Module 4: Structural Dynamics
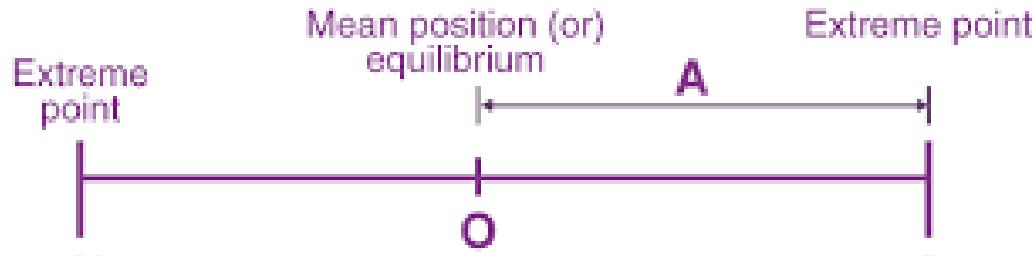
ANALYSIS OF STRUCTURES

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Structural Dynamics

- Structural dynamics deals with the effect of dynamic loading on structures.
- Vibration → To & fro motion of a particle or an assemblage of particles from equilibrium position.
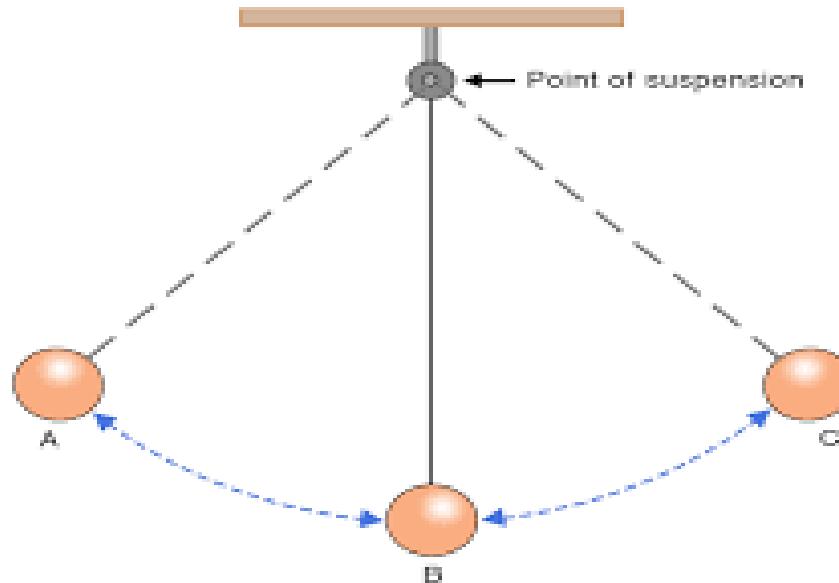
Structural Dynamics

Causes of Vibration
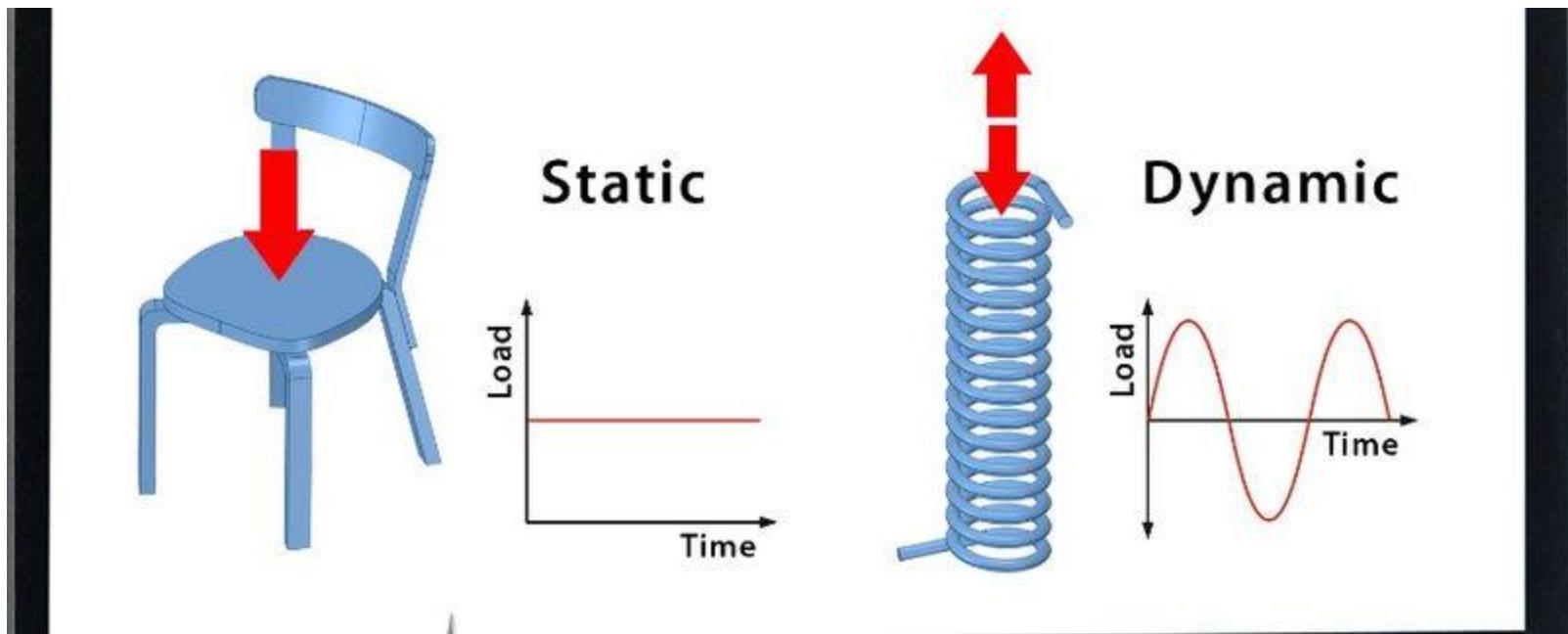
- - Seismic load
- - Wind load
- - Impact load
- - Machine load
- - Blast loading
- - Wave load

Dynamic Analysis

- Load, direction & magnitude change with time.

- Response also changes with time.

- Major responses:

- -Displacement

- - Velocity

- - Acceleration

- - Frequency

Difference from Ordinary Analysis

- ordinary analysis : static equilibrium equations.
- For dynamic analysis: dynamic equilibrium equations → Inertia force comes into effect.

Sources of Dynamic Load
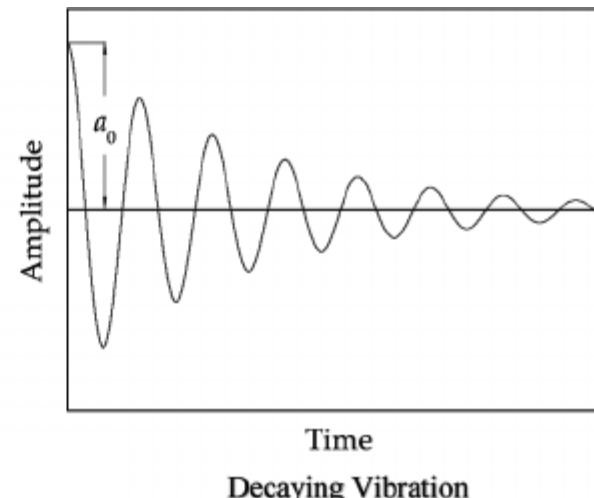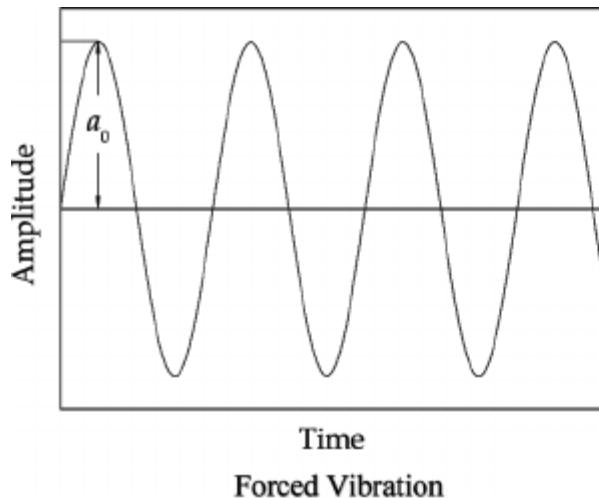
1. **Due to Initial Condition** → Initial pull in the form of initial velocity or initial acceleration causes continuous vibration.

