Module 1

ANALYSIS OF STRUCTURES

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Introduction to elasticity and plasticity

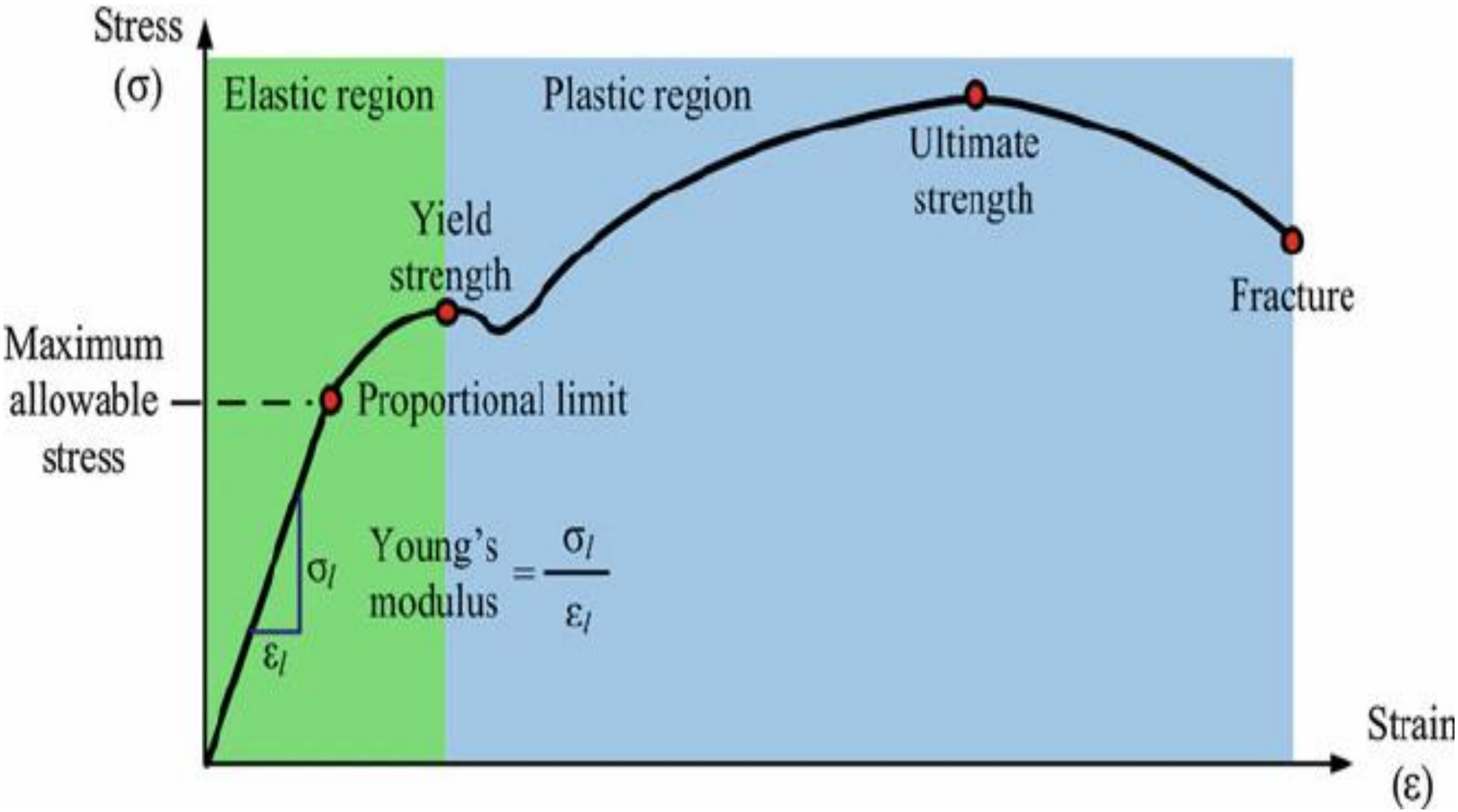
What is Elasticity?

- - Ability of a material to regain its original shape after unloading
- - Obeys Hooke’s Law
- - Steel under small loads

What is Plasticity?

- \- Property of a material to retain deformation after unloading
- \- Occurs beyond yield point
- \- Steel under large loads.

Stress-Strain Curve (Mild Steel)



Hooke's law :σ = E × ε is valid in elastic region

Elastic Constants

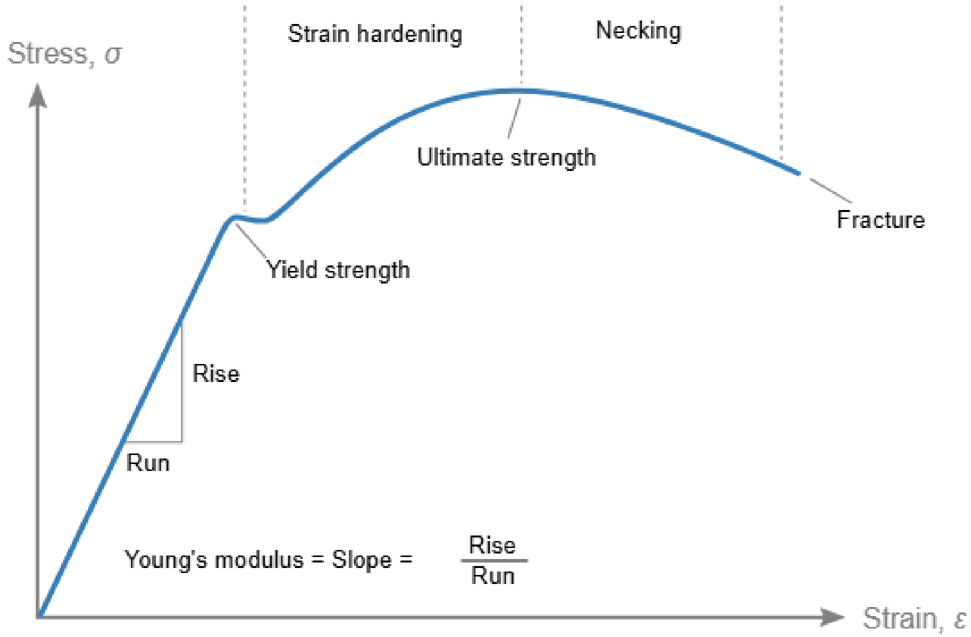
- Young's Modulus (E): Linear stress-strain
- Shear Modulus (G): Shear stress-shear strain
- Bulk Modulus (K): Volume change under pressure
- Poisson's Ratio (ν): Lateral/axial strain

Relations Between Constants

$$E = 2G(1 + \nu), \quad E = 3K(1 - 2\nu)$$

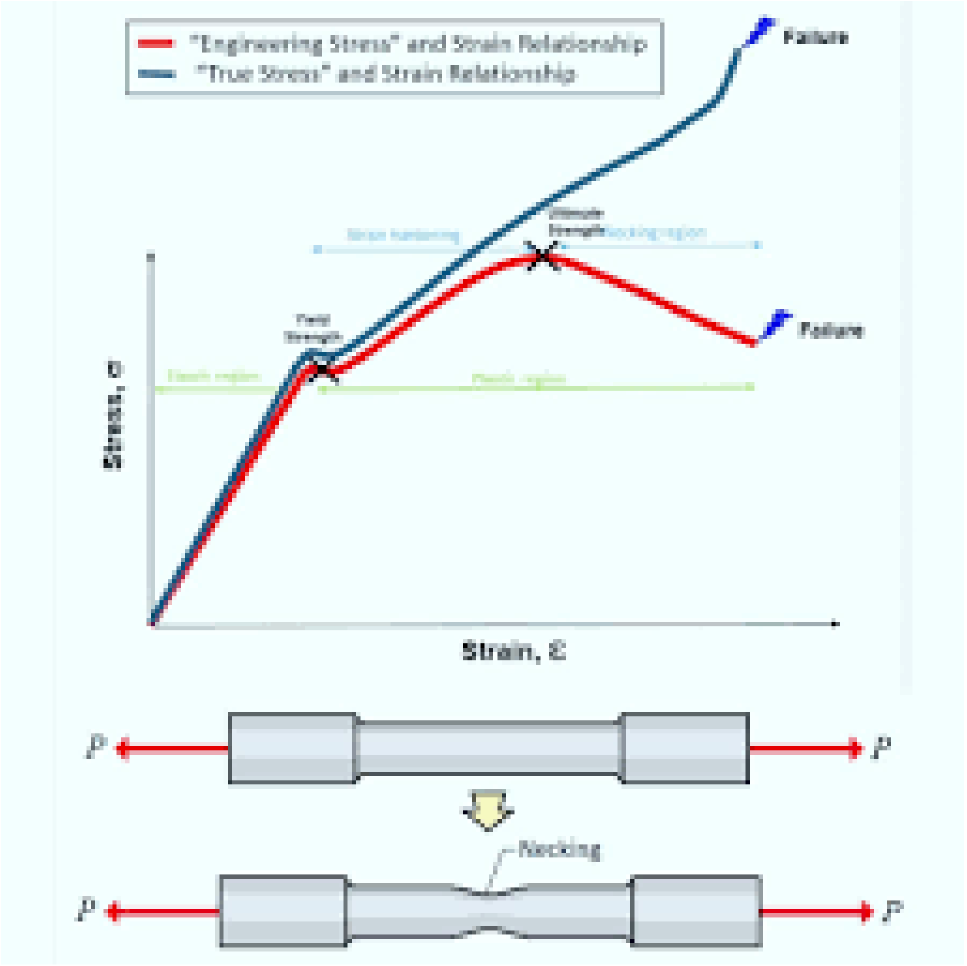
Yielding & Plastic Flow

- - Yield Point: Start of plastic deformation
- - Plastic Flow: Continuous deformation with constant stress
- - Strain Hardening: Material becomes stronger during plasticity



