Hence $S = \cosh(\cos\theta)\sin(\sin\theta)$

# MODULE :- 4

## Partial differential equation :-

Equations involving partial derivative is called P.D.E

eg:- (i) $\frac{\partial u}{\partial x} = c^2 \frac{\partial^2 u}{\partial x^2}$ (ii) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

(iii) $\frac{\partial u}{\partial x}\tan x + \frac{\partial u}{\partial y}\tan y = \tan u$

## FORMATION OF P.D.E

### [1] By eliminating arbitrary function

Eg: Form a P.D.E by eliminating arbitrary constants

① $ax + by + ab = z$

Ans:- $z = ax + by + ab \longrightarrow ①$ $\Rightarrow \frac{\partial z}{\partial x} = a, \frac{\partial z}{\partial y} = b$

ie $z = \frac{\partial z}{\partial x}x + \frac{\partial z}{\partial y}y + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = px + qy + pq$

we know $\frac{\partial u}{\partial x} = p$, $\frac{\partial u}{\partial y} = q$, $\frac{\partial^2 u}{\partial x^2} = r$

(ii) $z = (x^2+a)(y^2+b) \longrightarrow$ form P.D.E

Ans: $z = (x^2+a)(y^2+b) \longrightarrow ①$ $p = \frac{\partial z}{\partial x} = (y^2+b)2x$ $q = \frac{\partial z}{\partial y} = (x^2+a)2y$

ie we got $(x^2+a) = \frac{\partial z}{\partial y}\cdot\frac{1}{2y}$ & $(y^2+b) = \frac{\partial z}{\partial x}\cdot\frac{1}{2x}$

ie $\quad z = \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{1}{4xy} \Rightarrow 4xyz = pq = \frac{\partial^2 z}{\partial x \partial y}$.

(iii) Form PDE $\quad z = x(x-a) + y(y-b) + z = 1$

Ans : $\quad (x-a)x + (y-b)y + z = 1 \longrightarrow ①$

d.w.r.t $x \Rightarrow \quad \frac{\partial z}{\partial x} = 2x - a + \frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad a = 2x + p$

d.w.r.t $y = \quad 2y - b + \frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad b = 2y + q$

ie PDE $= x[x - (2x+p)] + y[y - (2y+q)] + z = 1$

ie $\quad x^2 + y^2 + px + qy + 1 = z$.

(iv) $\quad 1 = (x-a)^2 + (y-b)^2 + z^2$

Ans :- d.w.r.t $x$ : $\quad 0 = 2(x-a) + 2z\frac{\partial z}{\partial x} \quad \Rightarrow \quad (x-a) = -pz$

d.w.r.t $y$ : $\quad 0 = 2(y-b) + 2z\frac{\partial z}{\partial y} \quad \Rightarrow \quad (y-b) = -qz$

Hence above eqn becomes $\Rightarrow \quad 1 = z^2[p^2 + q^2 + 1]$

## [II] By eliminating arbitory $fx^n$

Eg: Form a partial-diff. eq$^n$ given

(i) $\quad z = f[x^2 - y^2] \longrightarrow ①$

Ans : $\quad \frac{\partial z}{\partial x} = p = f'(x^2-y^2)\,2x \longrightarrow ②$ $\quad q = \frac{\partial z}{\partial y} = f'(x^2-y^2) \cdot -2y \longrightarrow ③$

② / ③ $\Rightarrow \frac{p}{q} = \frac{-x}{y} \Rightarrow \underline{qx + py = 0}$.

(ii) $z = f(x+ay) + g(x-ay)$

Ans : $p = f'(x+ay) + g'(x-ay)$ $\quad q = \frac{\partial z}{\partial y} = f'(x+ay).a - ag'(x-ay)$

$= a\left[f'(x+ay) - g'(x-ay)\right]$

$r = \frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay)$ $\quad t = \frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 g''(x-ay)$

$= a^2\left[f''(x+ay) + g''(x-ay)\right]$

$= a^2 . r$

ie $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2} \Rightarrow t = a^2 r$.