2. **Environmental Effect** → Earthquake, wind, wave, etc.

3. **Mechanical Condition** → All types of machines.

Different Types of Vibration

1. Free Vibration & Forced Vibration

2. Periodic & Non-periodic Vibration

3. Deterministic & Non-deterministic Vibration

4. Longitudinal, Transverse & Torsional Vibration

5. Damped & Undamped Vibration

Free Vibration & Forced Vibration

- Free (Decaying) Vibration → Continues without external force after an initial displacement.

- Forced Vibration → Continues due to an external exciting force.

Time
Forced Vibration

Time
Decaying Vibration
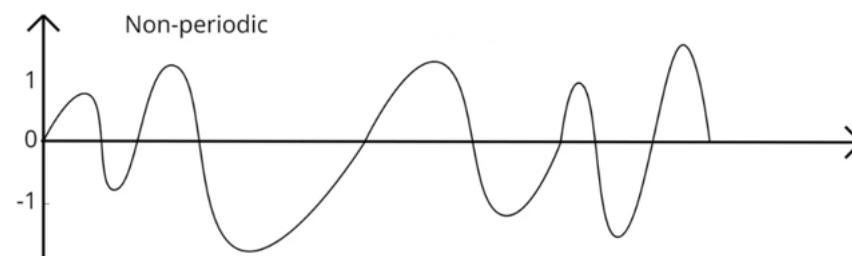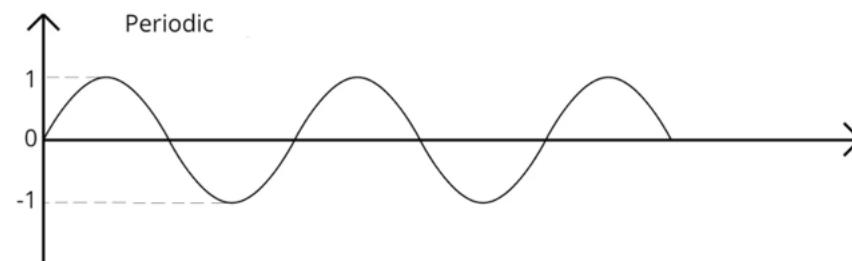
Periodic & Non-periodic Vibration

- Periodic → Follows a regular pattern (e.g., sine wave).

Example: Motion of simple pendulum

- Non-periodic → Irregular, no fixed pattern.

Example: Wave loads, earthquake loads.

Deterministic & Non-deterministic Vibration

- Deterministic → All characteristics, viz., velocity, acceleration, frequency etc. can be calculated.

- Non-deterministic → Random nature, difficult to calculate.

Longitudinal, Transverse & Torsional Vibration

- **Longitudinal** → Along the length.

Example: in bar, truss, any rod elements etc.
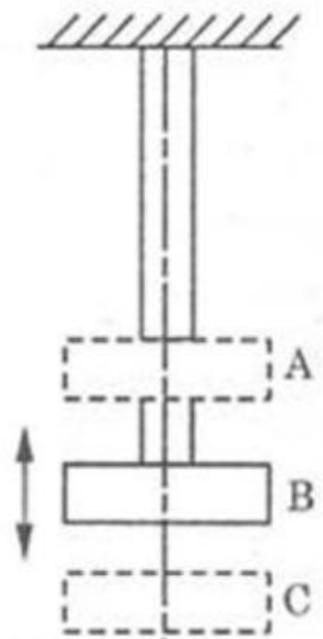
- **Transverse** → Perpendicular to the length.

Example: in column, beam etc.
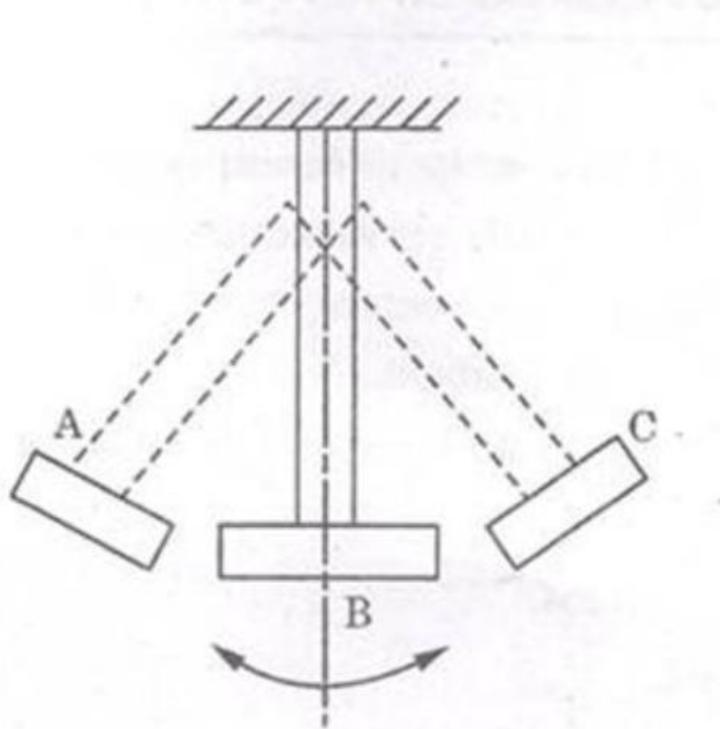
- **Torsional** → Due to twisting.

Example: in shafts etc.

Longitudinal, Transverse & Torsional Vibration
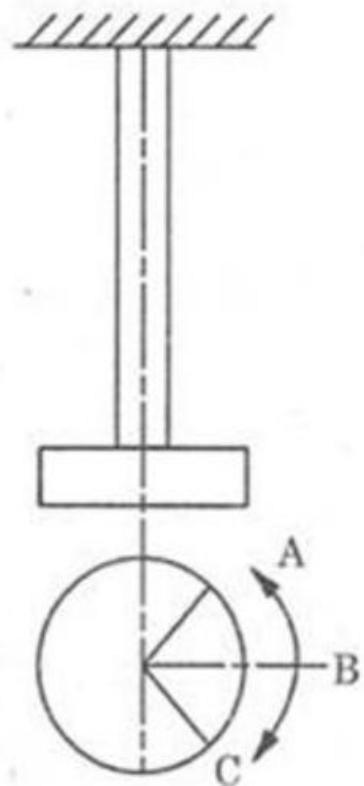
B-Mean position
A and C-Extreme positions

(a) Longitudinal Vibrations   (b) Transverse Vibrations   (c) Torsional Vibrations

Damped & Undamped Vibration

- Damped → Energy dissipates, vibration dies down.

- Undamped → No energy dissipation, vibration continues, amplitude of vibration remains constant.

Linear & Nonlinear Vibrations

-**Linear**: Displacements, velocity, frequency, and accelerations are **linearly related to the external exciting force**. The **governing differential eqn** will be **linear**.

-**Nonlinear**: The differential equation will be nonlinear.

Damping - Definition

- - Resistance encountered during vibration reduces amplitude over time.

- - Gradual reduction in amplitude over time.

- - Arises due to internal friction or external friction from surrounding fluid.

1) Viscous Damping

- Damping force proportional to velocity magnitude and opposite to direction of motion.

Formula: $F = C * \dot{x}$

(C = damping force constant and $\dot{x}$ = velocity).
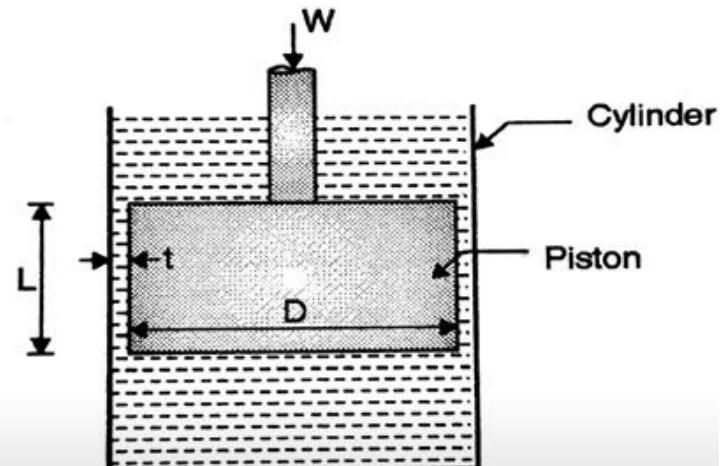
-Faster something moves, the more the damping force opposes it.

