26.11.2018 Probability and Random Processes 15MAT213

R.P = Probability + Time Component

Counting Principle:

1. Sum Rule

If task 1 is completed in n1 ways and task 2 is completed in n2 ways, then no. of ways in which either one task can be performed = $\boxed{n1 + n2}$ ↳ completed

2. Product Rule

If task 1 is completed in n1 ways, task 2 in n2 ways, then no. of ways in which both tasks can be completed = $\boxed{n_1 n_2}$

3. Arrangement

Given n objects, no. of arrangements of r objects is given by:

$$nP_r = \frac{n!}{(n-r)!} \quad \text{where } n! = 1 \times 2 \times 3 \ldots n$$

↳ $= \frac{n!}{n1!\,n2!\ldots}$ given n1, n2 are repetitions of certain symbol

4. Choice / Selection

No. of ways to choose r objects from a total of n objects:

$$nC_r = \frac{n!}{r!(n-r)!} = \frac{nP_r}{r!}$$

Alternative symbol for nCr = $\binom{n}{r}$

Now, $nCr = nC_{n-r}$

More clearly, No. of permutations of n objects of which n_1 are alike, n_2 are alike & n_3 are alike

$$= \frac{n!}{n_1!\,n_2!\,n_3!}$$

Problems:

1) No. of arrangements in word ENGINEERING

$$= \frac{11!}{3!3!2!2!}$$

2) Find no. of 2 digit even numbers

$$= 9 \times 5$$
$$= 45$$

→ 0, 2, 4, 6, 8

3) No. of 2 digit even numbers with no. of repetitions

$$= 45 - 4$$
$$= 41$$

→ 22
44
66
88

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27.11.18

Find the total number of integers n such that $2 \le n \le 2000$ and HCF of n and 36 is 1

$36 = 2 \times 2 \times 3 \times 3$

$= 2^2 \times 3^2$

Numbers which are divisible by 2 are 2, 4, 6, 8 2000

This is an A.P

$\Rightarrow$ No. of terms: $\underline{1000}$

$(T_n = a + (n-1)d)$

$2000 = 2 + 2n - 2$

$1000 = \cancel{1} + n \cancel{-1}$

$n = 1000 - 1 + 1$

$= 1000$

Numbers divisible by 3:

3, 6, 9 1998

$2000 = 2 + n - 1$

$n =$

$T_n = a + (n-1)d$

$1998 = 3 + 3n - 3$

$666 = \cancel{1} + n \cancel{-1}$

$\underline{n = 666}$

Now, to check numbers divisible by both 2 & 3, the GCD of n & 36 $\neq 1$

$\Rightarrow$ Numbers divisible by 6 : 6, 12, 18 ... 1998

$1998 = \cancel{6} + 6n \cancel{-6}$

$n = \frac{1998}{6} = 333$

$\therefore n = 1999 - (\underbrace{1000 + 666 - 333}_{A \cup B \text{ (by 2 or 3)}})$

$= 666$

$$\begin{array}{r} 1666 \\ -333 \\ \hline 1333 \end{array} \qquad \begin{array}{r} 1999 \\ -1333 \\ \hline 666 \end{array}$$

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• Find the sum of all numbers that can be formed with 2, 3, 4, 5 taken all at a time.

— — — —

No. of possible numbers

NOTE: With repetition, $= 4^4$

Without repetition $= 4!$

$= 4\times3\times2\times1 = \underline{24}$

Step A: Now, if the last digit is 2, 3! ways

3 3!

4 3!

5 3!

Total $= 4 \times 3! = \underline{24}$

Step B: Summing over unit places, 2 occurs 6 time

3 occurs 6 time

and so on

$\Rightarrow$ Sum of unit places

$= 6\times(2+3+4+5)$

$= 6\times 14 = \underline{\underline{84}}$

Step C: For a 4 digit number abcd,

Sum $= (a\times10^3)+(b\times10^2)+(c\times10^1)+(d\times10^0)$

a+b+c+d

place value

$\therefore$

Final: $\Rightarrow$ Sum $= 84(10^3+10^2+10^1+10^0)$

sum for each position

Note: If you consider with repetition then instead of 3! do 4^3

2×10

3×10 $\times6$

4×10

RANDOM EXPERIMENT

Def • Repeated experiment done under fixed conditions such that outcome is not predictable.

Every experiment has a sample space.

Def A sample space is defined as the set of all possible outcomes of an experiment.

Eg. • If experiment = tossing a coin
Then
$S =$
Sample space: $\{H, T\}$

• E = Rolling a die
$S = \{1, 2, 3, 4, 5, 6\}$

These have a finite sample space.

Sometimes, an experiment can have ∞ no. of elements in sample space.

Eg. • Tossing a coin until a head appears.

$S = \{H, TH, TTH, TTTH \ldots\ldots \infty\}$

• Lifetime of a bulb in hours. (not ordered in some way)

A sample space can be discrete, as above, or continuous.

Eg. • Choosing a point from unit disc

$S = \{(x, y) / x^2 + y^2 \leq 1\}$
→ ∞ sample space & continuous

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Def An EVENT is any subset of the sample space

A probability problem has:

1) Experiment
2) Sample space
3) Events

to get Outcome

For a set of n elements, there exist 2^n subsets (or) events

In rolling a die,
Let outcome = even prime

$\Rightarrow A = \{2\}$

event of getting even prime

Def Probability

If A is any event, the probability of A & S is its sample space

Classical Def

$$\boxed{p(A) = \frac{\text{No. of favourable outcomes}}{\text{Total no. of outcomes}}}$$

$$= \frac{\text{No. of elements in } A}{\text{No. of elements in } S}$$

Eg. In rolling a die, probability of getting even prime

$$= \frac{1}{6}$$

This definition is applicable to finite sample spaces and NOT for all events.
(exclusive & equally likely)

MUTUALLY EXCLUSIVE:

If occurrence of 1 event prevents that of another event, they are mutually exclusive

EXHAUSTIVE:
Union of ~~occurrence of~~ all events gives sample space.

EQUALLY LIKELY:
Chance of occurrence of all events must be equal.

Note: In set theory, the definition of probability is slightly different.

P(A) is a function from sample space to a range of numbers b/w 0 & 1.

A's sample space $\rightarrow [0,1]$

i.e. $f : S \rightarrow [0,1]$

Such that, / satisfying the condition.

- $p(S) = 1$
- $p(A_1 \cup A_2 \ldots \cup A_n) = p(A_1) + \ldots p(A_n)$

if $A_i \cap A_j = \phi$

This def is only for explanation

Check mutually exclusive

next page for actual def.

28.11.18 More experiments:

1. Tossing 2 coins

$S = \{HH, HT, TH, TT\}$
$n(S) = 4$

A = Event of getting no head
$P(A) = \frac{1}{4}$

2. Rolling a die

$S = \{1, 2, 3, 4, 5, 6\}$
$n(S) = 6$
$p(\text{getting prime}) = \frac{3}{6} = \frac{1}{2}$

Set Theory Proper Def: (Class def)

Def
Let S be a sample space
Probability is a set function from S to [0, 1] such that:

getting at least something in S

① $P(S) = 1$ and
② $P(A_1 \cup A_2 \cup A_3 \dots A_n) = \sum_{i=1}^{n} P(A_i)$ if $A_i \cap A_j = \emptyset$

PROPERTIES:

(a) $P(\phi) = 0$

~~Proof. S = A ∪ A^c~~
~~∴ P(S) = P(A) + P(A^c)~~

from property 2 above

complement

(b) $P(A^c) = 1 - P(A)$

For (a): Proof.

$p(\phi) = 1 - p(\phi^c) = 1 - p(S) = 1 - 1 = 0$ (property ①)

For (b): Proof:

$p(A^c) = 1 - p(A)$

Since $S = A \cup A^c$, then $P(S) = P(A) + P(A^c)$

From property ②,

$1 = P(A) + P(A^c)$

$\Rightarrow P(A^c) = 1 - P(A)$

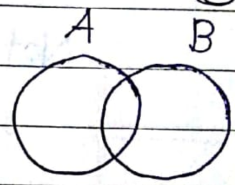
ⓒ If $A \subseteq B$ then $P(A) \leq P(B)$

Proof: Since $\phi \subseteq A \subseteq S$ then $0 \leq P(A) \leq 1$ } Note (not required in proof)

ⓓ $P(A \cap B^c) = P(A) - P(A \cap B)$

Start here

Now, if we take 2 sets A & B



PROOF for ⓓ

$A - B = \{x \in A \mid x \notin B\}$

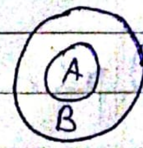
$= A \cap B^c$

$\therefore A = \underbrace{(A \cap B^c) \cup (A \cap B)}_{\text{disjoint events}}$

$P(A) = P(A \cap B^c) + P(A \cap B)$

$\therefore P(A \cap B^c) = P(A) - P(A \cap B)$

Proof for ⓒ:



Let $A \subseteq B$

$B = A \cup B = A \cup (A^c \cap B)$

$P(B) = P(A \cup B) = P(A) + P(A^c \cap B)$

$\Rightarrow P(B) \geq P(A)$ | since $P(A^c \cap B)$ is non-negative

Note: If ~~x~~ y is a non-negative number, $x+y \geq x$

From (d) we can also prove this:

(e) $\rightarrow$: $P(A^c \cap B) = P(B) - P(A \cap B)$

Proof is similar to (d)

(f) (6). $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\hookrightarrow$ prove from set theory definition & property (2)

$\hookrightarrow$ not recommended. Use method on next page.