Engineering Stress Strain vs True Stress Strain Curve

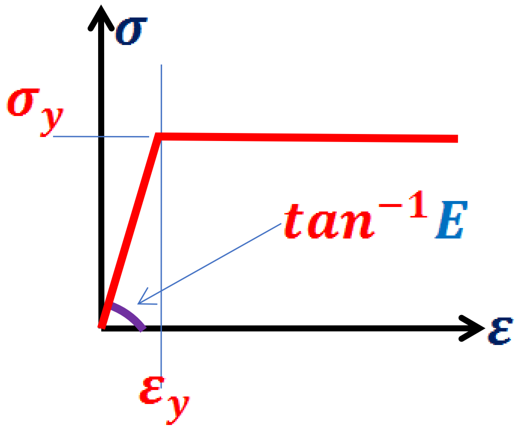
- Engineering stress-strain curves use the original cross-sectional area.
- True stress-strain curves use the instantaneous cross-sectional area.
- This leads to the true stress-strain curve consistently plotting higher stress values than the engineering curve for the same strain.



Introduction to plasticity

Plastic theory

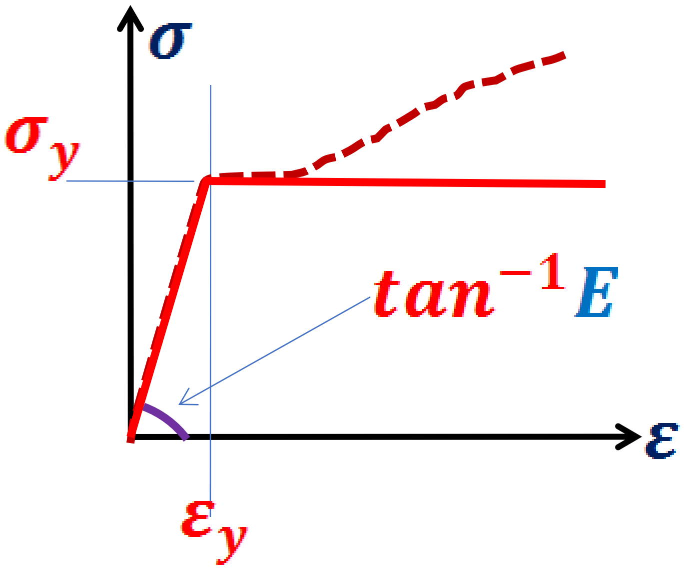
Idealised elastic-perfectly plastic stress-strain curve.



Plastic theory is based on this stress – strain curve.

Plastic theory

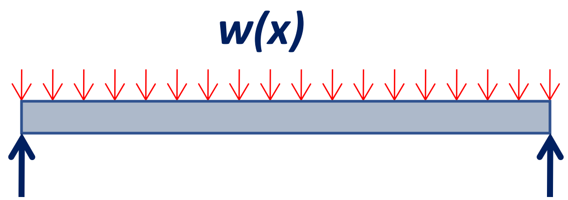
Idealised elastic-perfectly plastic stress-strain curve.



It is conservative since it ignores the subsequent strain-hardening.

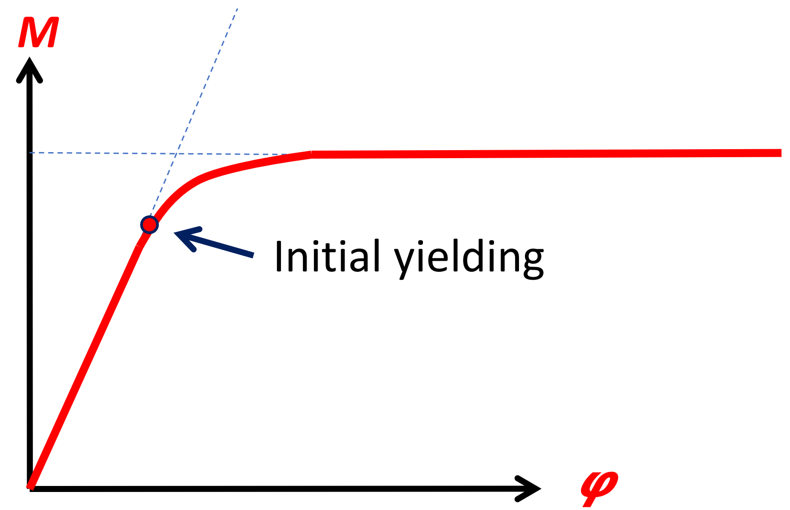
Plastic theory

Consider a simply supported beam with lateral load w(x) as shown.



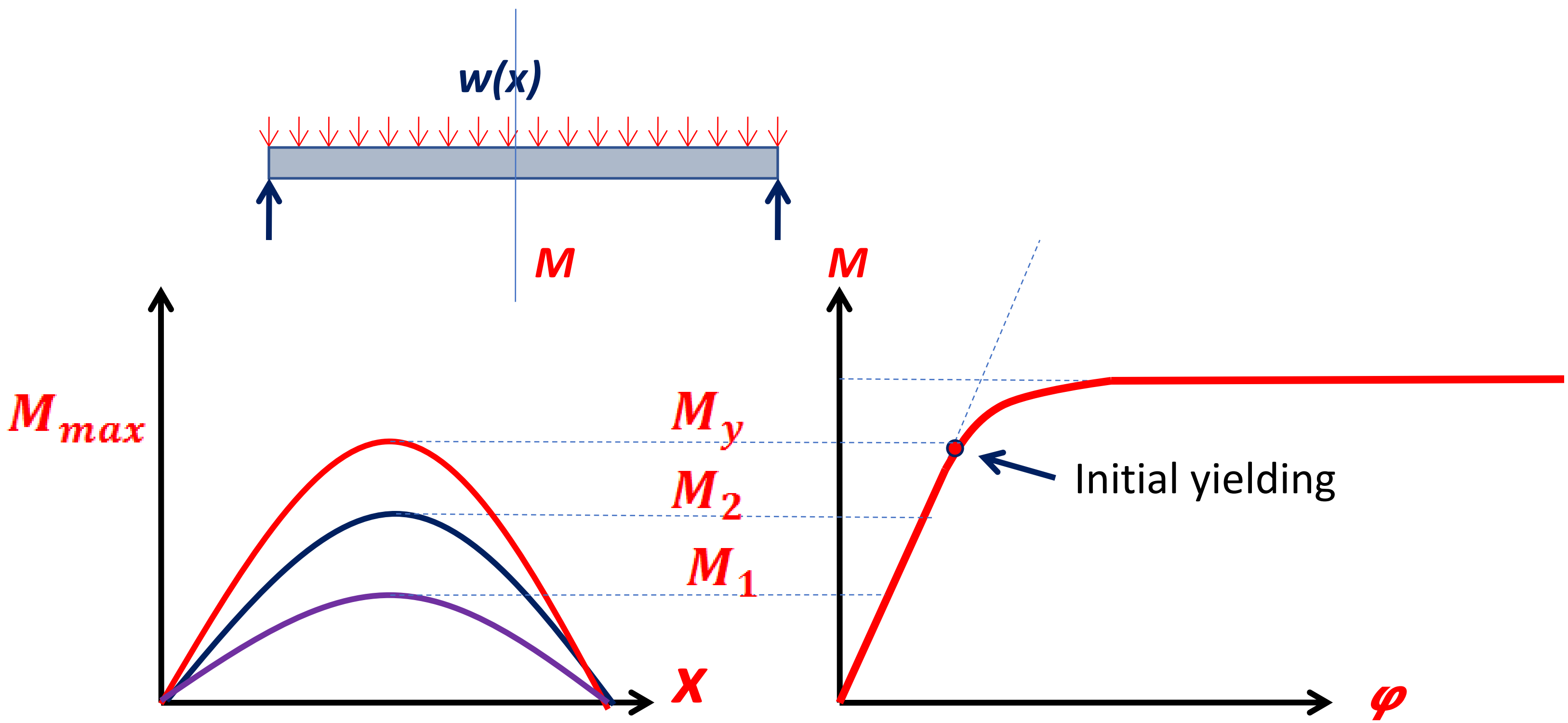
The bending moment , M, at any location is proportional to the local curvature, φ within the elastic limit.

The M-φ curve is non-linear beyond the elastic limit.