- Common in systems vibrating in a fluid.

- Advantages:
    - The differential equation is linear.
    - Allows easy analysis of the system.

Examples:
- Shock absorbers in cars: These use a fluid-filled cylinder and piston to dampen the oscillations of the suspension.
- When the weight W pushes the piston downwards, the piston tries to move inside the fluid-filled cylinder.
- Since the piston fits almost tightly, the fluid cannot escape easily — it is forced to flow through the thin clearance (t) between piston and cylinder.
- Flow through this narrow gap requires force due to viscous resistance.
- Faster motion creates more resistance, so the piston slows down automatically.

Dashpot Mechanism
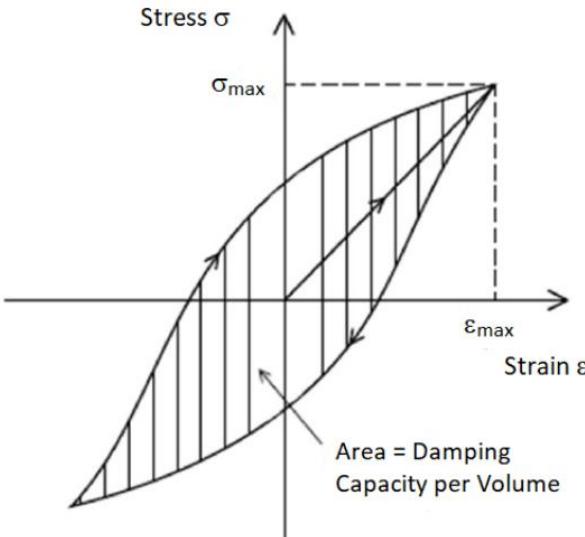
Dry or Coulomb Damping

- - Occurs when two machine parts rub against each other.

- - Practically constant resistance, independent of rubbing velocity.

- - Formula: $F = \mu * N$ (μ = coefficient of friction, N = normal pressure)

Solid or Structural Damping

- - Caused by internal molecular friction.

- - Stress-strain diagram forms a hysteresis loop.

- - Loop area represents damping capacity or energy dissipated per volume in one loading-unloading cycle.

- This type of curve is typical for materials like rubber, polymers, and metals etc. under plastic deformation.

- That straight line in the middle of the hysteresis loop represents the ideal elastic behavior of the material — i.e., what the stress–strain curve *would* look like if there were *no damping (no internal friction)*.

Stress σ

σ_{max}

ε_{max}

Strain ε

Area = Damping Capacity per Volume

Phase	Description
Loading (lower left → upper right path)	Stress increases with strain as the material is loaded.
Unloading (upper right → lower left path)	When the load is removed, the stress-strain path does **not** follow the same line.
Gap between paths	This difference is due to **internal friction / microstructural slip**.
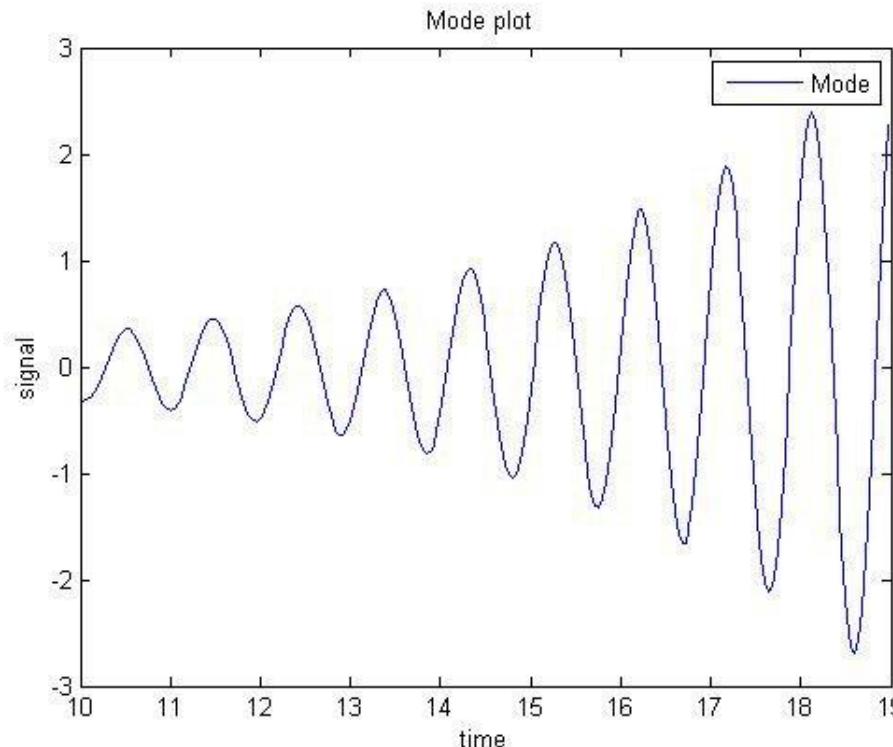
Negative Damping

• Instead of dissipating energy, it adds energy to the system.

• Results in progressively increasing amplitude.

• Leads to instability in vibration systems.


Mode plot

Degrees of Freedom (DOF)
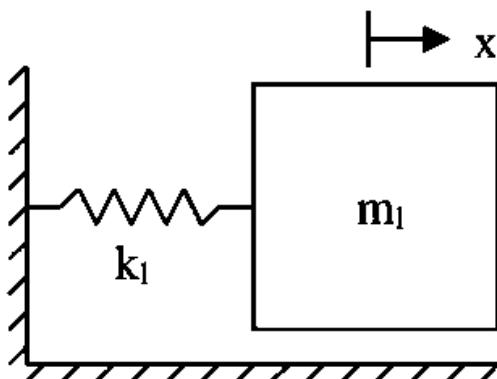
• Minimum number of coordinates required to define the position of mass points of a system at any instant.

• Or simply in how many ways it can move freely.

• Any continuous system can have infinite DOF.

• The process of idealization and suitable mathematical modeling helps reduce DOF to a discrete number.

Classification of Vibration Systems
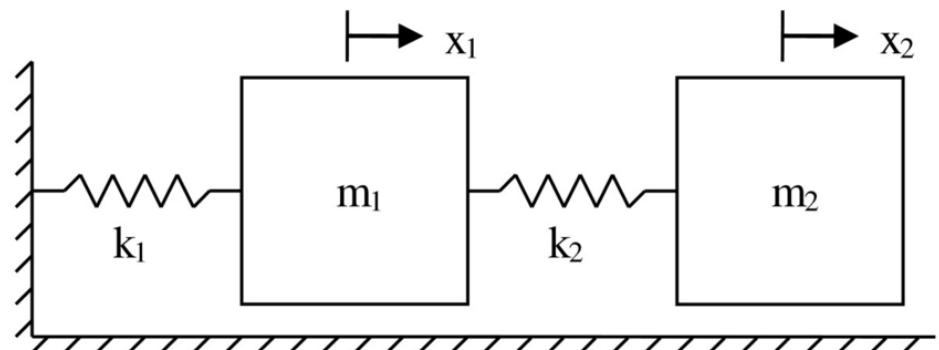
Vibration systems are classified based on the number of independent coordinates required:

1. Single DOF Systems - Described by a single coordinate.

2. Multiple DOF Systems - Require more than one coordinate.

3. Continuous Systems - Have an infinite number of DOFs.

• A mass supported by a spring constrained to move in one direction has single DOF.

• A two-mass spring system constrained to move in one direction has two DOFs.