Note: $(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$

$\Rightarrow P(A \cap B) \leq P(A)$ & $P(A \cap B) \leq P(B)$

$\therefore$

Example: $P(A) = 1/2$ $P(B) = 1/3$

$P(A \cap B) = 0.6$

Question is incorrect as $P(A \cap B) \geq P(A)$ & $P(B)$ both.

Now: for above example, given $P(A \cap B) = 0.25 = 1/4$

(1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 1/2 + 1/3 - 1/4$

(2) $P(A^c \cap B) = P(B) - P(A \cap B)$

$= 1/3 - 1/4$

(3) $P(A \cap B^c) = 1/2 - 1/4$

(4) $P(A^c \cup B^c) = P((A \cap B)^c) = 1 - 1/4 = 3/4$

use DeMorgan's laws

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Proving (f) by different property, since A & B info is not given (if they are mutually exclusive & so on)

$A \cup B = (A \cap B^c) \cup B$

$P(A \cup B) = P(A \cap B^c) + P(B)$ ← Substituting (d)
$= P(A) - P(A \cap B) + P(B)$

Hence proved.

Using this result,

(g) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

~~Take $A \cup B$ as one set.~~

(: from (f)

$\cancel{P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)}$
$= P(A \cup B) + \ldots$

---

1. A book containing 100 pages is opened at random. Find the probability that on the page,

a) Doublet is found
b) A number whose sum of digits is 10

Answer:
a) $A = \{11, 22, \ldots 99\}$
$n(A) = 9$
$P(A) = 9/100$

b) $A = \{19, 28, 37, 46, 55, 64, 73, 82, 91\}$
$n(A) = 9$ $P(A) = 9/100$

c) P (getting number in which HCF w/ 100 = 1)

Euler's function:

$\phi(n) = \{m \mid m<n \ \& \ gcd(m,n)=1\}$

$\Rightarrow \phi(n) = n(1-1/p)(1-1/q).....$

prime

In this case, $100 = 2^2 \times 5^2$

$\phi(n) = 100 \times (1-1/2) \times (1-1/5)$

$= \overset{10}{\cancel{100}} \times \frac{1}{2} \times \frac{4}{5} = 40$

$P(A) = \frac{40}{100} = \frac{4\cancel{0}}{10} = 0.4$

30.11.18 **Conditional probability**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Let A, B be two events in any sample space S.

Then, probability of A, given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

and $$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Example:

1. In rolling a die $S = \{1, 2, 3, 4, 5, 6\}$
A = getting prime number,
B = getting even prime

$P(A) = \frac{3}{6} = \frac{1}{2}$; $P(B) = \frac{1}{6}$

$P(A \cap B) = \frac{1}{6}$

$P(B|A) = \frac{1/6}{1/2} = \frac{2}{6} = \frac{1}{3}$
$\hookrightarrow \frac{P(A \cap B)}{P(A)}$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/6} = 1$ Sure event

If occurrence of A doesn't affect B,
$P(B|A) = P(B)$

Independent events:

If occurrence of an event A does not affect occurrence of an event B, then the probability of B given A = probability of B
&
$P(A \cap B) = P(A) . P(B)$

Thus A & B are independent events iff both above conditions are true.
1) mainly 2nd condition

Example:
In tossing a coin & rolling a die at the same time,
$S = \{ (H1), (H2)(H3,) \ldots, (H,6), (T,1), \ldots (T,6) \}$

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3.12.18

We know that $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Multiplication Rule:

$$P(A \cap B) = P(B) \cdot P(A|B)$$

Eg: If $P(A \cap B) = 0.15$ and $P(A \cup B) = 0.65$, and $P(A|B) = 0.5$, then
$P(B|A) = ?$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A) + P(B) = 0.15 + 0.65$$
$$= 0.80$$

Now, $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$P(B) = \frac{0.15}{0.5} = \frac{1.5}{5} = 0.3$$

$$\Rightarrow P(A) = 0.8 - 0.3 = 0.5$$

So $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.15}{0.5} = \dfrac{\overset{1.5}{\cancel{6.5}}}{5} = \underline{0.3}$

Note: Let A, B be independent events.
Then, A^c, B
or
A, B^c are independent
or
A^c, B^c

For $P(A^c \cap B)$, ~~then~~ ~~P(B~~
We know that it equals $P(B) - P(A \cap B)$

$\Rightarrow P(B) - P(A)P(B)$
$= P(B)(1 - P(A))$
$= P(B) \cdot P(A^c)$

Hence they are independent.

ii) A, B^c

$P(A \cap B^c) = P(A) - P(A \cap B)$
$= P(A) - P(A)P(B)$
$= P(A)[1 - P(B)]$
$= P(A) \cdot P(B^c)$

Hence they are independent.

iii) A^c, B^c

$P(A^c \cap B^c) = P((A \cup B)^c)$
$= 1 - P(A \cup B)$
$= 1 - P(A) \cdot P(B)$
$= (1 - P(A))(1 - P(B))$
$= P(A^c) \cdot P(B^c)$

$x + y z$
$(1 - P(A))($

(X) Baye's Theorem:

Let S be a sample space.
Let S be partitioned into K events $B_1, B_2, B_3 \dots B_K$ (Union gives S, any two events are disjoint)

Let A be any event in S.

$$\text{Probability } (B_s \mid A) = \frac{P(A \mid B_s) \times P(B_s)}{\sum_{i=1}^{k} P(A \mid B_i) \cdot P(B_i)}$$

(or)

$$\text{Probability } (B_s \mid A) = \frac{P(A \cap B_s)}{\sum_{i=1}^{K} P(A \cap B_i)}$$

This is called $\underbrace{\text{Law of Total Probability}}$.
denominator

i.e. $$\boxed{P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) \ldots + P(A \mid B_k)P(B_k)}$$

Example:

Bowl 1 contains 3 R chips and 7 Blue chips
Bowl 2 contains 8 R chips and 2 Blue chips

A die is cast. If 5 or 6, Bowl 1 else Bowl 2

$P(\text{Bowl } 1) = \frac{2}{6} = 1/3$

$P(\text{Bowl } 2) = \frac{4}{6} = \frac{2}{3}$

After choosing a bowl, a chip is picked and it is found to be red.

This event is A.

$P(A \mid B_1) = \frac{n(A)}{n(B_1)} = \frac{3}{10}$

$P(A \mid B_2) = \frac{8}{10}$

Material: Oliver Ibe's
Prob & Random Processes

2) Companies B_1, B_2, B_3 produce 30%, 45%, and 25% of the cars respectively. It is known that 2%, 3% and 2% of these cars produced ~~[illegible]~~ are defective. What is the probability that a car purchased is defective?

If a car purchased is found to be defective, what is the probability that this car is produced by B_1?

Answer:

a) Let A = event of defective car

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

$$= \underbrace{\frac{30}{100} \times \frac{2}{100}}_{B_1} + \underbrace{\frac{45}{100} \times \frac{3}{100}}_{B_2} + \underbrace{\frac{2}{100} \times \frac{25}{100}}_{B_3}$$

$$= \frac{60 + 135 + 50}{10000} = \frac{245}{10000} = 0.0245$$

$$\begin{array}{r} 135 \\ 110 \\ \hline 245 \end{array}$$

b) $P(B_1|A) = \dfrac{\frac{30}{100} \times \frac{2}{100}}{\frac{245}{10000}} = \dfrac{60}{245} = 0.2448$

or

$P(A \cap B) = P \times B$

4.12.18

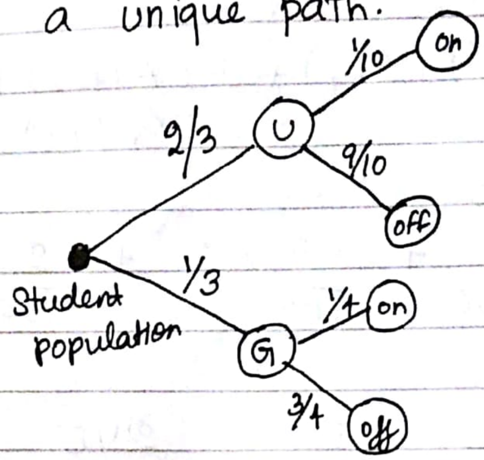
1) The university has 2x as many UG students as graduate students. 25% of the grad students live on campus & ~~25~~ 10% of the UG students live on campus.

i) If a student is chosen at random from the student population, what is the prob that the student is an on campus UG

ii) If a student living on campus is chosen at random, what is the prob that the student is a grad student?

Tree Diagram Method

Note: Any 2 pts can be connected by a unique path.