Moment- Curvature Curve in Plastic theory

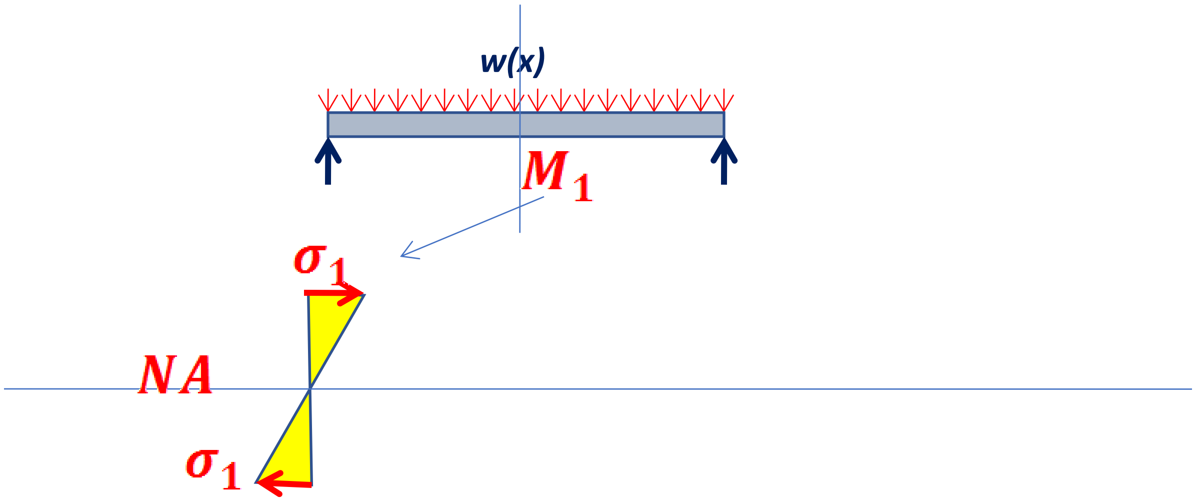
- ***Let the Maximum Bending Moment at midspan be M_1.***
- ***As the load increases, the maximum bending moment also increases to M_2, etc.***
- ***The load increases, till maximum bending moment reaches My. (the moment at which yielding starts.)***



Plastic theory

Consider a simply supported beam with lateral load w(x) as shown.

The bending stresses vary as follows:

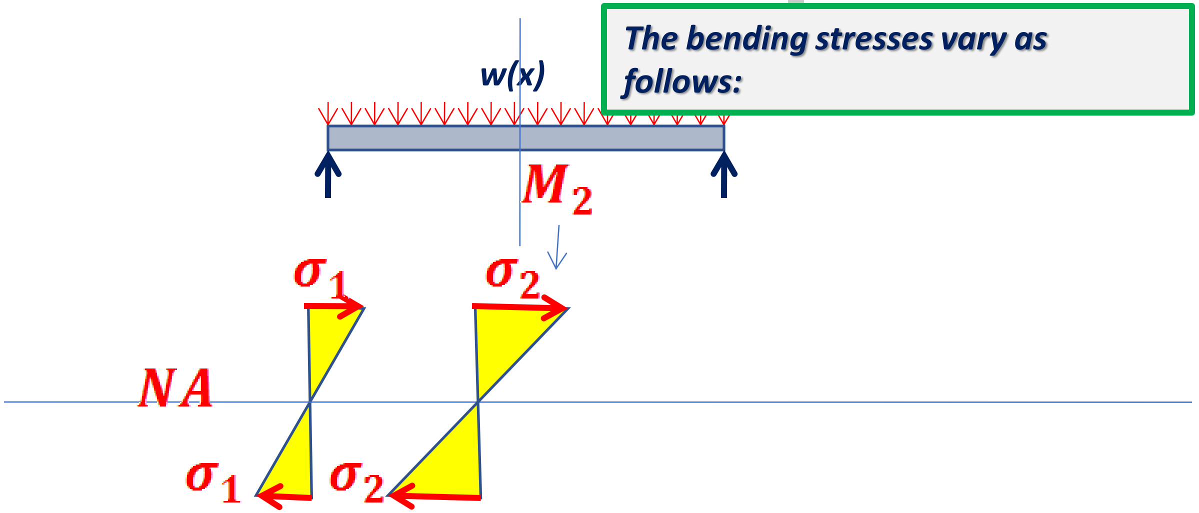


Bending stress distribution in a symmetric section.

Plastic theory

Consider a simply supported beam with lateral load *w(x)* as shown.

The bending stresses vary as follows:

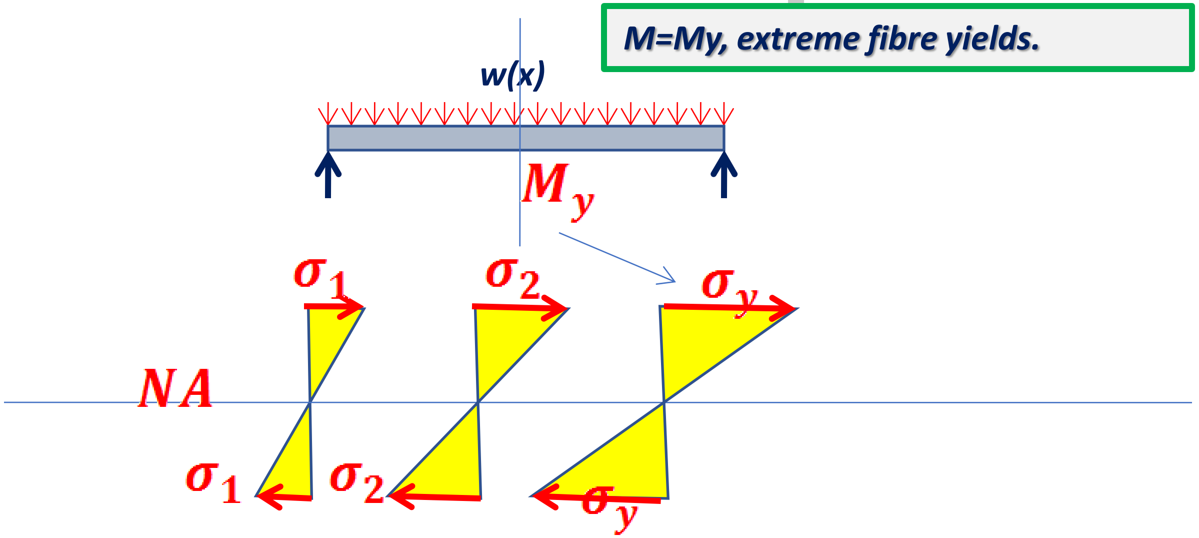


Bending stress distribution in a symmetric section.

Plastic theory

Consider a simply supported beam with lateral load *w(x)* as shown.

M=My, extreme fibre yields.

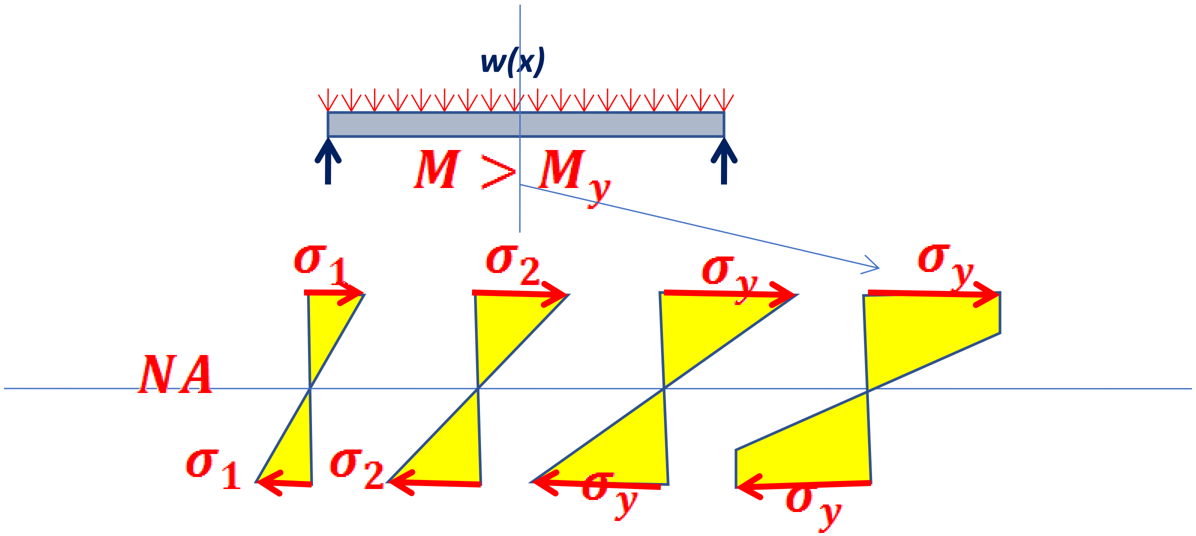


Bending stress distribution in a symmetric section.

Plastic theory

Consider a simply supported beam with lateral load *w(x)* as shown.

