Module 5: Matrix Method in Structural Analysis

ANALYSIS OF STRUCTURES

Dr. Vysakh K B
Assistant Professor
Dept. of Ship Technology
CUSAT

Introduction

- Indeterminate structures are crucial in structural engineering.
- Traditional methods are not suitable for complex structures.
- Matrix method is used for high-degree indeterminate structures.
- Two types: Flexibility Matrix Method & Stiffness Matrix Method.

Flexibility Matrix Method

- Developed from the consistent deformation method.
- Basic **unknowns are redundant forces**.
- Here, we need to identify basic determinate structure & thereby redundant forces.
- Number of redundant forces = degree of static indeterminacy.
- Method uses matrix equations to solve for redundant forces.
- Also known as **Force, or Compatibility Method**.

Stiffness Matrix Method

- Developed from the slope deflection method.
- Basic **unknowns are joint displacements (includes slope and deflection)**.
- Equations of equilibrium are formed and solved.
- Also known as **Displacement, or Equilibrium Method.**

Degree of Static Indeterminacy

- In structures, if there are more unknown forces (reactions) than the available equilibrium equations, the structure is statically indeterminate.
- In this case, extra equations other than the available static equilibrium equations are needed.
- This number of additional equations (or additional unknowns) is known as Degree of Redundancy.
  - **DSI=Total unknown support reactions−Available equilibrium equations**

How many extra supports or connections exist beyond what's necessary for stability?

Degree of Kinematic Indeterminacy

- The number of independent displacements allowed at a joint is known as degree of kinematic indeterminacy or degree of freedom.
- It essentially quantifies the structure's ability to move or deform.
- A structure is said to be kinematically indeterminate if the displacement components of its joints cannot be determined by compatibility equations alone.
- For these structures, additional equations based on equilibrium conditions must be formulated.
- DKI=Total independent joint displacements−Available constraint equations.

How Many Ways Can It Move?

Degrees of Freedom (DOF) at a Beam’s End

Each type of support allows or restricts movement differently:

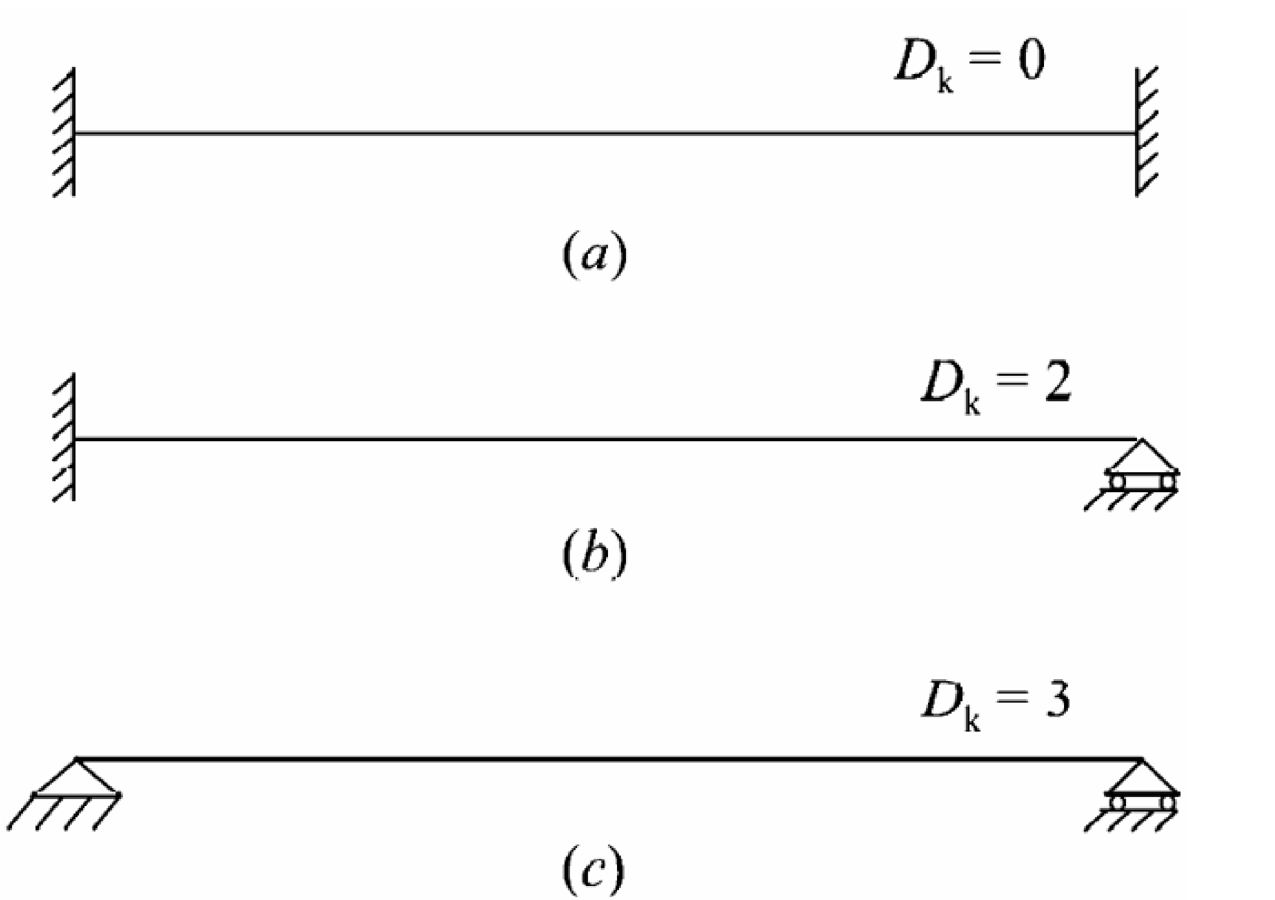
1️⃣ **Free end → 3 DOF** (can move left-right, up-down, and rotate)

2️⃣ **Roller → 2 DOF** (can rotate & move horizontally)

3️⃣ **Hinged end → 1 DOF** (can rotate but can’t move)

4️⃣ **Fixed end → 0 DOF** (no movement at all)

Kinematic Indeterminacy for Different Types of Beams



Kinematic Indeterminacy for Different Types of Frames

For pin-jointed frames

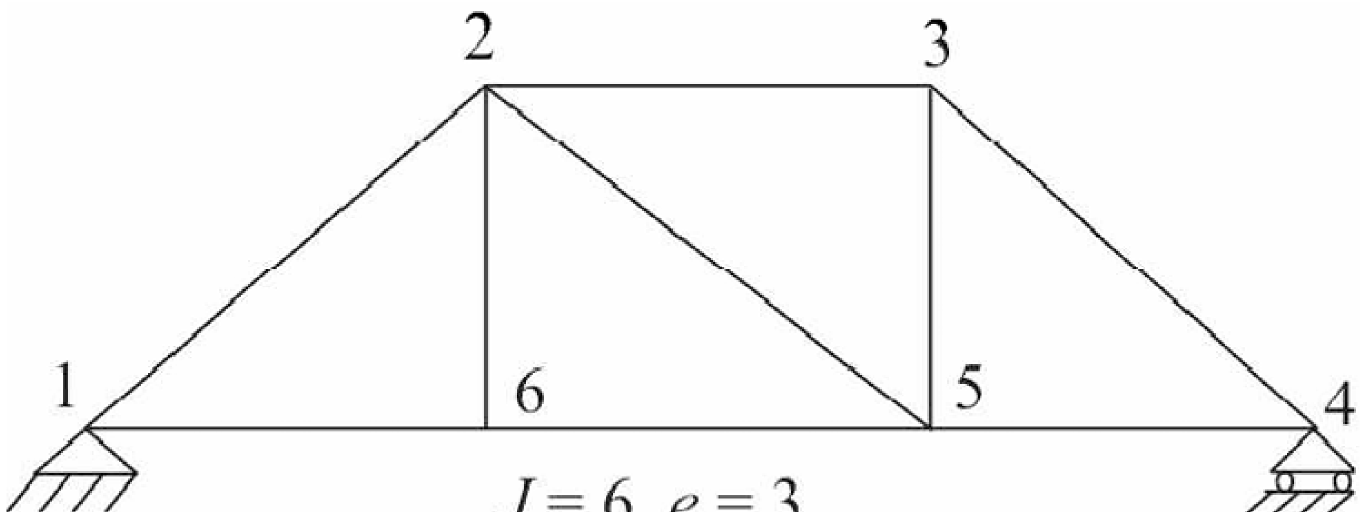
$$D_k = 2j - e \text{ for plane frames} = 3j - e \text{ for space frames}$$