U = Undergraduate student
G = Graduate student

∴ For Qn 1,

$P(UG) = 2/3$

$P(\text{On campus} \cap UG) = \frac{\cancel{2}}{3} \times \frac{1}{\cancel{10}\,5} = \frac{1}{15} = 0.06$

$\hookrightarrow \frac{P(\text{on campus}) \cdot P(UG)}{P(UG)}$

$= P(\text{ON campus} | UG) \times P(UG)$

$5\overline{)100}$ 0.06, 90, 10

i.e. $P(UG \cap \text{ONcampus}) = P(\text{ON campus} | UG) \cdot P(UG)$

$= \frac{2}{3} \times \frac{1}{10} = 0.06$

Qn: 2

Student living on campus is chosen,
$\hookrightarrow C$

Applying Baye's Theorem:

Total probability = Probability of on-campus student

$$= \frac{2}{3} \times \frac{1}{10} + \frac{1}{3} \times \frac{1}{4}$$

$$= \frac{1\times 4}{15\times 4} + \frac{1\times 5}{12\times 5} = \frac{9}{60} = \frac{3}{20}$$

A : event that student is a grad student

$$P(A|C) = \frac{P(C|A) \cdot P(A)}{\text{Total}}$$

$$= \frac{\frac{1}{4} \times \frac{1}{3}}{3/20} = \frac{\frac{1}{12}}{\frac{3}{20}} = \frac{5}{9} = 0.55$$

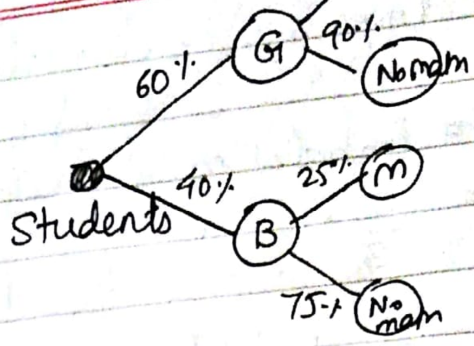
Problems:

1) ~~A~~ In certain college, 25% boys & 10% girls are studying Math. The girls constitute 60% of students.

a) What is the prob. that Mathematics is being studied?

b) If a student is selected at random and it's found to be studying Math, find the prob. that the student is a girl.

c) "- student is a boy.



i) P (Math being studied) = Total probability

P(G∩m)

= ~~P(...)~~ × P(G

$$= \frac{60}{100} \times \frac{10}{100} + \frac{40}{100} \times \frac{25}{100}$$

$$= \frac{600}{10000} + \frac{1000}{10000}$$

$$= \frac{16\cancel{00}}{100\cancel{00}} = 0.16$$

ii) m : event of studying math

G : event of being a girl

$$P(G|m) = \frac{P(m|G) \cdot P(G)}{\text{Total prob. of Math being studied}}$$

$$= \frac{\frac{6\cancel{0}}{10\cancel{0}} \times \frac{1\cancel{0}}{\cancel{100}}}{\frac{16}{\cancel{100}}} = \frac{6}{16} = \frac{3}{8} = 0.375$$

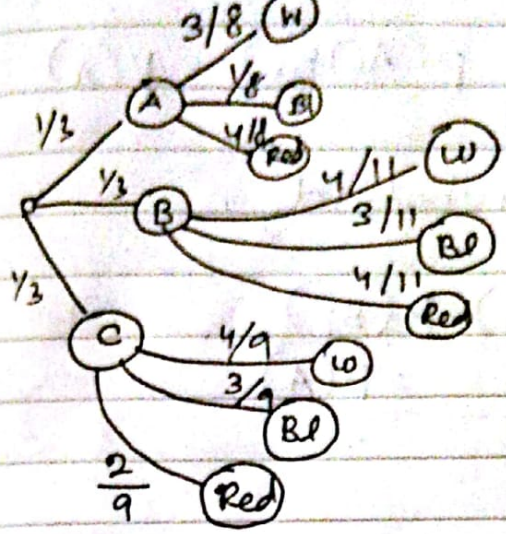
0.37
8)30
24
60
56
40

iii) $$P(B|m) = \frac{P(m|B) \cdot P(B)}{\text{Total}}$$

$$= \frac{\frac{\cancel{40}^{10}}{\cancel{100}} \times \frac{\cancel{25}}{\cancel{100}}}{\frac{16}{\cancel{100}}} = \frac{10}{16} = \frac{5}{8} = 0.625$$

1.000
0.375
0.625

3. Urn A contains 3 W, 1 B & 4 R balls
B contains 4 W, 3 B & 4 R balls
C contains 4 W, 3 B & 2 R balls
one urn is chosen at random & two balls are drawn. They happen to be red & black. What is the prob that both come from urn B?



P(getting red & black)

$$= \frac{1}{3} \times \left[\left(\frac{1}{8} \times \frac{4}{7}\right) + \left(\frac{4}{8} \times \frac{1}{7}\right)\right]$$

$$+ \frac{1}{3} \times \left[\left(\frac{3}{11} \times \frac{4}{10}\right) + \left(\frac{4}{11} \times \frac{3}{10}\right)\right]$$

$$+ \frac{1}{3} \times \left(\left(\frac{3}{9} \times \frac{2}{8}\right) + \left(\frac{3}{8} \times \frac{2}{9}\right)\right)$$

$$P(\text{urn } B \mid R\&B) = \frac{P(R\&B \mid \text{urn } B) \times P(\text{urn } B)}{\text{Total Prob of getting } R\&B}$$

5.12.18 Proof of Baye's Theorem:

both exclusive & exhaustive

Let S be a sample space.
Let $B_1, B_2, B_3 \ldots B_K$ be a partition on S.

i.e. $B_1 \cup B_2 \ldots B_K = S$
and $B_i \cap B_j = \phi$ if $i \neq j$

Let A be any event in S

$$A \subset \bigcup_{i=1}^{K} B_i$$

~~and~~ ~~$A \cap S$~~ $\cancel{\bigcup_{i=1}^{K}} (\cancel{B_i} \cap S)$

$$A = A \cap S = A \cap \left(\bigcup_{i=1}^{K} B_i \right) = (A \cap B_1) \cup (A \cap B_2) \ldots$$

$$= \bigcup_{i=1}^{K} (A \cap B_i) \quad - (1)$$

From (1), we apply Probability on both side. Also, it is seen that all events $A \cap B_i$ are exclusive.

$$P(A \cap S) = \sum_{i=1}^{K} P(A \cap B_i) = P(A)$$

By conditional probability,

$$P(A \cap S) = \sum_{i=1}^{K} P(A \mid B_i) \cdot P(B_i) = P(A) \quad - (2)$$

total probability of A

(2) is called Total Probability of A

To find: $P(B_i \mid A)$

By conditional probability,

$$P(B_i \mid A) = \frac{P(A \cap B_i)}{P(A)}$$

$$\Rightarrow P(B_s \mid A) = \frac{P(A \mid B_s) \cdot P(B_s)}{\sum_{i=1}^{K} P(A \mid B_i) \cdot P(B_i)}$$

where s is an index b/w 1 & K

Note: A, B are independent iff
$P(A \cap B) = P(A) \cdot P(B)$

$A \cap B = \phi$ so mutually exclusive

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$\frac{6}{16}$ $\quad \frac{3}{4} \times \frac{2}{4} = \frac{6}{16}$

Eg. Rolling a die $S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 5\}$

$B = \{2, 4, 6\}$

$P(A \cap B) = 0$ as $A \cap B = \phi$

↳ exclusive

and

not $P(A) \cdot P(B)$

⇒ Exclusive but dependent.

~~Dependency need not imply exclusivity~~ & V.V.

Exclusivity need not imply dependence

i.e. When independent, they must be exclusive

When dependent, not exactly necessary.

$P(A \cap B) =$

Tutorial 1 on these portions : Friday 7th Dec 2018

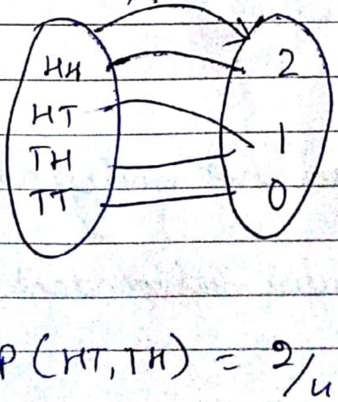
Random variable: Any function from Sample Space to real numbers

$X : S \to \mathbb{R}$

Let X : number of heads

eg. $S = \{HH, HT, TH, TT\}$

$X : S \to \mathbb{R}$ defined by no. of heads (in elements in S.)



