Module 5

FLUID MECHANICS I

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Dimensional Analysis

Dimensional analysis

- Dimensional analysis is a method used to express physical quantities in terms of their basic dimensions (Length, Mass, Time, Temperature, etc.).
- It helps in understanding the relationship between different physical quantities and checking the correctness of equations.

Fundamental Dimensions

- Length – **L**
- Mass – **M**
- Time – **T**
- Temperature – **θ** (if heat is involved)

Need for Dimensional Analysis

To convert one system of units to another

To derive relations between physical quantities

To check the dimensional homogeneity of equations

To develop model laws for experiments (model–prototype relationships)

Types of Quantities

- **Fundamental Quantities:**
  Quantities that are independent of others.
  Examples: Length (L), Mass (M), Time (T), Temperature (θ).
- **Derived Quantities:**
  Quantities that depend on fundamental quantities.
  Examples:
  - Velocity = L/T
  - Acceleration = L/T^2
  - Force = $M \cdot L/T^2$
  - Pressure = $M/(L \cdot T^2)$
  - Density = M/L^3

Dimensional Formula

- A **dimensional formula** expresses a physical quantity in terms of the fundamental dimensions.

Quantity	Dimensional Formula
Velocity	$L\,T^{-1}$
Acceleration	$L\,T^{-2}$
Force	$M\,L\,T^{-2}$
Work/Energy	$M\,L^{2}\,T^{-2}$
Pressure	$M\,L^{-1}\,T^{-2}$
Power	$M\,L^{2}\,T^{-3}$

Principle of Homogeneity

- Every correct physical equation must be **dimensionally homogeneous**, i.e.,
- The dimensions on both sides of an equation must be the same.
- **Example:**

$$s = ut + \frac{1}{2}at^2$$

Dimensions of each term = L (length) → Hence, the equation is dimensionally correct.

Buckingham's π Theorem

What is Buckingham's π Theorem?

- • It is a method used in dimensional analysis.
- • Helps to reduce the number of variables in a physical problem.
- • Converts variables into dimensionless groups (π terms).
- • Makes experiments and data analysis easier.

Why Do We Use It?

- To simplify complex equations.
- To find relationships between physical quantities.
- To design experiments with fewer tests.
- Common in fluid mechanics, heat transfer, and aerodynamics.

The Theorem Statement

- If a physical problem has:

❖n variables

❖involving m fundamental dimensions (like M, L, T).

- Then the problem can be expressed in terms of (n – m) dimensionless π terms.

Procedure

- Let $X_1, X_2, X_3, \ldots, X_n$ be the variables involved in a physical problem.
- Then, the physical phenomenon is mathematically represented as:

$$f_1(X_1, X_2, X_3, \ldots, X_n) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

- Equation (1) is a dimensionally homogeneous equation containing n variables.
- Then according to Buckingham's π-theorem, equation (1) can be written in terms of (n-m) dimensionless groups or π-terms as follows:

$$f(\pi_1, \pi_2, \ldots, \pi_{n-m}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

- Then each π-term is written as:

$$\pi_1 = X_2^{a_1} X_3^{b_1} X_4^{c_1} . X_1$$

$$\pi_2 = X_2^{a_2} X_3^{b_2} X_4^{c_2} . X_1$$

$$\pi_{n-m} = X_2^{a_{n-m}} X_3^{b_{n-m}} X_4^{c_{n-m}} . X_n$$

- Here X_2, X_3 and X_4 are repeating variables (mostly the independent variables).
- Each equation is **solved by the principle of dimensional homogeneity** and values of a_1, b_1, c_1, etc. are obtained.
- These values of π's are substituted in equation (2).
- The final equation for the phenomenon is obtained by expressing any one of the π-terms as a function of others as:

$$\pi_1 = \phi\,[\pi_2, \pi_3, \ldots, \pi_{n-m}]$$

or

$$\pi_2 = \phi_1\,[\pi_1, \pi_3, \ldots, \pi_{n-m}]$$

Method of Selecting Repeating Variables

The number of repeating variables are equal to the number of fundamental dimensions of the problem. The choice of repeating variables is governed by the following considerations :

1. As far as possible, the dependent variable should not be selected as repeating variable.
2. The repeating variables should be chosen in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.

Variables with Geometric Property are
(i) Length, *l* (ii) *d* (iii) Height, *H* etc.

Variables with flow property are
(i) Velocity, *V* (ii) Acceleration etc.

Variables with fluid property : (i) μ, (ii) ρ, etc.