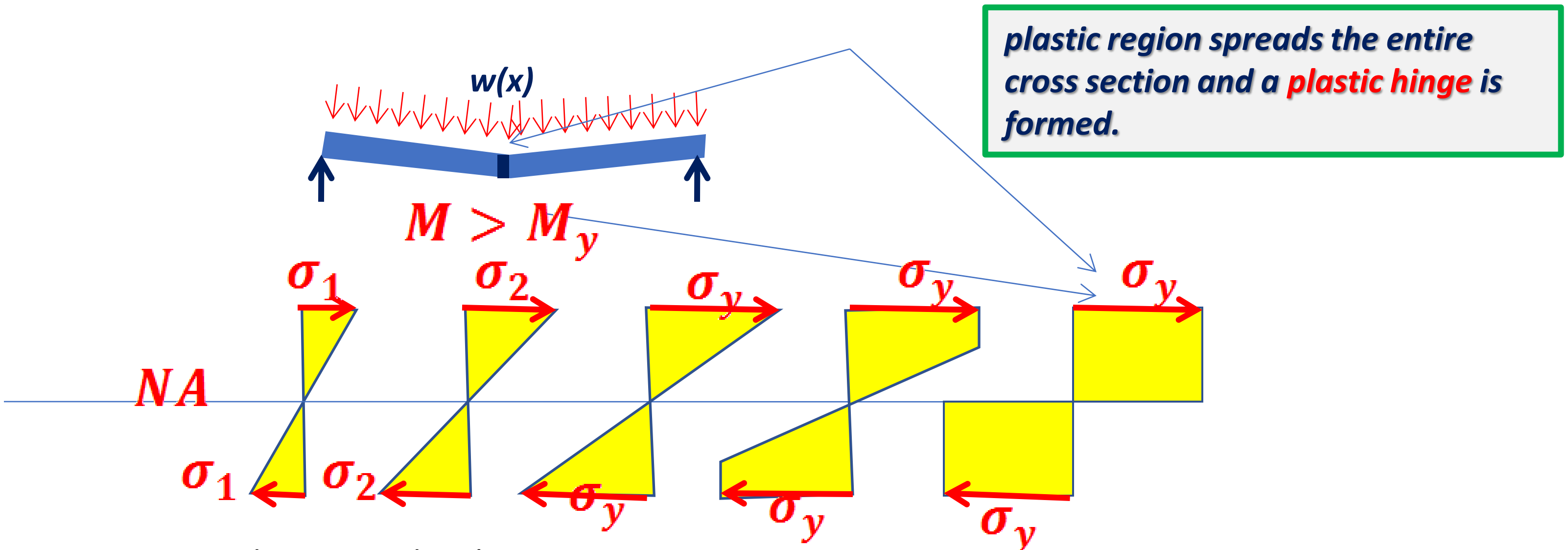
The bending stresses vary as follows:



When M=My, extreme fibre yields and enter the plastic region. The region will slowly spread thereafter.

Plastic theory

*Consider a simply supported beam with lateral load **w(x)** as shown.*

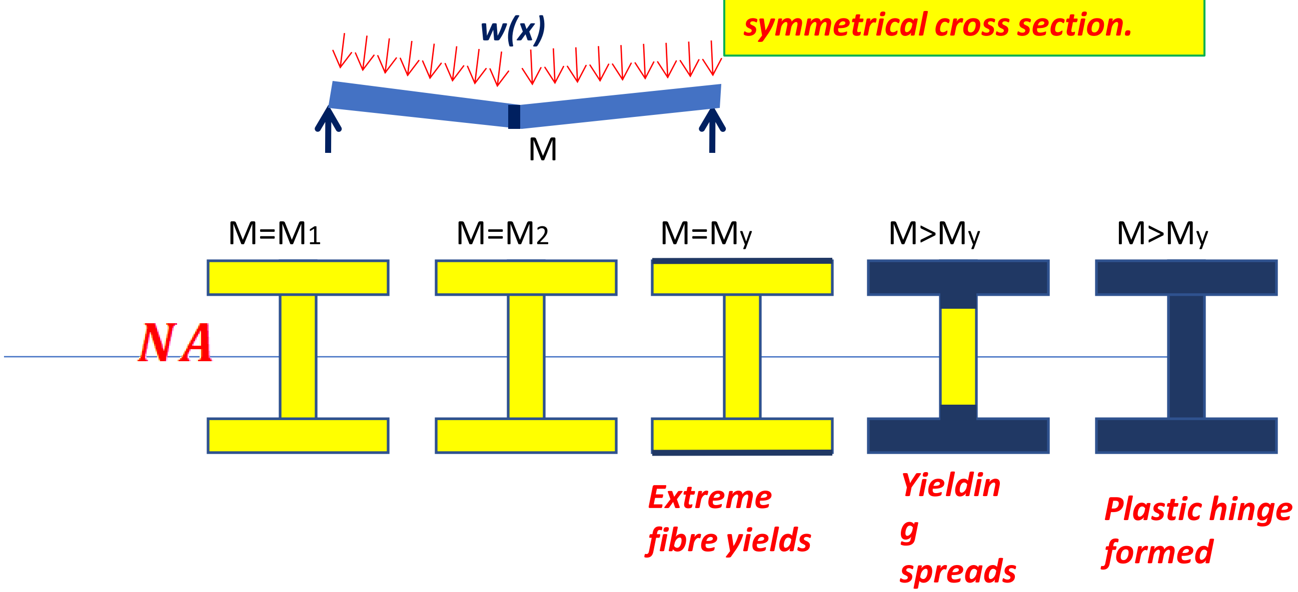


Bending stress distribution in a symmetric section.

Plastic theory

Consider a simply supported beam with lateral load *w(x)* as shown.

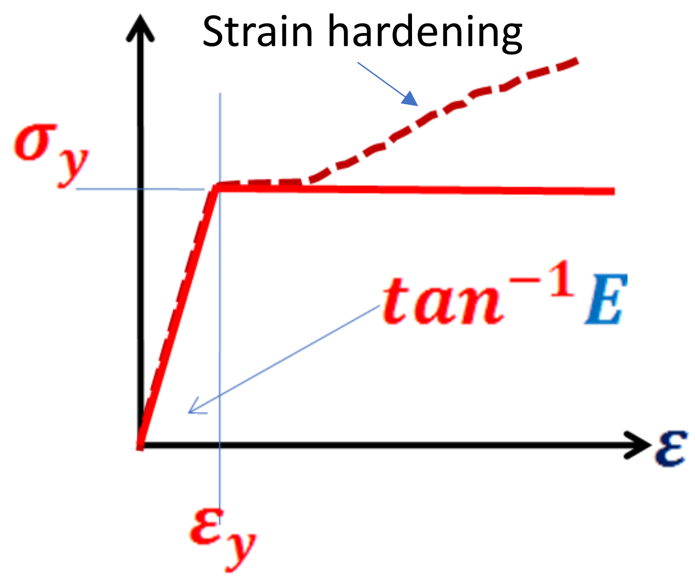
Plastic hinge formation in a symmetrical cross section.



The cross section is now fully plastic. The section cannot take any more bending moment. It is as if a hinge is inserted at the location. Hence the name plastic hinge.

Plastic theory

- The bending moment at the location corresponding to the condition is called the plastic moment, Mp of the section.
- The simply supported beam, as in this case, will collapse if such a hinge is formed. Probably, the strain hardening can delay the collapse.



Plastic theory

Assumptions used for finding the plastic moment of a section:

-***The material obeys Hooke's Law until the stress reaches the yield value.***

-***The yield stresses and the modulus of elasticity have the same value in compression as in tension.***

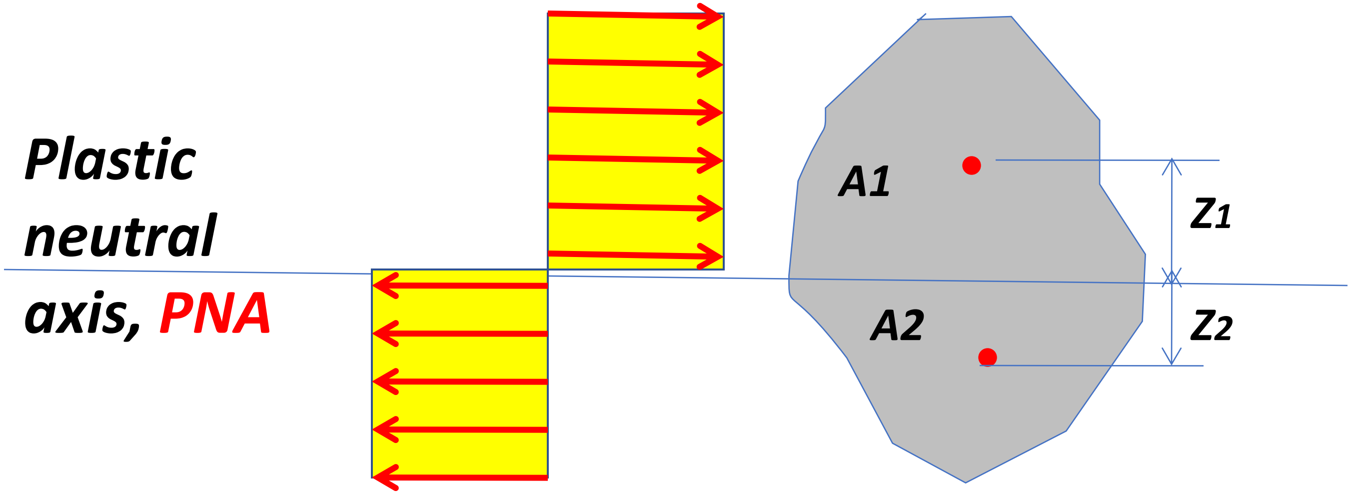
-***The material is homogeneous and isotropic in both the elastic and plastic states.***

-***There is no resultant axial force on the beam***

-***The plane transverse sections remain plane and normal to longitudinal axis after bending, the effect of shear being neglected.***

Plastic theory: Location of neutral axis

Beam with arbitrary cross section.