⇒ Now, sample space $A = \{0, 1, 2\}$

$P(HH) = P(X=2) = 1/4$

~~$P(HT) = P(TH) =$~~

$P(X=1) = P(HT, TH) = 2/4 = 1/2$ $\quad P(X=0) = P(TT) = 1/4$

Let X = no. of heads in tossing 2 coins

$X = x$	0	1	2
$P(X=x)$	$1/4$	$1/2$	$1/4$

12/18

Let Experiment : Rolling 2 dice

$$S = \{(1,1), (1,2) \ldots\ldots (1,6),$$
$$(2,1) \ldots\ldots\ldots (2,6)$$
$$\ldots\ldots\ldots\ldots (6,6)\}$$

Euent : {First die is odd}
A

B : {Second die is odd}

C : {Sum is odd}

$$P(A) = \frac{6+6+6}{36} = \frac{18}{36} = 1/2$$

1,2 1,4 1,6
2,1 2,3 2,5

$$P(B) = 1/2$$

$$P(C) = \frac{18}{36} = 1/2 \qquad \text{(at least 1 is odd)}$$

Now, $P(A \cap B \cap C) = 0$

$\neq P(A) \cdot P(B) \cdot P(C)$

$\Rightarrow$ The events A, B, C are not independent

But they are pairwise independent

i.e. $P(A \cap B) = \frac{1}{4}$ $\quad P(B \cap C) = \frac{1}{4}$ $\quad P(A \cap C) = \frac{1}{4}$

$= P(A) \cdot P(B)$ $\quad \frac{1}{4}, \frac{9}{36}$ $\quad = P(B) \cdot P(C)$ $\quad = P(A) \cdot P(C)$

Consider experiment E : Tossing 3 coins.

$S = \{HHH, HHT, HTH \dots TTT\}$ $\quad 2^3$ events

We define a random variable X to denote number of heads.

$X = x$	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$

→ probability mass function

. Given

$X = x$	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$

Compute :

i) $P(1 < x \leq 2) = 3/8$

ii) $P(X > 0) = 1 - 1/8 = 7/8$

iii) $P(X \leq 1) = \underset{P(X=0)}{1/8} + \underset{P(X=1)}{3/8} = 1/2$

A function $P(x)$ is said to be a probability mass function if

i) $p(x) \geq 0$

ii) $\sum_{x \in A} p(x) = 1$

A → set of numbers by random variable

where ~~~~ X is a ~~~~ random variable from $S \to \mathbb{R}$

and $A = \{ x(s) \mid s \in S \} \subseteq \mathbb{R}$

eg If

$X = x$	0	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	K	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{5}{6}$

All these numbers vary from 0 to 1 $(P(X=x))$

In the above case, you cannot have a true K value such that summation = 1

No probability function exists for this.

~~ea~~

g. In rolling 2 dice,

Let $S = \{(1,1) \dots (6,6)\}$

$n(S) = 36$

We define 1 random variable $X((a,b)) = \text{Max}(a,b)$

$X = x$	1	2	3	4	5	6
$P(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

2,1
1,2

This is a probability mass function

1,3 3,1
2,3 3,2
3,3

2
11
9
7
5
3
1

36

i) $P(X \geq 4) = \frac{27}{36}$

ii) $P(2 \leq X < 5) = \frac{3+5+7}{36} = \frac{15}{36}$

iii) $P(X < 5)$

,4 4,1
,4
,4

$\hookrightarrow 1 - \left(\frac{11}{36} + \frac{9}{36}\right)$

$= 1 - \frac{20}{36} = \frac{16}{36}$

Random Variable

Discrete random variable

Continuous random variable

Let S be a sample space.
Any function $X : S \to \mathbb{R}$ (real) is called a random variable.

$A = \{\underbrace{X(s)}_{\text{numbers}} \mid s \in S\}$ is called the induced sample space.

If $X(s)$ take ~~integer~~ real values $\Rightarrow$ discrete random variable
i.e. $A \subseteq \mathbb{R}$

If values of $X(s)$ are intervals $\Rightarrow$ continuous random variable

1) A function $P(x)$ is a pmf if
   i) $P(x) \geq 0 \ \forall x$
   ii) $\sum_{x \in A} P(x) = 1$

Probability density function:

A function $f(x)$ is pdf if
- $f(x) \geq 0 \quad \forall x$
- $\int_A f(x)\,dx = 1$

	DISCRETE	CONTINUOUS
Mean	$E(x) = \sum_{x \in A} x \cdot p(x)$	$E(x) = \int_{-\infty}^{\infty} x f(x)$
Variance	$V(x) = E(x^2) - (E(x))^2$ where $E(x^2) = \sum_{x \in A} x^2 p(x)$	$V(x) = E(x^2) - (E(x))^2$ where $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

Cumulative distribution function / Distribution function

Let S be the sample space & X be the random variable

$$A = \{X(s) \mid s \in S\}$$

We define distribution function by $F : A \to \mathbb{R}$ such that

$$\forall x, \quad F(x) = P(X \le x)$$

$\downarrow$ probability

Notation:

$$P_r(X = x) = P_r\{s \in S \mid X(s) = x\}$$

$$P_r(X \le x) = P_r\{s \in S \mid X(s) \le x\}$$

$$F(x) = P_r(X \le x)$$

$$= \sum_{\omega \le x} p(\omega) \quad \text{(if } X \text{ is discrete)}$$

$\downarrow$ pmf

$$= \int_{-\infty}^{x} f(\omega)\,d\omega \quad \text{(if } X \text{ is continuous)}$$

If pmf is as follow

$X = x$	0	1	2	3	4	5	6	7
$p(X=x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

i) Find the value of k

ii) $P(X < 6)$, $P(X >= 6)$

iii) $P(0 < X < 5)$

iv) Find distribution function

$X =$ 0 1 2 3

~~Discrete~~ 0.5 1.5 2.5

Continuous $X =$ 0 – 10 10 – 20

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Distribution function:

Def: Let X be discrete random variable.
We define $F(x) = Pr(X \leq x)$ for all real numbers x
(or)
$F(x) = \sum_{\omega \leq x} p(\omega)$ if $p(x)$ is p.m.f

always $p(\omega) \leq 1$

Note: Let X be discrete

i) $P(a < X \leq b) = P(X \leq b) - P(X \leq a)$
$= F(b) - F(a)$ distribution function

ii) $P(a \leq X \leq b) = P(X = a) + F(b) - F(a)$

iii) $P(a \leq X < b) = P(X = a) + P(a < X \leq b) - P(X = b) + P(X = b)$
$= P(X=a) + F(b) - F(a) - P(X = b)$

A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective. Find the expected number E of defective items.

Let S = sample space
$n(S) = 12C4$

Among 12, 5 items are defective & the rest are not.

E = average probability

Let X denote no. of defective items.

X takes 0, 1, 2, 3, 4

$P(X=0) = \frac{7C4}{12C4} = 7/99$ $P(X=1) = \frac{7C_3 \times 5C_1}{12C_4} = 35/99$

Properties:

i) F is non-decreasing function

i.e. $x < y \Rightarrow F(x) < F(y)$

ii) $F(\infty) = \lim_{x \to \infty} F(x) = 1$

iii) $F(-\infty) = 0$

2) A discrete random variable X has the following distribution function.

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/3 & 1 \le x \le 4 \\ 1/2 & 4 \le x < 6 \\ 5/6 & 6 \le x < 10 \\ 1 & x \ge 10 \end{cases}$$

x	0	1	2	3	4	5	6	7	8	9	10
$F(x)$	0	1/3	1/3	1/3	1/2	1/2	5/6	5/6	5/6	5/6	1
$P(x)$	0	1/3	1/3	1/3	1/6	1/6	2/6	2/6	2/6	2/6	1/6

$1/2 - 1/3 = 1/6$

$5/6 - 1/3 - 1/2$ $\quad$ $1 - \frac{4}{6} =$

Find:

i) $P(2 < x \le 6)$

ii) $P(x = 5)$ $\quad$ $\frac{5}{6} -$

iii) $P(x = 4)$ $\quad$ $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{1}{6}$

iv) $P(x \le 6)$

v) $P(x = 6)$ $\quad$ $17/6$

i) Using formula, $P(2 < x \le 6) = F(6) - F(2)$

$= 5/6 - \frac{1 \times 2}{3 \times 2} = 3/6 = 1/2$

ii) $P(x=5) = P(x \le 5) - P(x<5)$
$= F(5) - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0$

iii) $P(x=4) = P(x \le 4) - P(x<4)$

12.12.18 Continuous Random Variable:

A random variable $X : S \to R$ such that $\{X(s) \mid s \in S\}$ is uncountable

Probability Density Function (pdf)

A function $f(x) \ge 0$ is a pdf of X if $\int_A f(x)\,dx = 1$.

Example. $f(x) = \begin{cases} e^{-x} & x>0 \\ 0 & x \le 0 \end{cases}$

Now, $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} \underbrace{f(x)}_{0}\,dx + \int_{0}^{\infty} e^{-x}\,dx$

$= 0 + 1 = 1$

Distribution function (cumulative)

$F(x) = Pr(X \le x) = \int_{w \le x} f(w)\,dw$

$= \int_{-\infty}^{x} f(w)\,dw$

Properties:

i) $P(x=a) = \int_a^a f(x)\,dx = 0$

ii) $P(a<x<b) = \int_a^b f(x)\,dx$

$P(a \le x < b) = P(a < x \le b) = P(a \le x \le b)$

$= \int_a^b f(x)\,dx$

iii) $\frac{d}{dx} F(x) = \frac{d}{dx}\left[\int_{-\infty}^{x} f(\omega)\,d\omega\right] = f(x)$

iv) F is an increasing function

Consider a unit disc $= \{(x,y) \mid x^2+y^2<1\}$

Experiment: select a point from unit disc

S = set of (x,y) such that $x^2+y^2<1$

For $C \subset S$,

$P(C) = \frac{\text{Area of } C}{\text{Area of } S}$

$P(\text{getting a point from } 1^{st} \text{ quadrant}) = \frac{\frac{1}{4}\pi}{\pi} = \frac{1}{4}$

For a line or pt, area = 0

=> Probability anywhere for them = 0

2) pdf: $f(x) = \begin{cases} \frac{1}{2}\sin x & 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$

Find mean, variance & $P(0 < x < \pi/2)$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \text{if pdf}$$

$$= \int_{-\infty}^{0} 0 + \frac{1}{2}\int_{0}^{\pi} \sin x + \int_{\pi}^{\infty} 0 = 1$$

$$\frac{1}{2}\left[-\cos x\right]_0^{\pi} = \frac{1}{2}\left(-\cos\pi - (-\cos 0)\right)$$

$$= \frac{1}{2}\left((-1\times -1) + 1\right) = \frac{2}{2} = 1$$

It is a pdf.

