18/3/19 ERGODIC RANDOM PROCESS

Let $X(t)$ be a random process and $x(t)$ be a sample.
Time average of the function of $x(t)$ is defined by:

$$\bar{x} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

In real life, it is not possible to know all elements of r.p. We need to use the few that we have available, to analyse behaviour of process.

We can use time average to determine statistical average $E[X(t)]$

Statistical average

$$E[X(t)] = \begin{cases} \sum_{-\infty}^{\infty} x\, p(x,t) \\ \int_{-\infty}^{\infty} x\, f(x,t)\,dx \end{cases}$$

Definition: A stationary process $X(t)$ is ergodic if

$$E[X(t)] = \bar{x}$$

$E[X(t)]$ is also called the ensemble average.

Mean Ergodic Theorem: $\bar{x}_T$ = average in a particular time interval

If $X(t)$ is a stationary random process and $\bar{x}_T = \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$, then $X(t)$ is said to be mean ergodic if

$$\lim_{T \to \infty} V(\bar{X}_T) = 0$$

①

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Example:

$X(t) = A \cos(\omega t + \phi)$
ω is constant. A is a random variable which has a magnitude of +1, -1 with equal probabilities.
ϕ is a random variable uniformly distributed over $(0, 2\pi)$
A and ϕ are independent.

i) Check if it is a mean ergodic process

Stationary process check: (E independent of t
R_{xx} dependent on difference t_1, t_2)

a) $E[x(t)] = E[A \cos(\omega t + \phi)]$

$= E[A] \cdot E[\cos(\omega t + \phi)]$
$= \underbrace{\left(1 \times \frac{1}{2} + (-1)\frac{1}{2}\right)}_{\to 0} \cdot E[\cos(\omega t + \phi)]$
$= 0$

Expectation is thus independent of t.

Computing the autocorrelation function,

$$R_{xx}(t, t+\tau) = E[x(t)\,x(t+\tau)]$$

$= E[A\cos(\omega t + \phi) \cdot A\cos(\omega(t+\tau) + \phi)]$

$= E[A^2] \cdot E\left[\frac{1}{2}\left(\cos(\cancel{\omega t} + \cancel{\phi} - \cancel{\omega t} - \omega\tau - \cancel{\phi}) + \cos(2(\omega t + \phi) + \omega\tau)\right)\right]$

$= \frac{1}{2}E[A^2] \cdot E[\cos(-\omega\tau) + \cos(2(\omega t + \phi) + \omega\tau)]$

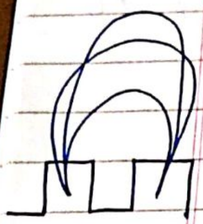
$$= \frac{1}{2}\left[1\times\frac{1}{2}+1\times\frac{1}{2}\right]\left(\cos\omega t + 0\right) = \frac{1}{2}\cos\omega t$$

Second term there is 0 because $E[\cos(2(\omega t+\phi)+\omega\tau)]$

$$= \int_0^{2\pi} \cos(2(\omega t+\phi)+\omega\tau)\times\frac{1}{2\pi}\,d\phi$$ (uniformly distributed)

↳ uniform distribution pdf

$$= \frac{1}{2\pi}\,\frac{\sin[2(\omega t+\phi)+\omega\tau]}{2}$$



i.e. let $2\omega t + 2\phi + \omega\tau = u$

$du = 2\,d\phi$

$$\text{Integral} = \frac{1}{2\pi\times 2}\int_{2\omega t+\omega\tau}^{2\omega t+4\pi+\omega\tau} \cos u\,du = \frac{1}{2\pi\times 2}\left[\sin u\right]_{2\omega t+\omega\tau}^{2\omega t+4\pi+\omega\tau}$$

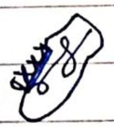
Elefontz

Hyppoe

$$= \frac{1}{4\pi}\left[\sin(2\omega t+4\pi+\omega\tau) - \sin(2\omega t+\omega\tau)\right]$$

$\equiv 2\omega t+\omega\tau$ (periodic)

↳ $\sin(2n\pi+\theta) = \sin\theta$



$$= \frac{1}{4\pi}(0) = 0$$

$\Rightarrow R_{xx}$ is dependent on ωt

Now to check if it is a mean ergodic process

$$\bar{x} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

Zephyr

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\cos(\omega t+\phi)\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\times A\left[\frac{\sin(\omega t+\phi)}{\omega}\right]_{-T}^{T}$$

$$= \lim_{T\to\infty}\frac{A}{2T}\left(\frac{\sin(\omega T+\phi) - \sin(-\omega T+\phi)}{\omega}\right)$$

$\therefore \lim_{T \to \infty} \frac{A}{2\omega T}$ (since $|\sin(\omega T+\phi) - \sin(\omega T+\phi)| \le 2$)

