

Bennett's inequality from Maurer and Pontil: For Y_1, \dots, Y_m IID in $[0, 1]$ and $\delta > 0$ with variance σ^2 , there is probability at least $1 - \delta$ that

$$\mathbb{E}(Y) - \frac{1}{m} \sum_{i=1}^m Y_i \leq \sqrt{\frac{2\sigma^2 \log(1/\delta)}{m}} + \frac{\log(1/\delta)}{3m}.$$

In RQMC we take $m = R$ replicates of n RQMC points. So the width of the one sided interval is at most

$$\frac{\sigma_n}{\sqrt{R}} \sqrt{2 \log(1/\delta)} + \frac{\log(1/\delta)}{3R}$$

where σ_n^2 is the RQMC variance using n sample points.

Suppose that $\sigma_n = An^{-\theta}$. For smooth integrands and $d = 1$, we have $\theta = 3/2$. For RQMC we take $\theta > 1/2$. For integrands of bounded variation in the sense of Hardy and Krause, but not smooth, $\theta \approx 1$ is reasonable. Let the budget be $B = nR$. Then the width is asymptotic to

$$\frac{An^{-\theta}}{\sqrt{B/n}} \sqrt{2 \log(1/\delta)} + \frac{\log(1/\delta)}{3B/n} = \frac{A}{\sqrt{B}} n^{1/2-\theta} \sqrt{2 \log(1/\delta)} + \frac{n \log(1/\delta)}{3B}.$$

For $\theta = 1/2$, the best n is $n = 1$. We assumed that $\theta > 1/2$. Then the derivative with respect to n is

$$(1/2 - \theta) \frac{A}{\sqrt{B}} n^{-1/2-\theta} \sqrt{2 \log(1/\delta)} + \frac{\log(1/\delta)}{3B}$$

which vanishes at

$$n^{-\theta-1/2} = \frac{\frac{\log(1/\delta)}{3B}}{(\theta - 1/2) \frac{A}{\sqrt{B}} \sqrt{2 \log(1/\delta)}}$$

that is

$$\begin{aligned} n &= \left(\frac{(\theta - 1/2) \frac{A}{\sqrt{B}} \sqrt{2 \log(1/\delta)}}{\frac{\log(1/\delta)}{3B}} \right)^{1/(\theta+1/2)} \\ &= \left(\frac{3(\theta - 1/2) A \sqrt{2B}}{\sqrt{\log(1/\delta)}} \right)^{1/(\theta+1/2)} \\ &\propto B^{1/(2\theta+1)}. \end{aligned}$$

For $\theta = 3/2$ we get $n = O(B^{1/4})$ while for $\theta = 1$, we get $n = O(B^{1/3})$. Note that with a better rate for σ_n^2 we take smaller n . I think this is because the constant term $\log(1/\delta)/(3R)$ does not respond to n . With small σ_n^2 we can take some larger R .