a+b}{a-b} = \dfrac{c+d}{c-d} \right.$

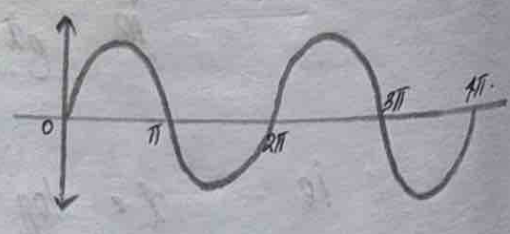
$$= \frac{e^y}{e^{-y}} = \underline{\underline{e^{2y}}}$$

therefor $2y = \log\left(\frac{1+x}{1-x}\right) \Rightarrow \underline{\underline{y = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)}}$.

Period of hyperbolic function

If $f(x) = f(T+x)$ then we say $f(x)$ has a period of T

eg: * $\sin x = \sin(2\pi + x) \Rightarrow$ Period of $\sin x = 2\pi$



* $\cos x = \cos(2\pi + x) \Rightarrow \cos$ has period 2π.

* $\tan x = \tan(\pi + x) \Rightarrow \tan$ has period π.

- $\sinh x = \frac{1}{i} \sin(ix)$ $\quad\quad\quad$ ★ $\sin(ix) = i \sinh x$

$$= -i \sin(ix) = -i \sin(-2\pi + ix)$$
$$= -i \sin(i^2 2\pi + ix)$$
$$= -i \sin(i[2\pi i + x]) = -i^2 \sinh(2\pi i + x)$$
$$= \sinh(2\pi i + x).$$

ie, $\sinh x = \sinh(2\pi i + x) \Rightarrow \boxed{\text{Period} = 2\pi i}$

<u>Period of $\cosh x$</u>

$$\cosh x = \cos(ix)$$
$$= \cos(-2\pi + ix) = \cos(i^2 2\pi + ix) = \cos(i(2\pi i + x))$$
$$= \cosh(2\pi i + x)$$

ie $\cosh x = \cosh(2\pi i + x) \rightarrow \boxed{\text{Period} = 2\pi i}$

<u>Period of $\tanh x$</u>

$\tan(ix) = i \tanh x \Rightarrow \tanh x = \frac{1}{i} \tan(ix) = -i \tan(ix)$

$$= -i \tan(-\pi + ix)$$
$$= -i \tan(i^2 \pi + ix)$$
$$= -i \tan(i(\pi i + x))$$
$$= -i^2 \tanh(\pi i + x) = \tanh(\pi i + x)$$

ie $\tanh x = \tanh(\pi i + x)$

$\Rightarrow \boxed{\text{Period} = \pi i}$

⊙ When $z = x + iy$ = Complex variable, $\sinh z$, the period will not change.

$\Rightarrow \sinh z = \sinh(x+iy) = \sin(2\pi i + (x+iy))$ $\quad \boxed{T = 2\pi i}$

$\Rightarrow$ Solved by the similar derivation.

Eg.:-

If $u = \log\tan(\pi/4 + \theta/2)$, then PT

(a) $\tanh u/2 = \tan\theta/2$

Ans:

$$U = \log\tan(\pi/4+\theta/2) \Rightarrow e^u = \tan(\pi/4+\theta/2)$$

$$= \frac{\tan\pi/4 + \tan\theta/2}{1 - \tan\pi/4\tan\theta/2} = \frac{1+\tan\theta/2}{1-\tan\theta/2}$$

ie
$$\tanh u/2 = \frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{e^{u/2} - 1/e^{u/2}}{e^{u/2} + 1/e^{u/2}} = \frac{e^u - 1}{e^u + 1}$$

$$= \frac{\frac{1+\tan\theta/2}{1-\tan\theta/2} - 1}{\frac{1+\tan\theta/2}{1-\tan\theta/2} + 1} =$$

$1 + \tan\theta/2 - 1 + \tan\theta/2$

$$= \frac{2\tan\theta/2}{2} = \tan\theta/2$$

$\Rightarrow \tanh u/2 = \tan\theta/2$ //

(b) $\cosh u = \sec\theta$

$$\Rightarrow \cosh u = \frac{e^u + e^{-u}}{2} \Rightarrow e^u = \frac{1+\tan\theta/2}{1-\tan\theta/2}$$

$$= \frac{\frac{1+\tan\theta/2}{1-\tan\theta/2} + \frac{1-\tan\theta/2}{1+\tan\theta/2}}{2} = \frac{1}{2}\frac{2[1+\tan^2\theta/2]}{(1-\tan^2\theta/2)}$$

$$= \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2} = \frac{1}{\left[\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right]} = \frac{1}{\cos 2[\theta/2]} = \frac{1}{\cos\theta} = \underline{\underline{\sec\theta}}$$

(c) $\sin hu = \tan\theta$

$$\sin hu = \frac{e^u - e^{-u}}{2} = \frac{\frac{1+\tan\theta/2}{1-\tan\theta/2} - \frac{1-\tan\theta/2}{1+\tan\theta/2}}{2} = \frac{4\tan\theta/2}{2[1-\tan^2\theta/2]}$$

$$= \frac{2\tan\theta/2}{1-\tan^2\theta/2}$$

$$= \tan 2\,\theta/2 = \underline{\underline{\tan\theta}}$$

(d) $\tan hu = \frac{\sin hu}{\cos hu} = \frac{\tan\theta}{\sec\theta} = \underline{\underline{\sin\theta}}$

Eg$_2$:- If $\tan\theta/2 = \tan h\, u/2$, P T $\sin h u = \tan\theta$

Ans :- $\tan\theta = \frac{2\tan\theta/2}{1-\tan^2\theta/2} = \frac{2\tan h\, u/2}{1-\tan^2 u/2} = \sin\theta$

$\Rightarrow$ Since $\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2} \Rightarrow \sin i\theta = \frac{2\tan(i\theta/2)}{1+\tan^2(\frac{i\theta}{2})}$

$$i\sinh\theta = \frac{2 i\tan h\,\theta/2}{1+i^2\tan^2\theta/2} = \underline{\frac{2\tan h\,\theta/2}{1-\tan^2\theta/2}}$$

Reverse of Eg.

Eg$_3$:- If $\tan\theta/2 = \tan h\, u/2$. P T $u = \log\tan[\pi/4 + \theta/2]$

Ans :- Given $\tan\theta/2 = \tan h\,\theta/2 = \frac{e^{+u/2} - e^{-u/2}}{e^{u/2}+e^{-u/2}} = \frac{e^u - 1}{e^u + 1}$

$$\tan\theta/2 = \frac{e^u-1}{e^u+1} \Rightarrow \frac{1}{\tan\theta/2} = \frac{e^u+1}{e^u-1}$$

$\Rightarrow$ Componendo dividendo

$$\frac{1+\tan\theta/2}{1-\tan\theta/2} = \frac{\cancel{2}e^u}{\cancel{2}} = e^u = \underline{\underline{\tan[\pi/4+\theta/2]}}$$

• $\dfrac{1+\tan\theta}{1-\tan\theta} = \tan\left[\pi/4 + \theta/2\right]$ $\quad \dfrac{1-\tan\theta}{1+\tan\theta} = \tan\left[\pi/4 - \theta/2\right]$

Eg$_4$:- If $\tan x/2 = \tan h(x/2)$, PT $\cos x \cdot \cosh x = 1$

Ans :- * $\cos x = \dfrac{1-\tan^2 x/2}{1+\tan^2 x/2} = \dfrac{1-\tan^2 h\, x/2}{1+\tan^2 h\, x/2} = \dfrac{1}{\left(\dfrac{1+\tan^2 hx}{1-\tan^2 hx}\right)}$ It is $\cosh x$ ↓ first prop

$\cos\theta = \dfrac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$

$\cos x = \dfrac{1}{\cosh x} \Rightarrow \cosh x \cdot \cos x = 1$

Eg$_5$:- If $\tan hx = \sin\theta$, PT I] $\sinh x = \tan\theta$
II] $\cosh x = \sec\theta$

Ans :- $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\tan h^2 x} = \sqrt{\sec h^2 x} = \sec hx$

II] $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{\sec hx} = \cosh x$

I] $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\tan hx}{\sec hx} = \sinh x$.

Qn :- Seperate in to real and imaginery part.

Ans :-> Qn: $\sin(\alpha + i\beta) = \sin\alpha \cos i\beta + \cos\alpha \sin i\beta$
$= \sin\alpha \cosh\beta + i\cos\alpha \sin h\beta$

② $\sin h(\alpha + i\beta) = \dfrac{1}{i}\sin(i(\alpha+i\beta)) = -i\sin[i\alpha - \beta]$
$= -i[\sin i\alpha \cos\beta - \cos i\alpha \sin\beta]$

$= -i \cdot i \sinh\alpha \cos\beta + i \cdot \cosh\alpha \sin\beta$

$= \sinh\alpha \cos\beta + [\cosh\alpha \sin\beta]\, i$

③ $\cosh[\alpha - i\beta] = \cos[i[\alpha - i\beta]]$ $\quad|\cosh x = \cos ix$

$= \cos[i\alpha + \beta]$

$= \cos i\alpha \cos\beta - \sin i\alpha \sin\beta = \cosh\alpha \cos\beta + i \sinh\alpha \sin\beta$

HW

1) Seperate to real and imaginary parts :-

(i) $\sin(\alpha - i\beta)$ $\quad$ (iii) If $\sin(A + iB) = x + iy$ PT

(ii) $\cos(\alpha + i\beta)$

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

Ans :- i) $\sin[\alpha - i\beta] = \sin\alpha \cos i\beta - \cos\alpha \sin i\beta = \sin\alpha \cosh\beta - i(\cos\alpha \sinh\beta)$

ii) $\cos(\alpha + i\beta) = \cos\alpha \cos i\beta - \sin\alpha \sin i\beta = \cos\alpha \cosh\beta - [\sin\alpha \sinh\beta]\, i$

iii) $\sin(A + iB) = \sin A \cos iB + \cos A \sin iB$

$= \sin A \cosh B + [\cos A \sinh B]\, i \Rightarrow$ $\quad x = \sin A \times \cosh B$

$y = \cos A \cdot \sinh B$.

$$\Rightarrow \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \frac{\sin^2 A \cosh^2 B}{\sin^2 A} - \frac{\cos^2 A \cdot \sinh^2 B}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1.$$

$$\Rightarrow \frac{\sin^2 B \cdot \cosh^2 B}{\cosh^2 B} + \frac{\cos^2 B \cdot \sinh^2 B}{\sinh^2 B} = \sin^2 B + \cos^2 B = 1$$

• Proving : ① $\sinh 2A = \frac{2\tanh A}{1+\tan^2 A}$

Ans : $\sin 2A = \frac{2\tan A}{1+\tan^2 A} \Rightarrow A \to iA \Rightarrow \sin 2iA = \frac{2\tan iA}{1+\tan^2(iA)}$

$i\sinh 2A = \frac{i\,2\tan Ah}{1+i^2\tan^2 Ah}$

$\Rightarrow \sinh 2A = \frac{2\tanh A}{1-\tan^2 hA}$

② $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A} \Rightarrow A \to iA \Rightarrow \cos 2iA = \frac{1-\tan^2 iA}{1+\tan^2 iA}$

$\Rightarrow \cos 2hA = \frac{1+\tan^2 hA}{1-\tan^2 hA}$

③ $\cos 3A = 4\cos^3 A - 3\cos A$

$A \to iA \Rightarrow \cos 3iA = 4\cos^3 iA - 3\cos iA$

$\cosh 3A = 4\cos^3 hA - 3\cosh A$

④ $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\Rightarrow A \to iA \quad B \to iB \Rightarrow \sin[iA - iB] = \sin iA\cos iB - \cos iA\sin iB$

$i\sinh(A-B) = i[\sinh A\cosh B - \cosh A\sinh B]$

$\sinh[A-B] = \sinh A\cosh B - \cosh A\sinh B$.

⑤ $\tan[A-B] = \frac{\tan A - \tan B}{1+\tan A\tan B} \Rightarrow \tan[i(A-B)] = \frac{\tan iA - \tan iB}{1+\tan iA\tan iB}$

$A \to iA \quad B \to iB \quad i\tanh(A-B) = \frac{i[\tanh A - \tanh B]}{1+i^2\tanh A\tanh B}$

$= \frac{\tanh A - \tanh B}{1-\tanh A\tanh B}$

Qn :- Seperate $\tan^{-1}(x+iy)$ in to real & imaginery parts :

Ans: Let $\tan^{-1}(x+iy) = A+iB \Rightarrow x+iy = \tan[A+iB]$

$x-iy = \tan[A-iB]$ - simily.

$$\Rightarrow \tan 2A = \tan[(A+iB)+(A-iB)] = \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$= \frac{2x}{1-[x^2-i^2y^2]} = \frac{2x}{1-x^2-y^2}$$

ie $\tan 2A = \frac{2x}{1-x^2-y^2} \Rightarrow A = \frac{1}{2}\tan^{-1}\left[\frac{2x}{1-[x^2+y^2]}\right]$ = Real part

$$\Rightarrow \tan(i2B) = \tan[(A+iB)-(A-iB)] = \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$= \frac{x+iy-(x-iy)}{1-(x+iy)(x-iy)} = \frac{2iy}{1-[x^2-i^2y^2]} = \frac{2iy}{1-x^2-y^2}$$

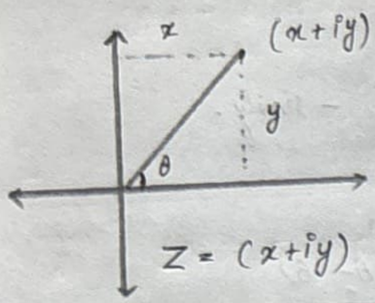
ie $\tan(2Bi) = \frac{i2y}{1-x^2-y^2} \Rightarrow i\tanh(2B) = \frac{i2y}{1-x^2+y^2}$

$$B = \frac{1}{2}\tanh^{-1}\left[\frac{2y}{1-x^2-y^2}\right]$$

= imaginery part //.

Qn : Seperate $\log[x+iy]$ in to real and imaginery part.

07 **Modulos- Amplitude formulae :-** $\boxed{x + iy = re^{i\theta}}$



r = modulous = $|z| = \sqrt{x^2+y^2}$

θ = amplitude = $\tan^{-1}(y/x)$ = Argument.

ie, $\log(x+iy) = \log(re^{i\theta}) = \log r + \log e^{i\theta} = \log r + i\theta = \underline{\underline{\log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)}}$

Qn :- if $\tan[A+iB] = x+iy$, P.T

(i) $x^2+y^2+2x\cot 2A = 1$ (ii) $x/y = \dfrac{\sin 2A}{\sinh 2B}$

Ans :-

$$\boxed{\begin{aligned} 2\sin A\cos B &= \sin(A+B) + \sin(A-B) \\ 2\cos A\sin B &= \sin(A+B) - \sin(A-B) \\ 2\sin A\sin B &= \cos(A-B) - \cos(A+B) \\ 2\cos A\cos B &= \cos(A+B) + \cos(A-B) \end{aligned} \quad \theta_A > \theta_B}$$

$$x+iy = \tan(A+iB) = \frac{\sin(A+iB)}{\cos(A+iB)} = \frac{2\sin(A+iB)\cos(A-iB)}{2\cos(A+iB)\cos[A-iB]}$$

$$= \frac{\sin[(A+iB)+(A-iB)] + \sin[(A+iB)-(A-iB)]}{\cos[(A+iB)+(\cos(A-iB)] + \cos[(A+iB)-(A-iB)]}$$

$$= \frac{\sin 2A + \sin(2iB)}{\cos 2A + \cos(2iB)}$$

$$= \frac{\sin 2A + i\sinh 2B}{\cos 2A + \cosh 2B}$$

$\Rightarrow$ Real part $= \dfrac{\sin 2A}{\cos 2A + \cosh 2B} = x$ Imagin. part $= \dfrac{\sinh 2B}{\cos 2A + \cosh 2B} = y$.

(i) $x^2 + y^2 + 2x \cot 2A = \left[\dfrac{\sin 2A}{\cos 2A + \cosh 2B}\right]^2 + \left[\dfrac{\sinh 2B}{\cos 2A + \cosh 2B}\right]^2 +$

$2 \cdot \dfrac{\cancel{\sin 2A}}{\cos 2A + \cosh 2B} \cdot \dfrac{\cos 2A}{\cancel{\sin 2A}}$

$$= \frac{\sin^2 2A + \sinh^2 2B + 2\cos 2A \cdot (\cos 2A + \cosh 2B)}{(\cos 2A + \cosh 2B)^2}$$

$$= \frac{(1-\cos^2 2A) + (\cosh^2 2B - 1) + 2\cos^2 2A + 2\cos 2A \cosh 2B}{(\cos 2A + \cosh 2B)^2}$$

$(a+b)^2$ format.

$$= \frac{\cos^2 2A + \cosh^2 2B + 2\cos 2A \cdot \cosh 2B}{(\cos 2A + \cosh 2B)^2} = 1$$

Qn 8- If $\cos(x+iy) = \cos\theta + i\sin\theta$. PT

(i) $\sin^2 x = \pm \sin\theta$

(ii) $\cos 2x + \cosh 2y = 2$

Ans :- Let $\cos\theta + i\sin\theta = \cos(x+iy)$

$= \cos x \cos(iy) - \sin x \sin(iy)$

$= \underbrace{\cos x \cosh y}_{\cos\theta} - i\,\underbrace{[\sin x \sinh y]}_{\sin\theta}$

ie $\cosh y = \dfrac{\cos\theta}{\cos x}$ $\sinh y = \dfrac{-\sin\theta}{\sin x}$, we have $\cosh^2 y - \sinh^2 y = 1$

ie $\dfrac{\cos^2\theta}{\cos^2 x} - \dfrac{\sin^2\theta}{\sin^2 x} = 1 \Rightarrow \cos^2\theta \sin^2 x - \sin^2\theta \cos^2 x = \sin^2 x \cos^2 x$

$(1-\sin^2\theta)\sin^2x - \sin^2\theta[1-\sin^2x] = (1-\sin^2x)\cdot\sin^2x$

$\sin^2x - \cancel{\sin^2\theta\sin^2x} - \sin^2\theta + \cancel{\sin^2x\sin^2\theta} = \sin^2x - \sin^4x$

$\cancel{\sin^2x} - \sin^2\theta = \cancel{\sin^2x} - \sin^4x \quad \Rightarrow \quad \underline{\underline{\sin\theta = \pm\sin^2x}}$

(ii) $\sin^2\theta + \cos^2\theta = 1 \quad \Rightarrow \quad 1 = \sin^2x\sinh^2\theta + \cos^2x\cosh^2y$

$= (1-\cos^2x)(\cosh^2y-1) + \cos^2x\cosh^2y$

$= \cosh^2y - 1 - \cos^2x\,\cancel{\cosh^2y} - \cos^2x + \cos^2x\,\cancel{\cosh^2y}$

$\Rightarrow \quad 2 = \cosh^2y - \cos^2x$

$= \left[\frac{1+\cosh^2 2y}{2}\right] + \left[\frac{1+\cos 2x}{2}\right]$

$\Rightarrow \quad \underline{\underline{2 = \cosh 2y + \cos 2x}}$

$\cos 2A = 2\cos^2A - 1$

$\cos^2A = \frac{1+\cos 2A}{2}$

$\cosh^2A = \frac{1+\cosh^2 2A}{2}$

Qn 8:- If $x+iy = \cosh[u+iv]$ PT (i) $\frac{x^2}{\cosh^2u} + \frac{y^2}{\sinh^2u} = 1$

(ii) $\frac{x^2}{\cos^2v} - \frac{y^2}{\sin^2v} = 1$

Ans: $x+iy = \cosh(u+iv)$

$= \cos[i[u+iv]] = \cos(iu-v) = \cos iu\cos v + \sin iu\sin v$

$= \cosh u\cos v + i[\sinh u\sin v]$

$\Rightarrow x = \cosh u\cos v \quad \Rightarrow y = \sinh u\sin v$

Hence (i) $\frac{\cancel{\cosh^2u}\cos^2v}{\cancel{\cosh^2u}} + \frac{\cancel{\sinh^2u}\sin^2v}{\cancel{\sinh^2u}} = \cos^2v + \sin^2v = 1$

(ii) $\frac{\cosh^2u\,\cancel{\cos^2v}}{\cancel{\cos^2v}} - \frac{\sinh^2u\,\cancel{\sin^2v}}{\cancel{\sin^2v}} = \cosh^2u - \sinh^2u = 1$ //

DERIVATIVES OF HYPERBOLIC FUNCTIONS

ⓐ $y = \sinh x = \dfrac{e^x - e^{-x}}{2} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}\left[e^x + e^{-x}\right] = \cosh x$

$$\boxed{\frac{d}{dx}\sinh x = \cosh x}$$

ⓑ $\cosh x = \dfrac{e^x + e^{-x}}{2} = y \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}\left[e^x - e^{-x}\right] = \sinh x$

$$\boxed{\frac{d}{dx}\cosh x = \sinh x}$$

ⓒ $y = \tanh x = \dfrac{\sinh x}{\cosh x} \Rightarrow \dfrac{dy}{dx} = \dfrac{\cosh x \cosh - \sinh x \cdot \sinh x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

$$\boxed{\frac{d}{dx}\tanh x = \operatorname{sech}^2 x}$$

$$\boxed{\frac{d}{dx}\left(a/b\right) = \frac{-ab' + ba'}{b^2}}$$

ⓓ $y = \operatorname{cosech} x = 1/\sinh x \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{\sinh^2 x} \cdot \left(\dfrac{d}{dx}\sinh x\right) = \dfrac{-1}{\sinh x} \cdot \dfrac{\cosh x}{\sinh x} = -\operatorname{cosech} x \coth x$

$$\boxed{\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x}$$

ⓔ $y = \operatorname{sech} x = 1/\cosh x \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{\cosh^2 x} \cdot \dfrac{d}{dx}(\cosh x) = \dfrac{-1}{\cosh x} \cdot \dfrac{\sinh x}{\cosh x} = -\operatorname{sech} x \cdot \tanh x$

$$\boxed{d/dx \operatorname{sech} x = -\operatorname{sech} x \cdot \tanh x.}$$

ⓕ $y = \coth x = 1/\tanh x \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{\tanh^2 x} \cdot \left[\dfrac{d}{dx}\tanh x\right] = \dfrac{-1}{\tanh^2 x} \cdot \operatorname{sech}^2 x$

$= -\operatorname{cosech}^2 x$

* $\boxed{\frac{d}{dx} \coth x = -\operatorname{cosech}^2 x}$

Integrals of hyperbolic functions

ⓐ $\int \sinh x \, dx = \cosh x + c$

ⓑ $\int \cosh x \, dx = \sinh x + c$

ⓒ $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

ⓓ $\int \operatorname{cosech} x \cdot \coth x \, dx = -\operatorname{cosech} x + c$

ⓔ $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$

ⓕ $\int \operatorname{cosech}^2 x \, dx = -\coth x + c$

DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

Basics : $\frac{d}{dx} \sin^{-1} x \Rightarrow$ $y = \sin^{-1} x$

$x = \sin y \Rightarrow \frac{dx}{dy} = \cos y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$

similarly :-

① $y = \sin^{-1} hx \Rightarrow x = \sinh^{-1} y \quad x' = \cosh y \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}}$

$= \frac{1}{\sqrt{1+x^2}}$

② $y = \cosh^{-1} x \Rightarrow x = \cosh y \quad \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\sinh x} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}$

③ $y = \tanh x \Rightarrow x = \tanh y \quad dy/dx = 1/\frac{dx}{dy} = \frac{1}{\operatorname{sech}^2 x} = \frac{1}{1-\tanh^2 y}$

$= \frac{1}{1-x^2}$ //

Module II

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x^{th} derivatives - Standard Forms

① $y = e^{ax} \Rightarrow y_1 = \frac{dy}{dx} = ae^{ax}$

$y_2 = \frac{dy_1}{dx} = a^2 e^{ax} \Rightarrow \boxed{y_n = a^n e^{ax}}$

② $y = (ax+b)^m \Rightarrow y_1 = \frac{dy}{dx} = am\,(ax+b)^{m-1}$

$y_2 = \frac{dy_1}{dx} = am(m-1)\,(ax+b)^{m-2} \cdot a = a^2 m(m-1)(ax+b)^{m-2}$

$$\boxed{y_n = a^n m(m-1)(m-2)\cdots(m-(n-1))(ax+b)^{m-n}}$$

Note :- when $y = (ax+b)^n$

$y_1 = an\,(ax+b)^{n-1}$

$y_2 = a^2 n(n-1)\,(ax+b)^{n-2} \Rightarrow y_n = a^n n(n-1)(n-2)\cdots(n-(n-1))(ax+b)^{n-n}$

$= a^n n(n-1)(n-2)\cdots \times 1$

$= a^n n!$

③ $y = \frac{1}{ax+b} = (ax+b)^{-1}$

$y_1 = (-1)\,(ax+b)^{-1-1} \cdot a$

$= a(-1)\,(ax+b)^{-2}$

$y_2 = a(-1)(-2)\,(ax+b)^{-3} \cdot a \Rightarrow y_n = a^n (-1)(-2)(-3)\cdots(-n)\,(ax+b)^{-[n+1]}$

$= a^2(-1)(-2)\,(ax+b)^{-3}$

III^{ly} :- $y_3 = a^3(-1)(-2)(-3)\,(ax+b)^{-4}$

this can be modified to $\Rightarrow$

$$y_n = a^n(-1)^n(1\cdot 2\cdot 3 \cdots\cdots n)\ (ax+b)^{-(n+1)} \Rightarrow \boxed{y_n = \frac{a^n(-1)^n n!}{(ax+b)^{+(n+1)}}}$$

④ $y = \log[ax+b] \Rightarrow$

$$y_1 = \frac{a}{ax+b} = a(ax+b)^{-1}$$

$$y_2 = a^2(-1)(ax+b)^{-2} \Rightarrow \boxed{y_n = \frac{a^n(-1)^{n-1}(n-1)!}{(ax+b)^n}}$$

|||ly $y_3 = a^3(-1)(-2)\ (ax+b)^{-3}$

⑤ $y = \sin(ax+b)$

$$y_1 = a\cos(ax+b)$$
$$= a\sin[\pi/2 + (ax+b)]$$

$$y_2 = a\cos[\pi/2 + (ax+b)]\cdot a$$
$$= a^2\sin\left[\frac{2\pi}{2} + (ax+b)\right]$$

$$y_3 = a^3\sin\left[\frac{3\pi}{2} + (ax+b)\right]$$

ie,

$$\boxed{y_n = a^n\sin[n\pi/2 + (ax+b)]}$$

⑥ $\cos[ax+b] = y$

$$y_1 = -\sin(ax+b)\cdot a$$
$$= a\cos[\pi/2 + (ax+b)]$$

$$y_2 = a\ {-\sin}[\pi/2 + (ax+b)]\cdot a$$
$$= a^2\cos\left[\frac{2\pi}{2} + (ax+b)\right]$$

$$y_3 = a^3\cos[3\pi/2 + (ax+b)]$$

$$\boxed{y_n = a^n\cos\left[\frac{n\pi}{2} + (ax+b)\right]}$$

⑦ $y = e^{ax}\sin[bx+c]$

$y_1 = ae^{ax}\sin[bx+c] + e^{ax}\cos[bx+c]\cdot b$

$= e^{ax}[a\sin(bx+c) + b\cos[bx+c]] \longrightarrow ①$

Let $a = r\cos\theta$, $b = r\sin\theta$ $\}$ $a^2+b^2 = r^2$

~~y_2~~ $\tan\theta = b/a \Rightarrow \theta = \tan^{-1}[b/a]$

$y_1 = e^{ax}[r\cos\theta\sin[bx+c] + r\sin\theta\cos[bx+c]]$

$= re^{ax}\sin[\theta + (bx+c)]$

$y_2 = r^2e^{ax}[\sin[2\theta + bx+c]] \Rightarrow y_n = r^n e^{ax}[n\theta + (bx+c)]$

ie $\boxed{y_n = (a^2+b^2)^{n/2} e^{ax}\sin[n\tan^{-1}[b/a] + (bx+c)]}$

⑧ $y = e^{ax}\cos[bx+c]$

$y_1 = ae^{ax}\cos[bx+c] + -\sin[bx+c]b\cdot e^{ax}$

$= e^{ax}[a\cos[bx+c] - b\sin[bx+c]] \longrightarrow$ $a = r\cos\theta$, $b = r\sin\theta$ $\}$ $a^2+b^2 = r^2$, $\tan\theta = b/a$

$= e^{ax}[r\cos\theta\cos[bx+c] - r\sin\theta\sin[bx+c]]$

$= re^{ax}[\cos\theta\cos[bx+c] - \sin\theta\sin[bx+c]]$

$= re^{ax}\cos[\theta + (bx+c)]$

Simly, $y_2 = r^2e^{ax}\cos[2\theta + (bx+c)] \Rightarrow y_n = r^n e^{ax}\cos[n\theta + (bx+c)]$

$\boxed{y_n = (a^2+b^2)^{n/2} e^{ax}[n\tan^{-1}(b/a) + (bx+c)]}$

No	Rational expression	Partial fraction form
1	$\frac{1}{(ax+b)(cx+d)}$	$\frac{A}{(ax+b)} + \frac{B}{cx+d}$
2	$\frac{1}{(ax+b)^2(cx+d)}$	$\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \frac{C}{(cx+d)}$
3	$\frac{1}{(ax^2+bx+c)(dx+e)}$	$\frac{Ax+b}{(ax^2+bx+c)} + \frac{C}{(cx+d\cdot)}$

Here, this can be only used when degree of numer less than that of denom.

eg: Find the n^{th} derivatives :

1. $\frac{1}{x^2-ax+b}$

Ans: $y = \frac{1}{x^2-ax+b}$

$= \frac{1}{(x-a)} * \frac{1}{(x-b)}$

$= \frac{A}{(x-a)} + \frac{B}{(x-b)}$

ie $\frac{1}{x^2-ax+b} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$

6 eg: $1 = A(x-3)+B(x-2) \Rightarrow$ $x=2 \Rightarrow A=-1$; $x=3 \Rightarrow B=1$

ie $1 = -1[x-3] + 1[x-2]$

ie $\frac{1}{(x-2)(x-3)} = \frac{-1}{(x-2)} + \frac{1}{(x-3)} = \frac{(-1)\,1^n\,(-1)^n n!}{(x-2)^{n+1}} + \frac{1^n(-1)^n n!}{(x-3)^{n+1}}$

$= n^{th}$ derivative.

(2) $y = \frac{1}{4x^2 - 1}$

$= \frac{1}{(2x+1)(2x-1)}$

$= \frac{A}{(2x+1)} + \frac{B}{(2x-1)}$

$A \Rightarrow$ Put $x = -1/2$

$\Rightarrow A = -1/2$

$B \Rightarrow B = 1/2$

$y = \frac{-1/2}{(2x+1)} + \frac{1/2}{(2x-1)}$

$y_n = \frac{(-1/2)\, 2^n (-1)^n n!}{(2x+1)^{(n+1)}} + (1/2)\frac{2^n (-1)^n n!}{(2x-1)^{(n+1)}}$

(3) $y = \frac{x^2+1}{x^2-6x+8}$

$$\begin{array}{r|l} & 1 \\ x^2-6x+8 & x^2+0x+1 \\ & x^2-6x+8 \\ \hline & 6x-7 \end{array}$$

$= \frac{x^2+1}{(x-2)(x-4)} = 1 + \frac{6x-7}{(x-2)(x-4)}$

$= 1 + \frac{-5/2}{(x-2)} + \frac{17/2}{(x-4)}$

$y_n = 1 + (-5/2)\frac{1^n (-1)^n n!}{(x-2)^{(n+1)}} + \frac{17}{2}\frac{1^n (-1)^n n!}{(x-4)^{(n+1)}}$

(4) $y = \log[x^2 - 7x + 10]$

$= \log([x-2][x-5])$

$= \log(x-2) + \log(x-5) \Rightarrow y_n = \frac{1^n (-1)^{(n-1)} (n-1)!}{(x-2)^n} + \frac{1^n (-1)^{n-1} (n-1)!}{(x-5)^n}$

(5) $\log\left(\frac{x-3}{x+5}\right) = y = \log(x-3) - \log(x+5)$

$y_n = \frac{1^n (-1)^{(n-1)} (n-1)!}{(x-3)^n} - \frac{1^n (-1)^{(n-1)} (n-1)!}{(x+5)^n}$

⑥ $y = \sin 6x \cos 4x$

$= \frac{1}{2}[2 \sin 6x \cos 4x] = [\sin 10x + \sin 2x] \cdot \frac{1}{2}$

ie, $y_n = \frac{1}{2}\left[10^n \sin\left[\frac{n\pi}{2} + 10x\right] + 2^n \sin\left[\frac{n\pi}{2} + 2x\right]\right]$

⑦ $y = \cos^2 2x \sin 3x$

$= \frac{1}{2}[1 + \cos(4x)] \sin 3x = \frac{1}{2} \sin 3x + \frac{1}{2} \sin(3x) \cos 4x$

$= \frac{1}{2} \sin 3x + \frac{1}{4}[\sin 7x - \sin x]$

$\Rightarrow y_n = \frac{1}{2} 3^n \sin\left[\frac{n\pi}{2} + 3x\right] + \frac{1}{4} 7^n \sin\left[\frac{n\pi}{2} + 7x\right] - \frac{1}{4} \sin\left(\frac{n\pi}{2} + x\right)$

⑧ $y = \cos 4x \cdot \sin 3x \cdot \cos x$

$= \frac{\cos 4x}{2}[2 \sin 3x \cos x] = \frac{1}{2} \cos 4x [\sin 4x + \sin 2x] = \frac{1}{4}(2 \sin 4x \cos 4x) + \frac{1}{4}(2 \sin 2x \cos 4x)$

$= \frac{1}{4} \sin 8x + \frac{1}{4}(\sin 6x - \sin 2x)$

$= \frac{1}{4}\left[8^n \sin\left[\frac{n\pi}{2} + 8x\right] + 6^n \sin\left(\frac{n\pi}{2} + 6x\right) - 2^n \sin\left(\frac{n\pi}{2} + 2x\right)\right]$

HW :- Find nth derivative of $\frac{x^4}{x^2 + 5x + 6}$, $\frac{2x^2 + 3x + 1}{x^2 - 3x + 2}$, $\log(x^2 - 1)$,

① $\frac{x}{x^2 + 5x + 6}$ $\sin^2 2x \cos 3x$, $\sin 5x \cos 3x$ $\sin 2x$

② $y = \dfrac{2x^2+3x+1}{x^2-3x+2}$

$$x^2-3x+2 \overline{\big) \begin{array}{r} 2 \\ 2x^2+3x+1 \\ 2x^2-6x+4 \\ \hline 9x-3 \end{array}}$$

$= 2 + \dfrac{3[3x-1]}{x^2-3x+2}$

ie $\dfrac{3(3x-1)}{x^2-3x+2} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-1)} = \dfrac{A(x-1)+B(x-2)}{(x-2)(x-1)}$

$\Rightarrow \quad 3(3x-1) = A(x-1) + B(x-2) \longrightarrow$ ① $x=1 \Rightarrow B=-6$

② $x=2 \Rightarrow A=15$

ie $y = 2 + \dfrac{15}{(x-2)} - \dfrac{6}{(x-1)}$ $\qquad y_n = 2 + (15)\dfrac{1^n(-1)^n n!}{(x-2)^{(n+1)}} + (-6)\dfrac{(-1)^n n!}{(x-1)^{(n+1)}}$

③ $y = \log(x^2-1) = \log((x-1)(x+1)) = \log(x-1) + \log(x+1)$

$y_n = \dfrac{1^n(-1)^{(n-1)}(n-1)!}{(x-1)^n} + \dfrac{1^n(-1)^{(n-1)}(n-1)!}{(x+1)^n}$

④ $y = \sin^2 2x \cos 3x = \frac{1}{2}[1+\cos 4x]\cos 3x$

$= \frac{1}{2}\cos 3x + \frac{1}{4}[2\cos 4x \cos 3x]$

$\Rightarrow \quad y_n = \frac{1}{2}3^n\cos[n\pi/2 + 3x] + \frac{1}{4}[7^n\cos[n\pi/2 + 7x] + \cos[\frac{n\pi}{2}+x]]$

⑤ $y = \sin 5x \cos 3x \sin 2x$

$= \frac{1}{2}\sin 5x[2\cos 3x \sin 2x]$

$= \frac{1}{2}\sin 5x[\sin 5x - \sin x]$

$= \frac{1}{2}\sin^2 5x - \frac{1}{2}\sin 5x \sin x = \frac{1}{2}\left[\frac{1-\cos 10x}{2}\right] - \frac{1}{2}\times\frac{1}{2}[\cos 4x - \cos 6x]$

$= \frac{1}{4}[1 - \cos 10x - \cos 4x + \cos 6x]$

$y_n = \frac{1}{4}[0 - 10^n$

Leibinitz's Rule

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If $y = U \cdot V$, then $y_n = U_n V + {}^nC_1 U_{n-1} V_1 + {}^nC_2 U_{n-2} V_2 + \cdots\cdots + UV_n$