PYQ *

(iii) $z = f\left(\frac{xy}{z}\right)$

Ans: $z = f\left(\frac{xy}{z}\right) \longrightarrow ①$ $\quad p = \frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) . \frac{y}{z} \longrightarrow$ random wrong . since z is a fxn of x & y.

ie $p = f'\left(\frac{xy}{z}\right)\left[\frac{-x\, \partial z/\partial x + z.1}{z^2}\right] y = f'(xy/z)\left[\frac{z - px}{z^2}\right] y . \longrightarrow ②$

$q = f'\left(\frac{xy}{z}\right)\left[\frac{z - y\, \partial z/\partial y}{z^2}\right] . x = f'\left(\frac{xy}{z}\right)\left[\frac{z - qy}{z^2}\right] x \longrightarrow ③$

$p/q = \frac{y(z - px)}{x(z - qy)} \Rightarrow px(z - qy) = qy(z - px)$

ie $px = qy$ is final PDE.

PYQ *

(iv) $\cancel{f(x^2+y^2)}$ $F(x^2+y^2, z - xy) = 0$

Ans: Giving substitutions for quantities : $r = x^2+y^2$ $\quad s = z - xy$

ie $f(x^2+y^2, z-xy) = f(r,s) = 0$

d.w.r.t $x$: $\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial f}{\partial r}\cdot 2x + \frac{\partial f}{\partial s}\cdot\left[\frac{\partial z}{\partial x} - y\right] = 0$

$= 2x\,\partial f/\partial r + \frac{\partial f}{\partial s}\cdot[p-y] \longrightarrow ②$

d.w.r.t. $y$ :-

$0 = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial f}{\partial r}[2y] + \frac{\partial f}{\partial s}\left[\frac{\partial z}{\partial y} - x\right] = 2y\frac{\partial f}{\partial r} + (q-x)\frac{\partial f}{\partial x}$

$= 0 \longrightarrow ③.$

---

* For eliminate $ax + by = 0$

$cx + dy = 0$

$acx + bcy = 0$
$acx + ady = 0$ } on substracting:

$(ad-bc)y = 0$ &

if $y \neq 0$, then $ad-bc=0$

which is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$

---

Applying the same here. So

$\begin{vmatrix} 2x & p-y \\ 2y & q-x \end{vmatrix} = 0 \rightarrow 2x(q-x) - 2y(p-y) = 0$

$\underline{qx - py = x^2 - y^2}$

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① $f(xy+z^2, x+y+z) = 0$.

Ans:- Simly : $r = xy + z^2$ , $s = x+y+z$.

then $f(r,s) = f(xy+z^2, x+y+z) = 0 \longrightarrow ①$

d.w.r.t $x$ :-

$\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} = 0$ them

$$\frac{\partial f}{\partial r}\left[y+2z\frac{\partial z}{\partial x}\right]+\frac{\partial f}{\partial s}\left[1+\partial z/\partial x\right]=[y+2pz]\frac{\partial f}{\partial r}+[1+p]\frac{\partial f}{\partial s}=0 \longrightarrow (2)$$

similarly d w r t y :- $\frac{\partial f}{\partial r}\left[x+2z\,\partial z/\partial y\right]+\frac{\partial f}{\partial s}\left[1+\frac{\partial z}{\partial y}\right]=[x+2qz]\frac{\partial f}{\partial r}+[1+q]\frac{\partial f}{\partial s}$

$$=0 \longrightarrow (3)$$

solving (2) & (3) $\rightarrow \begin{vmatrix} y+2pz & 1+p \\ x+2qz & 1+q \end{vmatrix}=0$

$$[y+2pz][1+q]-[1+p][x+2qz]=0$$

$$y+2pz+qy+\cancel{2pqz}-[x+2qz+xp+\cancel{2pqz}]=0$$

PYQ (4 mark)
$$\underline{\underline{2pz+(1+q)y=2qz+(1+p)x}}$$

(vi)

$$f(x+y+z,\ x^2+y^2+z^2)=0$$

Ans:- $r=x+y+z$, $s=x^2+y^2+z^2$ } from this we get

$\frac{\partial r}{\partial x}=1+\frac{\partial z}{\partial x}=1+p$

$\frac{\partial r}{\partial y}=1+\partial z/\partial y=1+q$

$\partial s/\partial x=2x+2z\,\partial z/\partial x=2[x+pz]$

$\frac{\partial s}{\partial y}=[y+qz]2.$

d w r t x: $\partial f/\partial r\,[1+p]+\partial f/\partial s\,2[x+pz]=0$, simly

d w r t y: $\partial f/\partial r\,[1+q]+\partial f/\partial s\cdot 2[y+qz]=0$, solving both

$$\begin{vmatrix} 1+p & 2(x+pz) \\ 1+q & 2(y+qz) \end{vmatrix}=0, \text{ then}$$

$$2[1+p][y+qz]-2[1+q][x+pz]=0$$

finally, $(1+p)y-(1+q)x+(q-p)z=0.$

or $x-y = p(y-z) + q(z-x)$.