Single mass - spring system   **Two masses - spring system**
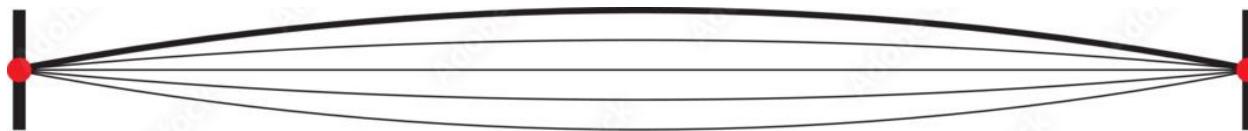
Vibration of Continuous Systems
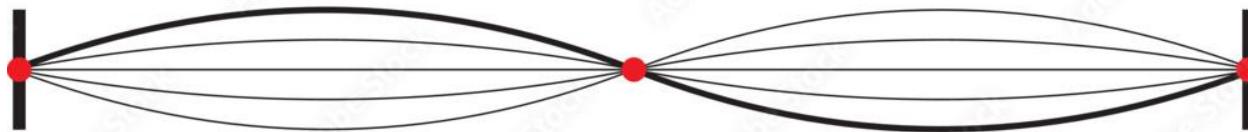
- Continuous systems have distributed mass and elasticity.

- Assumed to be homogeneous and isotropic, obeying Hooke's Law within the elastic limit.

- Infinite number of coordinates are needed to specify the position of every point. Their motion is defined at every point in the system.

- This is unlike discrete systems where motion is described at a finite number of points (e.g., masses in a multi-degree-of-freedom system).

- These systems possess an infinite number of degrees of freedom (DOF).

Normal modes of vibration

- A **normal mode** of vibration is a specific pattern in which all parts of a vibrating system oscillate **at the same frequency** and maintain a **fixed phase relationship** (a fixed shape pattern).

- In normal mode vibration, each particle undergoes simple harmonic motion at a specific frequency.
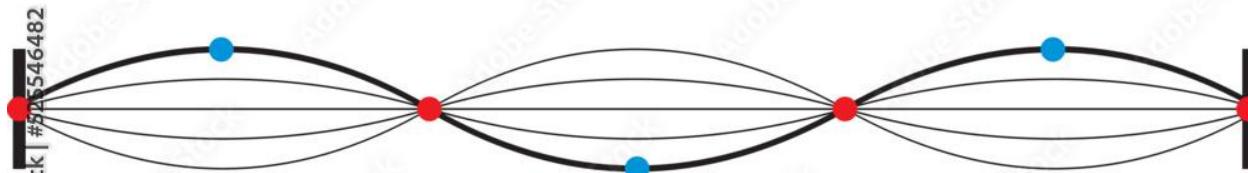
First Harmonic - Fundamental mode

Second Harmonic - First Overtone

Third Harmonic - Second overtone

● Nodes (no vibration)   ● Antinodes (Maximum vibration)

Key Features of Normal Modes:

✅ Each mode has a **fixed frequency** (natural frequency).

✅ The shape of vibration (mode shape) **does not change**, only the size of movement (amplitude) increases or decreases.

✅ The total motion of a system can be a **combination of multiple normal modes**

Equation of motion for a damped/free vibration system

**Spring-mass-damper model**
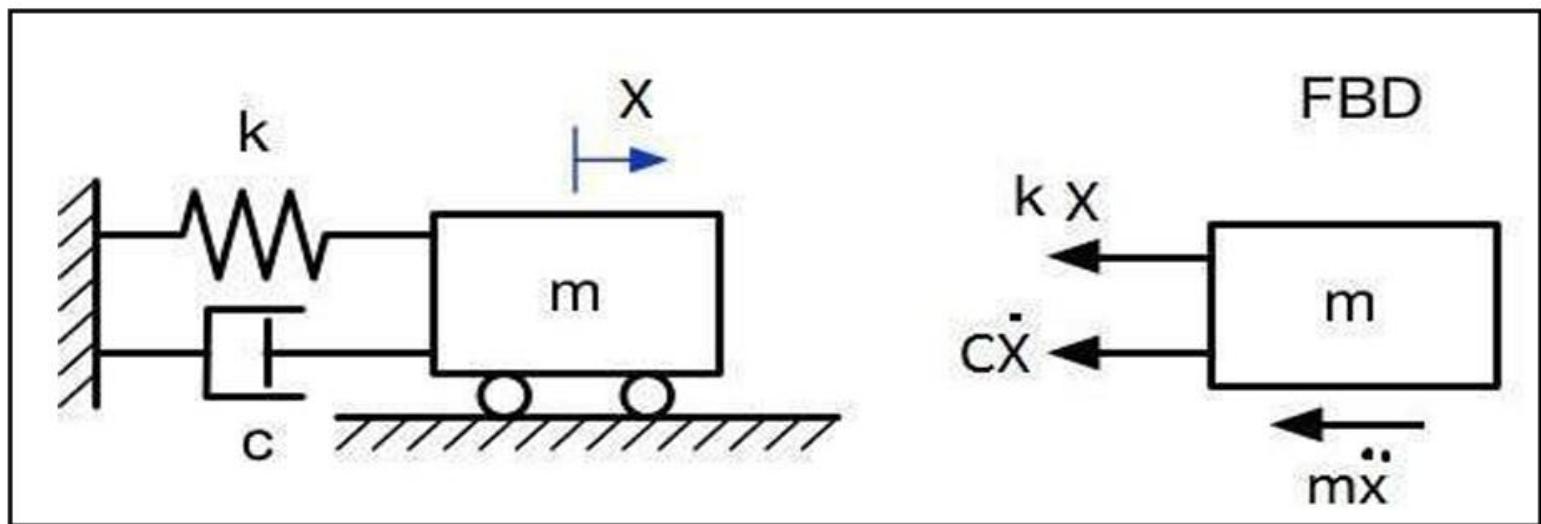
Figure 5 Mathematical model of SDOF system with damped free vibration

If $F(t) = 0$:

$$m\ddot{x} + c\dot{x} + kx = 0$$

This differential equation governs **free vibration** behavior.

Dynamic stiffness

- **Dynamic stiffness (k_d)** is the ratio of the amplitude of a harmonic (sinusoidal) or oscillatory force to the amplitude of the resulting harmonic displacement.

- It quantifies how a structure or component responds to a vibratory load.

- Unlike static stiffness, which only considers the force required to produce a certain displacement, dynamic stiffness also accounts for the effects of **inertia (mass)** and **damping (energy dissipation)**.

Dynamic stiffness (Contd.)

The equation of motion for a spring-mass-damper system under a harmonic force $F(t) = F_0 e^{i\omega t}$ is given by,

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

where:

- m is the **mass** of the system.

- c is the **damping coefficient**.

- k is the **static stiffness** of the spring.

- $x(t)$ is the displacement as a function of time.

- $\dot{x}(t)$ is the velocity.

- $\ddot{x}(t)$ is the acceleration.

- ω is the **angular frequency** of the applied force.

Dynamic stiffness (Contd.)

For a steady-state harmonic response, the displacement is also harmonic, $x(t) = x_0 e^{i\omega t}$.

Substituting this into the equation of motion and simplifying gives us the expression for dynamic stiffness:

$$k_d(\omega) = (k - m\omega^2) + i(c\omega)$$

This is a complex number, indicating that the displacement is not perfectly in phase with the force.

- The **real part**, $(k - m\omega^2)$, is the **storage stiffness**. It represents the elastic and inertial forces.

- The **imaginary part**, $c\omega$, is the **loss stiffness**. It represents the energy dissipated through damping.

The magnitude of the dynamic stiffness is often what is considered:

$$|k_d(\omega)| = \sqrt{(k - m\omega^2)^2 + (c\omega)^2}$$

Dynamic stiffness (Contd.)

Frequency Dependence

The behavior of dynamic stiffness changes significantly with the frequency of the applied force:

1. **Low-Frequency Range ($\omega \to 0$):**

   - As the frequency approaches zero, the inertial ($m\omega^2$) and damping ($c\omega$) terms become negligible.

   - $k_d \approx k$

   - In this region, the system's response is dominated by its **static stiffness**. The system behaves as if the load were applied statically.

Dynamic stiffness (Contd.)

2. **At Natural Frequency ($\omega = \omega_n$):**

   - The natural frequency (ω_n) is defined as $\sqrt{k/m}$.

   - At this frequency, the term $(k - m\omega^2)$ becomes zero.

   - $k_d(\omega_n) = i(c\omega_n)$

   - The dynamic stiffness is at its **minimum** value and is determined purely by the **damping** in the system. A very large displacement amplitude occurs for a given force, a phenomenon known as **resonance**.