$\boxed{{}^nP_r = \frac{n!}{(n-r)!} \quad {}^nC_r = \frac{n!}{(n-r)! \cdot r!}}$

eg: Find the n^{th} derivatives of the following:-

① $y = x^2 e^{3x} = e^{3x} \cdot x^2 = U \cdot V$

$y_n = U_n V + U_{n-1} V_1 + U_{n-2} V_2 + \cdots\cdots + UV_n$

$= 3^n e^{3x} x^2 + n \cdot 3^{n-1} e^{3x} \cdot 2x + \frac{n(n-1)}{2} 3^{n-2} e^{3x} \cdot 2 + 0$

$= x^2 e^{3x} \cdot 3^n + 2n\, e^{3x} x\, 3^{n-1} + n(n-1)\, e^{3x}\, 3^{n-2}$

② $y = x^2 \log x = \log x \cdot x^2 = U \cdot V$

$y_n = \frac{(-1)^{n-1}(n-1)!}{x^n} x^2 + (n) \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} 2x + \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} 2 \times \frac{n(n-1)}{2} + 0$

③ If $y = a \cos(\log x) + b \sin(\log x)$, PT

Ans: $y_1 = -a \sin(\log x) \cdot \frac{1}{x} + b \cos(\log x) \cdot \frac{1}{x}$

$xy_1 = -a \sin(\log x) + b \cos(\log x)$

$xy_2 + y_1 = -a \cos[\log x] \cdot \frac{1}{x} + -b \sin(\log x) \cdot \frac{1}{x}$

$x^2y_2 + xy_1 = -[[a\cos[\log x]] + b\sin(\log x)]$

$= -y$

$\Rightarrow \quad x^2y_2 + xy_1 + y = 0$

Taking n^{th} derivative :- $\quad Y_n = \left[y_{2+n}\, x^2 + n \cdot y_{2+n-1}\, 2x + {}^nc_2\, y_{2+n-2}\, 2\right] +$

$\left[y_{1+n}\, x + n \cdot y_{n-1+1} \times 1\right] + y_n = 0$

$\Rightarrow 0 = x^2 y_{2+n} + 2nx\, y_1 + \frac{n(n-1)}{2} \times 2\, y_n + x y_{n+1} + n y_n + y_n$

$= x^2 y_{n+2} + 2nx\, y_1 + \cancel{2}/\cancel{2}\, (n^2 - n)\, y_n + x y_{n+1} + n y_n + y_n$

$\Rightarrow \quad 0 = x^2 y_{en+1} + y_{(n+1)}\,(2n+1) + (n^2+1) y_n = 0$

(5) $\quad y = [\sin^{-1} x]^2$

$y_1 = 2\sin^{-1} x \times \frac{1}{\sqrt{(1-x^2)}} \quad \Rightarrow \quad (1-x^2)^{1/2} y_1 = 2\sin^{-1} x$

$(1-x^2)\, y_1^2 = 4[\sin^{-1}(x)]^2 = 4y$

ie

$(1-x^2)\, y_1^2 = 4y$

diff: $\quad (1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = 4y_2$

$\div 2y_2 \quad (1-x^2)\, y_1 + -xy_1 = 2 \quad \Rightarrow \quad (1-x^2)\, y_1 - xy_1 = 2$

taking n^{th} derivative:- $\left[y_{1+n}(1-x^2) + y_{1+n-1}(-2x) \cdot n + y_{1+n-2}(-2)\frac{n(n-1)}{2}\right] -$

$\left[y_{1+n}\, x + n\, y_{1+n-1}\right] + 0 = 0$

$0 = y_{n+1} - x^2 y_{n+1} + y_n(-2nx) - y_{n-1}\,(n)(n-1) + x y_{n+1} + n y_n$

$\Rightarrow \quad 0 = (1+x)y_{n+1} \times 1 + ny_n[1-2x] - n(n-1)y_{n-1} + x^2 y_{n+1}$

$0 = (1+x+x^2)y_{n+1} + ny_n(1-2x) - n(n-1)y_{n-1}$

(6) $y = e^{a\sin^{-1}x} \Rightarrow y_1 = e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$

$\Rightarrow ay = \sqrt{1-x^2}\,y_1 \Rightarrow (1-x^2)y_1^2 = a^2y^2$

diff $\Rightarrow \quad a^2 2yy_1 = (1-x^2)2y_1y_2 + y_1^2(-2x)$

$a^2y = y_2(1-x^2) + -xy_1 \Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0$

taking n^{th} deriv: $\left[y_{2+n}(1-x^2) + ny_{2+n-1}(-2x) + \frac{n(n-1)}{\cancel{2}} \times y_{2+n-2}(-\cancel{2})\right] +$

$-\left[y_{1+n}x + ny_{1+n-1}\right] - a^2y_n = 0$

$0 = (1-x^2)y_{2+n} - \cancel{2}nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n - a^2y_n$

$= (1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + \cancel{ny_n} - xy_{n+1} - \cancel{ny_n} - a^2y_n$

$= (1-x^2)y_{n+2} - y_{n+1}x[2n+1] - y_n[n^2+a^2]$

(7) If $y = \sin[m\sin^{-1}x]$ P.T. $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Ans :- $y_1 = \frac{\cos[m \sin^{-1}x]\, m}{\sqrt{1-x^2}}$ $\Rightarrow$ $(1-x^2) y_1^2 = m^2 \cos^2[m \sin^{-1}x]$

ie $(1-x^2) y_1^2 = m^2 [1 - \sin^2[m \sin^{-1}x]] = m^2[1-y^2]$

difference :- $(1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = m^2(-2yy_1)$

$\div 2y_1$ $(1-x^2) y_2 + -y_1 x = -m^2 y$ $\Rightarrow$ $(1-x^2)y_2 - xy_1 + m^2 y = 0$

apply Liebnitz :- $\left[(1-x^2) y_{2+n} + y_{n+1}(-2xn) + y_n\left(-\cancel{2}\frac{(n^2-n)}{\cancel{2}}\right)\right] - \left[xy_{n+1} + ny\right] + m^2 y_n = 0$

$= (1-x^2) y_{n+2} - 2nx\, y_{n+1} - n^2 y_n + \cancel{ny_n} - xy_{n+1} - \cancel{ny_n} + m^2 y_n = 0$

$\Rightarrow$ $(1-x^2) y_{n+2} - (2n+1)\, xy_{n+1} + (m^2-n^2) y_n = 0$

(8) If $y^{1/m} + y^{-1/m} = 2x$ PT $0 = (m^2+1) y_{n+2} + (2n+1) xy_{n+1} + (n^2-m^2) y_n$

Ans :- $y^{1/m} + 1/y^{1/m} = 2x$ $\Rightarrow$ $y^{2/m} + 1 = y^{1/m}\, 2x$ $\Rightarrow$ $(y^{1/m})^2 - 2x(y^{1/m}) + 1 = 0$

Since it is quad :- $y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2} = \underline{\underline{x \pm \sqrt{x^2-1}}}$

$\Rightarrow$ $y = [x \pm \sqrt{x^2+1}]^m$

Case (1) $\Rightarrow$ $y = [x + \sqrt{x^2-1}]^m$

$y_1 = m[x + \sqrt{x^2-1}]^{m-1}\left[1 + \frac{1}{2\sqrt{x^2-1}} \times 2x\right] = m(x+\sqrt{x^2-1})^{m-1}\left[1 + \frac{x}{\sqrt{x^2-1}}\right]$

$= \frac{m(x+\sqrt{x^2-1})^m}{\sqrt{x^2-1}} = my/\sqrt{x^2-1}$

111^{ly} for Case ② $\quad y_1 = \dfrac{-m(x-\sqrt{x^2-1})^m}{\sqrt{x^2-1}} = -my/\sqrt{x^2-1}$

Sq. both cases :- $\quad (x^2-1)y_1^2 = m^2y^2 \quad \& \quad (x^2-1)y_1^2 = m^2y^2$ (both same)

diff : $\quad (x^2-1)2y_1y_2 + y_1^2 2x = m^2 2yy_1$

$$(x^2-1)y_2 + xy_1 - m^2y = 0 \quad \text{[Relation involving } 2^{nd} \text{ derivative]}$$

taking n^{th} deri :- $\left[y_{n+2}(x^2-1) + ny_{n+1}2x + (n^2-n)y_n\right] + \left[y_{n+1}x + ny_n\right] - m^2y_n = 0$

$$\Rightarrow (x^2-1)y_{n+2} + 2nxy_{n+1} + n^2y_n \cancel{-ny_n} + xy_{n+1} \cancel{+ny_n} - m^2y_n = 0$$

$$\Rightarrow 0 = (x^2-1)y_{n+2} + xy_{n+1}(2n+1) + (n^2-m^2)y_n = 0$$

(HW) ① If $y = (x^2-1)^n$ PT $(x^2-1)y_{n+2} + 2nxy_{n+1} - n(n+1)y_n = 0$

Ans:- $\quad y = (x^2-1)^n \Rightarrow y_1 = n(x^2-1)^{n-1}2x = \dfrac{2n(x^2-1)^n x}{(x^2-1)} = xny/(x^2-1)$

$(x^2-1)y_1 = 2nyx \quad \Rightarrow$ taking 2^{nd} derivative

$$(x^2-1)y_2 + y_1 2x = 2n[xy_1 + y]$$

$$\Rightarrow 0 = (x^2-1)y_2 + 2xy_1 - 2nxy_1 - 2ny = 0$$

taking n^{th} derivative :- $\left[(x^2-1)y_{n+2} + ny_{n+1}2x + \dfrac{n(n-1)}{2} \times 2y_n\right] + 2\left[xy_{n+1} + ny_n\right]$

$$- 2n\left[xy_{n+1} + ny_n\right] - 2ny_n = 0$$

ie $0 = (x^2-1)y_{n+2} + \cancel{2nxy_{n+1}} + n^2y_n - ny_n + 2xy_{n+1}n + \cancel{2ny_n} - \cancel{2nxy_{n+1}} - 2n^2y_n - \cancel{2ny_n}$

$$0 = (x^2-1)y_{n+2} + n^2y_n - 2n^2y_n - ny_n + 2xy_{n+1}n$$

$$= (x^2-1)y_{n+2} - n^2y_n + 2nxy_{n+1} - ny_n$$

ie $\Rightarrow \underline{(x^2-1)y_{n+2} + 2nx(y_{n+1}) - n(n+1)y_n = 0}$

Qn :- If $\cos^{-1}(y/b) = \log(x/n)^n$ P.T $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$

Ans :-

$$\cos^{-1}(y/b) = n\log(x/n)$$

$$y = b\cos(n\log x/n) \Rightarrow y_1 = -b\sin(n\log(x/n))\, n^2/x \times 1/n$$

$$= -b\sin(n\log x/n)\, n/x$$

ie $xy_1 = -bn\sin(n\log(x/n))$

$$xy_2 + y_1 = -bn\cos(n\log(x/n))\, n\frac{1}{x/n}\times\frac{1}{n}$$

$$= -\frac{bn^2\cos(n\log(x/n))}{x} \Rightarrow x^2y_2 + xy_1 = -n^2y$$

Hence 2^{nd} derivative relation: $x^2y_2 + xy_1 + n^2y = 0$

Lebnitz rule: $\left[x^2y_{n+2} + 2y_{n+1}xn + \frac{n(n-1)}{2}y_n 2\right] + \left[xy_{n+1} + ny_n\right] + n^2y_n = 0$

$x^2 y_{n+2} + 2nx\, y_{n+1} + n^2 y_n - \cancel{n y_n} + x\, y_{n+1} \cancel{+ n y_n} + n^2 y_n = 0$]

ie $x^2 y_{n+2} + x y_{n+1}(2n+1) + 2n^2 y_n = 0$

Qn:- If $x = \sin t$, $y = \sin pt$ PT $(1-x^2)y_2 - xy_1 + p^2 y = 0$.

Hence PT $(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (p^2 - n^2) y_n = 0$.

Ans:-

$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{\cos t}{p \cos pt} \Rightarrow \text{ ie } dy/dt = \frac{p \cos tp}{\cos t}$$ (Parametric)

$$y_1 = \frac{p \cos pt}{\cos t} = \frac{p\sqrt{1-\sin^2 pt}}{\sqrt{1-\sin^2 t}} = \frac{p\sqrt{1-y^2}}{\sqrt{1-x^2}}, \text{ ie } y_1\sqrt{1-x^2} = p\sqrt{1-y^2}$$

sqr^n : $(1-x^2)y_1^2 = p^2(1-y^2)$

diff : $(1-x^2)2y_1y_2 + y_1^2(-2x) = p^2[-2yy_1]$

$\div 2y_1$ $(1-x^2)y_2 - xy_1 = -p^2 y \Rightarrow (1-x^2)y_2 - xy_1 + p^2 y = 0$

then n^{th} deriv: $\left[(1-x^2)y_{n+2} + ny_{n+1}(-2x) + \frac{n(n-1)}{\cancel{2}} y_n (-\cancel{2})\right] + -\left[x y_{n+1} + n y_n\right] + p^2 y_n = 0$

$(1-x^2)y_{n+2} + -2nx y_{n+1} - n^2 y_n + \cancel{n y_n} - x y_{n+1} - n y_n + p^2 y_n = 0$

$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (p^2 - n^2) y_n = 0$

TAYLOR SERIES & MACLAURN'S SERIES EXPANSION OF FXNS

1) Taylor Series :-

If $f(x)$ is differenciable succesively, then taylor series expansion of $f(x)$ about $x=a$ is

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots\cdots$$

2) Maclaurn Series :-

with $x = a = 0$, $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

Eg :- ① Expand $f(x) = x^4 - 3x^3 + 4x^2 + x + 1$ in powers of $(x-1)$

Ans :

$f(x) = x^4 - 3x^3 + 4x^2 + x + 1$	$f(a) = f(1) = 4$
$f'(x) = 4x^3 - 9x^2 + 8x + 1$	$f'(1) = 4$
$f''(x) = 12x^2 - 18x + 8$	$f''(1) = 2$
$f'''(x) = 24x - 18$	$f'''(1) = 6$
$f^{iv}(x) = 24$	$f^{iv}(1) = 24$

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$= 4 + \frac{4 \times (x-1)}{1!} + \frac{(x-1)^2}{\cancel{2!}} \times \cancel{2} + \frac{(x-1)^3}{\cancel{3!}} \times \cancel{6} + \frac{(x-1)^4}{\cancel{4!}} \times \cancel{24} + \cdots$$

$$= \underline{4 + 4(x-1) + (x-1)^2 + (x-1)^3 + (x-1)^4 + \cdots\cdots}$$

Qn: Expand $f(x) = e^x$ in powers of $(x-1)$.

Ans: $f(x) = e^x \Rightarrow f'(x) = f''(x) = f'''(x) = \dots = f^n(x) = e^x$

$f(a) = f(1) = e \qquad f'(1) = f''(1) = \dots = e^1 = e$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$= e + e(x-1) + e\frac{(x-1)^2}{2!} + e\frac{(x-1)^3}{3!} + \dots$$

$$= e\left[1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots\right]$$

Qn :- Expand $f(x) = \sin x$ in powers of $(x-a) = (x - \pi/2)$.

Ans: $f(x) = \sin x$

$f^{n}(x) = \sin x \qquad f(\pi/2) = 1$

$f'(x) = \cos x \qquad f'(\pi/2) = 0$

$f''(x) = -\sin x \qquad f''(\pi/2) = -1$

$f'''(x) = -\cos x \qquad f'''(\pi/2) = 0$

$f^{IV}[x] = \sin x \qquad f^{IV}(\pi/2) = 1$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$\sin x = 1 + \frac{(x-\pi/2)}{1!} \times 0 + \frac{(x-\pi/2)^2}{2!} \times (-1) + \frac{(x-\pi/2)^3}{3!} \times 0 + \frac{(x-\pi/2)^4}{4!} \times 1 + \dots$$

$$= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \frac{(x-\pi/2)^8}{8!} + \dots$$

Maclaurin's Series :-

(1) $f(x) = e^x \qquad f'(x) = f''(x) = f'''(x) = \dots = e^x$

$a = 0 \qquad f'(0) = f''(0) = f'''(0) = \dots = 1$

$$f(x) = f(0) + \frac{(x)}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots$$

$$\boxed{\begin{aligned} e^x &= 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + \cdots \\ e^{-x} &= 1 - x/1! + x^2/2! - x^3/3! + x^4/4! + \cdots \end{aligned}}$$

② $y = \sin x$ $\Rightarrow$

$f(x) = \sin x$		$f(0) = 0$
$f'(x) = \cos x$		$f'(0) = 1$
$f''(x) = -\sin x$	$a = 0$	$f''(0) = 0$
$f'''(x) = -\cos x$		$f'''(0) = -1$
$f^{iv}(x) = \sin x$		$f^{iv}(0) = 0$

$f(x) \Rightarrow \sin x$ ie $\sin x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!}(-1) + \frac{x^4}{4!} \times 0 + \frac{x^5}{5!} \times 1$

ie $\boxed{\sin x = x/1! - x^3/3! + x^5/5! - x^7/7! + \cdots}$

③ $f(x) = \cos x$

$f(x) = \cos x$	$f(0) = 1$
$f'(x) = -\sin x$	$f'(0) = 0$
$f''(x) = -\cos x$	$f''(0) = -1$
$f'''(x) = \sin x$	$f'''(0) = 0$
$f^{iv}(x) = \cos x$	$f^{iv}(0) = 1$

$$f(x) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + x^3/3! \, f'''(a)$$

$$\cos x = 1 + \frac{x}{1!} \times 0 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!} \times 0 + \frac{x^4}{4!} \times 1 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots}$$

(4) $y = \sinh x$

$f(x) = \sinh x \qquad f(0) = 0$

$f'(x) = \cosh x \qquad f'(0) = 1$

$f''(x) = \sinh x \qquad f''(0) = 0$

$f'''(x) = \cosh x \qquad f'''(0) = 1$

$$\left[\begin{array}{l} \sinh x = \dfrac{e^x - e^{-x}}{2} \\ \sinh 0 = 0 \end{array}\right]$$

$$f(x) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \cdots$$

$$\sinh x = 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 1 + \frac{x^4}{4!} \times 0 + \frac{x^5}{5!} \times 1 + \cdots$$

$$= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\boxed{\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$$

Note:

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Replace x with ix

$$\sin(ix) = \frac{ix}{1!} - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \frac{(ix)^7}{7!} + \cdots$$

$$i \sinh x = \frac{ix}{1!} + \frac{ix^3}{3!} + \frac{ix^5}{5!} + \frac{ix^7}{7!} + \cdots$$

$$= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

(5) $y = \cosh x$

$f(x) = \cosh x \qquad f(0) = 1$

$f'(x) = \sinh x \qquad f'(0) = 0$

$f''(x) = \cosh x \qquad f''(0) = 1$

$f'''(x) = \sinh x \qquad f'''(0) = 0$

$f^{iv}(x) = \cosh x \qquad f^{iv}(0) = 1$

$$\left[\begin{array}{l} \cosh x = \dfrac{e^x + e^{-x}}{2} \\ \cosh 0 = 2/2 = 1 \end{array}\right]$$

$\cos hx = 1 + \frac{x}{1!} \times 0 + \frac{x^2}{2!} \times 1 + \frac{x^3}{3!} \times 0 + \frac{x^4}{4!} \times 1 + \cdots$

$= 1 + x^2/2! + x^4/4! + x^6/6! + \cdots$

$$\boxed{\cos hx = 1 + x^2/2! + x^4/4! + x^6/6! + \cdots}$$

Qn :- Find Maclaurin's expansion of $\log(1+x)$. Also deduce that of $\log\left[\frac{1+x}{1-x}\right]$.

Ans :-

$$f(x) = \log[1+x] = \log(ax+b) = \frac{a^n (-1)^{n-1} (n-1)!}{(ax+b)^n}$$

ie $y_1 = \frac{1^n (-1)^{1-1} 0!}{(1+x)^1} = \frac{1}{(x+1)}$

$y_2 = \frac{1^1 (-1)^{2-1} (1!)}{(1+x)^2} = \frac{-1}{(1+x)^2}$

$y_3 = \frac{1^1 (-1)^{3-1} (2!)}{(1+x)^3} = \frac{2}{(1+x)^3}$

$y_4 = \frac{1 (-1)^{4-1} (3!)}{(1+x)^4} = \frac{-6}{(1+x)^4}$

$f(0) = 0$

$f'(0) = +1$

$f''(0) = -1$

$f'''(0) = 2$

$f^{iv}(0) = -6$

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + f'''(0) \frac{x^3}{3!} + \cdots$

$= 0 + \frac{x}{1!} \times 1 + \frac{x^2}{2!} \times -1 + \frac{x^3}{3!} \times 2 + \frac{x^4}{4!} (-6)$

$= 0 + x + x^2/2 (-1) + \frac{x^3}{3} - \frac{x^4}{4}$

$$\boxed{\log[1+x] = x - x^2/2 + x^3/3 - x^4/4 + x^5/5! + \cdots}$$

$$\boxed{\log[1-x] = -x - x^2/2 - x^3/3 - x^4/4 + \cdots}$$

ie $\log\left[{}^{1+x}/_{1-x}\right] = \log[1+x] - \log[1-x]$

$$= x - x^2/_2 + x^3/_3 - x^4/_4 + x^5/_5 + \cdots - \left[-x - x^2/_2 - x^3/_3 - x^4/_4 \cdots\right]$$

$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \cdots$$

ie $$\boxed{\tfrac{1}{2}\log\left[\frac{1+x}{1-x}\right] = x + x^3/_3 + x^5/_5 + \cdots}$$

Qn 8:- $y = \tan x$

$$f(x) = \tan x$$

$$f'(x) = \sec^2 x$$

$$f''(x) = \frac{d[1 + f(x)^2]}{dx} = 2f(x)\,f'(x) = 2\tan x \sec^2 x$$

$$f'''(x) = 2[f(x)f''(x) + f'(x)f'(x)]$$

$$= 2[f(x)f''(x) + f'(x)^2]$$

$$f^{IV}(x) = 2[f(x)f'''(x) + f''(x)f'(x) + 2f'(x)f''(x)]$$

$$= 2[f(x)f'''(x) + 3f'(x)f''(x)]$$

$$f^{V}(x) = 2[f(x)f^{IV}(x) + f'''(x)f'(x) + 3[f'(x)f'''(x) + f''(x)f''(x)]]$$

now $f(0) = 0 \quad f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = 2 \quad f^{IV} = 0 \quad f^{V} = 16$

ie $f(x) = f(a) + \frac{x}{1!}f'(a) + x^2/_{2!}\,f''(a) + x^3/_{3!}\,f'''(a) + \cdots$

$$\tan x = x/_{1!} + \frac{x^3}{3!}2 + \frac{x^5}{5!}16 + \cdots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

$$\boxed{\tan x = x + x^3/_3 + {}^2/_{15}\,x^5 + \cdots}$$

Assignment :- ① $y = \tan^{-1}x$, PT $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

② If $x = \sin t$ $y = \cos pt$ P.T. $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-p^2)y_n = 0$

③ Expand $y = \sin x$ in powers of $x - \pi/4$

④ Expand xe^{-x} using Maclauran's Series

⑤ PT (i) $\sinh(-x) = -\sinh x$

(ii) $\cosh(-x) = \cosh x$

⑥ Expand $\sin^2 x$ using Maclaurin's series.

⑦ solve $\sinh x = 3/4$

⑧ Solve $2\cosh 2x + 10\sinh 2x = 5$

⑨ If $\sinh x = 5/12$

Find (i) $\cosh x$ (ii) $\sinh 2x$ (iii) $\tanh x$ (iv) $\cosh 2x$.

Submit on 25/10/24

Qn: Find Maclaurin's expansion for x^2e^{2x}.

Ans: $y = f(x) = x^2e^{2x}$

* $f'(x) = 2x^2e^{2x} + 2e^{2x}x$
$= 2e^{2x}[x^2+x]$
$= 2f(x) + 2xe^{2x}$

* $f''(x) = 2f'(x) + 2[x(2e^{2x}) + e^{2x}]$
$= 2f'(x) + 4xe^{2x} + 2e^{2x}$
$= 2f'(x) + 2[2e^{2x}] + 2e^{2x}$
$= 2f'(x) + 2[f'(x) - 2f(x)] + 2e^{2x}$
$= 4f'(x) - 4f(x) + 2e^{2x}$

* $f'''(x) = 4f''(x) - 4f'(x) + 2\times2e^{2x}$

* $f^{IV}(x) = 4f'''(x) - 4f''(x) + 8e^{2x}$

$f(0) = 0$

$f'(0) = 0$

$f''(0) = 2$

$f'''(0) = 8+4 = 12$

$f^{IV}(0) = 48 + \cancel{-8} \cancel{8} = \underline{48}$

$$f(x) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \frac{x^3}{3!} f'''(a) + \cdots$$

$$= 0 + 0 + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 12 + \frac{x^3}{4!} \times 48 + \cdots$$

$$= \underline{x^2 + 2x^3 + 2x^4 + \cdots}$$

Qn: Find Maclaurian expansion of $f(x) = \cos^2 x$.

Ans: $f(x) = \cos^2 x$

$= \frac{1}{2} + \frac{1}{2}\cos 2x$

$f'(x) = \frac{1}{2} \times -\sin 2x \times 2 = -\sin 2x$ $\quad f'(0) = 0$

$f''(x) = -2 \times \cos 2x$ $\quad f''(0) = -2$

$f'''(x) = 4\sin 2x$ $\quad f'''(0) = 0$

$f^{IV}(x) = 8\cos 2x$ $\quad f^{IV}(0) = 8$

$f^{V}(x) = -16\sin 2x$ $\quad f^{V}(0) = 0$

$f^{VI}(x) = -32\cos 2x$ $\quad f^{VI}(0) = -32$

$$f(x) = 1 + \frac{x}{1!} \times 0 + \frac{x^2}{2!} \times (-2) + \frac{x^3}{3!} \times 0 + \frac{x^4}{4!} \times 8 + \frac{x^5}{5!} \times 0 + \frac{x^6}{6!} \times (-32)$$

$$= 1 - x^2 + 3^{-1} x^4 - \frac{2}{30} x^6 + \cdots$$

$$= \underline{\underline{1 - x^2 + \tfrac{1}{3} x^4 - \tfrac{1}{30} x^6 + \cdots}}$$

Qn: Find Maclaurin expansion of $e^{\sin x}$.

Ans: $f(x) = e^{\sin x}$

$f'(x) = e^{\sin x} \cdot \cos x = f(x)\cos x$

$f''(x) = \cos x\, f'(x) + f(x)(-\sin x)$

$= \cos x\, f'(x) - f(x)\sin x$

$f'''(x) = f''(x)\cos x - f(x)\sin x - f'(x)\sin x - f(x)\cos x$

$= -f(x)\cos x - 2f'(x)\sin x + f''(x)\cos x$

$= -f'(x) - 2f'(x)\sin x + f''(x)\cos x$

$f^{iv}(x) = -f''(x) - 2[f'(x)\cos x + \sin x\, f''(x)] + [f''(x)[-\sin x] + \cos x\, f'''(x)]$

$f(0) = 1 \quad , \quad f'(0) = 1 \quad f''(0) = 1 \quad f'''(0) = -1 - 2\times1\times0 + 1\times1$
$= \underline{0}$

$f^{iv}(0) = -\cancel{0}1 - 2[1\times1 + 0] + [0+0] = -3 //$

ie $\quad f(x) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \cdots\cdots$

$= 1 + \frac{x}{1!} + \frac{x^2}{2} \times 1 + \frac{x^3}{6} \times 0 + \frac{x^4}{24} \times -3$

$= \underline{1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \cdots\cdots}$

Qn:- Find Maclaurian expansion of $\log[1+\sin x]$.

Ans: $f(x) = \log[1+\sin x] \qquad f'(x) = \frac{\cos x}{1+\sin x} = \frac{1}{e^{f(x)}}\cos x \longrightarrow f'(0) = 1$

$f''(x) = \frac{(1+\sin x)(-\sin x) - \cos x \cdot \cos x}{[1+\sin x]^2} = \frac{-\sin x - 1}{[1+\sin x]^2} = \frac{-1}{[1+\sin x]}$

$f'''(x) = \frac{-\cos x}{(1+\sin x)^2} = \frac{-\cos x}{(1+\sin x)^2} //$

$f''(0) = -1$ and $f'''(0) = 1$

then $\quad f(x) = 0 + x\times1 + \frac{x^2}{2}\times(-1) + \frac{x^3}{6}\times1 + \cdots$

$= \underline{x - \frac{1}{2}x^2 + \frac{1}{6}x^3}$

MODULE :- 5 : Partial Differenciation

5

- when the function is dependant on more than,

Let z be a function of 2 independant variables x and y. $\Rightarrow z = f(x,y)$

we can differenciate z with resp. to x keeping y as a constant. The derivative so obtained is called partial derivative of z w.r.to x, and is denoted by

$$\boxed{\frac{\partial z}{\partial x}}$$

ie $z = f(x,y) \longrightarrow ①$

$z + \Delta z = f(x+\Delta x, y) \rightarrow ②$ } ② − ① $\Rightarrow$ $\Delta z = f(x+\Delta x, y) - f(x,y)$

$$\frac{\Delta z}{\Delta x} = \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

now taking limit :- $\lim\limits_{x \to 0} \frac{\Delta z}{\Delta x} = \lim\limits_{x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

If the limit on the RHS exits, it is called partial differentiation w.r.t. x.

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= \lim_{x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \\ \text{simly :- } \frac{\partial z}{\partial y} &= \lim_{y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}. \end{aligned}}$$

Now len $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are again functions of x and y, then we can have

① $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ ③ $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$

③ $\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x}$ ④ $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\, \partial y}$

but in general $\frac{\partial^2 z}{\partial x\, \partial y}$ is same as $\frac{\partial^2 z}{\partial y\, \partial x}$. ie this is 2^{nd} order partial derivatives.

Note :- we use the following notations

$p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$ $r = \frac{\partial^2 z}{\partial x^2}$ $s = \frac{\partial^2 z}{\partial x \partial y} \Big/ \frac{\partial^2 z}{\partial y\, \partial x}$ $t = \frac{\partial^2 z}{\partial y^2}$

Examples :-

① Find p and q of

ⓐ $z = x^3 + y^3 - 3xy$

ⓑ $z = x^2 y - x \sin(xy)$

ⓒ $z = \log(x^2 + y^2)$

ⓓ $z = xy/x+y$.

Ans :- ⓐ $z = x^3 + y^3 - 3xy$

$p = \frac{\partial z}{\partial x} = 3x^2 + 0 - 3y = 3x^2 - 3y$ //

$q = \frac{\partial z}{\partial y} = 0 + 3y^2 - 3x = 3y^2 - 3x$ //

ⓑ $z = x^2 y - x \sin(xy)$

$p = \frac{\partial z}{\partial x} = 2xy - [xy \cos(xy) + \sin(xy) \cdot 1]$ //

$q = \frac{\partial z}{\partial y} = x^2 - [x^2 \cos(xy)] = x^2[1 - \cos(xy)]$ //

(c) $\log(x^2+y^2) = z \Rightarrow p = \frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}$ $\qquad q = \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$

(d) $z = \frac{x-y}{x+y}$; $\quad p = \frac{\partial z}{\partial x} = \frac{(x+y)\cdot 1 - (x-y)\cdot 1}{(x+y)^2} = \underline{\underline{2y/(x+y)^2}}$

$$q = \frac{\partial z}{\partial y} = \frac{(x+y)\cdot(-1) - (x-y)\cdot 1}{(x+y)^2} = \underline{\underline{-2x/(x+y)^2}}$$

* when x and y are interchanged in a fx^n, and the fx^n dose not changes, then the fx^n is called <u>symmetric fx^n</u> in x and y.

for finding derivative of such functions, we can use simly, in 2^{nd} step.

Qn :- If $z = e^{ax+yb} \cdot f(ax-by)$, P T $\; b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\,z$

Ans :- $z = e^{ax+by} f(ax-by)$.

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by)\cdot a + f(ax-by)\cdot e^{ax+by} a$$

ie $\; b\frac{\partial z}{\partial x} = ab\left[e^{ax+by}\left(f'(ax-by) + f(ax-by)\right)\right] = ab\,e^{ax+by}\left(f'(ax-by) + f(ax-by)\right) \to (1)$

$$\frac{\partial z}{\partial y} = e^{ax+by}\cdot f'(ax-by)\cdot(-b) + f(ax-by)\, e^{ax+by}\cdot b$$

ie $\; a\frac{\partial z}{\partial y} = ab\,e^{ax+by}\left[f(ax-by) - f'(ax-by)\right] \longrightarrow (2)$

(1) + (2) $= 2ab\,e^{ax+by} f(ax-by) = 2ab\,z$

Hence $\; b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\,z$

Qn 8 If $Z = \log(e^x + e^y)$, PT $rt - s^2 = 0$

Ans : a symmetric fx^n:

$p = \frac{\partial z}{\partial x} = \frac{e^x}{e^x + e^y}$ simly $q = \frac{e^y}{e^x + e^y}$

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{e^x}{e^x + e^y}\right] = \frac{(e^x + e^y)e^x - (e^x \cdot e^x)}{(e^x + e^y)^2} = \frac{e^x e^y}{(e^x + e^y)^2}$ //

simly $t = \frac{\partial^2 z}{\partial y^2} = \frac{e^{x+y}}{(e^x + e^y)^2}$

$s = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{e^x}{e^x + e^y}\right] = e^x \cdot \left[\frac{-1}{(e^x + e^y)^2} \cdot e^y\right] = \frac{-e^{x+y}}{(e^x + e^y)^2}$ //

ie $rt - s^2 = \frac{e^{x+y}}{(e^x + e^y)^2} \cdot \frac{e^{x+y}}{(e^x + e^y)^2} - \left[\frac{e^{x+y}}{(e^x + e^y)^2}\right]^2 = 0$

Qn 8- If $Z = x^2 - y^2$, Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

Ans 8- $p = \frac{\partial z}{\partial x} = 2x$ $r = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial^2 z}{\partial x^2} = 2$

$q = \frac{\partial z}{\partial y} = -2y$ $t = \frac{\partial^2 z}{\partial y^2} = -2$

ie $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2 + (-2) = 0$ //

HW *

If $Z = \tan(y + ax) - [y - ax]^{3/2}$, PT $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial y^2}$

o) Laplace differential eq^n : $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ => Sol^n = $x^2 - y^2$.