- The repeating variables selected should not form a dimensionless group.
- The repeating variables together must have the same number of fundamental dimensions.
- No two repeating variables should have the same dimensions.
- In most of fluid mechanics problems, the choice of repeating variables may be
  (i) d, v, ρ (ii) l, v, ρ (iii) l, v, μ or (iv) d, v, μ.

Example

Qn : The efficiency η of a fan depends on density ρ, dynamic viscosity μ of the fluid, angular velocity ω, diameter D of the rotor and the discharge Q. Express η in terms of dimensionless parameters.

Soln:

Given: η is a function of ρ, μ, ω, D and Q.

$$\eta = f(\rho, \mu, \omega, D, Q)$$

or

$$f_1(\eta, \rho, \mu, \omega, D, Q) = 0$$

Step 1: Find the number of π-terms by writing the dimensions of each variable.

Dimensions of each variable are:

$$\eta = Dimensionless, \rho = ML^{-3}, \mu = ML^{-1}T^{-1},$$
$$\omega = T^{-1}, D = L, Q = L^3T^{-1}$$

$$\therefore \text{No. of fundamental dimenions}, m = 3$$

Thus Number of π-terms $= n - m = 6 - 3 = 3$

Step 2: Write the physical problem (eqn 1) as π-terms

$$\text{i.e., } f_1(\eta, \rho, \mu, \omega, D, Q) = 0$$

Is written as,

$$f_1(\pi 1, \pi 2, \pi 3) = 0.$$

Step 3: Define each π-term by proper selection of repeating variables (chosen such that at least one flow property, one fluid property and one geometrical property)

Choosing D, ω and ρ as repeating variables, the π-terms are written as:

$$\pi_1 = D^{a_1} \omega^{b_1} \rho^{c_1} \eta$$
$$\pi_2 = D^{a_2} \omega^{b_2} \rho^{c_2} \mu$$
$$\pi_3 = D^{a_3} \omega^{b_3} \rho^{c_3} Q$$

Step 4 : Evaluation of π-terms

First π-term

$$\pi 1 = D^{a1}\omega^{b1}\rho^{c1}\eta$$

Substituting dimensions on both sides of π_1:

$$M^0L^0T^0 = L^{a_1}\left(T^{-1}\right)^{b_1}\left(ML^{-3}\right)^{c_1}$$

$$M^0L^0T^0 = M^{c_1}L^{a_1-3c_1}T^{-b_1}$$

Equating the powers of M, L, T on both sides:

Power of M: $0 = c_1 + 0 \Rightarrow c_1 = 0$

Power of L: $0 = a_1 - 3c_1 + 0 \Rightarrow a_1 = 0$

Power of T: $0 = -b_1 + 0 \Rightarrow b_1 = 0$

Substituting the values of a_1, b_1 and c_1 in π_1, we get:

$$\pi_1 = D^0\omega^0\rho^0\eta = \eta$$

Second π-term

$$\pi_2 = D^{a_2}\omega^{b_2}\rho^{c_2}\mu$$

Substituting the dimensions on both sides:

$$M^0L^0T^0 = L^{a_2}\left(T^{-1}\right)^{b_2}(ML^{-3})^{c_2}(ML^{-1}T^{-1})$$

$$M^0L^0T^0 = M^{c_2+1}L^{a_2-3c_2-1}T^{-b_2-1}$$

Equating the powers of M, L, T on both sides:

Power of M, $0 = c_2 + 1, \quad \therefore \quad c_2 = -1$

Power of L, $0 = a_2 - 3c_2 - 1, \quad \therefore \quad a_2 = 3c_2 + 1$

$$= -3 + 1 = -2$$

Power of T, $0 = -b_2 - 1, \quad \therefore \quad b_2 = -1$

Substituting the values of a_2, b_2 and c_2 in π_2,

$$\pi_2 = D^{-2}\cdot\omega^{-1}\cdot\rho^{-1}\cdot\mu = \boldsymbol{\frac{\mu}{D^2\omega\rho}}$$

Third π-term

- π3=Da3·ωb3·ρc3·Q
- Substituting the dimensions on both sides:
- $M^0L^0T^0 = L^{a_3} \cdot \left(T^{-1}\right)^{b_3} \cdot \left(ML^{-3}\right)^{c_3} \cdot L^3T^{-1}$
- Equating the powers of M, L and T on both sides:
  - **Power of M,** $0 = c_3, \quad \therefore \quad c_3 = 0$

Power of L, $0 = a_3 - 3c_3 + 3, \quad \therefore \quad a_3 = 3c_3 - 3 = -3$

Power of T, $0 = -b_3 - 1, \quad \therefore \quad b_3 = -1$

- Substituting the values of a_3, b_3 and c_3 in π_3,
- $\pi_3 = D^{-3} \cdot \omega^{-1} \cdot \rho^0 \cdot Q = \frac{Q}{D^3\omega}$

Step 5: Substitute the π-terms and formulate the final functional relation.