# SOLUTION OF PARTIAL DIFF. EQN.

**Type ① :- Equations solvable by direct integration :**

① Solve $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\partial z/\partial x\right) = xy$.

Ans: on integ. w.r.t $x$ : $\partial z/\partial x = \int xy\,dx = \frac{x^2 y}{2} + f_1(y)$

again integrating : $z = \frac{yx^3}{6} + x f_1(y) + f_2(y)$

⑪ $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$.

Ans :- $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{x}{y} + a \Rightarrow$ integ: $\frac{\partial z}{\partial y} = \int\left(\frac{x}{y} + a\right)dx = \frac{x^2}{2y} + ax + f_1(y)$

again inte. w.r.t. $y \Rightarrow z = \int\left[\frac{x^2}{2 \cdot y} + ax + f_1(y)\right]dy$

$z = \log y \, x^2/2 + axy + f_2(y) + f_3(x)$

PYQ ⑪ $\frac{\partial^2 z}{\partial x^2} + z = 0$, given $x = 0$ ; $z = e^y$ and $\partial z/\partial y = 1$, solve.

Ans :- For convenience, we write $\partial^2 z/\partial x^2 + z = 0$ as $\frac{d^2 z}{dx^2} + z = 0$, by assuming $z$ only depends on $x$. Then

$\frac{d^2 z}{dx^2} + z = 0 \Rightarrow (D^2+1)z = 0 \Rightarrow$ A.E. $m^2 + 1 = 0$ $\quad m = \pm i$
$= \alpha \pm \beta i$

Complem. fxⁿ : CF $\Rightarrow y = e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right] = c_1 \cos x + c_2 \sin x$

Now we say both $C_1$ & $C_2$ depends on $y$ (or $f(y) = C_1$ & $C_2$). Then

$z = f_1(y)\cos x + f_2(y)\sin x$ , But when $x=0$, $z = e^y \longrightarrow$ Result $= f_1(y)$

then $e^y = f_1(y)$ , also $\partial z/\partial x = 1$ when $x=0$

then $\frac{\partial z}{\partial x} = -f_1(y)\sin x + f_2(y)\cos x \Rightarrow 1 = f_2(\sin x) \longrightarrow$ Result

then $z = e^y \cos x + \sin x$.

(iv) Solve $\partial^2 z/\partial x^2 = z$ , Given $z = e^y$ & $\partial z/\partial x = e^{-y}$ when $x=0$

Ans:- Let's assume $z$ dep. only on $x$. then

$$\frac{d^2 z}{dx^2} = z \Rightarrow (D^2-1)z = 0 \Rightarrow m^2-1=0 \Rightarrow m = \pm 1$$

ie Complementary fn = CF => "$C_1 e^{m_1 x} + C_2 e^{m_2 x} = y$"

ie $z = f_1(y)e^x + f_2(y)e^{-x} \longrightarrow$ (a)

Given Condition (1): $z = e^y$ when $x=0$, => applying on (a) gives

$$e^y = f_1(y) + f_2(y) \longrightarrow (1)$$

Cond$^n$ (2): $\cancel{\partial z/\partial y}$ $\partial z/\partial x = e^{-y}$ when $x=0$

$\partial z/\partial x = f_1(y)e^x - f_2(y)e^{-x} \Rightarrow e^{-y} = f_1(y) - f_2(y)e^0 \longrightarrow$ (2)

Solving (1) & (2) => $f_1(y) = \cosh y = \frac{e^y+e^{-y}}{2}$

$f_2(y) = \sinh y = \frac{e^y - e^{-y}}{2}$

Hence (a) => $z = e^x \cosh y + e^{-x}\sinh y$.