HW ans:-

$$z = \tan[y+ax] - [y-ax]^{3/2}$$

$$\frac{\partial z}{\partial x} = a\sec^2[y+ax] + 3/2\,[y-ax]^{1/2}a = a\left[\sec^2[y+ax] + \frac{3a}{2}[y-ax]^{1/2}\right]$$

$$\frac{\partial^2 z}{\partial x^2} = a\left[2\sec[y+ax]\sec x[ax+y]\tan[y+ax]\,a + \frac{3a}{2}[y-ax]^{-1/2}\times(-a)\right]$$

$$= a^2\left[2\sec^2[y+ax]\tan[ax+y] - 3/4\,[y-ax]^{-1/2}\right]$$

Now :

$$\frac{\partial z}{\partial y} = \sec^2[y+ax] - \frac{3}{2}[y-ax]^{1/2}$$

$$a^2\frac{\partial^2 z}{\partial y} = a^2\left[2\sec^2[y+ax]\tan[y+ax] - \frac{3}{4}[y-ax]^{-1/2}\right]$$

$$= \frac{\partial^2 z}{\partial x^2}$$

ie

$$\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$$

Qn : If $v = \frac{xz}{x^2+y^2}$, PT $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$

Ans :-

$$\frac{\partial v}{\partial x} = \frac{z(y^2-x^2)}{x^2+y^2} \qquad \frac{\partial^2 v}{\partial x^2} = \frac{2xz\,(x^2-3y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = xz\left[\frac{-1}{(x^2+y^2)^2}\times 2y\right] = \frac{-2xyz}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = -2yz\left[\frac{(x^2+y^2)^2\cdot 1 - 2(x^2+y^2)2y}{(x^2+y^2)^4}\right] = \frac{-2yz(x^2+y^2)\left[(x^2+y^2) - 2xy\right]}{(x^2+y^2)^4}$$

$$= \frac{-2xy\,(x^2-3y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial V}{\partial z} = \frac{x}{x^2+y^2} \qquad \frac{\partial^2 V}{\partial z^2} = 0 //$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{2xz\,(x^2-3y^2)}{(x^2+y^2)^3} - \frac{2xz\,(x^2-3y^2)}{(x^2+y^2)^3} + 0 = 0 //$$

Qn :- If $V = (x^2+y^2+z^2)^{-1/2}$, PT $\sum \frac{\partial^2 V}{\partial x^2} = 0$

Ans :- Given $V = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial V}{\partial x} = -\tfrac{1}{2}(x^2+y^2+z^2)^{-3/2} \times 2x = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 V}{\partial x^2} = -\left[x \times \frac{-3}{2}(x^2+y^2+z^2)^{-5/2} \cdot 2x + (x^2+y^2+z^2)^{-3/2} \cdot 1 \right]$$

$$= 3x^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} = (x^2+y^2+z^2)^{-3/2}\left[3x^2(x^2+y^2+z^2)^{-1/2} - 1\right] \rightarrow ①$$

Similary
$$\frac{\partial^2 V}{\partial y^2} = (x^2+y^2+z^2)^{-3/2}\left[3y^2(x^2+y^2+z^2)^{-1/2} - 1\right] \rightarrow ②$$

$$\frac{\partial^2 V}{\partial z^2} = (x^2+y^2+z^2)^{-3/2}\left[3z^2(x^2+y^2+z^2)^{-1/2} - 1\right] \rightarrow ③$$

$1+2+3 \rightarrow$
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \sum \frac{\partial^2 V}{\partial x^2} = 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2) - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$

Qn :- If $u = \log[x^3+y^3+z^3-3xyz)$, PT $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u$

$$= -9[x+y+z]^{-2}$$

Ans: $V = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial V}{\partial x} = \frac{1\times(3x^2-3yz)}{(x^3+y^3+z^3-3xyz)} \quad \text{simly} \quad \frac{\partial V}{\partial y} = \frac{3y^2-3xz}{(x^3+y^3+z^3-3xyz)}$$

$$\frac{\partial V}{\partial z} = \frac{3z^2-3xy}{(x^3+y^3+z^3-3xyz)}$$

ie
$$\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial V}{\partial z} = \frac{3[x^2+y^2+z^2]-3[yz+xz+xy]}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-xz-yz)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-xz-yz)}} = \frac{3}{(x+y+z)}$$

$$* \boxed{x^3+y^3+z^3-3xyz = (x+y+z)[x^2+y^2+z^2-xy-yz-xz]}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right](u) = \frac{\partial}{\partial x}\left[\frac{3}{(x+y+z)}\right] = \frac{-3}{(x+y+z)^2}$$

simly
$$\frac{\partial}{\partial y}\left[\frac{3}{(x+y+z)}\right] = \frac{-3}{(x+y+z)^2} \qquad \frac{\partial}{\partial z}\left[\frac{3}{(x+y+z)}\right] = \frac{-3}{(x+y+z)^2}$$

$$\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]^2(v) = -9\,(x+y+z)^{-2} \;//$$

HOMOGENOUS FUNCTIONS

*

Considering the polynomial in x, y every term is of the same degree

eg: $f(x,y) = a_0x^n + a_1x^{n-1}y + a_3x^{n-2}y^2 + \dots + a_ny^n$

(i) $f(x,y) = \underbrace{x^3}_{d=3} + \underbrace{7x^2y}_{d=3} - \underbrace{5xy^2}_{d=3} + \underbrace{4y^3}_{d=3}$ is homogenous degree = d

(ii) $f(x,y) = x^2 - xy - y^2$ is homogenous

(iii) $f(x,y) = \underbrace{x^3}_{d=3} + \underbrace{3xy}_{d=2} + \underbrace{y^2}_{d=2}$ is not homogenous

Note :- $f(x,y) = x^3 + y^3 + x^2y + xy^2 +$

$= x^n f(y/x)$

ie every homogenous functions can be written in the form :- $f(x,y) = x^n f(y/x)$

$\Rightarrow$ Here,

(a) $f(x,y) = x^2 \sin(y/x)$ is homogenous fx^n of degree 2

(b) $f(x,y) = \sin\left(\frac{x+y}{x-y}\right) = \sin\left(\frac{\cancel{x}(1+y/x)}{\cancel{x}(1-y/x)}\right)$

$= \sin\left(\frac{1+y/x}{1-y/x}\right) = (y/x)^0 \sin\left(\frac{1+y/x}{1-y/x}\right)$

ie this is a homogenous fx^n with degree = 0

To check wether a fx^n is homogenous :-

$f(x,y) =$ is homogenous fx^n of degree n, then $f(tx, ty) = t^n f(x,y)$

② $f(x,y) = x^2 \sin(y/x)$

$f(tx,ty) = (tx)^2 \sin\left(\frac{ty}{tx}\right) = t^2 x^2 \sin(y/x) = t^2 f(x,y)$

ie homogenous of degree 2.

EULER'S THEORUM [on homogenous functions]

If $f(x,y)$ is homogenous of degree n, then

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n.f}$$

eg: $f(x,y) = x^2 + y^2 + xy$ – Homogenous

then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2(x^2+y^2+xy)$

PROOF FOR EULER THEOREM :

Let f be a homogenous fxⁿ in x,y of degree n, $\Rightarrow f = x^n f(y/x)$

$\frac{\partial f}{\partial x} = x^n\left[f'(y/x) \cdot -y/x^2\right] + f(y/x)\, n x^{n-1}$

$= -x^{n-2} y f'(y/x) + n x^{n-1} f(y/x)$

now "x" with x $\Rightarrow$ $x\frac{\partial f}{\partial x} = -y x^{n-1} f'(y/x) + n x^n f(y/x) \longrightarrow ①$

$\frac{\partial f}{\partial y} = x^n f'(y/x)\, 1/x \Rightarrow y\frac{\partial f}{\partial y} = y x^{n-1} f'(y/x) \longrightarrow ②$

① + ② $x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y = n x^n f(y/x) = nf$ Hence proved.

Note :- we have $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf \longrightarrow$ ①

d.w.r.t. x $\Rightarrow \left[x\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}\cdot 1\right]+y\frac{\partial^2 f}{\partial x\partial y}=n\frac{\partial f}{\partial x}$

"x" with x $\Rightarrow x^2\frac{\partial^2 f}{\partial x^2}+\frac{x\,\partial f}{\partial x}+xy\frac{\partial^2 f}{\partial x\partial y}=nx\frac{\partial f}{\partial x} \longrightarrow$ ②

d w r t $y_{(\text{of } ①)}$ $\Rightarrow x\frac{\partial f}{\partial x\partial y}+\left[y\frac{\partial^2 f}{\partial y^2}+1\times\frac{\partial f}{\partial y}\right]=n\frac{\partial f}{\partial y}$.

"x" with y $\Rightarrow xy\frac{\partial f}{\partial x\partial y}+\frac{y^2\partial^2 f}{\partial y^2}+\frac{y\,\partial f}{\partial y}=ny\frac{\partial f}{\partial y}$. $\longrightarrow$ ③

② + ③ $\Rightarrow$ $x^2\frac{\partial^2 f}{\partial x^2}+\frac{x\,\partial f}{\partial x}+xy\frac{\partial^2 f}{\partial x\partial y}+xy\frac{\partial f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}+y\frac{\partial f}{\partial y}$

$=nx\frac{\partial f}{\partial x}+ny\frac{\partial f}{\partial y}$.

$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=nx\frac{\partial f}{\partial x}+ny\frac{\partial f}{\partial y}-x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial y}$

$=(n-1)\left[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right]$

$=n(n-1)f$

★ $$\boxed{x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f}$$

Qn :- Verify eulers theorem for :

① $ax^2+2hxy+by^2$

Ans : this is homogenous fn of degree 2.

now:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \Rightarrow x\frac{\partial f}{\partial x} = 2ax^2 + 2hyx$$

$$y\frac{\partial f}{\partial y} = \cancel{x}\, y[2hx + 2by]$$

ie $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2ax^2 + 2bxy + 2hxy + 2by^2 = 2[ax^2 + 2hxy + by^2]$

$$= \underline{\underline{2f}}$$

(2) $f = \sin\left(\frac{x+y}{x-y}\right) \Rightarrow \sin\left(\frac{tx+ty}{tx-ty}\right) = \sin\left(\frac{x+y}{x-y}\right) \cdot t^0$

replace with tx/ty $\Rightarrow f =$ homogenous fx^n of degree 0

$$x\frac{\partial f}{\partial y} = x\left[\cos\left[\frac{x+y}{x-y}\right]\left(\frac{(x-y).1 - (x+y).1}{(x-y)^2}\right)\right]$$

$$= x\left[\cos\left[\frac{x+y}{x-y}\right]\left(\frac{\cancel{x}\cancel{-x} - y - y}{(x-y)^2}\right)\right] = \frac{-2xy}{(x-y)^2}\cos\left[\frac{x+y}{x-y}\right] \longrightarrow (1)$$

$$y\frac{\partial f}{\partial y} = y\left[\cos\left[\frac{x+y}{x-y}\right]\left(\frac{(x-y).1 - (x+y).(-1)}{(x-y)^2}\right)\right] = y\left[\cos\left(\frac{x+y}{x-y}\right)\cdot\frac{2x}{(x-y)^2}\right]$$

$$= \frac{2xy}{(x-y)^2}\cos\left[\frac{x+y}{x-y}\right] \longrightarrow (2)$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0 = \underline{\underline{0 \times f}}$$

(3) $f = x\cos[y/x] = x^1\cos(y/x) \Rightarrow$ Hom. fx^n with degr = 1

$$x\frac{\partial f}{\partial x} = x\left[x \times (-1)\sin(y/x)\cdot\frac{-y}{x^2} + \cos(y/x)\cdot 1\right] = y\sin(y/x) + x\cos(y/x)$$

$$y\frac{\partial f}{\partial y} = xy\left[(-1)\sin(y/x)\cdot\frac{1}{x}\right] = -y\sin(y/x)$$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x\cos(y/x) = x^1\cos(y/x) =$ function itself //.

(4) $f(x) = \tan^{-1}(y/x) = u$

Ans: $u = x^0 \tan^{-1}(y/x)$, It is homogeneous with power[degree] 0

$$x\frac{\partial u}{\partial x} = x\left[\frac{x^2}{x^2+y^2} \times y \times \left[\frac{-1}{x^2}\right]\right] = \frac{-xy}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = y\left[\frac{x^2}{x^2+y^2} \times \frac{1}{x}\right] = \frac{xy}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-xy}{x^2+y^2} + \frac{xy}{x^2+y^2} = 0$$

Qn: Verify Eulers theorem

(1) If $u = \sin^{-1}\sqrt{x^2+y^2}$, PT $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Also derive that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u$

Ans:- $u = \sin^{-1}(\sqrt{x^2+y^2}) = \sin^{-1}\sqrt{t^2(x^2+y^2)} = \sin^{-1}(t\sqrt{x^2+y^2})$

= not homogenous fn.

Hence let $z = \sin u = \sqrt{x^2+y^2} \longrightarrow (1)$ $\Rightarrow f(tx, ty) = \sqrt{t^2(x^2+y^2)} = t^1\sqrt{x^2+y^2}$

= Homogenous with degree 1.

now eulers theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nf.$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1\times z = \sin u$$

$$x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = \sin u \Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$\Rightarrow \tan u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \longrightarrow (2)$$

diff. (2) w.r.to x.

$$\left[x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\times 1\right] + y\frac{\partial^2 u}{\partial x\partial y} = \sec^2 u\frac{\partial u}{\partial x}$$

"x" with x $\Rightarrow$

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = x\sec^2 u\frac{\partial u}{\partial x} \longrightarrow (3)$$

diff (2) w.r.to y:

$$x\frac{\partial^2 u}{\partial x\partial y} + \left[y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\times 1\right] = \sec^2 u\frac{\partial u}{\partial y}$$

"x" with y $\Rightarrow$

$$xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + 0 = y\sec^2 u\, y\frac{\partial u}{\partial y} \longrightarrow (4)$$

$$(3) + (4) \Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \tan u = \sec^2 u\tan u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (\sec^2 u - 1)\tan u = \tan^2 u\cdot\tan u = \tan^3 u$$

Qn:- If $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$, PT $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$. Also

derive $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial x\partial y} = 3\tan u\,(3\sec^2 u - 1)$.

Ans:- $u = \sin^{-1}\left(\frac{t^4 x^2y^2}{t(x+y)}\right) = \sin^{-1}\left[t^3\frac{x^2y^2}{(x+y)}\right]$ = not homogenous

So $\Rightarrow$ $z = \sin u = \sin\sin^{-1}\left(t^3\frac{x^2y^2}{(x+y)}\right) = t^3\left(\frac{x^2y^2}{(x+y)}\right) \longrightarrow ①$

= Homogenous with degree 3

From homogenous eq^n & Eulers theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3\sin u$$

$\Rightarrow$ $x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = 3\sin u \Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 3\sin u$

$\div$ with $\cos u$ $\Rightarrow$ $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u\cdot 3 \longrightarrow ②$

diff ② w.r.t. x

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = 3\sec^2 u\frac{\partial u}{\partial x}$$

"x" with x $\Rightarrow$ $x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} + x\frac{\partial u}{\partial x} = 3x\sec^2 u\frac{\partial u}{\partial x} \longrightarrow ③$

diff. ② w.r.t y:

$$x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 3\sec^2 u\frac{\partial u}{\partial y}$$

"x" with y $\Rightarrow$ $xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 3y\sec^2 u\frac{\partial u}{\partial y} \longrightarrow ④$

③+④ $\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 3\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$

$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + 3\tan u = 3\sec^2 u \times 3\tan u$

$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 3\tan u.\left(3\sec^2 u - 1\right)$

Qn :- If $u = e^{x^3+y^3}$, PT $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u$

Ans: $f(tx, ty) = e^{t^3[x^3+y^3]}$ = not homogenous

let $z = \log u = x^3+y^3 \Rightarrow f(tx, ty) = t^3(x^3+y^3)$

= Homogenous with degree 3.

from Eulers law :- $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3.z$

$x\frac{\partial}{\partial x}\log u + y\frac{\partial}{\partial y}\log u = 3\log u \Rightarrow x \times \frac{1}{u}\frac{\partial u}{\partial x} + y.\frac{1}{u}\frac{\partial u}{\partial x} = 3\log u$

$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u.$

Qn: If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, PT $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{2}\cot u$, Also derive that $x^2\,\partial^2 u/\partial x^2 + 2xy\,\partial^2 u/\partial x\partial y + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\cot u}{2}\left[1 - \frac{\operatorname{cosec}^2 u}{2}\right]$

Ans: since u is not homogenous $\Rightarrow z = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} \longrightarrow$ ①

$f(tx, ty) = \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})} = t^{1/2}\frac{(x+y)}{\sqrt{x}+\sqrt{y}}$

By Eulers theorem $\Rightarrow$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1/2\, z \Rightarrow x\frac{\partial \cos u}{\partial x} + y\frac{\partial \cos u}{\partial y} = \frac{1}{2}\cos u$$

$$\Rightarrow -x\sin u\frac{\partial u}{\partial x} + -y\sin u\frac{\partial u}{\partial y} = \frac{\cos u}{2} \quad (\div \text{ with } -\sin u)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-\cot u}{2} \longrightarrow ②$$

diff ② w.r.t. x

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = -1/2\,(-\operatorname{cosec}^2 u)\frac{\partial u}{\partial x} \longrightarrow (\text{"}x\text{" with } x)$$

$$x^2\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = \frac{+x}{2}\operatorname{cosec}^2 u\frac{\partial u}{\partial x} \longrightarrow ③$$

Simly : $$y^2\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} + xy\frac{\partial^2 u}{\partial xy} = \frac{y}{2}\operatorname{cosec}^2 u\frac{\partial u}{\partial y} \quad (\text{When diff } ③ \text{ w.r.t } y) \longrightarrow ④$$

$$③ + ④ \Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{\operatorname{cosec}^2 u}{2}\left[x\,\partial u/\partial x + y\frac{\partial u}{\partial y}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(\frac{-\cot u}{2}\right) = \frac{\operatorname{cosec}^2 u}{2}\left[\frac{-\cot u}{2}\right]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\cot u}{2}\left[1 - \frac{\operatorname{cosec}^2 u}{2}\right]$$

HW

Qn: If $u = \tan^{-1}\left[\frac{x^2+y^2}{x+y}\right]$, PT $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$. Then PT

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$$

TOTAL DERIVATIVE

If $z = f(x,y)$ where $x = \phi(t)$ and $y = \psi(t)$, then

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}}$$

eg: let $z = x^2 + xy + y^2$ and $x = t^2$, $y = 2t$

① $z = x^2 + xy + y^2 = (t^2)^2 + (t^2)(2t) + (2t)^2$
$= t^4 + 2t^3 + 4t^2$ $\Big\}$ $\frac{\partial z}{dt} = 4t^3 + 6t^2 + 8t$

OR

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{dt} = (2x+y)\,2t + (2y+x)\,2$$

$$= 4xt + 2ty + 4y + 2x$$

$$= 4\times t^2 \cdot t + 2t \cdot 2t + 4\times 2t + 2\times t^2$$

$$= \underline{4t^3 + 6t^2 + 8t}$$

Note ① :

i) In this case $\frac{dz}{dt}$ is called total derivative

ii) If $u = f(x,y,z)$ where $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$, then

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}}$$

iii) Let $z = f(x,y)$ and $x = x$; $y = y(x)$ is a special case with $t = x$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad \text{becomes}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{\partial x}$$

iv) If $z = f(x,y) = C$; a Constant , then $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$ becomes

$$0 = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} \quad \Rightarrow \quad -\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{dy}{dx}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = -\left[\frac{\partial z/\partial x}{\partial z/\partial y}\right]}$$

eg: If $z = x^2 + xy + y^2 = \phi$

Method ①: $\frac{dz}{dx} = 2x + \left[x\frac{dy}{dx} + y\right] + 2y\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx}(2y+x) = -(2x+y) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-(2x+y)}{2y+x}$$

Method ②: $\frac{dz}{dx} = -\frac{\partial z/\partial x}{\partial z/\partial y} = \frac{-(2x+y)}{(2y+x)}$

Note ⑤ :- If $f = f(x,y)$, then $\frac{\partial f}{\partial x} = f_x$; $\frac{\partial f}{\partial y} = f_y$

$$\boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}}$$

Qn 8- If $u = x^2+y^2$; $x = e^t$, $y = e^{-t}$ find du/dt

Ans 8- $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = 2x\,e^t + 2y\,(-e^{-t})$$

$$= [xe^t + y(-e^{-t})]\,2 = 2[e^{2t} - e^{-2t}]$$

Qn 8- $u = \sin(x^2+y^2)$, when $x = t + 1/t$, $y = t - 1/t$, find $\frac{du}{dt}$

Ans :- $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \cos(x^2+y^2)\,2x\left(1 - 1/t^2\right) + \cos(x^2+y^2)\,2y\,(1+1/t^2)$$

$$= 2\cos(x^2+y^2)\left[x\left(\frac{t^2-1}{t^2}\right) + y\left(\frac{t^2+1}{t^2}\right)\right]$$

$$= 2\cos\left[(t+1/t)^2 + (t-1/t)^2\right]\left[x\left(\frac{t^2-1}{t^2}\right) + y\left[\frac{t^2+1}{t^2}\right]\right]$$

Change of Variables :-

Let $u = f(x,y)$ where $x = \phi(s,t)$, $y = \psi(s,t)$, then

$$\boxed{\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\end{aligned}}$$

OR

Simly :- $u = f(x, y, z)$ where $f_1(s,t) = x$, $y = f_2(s,t)$, $z = f_3(s,t)$

Then $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}.$$

Qn: If $u = f(x-y, y-z, z-x)$ P.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Ans:- $u = f(x-y, y-z, z-x)$ and

$$\left.\begin{aligned} r &= x - y + 0z \\ s &= 0x + y - z \\ t &= -x + 0y + z \end{aligned}\right\} f(r,s,t)$$

Hence $\frac{\partial r}{\partial x} = 1$ $\frac{\partial r}{\partial y} = -1$ $\frac{\partial r}{\partial z} = 0$, $\frac{\partial s}{\partial x} = 0$ $\frac{\partial s}{\partial y} = 1$ $\frac{\partial s}{\partial z} = -1$,

$\frac{\partial t}{\partial x} = -1$ $\frac{\partial t}{\partial y} = 0$ $\frac{\partial t}{\partial z} = 1$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial r} + 0 + -\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} = \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1) \longrightarrow ③$$

$① + ② + ③ \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ //

Qn:- If $u = f[(2x-3y),(3y-4z),(4z-2x)]$ PT $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$

Ans:- Let $r = 2x - 3y + 0z$ $\quad s = 0x + 3y - 4z$ $\quad t = -2x + 0y + 4z$

Hence $f(r,s,t)$ will be u.

$\frac{\partial r}{\partial x} = 2 \quad \frac{\partial r}{\partial y} = -3 \quad \frac{\partial r}{\partial z} = 0$, $\quad \frac{\partial s}{\partial x} = 0 \quad \frac{\partial s}{\partial y} = 3 \quad \frac{\partial s}{\partial z} = -4$, $\quad \frac{\partial t}{\partial x} = -2 \quad \frac{\partial t}{\partial y} = 0 \quad \frac{\partial t}{\partial z}$,

Now, $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= 2\frac{\partial u}{\partial r} + (-2)\frac{\partial u}{\partial t} = 2\left[\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right] \longrightarrow ①$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} = -3\frac{\partial u}{\partial r} + 3\frac{\partial u}{\partial s} = 3\left[\frac{\partial u}{\partial s} - \frac{\partial u}{\partial r}\right] \longrightarrow ②$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z} = -4\frac{\partial u}{\partial s} + 4\frac{\partial t}{\partial z} = 4\left[\frac{\partial t}{\partial z} - \frac{\partial u}{\partial s}\right] \longrightarrow ③$$

Hence $\frac{1}{2}\times① + \frac{1}{3}\times② + \frac{1}{4}\times③ \Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$ //

Qn:- If $u = f(x/y, y/z, z/x)$, PT $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Ans 8- $r = x/y \quad s = y/z \quad t = z/x$ Hence $f(r,s,t) = u$

* $\frac{\partial r}{\partial x} = \frac{1}{y}, \quad \frac{\partial r}{\partial y} = \frac{-x}{y^2}, \quad \frac{\partial r}{\partial z} = 0$ * $\frac{\partial s}{\partial x} = 0, \quad \frac{\partial s}{\partial y} = \frac{1}{z}, \quad \frac{\partial s}{\partial z} = \frac{-y}{z^2}$

* $\frac{\partial t}{\partial x} = \frac{-z}{x^2} \quad \frac{\partial t}{\partial y} = 0 \quad \frac{\partial t}{\partial z} = \frac{1}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = \frac{1}{y} \times \frac{\partial u}{\partial r} + 0 + \frac{\partial u}{\partial t}\left(\frac{-z}{x^2}\right)$$

Hence $x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\cdot\frac{\partial u}{\partial t} = r\frac{\partial u}{\partial r} - t\frac{\partial u}{\partial t} \longrightarrow (1)$

Simly $y\frac{\partial u}{\partial y} = s\frac{\partial u}{\partial s} - r\frac{\partial u}{\partial r} \longrightarrow (2)$

$z\frac{\partial u}{\partial z} = t\frac{\partial u}{\partial t} - s\frac{\partial u}{\partial s} \longrightarrow (3)$

$(1) + (2) + (3) \Rightarrow \cancel{r\frac{\partial u}{\partial r}} - \cancel{t\frac{\partial u}{\partial t}} + \cancel{s\frac{\partial u}{\partial s}} - \cancel{r\frac{\partial u}{\partial r}} + \cancel{t\frac{\partial u}{\partial t}} - \cancel{s\frac{\partial u}{\partial s}} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

JACOBIANS *

If $u = f(x,y)$ and $v = g(x,y)$, then we can find 4 1st order partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

The Jacobian is denoted by $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$\lim$ly :- $u = f_1(x,y,z)$ $v = f_2(x,y,z)$ $w = f_3(x,y,z)$

Jacobian's $\Rightarrow$ $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Note :- ① If $J = \frac{\partial(u,v)}{\partial(x,y)}$ $J' = \frac{\partial(x,y)}{\partial(u,v)}$ $\Rightarrow$ $\boxed{J \cdot J' = +1}$

Note ② :- If u and v are functions of r & s and r & s and functions of x & y,

then $$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \frac{\partial(r,s)}{\partial(x,y)}}$$

Qn :- If $x = r\cos\theta$ $y = r\sin\theta$ find $J = \frac{\partial(x,y)}{\partial(r,\theta)} = ?$

Ans :- $x = r\cos\theta$ $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ $\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\frac{\partial y}{\partial r} = \sin\theta$ $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = \underline{r}$$

Qn :- $x = u(1-v)$ $y = uv$. Then verify $J \cdot J' = 1$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} (1-v) & -u \\ v & u \end{vmatrix}$$

$\frac{\partial x}{\partial u} = (1-v)$

$\frac{\partial x}{\partial v} = -u$

$\partial y/\partial u = v$

$\partial y/\partial v = u$

$J = u(1-v) + uv = u - uv + uv = \underline{\underline{u}}$

for finding transpose, express u & v in terms of x & y.

$\Rightarrow \quad x+y = u - uv + uv = u \quad \Rightarrow \quad u = x+y$

$v = y/u = y/x+y.$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = 1$

$\frac{\partial u}{\partial y} = 1$

$\frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2}$

$\frac{\partial v}{\partial y} = \frac{(x+y) - y}{(x+y)^2}$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{(x+y)} = \frac{+1}{u}$$

Hence $J \times J' = u \times \frac{1}{u} = \underline{\underline{1}}$

Qn 8- If $u = x^2 - 2y$, $v = x+y+z$, $w = x-2y+3z$ find $\left[\frac{\partial(x,y,z)}{\partial(u,v,w)}\right]^T$

Ans :- * $\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = -2 \quad \frac{\partial u}{\partial z} = 0$

* $\frac{\partial v}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = 1 \quad \frac{\partial v}{\partial z} = 1$

* $\frac{\partial w}{\partial x} = 1 \quad \frac{\partial w}{\partial y} = -2 \quad \frac{\partial w}{\partial z} = 3$

$$\left[\frac{\partial(x,y,z)}{\partial(u,v,w)}\right]^T = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \underline{\underline{10x+4}}$$

Qn 8 If $u = x+y+z$, $uv = y+z$, $uvw = z$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Ans :- $\frac{\partial x}{\partial u}$ [illegible]. <u>Hence</u> express x, y, z interms of u, v, w

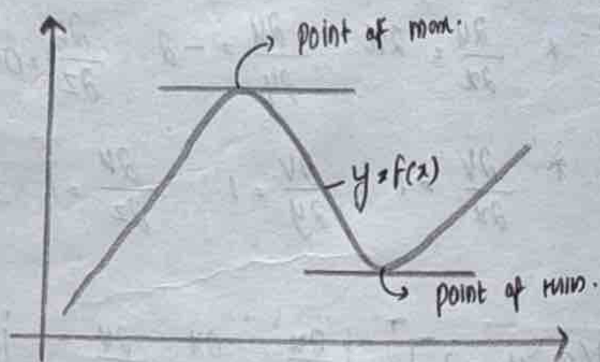
$\Rightarrow \quad z = uvw \qquad y = uv - z = uv - uvw = uv(1-w)$

$x = u - (z+y) = u - uv(1-w) - uvw = u - uv = u(1-v)$

$$\left.\begin{array}{l} * \ \frac{\partial x}{\partial u} = (1-v), \ \frac{\partial y}{\partial v} = -u, \ \frac{\partial x}{\partial w} = 0 \\ * \ \frac{\partial y}{\partial u} = v(1-w), \ \partial y/\partial v = u(1-w) \quad \partial y/\partial w = -uv \\ * \ \frac{\partial z}{\partial u} = vw \quad \frac{\partial z}{\partial v} = uw \quad \frac{\partial z}{\partial w} = uv \end{array}\right\} \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \rightarrow (1)$$

$$(1) = \begin{vmatrix} (1-v) & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} = uv \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -1 \\ vw & uw & 1 \end{vmatrix} = u^2v \begin{vmatrix} 1-v & -1 & 0 \\ v(1-w) & 1-w & -1 \\ vw & w & 1 \end{vmatrix}$$

$$= u^2v\left[(1-v)\left[(1-w)+w\right] + \left[v(1-w)+vw\right] + 0\right] = u^2v\left[(1-v)+v\right] = \underline{\underline{u^2v}}$$



* Maxima & minima of 1 Variable $f x^n$:-

(a) Point of max $\Rightarrow$ i) $dy/dx = 0$

ii) $\frac{d^2y}{dx^2} = -ve$

(b) Point of ~~minima~~ i) $dy/dx = 0$ ii) $\frac{d^2y}{dx^2} = +ve$.

ERRORS

*

Consider area $A = l \cdot b$ of a rectangle. When some error is made in l or b, it will reflect in the computed value of A. Such error's can be computed using partial derivatives.

Taylor Series Expansion Of $f(x,y)$

$$f(x+h, y+k) = f(x,y) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)^2 + \cdots$$

$$f(x+h, y+k) - f(x,y) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)^2\frac{1}{2!}$$

when h and k are small, we can neglect h^2, k^2 and higher powers.

$$\therefore \quad f(x+h, y+k) - f(x,y) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \quad \text{(approximatly)}.$$

* Here h = increment in h and k = increment in y.

So we replace h by δx and k by δy

$$\boxed{\delta f = \delta x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\delta y}$$ and given $F = f(x,y)$

We take differentials using $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

dx is taken as Δx, error in x and dy is taken as Δy, error in y.

df = Taken for Δf, error in function.

A1.

$$\boxed{\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.}$$

Note :- If x is the variable error in Δx

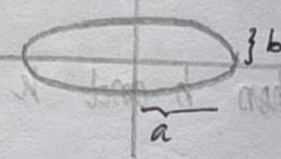
* $\frac{\Delta x}{x}$ = Relative error $\qquad \frac{\Delta x}{x} \times 100$ = Relative error.

* * To find relative error, we take logarithms and take derivatives.

* Qn :- Find the error in the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If 1% error is made in major axis and minor axis.

Ans :- Area = $\pi ab = A$ $\qquad$ given $\frac{\Delta a}{a} \times 100 = 1$

$\frac{\Delta b}{b} \times 100 = 1$



To find $\frac{\Delta A}{A} \times 100 \Rightarrow \log A = \log \pi + \log a + \log b$

$$\frac{1}{A} dA = 0 + \frac{1}{a} da + \frac{1}{b} db$$

Qn: Find the possible % error in computing resistance r from $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ if error in r_1 and r_2 are 2%.

Ans :- $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ $\qquad \frac{\Delta r_1}{r_1} \times 100 = \frac{\Delta r_2}{r_2} \times 100 = 2\%$

diffrentials :- $\frac{-1}{r^2} dr = \frac{-1}{r_1^2} dr_1 + \frac{-1}{r_2^2} dr_2 \Rightarrow \frac{1}{r} \frac{dr}{r} = \frac{1}{r_1} \frac{dr_1}{r_1} + \frac{1}{r_2} \frac{dr_2}{r_2}$

Error relation :- $\frac{1}{r}\frac{\Delta r}{r} = \frac{1}{r_1}\frac{\Delta r_1}{r_1} + \frac{1}{r_2}\frac{\Delta r_2}{r_2}$

$$\frac{1}{r}\frac{\Delta r}{r}\times 100 = \frac{1}{r_1}\frac{\Delta r_1}{r_1}\times 100 + \frac{1}{r_2}\frac{\Delta r_2}{r_2}\times 100$$

$$= \frac{2}{r_1} + \frac{2}{r_2} = 2\left[\frac{1}{r_1} + \frac{1}{r_2}\right] = 2\times \frac{1}{r}$$

$$\Rightarrow \frac{\Delta r}{r}\times 100 = 2$$ Relative

Qn :- If $PV^2 = k$ and the relative error in P & V are 0.05 & 0.025, Solve that error in k is 10 %.

Ans : $K = PV^2 \longrightarrow ①$ $\qquad \frac{\Delta P}{P} = 0.05 \qquad \frac{\Delta V}{V} = 0.025$

$\log k = \log P + 2\log V \longrightarrow ②$ $\qquad \frac{\Delta P}{P}\times 100 = 5 \qquad \frac{\Delta V}{V}\times 100 = 2.5$

differential ② $\Rightarrow$ $\frac{1}{k}dk = \frac{1}{P}dP + 2\frac{1}{V}dv$

Error relative :- $\frac{\Delta k}{k} = \frac{\Delta P}{P} + 2\frac{\Delta V}{V}$

$$\frac{\Delta k}{k}\times 100 = \frac{\Delta P}{P}\times 100 + 2\frac{\Delta V}{V}\times 100 = 5 + 2\times(2.5) = 10$$

Qn :- The work that must be done to propel a ship of displacement D for a distance s in time t in proportional to $\frac{s^2 D^{2/3}}{t^2}$. Find the increase in work done. D is increased by 1%, time is ~~decreased~~ reduced by 1 %, distance is diminished by 2 %.

Ans : $W \propto \frac{s^2 D^{2/3}}{t^2} \Rightarrow W = k\frac{s^2 D^{2/3}}{t^2}$ $\qquad \frac{\Delta s}{s}\times 100 = -2$

$\frac{\Delta t}{t}\times 100 = -1$

$\frac{\Delta W}{W}\times 100 = ?$ $\qquad \frac{\Delta D}{D}\times 100 = -1$

$$\log w = \log k + 2\log s + \frac{2}{3}\times \log D - 2\log t$$

diff :- $\frac{1}{w}dw = 0 + \frac{2}{3}\frac{dD}{D} + 2\frac{ds}{s} - \frac{2}{t}dt$

Error relation :- $\frac{\Delta W}{W}\times 100 = \frac{2}{3}\frac{\Delta D}{D}\times 100 + 2\frac{\Delta S}{S}\times 100 - 2\frac{\Delta t}{t}\times 100$

$$= 2(-2) + \frac{2}{3}\times(1) - 2(-1)$$

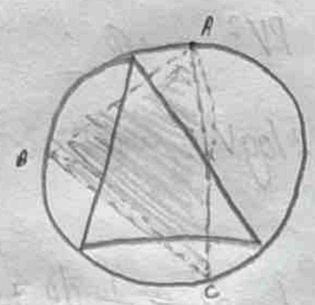
$$= \underline{\underline{-4/3}}$$

Qn:- The size of a triangle ABC vary in such a way that its circum radius remain constant. P.T $\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$.

Ans:-

Circum Circle

incircle



Relation of side, angle, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\therefore a = 2R\sin A$

$da = 2R\, d(\sin A)$

$= 2R\cos A\, dA$

$\frac{da}{\cos A} = 2R\, dA$

simly,

$\frac{db}{\cos B} = 2R\, dB$

$\frac{dc}{\cos C} = 2R\, dC$

$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} =$

$2R[dA + dB + dC]$

$= 2R\, d(A+B+C)$

$= 2R\, d(2\pi) = 2R\times 0$

$\therefore$ Error relation $\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0$.

MAXIMA & MINIMA OF FUNCTIONS OF 2 VARIABLES

Consider a function $f = f(x,y)$. $f(x,y)$ is maximum at $x=a$ & $y=b$ when $f(a+h, b+k) < f(a,b)$. Similarly f is minimum at $x=a$, $y=b$ $f(a,b) <$ $f(a+h, b+k)$ for +ve and -ve value of k.

* A point on the surface where the tangent plane is horizontal and the surface is falling for displacement in certain direction and rises for displacement in another direction is called a saddle point.

Working Rule For Maxima And Minima

Let $f(x,y)$ is the given function :-

(i) Find the partial derivatives

$$p = \frac{\partial f}{\partial x} \quad q = \frac{\partial f}{\partial y} \quad r = \frac{\partial^2 f}{\partial x^2} \quad s = \frac{\partial^2 f}{\partial x \partial y} \quad t = \frac{\partial^2 f}{\partial y^2}$$

(ii) Equate $p=0$; $q=0$ (solving for x & y)

Let $(a_1, b_1), (a_2, b_2)$ ... be the solution.

(iii) Find r, s, t at $f(a,b)$. ie value of 2^{nd} derivatives at these points.

(iv) If $rt - s^2 > 0$; $r<0$ f is maximum at (a,b)

If $rt - s^2 > 0$; $r > 0$ f is minimum at (a,b)

If $rt - s^2 < 0$ it is a saddle point.

If $rt - s^2 = 0$, the case is doubtful so that further investigation is required.

Qn8- Find the extreme points (Max, Min) $f = x^3y^2(1-x-y)$

$f = x^3y^2 - x^4y^2 - x^3y^3$

Ans: (i) $p = \frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3y^3x^2$

$q = \frac{\partial f}{\partial y} = 2yx^3 - 2x^4y - 3x^3y^2$

$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$

$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y.$

(ii) Solve ⓐ $p = 0$, ⓑ $q = 0$

$p = 0 \Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$x^2y^2[3 - 4x - 3y] = 0 \longrightarrow ①$

$q = 0 \Rightarrow 2yx^3 - 2yx^4 - 3x^3y^2 = 0 \Rightarrow x^3y[2 - 2x - 3y] = 0 \longrightarrow ②$

one chance $x = 0$ $y = 0$ OR $4x + 3y = 0$, $2x + 3y = 0$ $\Rightarrow 2x = 1$ $x = 1/2$ $y = 1/3$

Solutions are $(0,0)$ or $(1/2, 1/3)$.

(iii) Case ① $\Rightarrow (1/2, 1/3)$

$r = 6xy^2 - 12x^2y^2 - 6xy^3$

$= 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6\frac{1}{2}(1/3)^3$

$= \frac{1}{9} - \frac{1}{9} - \frac{1}{9} = \underline{\frac{-1}{9}}$

$s = 6x^2y - 8x^3y - 9x^2y^2 = 6(\frac{1}{2})^2\frac{1}{3} - 8(\frac{1}{2})^3\frac{1}{3} - 9(\frac{1}{2})^2(\frac{1}{3})^2 = \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = \underline{\frac{-1}{12}}$

$t = 2x^3 - 2x^4 - 6x^3y = 2\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)^3\frac{1}{3}$

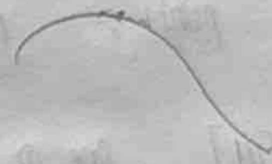
$= \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = \underline{\underline{-\frac{1}{8}}}$

Here $r < 0 \Rightarrow (r = -1/9)$ and $rt - s^2 = \frac{-1}{9} \times \frac{-1}{8} - \left(\frac{-1}{12}\right)^2$

$= \frac{-1}{72} - \frac{1}{144} = \frac{2-1}{144} = \underline{\underline{\frac{1}{144}}} > 0$

Hence $rt - s^2 > 0$ $r < 0$, f is maximum @ $(1/2, 1/3)$

$\Rightarrow f_{max} = x^3y^2[1-x-y] = \frac{1}{8} \times \frac{1}{9} \times \frac{1}{6} = \underline{\underline{1/432}}$



Case ② $\Rightarrow (0,0)$ $r = 0$ $s = 0$ $t = 0$

$\Rightarrow rt - s^2 = 0$, $r = 0$ $\Rightarrow$ Further investigation required.