Substituting the values of π_1, π_2 and π_3 in the equation:

$f_1(\pi 1, \pi 2, \pi 3) = 0$, we get

$$\boldsymbol{f_1\left(\eta, \frac{\mu}{D^2\omega\rho}, \frac{Q}{D^3\omega}\right) = 0}$$

or

$$\boldsymbol{\eta = \varphi\left(\frac{\mu}{D^2\omega\rho}, \frac{Q}{D^3\omega}\right)}$$

Dimensionless numbers

- **Dimensionless numbers** are those numbers which are obtained by dividing the inertia force by viscous force or gravity force or pressure force or surface tension force or elastic force.
- As this is a ratio of one force to the other force, it will be a dimensionless number.
- These dimensionless numbers are also called non-dimensional parameters.

Important dimensionless numbers

- The followings are the important dimensionless numbers :

1. Reynold's number,
2. Froude's number,
3. Euler's number,
4. Weber's number,
5. Mach's number.

1. Reynold’s number

Inertia force:

$$F_i = Mass \times Acceleration = \rho A V^2$$

Viscous force:

$$F_v = \tau A = \mu \frac{du}{dy} A = \mu \frac{V}{L} A$$

Reynold’s Number:

$$R_e = \frac{F_i}{F_v} = \frac{\rho A V^2}{\mu \frac{V}{L} A} = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$$

In **pipe flow**, the linear dimension $L$is taken as the **diameter (d)**:

$$R_e = \frac{Vd}{\nu} = \frac{\rho V d}{\mu}$$

Flow Regimes:

$R_e < 2000$:**Laminar flow**

$2000 < R_e < 4000$:**Transitional flow**

$R_e > 4000$:**Turbulent flow**

Significance:

Determines whether flow is **laminar or turbulent**.

Indicates the dominance of **viscous** or **inertial** forces.

Important in **pipe flow, boundary layers,** and **fluid machinery.**

2. Froude's Number (Fr)

Definition:
It is the square root of the ratio of the **inertia force** to the **gravity force** of a flowing fluid.

Formula:

$$\boldsymbol{F_r} = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho A V^2}{\rho A L g}} = \frac{\boldsymbol{V}}{\sqrt{\boldsymbol{Lg}}}$$

Where:

V: Velocity of flow

L: Characteristic length

g: Acceleration due to gravity

Significance:
Used to compare **inertial** and **gravitational** effects — important in **open channel flow** and **ship hydrodynamics**.

3. Euler's Number

Definition:
It is the square root of the ratio of the **inertia force** to the **pressure force** of a flowing fluid.

Formula:

$$\boldsymbol{E_u} = \sqrt{\frac{F_i}{F_p}} = \sqrt{\frac{\rho A V^2}{p \times A}} = \frac{\boldsymbol{V}}{\sqrt{\boldsymbol{p}/\boldsymbol{\rho}}}$$

Where:

p: Pressure

ρ: Density of fluid

Significance:
Represents the relationship between **pressure** and **inertial** forces — used in **pressure drop** and **fluid machinery** analysis.

4. Weber's Number (We)

Definition:
It is the square root of the ratio of the **inertia force** to the **surface tension force** of a flowing fluid.

Formula:

$$\boldsymbol{W_e} = \sqrt{\frac{F_i}{F_s}} = \sqrt{\frac{\rho L^2 V^2}{\sigma L}} = \boldsymbol{\frac{V}{\sqrt{\sigma/(\rho L)}}}$$

Where:

σ: Surface tension

L: Characteristic length

Significance:
Used in flows involving **surface tension**, such as **droplet formation**, **jets**, and **bubbles**.