3. **High-Frequency Range ($\omega \gg \omega_n$):**
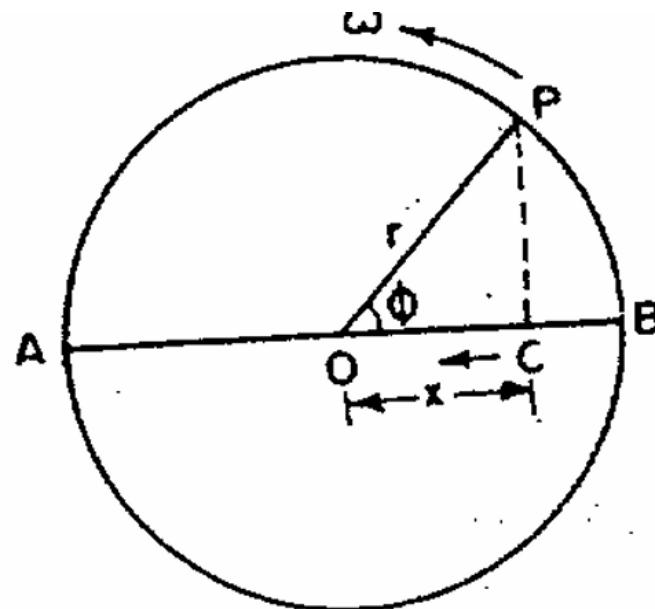
   - At frequencies well above the natural frequency, the inertial term ($m\omega^2$) becomes dominant.

   - $k_d \approx -m\omega^2$

   - The system's response is controlled by its **mass**. The stiffness increases rapidly with frequency, meaning it becomes very difficult to displace the system with a high-frequency force.

Free Vibration of Continuous Systems

• If motion starts at a shape matching a normal mode, only that mode is excited.

• If initiated by a sudden force, multiple modes are usually excited.

• Constraints and external forces influence which modes are excited.

• Specific modes can be selectively excited by proper initial conditions.
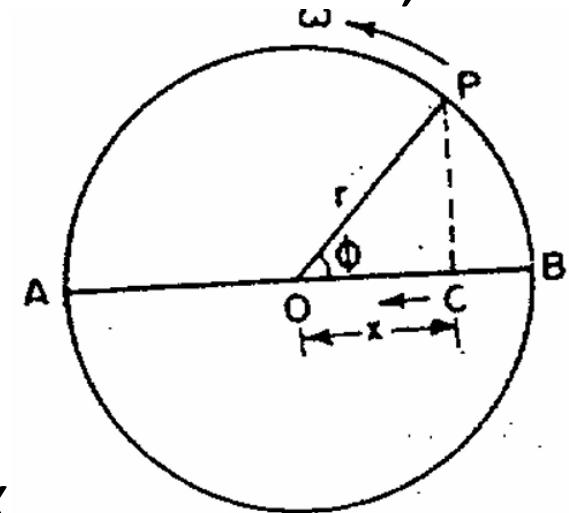
Introduction to Linear Vibrations and SHM

- A point P moves along the circumference of a circle with constant angular velocity (ω).

- As P rotates, the projection of P on a diameter (AB) oscillates, forming Simple Harmonic Motion (SHM).

Mathematical Representation of SHM

- - Consider a point C as the projection of P.
- It time is measured from the instant when P is at B, then φ = ωt
- - Displacement of C: x = rcos(ωt)
- - Velocity: v = dx/dt = -rω sin(ωt)
- - Acceleration: f = dv/dt

$$= -r\omega^2 \cos(\omega t) = -\omega^2 x$$

- - Indicates that acceleration is directly proportional to displacement. And the negative sign indicates acceleration is directed towards the mean position.

$$\omega^2 = \frac{\text{acceleration of } C}{\text{displacement of } C \text{ from the centre}} = \frac{f}{x}$$

or
$$\omega = \sqrt{\frac{\text{acceleration}}{\text{displacement}}} = \sqrt{\frac{f}{x}}$$

Let T = time of one complete 'to and fro' vibration in seconds

Then
$$\omega = \frac{2\pi}{T}$$

or
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi\sqrt{\frac{x}{f}}$$

Number of complete vibrations, n, is given by

$$n = \frac{1}{T} \text{ per second}$$

If W = weight of the vibrating body

Then   force = mass $\times$ acceleration

$$= \frac{W}{g} \times \text{acceleration}$$

Let k = stiffness of the supports $= \dfrac{\text{force}}{\text{displacement}}$

$\therefore$ $k = \dfrac{\text{force}}{\text{displacement}} = \dfrac{W}{g} \times \dfrac{\text{acceleration}}{\text{displacement}} = \dfrac{W}{g} \times \omega^2$

(acceleration $f = \omega^2 x$)

or $\omega = \sqrt{\dfrac{k.g}{W}}$

Hence $T = \dfrac{2\pi}{\omega} = 2\pi \sqrt{\dfrac{W}{k.g}}$

$\therefore$ $n = \dfrac{1}{T} = \dfrac{1}{2\pi} \sqrt{\dfrac{k.g}{W}}$
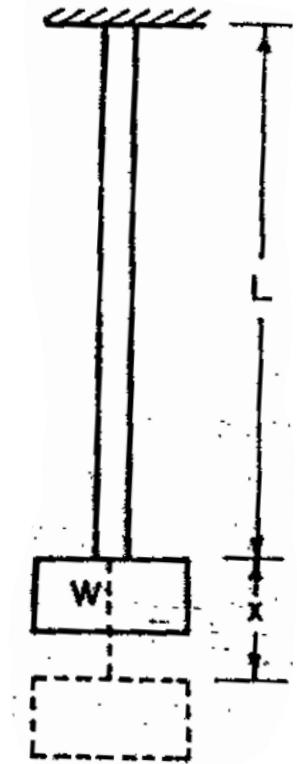
# Vibration of rods (Longitudinal)

- Consider an elastic rod subjected to the downward load W.

- Let the self weight of the rod be negligible in comparison to the weight W.

- If the weight W is given a displacement of x from the equilibrium position and then released, vibrations will be set up in the rod.

- Let **P = restoring force**

From Hooke's law,
  x = PL / AE
  P = AEx / L ...**(i)**

Vibration of rods (Contd.)

- But P = mass $\times$ acceleration

  Also, we know that T = $2\pi \sqrt{(x/f)}$,

  This gives acceleration, f = $(4\pi^2 x) / T^2$

  therefore, P = $(W / g) \times (4\pi^2 x / T^2)$ **...(ii)**

- Equating (i) and (ii), we get

  $(AEx / L) = (W / g) \times (4\pi^2 x / T^2)$

  or

  $(1 / T^2) = (EAg / 4LW\pi^2)$

Vibration of rods (Contd.)

- Also, n = (1 / T)

$$= (1 / 2\pi) \sqrt{(EAg / WL)}$$

If E is in N/mm^2, W is in N, A is in mm^2, g = 9810 mm/sec^2, and L is in mm, we get

n = (1 / 2π) $\sqrt{(9810 \ EA / WL)}$

This gives **n ≈ 15.8 $\sqrt{(EA / WL)}$ vibrations/ sec**
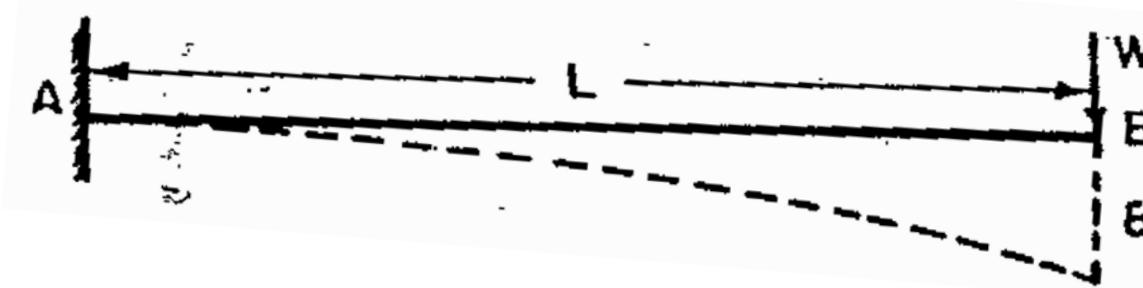
And from N = n X 60,

$$\textbf{N = 946 } \sqrt{\textbf{(EA / WL)}} \textbf{ vibrations/min.}$$
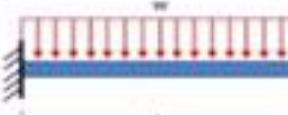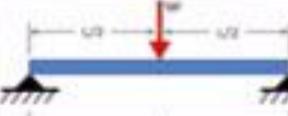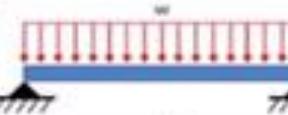
Vibration of beams (transverse)

Cantilever beam (or rod)

- A bar/ beam is said to have transverse vibrations when all the particles of the beam move along a **path perpendicular to the longitudinal axis of the beam.**

- Consider a cantilever AB of length L, subjected to a point load W.