Now checking the behaviour of the function @ $(0,0)$ in the neighbourhood.

So considering the line $y = x$ (pass through $(0,0)$).

$f = x^5(1-2x)$ $\Rightarrow$ i) $f(.1, .1) = f > 0$

ii) $f(-.1, -.1) = f < 0$

It is neither maximum or minimum at $(0,0)$

Qn:- $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. Find the extreme points.

Ans:- (1) $\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y = p$ $\quad r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$q = \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$ $\quad s = 4$

$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

(ii) Solve $p = q = 0 \Rightarrow$ $4y^3 + 4x - 4y = 0$ $y^3 + x - y = 0 \longrightarrow$ ②

① + ② $\Rightarrow x^3 + y^3 = 0 \Rightarrow y^3 = -x^3 \Rightarrow \underline{y = -x}$

① $\Rightarrow x^3 - x - x = 0 \Rightarrow x^3 - 2x = 0, \; x^3 = 2x \Rightarrow x = 0 / x = \sqrt{2} \pm$

when $x = 0 \Rightarrow y = 0$

$x = \pm\sqrt{2} \Rightarrow x = +\sqrt{2} \Rightarrow y = -\sqrt{2}$, $x = -\sqrt{2}$, $x = \sqrt{2}$

points :- $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

[iii] <u>Case ① :-</u> $(\sqrt{2}, -\sqrt{2})$

$r = 12x^2 - 4 = 12(\sqrt{2})^2 - 4 = 20$ (+ve)

$t = 12y^2 - 4 = 12(-\sqrt{2})^2 - 4 = 20$ (+ve)

$s = 4$ (always)

$rt - s^2 = 400 - 16 = 386 > 0$

Hence $f(x,y)$ is minimum @ $(\sqrt{2}, -\sqrt{2})$

minimum value $= -8$

<u>Case ② :</u> $(-\sqrt{2}, \sqrt{2})$

$r = 12x^2 - 4 = 12(-\sqrt{2})^2 - 4 = 20$

$t = 12y^2 - 4 = 12(\sqrt{2})^2 - 4 = 20$

$s = 4$ (always)

$rt - s^2 = 386 > 0$

Hence $f(x,y)$ is point of minimum.

<u>Case ③ :-</u> $(0,0)$

$r = -4 \quad t = -4 \quad s = 4 \quad * \; rt - s^2 = -16 + 16 = \underline{0}$

case is doubtful and further investigation required.

- $f = x^4 + y^4 - 2x^2 + 4xy - 2y^2 \Rightarrow f(0,0)$ and giving $y = x$

then $f = x^4 + x^4 - 2x^2 + 4x^2 - 2x^2 = 2x^4 > 0$

$\Rightarrow$ given ...

$f(x,y) = x^4 - 2x^2 = x^2(x^2-2)$ when x = very less (eg $= -1, 1$) $f(x,y) < 0$

Hence $f(0,0)$ is neither maximum nor minimum.

Qn: $f = 2x^2 - 4xy + y^4 - 2$. Find extreme points.

Ans:- (i) $p = \frac{\partial f}{\partial x} = 4x - 4y$ $\qquad s = \frac{\partial^2 f}{\partial x \partial y} = -4$

$q = \frac{\partial f}{\partial y} = -4x + 4y^3$ $\qquad t = \frac{\partial^2 f}{\partial y^2} = 12y^2$

$r = \frac{\partial^2 f}{\partial x^2} = 4$

Solve :-

(ii) $p = 0 = q \Rightarrow 4x - 4y = 0 \Rightarrow x = y.$

$4y^3 - 4x = 0 \Rightarrow y^3 = x$

$x^3 = x \longrightarrow x(x^2-1) = 0 \Rightarrow x = 0 / x = \pm 1$

Solutions: $(0,0), (1,1) (-1,-1)$

(iii) Case ① :- $(1,1)$

$r = 4 \quad t = 12 \quad s = -4$

$rt - s^2 = 12 \times 4 - 16 > 0$ $\qquad (1,1)$ is a point of minimum

minimum value: $2 \times 1 - 4 + 1 - 2 = \underline{\underline{-3}}$

Case ② :- $(-1,-1)$

$r = 4 \quad t = 12 \quad s = -4$ $\quad \therefore (-1,-1)$ is a point of minimum.

$rt - s^2 > 0$

min value $= \underline{\underline{-3}}$

Case ③ :- $(0,0)$ $\quad rt - s^2 = 4 \times 0 - 16 = -16 < 0$ $\quad (0,0)$ point of ~~max~~ sadelle.

~~max value :- -2~~

(1,1)	minim.	$f_{min} = 3$
(-1,-1)	minim	$f_{min} = -3$
(0,0)	saddle point	

Qn 8- Find extreme points of $f(x,y) = x^3 + y^3 - 3axy$.

Ans :- (i) $p = \frac{\partial f}{\partial x} = 3x^2 - 3ay$ $\quad s = \frac{\partial^2 f}{\partial x \partial y} = -3a$

$q = \frac{\partial f}{\partial y} = 3y^2 - 3ax$ $\quad t = \frac{\partial^2 f}{\partial y^2} = 6y$.

$r = \frac{\partial^2 f}{\partial x^2} = 6x$

(ii) $p = q = 0$ (solving) :- $\quad 3x^2 = 3ay \Rightarrow x^2 = ay \longrightarrow x^4 = a^2y^2$
$= a^2 ax$
$= ax^3$

$3y^2 = 3ax \Rightarrow y^2 = xa$

Hence $\quad x^4 - ax^3 = 0 \quad x^3[x-a] = 0 \Rightarrow x = 0 / x = a+$

points :- $(0,0)$, $(a,a) \rightsquigarrow$ only $(+a,+a)$ because $\frac{x^2}{y^2} = \frac{y}{x} \Rightarrow x^3 = y^3 \Rightarrow x = y$

(iii) Case ① :- $(0,0)$

$r = 0 \quad t = 0 \quad s = -3a \quad , \quad rt - s^2 = 0 - 9a^2 = -9a^2 < 0$

point of ~~minima~~ saddle $\Rightarrow$ ~~min value = 0~~

Case ② :- (a,a)

$r = 6a \quad t = 6a \quad s = -3a \quad , \quad rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

But 2 subcases arise here ;

Subcase ① :- $r = 6a > 0 \quad rt - s^2 > 0 \quad f$ is minimum when $a > 0$ at (a,a)

Subcase ② :⇒ $rt - s^2 < 0$, where $r = 6a < 0$ and $r < 0$

Hence f is max when $a < 0$ @ (a,a).

(a,a)	$a > 0$	f_{min}	$f_{min} = -a^3$
(a,a)	$a < 0$	f_{max}	$f_{max} = -a^3$ → but $a = -ve$ (like $(1,-1)$)
$(0,0)$		saddle point.	

~~LAGRANGE'S METHOD~~

Qn :- In a plane $\triangle ABC$, find the maximum value of $\cos A \times \cos B \times \cos C$

Ans :- $f(x,y,z) = f(A,B,C) = \cos A \cos B \cos C$

$= -\cos A \cos B \cos(A+B)$

$A+B+C = \pi$

$C = \pi - (A+B)$

$\cos C = \cos(\pi - (A+B))$

$= -\cos(A+B)$

①

$$p = \frac{\partial f}{\partial A} = -\cos B\left[\cos A(-\sin(A+B)) + \cos(A+B)\cdot(-\sin A)\right]$$

$$= \cos B\left[\sin(A+B)\cos A + \cos(A+B)\cdot \sin A\right]$$

$$= \cos B \cdot \sin(2A+B)$$

simly :

$$q = \frac{\partial f}{\partial B} = \sin(2B+A)\cos A.$$

$$r = \frac{\partial^2 f}{\partial A^2} = \cos B \cdot \cos(2A+B)\cdot 2 = 2\underline{\cos(A2+B)}\cdot \cos B$$

$$s = \frac{\partial^2 f}{\partial A \partial B} = \cos B\cdot\left(\cos(2A+B)\right) + \sin(2A+B)\cdot(-\sin B) = \cos(2A+B)\cos B - \sin(2A+B)\cdot\sin B$$

$$= \cos(2(A+B))$$

simly :-

$$t = \frac{\partial^2 f}{\partial B^2} = 2\cos(A+2B)\cdot\cos A.$$

② Solve $p = q = 0$ ⇒

$\cos B \sin(2A+B) = 0$ → $\cos B = 0$ ⇒ $B = \pi/2$

→ $\sin(2A+B) = 0$ $2A = -B$

$A = -\pi/4$

this is not possible since $f x^n$ value can't be 0.

So, $\cos A \neq \cos B \neq 0 \Rightarrow \sin(2A+B) \neq 0$ (Sum of 2 angles of a $\triangle$ never 0)

$2A+B = 180$ (can be possible).

$2A+B = 180 \longrightarrow ①$

$\sin(A+2B) = 0 \longrightarrow A+2B = 180 \longrightarrow ②$

This implies $A=B=\pi/3$ (only possibility).

At $(\pi/3, \pi/3) \Rightarrow$

$$\left.\begin{aligned} r &= 2\cos 180 \cos \pi/3 = -2\times 1/2 = -1 \\ t &= +2\cos 180 \cos \pi/3 = -1 \\ s &= \cos(2(A+B)) = \cos 240 = -1/2 \end{aligned}\right\} \begin{aligned} rt - s^2 &= 3/4 > 0 \\ r &< 0 \end{aligned}$$

ie f is maximum @ $A=B=\pi/3$ $\quad C = 2\times[A+B] = \frac{2\pi}{3}$.

Max value of $\cos A \cos B \cos C = \cos\pi/3 \cos\pi/3 \cos\frac{2\pi}{3} = \frac{-1}{2}\times -1/2 \times 1/2 = \underline{1/8}$

LAGRANGE'S METHOD OF UNDETERMIND MULTIPLIES *

To find the maximum value of a fx^n or min value of a fx^n $f=f(x,y,z)$ Subjected to $\phi(x,y,z)=0$, we take $F = f + \lambda\phi$

$= f(x,y,z) + \lambda(\phi(x,y,z))$.

The necessary cond^n for max. or min is ; $\frac{\partial F}{\partial x}=0 \quad \frac{\partial F}{\partial y}=0 \quad \frac{\partial F}{\partial \lambda}=0 \quad \frac{\partial F}{\partial z}=0$

* Point of max. or minimum depending with physical condition.

λ = Lagrange's multiplier.

Qn:- Find the minimum Value of $x^2+y^2+z^2$ Subjected to Cond^n $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

Ans:-

$$f = x^2+y^2+z^2 \qquad \phi = 1/x + 1/y + 1/z - 1$$

$$F = f + \lambda\phi = x^2+y^2+z^2 + \lambda\left[1/x + 1/y + 1/z - 1\right]$$

★ $\frac{\partial F}{\partial x} = 2x + \lambda\left(-1/x^2\right)$ $\qquad \frac{\partial F}{\partial y} = 2y + \lambda\left(\frac{-1}{y^2}\right)$ $\qquad \frac{\partial F}{\partial z} = +2z + \lambda\left(\frac{-1}{z^2}\right)$

$\frac{\partial F}{\partial \lambda} = \left(1/x + 1/y + 1/z - 1\right)$

necessory Cond^n $\Rightarrow \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = \frac{\partial F}{\partial \lambda} = 0$ $\Rightarrow 2x = \frac{+\lambda}{x^2}$, $2y = \frac{+\lambda}{y^2}$ $\quad 2z = \frac{+\lambda}{z^2}$

$\Rightarrow 2x^3 = \lambda \Rightarrow x = (\lambda/2)^{1/3}$, $\quad y = (\lambda/2)^{1/3}$, $\quad z = \left[\lambda/2\right]^{1/3}$

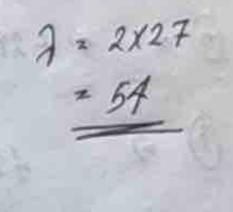
$\frac{\partial f}{\partial \lambda} = 0 \Rightarrow \left(\frac{2}{\lambda}\right)^{1/3} + \left(\frac{2}{\lambda}\right)^{1/3} + \left(\frac{2}{\lambda}\right)^{1/3} = 1$

$3\left(2/\lambda\right)^{1/3} = 1 \longrightarrow \frac{2}{\lambda} = (1/3)^3 = \frac{1}{27}$ $\qquad \lambda = 2\times 27 = \underline{\underline{54}}$

Hence $x = \left(\frac{54}{2}\right)^{1/3} = 3 = y = z \Rightarrow x = y = z = 3$//

★ Minimum value of $x^2+y^2+z^2 = 3\times 9 = 27$//

Qn :- A rectangular box opened @ top is to have a Volume 32 Cub Cm. Find dimensions of box req. least material for it's Construction.



Ans :-

Given $x \cdot y \cdot z = 32$

To find x, y, z such that Surf. area min.

Surf. are of this box = ~~xyz~~ $xy + 2xz + 2yz$.

ie $f = xy + 2xz + 2yz$ $\quad \phi = xyz - 32$

$F = xy + 2xz + 2yz + \lambda[xyz - 32]$

① $\frac{\partial F}{\partial x} = y + 2z + \lambda[yz] = 2z + \lambda[yz] + y$ $\quad \frac{\partial F}{\partial z} = 2x + 2y + \lambda(xy)$

$\frac{\partial F}{\partial y} = x + 2z + \lambda[xz]$ $\quad \frac{\partial F}{\partial \lambda} = (xyz - 32).$

* $y + 2z + \lambda yz = 0 \rightarrow ①$

$x + 2z + \lambda xz = 0 \rightarrow ②$

$2x + 2y + \lambda xy = 0 \rightarrow ③$

$xyz - 32 = 0 \rightarrow ④$

from ④ : $xyz = 32$

①$x \Rightarrow xy + 2zx + \lambda(xyz) = 0 \rightarrow ⑤$

② $y \Rightarrow xy + 2yz + \lambda(xyz) = 0 \rightarrow ⑥$

③ $z \Rightarrow 2xz + 2yz + \lambda(xyz) = 0 \rightarrow ⑦$

⑤ - ⑥ = $2xz - 2yz = 0 \longrightarrow x = y$

⑤ $\Rightarrow x^2 + 2xz + \lambda x^2 z = 0 \Rightarrow x[x + 2z + \lambda xz] = 0$ (x cant be 0)

③ $\Rightarrow 4x + \lambda x^2 = 0 \quad x[4 + \lambda x] = 0 \Rightarrow 4 = -\lambda x \quad \lambda = -4/x$

④ $\Rightarrow xyz = 32 \Rightarrow z = \frac{32}{xy} = \frac{32}{\frac{-4}{\lambda} \times \frac{-4}{\lambda}} = \underline{\underline{2\lambda^2}}$

$x + 2z + \lambda xz = 0$ (only possib) $\Rightarrow \frac{-4}{\lambda} + 2z + \lambda\left(\frac{-4}{\lambda}\right)z = 0 \Rightarrow \lambda = \frac{-2}{z}$

$\Rightarrow \frac{-4}{x} = \frac{-2}{z} \Rightarrow z = \frac{x}{2} \Rightarrow 2z = x$

ie $xyz = 32 \Rightarrow x \cdot x \cdot \frac{x}{2} = 32 \Rightarrow x^3 = 64 \quad x = \underline{4}$

ie dimensions $\Rightarrow$ 4 4 2 //.

* Maclaurian expansion of $y = \tan^{-1}x$, $y = \log(4x+3)$ to find $\log 7$.

Ans:- $y = \tan^{-1}x \Rightarrow y_1 = \frac{1}{1+x^2} \Rightarrow (1+x^2)y_1 = 0$

diff $\Rightarrow y_2(1+x^2) + y_1 \times 2x = 0$

apply Leibnitz rule :- $\left[y_{n+2}(1+x^2) + n\,y_{n+1}\,2x + n(n-1)x\right] + 2\left[y_{n+1}x + ny_1\right] = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + n^2x - nx + 2xy_{n+1} + 2ny_1 = 0$

$= (1+x^2)y_{n+2} + 2xy_{n+1}(n+1) + (n^2+n)y_n = 0 \longrightarrow ①$

Putting $x=0$ in ① $\Rightarrow y_{n+2} + (n^2+n)y_n = 0 \Rightarrow y_{n+2} = -(n^2+n)y_n \longrightarrow ②$

$y(0) = \tan^{-1}(0) = 0 \qquad y_1(0) = \frac{1}{1+0} = 1 \qquad y_2(0) = 0$

now puting values for "n" in ② $\rightarrow$

$y_3 = -2y_1 = -2 \quad (n=1)$

$y_4 = -6y_2 = 0 \quad (n=2)$

$y_5 = -12y_3 = 24$

$f(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0)$

$f(x) = 0 + x \times 1 + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}[0] + \frac{x^5}{5!}(24) + \dots$

$= x - x^3/3 + \frac{x^5}{5} + \dots$

★ $\boxed{\tan^{-1}x = x - x^3/3 + x^5/5}$

Qn :- Find the greatest and least distances of $(3,4,12)$ from the sphere $x^2+y^2+z^2=1$.

Ans :- Consider $(x\ y\ z)$ be any point on the sphere, Hence distance from $(3\ 4\ 12)$

$$d = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$$

Consider $f = D^2 = (x-3)^2+(y-4)^2+(z-12)^2 \longrightarrow ①$ such that $(x^2+y^2+z^2)=1 \longrightarrow ②$

$$F = f+\lambda\phi = (x-3)^2+(y-4)^2+(z-12)^2 + \lambda[x^2+y^2+z^2-1] = 0$$

* $\frac{\partial F}{\partial x} = 2(x-3)+2\lambda x = 0$ * $\frac{\partial F}{\partial y} = 2(y-4)+2\lambda y = 0$ * $\frac{\partial F}{\partial z} = 2(z-12)+2\lambda z = 0$

* $\frac{\partial F}{\partial \lambda} = (x^2+y^2+z^2-1) = 0$

@ $\frac{\partial F}{\partial x} = 0 \rightsquigarrow 2(x-3)+2\lambda x = 0$

$(x-3)+\lambda x = 0$

$x(1+\lambda) = 3$

$\Rightarrow x = \frac{3}{1+\lambda}$

Simly $\frac{\partial F}{\partial y} = 0$ $\quad y = \frac{4}{1+\lambda}$

$\frac{\partial F}{\partial z} = 0 \rightarrow z = \frac{12}{1+\lambda}$

But $x^2+y^2+z^2 = 1$

$\rightarrow \frac{1}{(1+\lambda)^2}[9+16+144] = 1 \rightarrow 1+\lambda = \sqrt{169} = \pm 13$ $\quad \lambda = 12$ OR $\lambda = -14$

* When $\lambda = 12 \longrightarrow (x\ y\ z) = (3/13,\ 4/13,\ 12/13) = A$

$\lambda = -14 \longrightarrow (x\ y\ z) = \left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right) = B$

given point $= (3\ 4\ 12) = P \longrightarrow$ $AP = \sqrt{(3-3/13)^2+(4-4/13)^2+(12-12/13)^2} = \sqrt{144}$

$= \underline{12 \text{ unit}}$

$BP = \sqrt{(3+3/13)^2 + (4+4/13)^2 + (12+12/13)^2} = \sqrt{196} = \underline{14}$

Minimum dist = 12 unit , Max dist = 14 unit

Qns- Dividing 24 in to 3 parts such that the continued product of 1^{st}, squ. of 2^{nd}, cube of 3^{rd} may be maximum.

Ans : Given :- $x+y+z = 24$ and xy^2z^3 = Maximum.

$f = xy^2z^3 = \text{Max}^m$ $\phi = x+y+z-24 = 0$

$F = f + \lambda\phi = xy^2z^3 + \lambda[x+y+z-24] =$

$\frac{\partial F}{\partial x} = y^2z^3 + \lambda = 0$, $\frac{\partial F}{\partial y} = 2xyz^3 + \lambda = 0$, $\frac{\partial F}{\partial z} = 3xy^2z^2 + \lambda = 0$, $\frac{\partial F}{\partial \lambda} = [x+y+z-24]$

$\Rightarrow \lambda = -y^2z^3 \longrightarrow ①$ $\lambda = -2xyz^3 \longrightarrow ②$ $\lambda = -3xy^2z^2 \longrightarrow ③$

$+y^2z^3 = +3xy^2z^2 \Rightarrow z = 3x$

$① \cdot ② \Rightarrow +y^{\cancel{2}}\cancel{z^3} = +2x\cancel{y}\cancel{z^3}$

$y = 2x$

$(x, y, z) = (x, 2x, 3x) \rightsquigarrow x+y+z = 24$

$6x = 24 \longrightarrow x = 4$

ie Values = 4, 8, 12

Max. value $= 4 \times 64 \times 1728 = \underline{\underline{4.42 \times 10^5}}$

5-1

MODULE - IV

Consider the eq^n, $y = mx + a/m$, we know that this eq^n represents tng^s to the parabola $y^2 = 4ax$ for diffrent values of m.

Therfore $y = mx + a/m$ may be called one parameter family of straight line where parameter is "m" and so that every member of this family touches the parameter $y^2 = 4ax$. ie parabola is called <u>envolop</u> of of <u>one parameter family</u> of straight lines $y = mx + a/m$.

* In general :- comisider a oneparameter family $f(x, y, \alpha) = 0$; "α" being the parameter.

So an <u>envelop</u> is defined as a curve such that every member of the one parameter family $f(x, y, \alpha) = 0$ touches it.

Result :- If $f(x, y, \alpha) = 0$ be a one parameter family. Envelop is obtained by eliminating α between,

$$f(x\ y\ \alpha) = 0 \quad \& \quad \frac{\partial F}{\partial \alpha} = 0$$

Examples :-

1) Find the envelop of the family $y = mx + a/m$

Ans :- Given $y = mx + a/m$ (m = parameter). $\longrightarrow$ ①

part. diff. w.r.to "m" $\Rightarrow$ $0 = x + \frac{-a}{m^2}$ $\Rightarrow$ $\frac{a}{m^2} = x$

hence $m = \sqrt{a/x}$ now ① $\Rightarrow$ $y = \sqrt{\frac{a}{x}} \cdot x + \frac{a}{\sqrt{a/x}} = \sqrt{ax} + \sqrt{ax}$

we got $y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$ (envelop of $y = mx + a/m$).

② Envelop of $y = mx \pm \sqrt{a^2m^2 - b^2}$

Ans:- Given $y = mx \pm \sqrt{a^2m^2 - b^2}$ (m = parameter) $\longrightarrow$ ①

part. diff. w.r. to m $\Rightarrow$ $0 = x \pm \frac{1}{\cancel{2}\sqrt{a^2m^2 - b^2}}(\cancel{2}a^2m) \longrightarrow$ ②

ie we have $x = \pm \frac{a^2m}{\sqrt{a^2m^2 - b^2}} \rightsquigarrow x^2 = \frac{a^4m^2}{(a^2m^2 - b^2)}$

$a^2m^2x^2 - b^2x^2 = a^4m^2 \rightsquigarrow m^2(a^2x^2 - a^4) = b^2x^2 \rightsquigarrow m^2 = \frac{b^2x^2}{a^2x^2 - a^4} \longrightarrow$ ③

$a^2m^2 = \frac{b^2x^2}{x^2 - a^2} \rightsquigarrow a^2m^2 - b^2 = \frac{b^2x^2}{x^2 - a^2} - b^2$

$= \frac{\cancel{b^2x^2} - \cancel{b^2x^2} + a^2b^2}{x^2 - a^2}$

$= \frac{a^2b^2}{x^2 - a^2}$

now substituting in ① $\Rightarrow$ $y = mx \pm \sqrt{a^2m^2 - b^2} \Rightarrow y - mx = \pm\sqrt{a^2m^2 - b^2}$

on squaring $\Rightarrow$ $(y - mx)^2 = (a^2m^2 - b^2)$

$\Rightarrow \left(y - \frac{bx}{a\sqrt{x^2 - a^2}} \times x\right)^2 = \frac{a^2b^2}{x^2 - a^2}$

$= \frac{(ay\sqrt{x^2 - a^2} - bx^2)^2}{a^2(x^2 - a^2)} = \frac{a^2b^2}{(x^2 - a^2)} \Rightarrow$

ഇങ്ങനെ ചെയ്താൽ ബുദ്ധിമുട്ടാണ് simplification

(3) => $m = \dfrac{bx}{a\sqrt{x^2-a^2}} \rightsquigarrow$ (1) => $y = mx \pm \sqrt{a^2m^2-b^2}$

$$= \frac{bx}{a\sqrt{x^2-a^2}}\, x \pm \frac{ab}{\sqrt{x^2-a^2}}$$

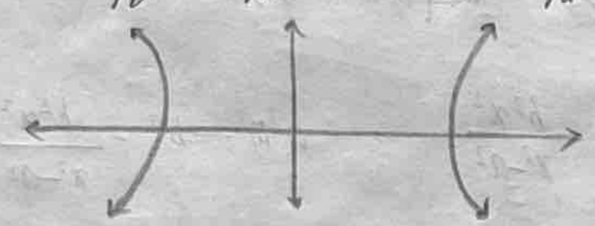
$$= \frac{b}{\sqrt{x^2-a^2}}\left[\frac{x^2}{a} \pm a\right]$$

$$= \frac{b}{\sqrt{x^2-a^2}}\left[\frac{x^2 \pm a^2}{a}\right] \longrightarrow (4)$$

now we are taking -ve value for the easier simplification

(4) => $y = \dfrac{b}{\sqrt{x^2-a^2}}\left[\dfrac{x^2-a^2}{a}\right] \rightsquigarrow ay = b\sqrt{x^2-a^2} \rightsquigarrow a^2y^2 = b^2x^2 - b^2a^2$

÷ with a^2b^2 => $y^2/b^2 = x^2/a^2 - 1 \Rightarrow x^2/a^2 - y^2/b^2 = 1$



Note- If $f(x,y,\alpha) = 0$ is $A\alpha^2 + B\alpha + C = 0 \longrightarrow (1)$

diff w.r.t. α => $2A\alpha + B = 0$ => (1) => $A\left[\frac{-B}{2A}\right]^2 + B\left[\frac{-B}{2A}\right] + C = 0$

on simplifing : $B^2 - 4AC = 0$.

So if the family is $A\alpha^2 + B\alpha + C = 0 \rightsquigarrow$ The Envelop is $\boxed{B^2 - 4AC = 0}$

Qn (2) : The qn can be done in a more easier way with (this) eqn.

Ans :- $y = mx \pm \sqrt{a^2m^2-b^2}$ => $(y - mx) = \pm\sqrt{a^2m^2-b^2}$

$$(y-mx)^2 = (a^2m^2-b^2)$$

$y^2 - 2ymx + m^2x^2 = a^2m^2 - b^2$

$m^2(x^2-a^2) + (-2xy)m + (y^2+b^2) = 0 \longrightarrow$ now envelop $\Rightarrow B^2 - 4AC = 0$

$A\alpha^2 + B\alpha + C = 0$

$(-2xy)^2 - 4(x^2-a^2)(y^2+b^2) = 0 \longrightarrow$ on simplifying $\underline{x^2/a^2 - y^2/b^2 = 1}$

HW *

③ Envelop of $y = mx \pm \sqrt{a^2m^2+b^2}$

Ans :- $y = mx \pm \sqrt{a^2m^2+b^2} \Rightarrow (y-mx) = \pm\sqrt{a^2m^2+b^2}$

$(y-mx)^2 = (a^2m^2+b^2) \Rightarrow y^2 + m^2x^2 - 2mxy - a^2m^2 - b^2 = 0$

Hence we get :- $m^2[x^2-a^2] + (-2xy)m + (y^2-b^2) = 0$

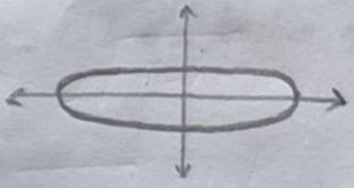
in the form :- $A\alpha^2 + B\alpha + C = 0$

Hence envelop $\Rightarrow B^2 = 4AC$

$\cancel{4}x^2y^2 = \cancel{4}(x^2-a^2)(y^2-b^2)$

$\cancel{x^2y^2} = \cancel{x^2y^2} - x^2b^2 - y^2a^2 + a^2b^2 \Rightarrow a^2b^2 = x^2b^2 + y^2a^2 \longrightarrow ①$

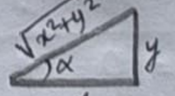
dividing ① with $a^2b^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ellipse)



Qn :- Find the envelop of $x\cos\alpha + y\sin\alpha = 1$; α = parameter.

Ans :- $x\cos\alpha + y\sin\alpha = 1 \longrightarrow ①$

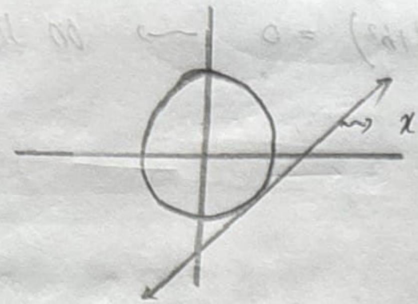
Par. Diff w.r.t. m $\Rightarrow -x\sin\alpha + y\cos\alpha = 0 \Rightarrow x\sin\alpha = y\cos\alpha \Rightarrow \tan\alpha = y/x$



$\Big\}$ $\sin\alpha = \frac{y}{\sqrt{x^2+y^2}}$ $\cos\alpha = \frac{x}{\sqrt{x^2+y^2}}$

① $x\cdot\frac{x}{\sqrt{x^2+y^2}} + y\cdot\frac{y}{\sqrt{x^2+y^2}} = 1 \Rightarrow \frac{x^2+y^2}{\sqrt{x^2+y^2}} = 1$

ie $x^2+y^2 = 1$ ⇒ ie $x\cos\alpha + y\sin\alpha = 1$ is a tangent to unit circle.



$x\cos\alpha + y\sin\alpha = 1$

Qn:- Find envelop of $y = mx - 2am - am^3$

Ans:- $y = mx - 2am - am^3 \longrightarrow ①$

Partial diff. w.r.to m:- $\partial y/\partial m = x - 2a - 3am^2 = 0 \Rightarrow 3am^2 = x - 2a$

$$m = \sqrt{\frac{x-2a}{3a}}$$

① ⇒ $y = m(x-2a) - am^3 = \sqrt{\frac{x-2a}{3a}}(x-2a) - a\left(\sqrt{\frac{x-2a}{3a}}\right)^3$

$$= \frac{(x-2a)^{3/2}}{\sqrt{3a}} - \cancel{a}\,\frac{(x-2a)^{3/2}}{3\cancel{a}\sqrt{3a}}$$

$$y = \frac{3(x-2a)^{3/2} - (x-2a)^{3/2}}{3\sqrt{3a}}$$

ie $3\sqrt{3a}\,y = 2(x-2a)^{3/2} \Rightarrow \underline{27ay^2 = 4(x-2a)^3}$ is envelop.

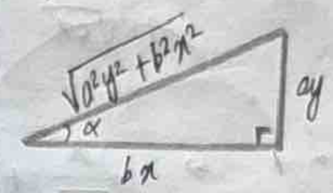
normal to $y^2 = 4xa$

Qn:- Find envelop of $x/a\cos\alpha + y/b\sin\alpha = 1$, α = parameter.

Ans:- $x/a\sin\alpha + y/b\sin\alpha = 1 \longrightarrow ①$

Partial diff. w.r.to α $\Rightarrow$ $-\frac{x}{a}\sin\alpha + \frac{y}{b}\cos\alpha = 0$ $\Rightarrow$ $\frac{y}{b}\cos\alpha = \frac{x}{a}\sin\alpha$

$$\tan\alpha = \frac{ay}{bx}$$



ie $\sin\alpha = \frac{ay}{\sqrt{a^2y^2+b^2x^2}}$, $\cos\alpha = \frac{bx}{\sqrt{b^2x^2+a^2y^2}}$, now putting in ① $\Rightarrow$

$$\frac{x}{a}\cdot\frac{bx}{\sqrt{b^2x^2+a^2y^2}} + \frac{y}{b}\cdot\frac{ay}{\sqrt{b^2x^2+a^2y^2}} = 1$$

$$\frac{b^2x^2 + a^2y^2}{ab\sqrt{b^2x^2+a^2y^2}} = 1$$

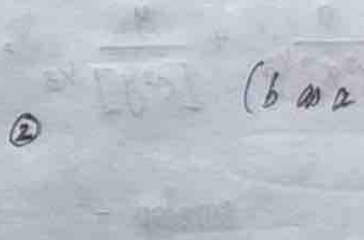
ie $b^2x^2 + a^2y^2 = a^2b^2$ $\Rightarrow$ $\div a^2b^2$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ellipse)

Qn 8 Find envelop of the family of tangents $\frac{x}{a} + \frac{y}{b} = 1$ where

i) $a+b = c$, c, const

ii) $a^2+b^2 = c^2$, c, const

iii) $ab = c^2$, c, cost.



Ans :- ① $\frac{x}{a} + \frac{y}{b} = 1$ → ① and $a+b = c$ → ② (b as a function of a)

Diff. partially ① & ② w.r.t. "a" $\Rightarrow$ ① $\Rightarrow$ $\frac{-x}{a^2} + \frac{-y}{b^2}\frac{db}{da} = 0$ → ③

② $\Rightarrow$ $1 + \frac{db}{da} = 0$ → ④

③ $\Rightarrow$ $-\frac{x}{a^2} + -\frac{y}{b^2}(-1) = 0$ $\Rightarrow$ $\frac{x}{a^2} = \frac{y}{b^2}$

$\Rightarrow$ $\frac{x/a}{a} = \frac{y/b}{b}$ (use componendo divi)

$\Rightarrow$ $\frac{x/a + y/b}{a+b} = \frac{1}{c}$

Hence $\frac{x}{a^2} = \frac{y}{b^2} = \frac{1}{c} \longrightarrow$ $a^2 = xc$, $b^2 = cy$

$a = \sqrt{cx}$ $\quad b = \sqrt{cy}$

① $\Rightarrow$ $x/a + y/b = 1 \Rightarrow \frac{x}{\sqrt{c\,x}} + \frac{y}{\sqrt{cy}} = 1 \Rightarrow \underline{\underline{\sqrt{x} + \sqrt{y} = \sqrt{c}}}$

ii) $x/a + y/b = 1$ and $a^2 + b^2 = c^2$

$\quad\;\; \hookrightarrow ①$ $\qquad\qquad\qquad \hookrightarrow ②$

Part. diff. of ① & ② w.r.to a (b is a fx^n of a) } ① $\Rightarrow$ $\frac{-x}{a^2} + \frac{-y}{b^2}\, db/da = 0 \longrightarrow ③$

② $\Rightarrow$ $2a + 2b\, db/da = 0 \Rightarrow \frac{db}{da} = -a/b \rightarrow ④$

Hence ③ $\Rightarrow$ $\frac{-x}{a^2} + \frac{-y}{b^2} \cdot \frac{-a}{b} = 0$ $\qquad \frac{ay}{b^3} = \frac{x}{a^2} \Rightarrow \frac{x}{a^3} = \frac{y}{b^3}$

$\hookrightarrow ⑤$

apply compon. divid. law: ⑤ $\frac{x/a + y/b}{a^2 + b^2} = \frac{1}{c^2} \Rightarrow \frac{x}{a^3} = \frac{y}{b^3} = \frac{1}{c^2}$

from this $a = [c^2 x]^{1/3}$ $\quad b = [c^2 y]^{1/3}$

① $\Rightarrow$ $\frac{x}{[c^2x]^{1/3}} + \frac{y}{[c^2y]^{1/3}} = 1 \Rightarrow \frac{x^{2/3}}{c^{2/3}} + \frac{y^{2/3}}{c^{2/3}} = 1$

envelop = $\underline{x^{2/3} + y^{2/3} = c^{2/3}}$

iii) $x/a + y/b = 1 \longrightarrow ①$ and $ab = c^2 \longrightarrow ②$

part. diff. of ① & ② w.r.to a since b = fx^n of a } ① $\Rightarrow$ $\frac{-x}{a^2} + \frac{-y}{b^2}\frac{db}{da} = 0 \longrightarrow ③$

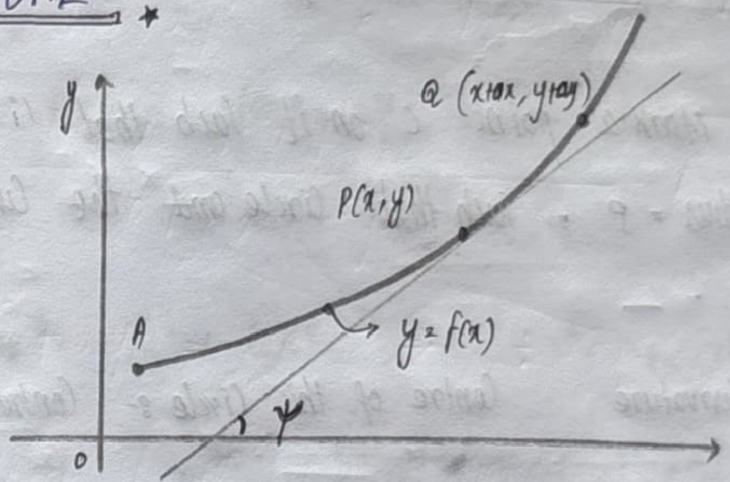
② $\frac{a\,db}{da} + b = 0 \Rightarrow \frac{db}{da} = \frac{-b}{a}$

③ $\Rightarrow \quad \frac{-x}{a^2} + \frac{-y}{b^2}\left(\frac{-b}{a}\right) \Rightarrow \frac{x}{a^2} = \frac{y}{ab} \Rightarrow \frac{x}{a} = \frac{y}{b}$

ie Comp. Divid. rule $\Rightarrow \quad \frac{x/a + y/b}{2} = \frac{1}{2} \quad x/a = y/b = 1/2$

Hence $\Rightarrow \quad a = 2x, \; b = 2y \longrightarrow$ ② $\Rightarrow \quad \underline{\underline{4xy = c^2}} \longrightarrow$ Rectangular Hyperbo *

CURVATURE *



Consider $y = f(x)$. Let A be a fixed point on the Curve. Let $P(x,y)$ & $Q(x+\Delta x, y+\Delta y)$ be 2 neighbouring (closer) point on $y = f(x)$. Let the arc length $AP = s$ and arclength $AQ = s + \Delta s$.