For rigid-jointed frames

$$D_k = 3j - e \text{ for plane frames}$$
$$= 6j - e \text{ for space frames}$$

Where e is the number of support reactions and j is the number of joints

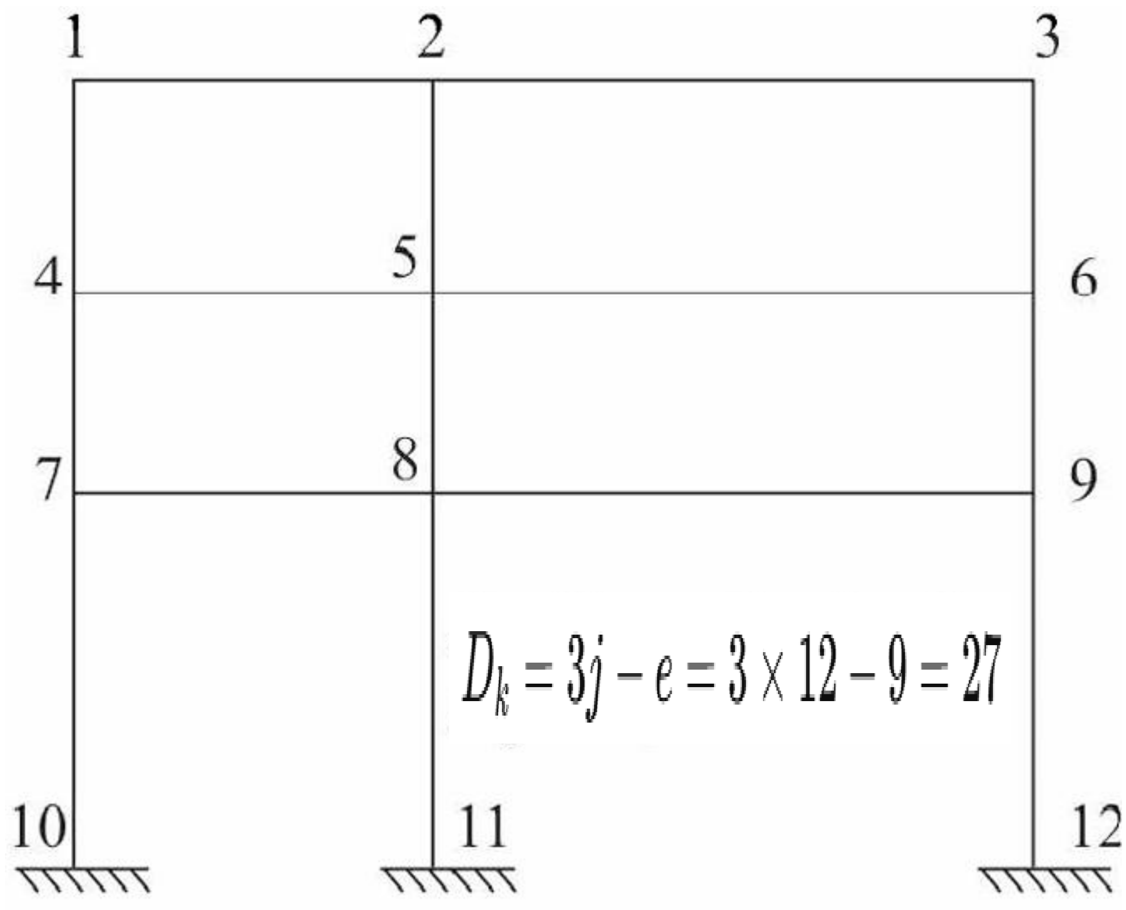
Example 1



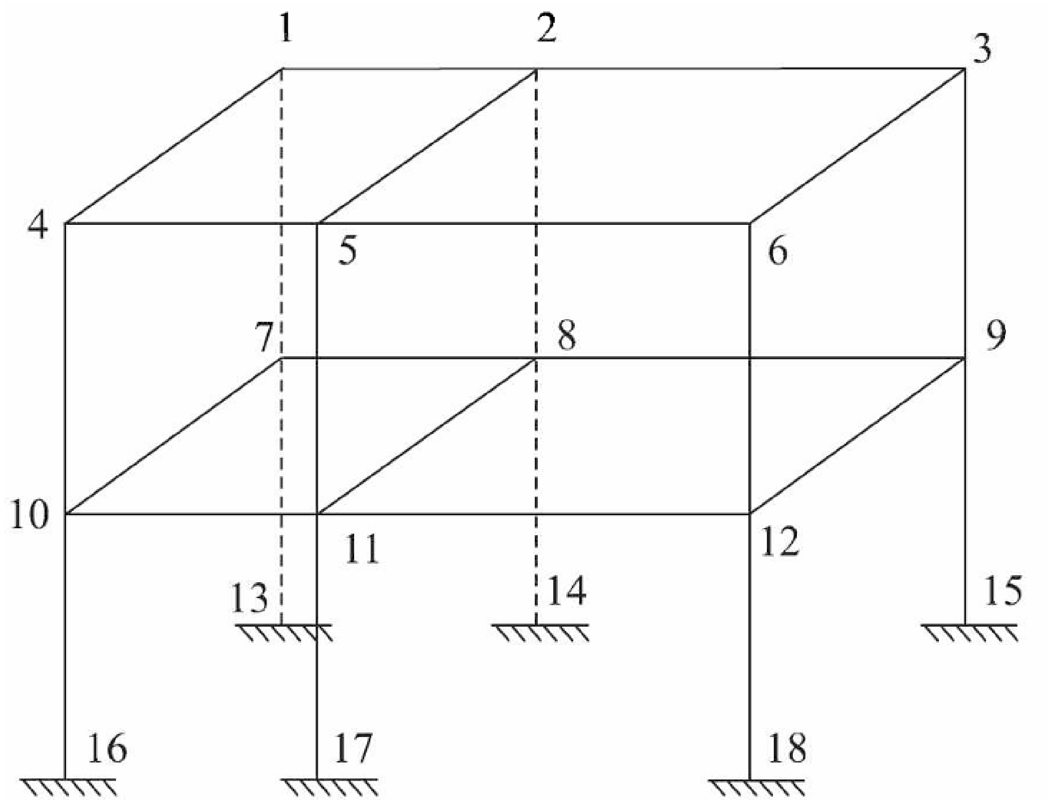
$$J = 6,\ e = 3$$

$$D_k = 6 \times 2 - 3 = 9$$

Example 2



Example 3



(*f*)

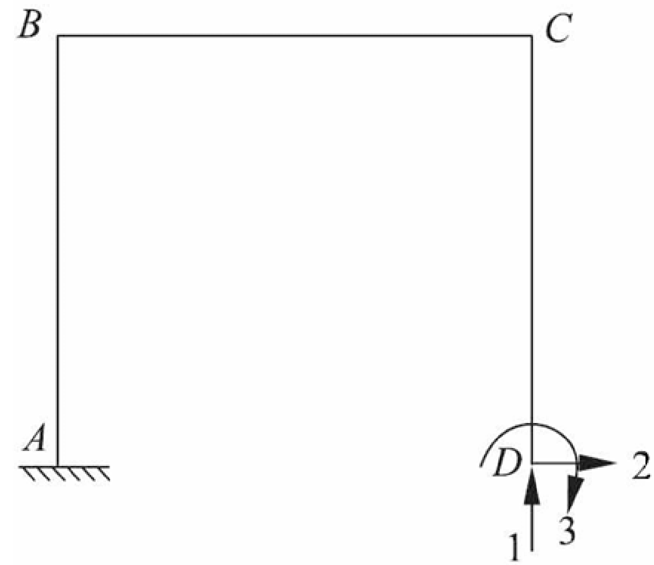
$J = 18,\ e = 6 \times 6 = 36$

$D_k = 18 \times 6 - 36 = 72$

GENERALIZED COORDINATE SYSTEMS

- The meaning of coordinate system in matrix methods of structural analysis is different from the cartesian or polar coordinate system.
- The **directions of unknown forces or displacements** to determine the structural systems are known as ***generalized coordinates***.

- For example, in the analysis of a single bay single storey frame shown in Figure, the coordinates selected are 1, 2, and 3.
- Then, redundant force vector [P] and displacement vector [Δ] are given by,



$$[P] = \begin{bmatrix} V_D \\ H_D \\ M_D \end{bmatrix} \text{ and } [\Delta] = \begin{bmatrix} \Delta V_D \\ \Delta H_D \\ \theta_D \end{bmatrix}$$

FLEXIBILITY MATRIX

- If a structure has *n* number of coordinates, its displacement response to the forces is represented by,

$$[\delta] = \begin{bmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \delta_{n1} & \delta_{n2} & \dots & \delta_{nn} \end{bmatrix}$$

- This is known as the **flexibility matrix.**
- The element δij of a flexibility matrix is the displacement at coordinate i due to a unit force at coordinate j.

- Hence, $\boldsymbol{\delta_{ii}P_i}$ is the displacement at i due to force Pi. Similarly, $\boldsymbol{\delta_{i2}P_2}$ is the displacement at coordinate i due to force P2 at coordinate 2.
- Thus, to develop the entire flexibility matrix, unit force should be applied successively at coordinates 1, 2, 3, ... n and the displacement at all the coordinates are computed.
- From Maxwell’s reciprocal theorem, $\boldsymbol{\delta_{ij}=\delta_{ji}}$.

STIFFNESS MATRIX

- If a structure has *n* coordinates, its force response to the displacement is represented by,

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{bmatrix}$$

- This is known as the **stiffness matrix.**
- The element kij is the force at coordinate i due to a unit displacement at coordinate j.

- Hence, **kiiΔi** is the force developed at coordinate i due to displacement Δi at i. Similarly, **kijΔj** is the force developed at coordinate i due to displacement Δj at coordinate j.
- Thus, to develop the stiffness matrix, unit displacement should be given successively in coordinates 1, 2, ... n and the forces developed at all the coordinates are computed.
- From Maxwell's theorem, $\mathbf{k_{ij} = k_{ji}}$.

FLEXIBILITY MATRIX METHOD

- Basic unknowns to be determined are the redundant forces.
- Hence, the degree of static indeterminacy of the structure is identified first and then the **coordinate number is assigned to each redundant force direction.**
- Then **redundants are removed**, the resulting structure is called as **basic determinate structure or released structure**.
- From the principle of superposition, the net displacement at any point in a determinate structure is the **sum of the displacements due to** the **applied loads in the basic determinate structure** and **redundant forces**.

$$\Delta_1 = \Delta_{1L} + \delta_{11}P_1 + \delta_{12}P_2 + \ldots + \delta_{1n}P_n$$
$$\Delta_2 = \Delta_{2L} + \delta_{21}P_1 + \delta_{22}P_2 + \ldots + \delta_{2n}P_n$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\Delta_n = \Delta_{nL} + \delta_{n1}P_1 + \delta_{n2}P_2 + \ldots + \delta_{nn}P_n$$

In matrix form

$$[\Delta] = \Delta_L + [\delta][P]$$
$$[P] = [\delta]^{-1}\ [(\Delta) - (\Delta_L)]$$

Δ_i = displacement in i^{th} coordinate direction

δ_{ij} = displacement at i due to unit force at j (flexibility matrix element)

Δ_{iL} = displacement at i due to given loading in released structure in coordinate direction i

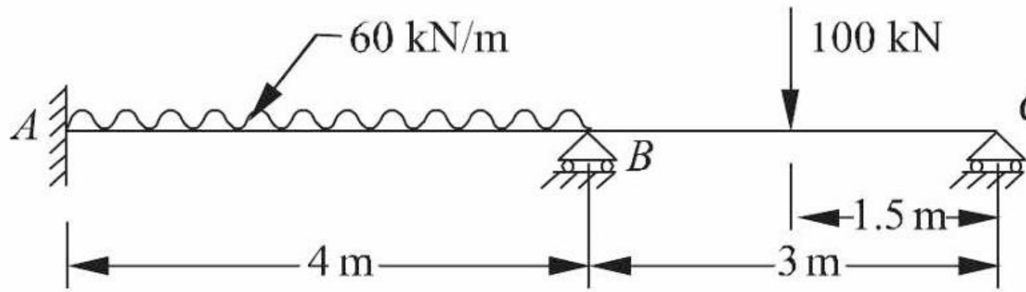
- In the indeterminate structure, the final displacements [Δ] are either zero or known values.
- The solution for [P] from the above equation gives all the redundant forces. Then, using equations of static equilibrium: bending moment, and shear forces required at any point may be found.

Steps in Flexibility Method

1. Determine the degree of static indeterminacy n and choose the redundants.
2. Release redundant forces and get the basic determinate structure. Then assign the coordinates to the redundant force directions.
3. Determine the deflections in coordinate directions due to given loading.
4. Formulation of flexibility matrix
5. Apply the flexibility equation

Example 1

- Analyse the continuous beam shown in Figure by the flexibility matrix method.