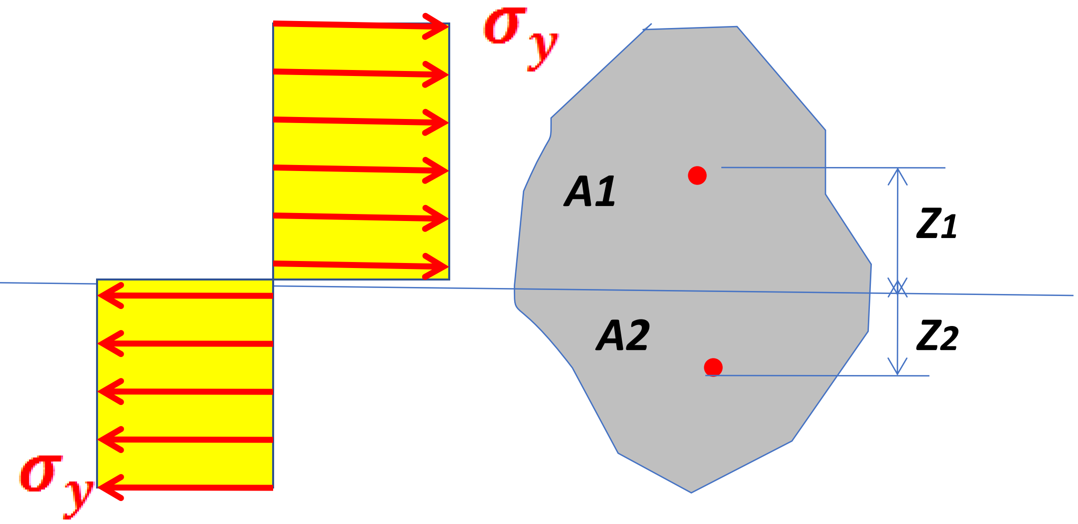


Let A_1 and A_2 be the compressive and tensile regions in a cross section of plastic hinge. Z_1 and Z_2 are the distance to their centroids from the plastic neural axis.

Plastic theory: Location of neutral axis

Beam with arbitrary cross section.

Plastic neutral axis, PNA



-Since there is no resultant axial force on the cross-section: $\sigma_y A_1 = \sigma_y A_2$

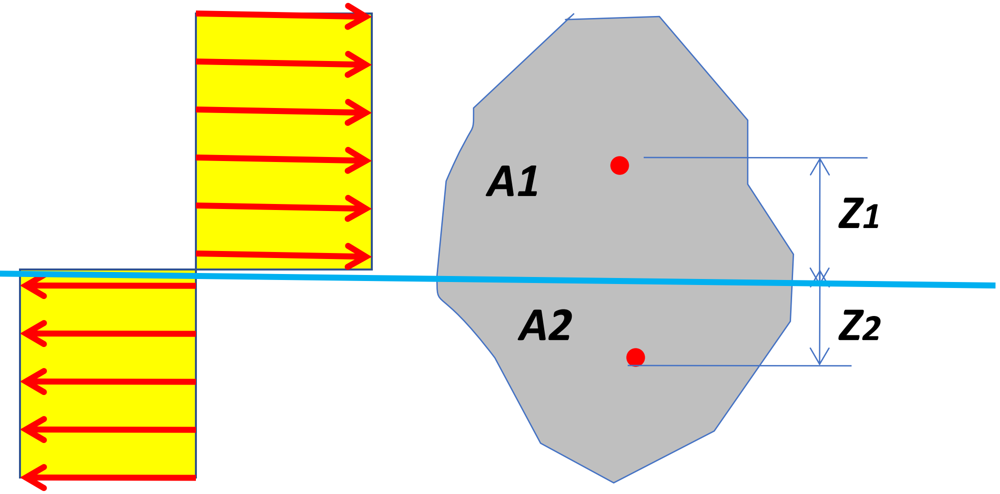
$$\Rightarrow \quad A_1 = A_2$$

PNA divides the area of cross section equally.

Plastic theory

Beam with arbitrary cross section.

Plastic neutral axis, PNA.



$$M_p = \sigma_y A_1 z_1 + \sigma_y A_2 z_2$$

$$M_p = \sigma_y \frac{A}{2} \left(z_1 + z_2\right)$$

This is just an **analogy** to the elastic formula:

$$M_p = \sigma_y z_p$$

So comparing both forms:

$$z_p = \frac{A}{2}(z_1 + z_2)$$

This is the plastic section modulus

- Thus, the modulus of section in the plastic analysis can be obtained again by taking the static moment of the tensile and compression areas about the neutral axis.
- Zp for a particular cross section is constant.
- Thus, Mp for a particular cross section and material is constant.

Plastic theory: Shape factor

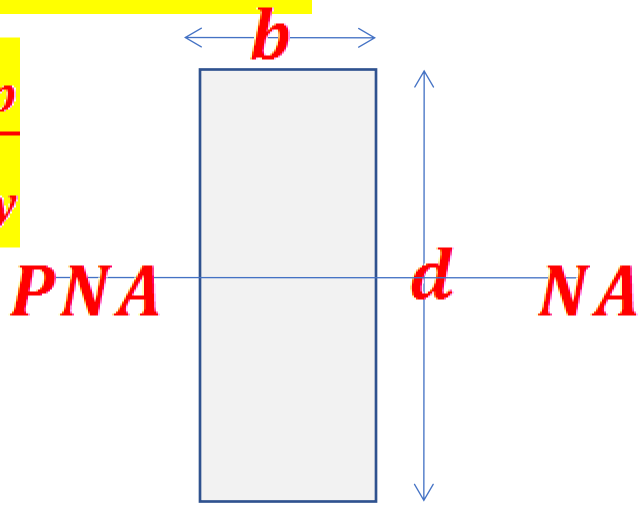
- Shape factor is defined by:

$$\boxed{shape\ factor = \frac{z_p}{z_y} = \frac{M_p}{M_y}}$$

Plastic theory

Shape factor for rectangular cross section:

$$shape\ factor = \frac{z_p}{z_y} = \frac{M_p}{M_y}$$



$$z_y = \frac{bd^3}{12\left(\frac{d}{2}\right)} = \frac{bd^2}{6}$$

$$z_p = \frac{1}{2} A\left(z_1 + z_2\right) = \frac{bd^2}{4}$$

$$shape\ factor = \frac{z_p}{z_y} = 1.5$$

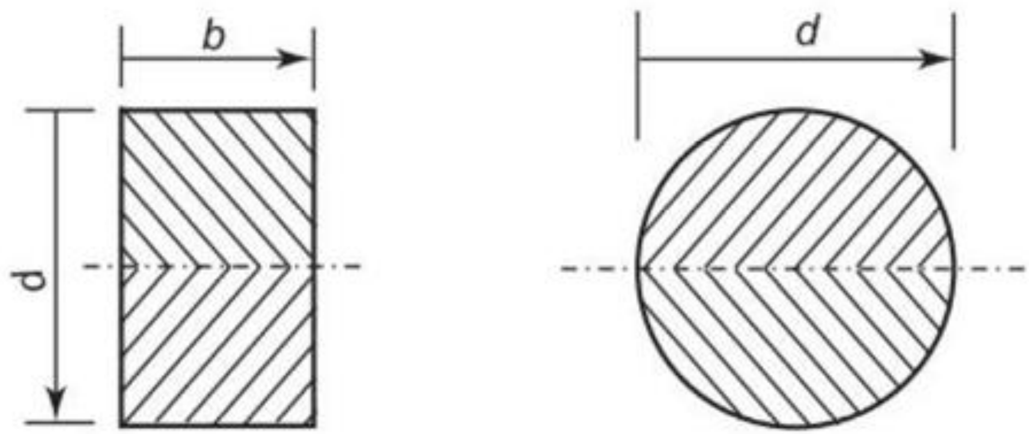
Significance of shape factor

- The **shape factor S** tells us how much **extra moment** a cross-section can carry **after yielding starts**, before forming a plastic hinge.
- A **rectangular section** has S=1.5
  → It can carry **50% more moment** plastically than what elastic analysis predicts.

Shape factor for circular cross section:

$\therefore$ Shape factor: $$S = \frac{Z_p}{Z_e}$$

$$= \frac{d^3}{6} \div \frac{\pi}{32} d^3 = 1.7$$



(a) (b)

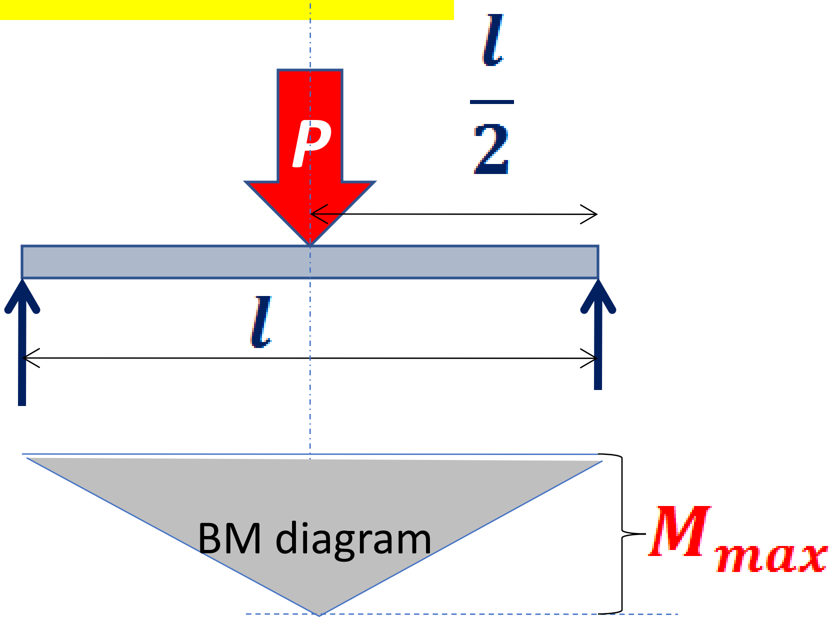
Plastic theory

Collapse of beams-ultimate load

Simply supported beam with point load at midspan.

$$M_{max} = \frac{Pl}{4}$$

$$\frac{4M_{max}}{l} = P$$



As we increase load P, Mmax increases. At Mmax=My, the outer fiber starts yielding.