Type ② :- LAGRANGE'S EQUATION *

An equation of the form $P_p + Qq = R$ when $P, Q, R$ are functions of $x, y, z$ and $p = \partial z/\partial x$, $q = \partial z/\partial y$

Method :- (i) * write associate equations : $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ii) * Solve these equations $u = a$ & $v = b$

(iii) * write the complete integral $f(u,v) = 0$

① $6p + 7q = 8$, solve the given d.e

Ans : $P = 6$ $Q = 7$ $R = 8$.

Now Associate eqn : As. E = $\frac{dx}{6} = \frac{dy}{7} = \frac{dz}{8}$

Now Consider $dx/6 = dy/7 \Rightarrow 7dx = 6dy \Rightarrow 7x = 6y + a$ (variable separ.)

$7x - 6y = a$

Similarly $dy/7 = dz/8 \Rightarrow 8dy = 7dz \Rightarrow 8y = 7z + b$

$8y - 7z = b$

ie complete integral is $= f(u,v) = f([7x-6y], [8y-7z]) = 0$

Verification: $f((7x-6y), (8y-7z)) \Rightarrow f(r,s)$ then proceed.

(ii) $p + q = \cos x$, solve given D.E :

Ans: $P_p + Q_q = R \Rightarrow P = 1$ $Q = 1$ $R = \cos x$ Also $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

ie $dx = dy = dz/\cos x$

Hence $dz = \cos x \, dx \Rightarrow$ on integ. $z = \sin x + b$

$dx = dy \Rightarrow$ on integ. $x = y + a$

ie $b = \sin x - z$ and $a = x - y$.

Solution $f(a, b) = f(x-y, \sin x - z)$.

③ $p \tan x + q \tan y = \tan z$, Solve this DE.

Ans: $P = \tan x$, $Q = \tan y$, $R = \tan z$ & Asso. eq$^n$ $= \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$

we have $\frac{1}{\tan x} dx = \frac{1}{\tan y} dy \Rightarrow \cot x \, dx = \cot y \, dy$

on integrating: $\log(\sin x) = \log(\sin y) + a \Rightarrow \frac{\sin x}{\sin y} = a$ (a can be log a)

similarly we have $\sin y / \sin z = b$

ie Solution $= f(a, b) = f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

④ $y^2 z p + x^2 z q = y^2 x$, Solve this DE.

Ans: $P = y^2 z$ $Q = x^2 z$ $R = y^2 x$ & $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{y^2 x}$

ie, $x^2 z \, dx = y^2 z \, dy \rightarrow x^3/3 = y^3/3 + a \Rightarrow a = 1/3 (x^3 - y^3)$

similarly, $y^2 x \, dy = x^2 z \, dz \rightarrow \frac{xy^3}{3} = \frac{x^2 z^2}{2} + b$

$\Rightarrow \frac{xy^3}{3} - \frac{x^2 z^2}{2} = b$ (Meaning less)

taking ① & ③ and integrating $x^2 - z^2 = b$

ie Sol$^n$: $f(a, b) = f(x^2 - y^2, x^2 - z^2)$.

⑤ $(x-z)q + (z-y)p = y-x$, solve (P & Q)

Ans: Given $P = (z-y)$ also $\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$

$Q = (x-z)$

$R = (y-x)$

property: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then $a/b = c/d = e/f = a+c+e/b+d+f$

then $dx+dy+dz / (z+-z)+(x-x)+(y-y) = \frac{dx+dy+dz}{0}$

and on cross multip. gives $dx+dy+dz = a$

int. above get $x+y+z = a$ //

Again each terms are equal to $\frac{xdx + ydy + zdz}{(z-y)x + y(x-z) + z(y-x)} = \frac{xdx + ydy + zdy}{0}$

on cross multip: $x\,dx + y\,dy + z\,dz = 0$, on integ.

$x^2 + y^2 + z^2 = b$

solution $= f(a,b) = f(x+y+z,\ x^2+y^2+z)$.

⑥ $(y+z)p - (x+z)q = x-y$. Solve

Ans: $P = (y+z)$, $Q = -(x+z)$, $R = (x-y)$ & $\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y}$.

similar to above method $a = x+y+z$.

Now if multip. are $[x, y, -z] \Rightarrow xdx + ydy - zdz = b$

then $x^2+y^2-z^2 = b$

Solution = $f(a,b) = f(x+y+z,\ x^2+y^2-z)$

PYQ

(7) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$, solve.