Step 1: Determine the degree of static indeterminacy n and choose the redundants

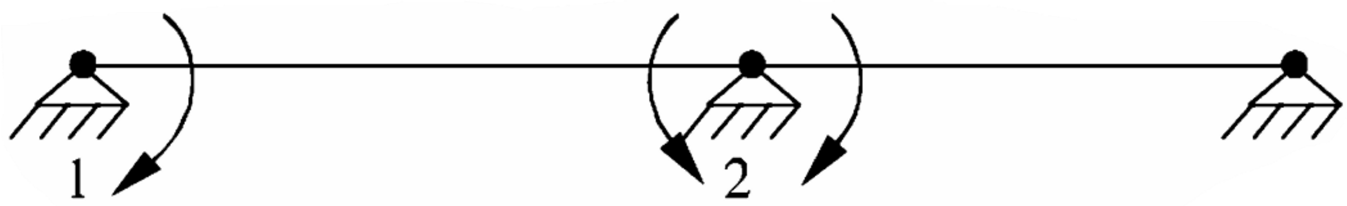
Number of reaction components = 5

Number of independent equations of equilibrium = 3

Degree of static indeterminacy = 5 - 3 = 2

Select M_A and M_B as redundant forces.

Step 2: Release redundant forces and get the basic determinate structure. Then assign the coordinates to the redundant force directions.



Released beam.

Therefore, the released structures are the two independent simply supported beams AB and BC.

Step 3: Determine the deflections in coordinate directions due to given loading

For this, we can use conjugate beam method.

- The **M/EI diagram** of the actual beam is treated as a load in the conjugate beam (here M is the **free** BMD).
- The shear force in the conjugate beam at any point gives the slope at the corresponding point in the actual beam.
- The BM in the conjugate beam at any point gives the deflection at the corresponding point in the actual beam.

- In the conjugate beam method, support conditions of the real beam are transformed to the conjugate beam as follows: **a fixed support becomes a free support, a free support becomes a fixed support, and a simple support (hinge or roller) remains a simple support.**

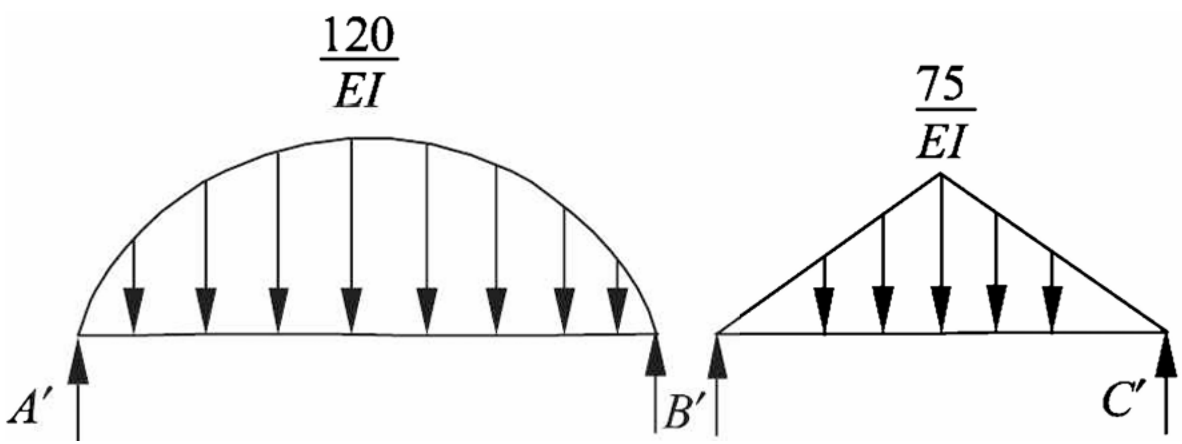
In portion AB, the free moment diagram is a symmetric parabolic curve with maximum ordinates

$$= \frac{60 \times 4^2}{8} = 120 \text{ kNm}$$

In portion BC, the free moment diagram is a symmetric triangle with maximum ordinate

$$= 100 \times \frac{3}{4} = 75 \text{ kNm}$$

Conjugate beam with load as M/EI diagram



Rotation at coord. 1 **due to given loading**

$$\Delta_{1L} = \text{Shear in conjugate beam at } A = \frac{1}{2} \times \frac{2}{3} \times \frac{120}{EI} \times 4 = \frac{160}{EI}$$

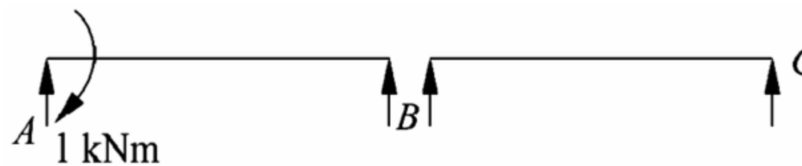
Rotation at coord. 2 **due to given loading**

$$\begin{aligned}\Delta_{2L} &= \text{Rotation at } B \text{ in } A'B' + \text{Rotation at } B \text{ in } B'C' \\ &= \text{Shear at } B \text{ in } A'B' + \text{Shear at } B \text{ in } B'C' \\ &= \frac{1}{2}\left(\frac{2}{3} \times \frac{120}{EI}\right) \times 4 + \frac{1}{2}\left(\frac{1}{2} \times \frac{75}{EI} \times 3\right) = \frac{216.25}{EI}\end{aligned}$$

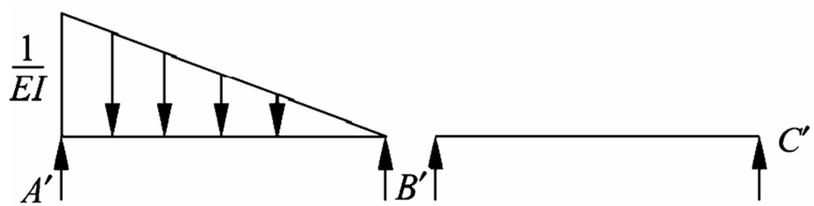
Step 4: Formulation of flexibility matrix

- To find the flexibility matrix, a **unit force is applied in one coordinate direction and the resulting displacements in all other coordinate directions are found.**
- To find out the displacements in coord. 1 and 2 due to forces at 1 and 2 each, we can **again use conjugate beam method.**

First applying a 1KNm moment in coord direction 1.



M/EI diagram



(For a simply supported beam with a triangular distributed load, the S.F at the supports are:

S.F at support with max. load intensity = 2/3 (Total Load)

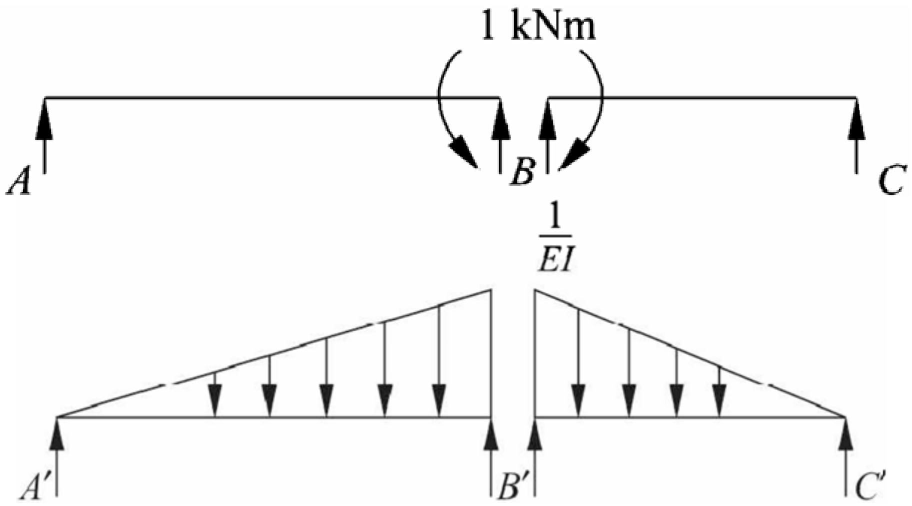
S.F at support with zero load intensity = 1/3 (Total Load))

Therefore

$$\delta_{11} = \frac{2}{3}\times\frac{1}{2}\times\frac{1}{EI}\times 4 = \frac{4}{3EI}$$

$$\delta_{21} = \frac{1}{3}\times\frac{1}{2}\times\frac{1}{EI}\times 4 = \frac{2}{3EI}$$

Then applying a 1KNm moment in coord direction 2



M/EI diagram

$$\delta_{12} = \frac{1}{3}\times\frac{1}{2}\times\frac{1}{EI}\times 4 = \frac{2}{3EI}$$

$$\delta_{22} = \frac{2}{3}\times\frac{1}{2}\times 4\times\frac{1}{EI}+\frac{2}{3}\times\frac{1}{2}\times\frac{1}{EI}\times 3 = \frac{7}{3EI}$$

Shear at B in $A'B'$ + Shear at B in $B'C'$

Step 5: Apply the flexibility equation

From consistency condition, the displacements in the coordinate directions in the actual continuous beam,

$$\Delta_1 = 0, \quad \Delta_2 = 0$$

(Here $\Delta_2 = 0$ since the net rotation b/w BA and BC is zero)

The flexibility matrix equation is,

$$[\delta]\ [P] = [\Delta] - [\Delta_L]$$

$$[\delta]\ [P] = [\Delta] - [\Delta_L]$$

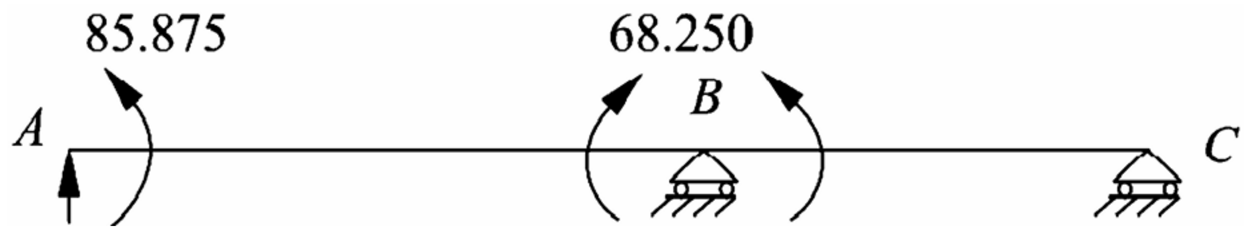
$$\begin{bmatrix} \frac{4}{3EI} & \frac{2}{3EI} \\ \frac{2}{3EI} & \frac{7}{3EI} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{160}{EI} \\ \frac{216.5}{EI} \end{bmatrix}$$