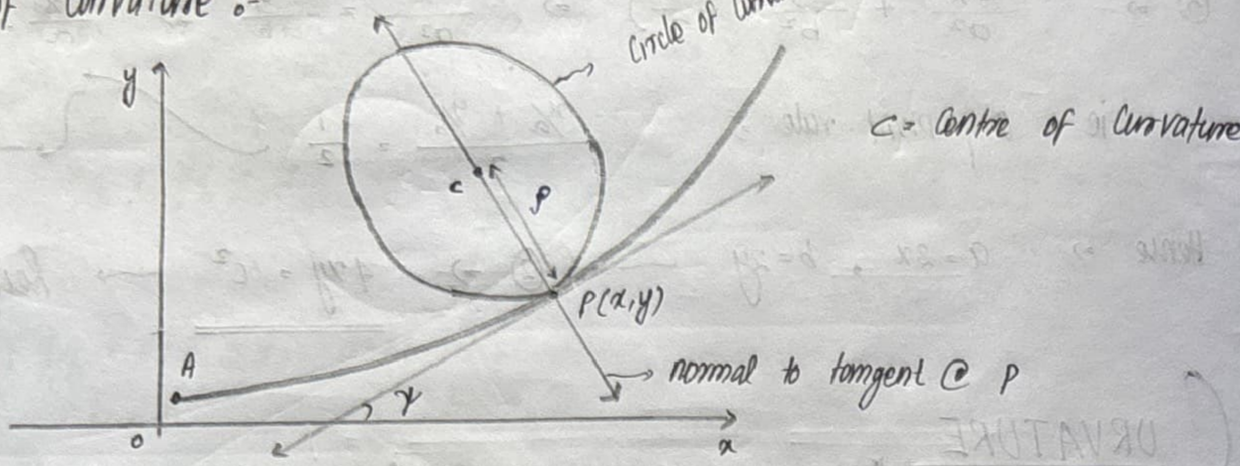
Let tangent at P makes an angle Ψ with ox. Then $\frac{d\Psi}{ds}$ is called Curvature and it is denoted by K (kappa).

$$\boxed{K = \frac{d\Psi}{ds}}$$

Radius of Curvature :- Reciprocal of the curvature

$$\boxed{\rho = \frac{1}{K} = \frac{1}{d\Psi/ds} = \frac{ds}{d\Psi}}$$

- Centre of Curvature :-



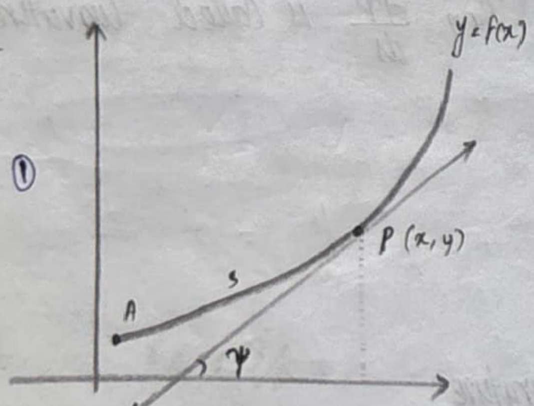
Consider the normal at "P". Mark a point "C" on it such that (i) $CP = \rho$. Draw a circle at C and radius $= \rho$, such that circle and the curve lie on the same side of the tangent.

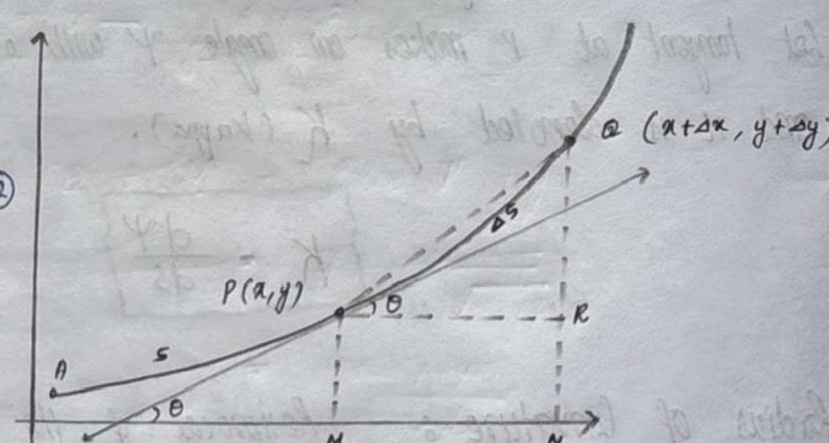
* This Circle = Circle of Curvature Centre of this Circle :- Centre of Curvature.

Prove :-

① Radius of Curvature, $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$ where $y_1 = \dfrac{dy}{dx}$, $y_2 = \dfrac{d^2y}{dx^2}$ at $P(x,y)$

Ans :-

① 

② 

① Curvature $= \dfrac{d\psi}{ds}$, $K = \dfrac{ds}{d\psi}$ $\longrightarrow$ ①

$\tan\psi$ = slope of the tangent $= dy/dx = y_1$

$\Rightarrow \psi = \tan^{-1}(y_1)$

Hence $\frac{d\psi}{dx} = \frac{d}{dx}(\tan^{-1} y_1) = \frac{1}{1+y_1^2}\frac{d}{dx}(y_1) \Rightarrow \frac{d\psi}{dx} = \frac{y_2}{1+y_1^2}$

now ⇒ ① ⇒ $\rho = \frac{ds}{d\psi} = \frac{ds}{dx}\cdot\frac{dx}{d\psi}$

$= \frac{ds/dx}{d\psi/dx} = \frac{ds/dx}{\frac{y_2}{1+y_1^2}} = \frac{1+y_1^2}{y_2}\cdot\frac{ds}{dx} \longrightarrow$ ②

② Arc length $PQ = \Delta s$, Let chord $PQ = \Delta c$

From $\triangle PQR$, $\tan\theta = \frac{QR}{PR} = \frac{QN - PM}{MN} = \frac{QN - PM}{ON - OM} = \frac{y+\Delta y - y}{x+\Delta x - x}$

$= \Delta y/\Delta x$

Also $PQ^2 = PR^2 + QR^2 \Rightarrow (\Delta c)^2 = (\Delta x)^2 + (\Delta y)^2$

$\div$ with $\Delta x^2 \Rightarrow \left(\frac{\Delta c}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2 \longrightarrow$ ③

As $\Delta x \to 0$:- $(Q \to P)$ chord $PQ \longrightarrow$ tangent @ P

$\theta \longrightarrow \psi$

$\Delta c \longrightarrow \Delta s$

now ③ ⇒ $\lim_{x\to 0}\left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \lim_{x\to 0}\left(\Delta y/\Delta x\right)^2$

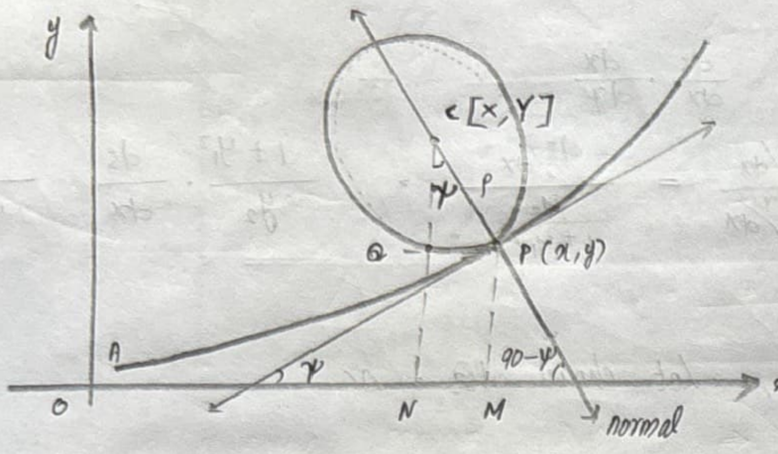
$[ds/dx]^2 = 1 + [dy/dx]^2 = 1 + y_1^2 \Rightarrow \frac{ds}{dx} = \sqrt{1+y_1^2}$

Hence eq^n ② ⇒ $\rho = \frac{1+y_1^2}{y_2}\cdot\frac{ds}{dx} = \frac{1+y_1^2}{y_2}\cdot\sqrt{1+y_1^2}$

$= \frac{(1+y_1^2)^{3/2}}{y_2}$

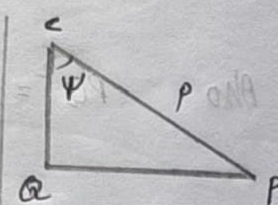
$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}$$

(2) To Find the Centre of Curvature $c[X, Y]$



Ans: Let $P = (x, y)$, ρ = radius of Curvature, $C = [X, Y]$ = Centre of Curvature

$$X = ON = OM - NM = x - PQ \longrightarrow (1)$$



$\sin\psi = \frac{PQ}{\rho}$

$\cos\psi = \frac{CQ}{\rho}$

$$* \quad \sin\psi = \frac{PQ}{\rho} = \cos\psi \cdot \tan\psi = \frac{1}{\sec\psi} \cdot \tan\psi = \frac{1}{\sqrt{1+\tan^2\psi}} \cdot \tan\psi$$

$$(1) \Rightarrow \quad X = x - PQ = x - \rho\sin\psi = x - \frac{(1+y_1^2)^{3/2}}{y_2} \frac{y_1}{\sqrt{1+y_1^2}}$$

Hence x-Coordinate :- $\quad X = x - \frac{y_1}{y_2}(1+y_1^2)$

Now, $\quad Y = CN = CQ + QN = CQ + PM = \rho\cos\psi + y$

$$= y + \rho\frac{1}{\sqrt{1+y^2}}$$

$$= y + \frac{(1+y_1^2)}{y_2}$$

Hence $\boxed{c(X, Y) = \left(x - \frac{y_1}{y_2}(1+y_1^2),\ y + \frac{(1+y_1^2)}{y_2}\right)}$

Qn :- Find radius of Curvatures of following :-

(i) $y^2 = 4ax$ at $(at^2, 2at)$

Ans :- $y^2 = 4ax \Rightarrow 2y \, dy/dx = 4a \Rightarrow y' = \frac{2a}{y}$

$$\frac{d^2y}{dx^2} = 2a \cdot \frac{-1}{y^2} \frac{dy}{dx} = \frac{-2a}{y^2} y'$$

@ $(at^2, 2at) \Rightarrow$ $y_1 = \frac{2a}{2at} = \frac{1}{t}$

$$y_2 = \frac{-2a}{4a^2t^2} \cdot \frac{1}{t} = \frac{-1}{2at^3}$$

Hence : $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1/t^2)^{3/2}}{-1/2at^3} = \frac{-2a\cancel{t^3} \cdot (1+t^2)^{3/2}}{\cancel{t^3}} = \underline{\underline{-2a(1+t^2)^{3/2}}}$

(ii) $xy = c^2$ at point (c, c)

Ans :- $xy_1 + y = 0 \Rightarrow dy/dx = \frac{-y}{x}$

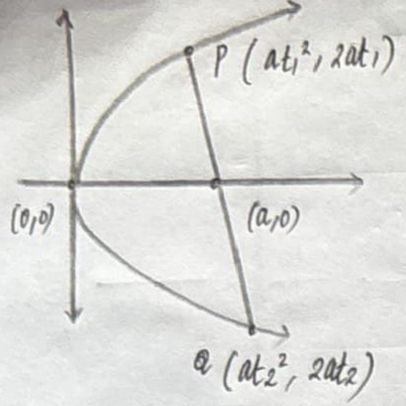
$$\frac{d^2y}{dx^2} = \cancel{(-y) \cdot (-1/x^2)} = -\left[\frac{x \, dy/dx - y}{x^2}\right]$$

$\therefore$ $y_1 (c,c) = \frac{-c}{c} = -1$ $\quad$ $y_2 = -\left[\frac{-c-c}{c^2}\right] = 2/c$

$\therefore$ $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{c}{2} \cdot 2^{3/2} = \underline{\underline{\sqrt{2}c}}$

(iii) If ρ_1 and ρ_2 are radius of Curvature at the ends of focal chord of $y^2 = 4ax$, P.T $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$

Ans : Any chord passing through focus : focal chord



Let "PQ" be a focal chord

Eqn of PQ :- $y - y_1 = \left[\dfrac{y_2 - y_1}{x_2 - x_1}\right](x - x_1)$

$\therefore\ y - 2at_1 = \dfrac{2at_2 - 2at_1}{at_2^2 - at_1^2} \cdot (x - at_1^2)$

$\Rightarrow\ \boxed{y - 2at_1 = \dfrac{2}{t_2 + t_1}(x - at_1^2)}$

Hence :- "PQ" is passing through $(a,0)$ $\Rightarrow\ 0 - 2at_1 = \dfrac{2}{t_1 + t_2}(a - at_1^2)$

on solving $\boxed{t_1 \cdot t_2 = -1}$

$*\ \ y^2 = 4ax \ \Rightarrow\ y_1 = 2a/y \qquad y_2 = \dfrac{d^2y}{dx^2} = \dfrac{-2a}{y^2}\, y_1$

At $P(at_1^2, 2at_1)$: $\quad y_1 = \dfrac{2a}{2at_1} = 1/t_1$

$y_2 = \dfrac{-2a}{4a^2t_1^2} \cdot \dfrac{1}{t_1} = \dfrac{-1}{2at_1^3}$

ie $\quad \rho_1 = \dfrac{(1 + y_1^2)^{3/2}}{y_2} = \dfrac{(1 + 1/t_1^2)^{3/2}}{-1} \cdot 2at_1^3 = -2a \cdot (1 + t_1^2)^{3/2}$

simly :- $\rho_2 = -2a(1 + t_2^2)^{3/2}$

Req. expression :- $\ 1/\rho_1^{2/3} + 1/\rho_2^{2/3} = \dfrac{1}{(-2a(1+t_1^2)^{3/2})^{2/3}} + \dfrac{1}{(-2a(1+t_2^2)^{3/2})^{2/3}}$

$= \dfrac{1}{[2a]^{2/3}}\left[\dfrac{1}{1+t_1^2} + \dfrac{1}{1+t_2^2}\right] \qquad t_2 = -1/t_1$

$= \dfrac{1}{(2a)^{2/3}} \cdot \left[\dfrac{1}{t_1^2+1} + \dfrac{t_1^2}{t_1^2+1}\right]$

$= \dfrac{1}{(2a)^{2/3}}\left[\dfrac{1+t_1^2}{t_1^2+1}\right]$

ie $\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}$

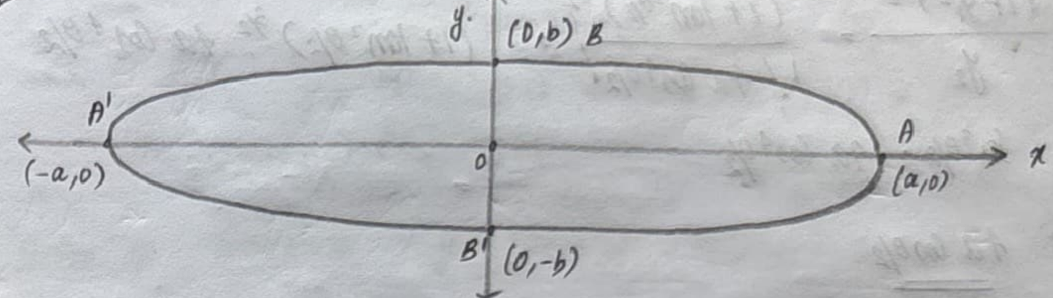
Problem with a^2

Qn 8:- P.T ρ at the end of the ~~major~~ minor axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is is equal to semilatus - rectum.

$b^2 = a^2[1-e^2]$

$L\cdot R = \frac{2b^2}{a}$

Ans:



Major axis = AA' and OA = a , minor axis = BB' and OB = b

* To find ρ at A = (a,0) :- $x^2/a^2 + y^2/b^2 = 1$ ∴ $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = \frac{-2x/a^2}{2y/b^2} = \frac{-b^2x}{a^2y}$$

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\left[\frac{y\cdot 1 - xy_1}{y^2}\right] = \frac{-b^2}{a^2}\left(\frac{y - xy_1}{y^2}\right)$$

* At B(0,b)

$y_1 = 0 \qquad y_2 = +b/a^2$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{1}{b/a^2} = a^2/b$$

Qn :- Find ρ at any point "(θ)" on $x = a[\theta + \sin\theta]$, $y = a[1-\cos\theta]$

Ans :- $x = a[\theta + \sin\theta] \Rightarrow \frac{dx}{d\theta} = a[1+\cos\theta]$

$y = a[1-\cos\theta] \Rightarrow dy/d\theta = a\sin\theta \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a[1+\cos\theta]}$

$$= \frac{a\cdot 2\sin\theta/2\,\cancel{\cos\theta/2}}{a\cdot 2\cos^2\theta/2}$$

$$= \tan\theta/2$$

$$y_2 = d^2y/dx^2 = \frac{d}{dx}(\tan\theta/2) = \frac{1}{2}\cdot\sec^2\theta/2\cdot\frac{d\theta}{dx} = \frac{1}{2\cos^2\theta/2}\cdot\frac{1}{dx/d\theta} = \frac{1}{2\cos^2\theta/2}\cdot\frac{1}{a[1+\cos\theta]}$$

$$= \frac{1}{2a\cos^2\theta/2\cdot 2\cos^2\theta/2} = \frac{1}{4a\cos^4\theta/2}$$

$$\therefore\ \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2\theta/2)^{3/2}}{1/4a\cos^4\theta/2} = (1+\tan^2\theta/2)^{3/2}\,4a\cos^4\theta/2$$

$$= \sec^3\theta/2\cdot 4a\cos^4\theta/2$$

$$= \underline{\underline{4a\cos\theta/2}}$$

Qn 8- Find the ρ @ any "θ" on $x = a\cos^3\theta,\ y = a\sin^3\theta$

Ans:- $dx/d\theta = a\,3\cos^2\theta\cdot(-\sin\theta)$ $\quad dy/d\theta = 3a\sin^2\theta\cdot\cos\theta$

$$y_1 = dy/dx = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = \underline{\underline{-\tan\theta}}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}(-\tan\theta) = -\sec^2\theta\cdot\frac{d\theta}{dx} = -\sec^2\theta\cdot\frac{1}{dx/d\theta} = \frac{-1}{\cos^2\theta}\cdot\frac{1}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{1}{3a\cos^4\theta\cdot\sin\theta}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{[1+\tan^2\theta]^{3/2}}{1/3a\sin\theta\cos^4\theta} = \frac{\sec^3\theta}{3a\sin\theta\cos^4\theta^{-1}} = 3a\sin\theta\cdot\cos\theta$$

$$= \underline{\underline{1/2\ 3a\sin2\theta}}$$

Qn 8- Find the Centre of Curvature of $xy = c^2$ at (c,c)

Ans 8- $xy = c^2$ @ (c,c) $\Rightarrow$ $xy_1 + y = 0$ $\rightarrow$ $y_1 = -y/x$

$$y_2 = -\left[\frac{xy_1 - y}{x^2}\right]$$

At $(c,c) \Rightarrow y_1 = -1$ and $y_2 = -\left[\frac{c(-1) - c}{c^2}\right] = 2/c$

Co-ordinates :-
$$X = x - \left[\frac{y_1}{y_2}(1+y_1^2)\right]$$
$$= c - \left(\frac{-1}{2/c}(1+1)\right) = c - (-c) = 2c$$
$$Y = y + \frac{1}{y_2}(1+y_1^2) = c + \frac{c}{2}(1+1) = 2c$$

Qn :- Find Centre of Curvature $y^2 = 4ax$ at $(at^2, 2at)$

Ans :- $2y\,y_1 = 4a \Rightarrow y_1 = \frac{2a}{y}$

$y_2 = -2a/y^2 \cdot dy/dx$

at $(at^2, 2at)$

$y_1 = 1/t \qquad y_2 = \frac{-1}{2at^3}$

$$X = x - \left[\frac{y_1}{y_2}(1+y_1^2)\right] = at^2 - \left[\frac{1/t}{-1/2at^3}\cdot\left[1+1/t^2\right]\right] = at^2 + \left[\frac{t^2+1}{t^3}\right]2at^3 = at^2 + 2a(1+t^2)$$
$$= 3at^2 + 2a$$

$$Y = y + \frac{1}{y_2}(1+y_1^2) = 2at + (-2at^3)\frac{(1+t^2)}{t^2} = 2at - 2at(1+t^2) = -2at^3$$

EVOLUTE :-

Evolute is the locus of Centre of Curvature. The given curve is the Involute of the evolute.

Qn :- Find evolute of the parabola $y^2 = 4ax$

We have (from prev. example) :- $X = 3at^2 + 2a \to ①$ $\qquad Y = -2at^3 \to ②$

evolute is the locus of $[X, Y]$

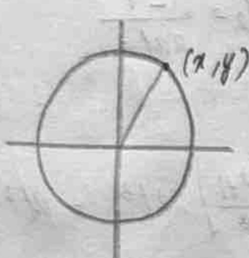
From ① $\Rightarrow$ $t = \left[\frac{x-2a}{3a}\right]^{1/2}$

② $\Rightarrow$ $Y = -2a\left[\frac{x-2a}{3a}\right]^{3/2} = \frac{-2a\,(x-2a)^{3/2}}{(3a)^{3/2}} \longrightarrow ③$

squ' of ③ $\Rightarrow$ $Y^2 = 4a^2\left[\frac{(x-2a)}{(3a)}\right]^3 = \frac{4a^2}{27a^3}(x-2a)^3 = \frac{4}{27a}(x-2a)^3$

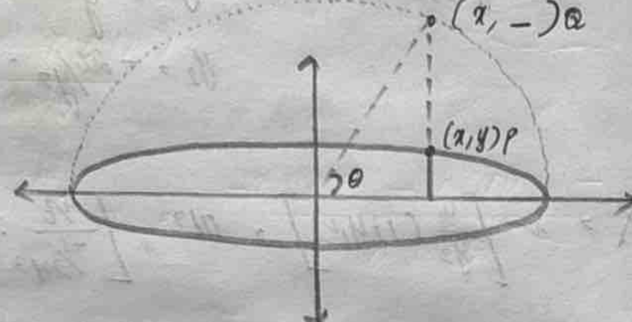
ie evolute :- $27aY^2 = 4(X-2a)^3$ (in general)

Qn :- Find the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$



Parametric of circle

$x = a\cos\theta$
$y = b\sin\theta$



Parametric of ellipse :- $x = a\cos\theta$
$y = b\sin\theta$ (by putting in eq^n^ $x^2/a^2 + y^2/b^2 = 1$)

Ans :- Consider the parametric equations :- $\left.\begin{array}{l} x = a\cos\theta \\ y = b\sin\theta \end{array}\right\}$ for $x^2/a^2 + y^2/b^2 = 1$

$dx/d\theta = -a\sin\theta, \quad dy/d\theta = b\cos\theta \Rightarrow y_1 = dy/dx = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left[-b/a \cot\theta\right] = \frac{-b}{a}\cdot -\mathrm{Cosec}^2\theta\, d\theta/dx = \frac{b}{a\sin^2\theta}\cdot\frac{1}{-a\sin\theta} = \underline{\underline{\frac{-b}{a^2\sin^3\theta}}}$$

$$X = x - \left[\frac{y_1}{y_2}(1+y_1^2)\right] = a\cos\theta - \left[\frac{\frac{-b\cot\theta}{a}}{\frac{-b}{a^2\sin^3\theta}}\left(1 + \frac{b^2\cot^2\theta}{a^2}\right)\right]$$

$$= a\cos\theta - \left(a\sin^2\theta\cos\theta\,\frac{(a^2 + b^2\cot^2\theta)}{a^2}\right) \quad \frac{\cos^2\theta}{\sin^2\theta}$$

$$= a\cos\theta - 1/a\,\cos\theta\,(a^2\sin^2\theta + b^2\cos^2\theta)$$

$$aX = a^2\cos^2\theta - \cos\theta\,(a^2\sin^2\theta + b^2\cos^2\theta)$$

$$= \cos\theta\left[a^2 - a^2\sin^2\theta - b^2\cos^2\theta\right] = \cos\theta\left[a^2\cos^2\theta - b^2\cos^2\theta\right] = \cos^3\theta\left[a^2 - b^2\right]$$

$$\text{ie}\quad \cos\theta = \left[\frac{aX}{(a^2-b^2)}\right]^{1/3} \longrightarrow ①$$

$$Y = y + \frac{1}{y_2}(1+y_1^2) = b\sin\theta + \frac{-a^2\sin^3\theta}{b}\left(1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right) = b\sin\theta + \frac{-\cancel{a^2}\sin^{\cancel{3}}\theta}{b}\cdot\frac{(a^2\sin^2\theta + b^2\cos^2\theta)}{\cancel{a^2\sin^2\theta}}$$

$$bY = b^2\sin\theta + -\sin\theta\,(a^2\sin^2\theta + b^2\cos^2\theta)$$

$$= \sin\theta\left[b^2(1-\cos^2\theta) - a^2\sin^2\theta\right] = \sin\theta\left[b^2\sin^2\theta - a^2\sin^2\theta\right]$$

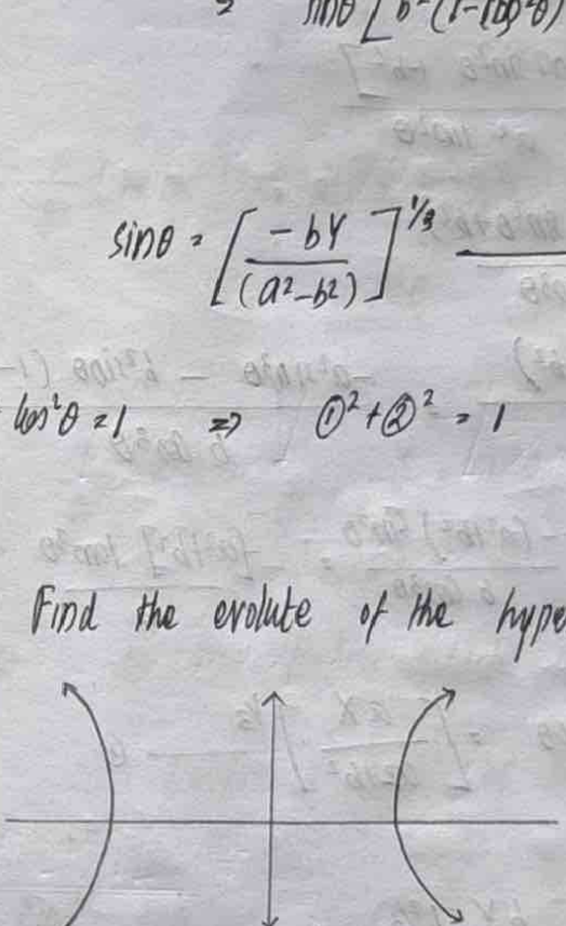
$$= -\sin^3\theta\,(a^2-b^2)$$

$$\text{ie}\quad \sin\theta = \left[\frac{-bY}{(a^2-b^2)}\right]^{1/3} \longrightarrow ②$$

$$\sin^2\theta + \cos^2\theta = 1 \implies ①^2 + ②^2 = 1 \implies \underline{\underline{(aX)^{2/3} + (bY)^{2/3} = (a^2-b^2)^{2/3}}}$$

Qn :- Find the evolute of the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

Ans :-



Parametric eqn of hyperbola :- $x = a\sec\theta$, $y = b\tan\theta$

$$dx/d\theta = a\sec\theta\tan\theta \qquad dy/d\theta = b\sec^2\theta \implies dy/dx = \frac{b\sec\theta\sec\theta}{a\cancel{\sec\theta}\tan\theta} = \frac{b\,\cancel{\cos\theta}}{a\,\cancel{\cos\theta}\sin\theta}$$

$$\underline{\underline{y_1 = b/a\,\operatorname{cosec}\theta}}$$

$$y_2 = \frac{-b}{a}\operatorname{cosec}\theta\cot\theta \cdot d\theta/dx = \frac{-b}{a^2}\operatorname{cosec}\theta\cot\theta\cdot\frac{1}{a\sec\theta\tan\theta} = \frac{-b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$$

$$X = x - \frac{y_1}{y_2}\left[1 + y_1^2\right] = a\sec\theta - \frac{\cancel{b/a}\operatorname{cosec}\theta}{-\cancel{b/a}\,a\,\frac{\cos^3\theta}{\sin^3\theta}}\left[1 + b^2/a^2\operatorname{cosec}^2\theta\right] = \overset{a\sec\theta}{}\;\frac{-\sin^2\theta}{\cos^3\theta}\cdot\frac{[a^2\sin^2\theta + b^2]}{a^2\sin^2\theta}$$

$$= \frac{a}{\cos\theta} + \frac{b}{\sin\theta}\left(\frac{a^2\sin^2\theta + b^2}{a^2\sin^2\theta}\right)\cdot\frac{a^2\sin^3\theta}{-b\cos^3\theta}$$

$$= \frac{a}{\cos\theta} + \frac{1}{a}\left[\frac{a^2\sin^2\theta + b^2}{\cos^3\theta}\right] = \frac{a^2\cos^2\theta + a^2\sin^2\theta + b^2}{a\cos^3\theta} = \frac{a^2+b^2}{a\cos^3\theta}$$

$$X = \frac{a^2+b^2}{a\cos^3\theta}$$

$$Y = y + \frac{1}{y_2}[1+y_1^2] = b\tan\theta + \frac{a^2\sin^3\theta}{-b\cos^3\theta}\left[1 + \frac{b^2}{a^2}\cdot\frac{1}{\sin^2\theta}\right]$$

$$= b\tan\theta + \frac{-a^2\sin^3\theta}{b\cos^3\theta}\cdot\left[\frac{a^2\sin^2\theta + b^2}{a^2\sin^2\theta}\right]$$

$$= b\tan\theta - \frac{1}{b}\sin\theta\left(\frac{a^2\sin^2\theta + b^2}{\cos^3\theta}\right)$$

$$= \frac{b^2\sin\theta\cos^2\theta - \sin\theta(a^2\sin^2\theta + b^2)}{b\cos^3\theta} = \frac{-a^2\sin^3\theta - b^2\sin\theta(1-\cos^2\theta)}{b\cos^3\theta}$$

$$Y = \frac{-a^2\sin^3\theta - b^2\sin^3\theta}{b\cos^3\theta} = \frac{-(a^2+b^2)\sin^3\theta}{b\cos^3\theta} = \frac{-[a^2+b^2]\tan^3\theta}{b}$$

$$\Rightarrow \tan\theta = \left[\frac{-bY}{a^2+b^2}\right]^{1/3} \longrightarrow ② \quad \sec\theta = \left[\frac{aX}{a^2+b^2}\right]^{1/3} \longrightarrow ①$$

But $\sec^2\theta - \tan^2\theta = 1 \Rightarrow \left[\frac{aX}{b^2+a^2}\right]^{2/3} - \left[\frac{bY}{a^2+b^2}\right]^{2/3} = 1$

evolute :- $(aX)^{2/3} - (bY)^{2/3} = [a^2+b^2]^{2/3}$.

Qn 8- Find the evolute of $a[\cos\theta + \theta\sin\theta] = x \quad y = a[\sin\theta - \theta\cos\theta]$

Ans :- $dx/d\theta = a[-\sin\theta + \theta\cos\theta + \sin\theta]$ $\quad dy/d\theta = a[\cos\theta + \theta\sin\theta - \cos\theta]$

$= a\theta\cos\theta \qquad\qquad = a\theta\sin\theta$

$$\frac{dy}{dx} = \frac{a\theta \sin\theta}{a\theta \cos\theta} = \tan\theta \quad \Big\} \quad y_1 = \tan\theta \qquad y_2 = \sec^2\theta \cdot \frac{d\theta}{dx} = \frac{1}{\cos^2\theta} \cdot \frac{1}{a\theta \cos\theta}$$

$$= \frac{1}{a\theta \cdot \cos^3\theta}$$

$$X = x - \frac{y_1}{y_2} \cdot [1 + y_1^2] = a(\cos\theta + \theta\sin\theta) - \frac{\tan\theta \cdot a\theta \cos^3\theta}{1}[1 + \tan^2\theta]$$

$$= a(\cos\theta + \theta\sin\theta) - \frac{\sin\theta}{\cos\theta} \cdot a\theta \cdot \cancel{\cos^3\theta} \cdot \frac{1}{\cancel{\cos^2\theta}}$$

$$= a\cos\theta + a\theta\sin\theta - a\theta\sin\theta$$

$$= \underline{\underline{a\cos\theta}}$$

$$Y = y + \frac{1}{y_2}[1 + y_1^2] = a\sin\theta - a\theta\cos\theta + a\theta\cos^3\theta \cdot \frac{1}{\cos^2\theta}$$

$$= \underline{\underline{a\sin\theta}}$$

ie $\frac{X}{a} = \cos\theta$, $\frac{Y}{a} = \sin\theta$ $\Rightarrow$ $\sin^2\theta + \cos^2\theta = 1$

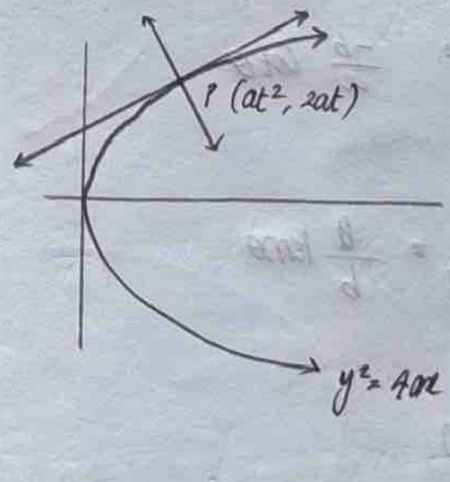
$x^2 + y^2 = a^2 \Rightarrow$ evolute.

Note :- we can define evolute in term of normal.

Evolute = Envelop of the normals.

eg: Find the eqn of the normal to $y^2 = 4ax$ at $(at^2, 2at)$ and hence find evolute.

Ans :-



$y^2 = 4ax$ $\qquad yy_1 = 4ax + 4ax_1$

$2y \, dy/dx = 4a \Rightarrow dy/dx = 2a/y$ $\qquad x = 4ax_1 - yy_1$

$$m = \frac{2a}{y} = \frac{2a}{2at} = \underline{\underline{1/t}}$$

Eqn of normal : $y - y_1 = m(x - x_1)$

when 2 line's with slopes m_1 & m_2 are $\perp^r$, then $\boxed{m_1 \cdot m_2 = -1}$

ie slope of the normal $\frac{1}{t} m_n = -1 \Rightarrow m_n = -t$

eq^n of normal :- $y - 2at = -t(x - at^2)$

$y - 2at = -tx + at^3 \Rightarrow y + xt = 2at + at^3 \longrightarrow (1)$

St evolute is the envelop of the normal. ie Parti. diff. of (1) w.r.t. 't' gives

$0 + x = 2a + a3t^2 \Rightarrow t = \left[\frac{x-2a}{3a}\right]^{1/2}$

(1) $\Rightarrow$ $y + t(x-2a) = at^3 \Rightarrow y + \left(\frac{x-2a}{3a}\right)^{1/2}(x-2a) = a\left(\frac{x-2a}{3a}\right)^{3/2}$

ie $y + \frac{(x-2a)^{3/2}}{\sqrt{3a}} = \frac{a(x-2a)^{3/2}}{3a\sqrt{3a}}$

$y = \frac{(x-2a)^{3/2}}{3\sqrt{3a}} - \frac{(x-2a)^{3/2}}{\sqrt{3a}} = \frac{(x-2a)^{3/2}}{\sqrt{3a}}\left[\frac{1}{3} - 1\right]$

$27ay^2 = 4(x-2a)^3 \Rightarrow$ evolute

Qn 8- Find the eq^n to the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at $(a\cos\theta, b\sin\theta)$ and hence find evolute.