$P(0 < x < \pi/2)$

$$= \int_0^{\pi/2} f(x)\,dx = \frac{1}{2}\int_0^{\pi/2} \sin x$$

$$= \frac{1}{2}\left(-\cos \pi/2 - (-\cos 0)\right)$$

$$= \frac{1}{2}\left(0 - (-1)\right) = 1/2$$

Mean $E(x)$ $= \int_{-\infty}^{\infty} x\, f(x)\,dx$

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$$= \int_{-\infty}^{0} 0\,x + \frac{1}{2}\int_{0}^{\pi} x \sin x \, dx + \int_{\pi}^{\infty} 0$$

$u = x$
$dv = \sin x\, dx$
$\int u\, dv = uv - \int v\, du$

$$= \frac{1}{2}\left[(-x\cos x)_0^{\pi} - \int_0^{\pi} -\cos x\, dx\right]$$

$$= \frac{1}{2}\left[\left(-\pi\cos\pi - (\cancel{0})\right) - \left(-\sin x\right)_0^{\pi}\right]$$

$$= \frac{1}{2}(\pi + 0) = \pi/2$$

Variance : $\cancel{\sqrt{E(x^2) - [E(x)]^2}}$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^{\pi} x^2 \frac{\sin x}{2}\, dx = \frac{1}{2}\int_0^{\pi} \underset{u}{x^2}\, \underset{dv}{\sin x\, dx}$$

$$= \frac{1}{2}\left[(x^2 \cdot -\cos x)_0^{\pi} - \int_0^{\pi} -\cos x \cdot 2x\, dx\right]$$

$$= \frac{1}{2}\left(\left(-x^2\cos x\right)_0^{\pi} + 2\int_0^{\pi} x\cos x\, dx\right)$$

$$= \frac{1}{2}\left(\left(-x^2\cos x\right)_0^{\pi} + 2x(-\sin x)_0^{\pi} + 2\cos x_0^{\pi}\right)$$

$$= \frac{1}{2}\left(\pi^2 + 2 - 2\right) = \pi^2/2$$

$-\pi^2\cos\pi$

$$Var(x) = \pi^2/2 - \pi^2/4 = \pi^2/4$$

7-12-18

$\ulcorner$ function of x

Expectation for $u(x)$

Let X be distribution

$$E(u(x)) = \begin{cases} \sum_{x=-\infty}^{\infty} u(x) \cdot f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} u(x) f(x) & \text{if } X \text{ is continuous} \end{cases}$$

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Moments of a distribution X:

Central moment (r^{th}):

$$\mu_r = E[(x-\mu)^r] \quad r = 0, 1, 2, 3, \ldots$$

$$\mu = E(X) = \overline{X}$$

↳ weighted average
or
mean

Every distribution can be uniquely defined (rather, determined) by its moments.

If $\mu = 0$,

$$\mu_r' = E(x^r)$$

↓

is moment about origin

We know the values of $E(X)$ for a distribution X,

i.e. $E(x) = \sum x\, p(x)$ or $\int x\, f(x)\, dx$

- For a constant c,

$$E(c) = c \sum p(x) \quad \text{or} \quad c \int f(x)\, dx$$
$$= c \qquad\qquad\qquad = c$$

- $E(X+Y) = \Sigma p(x) + \Sigma p(y)$ or $\int f(x) + \int f(y)$

$= E(X) + E(Y)$

- $E(aX+bY) = a\,E(X) + b\,E(Y)$

- $E(XY) \neq E(X) \cdot E(Y)$ necessarily

If variables X, Y are independent random variables (i.e they come from a collection of independent events)

then $E(XY) = E(X) \cdot E(Y)$

Variance of X: = 2^{nd} central moment of X

$$\text{Variance}(X) = E[(X-\mu)^2]$$

$$= E[X^2 - 2X\mu + \mu^2]$$

applying property, $= E(X^2) - 2\mu E(X) + \mu^2$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

$$\text{Var}(X) = E(X) - (E(X))^2$$

$$\text{Var}(cX) = E((cX)^2) - [E(cX)]^2$$

$$= E(c^2X^2) - [E(cX)]^2$$

$\hookrightarrow c \cdot E(X) . c : E(X)$

$$= c^2E(X^2) - c^2[E(X)]^2$$

$$= c^2\,\text{Var}(X)$$

$$\text{Var}(c) = E(c^2) - [E(c)]^2$$

$$= c^2 - c^2$$

$$= 0$$

deviation from mean

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Note:

Let $\mu_r' = E(X^r)$

$\mu_r = E[(X-\mu)^r]$

For r=1,

$\mu_1 = E[X-\mu]$
$= E(X) - \mu = 0$

$\mu_2 = E((X-\mu)^2)$
$= E(X^2) - (E(X))^2$
$= \mu_2' - \mu_1'^2$

$\mu_3 = E((X-\mu)^3)$
$= E(X^3 - \mu^3 - 3X^2\mu + 3X\mu^2)$
$= E(X^3) - \mu^3 - 3\mu E(X^2) + 3\mu^2 E(X)$
$= \mu_3' - \mu_1'^3 - 3\mu\mu_2' + 3\mu^2\mu$ $\to \mu^3$
$=$

$\mu_3' - 3\mu\mu_2' + \underbrace{3\mu^3 - \mu^3}_{2\mu^3}$

$E[X^r - \mu X^{r-1} + \mu'^2 X^{r-2} - \cdots \mu^r]$

Generalizing,

$$\boxed{\mu_r = \mu_r' - {}^rC_1 \cdot \mu_{r-1}'\mu + \cdots + (-1)^r\, {}^rC_j\, \mu_{r-j}'\mu^j + (-1)^r \mu_0' \mu^r}$$

$$= \frac{4}{81}\int_0^3 9x^3 - x^5\, dx \quad = \frac{4}{81} \times \left(\frac{9 \times 81}{4} - \frac{486}{6}\right)$$

$$= \frac{4}{81} \times \frac{729 - 324}{4} = \frac{405}{81}$$

& so on for $\mu_3' = \int_0^3 x^3 \cdot \frac{4x(9-x^2)}{81}\, dx$

$$= \frac{216}{35}$$

ii) $\mu_1 = E((x-\mu)) = \mu = 8/5$

$\mu_2 = E((x-\mu)^2) = \cancel{\text{afllo}} = 61/25$

$\mu_3 = E((x-\mu)^3) = 32/1375$

[redacted]

$$5 - \frac{64}{25} = \frac{125-64}{25}$$

17.12.18 Moment Generating Function (mgf)

Defined by:

$$E(e^{tx}) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx & \text{if } x \text{ is continuous} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x) dx & \text{if } x \text{ is discrete} \end{cases}$$

It is a special type of expectation

Notation: $M(t) = E(e^{tX})$

Ways to get the moments of $M(t)$:

1) $M(t)$ by Maclaurent Series.

$$M(t) = M(0) + \frac{M'(0)}{1!}t + \frac{M''(0)}{2!}t^2 \ldots\ldots \infty$$

Now,

$$\frac{d}{dt}M(t) = \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$1-x$

$$M'(t) = \int_{-\infty}^{\infty} x e^{tx} f(x)\,dx$$

$$M(0) = 1$$

$$M'(0) = \int_{-\infty}^{\infty} x f(x)\,dx = E(x)$$

$$\therefore M(t) = 1 + \frac{E(x)}{1!}t + \frac{E(x^2)}{2!}t^2 \ldots\ldots$$

Property: Every distribution of a random variable is uniquely determined by m.g.f.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\ldots$$

expansion of e^{tx}

Note: $E(e^{tx}) = E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \ldots\ldots\right]$

X is given random variable.

$$= E(1) + \frac{t\,E(x)}{1!} + \frac{t^2\,E(x^2)}{2!} \ldots\ldots$$

1

Breathe

Live

Radiate

1) Given pdf $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

i) Then find the distribution function

ii) Mean, variance

iii) MGF

Checking if $f(x)$ is a pdf,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx = \lambda \int_{0}^{\infty} e^{-\lambda x}\,dx$$

$$= \lambda \left[-\frac{e^{-\lambda x}}{\lambda} \right]_0^{\infty} = 1 \text{ if } \lambda > 0$$

$\hookrightarrow 0-(-1)$

Yes, it is a pdf for $\lambda > 0$

To find:

i) $F(x) = P(X \leq x)$

$$= \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{x} \lambda e^{-\lambda u}\,du$$

$$= \cancel{\lambda} \left[\frac{e^{-\lambda u}}{-\cancel{\lambda}} \right]_{0}^{x}$$

because for $-ve$, $f(u)du = 0$

$$= -\left[\frac{e^{-\lambda x}}{-1} \right]_0^{x}$$

$F(x) = 1 - e^{-\lambda x}$

$$\text{mgf} = M(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{0}^{\infty} e^{tx} \cdot \lambda e^{-\lambda x}\,dx$$

$$\frac{e^{-x(\lambda - t)}}{-(\lambda - t)}$$

$$\lambda \int_0^{\infty} e^{x(t-\lambda)}\,dx = \lambda \left[\frac{e^{x(t-\lambda)}}{(t-\lambda)} \right]_0^{\infty}$$

λ must be +ve for mgf to exist.