- If the system is given an initial displacement to end B in the direction of W, the system will be subjected to transverse vibratic
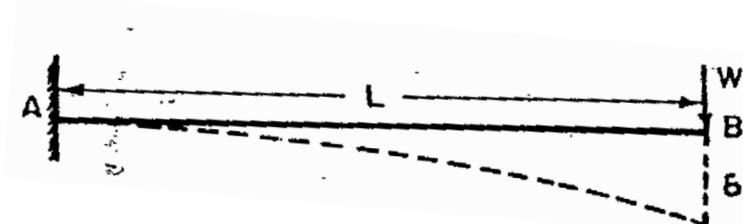
Standard formulae for deflection and slope

SR. NO.	TYPE OF BEAM	MAX. BM	SLOPE	DEFLECTON
1		M	$\theta = \dfrac{ML}{EI} = \dfrac{ML}{EI}$	$\delta = \theta \times \dfrac{L}{2} = \dfrac{ML^2}{2EI}$
2		WL	$\theta = \dfrac{ML}{2EI} = \dfrac{WL^2}{2EI}$	$\delta = \theta \times \dfrac{2L}{3} = \dfrac{WL^3}{3EI}$
3		$\dfrac{WL^2}{2}$	$\theta = \dfrac{ML}{3EI} = \dfrac{WL^3}{6EI}$	$\delta = \theta \times \dfrac{3L}{4} = \dfrac{WL^4}{8EI}$
4		$\dfrac{WL^2}{6}$	$\theta = \dfrac{ML}{4EI} = \dfrac{WL^3}{24EI}$	$\delta = \theta \times \dfrac{4L}{5} = \dfrac{WL^4}{30EI}$
5		$\dfrac{WL}{4}$	$\theta = \dfrac{ML}{4EI} = \dfrac{WL^2}{16EI}$	$\delta = \theta \times \dfrac{L}{3} = \dfrac{WL^3}{48EI}$
6		$\dfrac{WL^2}{8}$	$\theta = \dfrac{ML}{3EI} = \dfrac{WL^3}{24EI}$	$\delta = \theta \times \dfrac{5L}{16} = \dfrac{5WL^4}{384EI}$

- Let δ = deflection at the free end = $WL^3 / 3EI$
  or
  $W = 3EI\delta / L^3$

  $\therefore$ Stiffness $k = W / \delta = 3EI / L^3$ ….. **(1)**

- Also, we know $n = 1 / 2\pi \sqrt{(k.g / W)}$

$$= 1 / 2\pi \sqrt{(3EI\, g / L^3 W)} \qquad \text{(From eq. 1)}$$

If $g = 9810$ mm/sec^2 and δ is in mm
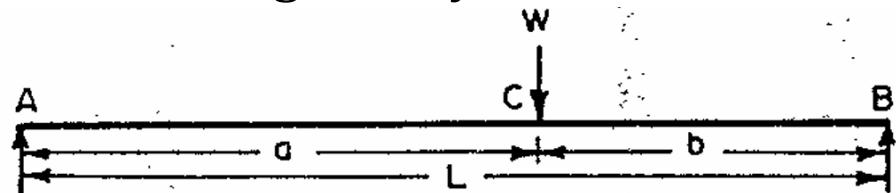
$n = 1 / 2\pi \sqrt{(9810 / \delta)}$

- On Solving **$n = 15.8 / \sqrt{\delta}$ per second**
- **$N = 60\, n \approx 946 / \sqrt{\delta}$ per minute**

Simply supported beam with point load

- Deflection under the concentrated load is given by

$$\delta = Wa^2b^2 / 3EIL$$

$$\therefore \text{Stiffness} = k = W / \delta = 3EI\,L / a^2b^2$$

We know $\quad n = 1 / 2\pi \sqrt{(k.g / W)}$

$$= 1 / 2\pi \sqrt{(g / \delta)}. \text{ or } = 1 / 2\pi \sqrt{(3EI\,L\,g / a^2b^2W)}.$$
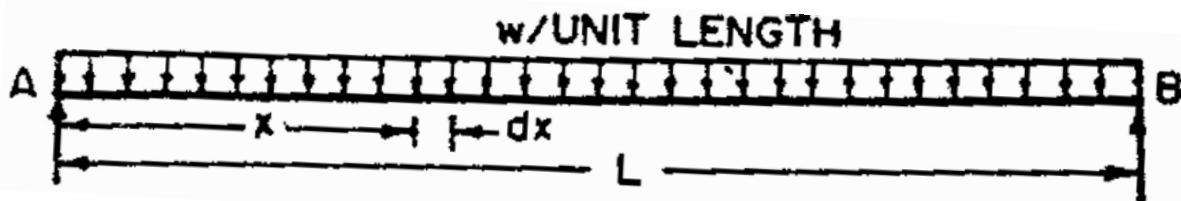
- Hence taking g = 9810 mm/sec² and y in mm

$$\mathbf{n \approx 15.8 / \sqrt{\delta} \text{ per second}}$$

$$\mathbf{N = 946 / \sqrt{\delta} \text{ per minute}}$$

Uniformly loaded beam (or rod or shaft)

- Consider a beam with udl subjected to transverse vibrations.

- The natural frequency can be obtained by **equating the strain energy stored in the static deflected position to the kinetic energy of transverse vibration**.

- The amplitude at any point = static deflection at that point.

w/UNIT LENGTH

A — x — |—dx, L

- Consider an elementary length at distance from one end.
- Let be the static deflection there, given by,

(General derived expression for a ss beam with udl)

$$y = \frac{w}{24EI}\left[\; x^4 - 2Lx^3 + L^3x \;\right]$$

Weight on the elementary length $= w\, dx$

∴ Strain energy of length of $dx = \frac{1}{2}(wdx)(y)$

∴ Strain energy of the whole beam

$$= \int_0^L \frac{1}{2}\, wdx\; y = \frac{1}{2}\, w \int_0^L \frac{w}{24EI}\left(x^4 - 2Lx^3 + L^3x\right) dx$$

$$= \frac{w}{48EI}\left[\; \frac{x^5}{5} - \frac{2Lx^4}{4} + \frac{L^3x^2}{2} \;\right]_0^L$$

$$= \frac{w^2 L^5}{240 EI} \qquad\qquad \dots(1)$$

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Again, the maximum velocity of the elementary length = **2πyn**

$\therefore$ K.E. of elementary length $= \dfrac{1}{2} \cdot \dfrac{wdx}{g} (2\pi \, yn)^2$ $K.E. = \dfrac{1}{2}mv^2$

$\therefore$ K.E. of the whole beam $= \dfrac{1}{2} \cdot \dfrac{w}{g} \displaystyle\int_0^L (2\pi \, yn)^2 \, dx$

$= \dfrac{2\pi^2 n^2 w}{g} \displaystyle\int_0^L \left[\dfrac{w}{24EI} \left\{ x^4 - 2Lx^3 + L^3 x \right\} \right]^2 dx$

$= \dfrac{\pi^2 n^2 w^3}{288 g E^2 I^2} \displaystyle\int_0^L \left(x^8 + 4L^2 x^6 - 4L x^7 + L^6 x^2 + 2L^3 x^5 - 4L^4 x^4 \right) dx$