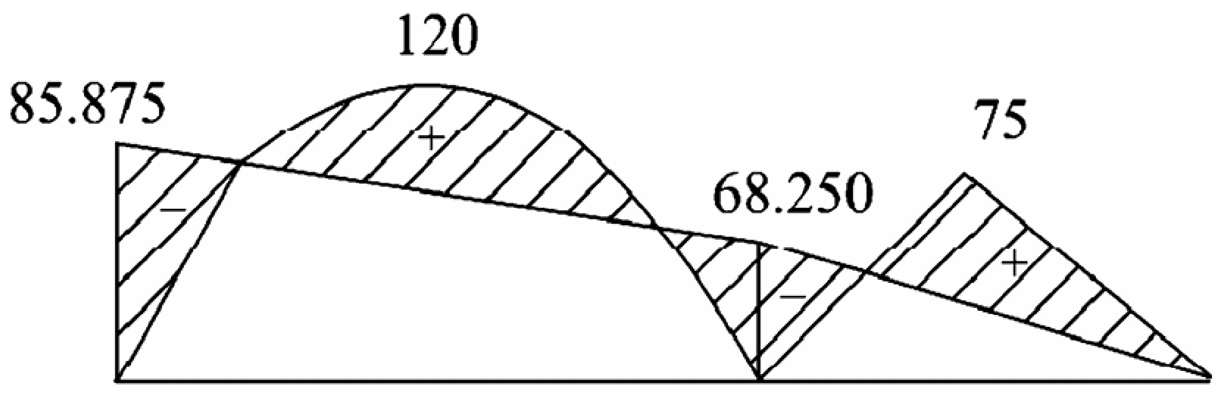
$$\frac{1}{3EI} \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} -160 \\ -216.5 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 3 \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}^{-1} \begin{bmatrix} -160 \\ -216.5 \end{bmatrix}$$

$$= \frac{3}{4 \times 7 - 2 \times 2} \begin{bmatrix} 7 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -160 \\ -216.5 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -687 \\ -546 \end{bmatrix} = \begin{bmatrix} -85.875 \\ -68.250 \end{bmatrix}$$

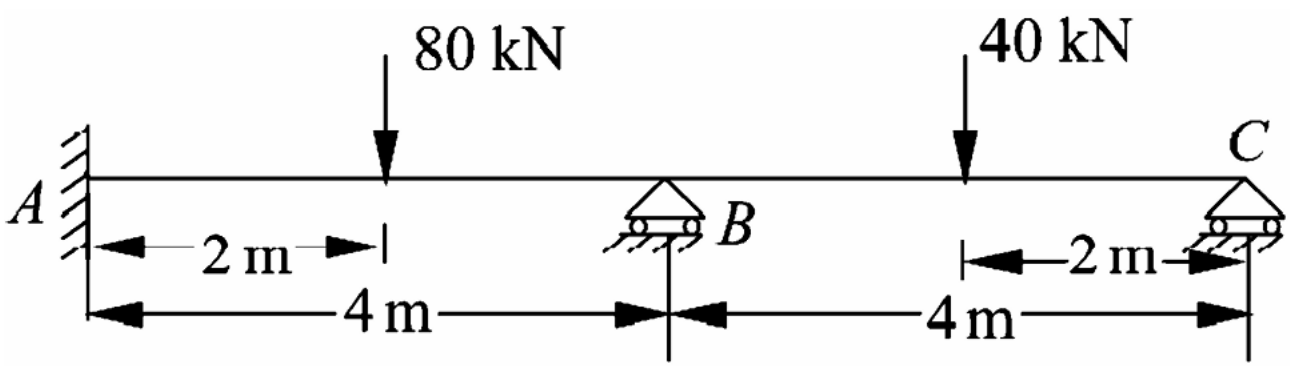
Final bending moment diagram:

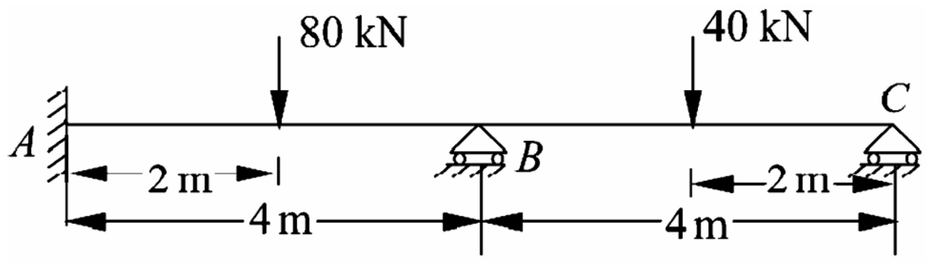




Example 2

Use flexibility matrix method, analyse the continuous beam shown in Figure, if the downward settlement of supports **B** and **C** are **10 mm** and **5 mm**, respectively. Take **EI = 184 × 10^{11} Nmm²**.





Step 1: Determine the degree of static indeterminacy n and choose the redundants.

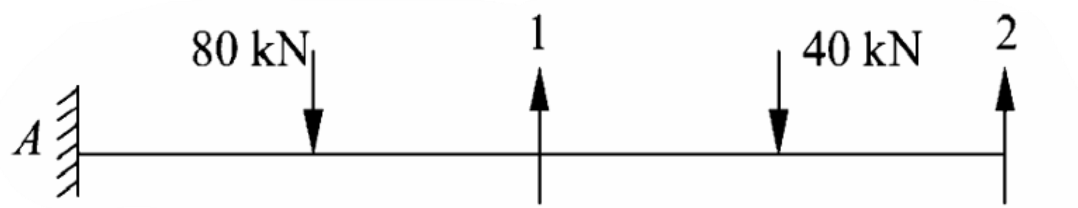
Number of reaction components = 5

Number of independent equations of equilibrium = 3

Degree of static indeterminacy = 5 - 3 = 2

Step 2: Release redundant forces and get the basic determinate structure. Then assign the coordinates to the redundant force directions.

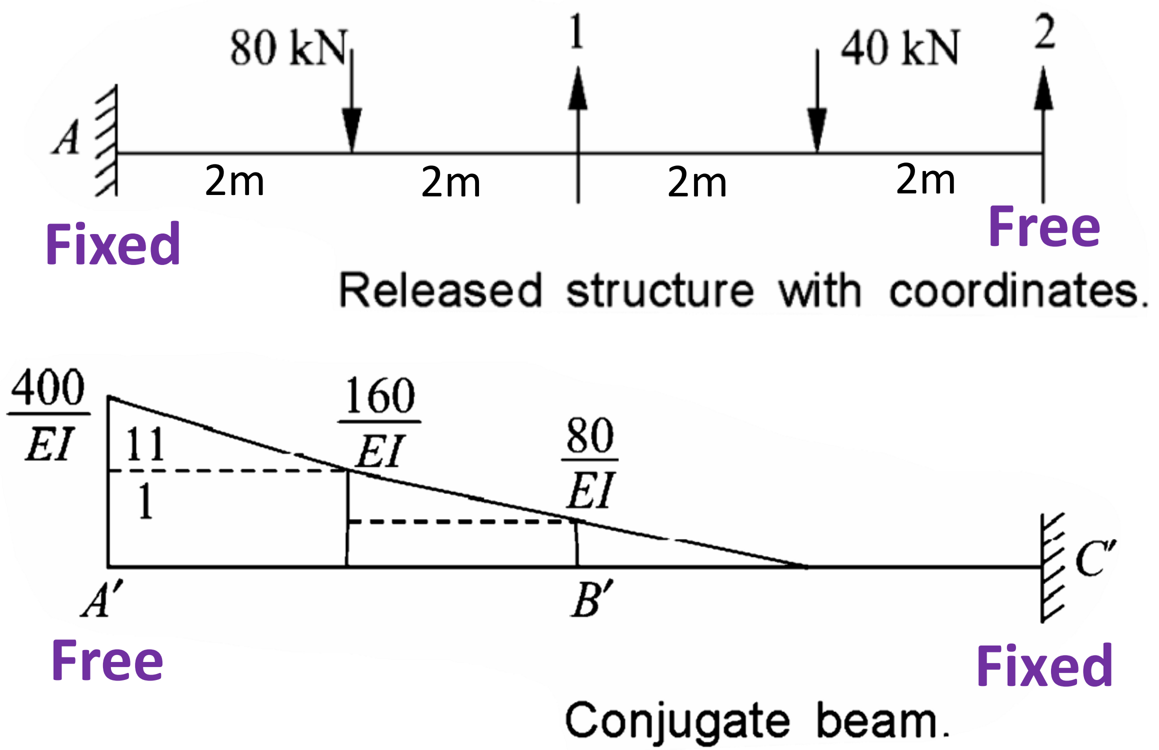
There is settlement at B and C. If vertical reaction components at **B** and **C** are removed, the resulting structure is a cantilever beam of span **8 m** with fixed end at **A.**

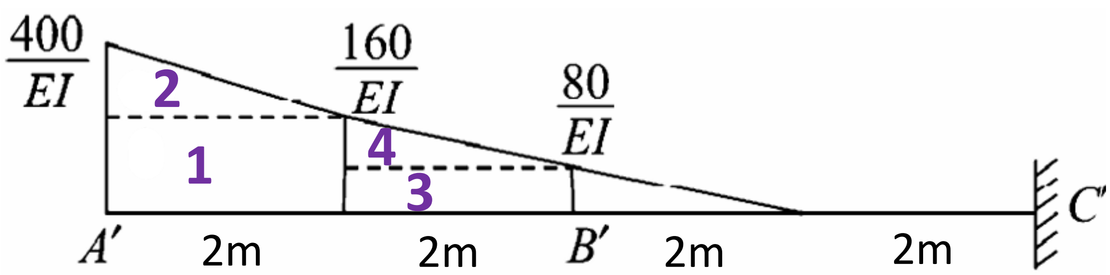


Released structure with coordinates.

Step 3: Determine the deflections in coordinate directions due to given loading

For this, we can use conjugate beam method.





Δ_{1L} = Displacement at 1 in released structure

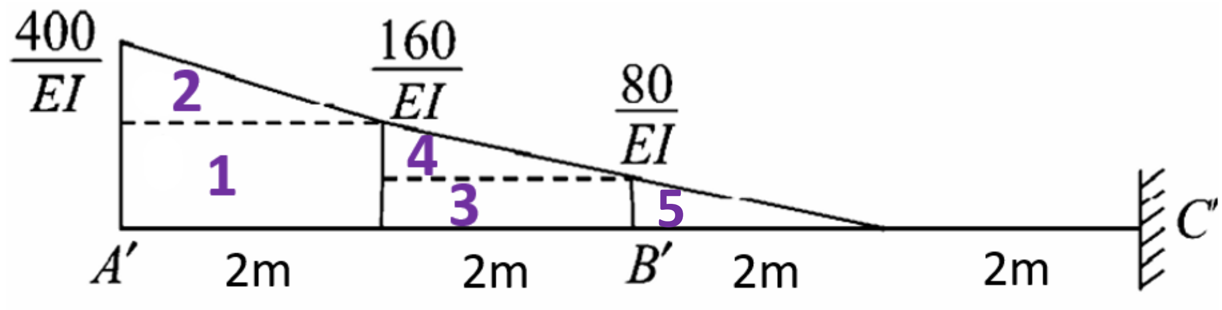
= Moment at B' in conjugate beam

$$= -\left[\underbrace{\frac{160}{EI}\times 2\times 3}_{\text{Moment due to Area 1}}+\underbrace{\frac{1}{2}\times 2\times \frac{(400-160)}{EI}\times\left(2+\frac{4}{3}\right)}_{\text{Moment due to Area 2}}+\underbrace{2\times\frac{80}{EI}\times 1}_{\text{Moment due to Area 3}}+\underbrace{\frac{1}{2}\times 2\times\frac{80}{EI}\times\frac{4}{3}}_{\text{Moment due to Area 4}}\right]$$

$$EI = 184 \times 10^{11}\ \text{Nmm}^2$$

$$= 18400\ \text{kNm}^2$$

$$\Delta_{1L} = -\frac{2026.67}{18400} = -\,0.110\ \text{m}$$



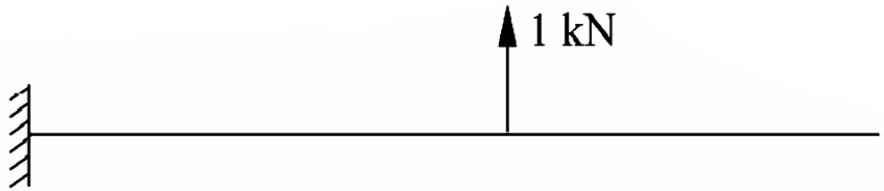
$$\Delta_{2L} = -\frac{1}{EI}\left[160 \times 2 \times 7 + \frac{1}{2} \times (400 - 160) \times 2 \times 7.33\right.$$

$$\left. +80 \times 2 \times 5 + \frac{1}{2} \times (160 - 80) \times 2 \times 5.33 + \frac{1}{2} \times 80 \times 2 \times 3.33\right]$$