When Mmax=My, P=Py;

$$P_y = \frac{4M_y}{l}$$

- ***When $M_{max}=M_p$, the beam collapses since it cannot take any load. The load corresponding to this is called ultimate load, P_{ult}.***
- Thus,

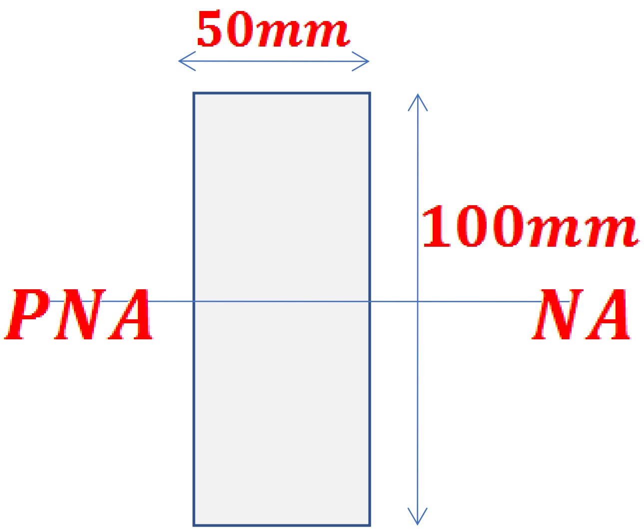
$$P_{ult} = \frac{4M_p}{l}$$

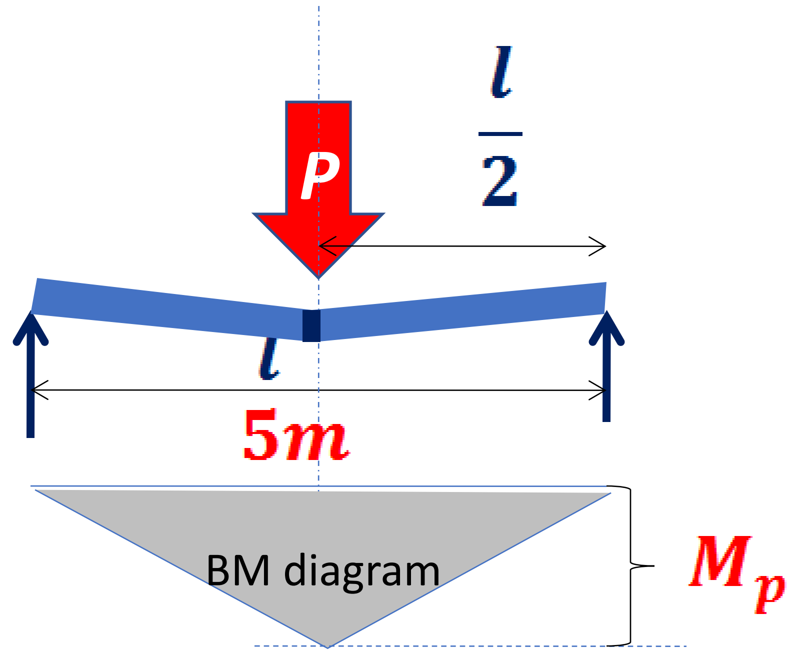
Where, $$M_p = z_p\sigma_y$$

For the rectangular cross section shown:

$$M_p = \frac{bd^2}{4}\sigma_y$$

Plastic theory





For the rectangular cross section shown:

$$M_p = 29.375kN - m \quad \Rightarrow \quad P_{ult} = 23.5kN$$

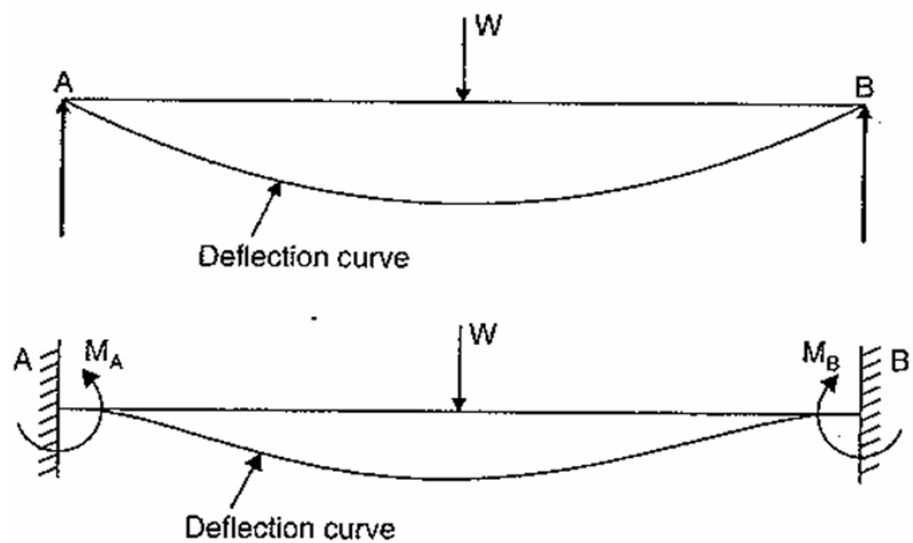
$$M_y = 19.583kN - m \quad \Rightarrow \quad P_y = 15.666kN$$

Fixed and Continuous Beams

Fixed Beams

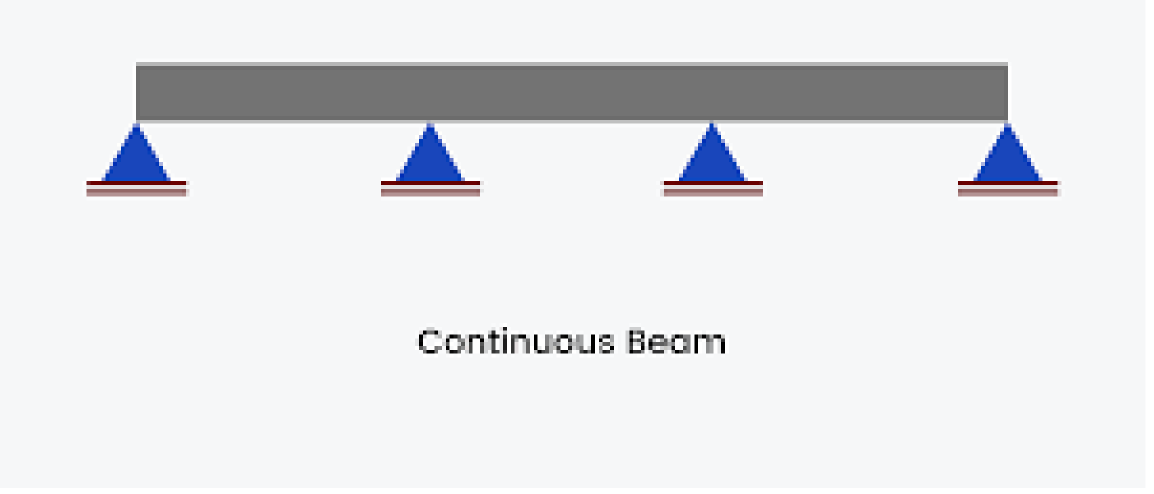
- A beam whose both ends are fixed is known as a fixed beam.
- Fixed beam is also called a *built-in* or *encaster* beam.
- In case of a fixed beam both its ends are rigidly fixed and the slope and deflection at the fixed ends are zero.
- But the fixed ends are subjected to end moments. Hence end moments are not zero in case of a fixed beam.

- In case of fixed beam, the deflection and slope are zero at the fixed ends as shown in Fig.
- The slope will be zero at the ends if the deflection curve is horizontal at the ends.
- To bring the slope back to zero (*i.e.*, to make the deflection curve horizontal at the fixed ends), the end moments MA will be acting anti-clockwise and MB will be acting clockwise as shown in Fig.



Continuous beams

- A beam which is supported on more than two supports is known as **continuous beam**.



Statically Determinate and Indeterminate Structures

(REFER CLASS NOTES)

Superposition of BMDs for a statically indeterminate structure
(Free and Fixed BMDs to get combined/Final BMD)

(REFER CLASS NOTES)

Displacement of beams

1. DOUBLE INTEGRATION METHOD

(REFER CLASS NOTES)

2. MACAULAY'S METHOD

- In cases where we **cannot have a single equation** to represent the BM variation for the whole beam, as in case of unsymmetrical loading or multiple point or uniformly distributed loads, DI method becomes more tedious.
- Due to this, a **revised form of successive integration** method, developed by M H Macaulay.
- Macaulay's method **simplifies the analysis by allowing a single equation** to represent the entire beam.
- This method, also known as the method of singularity functions, since the bending moment equation changes abruptly at different points along the beam.