Ans: $P = x^2(y-z)$
$Q = y^2(z-x)$ and $dx/x^2(y-z) = dy/y^2(z-x) = dz/z^2(x-y)$ = Associate eq$^n$.
$R = z^2(x-y)$

We will use multipl. as $(1/x^2, 1/y^2, 1/z^2) \Rightarrow$ $1/x^2\,dx + 1/y^2\,dy + 1/z^2\,dz = 0$

$-1/x + -1/y + -1/z = -a$

or $1/x + 1/y + 1/z = a$

Simly if multip. are $(1/x, 1/y, 1/z) \Rightarrow$ $1/x\,dx + 1/y\,dy + 1/z\,dz = 0$

$\Rightarrow \log(xyz) = \log b \Rightarrow b = \cancel{abc}\ xyz$

Solution $\Rightarrow f(a,b) = f(1/x + 1/y + 1/z,\ xyz)$.

Qn: Solve $(mz-ny)p + (nx-lz)q = ly-mx$

Ans: Associate eq$^n$: $dx/(mz-ny) = dy/(nx-lz) = dz/(ly-mx)$

If multip. are $(x,y,z) \Rightarrow$ $x\,dx + y\,dy + z\,dz = 0$

$\Rightarrow x^2+y^2+z^2 = a$

If multip. are $(l,m,n) \Rightarrow$ $l\,dx + m\,dy + n\,dz = 0$

$\Rightarrow lx + my + nz = b$

Sol$^n$: $f(a,b) = f(x^2+y^2+z^2,\ lx+my+nz) = 0$

# NON-LINEAR EQUATIONS.

Type ① :-

$f(p,q) = 0$ , an equation only involving $p$ & $q$ $\therefore$ ($x, y, z$ are absent).

The solution is $z = ax+by+c$, where $f(a,b) = 0$.

Example : solve $\sqrt{p} + \sqrt{q} = 1$

It is of the form

② $p^2 + q^2 = 1$

Ans:- It is of the form $f(p,q) = 0$ & sol$^n$ = $ax+by+c$ where $f(a,b) = 0$

$a^2+b^2 = 1 \Rightarrow b = \sqrt{1-a^2}$

sol$^n$ $\Rightarrow$ $z = ax + \sqrt{1-a^2}\, y + c$

③ $p + q + pq = 0$

Ans: of the form $f(p,q) = 0$. & sol$^n$ is $ax+by+c$ = & $f(a,b) = 0$

ie $a + b + ab = 0 \Rightarrow ab + b = -a$

$b(a+1) = -a \Rightarrow b = -a/(1+a)$

ie sol$^n$ : $z = ax - a/(1+a)\, y + c$

Type ② :- $f(z, p, q) = 0$ [equation contain $p, q$ & $z$, not $x, y$] $\longrightarrow$ ①

Sol$^n$ : Put $u = x + ay$ $\Rightarrow$ $f(u) = z$

$$\left.\begin{aligned} p &= \partial z/\partial x = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = 1 \times dz/du \\ q &= \partial z/\partial y = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = dz/du \cdot a \end{aligned}\right\}$$ subtituting in ① gives ordinary diff. eq$^n$ which can be solved.

PYQ

Eg① :- Solve $p(1+q) = qz$. (x, y are absent here).

Ans: Sol$^n$ : $u = x + ay$ & $f(u) = z$

$\therefore$ $\left.\begin{aligned} p &= dz/du \\ q &= a\, dz/du \end{aligned}\right\}$ subti. for $p$ & $q$ in given eq$^n$. ie.

$\cancel{dz/du}\left[1 + a\, dz/du\right] = a \cancel{dz/du} \cdot z$ $\longrightarrow$ from given eq$^n$.

$a\, dz/du + 1 = az$ $\Rightarrow$ $\frac{a}{az-1} dz = du$ (variable seperable)

on integrating: $\int \frac{a}{az-1} dz = \int du$ $\Rightarrow$ $\log(az-1) = u + \log b$

ie $\log\left(\frac{az-1}{b}\right) = u$

$\Rightarrow$ $\frac{1 + e^u b}{a} = z$.