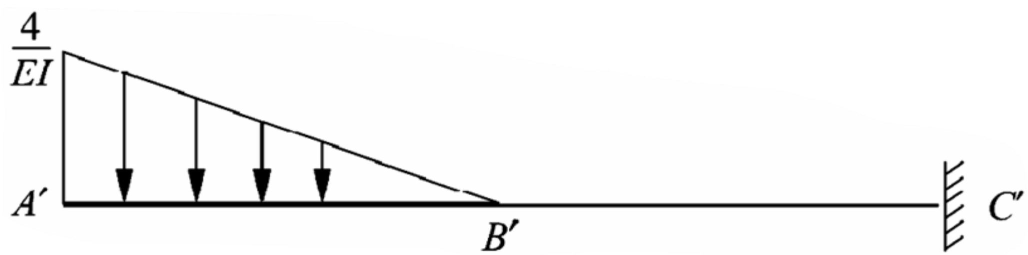
$$= -\frac{5492}{EI} = -0.298m$$

Step 4: Formulation of flexibility matrix

First applying a 1KN force in coord direction 1.



Released structure with unit load in coordinate direction 1.



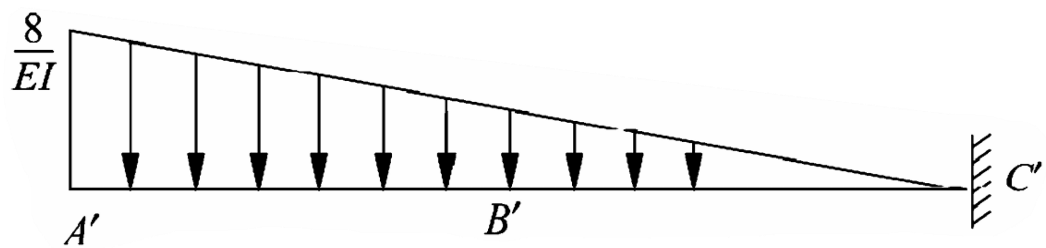
Conjugate beam with M/EI diagram

$$\delta_{11} = \frac{1}{2} \times \frac{4}{EI} \times 4 \times \left(\frac{2}{3} \times 4\right) = \frac{21.333}{EI}$$

$$\delta_{21} = \frac{1}{2} \times \frac{4}{EI} \times 4 \times \left(4 + \frac{2 \times 4}{3}\right) = \frac{53.33}{EI}$$

Then applying a 1KN force in coord direction 2.

Released structure with unit load in coordinate direction 2.



Conjugate beam with M/EI diagram

$$\delta_{12} = \frac{1}{2} \times \frac{4}{EI} \times 4 \times \frac{8}{3} + \frac{4}{EI} \times 4 \times 2 = \frac{53.33}{EI}$$

$$\delta_{22} = \frac{1}{2} \times \frac{8}{EI} \times 8 \times \frac{16}{3} = \frac{170.667}{EI}$$

Step 5: Apply the flexibility equation

The displacements in the coordinate directions in the actual continuous beam are:

$$\Delta_1 = -0.01$$
$$\Delta_2 = -0.005$$

- Flexibility matrix equation is

$$[\delta]\ [P] = [\Delta] - [\Delta_L]$$

$$\frac{1}{EI}\begin{bmatrix} 21.333 & 53.333 \\ 53.333 & 170.667 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -0.01 \\ -0.005 \end{bmatrix} - \begin{bmatrix} -0.110 \\ -0.298 \end{bmatrix}$$

This reduces to

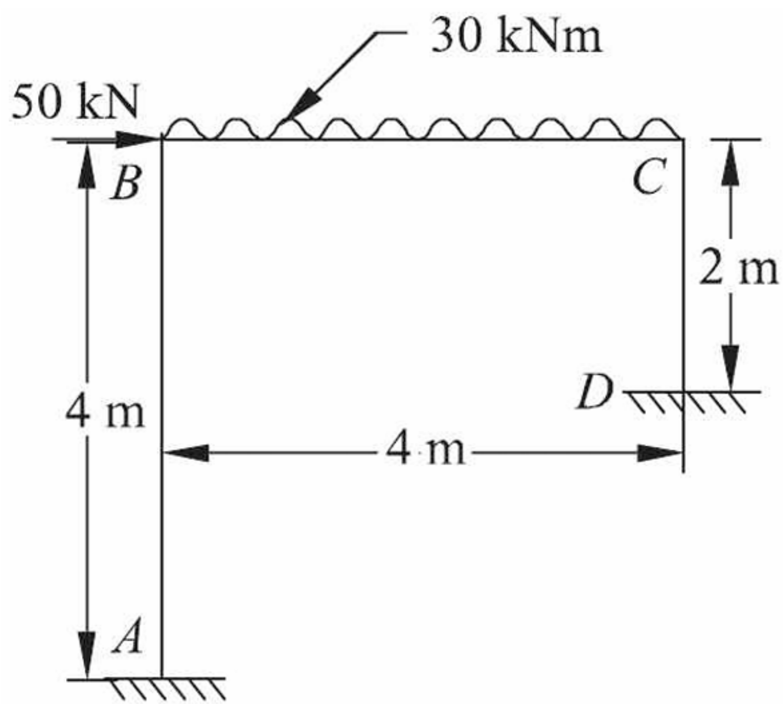
$$\frac{1}{EI}\begin{bmatrix} 21.333 & 53.333 \\ 53.333 & 170.667 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0.100 \\ 0.293 \end{bmatrix}$$

Finally, we will get the unknown forces as:

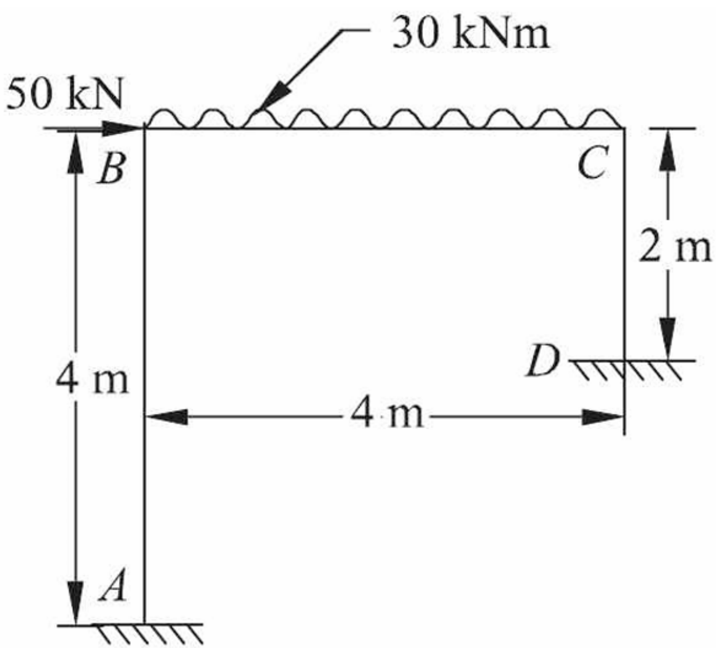
$$P_1 \approx 33.27 \quad \text{and} \quad P_2 \approx 21.19$$

Example 3

Analyse the portal frame ABCD shown in Figure by the flexibility matrix method. EI is constant throughout.



Step 1: Determine the degree of static indeterminacy n and choose the redundants

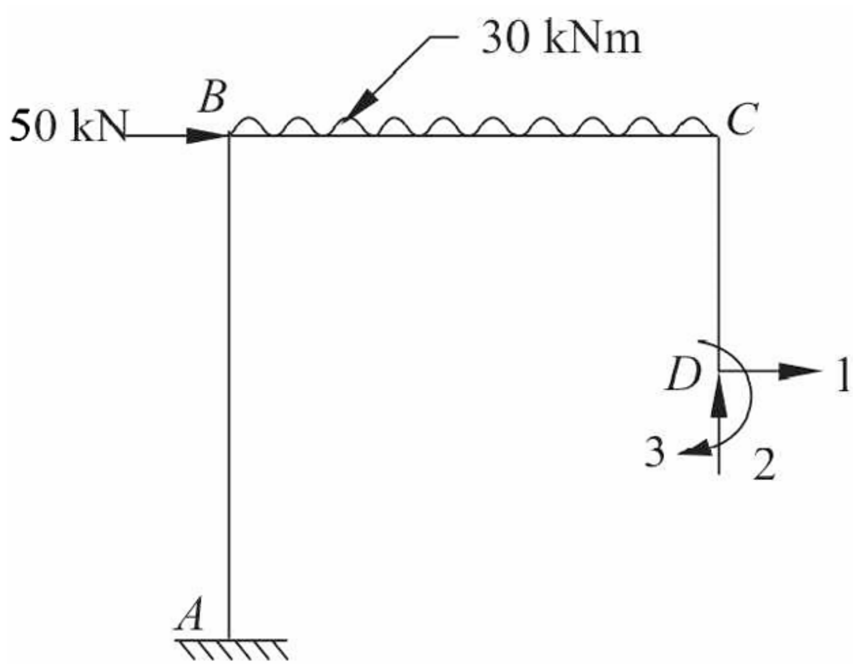


Degree of static indeterminacy = 6- 3 = 3

The redundants selected are:

$$[P] = \begin{bmatrix} H_D \\ V_D \\ M_D \end{bmatrix}$$

Step 2: Release redundant forces and get the basic determinate structure. Then assign the coordinates to the redundant force directions.