Ans: $x = a\cos\theta$ $\quad y = b\sin\theta$ $\quad$ $dy/dx = \frac{-b}{a}\cot\theta$

$dx/d\theta = -a\sin\theta$ $\quad dy/d\theta = b\cos\theta$

slope of the normal $\Rightarrow \frac{-1}{[dy/dx]} = \frac{a}{b}\tan\theta$

eq^n of normal :- $y - b\sin\theta = \frac{a}{b}\tan\theta(x - a\cos\theta)$

$$y - b\sin\theta = \frac{a}{b}x\tan\theta - \frac{a^2}{b}\sin\theta \Rightarrow by - b^2\sin\theta = -a^2\sin\theta + \frac{ax}{b}\tan\theta$$

$$by + \sin\theta\left[a^2 - b^2\right] = \frac{ax}{b}\tan\theta$$

normal :- $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2 \rightarrow ①$ (Part. diff. w.r.t. θ)

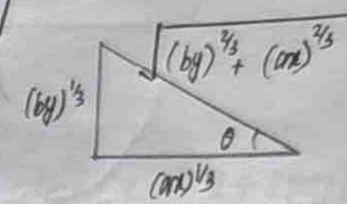
Evolute :- $0 = ax\sec\theta\tan\theta - by(-\csc\theta \cdot \cot\theta)$

$$0 = ax\sec\theta\tan\theta + by\csc\theta\cot\theta$$

$\tan\theta = -ve$

$\Rightarrow$ either $\sin$/$\cos$ $-ve$

$$\Rightarrow \tan^3\theta = \frac{-by}{ax} \Rightarrow \tan\theta = \left[\frac{-by}{ax}\right]^{1/3}$$



ie $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{-\sqrt{(ax)^{2/3} + (by)^{2/3}}}{(by)^{2/3}}$

$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\sqrt{(ax)^{2/3} + (by)^{2/3}}}{(ax)^{2/3}}$

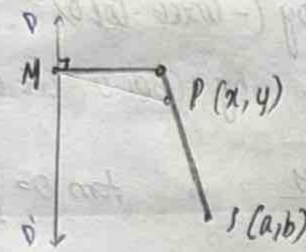
$$① \Rightarrow \quad ax \cdot \frac{\sqrt{(ax)^{2/3} + (by)^{2/3}}}{(ax)^{1/3}} - by\frac{-\sqrt{(ax)^{2/3} + (by)^{2/3}}}{(by)^{2/3}} = a^2 - b^2$$

$$\sqrt{(ax)^{2/3} + (by)^{2/3}}\left[(ax)^{1-1/3} + (by)^{1-1/3}\right] = a^2 - b^2$$

ie $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

Module :- 3 *

Conics :- Consider a fixed line DD' and a fixed point S. A Conic is defined as the locus of P such that $\frac{SP}{PM} = e$, a Constant



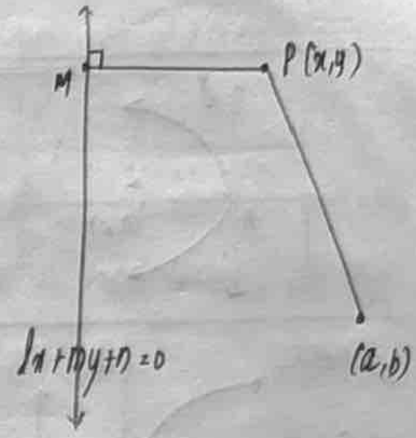
$lx + my + n = 0$

When SP is the distance of P from S. PM is $\perp^r$ distance to the line.

- Fixed point called focus :- Generally denoted by S
  Fixed line is called directrix
  Constant ratio is called eccentricity

If $e = 1$ Parabola
$e < 1$ ellipse
$e > 1$ Hyperbola.

General Eqⁿ of a Conic



$$SP = \sqrt{(x-a)^2 + (y-b)^2}$$

PM = ⊥r distance from P(x,y) to the line $lx + my + n = 0$

$$PM = \frac{lx + my + n}{\sqrt{l^2 + m^2}}$$

But $SP/PM = e \Rightarrow SP^2 = e^2 PM^2$

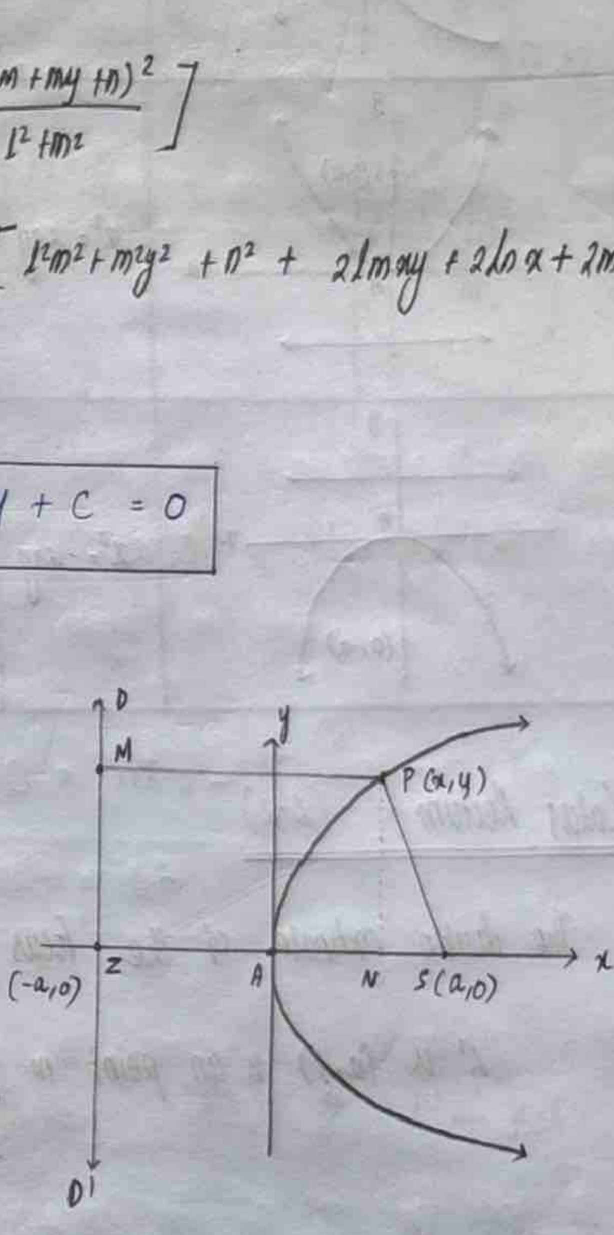
$$(x-a)^2 + (y-b)^2 = e^2 \left[\frac{(lx + my + n)^2}{l^2 + m^2} \right]$$

ie $(l^2+m^2)\left[(x-a)^2 + (y-b)^2\right] = e^2\left[l^2m^2 + m^2y^2 + n^2 + 2lmxy + 2lnx + 2mny\right]$

This can be put in the form:

$$\boxed{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$$

Standard Equation of the parabola



$\frac{SP}{PM} = e = 1 \Rightarrow SP = PM$

ie $\frac{SA}{AZ} = 1 \Rightarrow SA = AZ = a$

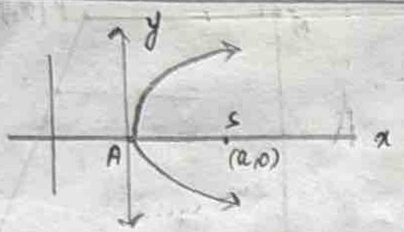
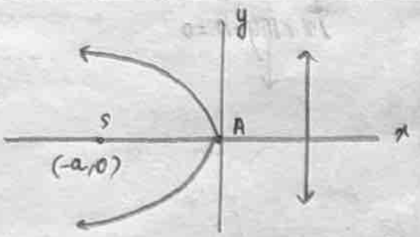
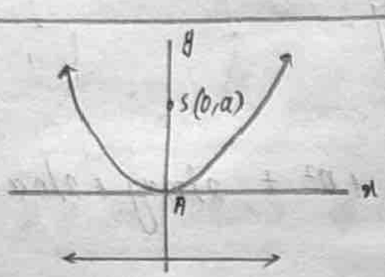
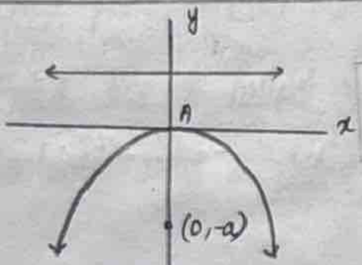
eqⁿ of the directrix is $x = -a$

$$x + a = 0$$

$$SP = \sqrt{(x-a)^2 + y^2}$$

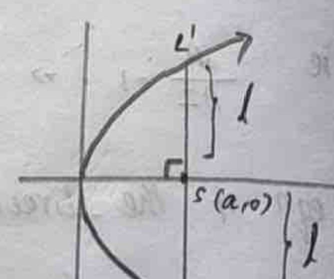
$PM = NZ = AN + AZ = x + a$ But $\frac{SP}{PM} = e = 1 \Rightarrow SP^2 = PM^2$

$$(x-a)^2 + y^2 = (x+a)^2 \Rightarrow y^2 = (x+a)^2 - (x-a)^2 = 4ax$$

	Eq^n	Focus (S)	Eq^n of Directrix	Vertex (A)
	$y^2 = 4ax$	$(a, 0)$	$x = -a$	$A(0,0)$
	$y^2 = -4ax$	$(-a, 0)$	$x = a$	$A(0,0)$
	$x^2 = 4ay$	$(0, a)$	$y = -a$	$A(0,0)$
	$x^2 = -4ay$	$(0, -a)$	$y = a$	$A(0,0)$

Latus Rectum :- (L·R)

The double ordinate of the focus is called latus rectum.



L' is (a, l) is an point in $y^2 = 4ax$

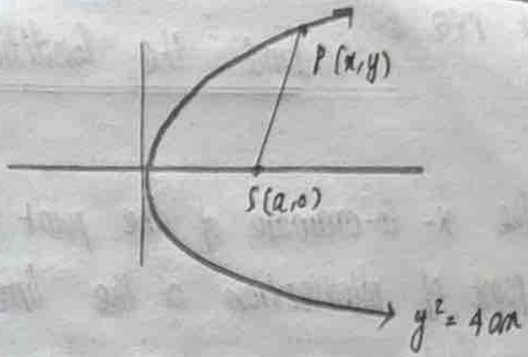
$$l^2 = 4a \cdot a$$

ie $\quad l = 2a \qquad$ Length of $L \cdot R = 4a$

Focal distance :- (FD)

Let $P(x,y)$ be any point on $y^2 = 4ax$, then the distance SP is called focal distance.

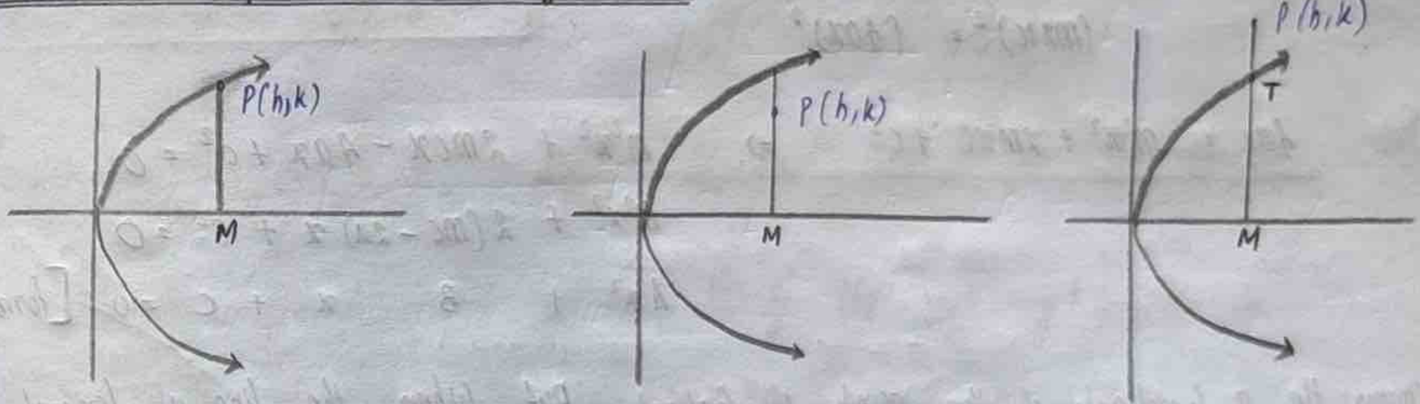
Focal distance :-



$$F.D = PS = \sqrt{(x-a)^2+y^2} = \sqrt{(x-a)^2+4ax}$$
$$= \sqrt{(x+a)^2}$$
$$= |x+a|$$

Focal distance $= |x+a|$

Position of a point w.r.t $y^2 = 4ax$



Case ① :- If $P(h,k)$ is a point in $y^2 = 4ax$ then $k^2 = 4ah$

ie $k^2 - 4ah = 0$

Case ② :- sk is a point outside the curve $PM > TM \longrightarrow ①$

But $PM = k$ $TM = \sqrt{4ah}$

① $\Rightarrow$ $k > \sqrt{4ah}$ $\Rightarrow$ $k^2 - 4ah > 0$

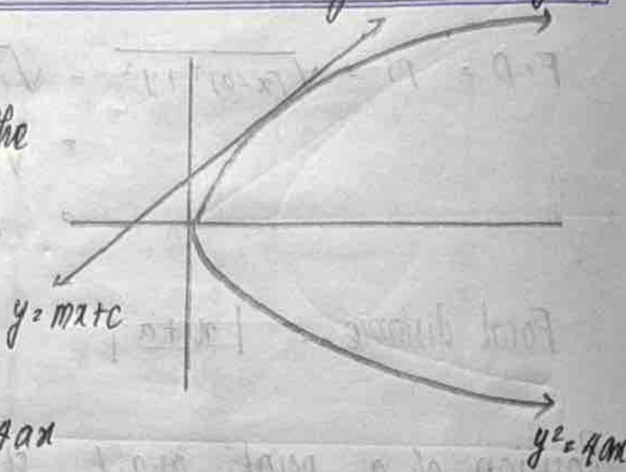
Case ③ :- $P(h,k)$ is a point inside the parabola $TM > PM \Rightarrow k^2 < 4ah$

ie $k^2 - 4ah < 0$

In general :- a point $P[(x)(y)]$ is
i) in the parabola if $y^2 - 4ax = 0$
ii) inside the parabola if $y^2 - 4ax < 0$
iii) outside the parabola if $y^2 - 4ax > 0$

* PYQ :- Find the Condition that $y = mx + c$ be the tangent to $y^2 = 4ax$

The x- Co-ordinate of the point of intersection of the point of intersection of the line $y = mx + c$ and parabola $y^2 = 4ax$ are given by eliminating y.



$$y = mx + c \Rightarrow y^2 = (mx+c)^2 \quad y^2 = 4ax$$

$$(mx+c)^2 = (4ax)$$

$$4ax = m^2x^2 + 2mxc + c^2 \Rightarrow m^2x^2 + 2mcx - 4ax + c^2 = 0$$

$$m^2x^2 + 2(mc - 2a)x + c^2 = 0$$

$$Ax^2 + Bx + C = 0 \quad [\text{form}]$$

Given the x-Coordinate of the point of Contact. But when the line is tangent, there is only one point of Contact

$\Rightarrow$ Roots are equal $\Rightarrow B^2 - 4AC = 0$

ie $4(mc - 2a)^2 - 4m^2c^2 = 0 \Rightarrow \cancel{m^2c^2} - 4amc + 4a^2 - \cancel{m^2c^2} = 0$

$$4a^2 = 4amc \Rightarrow c = a/m$$

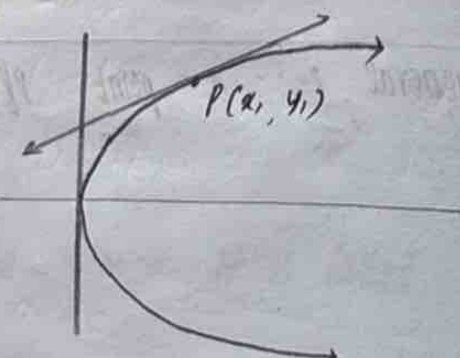
Note : $y = mx + c$ is a tangent to $y^2 = 4ax$ when $y = mx + a/m$

To find the eqⁿ of the tangent at (x_1, y_1)

$$y^2 = 4ax$$

$$2y\, dy/dx = 4a$$

$$dy/dx = 2a/y$$



$m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = \frac{2a}{y_1}$

Eq^n of the tangent : $y - y_1 = m(x - x_1)$

$y - y_1 = \frac{2a}{y_1}(x - x_1) \Rightarrow yy_1 - y_1^2 = 2ax - 2ax_1$

ie $yy_1 = 2ax + y_1^2 - 2ax_1$

$= 2ax + 4ax_1 - 2ax_1 = 2ax + 2ax_1 = 2a(x + x_1)$

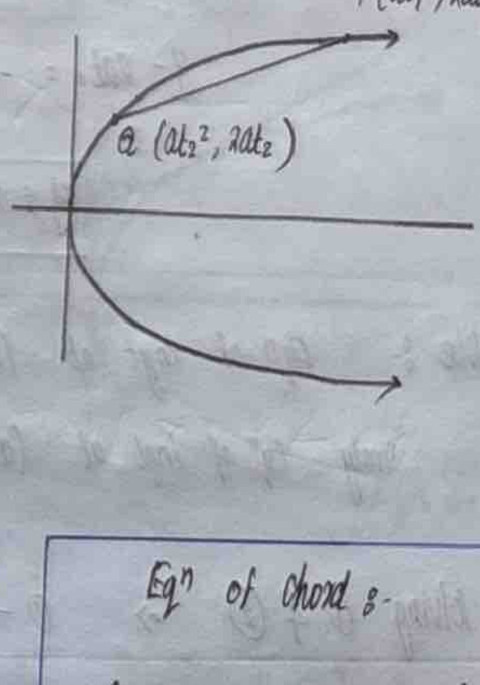
ie Tangent @ $(x_1, y_1) \Rightarrow$ $\boxed{yy_1 = 2a(x + x_1)}$

Parametric Eq^n of $y^2 = 4ax$

The 2 equation $x = at^2$ and $y = 2at$ satisfy the eq^n $y^2 = 4ax$

ie any point on parabola can be taken as $(at^2, 2at)$.

① **Eq^n of the chord joining 't_1' and 't_2'**



The eq^n of PQ is $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$

ie $y - 2at_1 = \frac{2a(t_2 - t_1)}{a(t_2 - t_1)(t_2 + t_1)}(x - x_1)$

$y - 2at_1 = \frac{2(x - x_1)}{(t_2 + t_1)}$

$(y - 2at_1)(t_2 + t_1) = 2(x - at_1^2)$

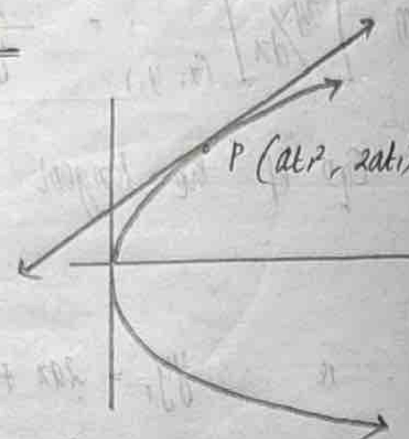
$y(t_1 + t_2) - 2at_1(t_1 + t_2) = 2ax - 2at_1^2$

$\Rightarrow y(t_1 + t_2) = 2x + 2at_1t_2$

Eq^n of chord :-

$y(t_1 + t_2) = 2(x + at_1t_2)$

(i) To find the eq^n of tangent at $(at_1^2, 2at_1)$



chord ~~and~~ eq^n : t_2 approaches t_1 ~~means~~ chord becomes tngt.

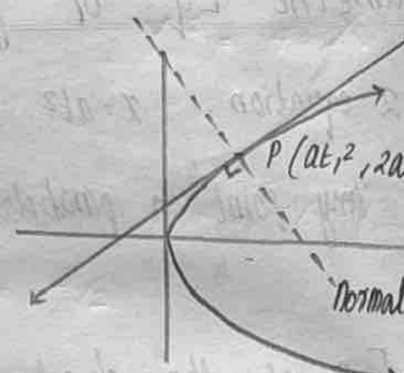
ie chord eq^n ~~a~~ replace t_2 with t_1.

ie Put $t_2 = t_1$ in the eq^n of the chord,

$$y(t_1+t_1) = 2x + 2at_1 t_2$$

$$2yt_1 = 2(x + at_1^2) \Rightarrow \boxed{eq^n \text{ of tngt : } yt_1 = x + at_1^2}$$

(ii) To find eq^n of normal at $(at_1^2, 2at_1)$



Eq^n of tangent is $yt_1 = x + at_1^2$

$$y = \frac{1}{t_1}x + at_1 \quad (\text{in form } y = mx + c)$$

slope of the tangent $= 1/t_1 \Rightarrow$ slope of normal $= -t_1$

ie normal at $(at_1^2, 2at_1)$

$$y - 2at_1 = -t_1(x - at_1^2) \Rightarrow y - 2at_1 = -t_1 x + at_1^3$$

$$\boxed{y + xt_1 = 2at_1 + at_1^3}$$

Note :- Eq^n of tngt at $(at_1^2, 2at_1)$ $\quad yt_1 = x + at_1^2 \longrightarrow (1)$

simly Eq^n of tngt at $(at_2^2, 2at_2)$ $\quad yt_2 = x + at_2^2 \longrightarrow (2)$

solving (1) & (2) $\Rightarrow$ (1) - (2) $\Rightarrow$ $yt_1 - yt_2 = at_1^2 - at_2^2$

$$y\cancel{(t_1 - t_2)} = a\cancel{(t_1 - t_2)}(t_1 + t_2)$$

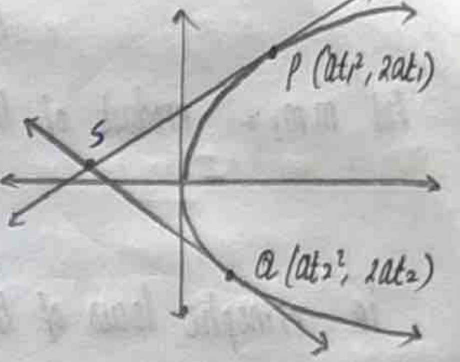
$$y = a(t_1 + t_2)$$

(1) $\Rightarrow$ $x = yt_1 - at_1^2 = a(t_1+t_2)t_1 - at_1^2$

$$= \cancel{at_1^2} + at_1t_2 - \cancel{at_1^2} = at_1t_2$$

ie Point of intersection of tangent at t_1 & t_2

$S\left(at_1t_2\ ,\ a(t_1+t_2)\right)$

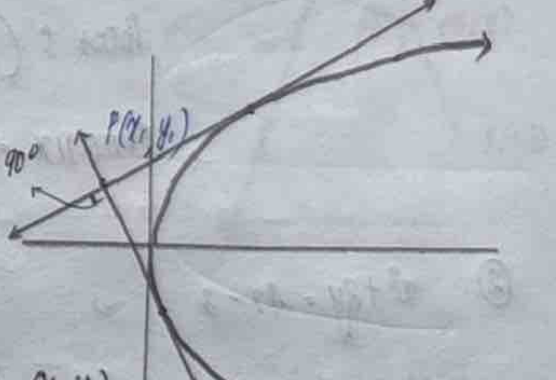


ORTHOPTIC LOCUS

It is the locus of the point of intersection of $\perp^r$ tangent to a curve

* QP : **Find the orthoptic locus of $y^2 = 4ax$**

We know that $y = mx + a/m$ represents a tngt to $y^2 = 4ax$. Let $P(x_1, y_1)$ be the point of intersection of 2 tngt's.



If $y = mx + a/m \longrightarrow$ ① passes through (x_1, y_1)

$y_1 = mx_1 + a/m \Rightarrow x_1m^2 - y_1m + a = 0 \longrightarrow$ ②

It is a quadratic in m. But m is the slope of the tngt passing through (x_1, y_1) But tngt's are $\perp^r$. Solving ② we get two values for m which are the slope of 2 tngt passing through (x_1, y_1). But tangt's are perpendicular.

$$m_1 \cdot m_2 = -1$$

But $m_1 m_2 =$ product of the roots $= c/a = a/x_1 = -1$

$$a = -x_1 \Rightarrow a + x_1 = 0$$

$ax^2 + bx + c = 0$
prod. of roots $= c/a$
Sum of roots $= -b/a$

ie orthoptic locus of the parabola is directrix.

Qn:- Find the vertex, focus and directrix of

① $y^2 + 6y - 2x + 5 = 0$

Ans: $y^2 + 6y = 2x - 5 \Rightarrow y^2 + 6y + 9 = 2x + 4 \Rightarrow (y+3)^2 = 2(x+2)$

$$Y^2 = 4aX$$

ie $4a = 2 \Rightarrow a = 1/2$

$Y = y+3 \quad X = x+2$ } w.r.t X & Y Vertex $= (0,0)$ Focus $= (-1/2, 0)$

Directrix $= -1/2$

w.r.t original chord :- Vertex $(0-2, 0-3) = (-2,-3)$

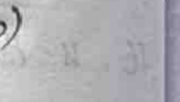
Focus $S\,(1/2 - 2, 0-3) = (-3/2, -3)$

Directrix $\Rightarrow x + 2 = -1/2 \Rightarrow x = -5/2$

② $x^2 + 2y = 4x - 3 \Rightarrow$

Ans : $x^2 - 4x + 4 = -2y - 3 + 4 \Rightarrow (x-2)^2 = -2y + 1$

$= -2(y - 1/2)$ } $x^2 = -4ay$



$4a = -2 \Rightarrow a = -1/2$

ie Vertex $= (+2, 1/2)$ Focus : $((2,0))$ Directrix $\Rightarrow 2y = 1$

(8) $y^2 - 2y - 6x + -17 = 0$

Ans: $\left.\begin{aligned} y^2 - 2y + 1 &= 6x + 18 \\ &= 6(x+3) \end{aligned}\right\}$ $\quad (y-1)^2 = 6(x+3)$ $\quad 4a = 6 \Rightarrow a = 3/2$

$y^2 = 4a\ x$

Vertex = $(1, -3)$ $\quad$ Focus : $(-3/2, 1)$ $\quad$ Directrix $\Rightarrow$ $x = -9/2$

$x = -\frac{3}{2} - 3/2$

(9) $x^2 + 4x + 4y + 16 = 0$ $\quad \Rightarrow \quad x^2 + 4x + 4 = -4y - 16 + 4$

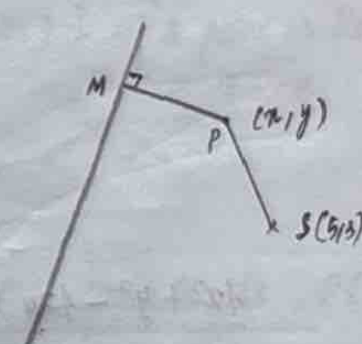
$(x+2)^2 = -4y - 12 = -4(y+3)$

$(x+2)^2 = -4(y+3)$ $\quad -4a = -4 \Rightarrow a = +1$ $\quad (0, -a)$

$x^2 = -4a\ y$

Vertex $(-2, -3)$ $\quad$ focus : $(-2, -4)$ $\quad$ directrix $\Rightarrow$ $Y = -2$

Qn 8- Find the eqn of the parabola with focus at $S(5,3)$ and directrix $3x - 4y + 1 = 0$



Ans:

$SP = \sqrt{(x-5)^2 + (y-3)^2}$

PM = $\perp^r$ distance to the directrix

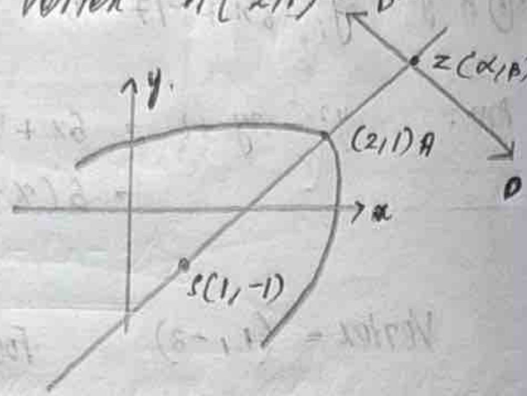
$= \dfrac{|3x - 4y + 1|}{\sqrt{3^2 + 4^2}} = \dfrac{3x - 4y + 1}{5}$

But $\dfrac{SP}{PM} = 1 \Rightarrow SP^2 = PM^2$ $\quad (x-5)^2 + (y-3)^2 = \dfrac{(3x - 4y + 1)^2}{25}$

$\Rightarrow x^2 - 10x + 25 + y^2 - 6y + 9 = \dfrac{9x^2 + 16y^2 + 1 - 24xy + 6x - 8y}{25}$

ie $16x^2 + 9y^2 + 24xy - 256x - 142y + 849 = 0.$

(7) Find the eqⁿ of parabola with focus $S(1,-1)$ and vertex $A(2,1)$



Ans: Let $Z(\alpha,\beta)$ be Co-ordinates. But A is midpoint of SZ.

ie $2 = \frac{\alpha+1}{2}$ $\quad 1 = \frac{\beta-1}{2}$

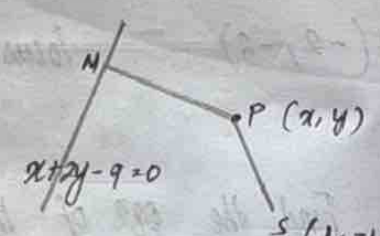
$\alpha = 3$ $\quad \beta = 3$ $\quad (\alpha,\beta) = (3,3)$.

- Here SZ is $\perp^r$ to DD' $\Rightarrow$ slop of $DD' = \frac{-1}{\text{slop of } SZ} = \frac{-(x_2-x_1)}{y_2-y_1} = \frac{-(2-1)}{1+1}$

$= -\frac{1}{2}$

eqⁿ of directrix = $y-3 = \frac{-1}{2}(x-3) \Rightarrow 2y - 6 = -x+3$

Eqⁿ of directrix = $x+2y-9=0$



$SP^2 = PM^2 \Rightarrow$ $SP = \sqrt{(x-1)^2+(y+1)^2}$

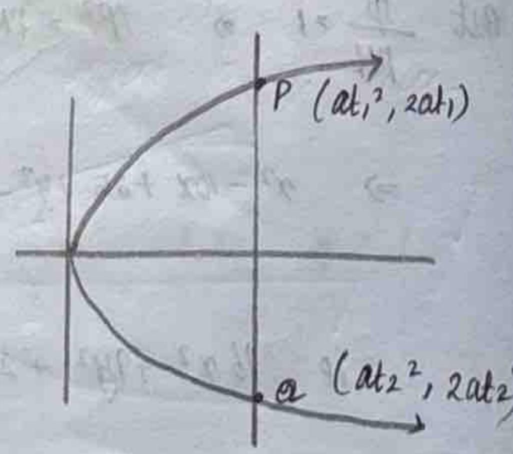
$PM = \frac{x+2y-9}{\sqrt{5}}$

$5((x-1)^2+(y+1)^2) = (x^2+4y^2+81-4xy+2x+4y)$

$4x^2+y^2-4xy+8x+46y-71=0$

Qn 8- If the chord joining 2 points t_1 & t_2 on $y^2=4ax$ passes through focus, PT $t_1 \cdot t_2 = -1$

Ans:



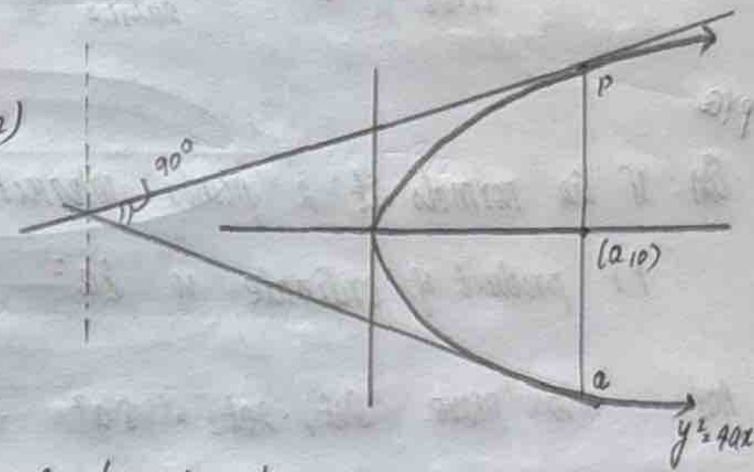
Eqⁿ of PQ :

$y(t_1+t_2) = 2x + 2at_1t_2$

It passes through $(a,0)$. ie

$0 = 2a + 2a\, t_1 t_2 \Rightarrow t_1 t_2 = -2a/2a = -1$ //

Qn: PT the tangent at the end point of focal chord of $y^2 = 4ax$ intersect at right angles on the directrix.



Ans: Let $P = (at_1^2, 2at_1)$ $Q = (at_2^2, 2at_2)$

Hence $t_1 t_2 = -1$

Eqn of tangent @ t_1 ;

$$yt_1 = x + at_1^2$$

$$y = \frac{1}{t_1} x + at_1 \rightarrow \text{slope of tangent}_1 = \frac{1}{t_1}$$

Simly slope of tngt @ $t_2 \Rightarrow \frac{1}{t_2} \rightarrow$ slope of tangent$_2$ = $\frac{1}{t_2}$

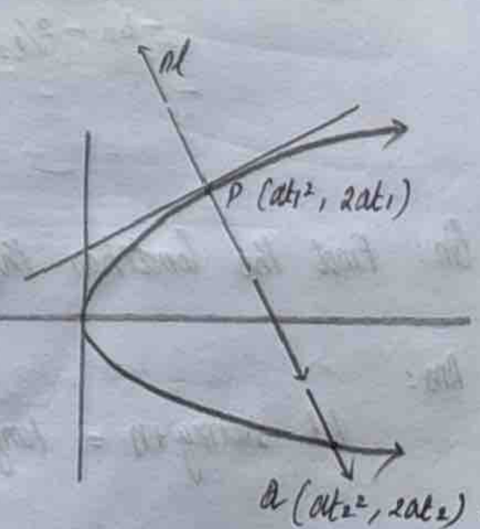
$$m_1 m_2 = \frac{1}{t_1 t_2} = \frac{1}{-1} = -1 \text{ //}$$ (intersects @ right angle).

points of intersection of tngts @ $t_1, t_2 \Rightarrow$ $(at_1t_2,\ a(t_1+t_2))$

$(-a,\ a(t_1+t_2)) \Rightarrow$ a point on line $x = -a$

$x = -a$ is the directrix @

Qn: If the normal at $at_1^2, 2at_1$ to $y^2 = 4ax$ meet the curve again at t_2, PT t_2, then $t_2 = -t_1 - 2/t_1$



Ans: Eqn of PQ as the normal at $(at_1^2, 2at_1)$

is $y + xt_1 = 2at_1 + at_1^3 \longrightarrow ①$

Eqn of PQ as the chord joining t_1 & t_2

$$y(t_1+t_2) - 2x = 2a\, t_1 t_2 \longrightarrow ②$$

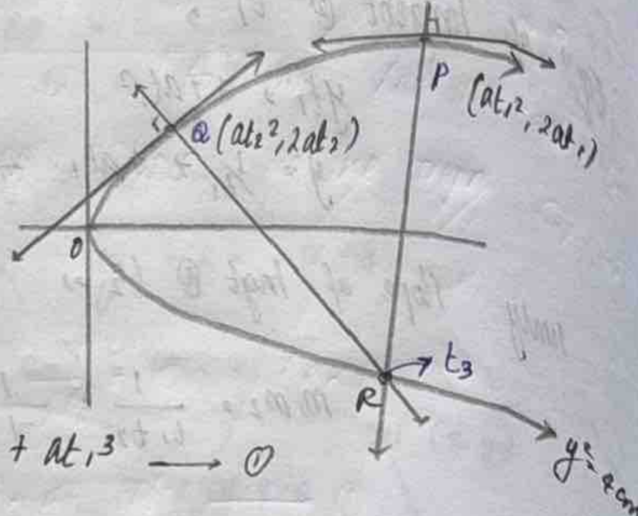
① & ② are identical as they represent the same line. Comparing Coeff.

ie $\frac{1}{t_1+t_2} = \frac{t_1}{-2} = \frac{2at_1+at_1^3}{2at_1t_2}$ ⇒ $-2 = t_1(t_1+t_2)$

$t_2 = -t_1 - 2/t_2$

PYQ

Qn: If the normals at 2 points intersect on the curve for a parabola $y^2 = 4ax$ PT product of ordinate is $8a^2$



Ans: To prove $2at_1 \cdot 2at_2 = 8a^2$

$4a^2 t_1t_2 = 8a^2$

To prove $t_1 t_2 = 2$

Eqⁿ of normal @ $(at_1^2, 2at_1)$ ⇒ $y + xt_1 = 2at_1 + at_1^3$ ⟶ ①

Eqⁿ of nl as chord joining t_1 & t_3 ⇒ $y(t_1+t_3) = 2x + 2at_1t_3$ ⟶ ②

① & ② are identical ⇒ Comparing Coeff.