$\int \frac{n'}{n} dx = \log n$

Eg②: Solve $z = p^2 + q^2$

Ans: $u = x + ay \Rightarrow \left.\begin{aligned} p &= dz/du \\ q &= a\, dz/du \end{aligned}\right\}$ $z = (dz/du)^2 + a^2 (dz/du)^2$

$= (1+a^2)(dz/du)^2$

$dz/du = \sqrt{\frac{z}{1+a^2}} \Rightarrow du = \sqrt{1+a^2} \cdot \frac{1}{\sqrt{z}} dz$ (on Integrating)

ie $2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + c \Rightarrow$ ~~$u^2 = 4(1+a^2) z$~~

$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + c$ //

Eg ③ : Solve $p^2 z^2 + q^2 = 1$

ans : $(x, y)$ are absent and the $u = x+ay$ $\left\langle \begin{array}{l} p = dz/du \\ q = a\, dz/du \end{array} \right.$

$$[a^2+z^2]\left(\frac{dz}{du}\right)^2 = 1 \Rightarrow dz/du = \frac{1}{\sqrt{a^2+z^2}}$$

$\sqrt{a^2+z^2}\, dz = du \Rightarrow$ on integrating

$$z/2\sqrt{a^2+z} + a^2/2 \log|x + \sqrt{a^2+z^2}| = u + c = (x+ay) + c$$

Type III

$$f_1(x,p) = f_2(y,q)$$

$z$ is absent. Terms in $x$ & $p$ can be seperated from term in $y$ & $z$. Let

$$f_1(x,p) = f_2(y,q) = a$$

From $f_1(x,p) = a$, $f_2(y,q) = a$ we can find $p$ & $q$.

Now $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \Rightarrow dz = p\,dx + q\,dy$

Substituting for

Qn ① :- Solve $p+q = x+y$

Ans : $p-x = -q+y = a$ ($z$ is absent) $\left\langle \begin{array}{l} p-x = a \\ y-q = a \end{array} \right.$

but $dz = p\,dx + q\,dy = (x+a)\,dx + (y-a)\,dy$

$\int dz = \int (x+a)\,dx + \int (y-a)\,dy \Rightarrow z = ax + x^2/2 + y^2/2 - ay + b$

② $p^2 + q^2 = x+y$ (Solve)

Ans :- $p^2 - x = y - q^2 = a \Rightarrow \left\langle \begin{array}{l} p^2 - x = a \\ y - q^2 = a \end{array} \right.$

But $dz = p\,dx + q\,dy = \sqrt{a+x}\,dx + \sqrt{y-a}\,dy$

$z = \int \sqrt{a+x}\,dx + \int \sqrt{y-a}\,dy = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b.$

③ $\sqrt{p} + \sqrt{q} = x+y$, the solve given eqn.

Ans : $\sqrt{p} - x = y - \sqrt{q} = a$ $\left\langle \begin{array}{l} p = (a+x)^2 \\ q = (y-a)^2 \end{array} \right.$

the $dz = p\,dx + q\,dy = (a+x)^2\,dx + (y-a)^2\,dy$

$z = \int (x+a)^2\,dx + \int (y-a)^2\,dy = \frac{(x+a)^3}{3} + \frac{(y-a)^3}{3} + b$

$x+a = u$

$du = dx$

## CHARPITS METHOD

8 mark qn. *

Let the partial differential equation be $f = f(x, y, z, p, q) \longrightarrow ①$

we will find another fxn $\phi = \phi(x, y, z, p, q) \longrightarrow ②$

Solving ① & ②, we will find $p$ & $q$ and $dz = p\,dx + q\,dy$ and on integrating we obtain the Solution.

To find $\phi$ :- (I) first we find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}$

(II) we consider a associative equation — $\frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dz}{-p\,\partial f/\partial p - q\,\partial f/\partial q}$

$$= \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{d\phi}{0}$$

from this, we will find the solution, a relation b/w $p$ & $q$.

Qn ① : Using charpit's method, Solve $pxy + pq + qy = yz$

Ans : We have $pxy + pq + qy - yz = 0$

* $\frac{\partial f}{\partial x} = py$ $\frac{\partial f}{\partial y} = px + q - z$ $\frac{\partial f}{\partial z} = -y$ $\frac{\partial f}{\partial p} = xy + q$ $\frac{\partial f}{\partial q} = p + y$.

