Betting with randomized quasi-Monte Carlo

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1 Introduction

2 Notation

We use $\varphi(\cdot)$ to denote the probability density function of $\mathcal{N}(0,1)$, the standard normal distribution on R. The cumulative distribution function (CDF) of $\mathcal{N}(0,1)$ is denoted by $\Phi(\cdot)$.

3 Test cases

We want some test cases that explore the effects of using RQMC points in betting and empirical Bernstein confidence intervals. It makes sense to have integrands bounded between 0 and 1 and either attaining or approaching those bounds. We want to look for differences depending on the smoothness of the integrand as well as on the dimension of the integrand. Ridge functions described below let us independently vary smoothness and dimension.

Let $\boldsymbol{x} \sim \mathcal{U}(0,1)^d$. Then $\boldsymbol{z} = \Phi^{-1}(\boldsymbol{x}) \sim \mathcal{N}(0,I)$, where the inverse CDF has been applied componentwise. Next

$$\frac{1}{\sqrt{d}}\sum_{j=1}^{d} z_j \sim \mathcal{N}(0,1)$$

for any dimension $d \geqslant 1$. We will use ridge functions of the form

$$f(\boldsymbol{x}) = g\Bigl(\frac{1}{\sqrt{d}}\sum_{j=1}^{d}\Phi^{-1}(x_j)\Bigr).$$

The mean and variance of f depends only on g (not d) as do critical aspects of the smoothness of g. Three interesting ridge functions for our case are

$$\begin{aligned}
g_{\mathrm{jmp}}(w) &= 1\{w \geqslant 1\},\\
g_{\mathrm{knk}}(w) &= \min(\max(-2,w),1) \quad \text{and}\\
g_{\mathrm{smo}}(w) &= \Phi(w).
\end{aligned}$$

The function g_{smo} is infinitely differentiable. The function g_{knk} has kinks where the derivative does not exist. The function g_{jmp} has a discontinuity. We can consider a range of dimensions $d \in \{1, 2, 4, 16, 64\}$. These are in geometric progression except that $d = 2$ has been inserted into the progression because it is the smallest dimension for which g_{jmp} has infinite variation in the sense of Hardy and Krause.

Ridge functions like this are studied in [1]. Based on similar functions there, it is reasonable to predict that g_{knk} and g_{smo} will have a low mean dimension while g_{jmp} might have a mean dimension growing roughly like $O(\sqrt{d})$. If so, then the first two functions remain reasonably QMC friendly as the dimension increases while g_{jmp} would not.

Another challenge that might be interesting is functions with an unbounded derivative like we see in some finance applications. For instance

$$g(w) = \min\bigl(1, \sqrt{\max(w + 2, 0)}/2\bigr)$$

unbounded derivative as $w \downarrow -2$. The mean dimension of this one might not have been studied.

Bassed on Aadit's computations so far, it looks like the interesting issues are what happens with R replicates of $n = 2^m$ RQMC points where m is small, such as $0 \leqslant m \leqslant 5$. If we ever see that $m = 5$ is optimal then we might have to try larger values.

The R replicates of RQMC are independent. Then RQMC with $m = 0$ is the same as plain MC. If we were to look at $B = nR$ consecutive RQMC points broken into R batches of n points each, then I don't think any of the theory we have fits that. It might be interesting to know what happens; if it is really good, then maybe some theory can be built to explain it. But that is for later.

We can keep score by the width and coverage of 95% confidence intervals. For the methods with guaranteed coverage, we expect widths to be greater than for the CLT intervals and the coverage is probably also well beyond the nominal level. The three coverage methods are

1. CLT: neither guaranteed or always valid
2. Empirical Bernstein: guaranteed but not always valid
3. Betting: guaranteed and always valid

Each improvement probably costs us some width. The best n for CLT will be very large, perhaps beyond what we look at. It might be enough to just have the CLT simulations go at least one step farther than whatever comes out best for E-B and betting.

References

[1] C. Hoyt and A. B. Owen. Mean dimension of ridge functions. *SIAM Journal on Numerical Analysis*, 58(2):1195–1216, 2020.