Procedure

1. Write the **bending moment expression for the extreme section considering the farthest side** of the section.
2. **Split each terms** in that expression by dotted line.
3. **Equate it to the differential expression** for B.M.
4. Integrate successively to get the expressions of slope and deflection. The **constant of integration is to be written along with the first term of the expression**. Note that the **brackets are to be integrated as a whole**, i.e., the **integration of (x-a) will be:**

$$\frac{(x-a)^2}{2} \text{ and not } \frac{x^2}{2} - ax$$

5. Apply the boundary conditions to get the constants of integration. While applying the boundary conditions, **consider only the respective part of the equation in which that corresponding boundary conditions is valid.**
6. Substitute the constants back into the respective equations to get the slope and deflections.

Numerical 1

A beam of length 6 m is simply supported at its ends and carries two point loads of 48 kN and 40 kN at a distance of 1 m and 3 m respectively from the left support. Find :

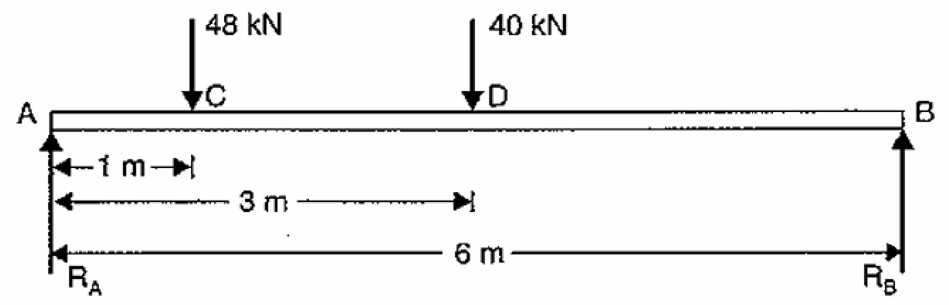
(i) deflection under each load,

(ii) maximum deflection, and

(iii) the point at which it occurs.

Given: $E = 2 \times 10^5 \ \mathrm{N/mm^2}$ and $I = 85 \times 10^6 \ \mathrm{mm^4}$

Solution:

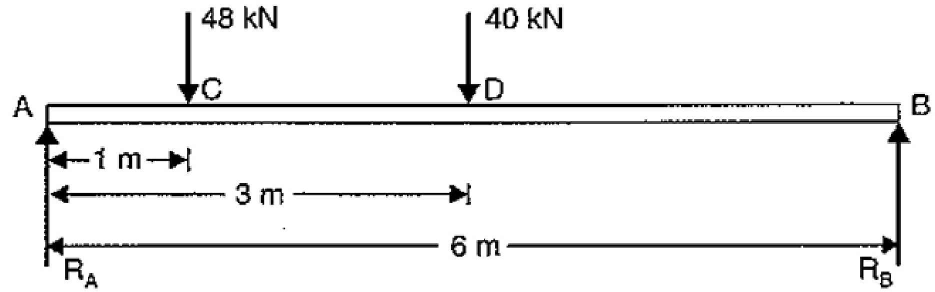


Taking moments about A, we get

$$R_B \times 6 = 48 \times 1 + 40 \times 3 = 168$$

$$\therefore \quad R_B = \frac{168}{6} = 28 \text{ kN}$$

$$\therefore \quad R_A = \text{Total load} - R_B = (48 + 40) - 28 = 60 \text{ kN}$$



Consider the section X in the last part of the beam (*i.e.*, in length DB) at a distance x from the left support A. The B.M. at this section is given by,

$$EI\frac{d^2y}{dx^2} = R_A.x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3)$$

$$= 60x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3)$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = \frac{60x^2}{2} + C_1 \;\vdots\; -48\frac{(x-1)^2}{2} \;\vdots\; -40\frac{(x-3)^2}{2}$$

$$= 30x^2 + C_1 \;\vdots\; -24(x-1)^2 \;\vdots\; -20(x-3)^2 \qquad \ldots(i)$$

Integrating the above equation again, we get

$$EIy = \frac{30x^3}{3} + C_1x + C_2 \;\vdots\; \frac{-24(x-1)^3}{3} \;\vdots\; \frac{-20(x-3)^3}{3}$$

$$= 10x^3 + C_1x + C_2 \;\vdots\; -8(x-1)^3 \;\vdots\; -\frac{20}{3}(x-3)^3 \qquad \ldots(ii)$$

To find the values of C_1 and C_2, use two boundary conditions. The boundary conditions are :

(*i*) at $x = 0, y = 0,$ and (*ii*) at $x = 6$ m, $y = 0$.

(*i*) Substituting the first boundary condition *i.e.*, at $x = 0$, $y = 0$ in equation (*ii*) and considering the equation upto first dotted line (as $x = 0$ lies in the first part of the beam), we get

$$0 = 0 + 0 + C_2 \quad \therefore \quad C_2 = 0$$

(*ii*) Substituting the second boundary condition *i.e.*, at $x = 6$ m, $y = 0$ in equation (*ii*) and considering the complete equation (as $x = 6$ lies in the last part of the beam), we get

$$0 = 10 \times 6^3 + C_1 \times 6 + 0 - 8(6-1)^3 - \frac{20}{3}(6-3)^3 \qquad (\because \quad C_2 = 0)$$

or

$$0 = 2160 + 6C_1 - 8 \times 5^3 - \frac{20}{3} \times 3^3$$

$$= 2160 + 6C_1 - 1000 - 180 = 980 + 6C_1$$

$$\therefore \qquad C_1 = \frac{-980}{6} = -163.33$$

Now substituting the values of C_1 and C_2 in equation (*ii*), we get

$$EIy = 10x^3 - 163.33x \;\vdots\; - 8(x-1)^3 \;\vdots\; - \frac{20}{3}(x-3)^3 \qquad \text{...(}iii\text{)}$$

(*i*) (*a*) *Deflection under first load i.e., at point C*. This is obtained by substituting $x = 1$ in equation (*iii*) upto the first dotted line (as the point C lies in the first part of the beam). Hence, we get

$$\begin{aligned} EI.\,y_c &= 10 \times 1^3 - 163.33 \times 1 \\ &= 10 - 163.33 = -\,153.33 \text{ kNm}^3 \\ &= -\,153.33 \times 10^3 \text{ Nm}^3 \\ &= -\,153.33 \times 10^3 \times 10^9 \text{ Nmm}^3 \\ &= -\,153.33 \times 10^{12} \text{ Nmm}^3 \end{aligned}$$

$$\therefore \qquad y_c = \frac{-\,153.33 \times 10^{12}}{EI} = \frac{-\,153.33 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} \text{ mm}$$

$$= -\,\mathbf{9.019\ mm.} \quad \textbf{Ans.}$$

(Negative sign shows that deflection is downwards).

(*b*) *Deflection under second load i.e. at point D*. This is obtained by substituting $x = 3$ m in equation (*iii*) upto the second dotted line (as the point D lies in the second part of the beam). Hence, we get

$$\begin{aligned} EI.y_D &= 10 \times 3^3 - 163.33 \times 3 - 8(3 - 1)^3 \\ &= 270 - 489.99 - 64 = -\,283.99 \text{ kNm}^3 \\ &= -\,283.99 \times 10^{12} \text{ Nmm}^3 \end{aligned}$$

$$\therefore \qquad y_D = \frac{-\,283.99 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -\,\mathbf{16.7\ mm.} \quad \textbf{Ans.}$$

(*ii*) *Maximum Deflection.* The deflection is likely to be maximum at a section between C and D. For maximum deflection, $\frac{dy}{dx}$ should be zero. Hence equate the equation (*i*) equal to zero upto the second dotted line.

$$\therefore \qquad 30x^2 + C_1 - 24(x-1)^2 = 0$$

or $\quad 30x^2 - 163.33 - 24(x^2 + 1 - 2x) = 0 \qquad (\because \quad C_1 = -163.33)$

or $\quad 6x^2 + 48x - 187.33 = 0$

The above equation is a quadratic equation. Hence its solution is

$$x = \frac{-48 \pm \sqrt{48^2 + 4 \times 6 \times 187.33}}{2 \times 6} = 2.87 \text{ m.}$$

(Neglecting – ve root)

Now substituting $x = 2.87$ m in equation (*iii*) upto the second dotted line, we get maximum deflection as

$$EIy_{max} = 10 \times 2.87^3 - 163.33 \times 2.87 - 8(2.87 - 1)^3$$
$$= 236.39 - 468.75 - 52.31$$
$$= 284.67 \text{ kNm}^3 = -284.67 \times 10^{12} \text{ Nmm}^3$$

$$\therefore \qquad y_{max} = \frac{-284.67 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6} = -\mathbf{16.745 \text{ mm.}} \quad \textbf{Ans.}$$

MOMENT AREA THEOREMS/MOHR'S THEOREM

The moment area method is based on the following two theorems (known as **Mohr's theorems**):