* Now Associative equation :-

$$\frac{dx}{-(\partial f/\partial p)} = \frac{dy}{-(\partial f/\partial q)} = \frac{dz}{-(\partial f/\partial p \cdot p + \partial f/\partial q \cdot q)} = \frac{dp}{\partial f/\partial x + p\,\partial f/\partial z} = \frac{dq}{\partial f/\partial y + q\,\partial f/\partial z}$$

$$\Rightarrow \frac{dx}{-(xy+q)} = \frac{dy}{-(p+q)} = \frac{dz}{-(pxy+pq) - (pq+qy)} = \frac{dp}{py + p(-y)} = \frac{dq}{px+q-z-qy}$$

ie we have $\frac{dp}{0}$ = each term $\Rightarrow dp = 0 \Rightarrow p = a \longrightarrow$ ①

Substit. $p = a$ in given eqⁿ $\Rightarrow axy + aq + qy - yz = 0$

$$q(y+a) = yz - axy \Rightarrow q = \frac{y(z-ax)}{(y+a)}$$

But, $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\,dx + q\,dy$

$$dz = a\,dx + \frac{y(z-ax)}{a+y} dy \Rightarrow dz - a\,dx = \frac{y(z-ax)}{(a+y)} dy.$$

ie $$\frac{dz - a\,dx}{(z-ax)} = \frac{y\,dy}{(a+y)}$$ (Integrating this)

$$\Rightarrow \log(z-ax) = \int \frac{a+y}{a+y} dy - \int \frac{a}{a+y} dy = y - a\log(a+y) + \log b$$

$a+y=u$

$= u - a\log u$

ie $$y = \log\left[\frac{(z-ax)(a+y)^{a}}{b}\right] = y \cdot y - a\log(a+y)$$

$a\,(y - a\log(a+y))$

$$(z-ax)(a+y)^{a} = b\,e^{y}.$$

PYQ

② Solve $2z + p^2 + qy + 2y^2 = 0$

Ans: we have $2z + p^2 + qy + 2y^2 = 0$

$\partial f/\partial x = 0$, $\partial f/\partial y = q+4y$, $\partial f/\partial z = 2$, $\partial f/\partial p = 2p$, $\partial f/\partial q = y$

Aux. eq$^n$ :- $$\frac{dx}{-(2p)} = \frac{dy}{-(y)} = \frac{dz}{-(p\times 0 + q^2 + 4qy)} = \frac{dp}{0+2p} = \frac{dq}{q+4y+2q}$$

we have $\Rightarrow$ $\frac{dx}{-2p} = \frac{dp}{2p} \Rightarrow dx = -dp$ ; $x = -p + a$

$P = a - x$

From given eq$^n$ : $2z + (a-x)^2 + qy + 2y^2 = 0$

$qy = -2z - 2y^2 - (a-x)^2 = -2z - 2y^2 - (a^2 - 2ax + x^2)$

$qy = -2z - 2y^2 - a^2 + 2ax - x^2$

$$q = \frac{2ax}{y} - \frac{(2z+2y^2+a^2+x^2)}{y}.$$

Also $dz = p\,dx + q\,dy = \frac{}{}(a-x)\,dx + \frac{-1}{y}\left(2z + (a-x)^2 + 2y^2\right)dy$

multip: above eq$^n$ with $2y^2$ :-

$2y^2 dz = 2y^2(a-x)\,dx - 2y\left(2z + (a-x)^2 + 2y^2\right)dy$

$= 2y^2(a-x)\,dx - 4yz\,dy - 2y(a-x)^2 dy - 4y^3\,dy$

ie $2y^2 dz + 4yz\,dy = -\left[(a-x)^2 2y\,dy - y^2\,2(a-x)\,dx\right] - 4y^3\,dy$.

$d(2y^2 z) = -\frac{d}{dx}\left((a-x)^2 y^2\right) - d(y^4)$ $\longrightarrow$ on integrating

$2y^2 z = -(a-x)^2 y^2 - y^4 + b \rightarrow \underline{b = 2y^2 z + \left(y^2(a-x)^2\right) + y^4}$

③ Solve $f = p^2 y + q^2 y - qz = 0$

Ans:- $\partial f/\partial x = 0$ $\quad \partial f/\partial y = p^2+q^2$ $\quad \partial f/\partial z = -q$ $\quad \partial f/\partial p = 2py$ $\quad \partial f/\partial q = 2qy - z$