$\frac{1}{t_1+t_3} = \frac{t_1}{-2}$ ⇒ $t_3 = -t_1 - 2/t_1$ llly $t_3 = -t_2 - 2/t_2$

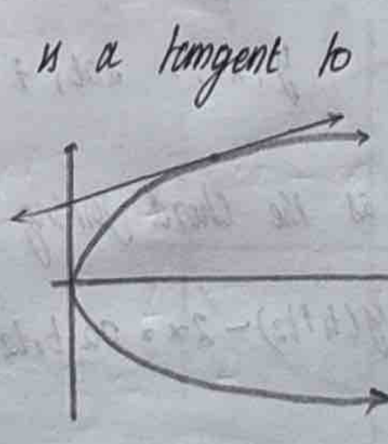
$-t_2 - 2/t_2 = -t_1 - 2/t_1$ ⇒ $t_1(t_2^2+2) = t_2(t_1^2+2)$

⇒ $t_1t_2 = 2$ //

Qn: Find the condition that $lx+my+n=0$ is a tangent to $y^2=4ax$.

Ans:

Let $lx+my+n$ = tanget @ t_1



$lx+my+n=0 \longrightarrow (1)$

Simly tngt is $yt_1 = x+at_1^2$

$-x+yt_1 = at_1^2 \longrightarrow (2)$

Comparing (1) & (2) $\Rightarrow$ $\frac{l}{-1} = \frac{m}{t_1} = \frac{n}{-at_1^2}$ $\Rightarrow$ $t_1 = \frac{-m}{l}$

$\frac{l}{-1} = \frac{n}{-at_1^2}$ $\Rightarrow$ $n = la\, m^2/l^2$

ie $ln = am^2$ $\Rightarrow$ $\underline{am^2 - ln = 0}$

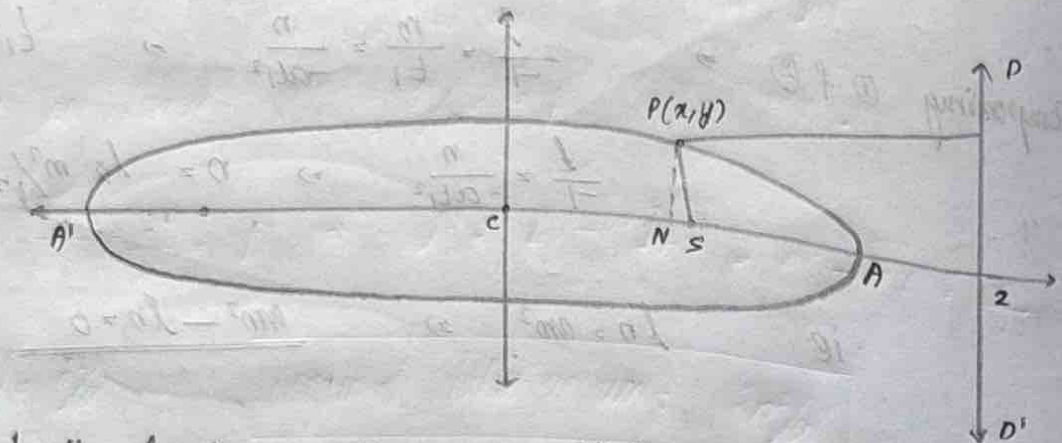
(HW) :- i) Find the equation of the parabola $S(1,1)$ and vertex $(-1,1)$

ii) Find the Condn that $lx+my+n=0$ is a normal to $y^2=4ax$.

iii) S.T the circle discribed on any focal chord of a parabola $y^2=4ax$ has diameter touches the directrix.

Ellipse *

Standard Equation :-



Let S be the focus, DD' be the directrix. Draw S2 $\perp^r$ to DD'. Let it meet the curve at A & A'. Let C be the midpoint of AA'. Choose C as the origin C2 as the x-axis. CA = CA' = a

For any point P(x,y) on the curve :- $\frac{SP}{PM} = e \quad (e<1)$

$$SP = e\,PM \Rightarrow SP^2 = e^2\,PM^2 \longrightarrow ①$$

Since A is a particular position of P :- $\frac{SA}{S2} = e \Rightarrow SA = e\,A2 \longrightarrow ②$

Simly for A' : $\frac{SA'}{A'2} = e \Rightarrow SA' = e\,A'2 \longrightarrow ③$

③ - ② $\Rightarrow$ $SA' - SA = e[A'2 - A2]$

$$[A'C + CS] - [CA - CS] = e\,AA'$$

$$a + cs - a + cs = e\,2a \qquad \Rightarrow \qquad 2cs = 2ae$$

$$\boxed{CS = ae}$$

②+③ $\Rightarrow$ $SA + SA' = e[A2 + A'2] \Rightarrow AA' = e[(C2 - CA) + CA' + C2]$

$$= e[C2 - a + a + C2]$$

$$2a = 2e\,C2$$

Hence $CZ = a/e$ $\boxed{CZ = a/e}$

$$\Rightarrow SP = \sqrt{(x-ae)^2+(y-0)^2} = \sqrt{(x-ae)^2+y^2}$$

$$PM = NZ = CZ - CN = a/e - x$$

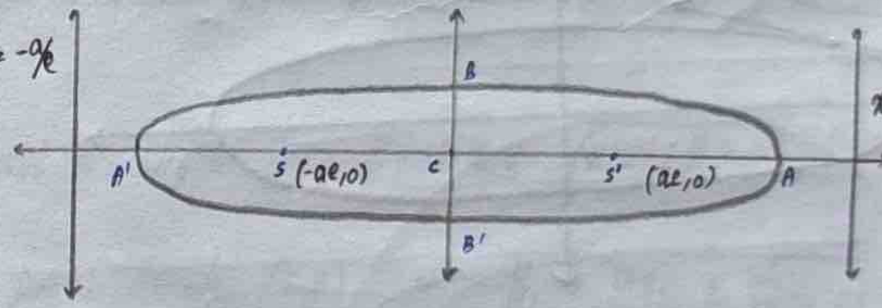
$$a^2/e^2 - \frac{2ax}{e} + x^2$$

now ① $\Rightarrow$ $$(x-ae)^2+y^2 = e^2\left(a/e - x\right)^2$$

$$x^2 - \cancel{2aex} + a^2e^2 + y^2 = e^2 \cdot a^2/e^2 - \cancel{2aex} + x^2e^2$$

$$x^2[1-e^2] + y^2 = a^2[1-e^2]$$

now $\div$ with $a^2[1-e^2]$: $\left.\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2[1-e^2]} = 1\right\}$ $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad b^2 = a^2[1-e^2]}$

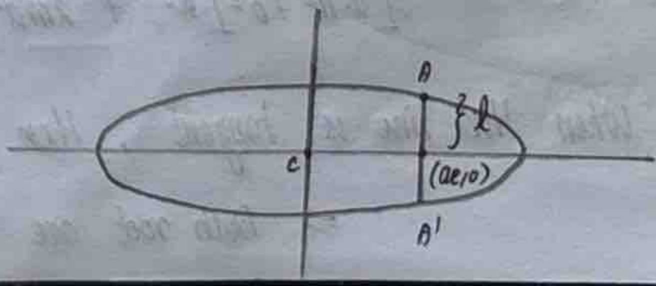


Note :- The Curve is symmetric about ox & oy. Hence it has 2 foci i) S(ae,0) ii) (-ae,0). It has 2 directrix $x = a/e$ and $x = -a/e$. C is called the Centre of the ellipse. C(0,0)

AA' is called major axis : Eq^n $\Rightarrow$ $y = 0$ and length $= 2a$

BB' is called minor axis : Eq^n $\Rightarrow$ $x = 0$ and length $= 2b$

- Length of the latus rectum:-



Double ordinate of the focus : $(ae, 0)$ is called Latus rectum

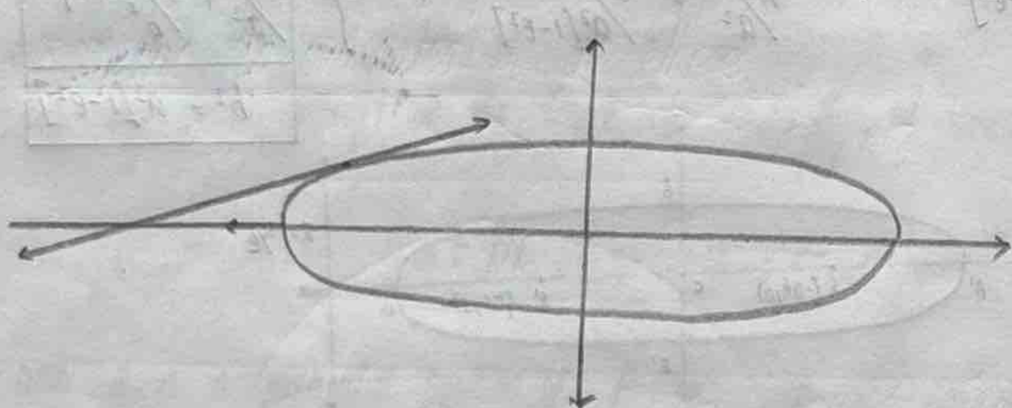
Let $AA' = 2l$, then point $m(AS, SA') = (ae, l)$

If m is a point on the Curve $x^2/a^2 + y^2/b^2 = 1$

$$\frac{a^2e^2}{a^2} + \frac{l^2}{b^2} = 1 \Rightarrow l^2 = [1-e^2]b^2 = b^2 - b^2e^2 =$$

ie we get $l = b^2/a$: $\boxed{\text{Length of latus rectum : } 2b^2/a}$

PYQ :- Find the Condition that $y = mx + c$ be a tngt for $x^2/a^2 + y^2/b^2 = 1$



The x-Coordinate of the point of Contact of $y = mx + c \longrightarrow ①$ of the Curve $x^2/a^2 + y^2/b^2 = 1$ are given by eliminating y from ① & ②

$$\frac{x^2}{a^2} + \frac{(y)^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\frac{b^2x^2 + a^2(m^2x^2 + 2cmx + c^2)}{a^2b^2} = 1$$

$$[a^2m^2 + b^2]x^2 + 2ma^2cx + [a^2c^2 - b^2] = 0$$

* When the line is tangent, then only one point of Contact

$\Rightarrow$ Both root are equal $\Rightarrow B^2 - 4AC = 0$.

$$[2ma^2c]^2 - 4(b^2+a^2m^2)a^2[c^2-b^2] = 0$$

$$\cancel{4m^2a^4c^2} = 4[a^2b^2c^2 - a^2b^4 + \cancel{a^4m^2c^2} - a^2m^2b^2]$$

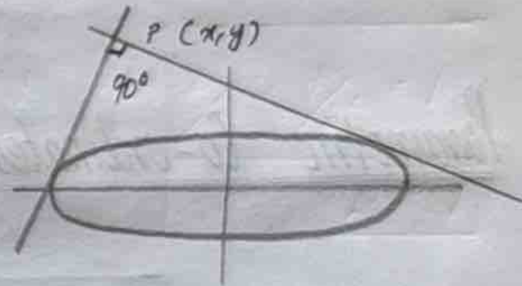
$$0 = b^2c^2 - a^2b^4 - a^2m^2b^2$$

$$= c^2 - a^2b^2 - a^2m^2 \Rightarrow c^2 = a^2m^2 + b^2$$

$$c = \sqrt{a^2m^2+b^2}$$

Note: $y = mx + c$ is a tangt. to ellipse $x^2/a^2 + y^2/b^2 = 1$ when $\boxed{c = \sqrt{a^2m^2+b^2}}$

PYQ :- To find the orthoptic locus of the ellipse $x^2/a^2 + y^2/b^2 = 1$



Let $P(x,y)$ be a point such that tngt through (x,y) are at right angle. To find locus of P

Any tangent to ellipse: $y = mx \pm \sqrt{a^2m^2+b^2} \longrightarrow ①$

If point passes through (x_1, y_1) $\Rightarrow$ $y_1 = mx_1 \pm \sqrt{a^2m^2+b^2}$

$$(y_1 - mx_1) = \pm\sqrt{a^2m^2+b^2}$$

$$y_1^2 - 2my_1x_1 + m^2x_1^2 = a^2m^2 + b^2 \Rightarrow x_1^2m^2 + y_1^2 - 2x_1y_1m - a^2m^2 - b^2 = 0$$

$$(x_1^2 - a^2)m^2 + (-2x_1y_1)m + (y_1^2 - b^2) = 0$$

$$A\,m^2 + B\,m + C = 0$$

on solving we get 2 tangt slopes having value m_1 & m_2. But these tngts are perpendicular. ie

$$m_1 m_2 = -1 \qquad \text{prod} = C/A$$

$$\frac{y_1^2 - b^2}{x_1^2 - a^2} = -1$$

Double ordinate of the focus (ae, 0) is called Latus rectum

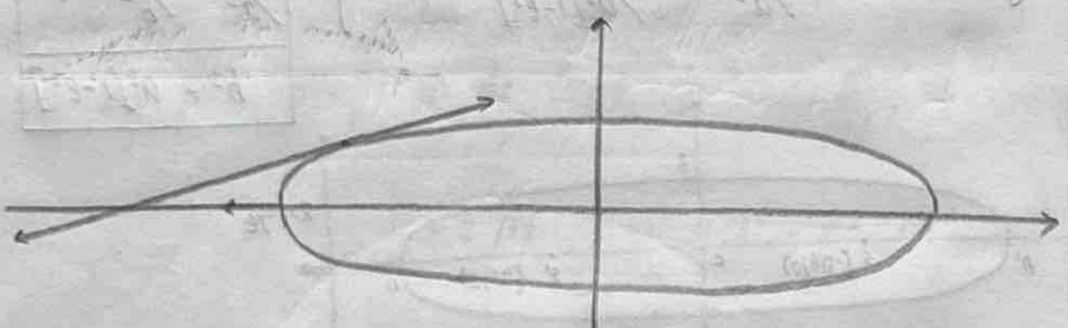
Let $AA' = 2l$, then point $M(AS, SA') = (ae, l)$

If M is a point on the Curve $x^2/a^2 + y^2/b^2 = 1$

$$\frac{a^2e^2}{a^2} + \frac{l^2}{b^2} = 1 \Rightarrow l^2 = [1-e^2]b^2 = b^2 - b^2e^2 = \frac{b^4}{a^2}$$

ie we get $l = b^2/a$: $\boxed{\text{Length of latus rectum} : 2b^2/a}$

PYQ :- Find the Condition that $y = mx + c$ be a tngt for $x^2/a^2 + y^2/b^2 = 1$



The x-Coordinate of the point of Contact of $y = mx + c \longrightarrow$ ① of the Curve $x^2/a^2 + y^2/b^2 = 1$ are given by eliminating y from ① & ②

$$\frac{x^2}{a^2} + \frac{(y)^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\frac{b^2x^2 + a^2(m^2x^2 + 2cmx + c^2)}{a^2b^2} = 1$$

$$[a^2m^2 + b^2]x^2 + 2ma^2cx + [a^2c^2 - b^2] = 0$$

* When the line is tangent, then only one point of Contact

$\Rightarrow$ Both root are equal $\Rightarrow B^2 - 4AC = 0$.

$$[2ma^2c]^2 - 4(b^2+a^2m^2)a^2[c^2-b^2] = 0$$

$$\cancel{4m^2a^4c^2} = 4[a^2b^2c^2 - a^2b^4 + \cancel{a^4m^2c^2} - a^2m^2b^2]$$

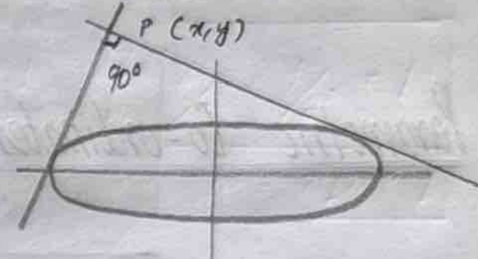
$$0 = b^2c^2 - a^2b^4 - a^2m^2b^2$$

$$= c^2 - a^2b^2 - a^2m^2 \Rightarrow \quad c^2 = a^2m^2 + b^2$$

$$c = \sqrt{a^2m^2+b^2}$$

Note: $y = mx + c$ is a tangt. to ellipse $x^2/a^2 + y^2/b^2 = 1$ when $\boxed{c = \sqrt{a^2m^2+b^2}}$

PYQ :- <u>To find the orthoptic locus of the ellipse $x^2/a^2 + y^2/b^2 = 1$</u>



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$$(y_1 - mx_1) = \pm\sqrt{a^2m^2+b^2}$$

$$y_1^2 - 2my_1x_1 + m^2x_1^2 = a^2m^2 + b^2 \Rightarrow \quad x_1^2m^2 + y_1^2 - 2x_1y_1m - a^2m^2 - b^2 = 0$$

$$(x_1^2 - a^2)m^2 + (-2x_1y_1)m + (y_1^2 - b^2) = 0$$

$$A\,m^2 + B\,m + C = 0$$

on solving we get 2 tangt slopes having value m_1 & m_2. But these tngts are perpendicular. ie

$$m_1m_2 = -1 \qquad | \text{ prod} = C/A$$

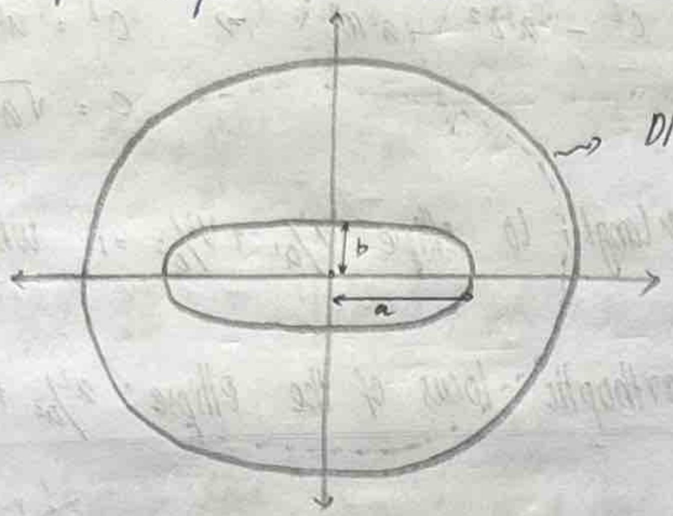
$$\frac{y_1^2 - b^2}{x_1^2 - a^2} = -1$$

ie $\quad x_1^2+y_1^2=a^2+b^2$

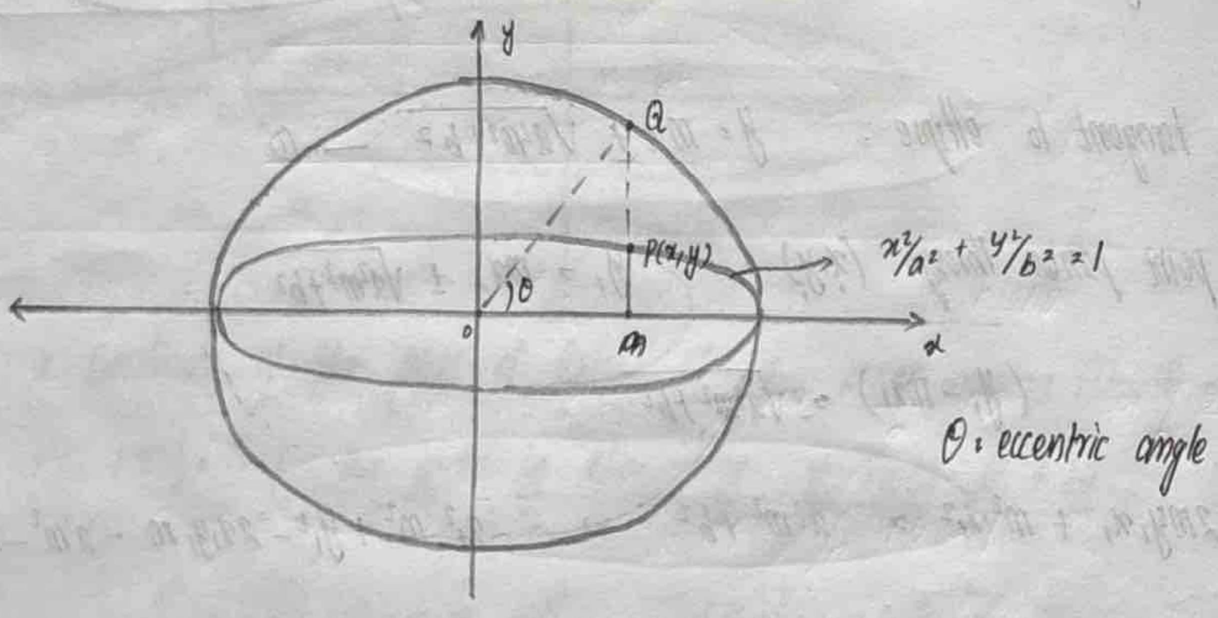
this is a circle with centre @ $(0,0)$ and $r=\sqrt{a^2+b^2}$

orthoptic locus of an ellipse is a circle called director circle.

Fig:



Parametric Co-ordinates of an ellipse



Let $P(x,y)$ is a point on ellipse $x^2/a^2+y^2/b^2=1$, draw PM $\perp^r$ to OX

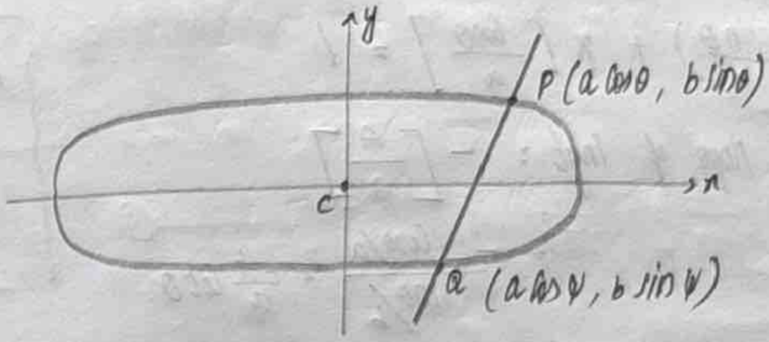
The $\perp^r$ MP met the circle $x^2+y^2=a^2$. Join OQ.

Let $\angle QOM=\theta \qquad OQ=a \Rightarrow \cos\theta=\dfrac{OM}{OQ}=x/a \Rightarrow x=a\cos\theta$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{a^2\cos^2\theta}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2[1-\cos^2\theta]=b^2\sin^2\theta$$

$$y=b\sin\theta$$

To find the chord joining 2 points where eccentric angle are θ & ψ



Eq^n of PQ : $y - y_1 = m(x - x_1) \Rightarrow y - b\sin\theta = \dfrac{b\sin\psi - b\sin\theta}{a\cos\psi - a\cos\theta}(x - a\cos\theta)$

$$y - b\sin\theta = \frac{b[\sin\psi - \sin\theta]}{a[\cos\psi - \cos\theta]}(x - a\cos\theta)$$

$$y - b\sin\theta = \frac{b\left[\cancel{2}\cos\left(\frac{\psi+\theta}{2}\right)\cancel{\sin\left(\frac{\psi-\theta}{2}\right)}\right]}{a\left[-\cancel{2}\sin\left[\frac{\psi+\theta}{2}\right]\cancel{\sin\left[\frac{\psi-\theta}{2}\right]}\right]}(x - a\cos\theta)$$

ie $[y - b\sin\theta]\left(-a\sin\left(\frac{\psi+\theta}{2}\right]\right) = b\cos\left[\frac{\psi+\theta}{2}\right](x - a\cos\theta]$

$$-ay\sin\left[\frac{\psi+\theta}{2}\right] + ab\sin\theta\sin\left[\frac{\psi+\theta}{2}\right] = bx\cos\left[\frac{\psi+\theta}{2}\right] - ab\cos\theta\cos\left[\frac{\psi+\theta}{2}\right]$$

$\div$ with ab: $\quad \frac{x}{a}\cos\left(\frac{\psi+\theta}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\psi}{2}\right) = \frac{ab}{ab}\left[\sin\theta\sin\left(\frac{\theta+\psi}{2}\right) + \cos\theta\cos\left[\frac{\theta+\psi}{2}\right]\right]$

$$= \cos\left[\left(\frac{\theta+\psi}{2}\right) + \theta\right]$$

ie $\boxed{\frac{x}{a}\cos\left(\frac{\theta+\psi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\psi}{2}\right) = \cos\left[\frac{\theta-\psi}{2}\right]}$

Not: when $Q \to P$, the chord becomes the tangent @ P. At P $\psi = \theta$, hence equation of tangent @ P having eccentric angle θ

$$\boxed{\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1}$$

Equation of the normal at $(a\cos\theta, b\sin\theta)$

Eq^n of tangent : $y\left(\frac{\sin\theta}{b}\right) + x\left[\frac{\cos\theta}{a}\right] = 1$

slope of tngt : $-\left[\frac{A}{B}\right]$

$= -\frac{\cos\theta/a}{\sin\theta/b} = \frac{-b}{a}\cot\theta$

$Ax + by + c = 0$

$by = -Ax - c$

$y = \frac{-A}{B}x + \frac{-c}{B}$

ie ~~cross~~ slope of normal $\Rightarrow$ $\frac{-a}{b}\tan\theta$

Eq^n of chord : $y - b\sin\theta = \frac{-a}{b}\frac{\sin\theta}{\cos\theta}(x - a\cos\theta)$

$b\cos\theta(y - b\sin\theta) = -a\sin\theta(x - a\cos\theta)$

$y\, b\cos\theta - b^2\sin\theta\cos\theta = -x\, a\sin\theta + a^2\sin\theta\cos\theta$

$ax\sin\theta + by\cos\theta = \sin\theta\cos\theta\left[a^2 + b^2\right]$

$\div \sin\theta\cos\theta$: $\boxed{\frac{ax}{\cos\theta} \mp \frac{by}{\sin\theta} = a^2 \mp b^2}$

- For ellipse :- Parametric eq^n : $x = a\cos\theta$ $y = b\sin\theta$

Eq^n of chord joining θ & φ : $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \cos\left[\frac{\theta+\varphi}{2}\right]$

Eq^n of tngt @ θ : $x/a\cos\theta + y/b\sin\theta = 1$

Eq^n of normal @ θ : $\left[\frac{a}{\cos\theta}\right]x \mp \left[\frac{b}{\sin\theta}\right]y = a^2 \mp b^2$

PYQ:

Qn: Find the Condition that $x\cos\alpha + y\sin\alpha = p$ a tngt to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ans: $x\cos\alpha + y\sin\alpha = p \longrightarrow ①$ be a tngt @ $(a\cos\alpha, b\sin\alpha)$

But tngt : $x/a\cos\theta + y/b\sin\theta = 1 \longrightarrow ②$

① & ② are identical, ie compare co-efficients

$$\frac{\cos\alpha}{\cos\theta/a} = \frac{\sin\alpha}{\sin\theta/b} = \frac{p}{1} \Rightarrow a\frac{\cos\alpha}{\cos\theta} = b\frac{\sin\alpha}{\sin\theta} = p$$

$$\sin\theta = \frac{b\sin\alpha}{p}, \quad \cos\theta = \frac{a\cos\alpha}{p}, \text{ so } \sin^2\theta + \cos^2\theta = 1$$

$$p^2 = b^2\sin^2\alpha + a^2\cos^2\alpha \Rightarrow p = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$$

HW PYQ

(i) Find the condⁿ that $lx + my + n = 0$ is a tngt to $x^2/a^2 + y^2/b^2 = 1$

(ii) Find the condⁿ that $lx + my + n = 0$ is a normal to to ellipse $x^2/a^2 + y^2/b^2 = 1$

⇒ Find envelop of $y = mx + am^3$ (from internal exam).

Ans:- $y = mx + am^3 \to$ ① $\Rightarrow$ d.p. w.r.t m gives $x + 3am^2 = 0$

$$m^2 = \frac{-x}{3a}$$

① ⇒ $y = m(x + am^2) \Rightarrow y^2 = m^2[x + am^2]^2$

$$= \frac{-x}{3a}\left[x + a\left(\frac{-x}{3a}\right)\right]^2 = \frac{-x}{3a}\left[x - x/3\right]^2$$

$$y^2 = \frac{-x}{3a}\cdot\frac{4x^2}{9} = \frac{-4x^3}{27a}$$

$$27ay^2 = -4x^3 \Rightarrow 27ay^2 + 4x^3 = 0$$

HW

(i) Let $lx + my + n = 0 \to$ ① be a tngt to $[a\cos\theta, b\sin\theta]$

but tngt is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 0 \longrightarrow$ ②

① & ② are identical ⇒ compare coeff: $\frac{l}{\left(\frac{\cos\theta}{a}\right)} = \frac{m}{\left(\frac{\sin\theta}{b}\right)} = \frac{-n}{1}$

ie $\cos\theta = \frac{-al}{n}$ $\sin\theta = \frac{-bm}{n}$

but $\sin^2\theta + \cos^2\theta = 1 \Rightarrow b^2m^2 + a^2l^2 = n^2$ (cond for tangency)

② $lx + my + n = 0$ is a normal at $(a\cos\theta, b\sin\theta)$ $\longrightarrow$ ①

But normal is $\frac{a}{\cos\theta}x - \frac{b}{\sin\theta}y = a^2 - b^2$ ——— ②

① & ② are identical $\Rightarrow \frac{l}{a/\cos\theta} = \frac{m}{-b/\sin\theta} = \frac{-n}{a^2-b^2}$

$\Rightarrow \sin\theta = \frac{-bn}{m(a^2-b^2)}$ $\cos\theta = \frac{-an}{l(a^2-b^2)}$

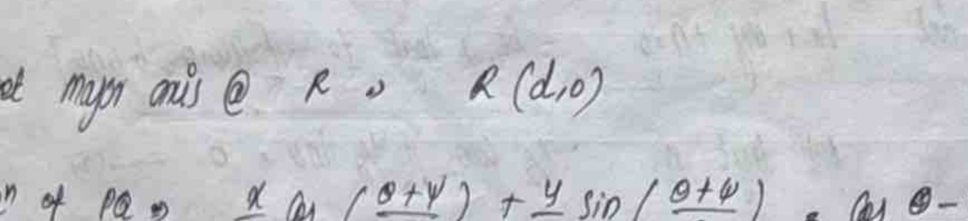
$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \frac{b^2n^2}{m^2(a^2-b^2)^2} + \frac{a^2n^2}{l^2(a^2-b^2)^2} = 1$

ie $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$

PYQ* If the chord joining 2 points whose eccentric angle α & β on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet the major axis at a distance d from the centre, PT

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{d-a}{d+a}$$

Ans.



Let PQ meet major axis @ R $\Rightarrow$ R (d,0)

Eqn of PQ $\Rightarrow \frac{x}{a}\cos\left(\frac{\theta+\varphi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\varphi}{2}\right) = \cos\frac{\theta-\varphi}{2}$

ie here $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\frac{\alpha-\beta}{2}$ $\longrightarrow$ ①

① passes through R (d, 0)

ie $\quad \frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right) \Rightarrow \frac{d}{a} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$

Compon. divid. rule $\Rightarrow$ $\quad \frac{d-a}{d+a} = \frac{\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)}$

$$= \frac{+2\sin\left(\frac{\alpha-\beta}{2}+\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}-\left(\frac{\alpha+\beta}{2}\right)\right)}{2\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}-\left(\frac{\alpha+\beta}{2}\right)\right)}$$

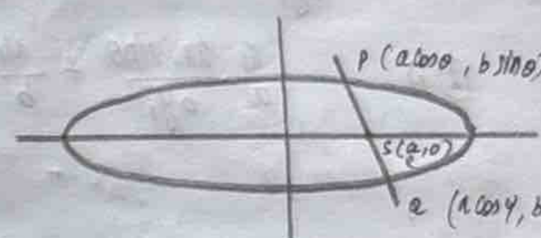
$$\frac{d-a}{d+a} = \frac{\sin\alpha/2\ \sin\beta/2}{\cos\alpha/2\ \cos\beta/2} = \underline{\tan\alpha/2\ \tan\beta/2}$$

~~Qn 8 The locus of the point of intersection of~~

joining $(\theta \,\&\, \varphi)$

Qn 8 If PQ is a focal chord of $x^2/a^2 + y^2/b^2 = 1$ prove that

$$\cos\left(\frac{\theta-\varphi}{2}\right) = e\cos\left(\frac{\theta+\varphi}{2}\right)$$



Ans:- Eq^n of PQ $\Rightarrow$

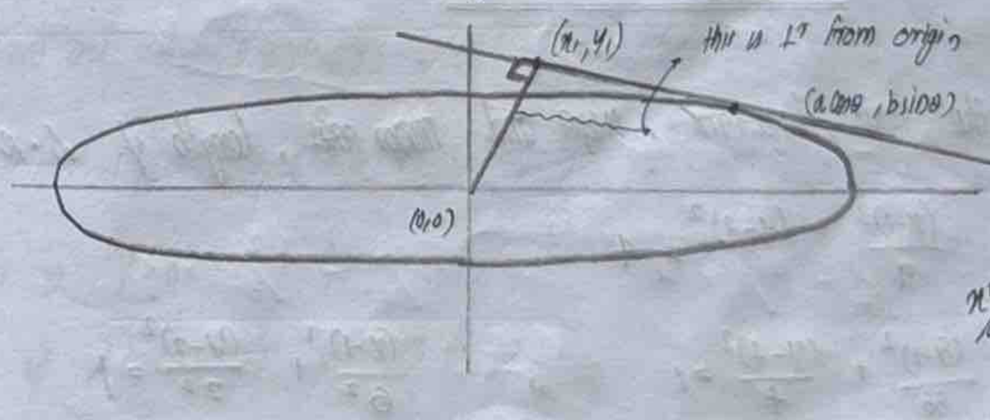
$$\frac{x}{a}\cos\left(\frac{\theta+\varphi}{2}\right) + \frac{y}{b}\sin\left(\frac{\theta+\varphi}{2}\right) = \cos\frac{\theta-\varphi}{2}$$

It passes through $[(ae), 0]$ $\Rightarrow$

$$\frac{ae}{a}\cos\left(\frac{\theta+\varphi}{2}\right) = \cos\frac{\theta-\varphi}{2} \quad \Rightarrow \quad e\cos\left[\frac{\theta+\varphi}{2}\right] = \cos\left[\frac{\theta-\varphi}{2}\right]$$

Qn:- If P is the foot of $\perp^r$ from the origin to any tangt. to the ellipse, PT locus of P is $(x^2+y^2)^2 = (a^2x^2+b^2y^2)$

Ans:



$x^2/a^2 + y^2/b^2 = 1$

Consider a tngt @ $(a\cos\theta, b\sin\theta)$ and eq^n of tangent :

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \longrightarrow ①$$

But (x_1, y_1) is a point on the line $\Rightarrow \frac{x_1}{a}\cos\theta + \frac{y_1}{b}\sin\theta = 1 \longrightarrow ②$

(*) Slop of tngt $\Rightarrow$ $-\frac{a}{b} = -\left(\frac{\cos\theta/a}{\sin\theta/b}\right) = \frac{-b\cos\theta}{a\sin\theta} = m_1$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1}$$

tangent and normal $\perp^r$ $(m_1 m_2 = -1)$ $\Rightarrow \frac{-b\cos\theta}{a\sin\theta} \cdot \frac{y_1}{x_1} = -1$

ie $\quad by_1\cos\theta = ax_1\sin\theta \longrightarrow ③$

$\cdot \cos\theta = \frac{ax_1\sin\theta}{by_1} \qquad \cdot \sin\theta = \frac{by_1\cos\theta}{ax_1}$

② $\Rightarrow \frac{x_1}{a}\cdot\frac{ax_1\sin\theta}{by_1} + \frac{y_1}{b}\sin\theta = 1 \Rightarrow \frac{\sin\theta}{b}\left[\frac{x_1^2}{y_1} + y_1\right] = 1$

$$\sin\theta\frac{(x_1^2+y^2)}{by_1} = 1$$

$\sin\theta = \frac{by_1}{(x_1^2+y_1^2)} \qquad$ simly $\cos\theta = \frac{ax_1}{(x_1^2+y_1^2)}$

But $\sin^2\theta + \cos^2\theta = 1 \qquad \frac{b^2y_1^2}{(x_1^2+y_1^2)^2} + \frac{a^2x_1^2}{(x_1^2+y_1^2)^2} = 1$

$$a^2x_1^2 + b^2y_1^2 = (x_1^2+y_1^2)^2$$

ie locus $\Rightarrow \quad \underline{\underline{a^2x^2 + b^2y^2 = (x^2+y^2)^2}}$

$\frac{y_1}{b}\frac{by_1\cos\theta}{ax_1}$

$\frac{y_1^2\cos\theta}{ax_1}$

$\frac{\cos\theta}{a}\left[x_1 + \frac{y_1^2}{x_1}\right]$

$= \frac{\cos\theta}{x_1 a}(x_1^2+y_1^2) = 1$

$(x_1^2+y_1^2) = \frac{ax_1}{\cos\theta}$

Qns- Find the focus, directrix, major and minor axis, length of L.R of

① $\frac{(x-1)^2}{9} + (y-2)^2 = 4$

Ans:- $\frac{(x-1)^2}{36} + \frac{(y-2)^2}{4} = 1 \quad \Rightarrow \quad \frac{(x-1)^2}{6^2} + \frac{(y-2)^2}{2^2} = 1$