5. Mach's Number (M)

Definition: Mach's number is defined as the **square root of the ratio of the inertia force** of a flowing fluid to the **elastic force** of the fluid.

$$M = \sqrt{\frac{F_i}{F_e}}$$

$F_i = \rho A V^2$)Inertia force)
$F_e = KA = K \times L^2$)Elastic force, K= Elastic stress)

Derivation:

$$M = \sqrt{\frac{\rho A V^2}{K L^2}} = \sqrt{\frac{\rho V^2}{K}} = \frac{V}{\sqrt{K/\rho}}$$

But

$$\sqrt{\frac{K}{\rho}} = C = \mathit{Velocity\ of\ sound\ in\ the\ fluid}$$

Hence,

$$\boxed{\boldsymbol{M = \frac{V}{C}}}$$

Significance:

- Indicates whether the flow is **subsonic**, **transonic**, **supersonic**, or **hypersonic**.
- Represents the ratio of **flow velocity to the speed of sound** in the medium.

Flow Type	Mach Number (M) Range
Subsonic	$M < 1$
Transonic	$M \approx 1$
Supersonic	$1 < M < 5$
Hypersonic	$M > 5$

- Similitude is defined as the similarity between the model and its prototype in every respect. When the model and prototype have similar properties, results from the model can predict prototype behavior.
- **Three types of similarity:**
- Geometric Similarity
- Kinematic Similarity
- Dynamic Similarity

Geometric Similarity

Definition: Geometric similarity exists between the **model** and **prototype** when the ratio of all corresponding linear dimensions is the same.

Let:

$L_m, b_m, D_m, A_m, \forall_m$ →model values

$L_P, b_P, D_P, A_P, \forall_P$ →prototype values

Linear ratio (Scale ratio):

$$\frac{L_P}{L_m} = \frac{b_P}{b_m} = \frac{D_P}{D_m} = L_r$$

L_r =Scale ratio

Area ratio:

$$\frac{A_P}{A_m} = L_r^2$$

Volume ratio:

$$\frac{\forall_P}{\forall_m} = L_r^3$$

Kinematic Similarity

Definition:
Kinematic similarity means the **similarity of motion** between the **model** and **prototype**.
It exists when the ratios of **velocity** and **acceleration** at corresponding points are equal.

Let:
V_{P1}, V_{P2} =Velocities of fluid at points 1 & 2 in prototype
a_{P1}, a_{P2} =Accelerations of fluid at points 1 & 2 in prototype
$V_{m1}, V_{m2}, a_{m1}, a_{m2}$ =Corresponding values in model

For kinematic similarity:

$$\frac{V_P}{V_m} = \frac{V_{P1}}{V_{m1}} = \frac{V_{P2}}{V_{m2}} = V_r$$

V_r =Velocity ratio (Eq. 12.9)

$$\frac{a_P}{a_m} = \frac{a_{P1}}{a_{m1}} = \frac{a_{P2}}{a_{m2}} = a_r$$

a_r =Acceleration ratio

Dynamic Similarity

Definition:
Dynamic similarity exists between a **model and prototype** if the **ratios of corresponding forces** acting at corresponding points are **equal** and their **directions are the same**.

Forces considered:

$(F_i)_P$ →Inertia force in prototype

$(F_v)_P$ →Viscous force in prototype

$(F_g)_P$ →Gravity force in prototype

$(F_i)_m$ $(F_v)_m$ $(F_g)_m$" →Corresponding forces in model

Condition for Dynamic Similarity:

$$\frac{(F_i)_P}{(F_i)_m} = \frac{(F_v)_P}{(F_v)_m} = \frac{(F_g)_P}{(F_g)_m} = F_r$$

where F_r =**Force ratio**

Significance of Non-Dimensionalisation of Fluid Flow Equations

- **1. Simplifies analysis:**
  Converts complex dimensional equations into simpler **dimensionless forms**, reducing the number of variables.
- **2. Reveals governing parameters:**
  Helps identify important **dimensionless numbers** (e.g., Reynolds, Froude, Mach, Weber numbers) that control the flow behavior.
- **3. Enables model–prototype similarity:**
  Non-dimensional equations allow **dynamic similarity** testing between small-scale models and full-scale prototypes.

- **4. Facilitates comparison:**
  Allows direct comparison of different fluid systems **independent of size, fluid, or units**.
- **5. Reduces experimental effort:**
  Once key dimensionless parameters are known, results from one set of conditions can be used to **predict** others.
- **6. Aids numerical simulations:**
  Improves stability and accuracy of **computational fluid dynamics (CFD)** solutions by keeping variables within manageable ranges.

(Refer numericals from class notes and textbook)

Reference

- Bansal, R. K. (2010). *A Textbook of Fluid Mechanics and Hydraulic Machines (Revised 9th Ed., SI Units)*. New Delhi: Laxmi Publications.