$7! \times 6!$

$$= \frac{\lambda}{t-\lambda}\,[0+1] = \frac{\lambda}{-(t-\lambda)} = \frac{\lambda}{\lambda - t}$$

if $\lambda - t > 0$

$\therefore M(t) = \dfrac{\lambda}{\lambda - t}$ for above condition.

ii) Since $E(x) = M'(0)$

$$\Rightarrow E(x) = \frac{d}{dt}\frac{\lambda}{\lambda - t} = -\lambda\left(\frac{1}{(\lambda - t)^2}\right) = \text{at } t=0$$

$$\frac{1}{x} = -\frac{1}{x^2}$$

$$= -\frac{1}{\lambda} \times -1 = \frac{1}{\lambda} \qquad \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

18/12/18 Properties of Mgf

Let X be a random variable.

Then

$$M(t) = E(e^{tx})$$

(or) $M_X(t) = E(e^{tX})$

① $M_X(ct) = E(e^{ctx}) = E(e^{(cx)t})$

$= M_{cx}(t)$

② $M_{X+C}(t) = E(e^{t(x+c)})$

(where c is constant)

$= E(e^{tc} \cdot e^{tx})$

$= e^{tc} \cdot E(e^{tx})$

$\Rightarrow M_{X+C}(t) = e^{tc} \cdot M_X(t)$

③ $M_{x_1+x_2 \dots x_n}(t) = E(e^{t(x_1+x_2 \dots x_n)})$

$= E(e^{tx_1} \cdot e^{tx_2} \dots e^{tx_n})$

If X_i's are independent,

then $M_{X_1+X_2 \dots x_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \dots M_{X_n}(t)$

Distribution Types

① Uniform Distribution

For discrete case, given some n

$$\Rightarrow p(x) = \begin{cases} 1/n & x = 1 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

Since $\sum_{x=1}^{n} 1/n = \frac{1}{n} \times \sum_{x=1}^{n} 1 = \frac{1}{n} \times n = 1$

Any distribution having (above) pmf $p(x)$ is called uniform distribution.

- Finding mean,

$$\mu = E(x) = \sum_{x=-\infty}^{\infty} x\, p(x)$$

But we restrict summation values

$$\mu = E(x) = \sum_{x=1}^{n} x \cdot 1/n$$

$$= \frac{1}{n} \sum_{x=1}^{n} x = \frac{1}{\cancel{n}} \frac{(\cancel{n})(n+1)}{2} = \frac{n+1}{2}$$

1 + 2 + 3 ..

- Finding variance,

$$E(x^2) = \sum_{x=1}^{n} x^2 p(x)$$

$$= \sum_{x=1}^{n} \frac{1}{n} x^2 = \frac{1}{n} \sum_{x=1}^{n} x^2 = \frac{1}{n} \times (1+4+9 \ldots)$$

$$= \frac{1}{\cancel{n}} \cdot \frac{\cancel{n} \cdot (n+1) \cdot 2n+1}{6}$$

$$\sigma^2 = Var(X) = E(X^2) - (E(X))^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$\frac{4(2n^2+2n+n+1) - 6(n^2+2n+1)}{24} = \frac{8n^2+12\cancel{n}+4 - 6n^2 - \cancel{12n} - 6}{24}$$

$$= \frac{2n^2-2}{24} = \frac{n^2-1}{12}$$

$Var(x) = \sigma^2 = \frac{n^2-1}{12}$

- To find mgf

$m(t) = E(e^{tx})$

$\sum_{x=1}^{n} e^{tx} \cdot p(x) = \frac{1}{n} \sum_{x=1}^{n} e^{tx}$

$= \frac{1}{n} \left(e^{t} + e^{2t} + e^{3t} \ldots\ldots e^{nt} \right)$

e^{2t} $= \frac{1}{n} \times \frac{e^{t}(1-e^{nt})}{1-e^{t}}$

G P

$S_n = \frac{a(1-r^n)}{1-r}$

Note:

Finding $m'(0)$:

$= \frac{d}{dt} \left(\frac{1}{n} \cdot \frac{e^{t}(1-e^{nt})}{1-e^{t}} \right)$

$= \frac{d}{dt} \cdot \frac{1}{n} \left(e^{t} + e^{2t} \ldots\ldots e^{nt} \right)$

$= \frac{1}{n} \left(e^{t} + 2e^{2t} + \ldots + ne^{nt} \right)$

At $t=0$ $= \frac{1}{n} (1 + 2 + \cdots n)$

$= \frac{1}{n} \times \frac{n(n+1)}{2} = \frac{n+1}{2} = E(X)$

Variance $= M''(0) - (M'(0))^2$

$= \frac{1}{n} \left(e^{t} + 4e^{2t} + \ldots n^2 e^{nt} \right) - \frac{(n+1)^2}{4}$

At $t=0$,

$$= \frac{1}{n}(1+4+9\ldots\ldots n^2) - \frac{(n+1)^2}{4}$$

$$= \frac{1}{\cancel{n}} \times \frac{(n+1)(2n+1)\cancel{n}}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$$

Case II: Continuous Uniform Distribution:

$$f(x) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Any distribution having this pdf $f(x)$ is called continuous uniform distribution.

Since $\int_{-\infty}^{\infty} f(x)\,dx = \int_{a}^{b} \frac{1}{b-a}\,dx$

$$= \frac{1}{b-a}\int_a^b dx = \frac{1}{b-a}(b-a) = 1$$

$\therefore f(x)$ is a pdf.

i) Mean:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

Here,

$$\mu = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a} \times \left(\frac{x^2}{2}\right)_a^b = \frac{1}{b-a} \times \frac{b^2-a^2}{2}$$

$$= \frac{1}{\cancel{b-a}} \frac{\cancel{(b-a)}(b+a)}{2} = \frac{a+b}{2}$$

Variance (x)

$$= E(x^2) - (E(x))^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$= \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b$$

$$= \frac{1}{b-a} \times \frac{b^3-a^3}{3} = \frac{1}{\cancel{b-a}} \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3}$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4} \quad \frac{a^2+b^2+2ab}{4}$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12} = \frac{(a-b)^2}{12}$$

Mgf:

$$M(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left(\frac{e^{tx}}{t}\right)_a^b$$

$$= \frac{1}{t(b-a)} \cdot (e^{bt} - e^{at})$$

$$= \frac{e^{tb} - e^{ta}}{tb - ta}$$

Now,

M'(0) : $M'(t) = \frac{1}{b-a}\left[\frac{1}{t}(be^{tb} - ae^{ta}) - \frac{(e^{tb} - e^{ta})}{t^2}\right]$

$M'(0)$

$= \frac{1}{b-a}\left[\frac{(b-a)}{t} - 1\right]$ Since t comes in denominator, we cannot evaluate M'(0) for this continuous distribution.

$\therefore$ For $t = 0$ it doesn't exist & hence mean cannot be evaluated using this final formula. (this is incorrect. refer next page)

19/12/18

$$f(x) = \begin{cases} 1/x^2 & 1 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

f(x) is a pdf because $\int_{-\infty}^{\infty} f(x)\,dx = \int_{1}^{\infty} \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^{\infty}$

$= 1$

But expectation E(X) doesn't exist

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int \cancel{x} \times \frac{1}{\cancel{x^2}\,x}\,dx = [\log x]_1^{\infty}$$

Note:

$$M(t) = \frac{e^{bt} - e^{at}}{t(b-a)} \quad — ①$$

$$M'(t) = \frac{1}{b-a} \quad \frac{t[be^{bt} - ae^{at}] - [e^{bt} - e^{a}}{t^2}$$

$$M'(t)\Big|_{t=0} = \frac{0}{0}$$ which is indeterminate.