AE :- $\frac{dx}{-2py} = \frac{dy}{-(2qy-z)} = \frac{dz}{-[2yq^2 - qz]} = \frac{dp}{0+(-pq)} = \frac{dq}{p^2+q^2-q^2}$

$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow \frac{dp}{q} = \frac{-dq}{p} \Rightarrow p^2 = -q^2 + a^2 \Rightarrow a^2 = p^2+q^2$

from given eq$^n$ :- $a^2 y - qz = f = 0 \rightarrow a^2 y = qz \rightarrow q = a^2 y/z$

$p^2 = a^2 - q^2 = a^2 - \frac{a^4 y^2}{z^2} = a^2\left[1 - \frac{a^2 y^2}{z^2}\right] \rightarrow p = \frac{a}{z}\sqrt{z^2 - a^2 y^2}$

But $dz = p\,dx + q\,dy = \frac{a}{z}\sqrt{z^2 - a^2 y^2}\,dx + \frac{a^2 y^2}{z}\,dy$

ie $z\,dz = a\sqrt{z^2 - a^2 y^2}\,dx + a^2 y^2\,dy \Rightarrow \frac{z\,dz - a^2 y^2\,dy}{\sqrt{z^2 - a^2 y^2}} = a\,dx$

If $u = \sqrt{z^2 - a^2y^2}^{\,2}$, then it is of the form $\frac{du}{2\sqrt{u}} = \sqrt{a}\, du$

then given integrating $\Rightarrow$ $\sqrt{u} = ax + b$

$\sqrt{z^2 - a^2y^2} = ax + b \Rightarrow (z^2 - a^2y^2) = (ax+b)^2$

④ Solve $1 + p^2 = qz$.

Ans: $\partial f/\partial x = 0$ $\partial f/\partial y = 0$ $\partial f/\partial z = -q$ $\partial f/\partial p = 2p$ $\partial f/\partial q = -z$.

$$AE = \frac{dx}{-2p} = \frac{dy}{-(-z)} = \frac{dz}{-[2p^2 + (-qz)]} = \frac{dp}{0 + (-pq)} = \frac{dq}{0 + (-q^2)}$$

So we have $\frac{-dp}{pq} = \frac{-dq}{q^2} \rightarrow \frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log q + \log a$

$p/q = a$.

then $p = aq$ (subti. in given eq$^n$)

$1 + (a^2q^2) = qz \Rightarrow 1 + q[qa^2 - z] = 0 \Rightarrow a^2q^2 - qz + 1 = 0$

$$\frac{z \pm \sqrt{z^2 - 4 \times a^2 \times 1}}{2a^2} = \frac{z \pm \sqrt{z^2 - 4a^2}}{2a^2} = q \quad \& \quad p = aq = \frac{z \pm \sqrt{z^2 - 4a^2}}{2a}$$

$$dz = p\,dx + q\,dy = \frac{z \pm \sqrt{z^2 - 4a^2}}{2a}dx + \frac{z \pm \sqrt{z^2 - 4a^2}}{2a^2}dy$$

$$\Rightarrow 2a^2 dz = a\left(z \pm \sqrt{z^2 - 4a^2}\right)dx + \left(z \pm \sqrt{z^2 - 4a^2}\right)dy.$$

$$\frac{2a^2\, dz}{z \pm \sqrt{z^2 - 4a^2}} = a\,dx + dy \rightarrow \text{on integrating}$$

$$ax + y = \int \frac{2a^2}{z \pm \sqrt{z^2 - 4a^2}}\, dz \Big\}\ 2 \text{ possible sol}^n.$$

Multiplying with Conjugate on RHS of above eq$^n$ $\Rightarrow$

$$ax+y = 2a^2 \int \frac{z - \sqrt{z^2 - 4a^2}}{4a^2}\, dz = \frac{1}{2}\left[\int z\, dz - \int \sqrt{z^2 - (2a)^2}\, dz\right]$$

$\Rightarrow$ we have $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2} \log \left| x - \sqrt{x^2 - a^2} \right|$

then

$$ax+b = \frac{1}{2}\left[ z^2/2 - z/2 \sqrt{z^2 - 4a^2} - \frac{4a^2}{2} \log \left| z - \sqrt{z^2 - 4a^2} \right| \right] //$$

HW : Using charpits method :- Solve (i) $z^2 = pqxy$ (ii) $q + px = p^2$