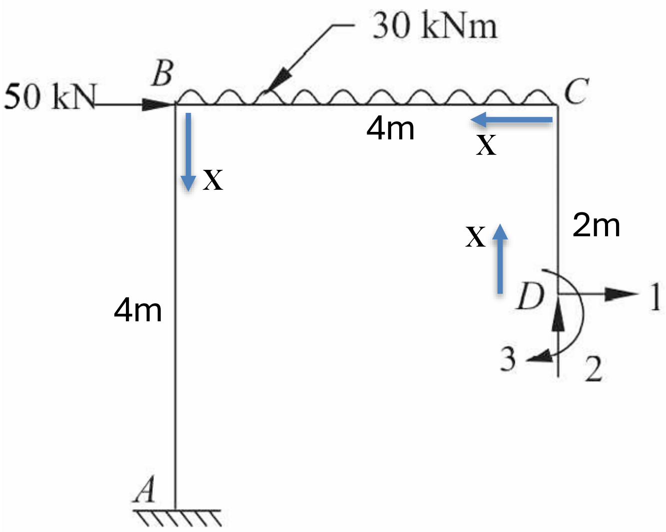


Released structure with coordinates.

The basic determinate structure is a cantilever frame.

Step 3: Determine the deflections in coordinate directions due to given loading

For this, we can use **unit load method**.



Moments due to given loads and unit loads

Portion	*DC*	*CB*	*BA*
Flexural rigidity	*EI*	*EI*	*EI*
Origin	*D*	*C*	*B*
Limit	0–2	0–4	0–4
M	0	$15x^2$	$50x + 240$
m_1	$-x$	-2	$-2 + x$
m_2	0	$-x$	-4
m_3	1	1	1

Displacement at coord. 1 due to given loading:

$$\Delta_{1L} = \int\left(\frac{Mm_1}{EI}\right)dx = \int_{BA}\frac{Mm_1}{EI}dx + \int_{CB}\frac{Mm_1}{EI}dx + \int_{DC}\frac{Mm_1}{EI}dx$$

$$= \int_0^4\left(\frac{15x^2(-2)}{EI}\right)dx + \int_0^4\left[\frac{(50x+240)\,(-2+x)}{EI}\right]dx$$

$$= \int_0^4\left(\frac{-30x^2}{EI}\right)dx + \int_0^4\left[\frac{(50x^2+140x-480)}{EI}\right]dx$$

$$= \frac{1}{EI}\left[-30\times\frac{x^3}{3}\right]_0^4 + \frac{1}{EI}\left[\frac{50x^3}{3}+70x^2-480x\right]_0^4$$

$$= -\frac{373.33}{EI}$$

Displacement at coord. 2 due to given loading:

$$\Delta_{2L} = \int \left(\frac{Mm_2}{EI} \right) dx$$

$$= \int_0^4 \left(\frac{-15x^3}{EI} \right) dx + \int_0^4 \left[\frac{(50x + 240)\,(-4)}{EI} \right] dx$$

$$= -\left(\frac{1}{EI} \right) \left[\frac{15x^4}{4} \right]_0^4 - \left(\frac{4}{EI} \right) \left[25x^2 + 240x \right]_0^4$$

$$= -\left(\frac{6400}{EI} \right)$$

Displacement at coord. 3 due to given loading:

$$\Delta_{3L} = \int \left(\frac{Mm_3}{EI} \right) dx$$

$$= \int_0^4 \left(\frac{15x^2}{EI} \right) dx + \int_0^4 (50x + 240) \frac{dx}{EI}$$

$$= \frac{1}{EI}[5x^3]_0^4 + \frac{1}{EI}[25x^2 + 240x]_0^4 = \frac{1680}{EI}$$

Step 4: Formulation of flexibility matrix

$$\delta_{11} = \int \left(\frac{m_1^2}{EI} \right) dx$$

$$= \int_0^2 \left(\frac{x^2}{EI} \right) dx + \int_0^4 4 \left(\frac{dx}{EI} \right) + \int_0^4 \left(\frac{(-2+x)^2}{EI} \right) dx$$

$$= \frac{1}{EI} \left[\frac{x^3}{3} \right]_0^2 + \frac{4}{EI} \left[x \right]_0^4 + \frac{1}{EI} \left[4x - \frac{4x^2}{2} + \frac{x^3}{3} \right]_0^4$$

$$= \frac{24}{EI}$$

$$\delta_{22} = \int m_2^2 \left(\frac{dx}{EI}\right) = \int_0^4 \frac{x^2}{EI}\,dx + \int_0^4 \frac{16}{EI}\,dx$$

$$= \frac{1}{EI}\left[\frac{x^3}{3}\right]_0^4 + \frac{16}{EI}\left[x\right]_0^4$$

$$= \frac{85.33}{EI}$$

$$\delta_{33} = \int \left(\frac{m_3^2}{EI}\right) dx = \int_0^2 \left(\frac{dx}{EI}\right) + \int_0^4 \left(\frac{dx}{EI}\right) + \int_0^4 \left(\frac{dx}{EI}\right)$$

$$= \frac{1}{EI}\left[x\right]_0^2 + \frac{1}{EI}\left[x\right]_0^4 + \frac{1}{EI}\left[x\right]_0^4 = \frac{10}{EI}$$

$$\delta_{21} = \delta_{12} = \int \left(\frac{m_1\, m_2}{EI} \right) dx = \int_0^4 \left(\frac{2x}{EI} \right) dx + \int_0^4 \left(\frac{(-4)(-2+x)}{EI} \right) dx$$

$$= \frac{1}{EI} \left[x^2 \right]_0^4 - \left(\frac{4}{EI} \right) \left[-2x + \frac{x^2}{2} \right]_0^4$$

$$= \frac{16}{EI}$$

$$\delta_{31} = \delta_{13} = \int \left(\frac{m_1 m_3}{EI} \right) dx = \int_0^2 \left(\frac{-x}{EI} \right) dx + \int_0^4 \left(\frac{-2}{EI} \right) dx + \int_0^4 \frac{(x-2)}{EI} dx$$

$$= \frac{1}{EI} \left[-\frac{x^2}{2} \right]_0^2 + \frac{1}{EI} \left[-2x \right]_0^4 + \frac{1}{EI} \left[\frac{x^2}{2} - 2x \right]_0^4$$

$$= -\frac{10}{EI}$$

$$\delta_{32} = \delta_{23} = \int \frac{m_2 m_3}{EI}\,dx$$

$$= \int_0^4 \frac{-x}{EI}\,dx + \int_0^4 \frac{-4}{EI}\,dx$$

$$= \frac{1}{EI}\left[-\frac{x^2}{2}\right]_0^4 - \left(\frac{4}{EI}\right)\left[x\right]_0^4 = -\frac{24}{EI}$$

Step 5: Apply the flexibility equation

From consistency condition, the displacements in the coordinate directions in the actual continuous beam,

$$\Delta_1 = 0, \quad \Delta_2 = 0 \quad \Delta_3 = 0$$

The flexibility matrix equation is,

$$[\delta] \; [P] = [\Delta] - \; [\Delta_L]$$

$$\frac{1}{EI}\begin{bmatrix} 24 & 16 & -10 \\ 16 & 85.33 & -24 \\ -10 & -24 & 10 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{EI}\begin{bmatrix} -373.33 \\ -6400.00 \\ 1680.00 \end{bmatrix}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 24 & 16 & -10 \\ 16 & 85.33 & -24 \\ -10 & -24 & 10 \end{bmatrix}^{-1}\begin{bmatrix} 373.33 \\ 6400.00 \\ -1680.00 \end{bmatrix}$$

$$= \left[\frac{1}{24\times 277.3 + 16\times 80 - 10\times 469.3}\right]\times\begin{bmatrix} 277.3 & 80 & 469.3 \\ 80 & 140 & 416.0 \\ 469.3 & 416 & 1791.92 \end{bmatrix}\begin{bmatrix} 373.33 \\ 6400.00 \\ -1680.00 \end{bmatrix}$$

$$= \begin{bmatrix} -53.33 \\ 70.01 \\ -53.30 \end{bmatrix}$$

STIFFNESS MATRIX METHOD

- Basic unknowns to be determined are the joint displacements.
- Hence, the degree of kinematic indeterminacy is identified first and then coordinate number is assigned to each unknown displacement components.
- To start with, the joint displacement in all directions are restrained. Let the forces developed due to applied loads in the restrained structure in the coordinate directions be $P_1, P_2, \ldots, P_n$.
- Determine the stiffness matrix of the structure by applying unit displacement in each of the coordinate directions and find the forces developed.

- From the principle of superposition and equilibrium condition, the forces developed in the coordinate directions are:

$$P_1 = P_{1L} + k_{11}\Delta_1 + k_{12}\Delta_2 + ... + k_{1n}\Delta_n$$

$$P_2 = P_{2L} + k_{21}\Delta_1 + k_{22}\Delta_2 + ... + k_{2n}\Delta_n$$

$$...$$

$$P_n = P_{nL} + k_{n1}\Delta_1 + k_{n2}\Delta_2 + ... + k_{nn}\Delta_n$$

$$[P] = [P_L] + [k][\Delta]$$

$$[k][\Delta] = [P] - [P_L] = [P - P_L]$$

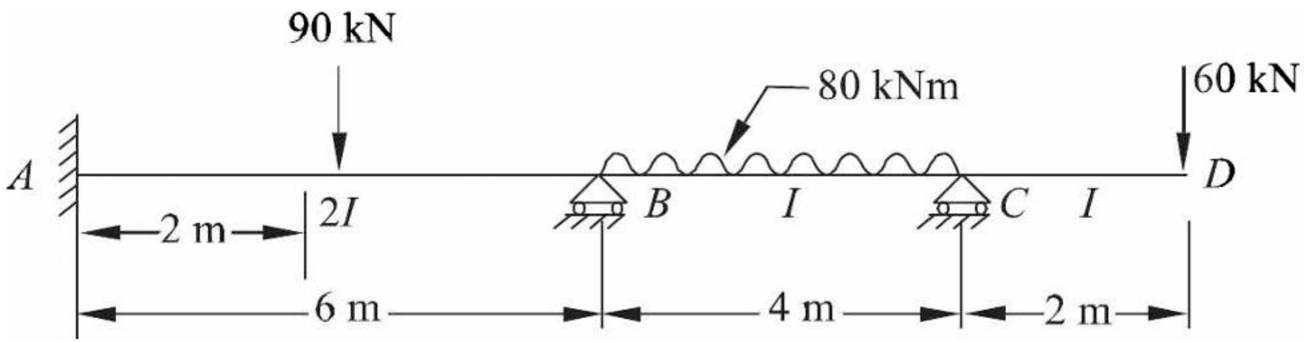
- Solving this stiffness equation, the displacements in all the coordinate directions can be found.
- Then using the equilibrium equations,

$$P_i = P_i L + \sum_{j=1}^{n} k_{ij} \Delta_i$$

all the required forces can be found.