∴ We may apply a different rule to find scope (because this is indeterminate but not ∞, we can find a value for $M'(t)$ at $t=0$)

Eqn ① can be rewritten as

$$M(t) = \frac{1}{t(b-a)}\left[\left[1 + \frac{bt}{1!} + \frac{(bt)^2}{2!} + \dots\right] - \left[1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \dots\right]\right]$$

$$\frac{1}{t(b-a)}\left[\frac{t(b-a)}{1!} + \frac{t^2(b^2-a^2)}{2!} + \frac{t^3(b^3-a^3)}{3!} \dots\right]$$

$$= \left[1 + \frac{b+a}{2}t + \frac{b^2+ab+a^2}{6}t^2 + \dots\right]$$

series

Now, $M'(t) = \frac{b+a}{2} + \frac{2x(b^2+ab+a^2)t}{6} + \dots$

$\boxed{M'(0) = \boxed{\frac{b+a}{2}} + \dots 0}$

expectation of x

$\boxed{M''(0) = \frac{a^2+ab+b^2}{3}} = E(x^2)$

Expellimo Chico

II Binomial Distribution

Poisson

Poisson

Assume an exp. w/ only 2 outcomes
Let p = Probability of success
then $1-p$ = Probability of failure $= q$

$p+q = 1$

Repeat this experiment n times.

Assume a random variable $X = x$, which denotes number of successes.

$Pr(X=x) : {}^nC_x p^x q^{n-x}$ $\quad x = 0, 1, 2 \cdots n$

This can be done only when trials are independent.

Any X having above pmf is called Binomial distribution.

All numbers/terms involved should be non-negative

$\sum_{-\infty}^{\infty} P(X=x) = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x}$

Summation of all terms should give 1

Note: Binomial distribution is always discrete.

$= (p+q)^n$

$= 1^n = 1$

Parameters : n, p

To find the mgf : $M(t) = E(e^{tx})$

We assume X is binomial.

$$m(t) = \sum_{-\infty}^{\infty} e^{tx} \, p(X=x)$$

$$= \sum_{x=0}^{n} e^{tx} \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x \cdot (pe^t)^x \cdot q^{n-x}$$

$$= (pe^t + q)^n$$

which is the required mgf for Binom.

To find Mean:

$E(x) = m'(0)$

$m'(t) = n(q + pe^t)^{n-1} \times pe^t$

At $t=0$,

$= n(q+p)^{n-1} \times p$

$= np$

$M''(t) = np[(q+pe^t)^{n-1} \times e^t + e^t(n-1)(q+pe^t)^{n-2} \times pe^t]$

$= npe^t [(q+pe^t)^{n-1} + pe^t(n-1)(q+pe^t)^{n-2}]$

$M''(0) = np[1 + p(n-1)] = E(x^2)$

$= \cancel{np}$

$Var(x) = E(x^2) - (E(x))^2$

$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$

$= np - np^2$

$= np(1-p) = npq$

21/12/18

1) The mean & variance of a Binomial distribution are 4 & 4/3 respectively. Find $p(x \geq 6)$

Given mean $= 4$, var $= 4/3$

$np = 4$

$npq = 4/3 \Rightarrow q = \frac{4/3}{4} = 1/3 \Rightarrow p = 1 - 1/3 = 2/3$

$P(x \geq 6) = nC_x p^x q^{n-x}$

$np = 4 \quad n = \frac{4}{p} = \frac{4 \times 3}{2} = 6$

$= 6C_6 p^6 q^0$

$= \left(\frac{2}{3}\right)^6$

2) 6 cards are drawn from a pack of 52 cards. Find prob. that:

i) There are at least 3 diamonds

ii) None is a diamond

3) The prob. of a man hitting a target is $1/3$.
i) If he fires 5 times, what is the prob. of his hitting target at least 2x ?
ii) How many times must he fire so that the prob. of his hitting the target at least once is more than 90% ?

$p = 1/3 \quad q = 2/3$

↳ prob. of hitting target

i) $n = 5$

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - [5C_0 \, 1 \times \left(\frac{2}{3}\right)^5 + 5C_1 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^4]$

$0.13 + 0.32$

$= 0.54$

ii) Given: $P(X \geq 1) > 90\%$

$1 - P(X=0) > 0.9$

~~x = 0x3 > 0.9~~

$1 - \left(nC_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^n\right) > 0.9$

by trial & error

$1 - \left(\frac{2}{3}\right)^n > 0.9 \qquad \boxed{n = 6}$

4) The mean of a Binomial Distribution is 3 & Var $= 9/4$. Find n.
ii) $P(X \geq 7)$
iii) $P(1 \leq x \leq 6)$

$np = 3 \quad npq = 9/4 \qquad q = \frac{9}{4} \times \frac{1}{3} = 3/4$

$p = 1/4 \qquad \frac{n}{4} = 3 \quad n = 12$

(2) Mgf of a r.v is $\left(\frac{1}{3}+\frac{2}{3}e^t\right)^5$.

Find $P(X=2$ or $3)$

Mgf of Binom. $= (q+pe^t)^n$

$n=5 \quad q=1/3 \quad p=2/3$

$P(X=2) = 5C_2 \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3$

$P(X=3) = 5C_3 \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2$

Adding we get the probability of $X=2$ or $X=3$

22/12/18 Note:

We know that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\Rightarrow \lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n = e^{-2}$$

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Page

Goal: To obtain Poisson Distribution from Binomial Distribution:

For a Binomial distribution,

$$P(X=r) = {}^nC_r\, p^r q^{n-r}$$

r varies from 0 to n

For large n, the computation of this is difficult.

=> We use Poisson distribution.

$$ {}^nC_r = \left(\frac{n(n-1)(n-2)\ldots\ldots(n-(r+1))}{1\times2\times3\times4\ldots\times r} \right) p^r q^{n-r}$$

Let np be a finite value = m

$n = \frac{m}{p}$ => For a large n, p will be small.

=> Poisson distribution will be used mostly for accidents & wrongly occurring events.

$$\Rightarrow {}^nC_r = \frac{\left(\frac{m}{p}\right)\left(\frac{m}{p}-1\right)\left(\frac{m}{p}-2\right)\ldots\left(\frac{m}{p}-r-1\right)}{1\times2\times\ldots r}$$

$$= \frac{m(m-p)(m-2p)\ldots(m-(r+1)p)\, \cancel{p^r}\, q^{n-r}}{\cancel{p^r}\times1\times2\times\ldots r}$$

$$= \frac{m(m-p)(m-2p)\ldots(m-(r+1)p)}{r!} \cdot \underbrace{\left(1-\frac{m}{n}\right)}_{q}{}^{n-r}$$

$1-p$

$$= \frac{m(m-p)\ldots(m-(r+1)p)\left(1-\frac{m}{n}\right)^n}{1\times 2\times \ldots r \times \left(1-\frac{m}{n}\right)^r}$$

As $n \to \infty$, $p = \left(\frac{m}{n}\right) \to 0$

Substituting,

$$= \frac{\underset{(r \text{ times})}{m\times m\times \ldots \times m} \times e^{-m}}{1\times 2\times \ldots r}$$

$$= \frac{m^r e^{-m}}{r!} \quad r = 0,1,2\ldots\infty$$

Mean from Binom = m

$\therefore$ Poisson pmf : $P(X=x) = \frac{m^x e^{-m}}{x!}$, $x = 0,1,2,3\ldots\infty$

P is pmf because,

$$\sum_{x=0}^{\infty} p(X=x) = e^{-m}\sum_{x=0}^{\infty}\frac{m^x}{x!}$$

$$= e^{-m}\times e^m = 1$$

To find mgf:

$$M(t) = E(e^{tX})$$

$$= \sum_{x=0}^{\infty} e^{tx}\cdot\frac{m^x e^{-m}}{x!}$$

$= (e^t m)^x$

$= e^{-m} \sum_{x=0}^{\infty} e^{tx} \cdot \frac{m^x}{x!} = e^{-m} \cdot (e^{me^t})$

$= e^{m(e^t - 1)}$

Taking $M'(t)$: To find mean

$\text{mean} = \frac{d}{dt} e^{m(e^t-1)} = e^{m(e^t-1)} \times me^t$

At $t=0$, Mean $= m$ (for binom distribution)

Variance : $M''(t)$

$= e^{m(e^t-1)} \times me^t \times me^t + me^t \cdot e^{m(e^t-1)}$

1 m m m

At $t=0$,

$= m^2 + m$

$Var(x) = \sigma^2 = M''(0) - (M'(0))^2$

$= m^2 + m - m^2$

$= m$

HW: Apply original formula to find mean & var.

Problems.