$= \dfrac{\pi^2 n^2 w^3}{288 g E^2 I^2} \left[\dfrac{L^9}{9} + \dfrac{4L^9}{7} - \dfrac{4L^9}{8} + \dfrac{L^9}{3} + \dfrac{2L^9}{6} - \dfrac{4L^9}{5} \right]$

$= \dfrac{31 \pi^2 n^2 w^3 L^9}{288 \times 630 E^2 I^2 g}$ ...(2)

Equating (1) and (2), we get

$$\frac{w^2 L^5}{240EI} = \frac{31\pi^2 n^2 w^3 L^9}{288 \times 630 E^2 I^2 g}$$

$$n^2 = 2.47 \frac{EIg}{wL^4}$$

$$n = 1.572 \sqrt{\frac{EIg}{wL^4}}$$

Substituting $\delta = $ central deflection $= \frac{5}{384} \frac{wL^4}{EI}$ $\boxed{\frac{EI}{wL^4} = \frac{5}{384\delta}}$

we get $\qquad n = 1.572 \sqrt{\frac{5g}{384\delta}} = 0.179 \sqrt{\frac{g}{\delta}}$ per sec
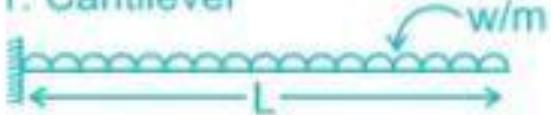
Taking $\qquad g = 9810$ mm/sec^2 and δ in mm, we get

$$n = \frac{17.73}{\sqrt{\delta}} \text{ per sec}$$
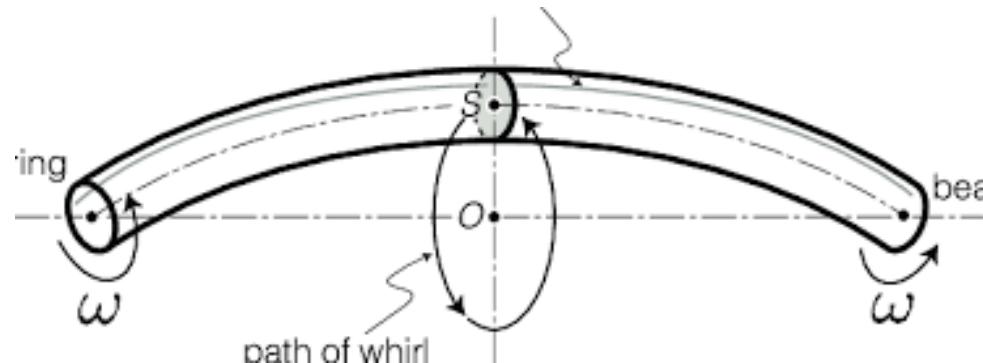
and $\qquad N = \frac{1064}{\sqrt{\delta}}$ per min

Expressions for strain energy

Beams	Strain Energy
1. Cantilever w/m, L	$U = \dfrac{w^2 L^5}{40 EI}$
2. Simply supported w/m, L	$U = \dfrac{w^2 L^5}{240 EI}$
3. Propped cantilever w/m, L	$U = \dfrac{w^2 L^5}{640 EI}$
4. Fixed beams w/m, L	$U = \dfrac{w^2 L^5}{1440 EI}$

Critical or Whirling Speed of a Shaft

- Imagine you have a rotating shaft (like in a motor or turbine). Ideally, it should spin perfectly along a straight axis.

- But in reality, due to imperfections (like slight bends, its own weight, or vibrations), the shaft doesn't rotate perfectly straight.

- Now, as the shaft rotates, centrifugal forces act on it, causing it to bend even more.

- However, the stiffness of the shaft tries to bring it back to its normal shape.

Whirling Speed (Contd.)

✓ At low speeds, the stiffness overcomes centrifugal force, and the shaft stays mostly stable.

✓ As the speed increases, the centrifugal force also increases.

✓ At a certain critical speed, the centrifugal force completely overcomes the stiffness, leading to unstable vibrations.

✓ At this stage, instability will follow and the deflection and stress, unless prevented, will increase until fracture occurs.

✓ The speed which just gives balance between the two sets of forces is called the **whirling speed.**

- Consider a shaft simply supported at the ends and carrying a central point load W. Let the weight of the shaft be negligible.

- Let e = initial difference between the geometrical axis and the axis of rotation (due to the initial imperfection)

- y = increase in the displacement due to rotation

The centrifugal force of the rotating mass

$$\frac{W}{g}(e+y)\omega^2 \quad ...(1)$$

$$F = mr\omega^2$$

where ω = angular speed of the shaft, in radians/sec

Restoring force = $k \cdot y$

where k = stiffness of the shaft

Equating the two, we get

$$\frac{W}{g}(e+y)\omega^2 = ky$$

$$y\left(k - \frac{W}{g}\omega^2\right) = \frac{W}{g}e\omega^2$$

$$y = \frac{We}{g}\frac{\omega^2}{k - \frac{W}{g}\omega^2} = \frac{e}{\frac{kg}{W\omega^2} - 1} \qquad (2)$$

The above equation gives the deflection due to rotation, at any angular speed ω. At the whirling speed $\omega = w_c$, the deflection y becomes infinitely great. This gives

$$\frac{kg}{W\omega_c^2} - 1 = 0$$

or

$$\omega_c^2 = \frac{kg}{W} \qquad \text{(3)}$$

But

$$\frac{W}{k} = \text{static deflection} = \delta$$

$\therefore$

$$\omega_c^2 = \frac{g}{\delta}$$

or

$$\omega_c = \sqrt{\frac{g}{\delta}}$$

or

$$n_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} \quad \text{revolutions per second}$$

or

$$n_c = \frac{30}{\pi}\sqrt{\frac{g}{\delta}} \quad \text{per minute}$$

This is the same expression for the natural frequency.

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- **Thus, we find the critical speed n_c is equal to the natural frequency of vibration of the system**.

- In a similar way, we can prove that the critical speed of an **unloaded shaft, taking into account its self-weight**, or the critical speed of **a shaft carrying uniformly distributed** load is also equal to the natural frequency of vibration.

- Taking g=9810 mm/sec², and δ in mm, the equation $n_c = \dfrac{30}{\pi}\sqrt{\dfrac{g}{\delta}}$ per minute reduces to:

$$n_c = \frac{946}{\sqrt{\delta}} \quad \text{R.P.M.}$$

Significance of whirling speed

- Substituting equation (3) in (2), we get
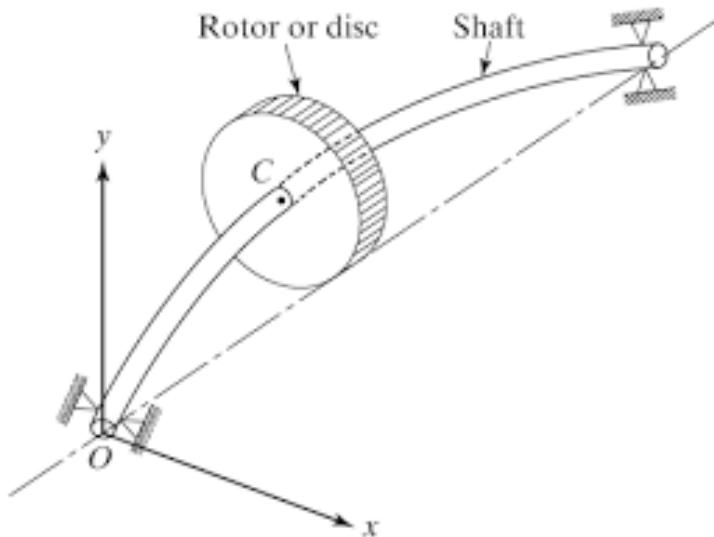
$$y = \frac{e}{\frac{\omega_c^2}{\omega^2} - 1} = \frac{-e\omega^2}{\omega^2 - \omega_c^2}$$

- It is evident from this eqn that **y becomes negative if** the speed of rotation **ω is greater than the critical speed ωc**. In other words, the **shaft tries to straighten out.**

- If the shaft is **accelerated past the critical speed quickly**, the whirling effect doesn't get time to fully develop, and it stabilize again by straightening out, as the speed increases.

- Also note that at very high speed ($w \gg w_c$), $y = -e$, and the geometrical axis and the axis of rotation will coincide.