$X = x-1$ } w.r.to new axis: $a^2 = 36$ $b^2 = 4$ $b^2 = a^2[1-e^2]$
$Y = y-2$ } $a = 6$ $b = 4$ $4 = 36[1-e^2]$ $e^2 = 8/9$

$e = \frac{2\sqrt{2}}{3}$

Focus: $S(ae, 0) = \left(6 \times \frac{2\sqrt{2}}{3}, 0\right) = (4\sqrt{2}, 0)$

Similarly $S' = (-4\sqrt{2}, 0)$

Directrix: $x = \frac{a}{e} = \frac{6 \cdot 3}{2\sqrt{2}} = 9/\sqrt{2} \Rightarrow x = \pm(+9/\sqrt{2})$

Eqn of major axis: $Y = 0 \Rightarrow 0 = y-2 \Rightarrow y = 2$

Minor axis: $X = 0 \Rightarrow x = 1$

Length of LR: $\frac{2b^2}{a} = \frac{2 \times 4}{6} = 4/3$

w.r.t original axis

$X = x-1 \Rightarrow x = X+1$
$Y = y-2 \Rightarrow y = Y+2$

$S = (4\sqrt{2}+1, 2)$

$x = 9/\sqrt{2}+1$ $y =$

② $4x^2 + 9y^2 - 48x + 72y = 144$ $8x - 48 = 0$

③ $5x^2 + 9y^2 - 3 \quad - 36y + 1 = 0$

②

$4(x^2 - 12x) + 9(y^2 + 8y) = 144$

$4(x^2 - 12x + 36 - 36) + 9(y^2 + 8y + 16 - 16) = 144$

$4(x-6)^2 + 9(y+4)^2 = 144 + 4 \times 36 + 9 \times 16 = 432$

$\frac{(x-6)^2}{108} + \frac{(y+4)^2}{48} = 1 \Rightarrow \frac{X^2}{a^2} + \frac{Y^2}{48} = 1$

$\sqrt{60} = \sqrt{2 \times 30} = \sqrt{2 \times 2 \times 15} = 2\sqrt{15}$

$b^2 = a^2[1-e^2] \Rightarrow 48 = 108[1-e^2]$ $e^2 = 1 - \frac{48}{108} = \frac{108-48}{108} = \frac{60}{108} \Rightarrow e = \sqrt{\frac{5}{9}}$

$= \sqrt{5}/3$

① WRT New axis: $S = (ae, 0) = \left(\sqrt{108} \cdot \frac{\sqrt{5}}{3}, 0\right) = \left(\sqrt{\frac{540}{9}}, 0\right) = (2\sqrt{15}, 0)$

$S' = (-2\sqrt{15}, 0)$

Major axis $\Rightarrow Y = 0$ Minor axis $= X = 0$

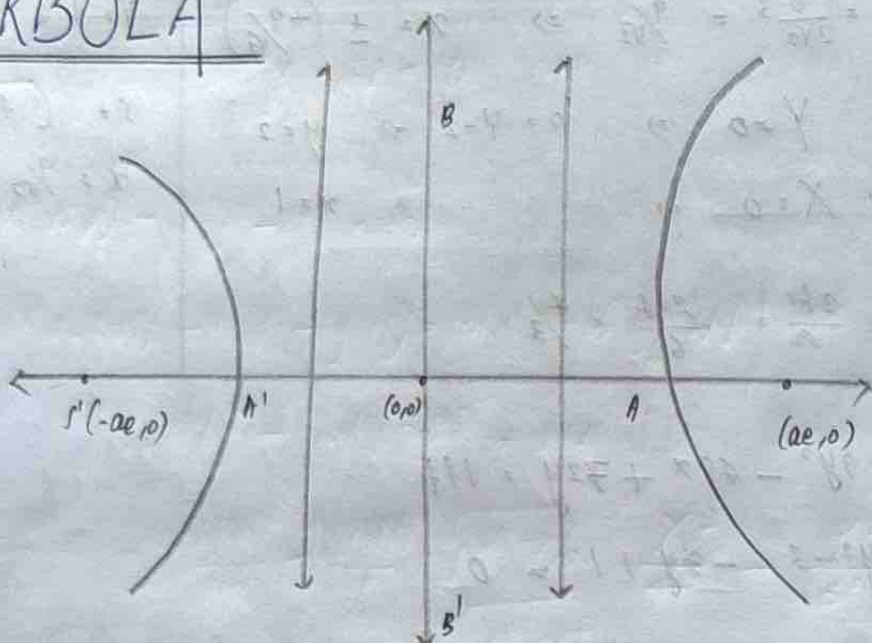
Length of LR $= 2b^2/a = \frac{2 \times 48}{\sqrt{108}} = \frac{96}{\sqrt{108}}$

① W.R.T original axis : $X = x - 6 \Rightarrow x = X + 6$

$y = Y + 4 \quad y = Y + 4 \Rightarrow y =$

$\Rightarrow S = (6 + 2\sqrt{15}, 0 - 4)$

HYPERBOLA



The equation is $x^2/a^2 - y^2/b^2 = 1$ where $b^2 = a^2[e^2 - 1]$

* Focus : $S(ae, 0)$, $S'(-ae, 0)$

Directrix : $x = a/e \quad x = -a/e$

AA' = Transverse axis $\Rightarrow$ Eq^n $\Rightarrow$ $y = 0$ and length AA' = 2a

BB' = Conjugate axis $\Rightarrow$ Eq^n $\Rightarrow$ $x = 0$ and length BB' = 2b

Qn:- Find focus, eq^n of the directrix, axes of the hyperbola for

① $9x^2 - 16y^2 = 144$

Ans : $x^2/16 - y^2/9 = 1 \Rightarrow x^2/4^2 - y^2/3^2 = 1$

$b^2 = a^2[e^2-1] \Rightarrow 16 = 9[e^2-1] \quad \frac{16}{9}+1 = e^2 \Rightarrow \frac{25}{16} = e^2 \Rightarrow e = 5/4$

$\Rightarrow$ Focus: $S = (ae, 0) = (4\times 5/4, 0) = (5, 0)$

$S' = (-ae, 0) = (-5, 0)$

Eq^n of the directrix : $x = a/e = \frac{4\times 4}{5} = 16/5 \qquad x = \pm 16/5$

Transverse axis : $y = 0$ Conjugate axis, $x = 0$

Length of LR = $2b^2/a = \frac{2\times 9}{4} = 18/4 = 9/2$

(2) $\frac{(x+1)^2}{9} - \frac{(y-2)^2}{7} = 1$

Ans: in the form $\frac{X}{3^2} - \frac{Y}{(\sqrt{7})^2} = 1 \qquad X = x+1 \quad Y = y-2$

$b^2 = a^2[e^2-1] \Rightarrow 7 = 9[e^2-1] \Rightarrow \frac{7}{9}+\frac{9}{9} = e^2 \Rightarrow e = 4/3$

(i) W.r.t new axis :- $S(ae, 0) = (3\times 4/3, 0), (4, 0)$

$S' = (-ae, 0) = (-4, 0)$

Directrix $\Rightarrow x = a/e = \frac{3\times 3}{4} = 9/4 \qquad x = \pm 9/4$

Transverse axis : $Y = 0$ Conjugate axis. $X = 0$

Length of latus rectum : $2b^2/a = \frac{2\times 7}{3} = \frac{14}{3}$

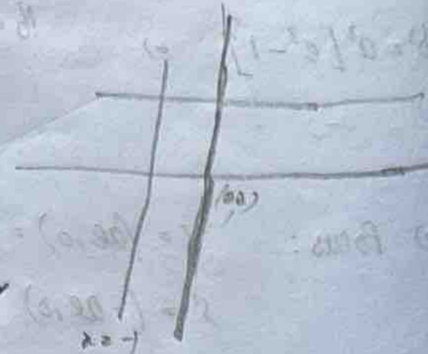
(ii) w.r.t original axis :- $X = x+1 \Rightarrow x = X-1 \qquad y = Y+2$

ie $x = 4-1 = 3 \quad y = 0+2 = 2 \qquad S(3, 2)$

$S'(-5, 2)$

• Directrix: $X = 9/4 \Rightarrow x = 9/4 - 1 = 5/4$

$X' = -9/4 \Rightarrow x' = \frac{-9}{4} - \frac{4}{4} = \frac{-13}{4}$



Transverse axis: $y = 2$ Conjugate axis: $x = -1$

③ $9x^2 - 16y^2 + 18x + 32y - 151 = 0$

Ans: $9(x^2 + 2x) - 16(y^2 - 2y) = 151$

$9(x^2 + 2x + 1) - 16(y^2 - 2y + 1) = 151 + 9 - 16 = 144$

$9(x+1)^2 - 16(y-1)^2 = 144$

$\frac{(x+1)^2}{16} - \frac{(y-1)^2}{9} = 1 \Rightarrow \frac{X^2}{4^2} - \frac{Y^2}{3^2} = 1$

$b^2 = a^2[e^2 - 1] \Rightarrow$ $9 = 16[e^2 - 1] \Rightarrow e^2 = \frac{25}{16}$ $e = 5/4$

• w.r.t new axis:- $S = (ae, 0) = (4 \times 5/4, 0) = (5, 0)$

$S' = (-ae, 0) = (-5, 0)$

Directrix $\Rightarrow x = a/e = \frac{4 \times 4}{5} = 16/5 \Rightarrow x = \pm 16/5$

Length of L.R $= 2b^2/a = \frac{2 \times 9}{4} = 9/2$

Transverse axis: $Y = 0$ Conju. axis: $X = 0$

• w.r.t original axis:- $X = x + 1$ $Y = y - 1$

$x = X - 1$ $y = Y + 1$

$S = (4, 1)$, $S'(-6, 1)$

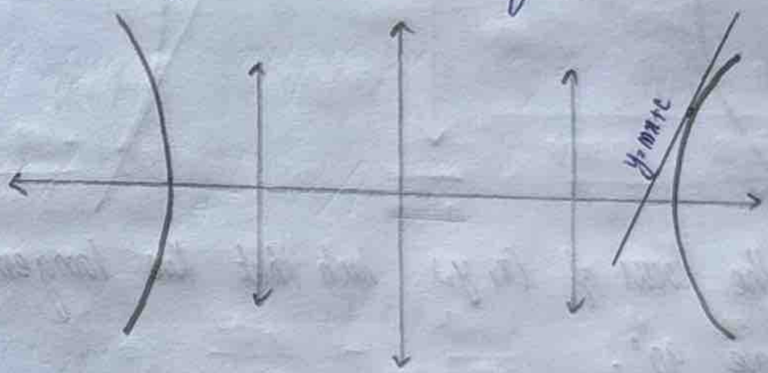
Directrix $\Rightarrow$ $x = 11/5$ $x' = \frac{-21}{5}$

Transverse axis : $y = 1$ Conjugate axis : $x = -1$

④ $3x^2 - y^2 + 10x + 2y - 1 = 0$

⇒ Find the condⁿ that $y = mx + c$ be a tangent to $x^2/a^2 - y^2/b^2 = 1$

Ans :-



The x-coordinate of the point of contact $x^2/a^2 - y^2/b^2 = 1$ to $y = mx + c$ are given by eliminating y.

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1 \Rightarrow b^2x^2 - a^2[m^2x^2 + 2mcx + c^2] = a^2b^2$$

$$b^2x^2 - a^2m^2x^2 - 2a^2mcx - a^2c^2 - a^2b^2 = 0$$

$$x^2[b^2 - a^2m^2] - x[2a^2mc] - a^2[c^2 + b^2] = 0$$

Since the x-co-ordinate of tngt have only one value, this quadratic have similar roots ⇒ $B^2 - 4AC$ of quad is zero.

$$\cancel{4}(a^{\cancel{4}\,2}m^2c^2) = -\cancel{4}(b^2 - a^2m^2)(c^2+b^2)a^2$$

$$\cancel{m^2c^2 = b^2c^2 + 2b^2 - a^2m^2c^2 - a^2m^2b^2}$$

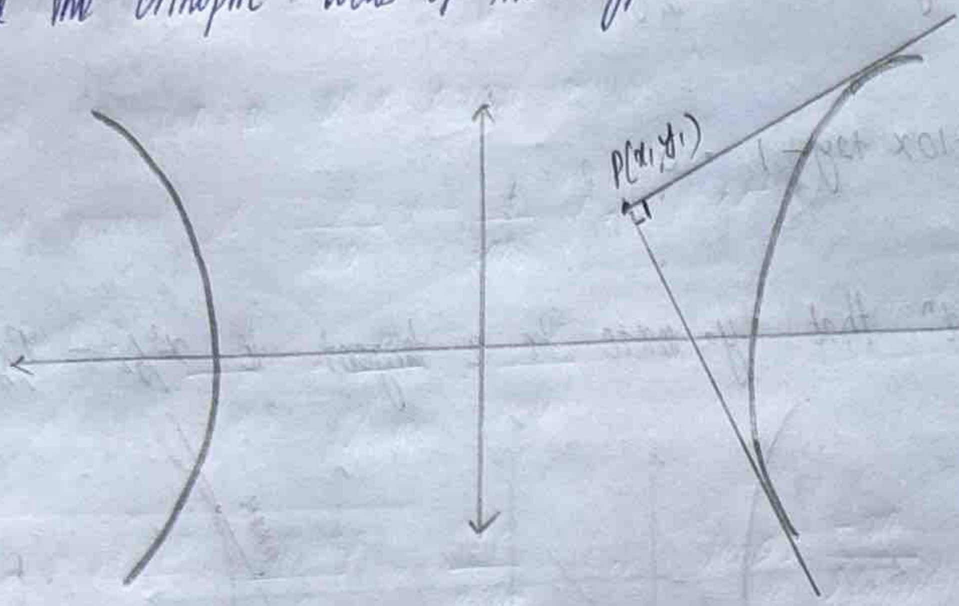
$$\cancel{m^2c^2[1+a^2] = b^2[c^2 + 2 - a^2m^2]}$$

$$c^2 = a^2m^2 - b^2 \Rightarrow c = \pm\sqrt{a^2m^2 - b^2}$$

Note : eqⁿ $y = mx \pm \sqrt{a^2m^2 - b^2}$ is a tangnet to hyperbola for diff. values of m.

⇒ To find the orthoptic locus of the hyperbola

Ans:



orthoptic locus is the locus of (x_1, y_1) such that the tangents from P to hyperbola meets @ angle $90°$.

Eq^n of tngt: $y = mx \pm \sqrt{a^2m^2 - b^2}$ (passes through x_1, y_1)

$y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2}$

$(y_1 - mx_1)^2 = a^2m^2 - b^2 \Rightarrow y_1^2 - 2mx_1y_1 + m^2x_1^2 = a^2m^2 - b^2$

$y_1^2 - 2x_1y_1m + m^2(x_1^2 - a^2) - b^2 = 0$

$m^2(x_1^2 - a^2) - 2x_1y_1 m + (y_1^2 + b^2) = 0$

forms: $(m^2 A + -B\ m + C = 0)$

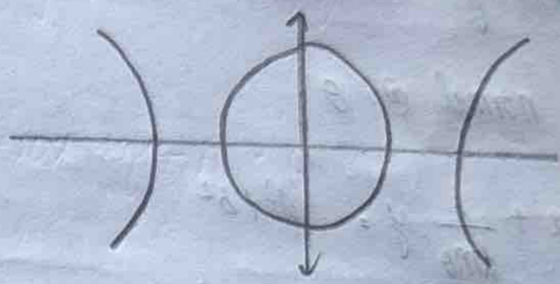
solving we get 2 values for m which are the slopes of 2 tangents. But tangents are $\perp^r$. ie $m_1 m_2 = -1$ (prod. of root = -1)

$C/A = -1 \Rightarrow \dfrac{(y_1^2 + b^2)}{x_1^2 - a^2} = -1 \Rightarrow y_1^2 + b^2 = a^2 - x_1^2$

$$\boxed{x_1^2 + y_1^2 = a^2 - b^2}$$

ie locus of (x_1, y_1) is $x_1^2 + y_1^2 = (\sqrt{a^2 - b^2})^2$

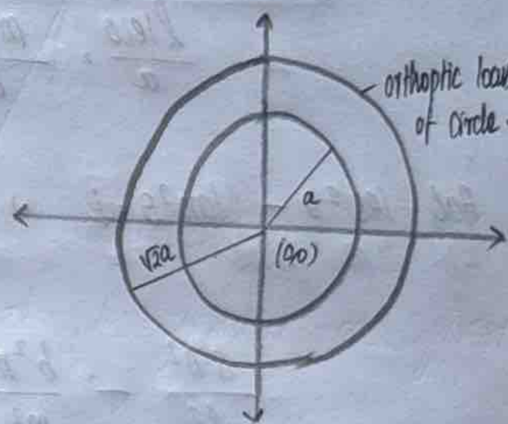
which is a circle with centre (0,0) and $r = \sqrt{a^2-b^2}$



★ orthoptic locus of ellipse : $x^2+y^2 = a^2+b^2$

But when $a=b$, the ellipse becomes a circle. So to find the orthoptic locus of circle,

$x^2+y^2 = a^2+a^2 = 2a^2$ $\Rightarrow (x^2+y^2) = (\sqrt{2}a)^2$



PARAMETRIC EQ^N OF HYPERBOLA

Any point on the hyperbola can be taken as $x = a\sec\theta$, $y = b\tan\theta$ which is parametric eq^n.

θ = eccentric angle.

→ Eq^n of the chord joining θ & ψ :-

$$\frac{x}{a}\cos\left(\frac{\theta-\psi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\psi}{2}\right) = \cos\left(\frac{\theta+\psi}{2}\right)$$

⇒ Eq^n of the tangent at θ

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

⇒ Eq^n of the normal @ θ

$$\frac{a}{\sec\theta}x + \frac{b}{\tan\theta}y = a^2+b^2$$

PYQ: Find the condition that $lx+my+n=0$ be a normal to hyperbola.

Ans: Let $lx+my+n=0$ is a normal @ θ

But normal is $\frac{a}{\sec\theta}x+\frac{b}{\tan\theta}y=a^2+b^2$

Compare coeff: $\frac{l}{a/\sec\theta}=\frac{m}{b/\tan\theta}=\frac{-n}{a^2+b^2}$

$\frac{l\sec\theta}{a}=\frac{m\tan\theta}{b}=\frac{-n}{a^2+b^2} \Rightarrow \sec\theta=\frac{-an}{l(a^2+b^2)}$ $\tan\theta=\frac{-bn}{m(a^2+b^2)}$

But $\sec^2\theta-\tan^2\theta=1 \Rightarrow \frac{a^2n^2}{l^2(a^2+b^2)}-\frac{b^2n^2}{m^2(a^2+b^2)}=1$

$\frac{a^2n^2}{l^2}-\frac{b^2n^2}{m^2}=a^2+b^2 \Rightarrow \frac{a^2}{l^2}-\frac{b^2}{m^2}=\frac{a^2+b^2}{n^2}$

Q. Prove that $x\cos\alpha+y\sin\alpha=p$ is a tangent to hyperbola $x^2/a^2-y^2/b^2=1$ if $p^2=a^2\cos^2\alpha-b^2\sin^2\alpha$

Ans: given $x\cos\alpha+y\sin\alpha=p$ is a tgt to hyperbola @ α

But tgt is $\cancel{\frac{a}{\sec\alpha}x+\frac{b}{\tan\alpha}y=a^2+b^2}$ $x/a\sec\theta-y/b\tan\theta=1$

Compare coeff: $\cancel{\frac{\cos\alpha}{a/\sec\alpha}}$ $\frac{\cos\alpha}{\sec\theta/a}=\frac{-b\sin\alpha}{\tan\theta}=-p$

$\sec^2\theta-\tan^2\theta=1 \Rightarrow \frac{a^2\cos^2\alpha}{p^2}-\frac{b^2\sin^2\alpha}{p^2}=1 \Rightarrow a^2\cos^2\alpha-b^2\sin^2\alpha=p^2.$

③ Find the Condⁿ that $y = mx + c$ be a normal to $x^2/a^2 - y^2/b^2 = 1$

Ans: Let $y = mx + c$ be a normal @ θ ⟶ ①

But normal is $\frac{a}{\sec\theta}x + \frac{b}{\tan\theta}y = a^2 + b^2$ ⟶ ②

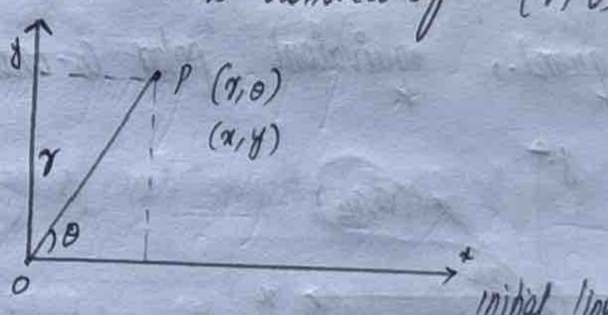
Compare Coeff: $\frac{-m}{a/\sec\theta} = \frac{1}{b/\tan\theta} = \frac{c}{a^2+b^2}$

$\sec^2\theta - \tan^2\theta = 1 \rightarrow \left(\frac{-ac}{m(a^2+b^2)}\right)^2 - \left(\frac{bc}{a^2+b^2}\right)^2 = 1$

$$\frac{a^2c^2}{m^2(a^2+b^2)^2} - \frac{b^2c^2}{(a^2+b^2)^2} = 1 \Rightarrow \frac{a^2}{m^2} - b^2 = \frac{(a^2+b^2)^2}{c^2}$$

POLAR CO-ORDINATES

Consider a fixed point O and a horizontal line through O which is called initial line. Let P be any point on the plane. Let OP = r and OP makes θ with initial line. Then P is denoted by (r, θ)



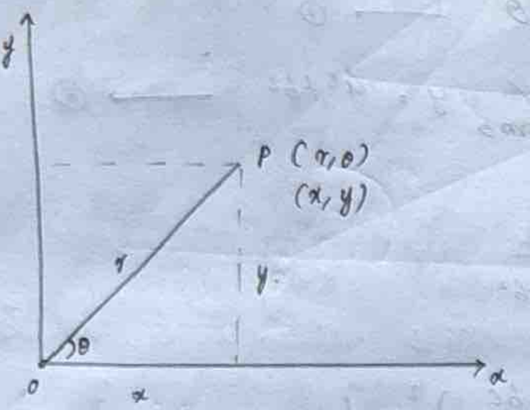
⇒ r is +ve if measured from O along the line bounding the Vectorial angle θ.

⇒ r is -ve if measured in the opposite dirⁿ.

⇒ θ is +ve if it is measured in the anticlock wise dirⁿ

⇒ θ is -ve if it measured in -ve dirⁿ.

Relation b/w Cartesian and polar Co-ordinates



$\cos\theta = x/r \qquad \sin\theta = y/r$

$\boxed{x = r\cos\theta} \quad \boxed{y = r\sin\theta}$

Also :- $x^2 + y^2 = r^2 \Rightarrow \boxed{r = \pm\sqrt{x^2+y^2}}$

$\boxed{\theta = \tan^{-1}(y/x)}$

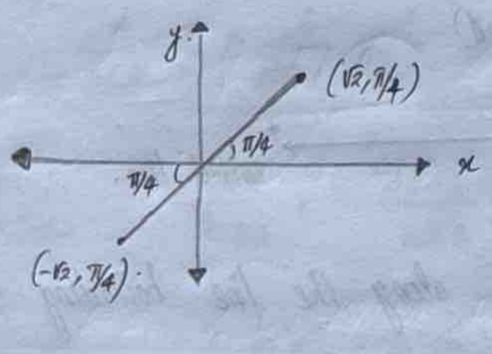
Eg: (i) in polar $P = (3, 45^\circ) \Rightarrow r = 3$ & $\theta = 45^\circ$

$\Rightarrow x = r\cos\theta = 3/\sqrt{2} \qquad y = r\sin\theta = 3/\sqrt{2}$ $\quad \underset{\text{in polars}}{P(3, 45^\circ)} = \underset{\text{in cartesian.}}{P(3/\sqrt{2}, 3/\sqrt{2})}$

(ii) Consider $(-1, -1)$ in Cartesian :-

$r = \sqrt{(-1)^2+(-1)^2} = \sqrt{2} \qquad \theta = \tan^{-1}(y/x) = \tan^{-1}(1) = 45^\circ$ //

Since $(-1, -1)$ is in the 3rd quad., equivalent polar co-ordinate is $(-\sqrt{2}, \pi/4)$



CURVES IN POLAR CO-ORDINATES

① Consider $x^2+y^2 = 2x \Rightarrow x^2+y^2+2x+0y+0 = 0$

which is a circle.

$x^2+y^2 = 2x \Rightarrow r^2 = 2(r\cos\theta) \Rightarrow \boxed{r = 2\cos\theta}$

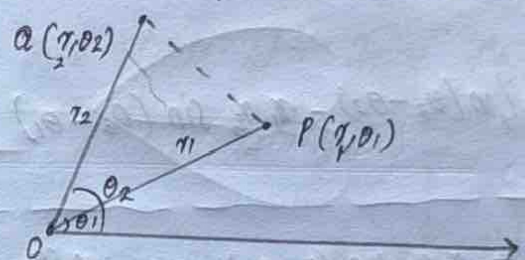
In polar Co-ordinate $r = 2\cos\theta$ represents a Circle.

① Consider $r = -8\sin\theta \Rightarrow r^2 = -8(r\sin\theta)$

$$x^2+y^2 = -8y \Rightarrow x^2+y^2-8y+0x+0 = 0$$

⑥ The Circle $r = 2\cos\theta$ in polar Corresponds to $x^2+y^2+0x+-8y+0$ in Condition.

② Distance formulae in polar Co-ordinate :-

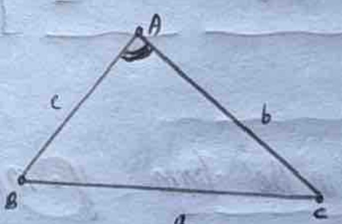


Let $P(r_1,\theta_1)$ $Q = (r_2,\theta_2)$

In $\triangle OPQ$ $OP = r_1$, $OQ = r_2$

$\angle POQ = \theta_2 - \theta_1$

(*)

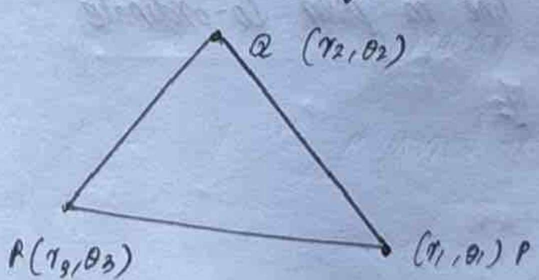


$$a^2 = b^2+c^2-2bc\cos A$$

$$PQ^2 = OP^2 + OQ^2 - 2OP \cdot OQ \cos(\angle POQ)$$

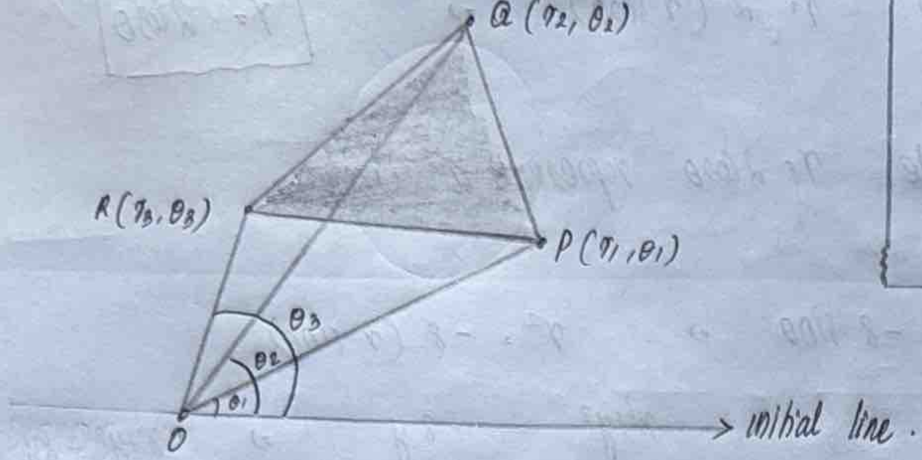
$$\boxed{PQ^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2-\theta_1)}$$

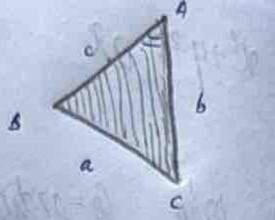
③ Area of a triangle :-



Area of the $\triangle$ PQR with vertices

$(r_1,\theta_1), (r_2,\theta_2), (r_3,\theta_3)$.





Area $= \frac{1}{2} bc \sin A$.

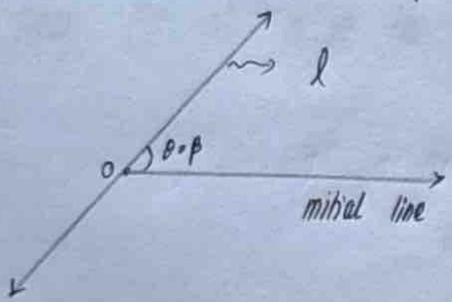
Area of $\triangle PQR$ = Area of $\triangle OPQ$ + Area of $\triangle OQR$ + [− Area of $\triangle OPR$]

$$= \frac{1}{2}(OP)(OQ)\sin(\angle POQ) + \frac{1}{2}(OQ)(OR)\sin(\angle QOR) + -\frac{1}{2}(OP)(OR)\sin(\angle POR)$$

$$= \frac{1}{2}\left[r_1r_2\sin(\theta_2-\theta_1) + r_2r_3\sin(\theta_3-\theta_2) - r_1r_3\sin(\theta_3-\theta_1)\right]$$

$$\boxed{\triangle = \frac{1}{2}\left[r_1r_2\sin(\theta_2-\theta_1) + r_2r_3\sin(\theta_3-\theta_2) - r_1r_3\sin(\theta_3-\theta_1)\right]}$$

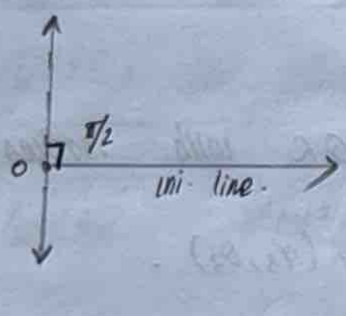
(7) (i) If $\theta = \beta \Rightarrow \beta = \tan^{-1}(y/x)$ $\quad y = \tan\beta\, x$ in the form $y = mx$



$l :- y = \tan\beta\, x$ is a line passing through origin

$\Rightarrow$ Eq^n of initial line :- $y=0$ $\quad r\sin\theta = 0 \Rightarrow \boxed{\theta = 0}$

(ii) $r\cos\theta = a \Rightarrow x = a$ is a vertical line



Note :- Eq^n of vertical line in polar Co-ordinate

$\theta = \pi/2$

(iii) $r\sin\theta = b \Rightarrow y = b$ is a horizontal line

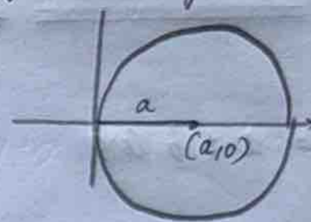
(iv) $r = a$ (θ can be any) $\Rightarrow r^2 = a^2$

is a Circle $\qquad x^2+y^2 = a^2$ is a Circle

(v) $r = 2a\cos\theta$

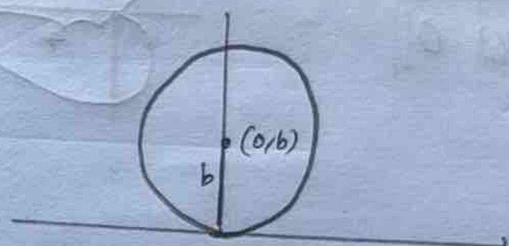
$r^2 = 2a\,(r\cos\theta) \Rightarrow x^2+y^2 = 2ax \Rightarrow x^2+y^2-2ax+0y+0 = 0$

is a Circle @ $(a,0)$ where $r = a$



(vi) $r = 2b\sin\theta \Rightarrow r^2 = 2b\,r\sin\theta \Rightarrow x^2+y^2+0x-2bx+0 = 0$

is a Circle at Centre $(0,b)$ & $r = b$.



Qn 8- Trace the Curve $r = 6\sin\theta$. Also write the Cartition sign.

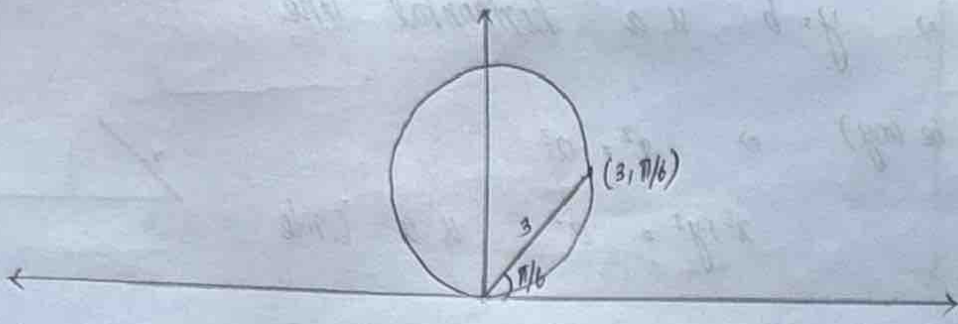
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	2π
$r = 6\sin\theta$	0	3	$3\sqrt{2}$	$3\sqrt{3}$	6	$3\sqrt{2}$	0	$-3\sqrt{2}$	0

} with the plot lines on polars Co-ordi.

OR

$r = 6\sin\theta \Rightarrow r^2 = 6\,(r\sin\theta) \Rightarrow x^2+y^2 = 6y$

$x^2+y^2+0x-2(3y)+0 = 0 \Rightarrow C\,(0,3)$ and $r = 3$

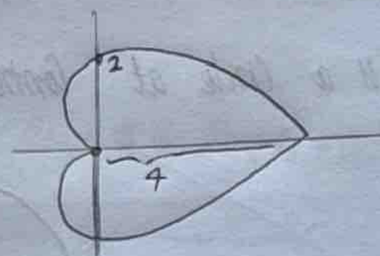


• CARDIOIDS *

$r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$ is called cardioids.

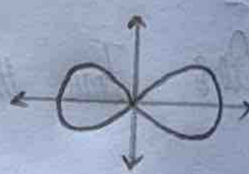
eg: $r = 2[1 + \cos\theta]$ where $a = b = 2$.

θ	0	$\pi/4$	$\pi/2$	$\frac{3\pi}{4}$	π
r	4	$2[1+1/\sqrt{2}]$	2	$2(1-1/\sqrt{2})$	0

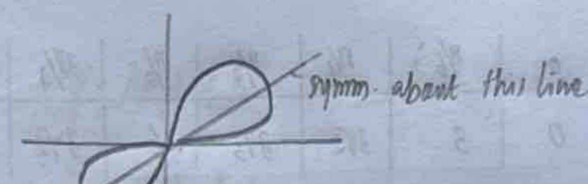


LEMNISCATES ⊛

It resembles the shape of infinity (∞). eg$_1$: $\boxed{r^2 = a^2\cos 2\theta}$

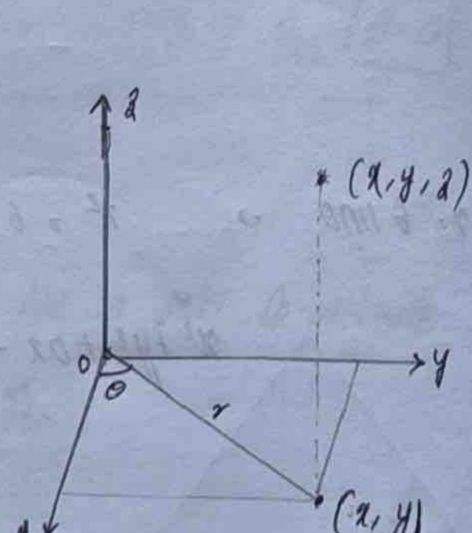


eg$_2$: $r^2 = a^2\sin 2\theta$:-



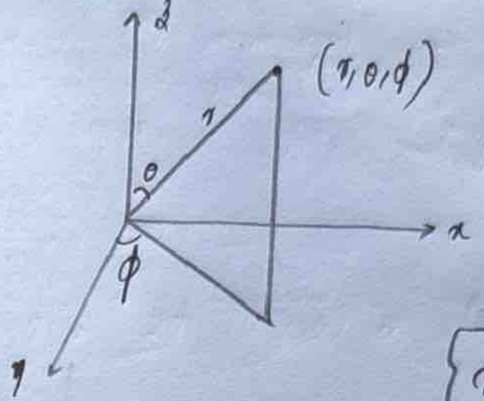
CYLINDRICAL CO-ORDINATES *

Here we are introducing a polar co-ordinate in x-y plane.



$\Rightarrow (x, y, z) = (r\cos\theta, r\sin\theta, z)$ [3 D system]

SPHERICAL CO-ORDINATES *



In cartesian Coordinates :-

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta .$$

$$\boxed{r = \sqrt{x^2+y^2+z^2}}$$

$$\boxed{\theta = \cos^{-1}(z/r) = \cos^{-1}\left(\frac{x}{r \sin\theta}\right)}$$