1. If X is Poisson distribution with $P(X=0) = P(X=1)$ then calculate $P(X=2)$
ii) $P(x=3)$
iii) $P(x \geq 4)$

$\frac{m^0 e^{-m}}{0!} = \frac{m^1 e^{-m}}{1!}$ $m = 1$

12/18 Exponential Distribution:

$$f(x) : \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \lambda > 0 \\ 0 & x < 0 \end{cases}$$

If pdf of a distribution X is as above $= f(x)$, that distribution is called an exponential distribution

Distribution function: (cumulative)

$$F(x) = P(X \leq x) = \int_{-\infty}^{\infty} f(t)\,dt = \int_0^x \lambda e^{-\lambda t}\,dt$$

$$= \lambda \left(\frac{e^{-\lambda t}}{-\lambda} \right)_0^x = \lambda \left(\frac{e^{-\lambda x}}{-\lambda} - \frac{1}{-\lambda} \right)$$

$$= -(e^{-\lambda x} - 1) = 1 - e^{-\lambda x}$$

To find Mgf:

$$M(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} \cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} e^{x(t-\lambda)}\,dx = \lambda \int_0^{\infty} e^{-x(\lambda - t)}\,dx$$

$$= \lambda \left[\frac{e^{-x(\lambda - t)}}{-(\lambda - t)} \right]_0^{\infty}$$

No mgf for $\lambda = t$
$\downarrow$
∞ value

If $\lambda - t$ is +ve, then numerator becomes 0

$$= \lambda \times \left[0 - \frac{1}{-(\lambda - t)} - (\lambda - t) \right] = \lambda \left(\frac{1}{\lambda - t} \right) = \frac{\lambda}{\lambda - t}$$

if $\lambda - t > 0$
or if $\lambda > t$

$M(t) = \frac{\lambda}{\lambda - t}$ => To compute $M'(t)$:

$$= \frac{d}{dt}\left(\frac{\lambda}{\lambda - t} \right) = \frac{-\lambda \times -1}{(\lambda - t)^2} = \frac{\lambda}{(\lambda - t)^2} \quad M'(0) = \frac{1}{\lambda}$$

$\frac{\lambda}{\lambda^2}$

$$M''(t) = \frac{-2\lambda \times (-1)}{(\lambda - t)^3} = M''(0) = \frac{2}{\lambda^2}$$

$\lambda^2 - 2\lambda t + t^2$

$$\text{Var } \sigma^2 = M''(0) - (M'(0))^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Note:

$$M(t) = \frac{\lambda}{\lambda - t} = \frac{\lambda}{\lambda(1 - t/\lambda)} = \left(1 - \frac{t}{\lambda} \right)^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} \ldots\ldots\ldots$$

$M'(t) = \frac{1}{\lambda} + \frac{2t}{\lambda^2} \ldots$ | $M''(t) = \frac{2}{\lambda^2} + \frac{6t}{\lambda^3} \ldots$

$\hookrightarrow$ For 0, $= \frac{1}{\lambda}$ $\quad = \frac{2}{\lambda^2}$ for 0

N = 120

50 25 10 5 1

70

20 $2^0\ 2^1\ 2^2 \ldots$

12.18 Normal Distribution

Any distribution with pdf

$$f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & -\infty < x < \infty \\ \end{cases}$$

is called Normal Distribution
(where σ & μ are some constants)

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma}$

$\Rightarrow dz = \frac{1}{\sigma} dx$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2)} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

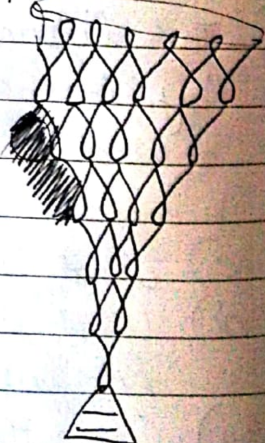
$= \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi}$ (since that value equivalent to $\sqrt{2\pi}$ by result,)

$= 1$

$u = \frac{z^2}{2}$

Mean:

$$= \int_{-\infty}^{\infty} x f(x)\, dx$$



$= \mu$ (as used in $f(x)$)

Var $= \sigma^2$

Eg: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}$ $\quad \sigma = 1 \quad$ Var $= 1$

$\mu = 1$

To find mgf:

$$E(e^{t}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)dx = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$x = \sigma z + \mu$

Assume $z = \frac{x-\mu}{\sigma}$ $\quad dz = \frac{1}{\sigma} dx$

$$\frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} \cdot e^{-\frac{z^2}{2}} \times \cancel{\sigma}\, dz$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\mu} \cdot e^{t\sigma z} \cdot e^{-z^2/2} dz$$

$a^2 - 2ab$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma t z - z^2/2} dz = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma t z)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 + 2\sigma t z + \sigma^2 t^2 - \sigma^2 t^2)} dz$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - \sigma t)^2} dz$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \times \sqrt{2\pi} = e^{\mu t + \frac{\sigma^2 t^2}{2}} \text{ . mgf}$$

$$M'(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \left(\mu + \frac{\sigma^2 \cdot 2t}{2}\right)$$

$$M'(0) = e^0 \cdot (\mu + 0)$$
$$= \mu$$

$$M''(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} (\sigma^2) + (\mu + \sigma^2 t) \cdot e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot (\mu$$

At $t = 0$,

$$M''(t) = e^0 (\sigma^2) + (\mu)(\mu)$$
$$= \sigma^2 + \mu^2$$

$$\text{Var} = M''(0) - (M'(0))^2$$
$$= \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

Standard Normal Distribution:

Let X be normal with mean μ, var σ^2

$$Z = \frac{X - \mu}{\sigma}$$

new random variable = standard normal distribution

Z is called standard normal distribution with mean 0 & variance 1.

Computing mean of Z:

Expectation of constant = constant

$$E(Z) = E\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\left[E(x) - E(\mu)\right]$$

$$= \frac{1}{\sigma}(\mu - \mu) = 0 \text{ for this transformation.}$$

Computing variance of Z = $\text{Var}\left(\frac{x-\mu}{\sigma}\right)$

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$= \frac{1}{\sigma^2} Var(x-\mu) = \frac{1}{\sigma}\left(E((x-\mu)^2) - (E(x-\mu))^2\right)$

$= \frac{1}{\sigma^2}\left(E(x^2) + E(\mu^2) - E(2\mu x) - \cancel{E(x-\mu)}\right)$ 0

$= \frac{1}{\sigma^2}\left(E(x^2) - 2\mu^2 + \mu^2\right) = \frac{1}{\sigma^2}\left(E(x^2) - \mu^2\right)$

$2\mu(E(x))$

$= \frac{1}{\sigma^2}(\sigma^2) = 1$ for this transformation

$Mgf = (e^{\mu t + \sigma^2 t^2/2}) = e^{0 + 1 \cdot t^2/2} = e^{t^2/2}$

$m'(t) = e^{t^2/2} \cdot \frac{\cancel{2}t}{\cancel{2}}$ at $0 \Rightarrow 0$

$m''(t) = e^{t^2/2} + t \cdot e^{t^2/2} \Rightarrow$ At $t=0$, $1 + 0$

$\sigma^2 = 1 - 0^2 = 1$

Ref Table

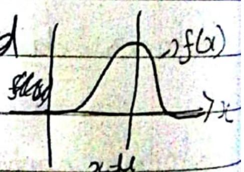
Distribution (x)	Mean	Var	Mgf	Parameters
Uniform disc	$\frac{n+1}{2}$	$\frac{n^2-1}{12}$	$\frac{1}{n}\left(\frac{e^t(1-e^{nt})}{1-e^t}\right)$	n
Uniform continuous	$a+b/2$	$\frac{(a-b)^2}{12}$	$\frac{1}{t(b-a)}(e^{bt}-e^{at})$	a, b
Binomial	np	npq	$(q+pe^t)^n$	(n, p)
Poisson	m	m	$e^{m(e^t-1)}$	m
Exponential	$1/\lambda$	$1/\lambda^2$	$\lambda/\lambda-t$ for $\lambda > t$	λ
Normal	μ	σ^2	$e^{\mu t + \frac{\sigma^2 t^2}{2}}$	μ, σ, t
Standard Normal	0	1	$e^{t^2/2}$	.

	Pmf / Pdf	
UD	$\frac{1}{n}$ $x=1 \dots n$, 0 otherwise	Sad
UC	$\frac{1}{b-a}$ $a \le x \le b$, 0 otherwise	
B	$p(x) = nC_x\, p^x q^{n-x}$ $x = 0, 1, 2 \dots n$	
P	$p(x) = \frac{e^{-m} \cdot m^x}{x!}$ $x = 0, 1, 2 \dots n$	
E	$\lambda e^{-\lambda x}$ $x \ge 0, \lambda > 0$, 0 for $x < 0$	σ^2
N	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ $-\infty < x < \infty$	
SN	$z = \frac{x-\mu}{\sigma}$	

28/12/18 Properties of normal distribution

Normal distribution: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

1) Curve of $f(x)$ is bell-shaped



2) $f(x)$ is symmetric about the $x = \mu$ line

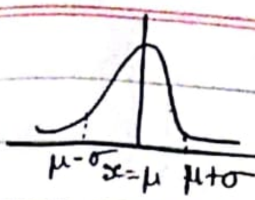
3) Maximum probability $= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(0)} = \frac{1}{\sigma\sqrt{2\pi}}$

4) Limiting case of Binomial is normal, provided p, q are not small.

5) $P(a < x < b) = \int_a^b f(x)\,dx$

substitute above $f(x)$

0.8.



when $x=\mu$, $z=0$ 0.8413

$P(\mu-\sigma < x < \mu+\sigma)$

$= P(-\sigma < x-\mu < +\sigma)$

$= P\left(-1 < \frac{x-\mu}{\sigma} < 1\right)$

$= P(-1 < Z < 1)$

$\int_{-1}^{1} \phi(z)\,dz$ where $\phi(z)$ is the pdf of standard normal distribution

$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz = 1$

Now, $= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{z^2}{2}}\,dz = 0.9534$

From ~~table~~, table you get 1.6826 $= 2\int_{0}^{1} \phi(x)\,dx$

$0.4767 \times 2 = 0.9534$

$= 2 \times 0.3413$

$= \cancel{0.9534}\ 0.682$