- This is why many machines are **designed to run above the whirling speed** to avoid instability

Numerical: Question

A shaft 20 mm diameter and 500 mm between the long bearing at its ends, carries a wheel weighing 100 N midway between the bearings. Neglecting the increase of the stiffness due to attachment of the wheel to the shaft, find the critical speed of rotation, and the maximum bending stress when the shaft is rotating at 4/5 of this speed, if the centre of gravity of the wheel is 0.4 mm from the centre of the shaft. Take $E = 2 \times 10^5$ N/mm^2.

Rotor or disc   Shaft

Solution:

When shaft is supported on long bearings, it has an effect of fixidity at the end. Hence the shaft may be considered to a fixed beam with a central point load. The central deflection δ for such a case is given by

$$\delta = \frac{WL^3}{192EI}$$

$\therefore$ Stiffness, $k = \dfrac{W}{\delta} = \dfrac{192EI}{L^3}$

$$I = \frac{\pi}{64} (20)^4 = 0.785 \times 10^4 \text{ mm}^4$$

Again,

$$N_c = \frac{60}{2\pi} \sqrt{\frac{k \cdot g}{W}}$$

$$= \frac{30}{\pi} \sqrt{\frac{192EIg}{WL^3}}$$

$$= \frac{30}{\pi} \sqrt{\frac{192 \times 2 \times 10^5 \times 0.785 \times 10^4 \times 9810}{100 \, (500)^3}}$$

$= 4645$ revolutions per minute.

We know that

$$y = \frac{e\omega^2}{\omega_c^2 - \omega^2}$$

Putting $\omega = 4/5\omega_c = 0.8\omega_c$ and $e = 0.4$ mm, we get

$$y = \frac{0.4(0.8\omega_c)^2}{\omega_c^2 - (0.8\omega_c)^2} = \frac{0.4 \times 0.64}{1 - 0.64} = 0.71 \text{ mm}$$

$\therefore$ Central centrifugal bending force

$$= k.y = 0.71k$$

$\therefore$ B.M. $M = \frac{1}{8}(0.71k)L$

$$= \frac{0.71L}{8} \times \frac{192EI}{L^3} = \frac{0.71 \times 24EI}{L^2}$$

$$f = \frac{M}{Z} = \frac{M}{I} \times 10 = \frac{0.71 \times 24E \times 10}{L^2}$$

(Z= I/y. Note that y here is distance from the neutral axis. i.e., 20/2=10 mm)

$$= \frac{7.1 \times 24 \times 2 \times 10^5}{500 \times 500}$$

$$= 136.3 \text{ N/mm}^2$$
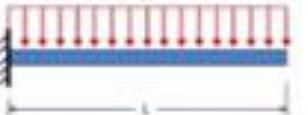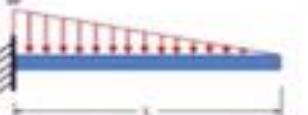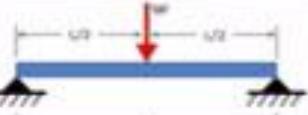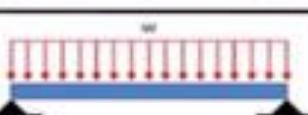
Note:
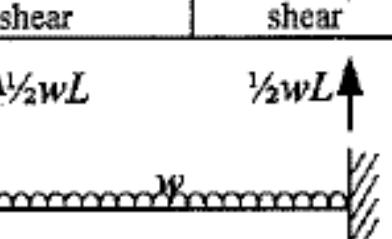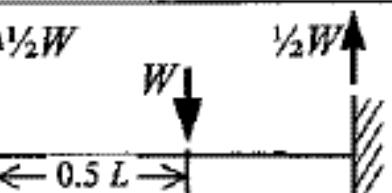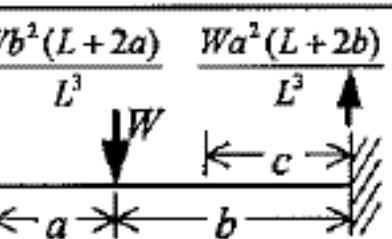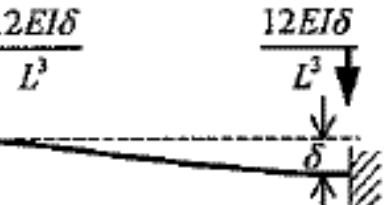
If the bearings are of short length, or if they have spherical seatings, it is taken as simply supported at the ends.
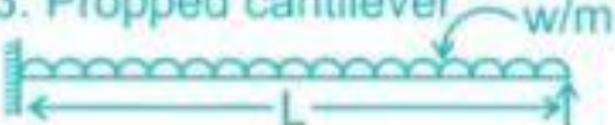
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1) Punmia, B. C. (2004). *SMTS-II Theory of Structures*. Firewall Media.

2) Thomson, W. (2018). *Theory of vibration with applications*. CrC Press.

Useful formulae

SR. NO.	TYPE OF BEAM	MAX. BM	SLOPE	DEFLECTON
1		M	$\theta = \dfrac{ML}{EI} = \dfrac{ML}{EI}$	$\delta = \theta \times \dfrac{L}{2} = \dfrac{ML^2}{2EI}$
2		WL	$\theta = \dfrac{ML}{2EI} = \dfrac{WL^2}{2EI}$	$\delta = \theta \times \dfrac{2L}{3} = \dfrac{WL^3}{3EI}$
3		$\dfrac{WL^2}{2}$	$\theta = \dfrac{ML}{3EI} = \dfrac{WL^3}{6EI}$	$\delta = \theta \times \dfrac{3L}{4} = \dfrac{WL^4}{8EI}$
4		$\dfrac{WL^2}{6}$	$\theta = \dfrac{ML}{4EI} = \dfrac{WL^3}{24EI}$	$\delta = \theta \times \dfrac{4L}{5} = \dfrac{WL^4}{30EI}$
5		$\dfrac{WL}{4}$	$\theta = \dfrac{ML}{4EI} = \dfrac{WL^2}{16EI}$	$\delta = \theta \times \dfrac{L}{3} = \dfrac{WL^3}{48EI}$
6		$\dfrac{WL^2}{8}$	$\theta = \dfrac{ML}{3EI} = \dfrac{WL^3}{24EI}$	$\delta = \theta \times \dfrac{5L}{16} = \dfrac{5WL^4}{384EI}$

L = overall length, W = point load, M = moment, w = load per unit length				Maximum deflection Δ	Max deflection position c (from RH end)	Max bending moment (modulus)
LH End		**RH End**				
moment	shear	shear	moment			
$\dfrac{wL^2}{12}$	$\tfrac{1}{2}wL$	$\tfrac{1}{2}wL$	$\dfrac{wL^2}{12}$	$\dfrac{wL^4}{384EI}$	$\dfrac{L}{2}$	$\dfrac{wL^2}{12}$
$\dfrac{WL}{8}$	$\tfrac{1}{2}W$ (at $0.5L$, load W)	$\tfrac{1}{2}W$	$\dfrac{WL}{8}$	$\dfrac{WL^3}{192EI}$	$\dfrac{L}{2}$	$\dfrac{WL}{8}$
$\dfrac{Wab^2}{L^2}$; $\dfrac{Wb^2(L+2a)}{L^3}$	(load W at a, b)	$\dfrac{Wa^2(L+2b)}{L^3}$	$\dfrac{Wa^2b}{L^2}$	$\dfrac{Wa^2bc^2}{6EIL^2}$	$\dfrac{2Lb}{L+2b}$	$\dfrac{Wab^2}{L^2}$
				$a \le b$		
$\dfrac{6EI\delta}{L^2}$; $\dfrac{12EI\delta}{L^3}$		$\dfrac{12EI\delta}{L^3}$	$\dfrac{6EI\delta}{L^2}$	δ	0	$\dfrac{6EI\delta}{L^2}$

Beams	Strain Energy
1. Cantilever w/m L	$U = \dfrac{w^2 L^5}{40 EI}$
2. Simply supported w/m L	$U = \dfrac{w^2 L^5}{240 EI}$
3. Propped cantilever w/m L	$U = \dfrac{w^2 L^5}{640 EI}$
4. Fixed beams w/m L	$U = \dfrac{w^2 L^5}{1440 EI}$