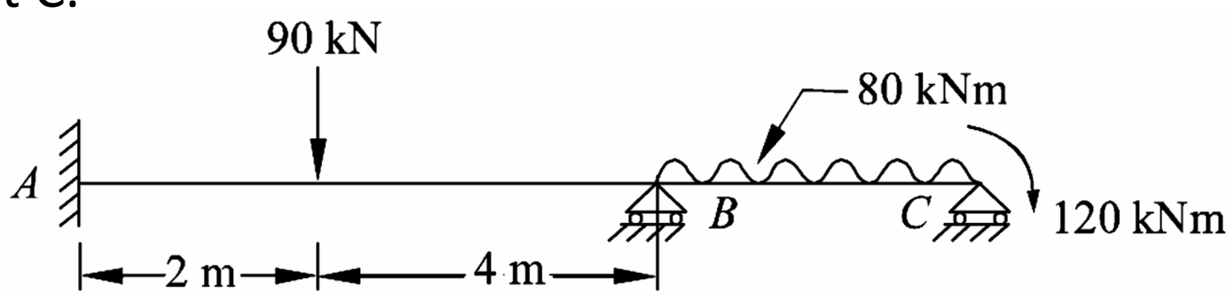
Example 4

- Analyse the beam shown in Figure by stiffness matrix method.



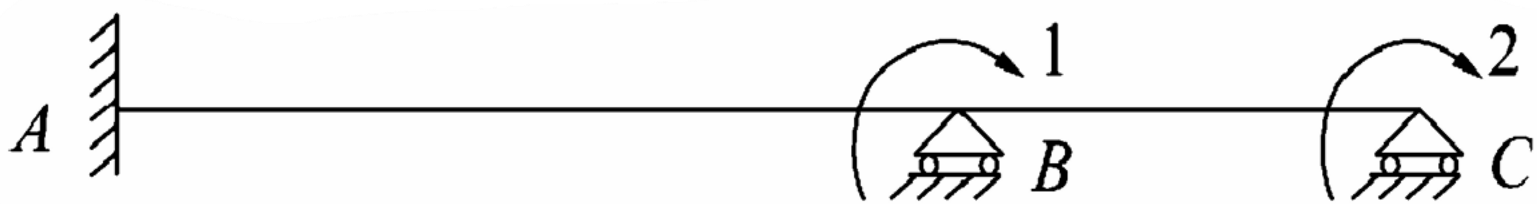
Solution:

The overhanging portion is accounted by providing it as a moment of 120kNm at C.



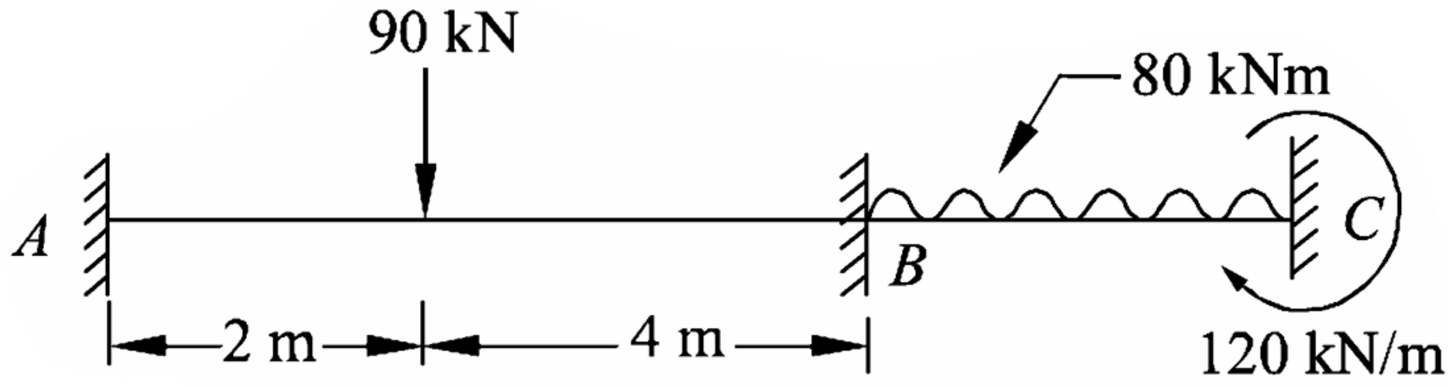
Step 1: Determine the degree of kinematic indeterminacy n and choose the coordinates.

Degree of kinematic indeterminacy = 2



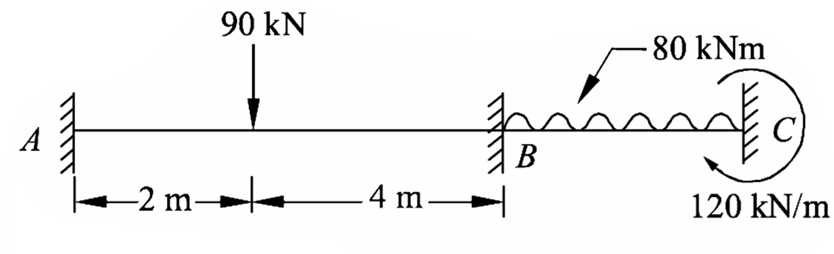
Coordinates selected.

Step 2: Restrain the joint displacements and obtain a fully restrained structure.



Fully restrained structure

Step 3: Determine the fixed end moments due to the given loading



$$M_{\mathrm{FAB}} = -\frac{90 \times 2 \times 4^2}{6^2} = -80\,\mathrm{kNm}$$

$$M_{\mathrm{FBA}} = \frac{90 \times 2^2 \times 4}{6^2} = 40\,\mathrm{kNm}$$

$$M_{\mathrm{FBC}} = -\frac{80 \times 4^2}{12} = -106.67\,\mathrm{kNm}$$

$$M_{\mathrm{FCB}} = 106.67\ \mathrm{kNm}$$

Therefore, the force matrix in the coordinate direction due to the given loading is:

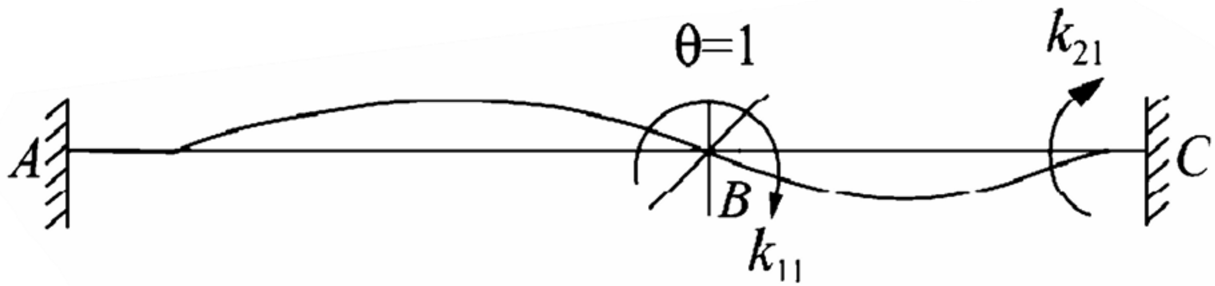
$$[P_{\mathrm{L}}] = \begin{bmatrix} M_{\mathrm{FBA}} + M_{\mathrm{FBC}} \\ M_{\mathrm{FCB}} \end{bmatrix} = \begin{bmatrix} 40 - 106.67 \\ 106.67 \end{bmatrix} = \begin{bmatrix} -66.67 \\ 106.67 \end{bmatrix}$$

Step 4: Formulation of stiffness matrix

- Stiffness matrix is obtained by applying unit displacement in each of the coordinate direction and find the forces developed in that directions.

Note: *Moment required to rotate a beam by 1 radian is the stiffness of the beam and its value is* $\frac{4EI}{L}$.

Moment developed at the other end when one end of fixed beam rotated by 1 radian is $\frac{2EI}{L}$.



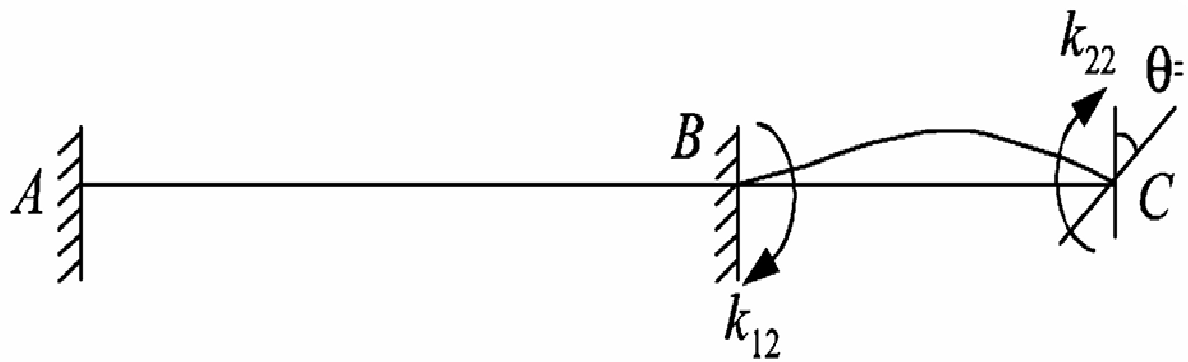
Unit displacement in coordinate direction 1:

k_{11} = Moment required to rotate *BA* and *BC* (joint *B*) by 1 radians.

$$= \frac{8EI}{6} + EI = \frac{7}{3}EI$$

$$k_{21} = \frac{2EI}{4}$$

Unit rotation at C: (Coordinate direction 2)



Restrained structure with unit displacement in coordinate direction 2.

$$k_{21} = \frac{2EI}{4} = 0.5EI$$

$$k_{22} = \frac{4EI}{4} = EI$$

Step 5: Apply the stiffness equation

Final force vector is:

$$P = \begin{bmatrix} 0 \\ 120 \end{bmatrix}$$

$$[k]\ [\Delta] = [P - P_{\mathrm{L}}]$$

$$\begin{bmatrix} \dfrac{7EI}{3} & 0.5EI \\ 0.5EI & EI \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 0 + 66.7 \\ 120 - 106.67 \end{bmatrix}$$

$$\frac{EI}{3}\begin{bmatrix} 7 & 1.5 \\ 1.5 & 3 \end{bmatrix}\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 66.67 \\ 13.33 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{3}{EI}\begin{bmatrix} 7 & 1.5 \\ 1.5 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 66.67 \\ 13.33 \end{bmatrix}$$

$$= \left(\frac{3}{EI}\right)\left(\frac{1}{7 \times 3 - 1.5^2}\right)\begin{bmatrix} 3 & -1.5 \\ -1.5 & 7 \end{bmatrix}\begin{bmatrix} 66.67 \\ 13.33 \end{bmatrix}$$

$$= \left(\frac{3}{EI} \times \frac{1}{18.75}\right)\begin{bmatrix} 180.015 \\ -66.95 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 28.802 \\ -1.070 \end{bmatrix}$$

$$\theta_B = \frac{28.802}{EI} \text{ and } \theta_C = \frac{-1.070}{EI}$$

- The unknown forces/moments can then be found out using slope deflection method.
- For a span AB,