Theorem 1

*The **change in the slope** between two points under flexure is equal to the **area of** **M/EI** **diagram** between those two points.*

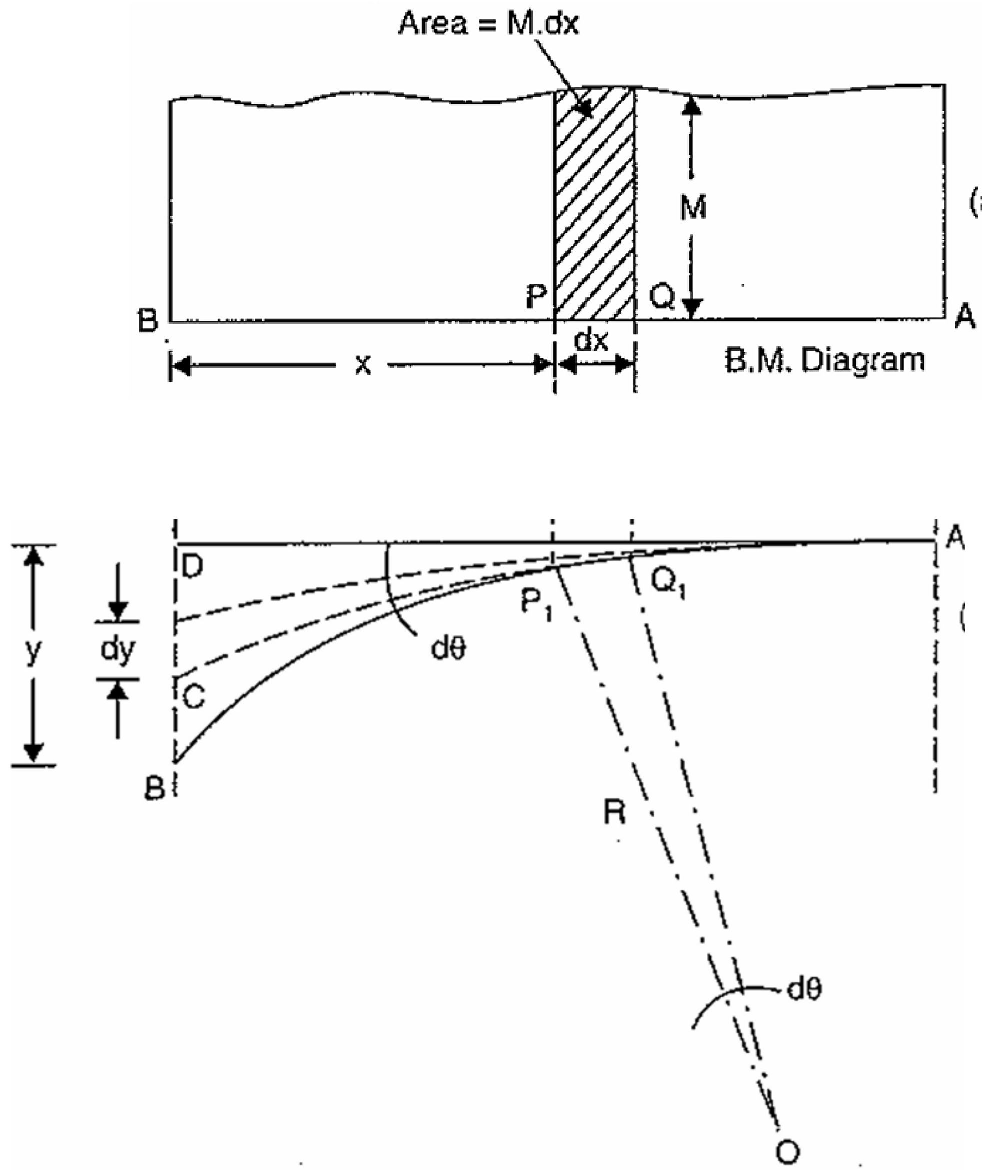
Theorem 2

*The **change in the deflection** between two points under flexure is equal to the **moment of area of** **M/EI** **diagram** between those two points.*

Proof

Consider the beam AB. Let P and Q be any two points on this beam.

The M/EI diagram and the elastic curve of the beam after loading are shown here.



For the deflected part P_1Q_1 of the beam, we have

$$P_1Q_1 = R.d\theta$$

But $$P_1Q_1 \approx dx$$

$$\therefore \quad dx = R.d\theta$$

$$\therefore \quad d\theta = \frac{dx}{R} \qquad \ldots(i)$$

But for a loaded beam, we have

$$\frac{M}{I} = \frac{E}{R} \quad \text{or} \quad R = \frac{EI}{M}$$

Substituting the values of R in equation (i), we get

$$d\theta = \frac{dx}{\left(\frac{EI}{M}\right)} = \frac{M\,dx}{EI} \qquad \ldots(ii)$$

Since the slope at point A is assumed zero, hence total slope at B is obtained by integrating the above equation between the limits 0 and L.

$$\therefore \quad \theta_A - \theta_B = \quad \theta = \int_0^L \frac{M.dx}{EI} = \frac{1}{EI}\int_0^L M.dx$$

But $M.dx$ represents the area of B. M. diagram of length dx. Hence $\int_0^L M.dx$ represents the area of B. M. diagram between A and B.

$$\therefore \quad \theta = \frac{1}{EI}\,[\text{Area of B. M. diagram between } A \text{ and } B]$$

Now the deflection, due to bending of the portion P_1Q_1 is given by

$$dy = x.d\theta$$

Substituting the value of $d\theta$ from equation (*ii*), we get

$$dy = x \, . \, \frac{M.dx}{EI} \qquad \ldots(iii)$$

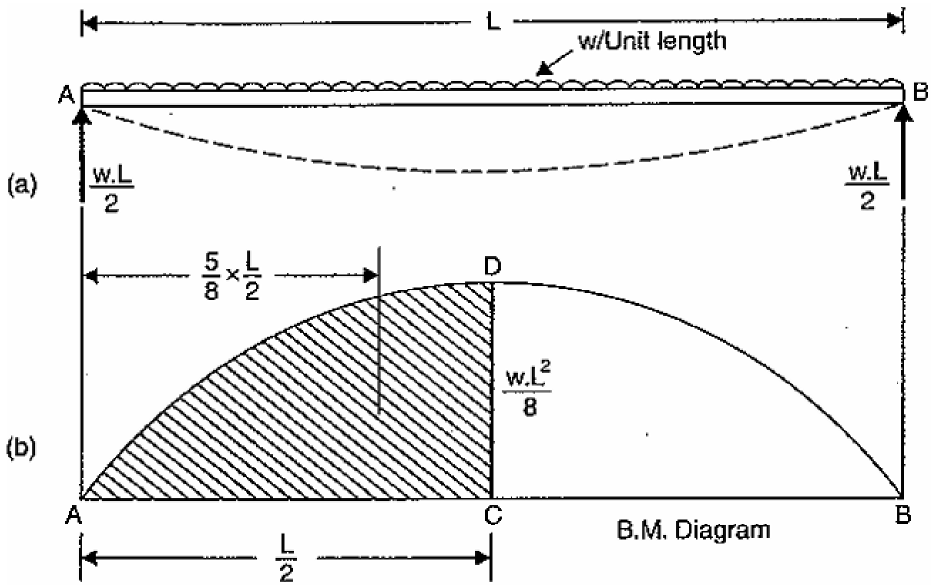
Since deflection at A is assumed to be zero, hence the total deflection at B is obtained by integrating the above equation between the limits zero and L.

$$\therefore \qquad y = \int_0^L \frac{xM.dx}{EI} = \frac{1}{EI} \int_0^L xM.dx$$

But $x \times M.dx$ represents the moment of area of the B.M. diagram of length dx about point B.

Thus, the whole term in the RHS is equal to the total area of M/EI diagram between B and A multiplied by the distance of the C.G. of the B.M. diagram area from B.

Slope and deflection of a simply supported beam carrying a uniformly distributed load by Mohr's theorem



(*i*) Now using Mohr's theorem for slope, we get

$$\text{Slope at} \quad A = \frac{\text{Area of B.M. diagram between } A \text{ and } C}{EI}$$

But area of B.M. diagram between A and C

$$= \text{Area of parabola } ACD$$

$$= \frac{2}{3} \times AC \times CD$$

$$= \frac{2}{3} \times \frac{L}{2} \times \frac{wL^2}{8} = \frac{w \,.\, L^3}{24}$$

$$\therefore \quad \text{Slope at} \quad A = \frac{w \,.\, L^3}{24EI}$$

(*ii*) Now using Mohr's theorem for deflection,

$$y = \frac{A\bar{x}}{EI}$$

where $\quad A$ = Area of B.M. diagram between A and C

$$= \frac{w \,.\, L^3}{24}$$

and $\quad \bar{x}$ = Distance of C.G. of area A from A

$$= \frac{5}{8} \times AC = \frac{5}{8} \times \frac{L}{2} = \frac{5L}{16}$$

$$\therefore \qquad y = \frac{\dfrac{w.L^3}{24} \times \dfrac{5L}{16}}{EI} = \frac{5}{384}\,\frac{w.L^4}{EI}$$

More Questions from Moment Area Method

(REFER CLASS NOTES)

Reference

- **Bansal R. K; Strength of Materials**; Lakshmi Publications; New Delhi, 4th edition, 2007.