$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3\Delta}{L}\right) + M_{AB}^{fixed}$$

$$M_{BA} = \frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3\Delta}{L}\right) + M_{BA}^{fixed}$$

From slope deflection equations:

$$M_{AB} = -80 + \frac{2E \times 2I}{6}(2\theta_A + \theta_B - 0)$$

$$= -80 + \frac{4}{6}EI\left[\frac{28.802}{EI}\right] = -60.80 \text{ kNm}$$

$$M_{BA} = 40 + \frac{2E(2I)}{6}\left[0 + 2 \times \frac{28.802}{EI} - 0\right] = 78.403 \text{ kNm}$$

$$M_{BC} = -106.67 + \frac{2EI}{4}(2\theta_B + \theta_C - 0)$$

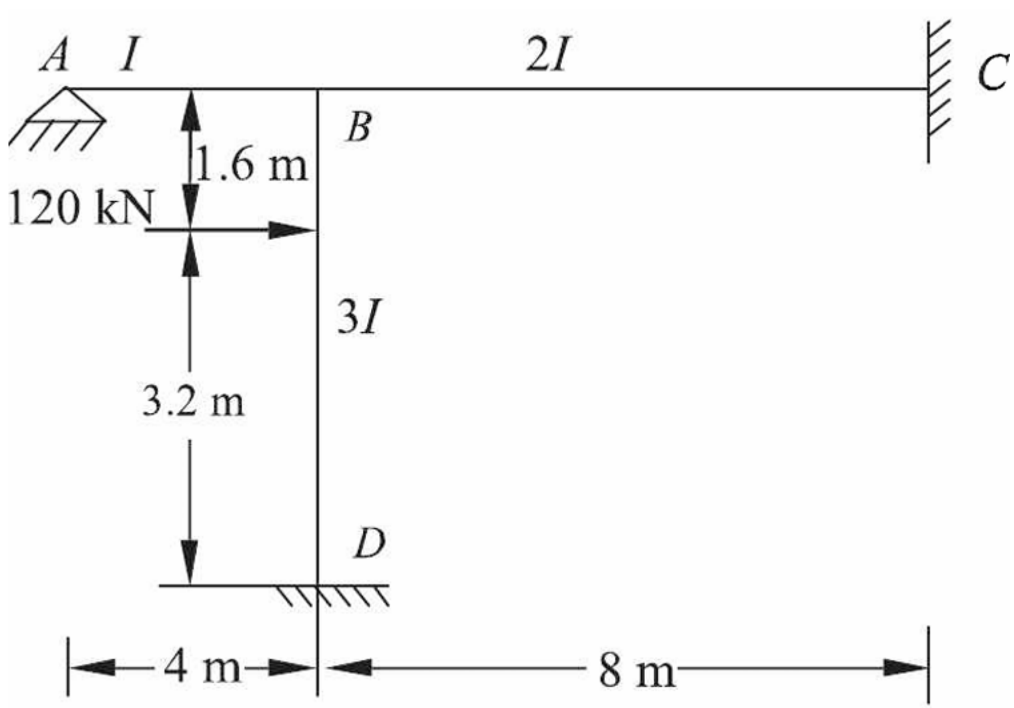
$$= -106.67 + \frac{2EI}{4}\left[\frac{2 \times 28.802}{EI} - \frac{1.071}{EI}\right] = -78.403 \text{ kNm}$$

$$M_{CB} = 106.67 + \frac{2EI}{4}(\theta_B + 2\theta_C - 0)$$

$$= 106.67 + \frac{2EI}{4}\left[\frac{28.802}{EI} - \frac{2 \times 1.071}{EI}\right] = 120 \text{ kNm}$$

Example 5

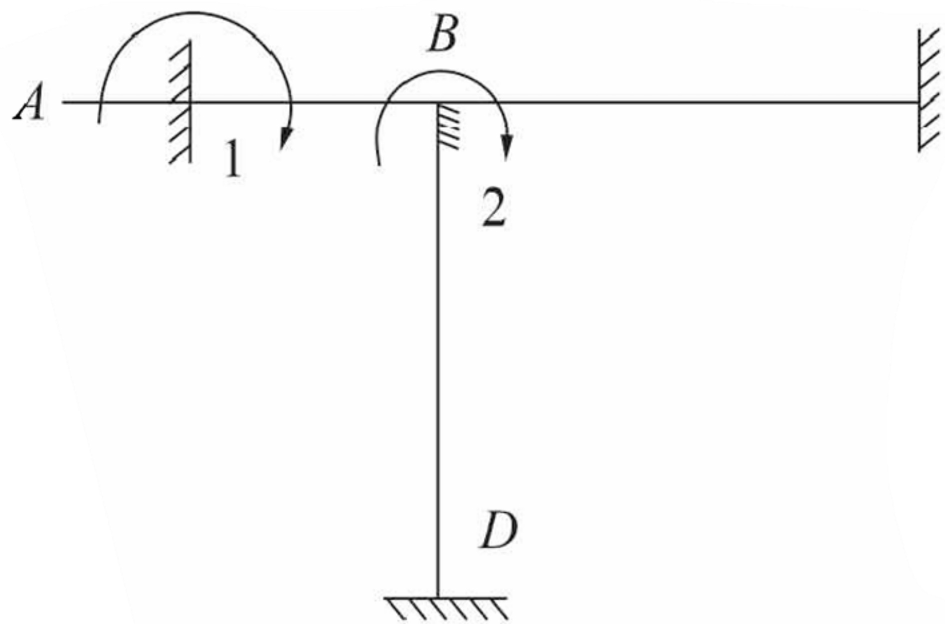
- Using the displacement method, analyse the frame shown in Figure.



Step 1: Determine the degree of kinematic indeterminacy n and choose the coordinates.

Degree of kinematic indeterminacy = 2

Step 2: Restrain the joint displacements and obtain a fully restrained structure.



Fully restrained structure and coordinates.

Step 3: Determine the fixed end moments due to the given loading

$$M_{\mathrm{FDB}} = -\frac{120 \times 3.2 \times 1.6^2}{4.8^2} = -42.667 \text{ kNm}$$

$$M_{\mathrm{FBD}} = \frac{120 \times 3.2^2 \times 1.6}{4.8^2} = 85.333 \text{ kNm}$$

All other fixed end moments are zero.

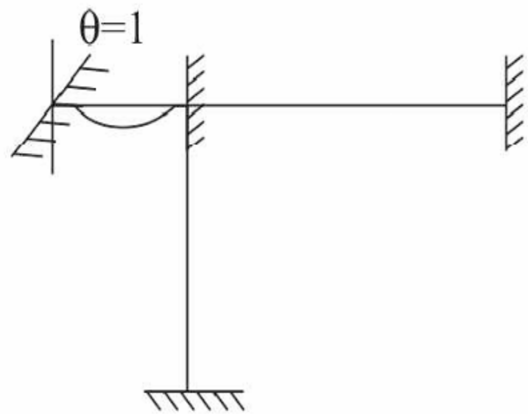
Therefore, the force matrix in the coordinate direction due to the given loading is:

$$[P_{\mathrm{L}}] = \begin{bmatrix} 0 \\ 85.333 \end{bmatrix}$$

Step 4: Formulation of stiffness matrix

Stiffness Matrix

(a) Unit displacement in coordinate direction 1:

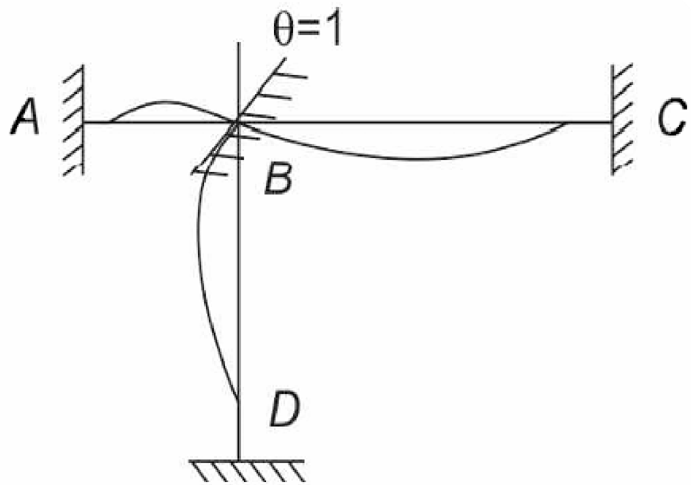


Fully restrained structure with unit displacement in coordinate direction 1.

$$k_{11} = \frac{4EI}{4} = EI$$

$$k_{21} = \frac{2EI}{4} = 0.5\,EI$$

Unit displacement in coordinate direction 2:



Fully restrained structure with unit displacement in coordinate direction 2.

$$k_{12} = \frac{2EI}{4} = 0.5\,EI$$

$$k_{22} = \frac{4EI}{4} + \frac{4\,E\,(2I)}{8} + \frac{4\,E\,(3I)}{4.8} = 4.5\,EI$$

Step 5: Apply the stiffness equation

Final force vector is:

$$[P] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, the stiffness equation is: $[k]\ [\Delta] = [P - P_L]$

$$EI\begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 4.5 \end{bmatrix}\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} 0 - \quad 0 \\ 0 - 85.33 \end{bmatrix}$$

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 4.5 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -85.33 \end{bmatrix}$$

$$= \left(\frac{1}{EI}\right)\left(\frac{1}{4.5 - 0.5^2}\right)\begin{bmatrix} 4.5 & -0.5 \\ -0.5 & 1.0 \end{bmatrix}\begin{bmatrix} 0 \\ -85.33 \end{bmatrix}$$

$$= \frac{1}{EI}\begin{bmatrix} 10.039 \\ -20.078 \end{bmatrix}$$

$$\theta_A = \frac{10.039}{EI} \text{ and } \theta_B = \frac{-20.078}{EI}$$

Using Slope Deflection method,

$$M_{AB} = 0 + \left(\frac{2EI}{4}\right)\left[\frac{(2 \times 10.039 - 20.078)}{EI}\right] = 0$$

$$M_{BA} = 0 + \left(\frac{2EI}{4}\right)\left(\frac{(10.039 - 2 \times 20.078)}{EI}\right) = -15.059 \text{ kNm}$$

$$M_{BC} = 0 + \frac{2E(2I)}{8}[2 \times (-20.078) + 0 - 0] = -20.078 \text{ kNm}$$

$$M_{CB} = 0 + \frac{2E(2I)}{8}(-20.078 + 0 - 0) = -10.039 \text{ kNm}$$

$$M_{BD} = 85.333 + \frac{2E(3I)}{4.8}[2 \times (-20.078) + 0 - 0] = 35.138 \text{ kNm}$$

$$M_{DB} = -42.667 + \frac{2E(3I)}{4.8}[(-20.078) + 0 - 0] = -67.76 \text{ kNm}$$

Reference

- Structural Analysis Vol-2 : S S Bhavikatti