$= 0$

$\therefore E[X(t)] = \bar{x} = 0$ $X(t)$ is mean ergodic

2) A binary transmission process $X(t)$ has zero mean and autocorrelation function $R(\tau) = 1 - \frac{|\tau|}{T}$. Find the mean and variance of the process $X(t)$ over $(0,T)$ and verify whether it is mean ergodic.

$$\text{Note: } V(\bar{X}_T) = \frac{1}{2T}\int_{-T}^{T} \underbrace{C(\tau)}_{\text{autocovariance}} \left(1 - \frac{|\tau|}{2T}\right) d\tau$$

even function

(or)

$$V(\bar{X}_T) = \frac{1}{T}\int_{0}^{T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right) d\tau$$

9/3/19 Note: Let $\bar{X}_T = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$

$$E[\bar{X}_T] = \frac{1}{2T}\int_{-T}^{T} E[x(t)]\,dt = E[x(t)]$$

(independent of t)

$$E[\bar{X}^2] = E\left[\left(\frac{1}{2T}\int_{-T}^{T} x(t_1)\,dt_1\right)\left(\frac{1}{2T}\int_{-T}^{T} x(t_2)\,dt_2\right)\right]$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E[x(t_1)\,x(t_2)]\,dt_1\,dt_2 \quad \text{— (1)}$$

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Now, computing $(E[\bar{X}(t)])^2$, $\hookrightarrow (E[\bar{X}_T])^2$

$= E[\bar{X}_T]E[\bar{X}_T]$

$= \frac{1}{2T}\int_{-T}^{T} E[X(t_1)]\,dt_1 \times$

$\frac{1}{2T}\int_{-T}^{T} E[X(t_2)]\,dt_2$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E[x(t_1)]\,E[x(t_2)]\,dt_1\,dt_2 \quad -(2)$$

If we perform (1) - (2), we get

$$Var(\bar{X}_T) = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}\left(E[x(t_1)x(t_2)] - E[x(t_1)]E[x(t_2)]\right)dt_1\,dt_2$$

$$\Rightarrow Var(\bar{X}_T) = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)\,dt_1\,dt_2$$

$\hookrightarrow$ autocovariance ~~autocovariance~~

$$= \frac{1}{4T^2}\int_{-2T}^{2T} C(\tau)(2T - |\tau|)\,d\tau$$

$\hookrightarrow$ this is done by Jacobian transformation.

$$= \frac{1}{2T}\int_{-2T}^{2T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau$$

(X) $\left\{ Var(\bar{X}_T) = \frac{1}{T}\int_{0}^{2T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau \right.$

Solution for p...

Given, $R(\tau) = 1 - \frac{|\tau|}{T}$ $E[x(t)] = 0$

Cross covariance $C(\tau) = R(\tau) - \underbrace{E[x(t)]\,E[x(t+\tau)]}$

$$Var(\bar{X_T}) = \frac{1}{T}\int_0^{2T}\left(1 - \frac{|\tau|}{T}\right)\left(1 - \frac{|\tau|}{T}\right)d\tau$$

$$= \frac{1}{T}\int_0^{2T}\left(1 - \frac{|\tau|}{T}\right)^2 d\tau$$

$$= \frac{1}{T}\left[\frac{\left(1 - \frac{\tau}{T}\right)^3}{3 \times -1/T}\right]_0^{2T}$$

Since $\tau \geq 0$, $|\tau| = \tau$

Cancelling $1/T$,

$$= \frac{\left(1 - \frac{2\cancel{T}}{\cancel{T}}\right)^3 - (1)}{-3} = \frac{(1-2)^3 - 1}{-3}$$

$$= \frac{-1-1}{-3} = \frac{2}{3}$$

$$\int (ax+b)^m dx = \frac{(ax+b)^{m+1}}{a(m+1)}$$

Since $Var(\bar{X_T})$ does not approaches 0 at $T \to \infty$

Given, $X(t)$ is not mean ergodic.

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when mean=0, autocorrelation = autocovariance.

(7)

3. If $X(t)$ is a zero mean WSS with $R_{xx}(\tau) = e^{-2|\tau|}$, show that $X(t)$ is mean ergodic.

Soln: $Var(\bar{X_T}) = \frac{1}{2T}\left(1 + \frac{e^{-4T}}{4T} - \frac{1}{4T}\right)$

$Var(\bar{X_T}) =$

More problems: (on mean ergodicity)

(1) $X(t) = a \sin(\omega t + \theta)$
a is constant
θ is uniformly distributed on $(0, 2\pi)$
Check if $X(t)$ is mean ergodic.

(2) $X(t)$ has mean 0.
$R(\tau) = e^{-2\lambda \tau}$

PINSELOG ON

20/3/19 Power Spectral Density: (PSD)
↳ characterization based on frequency domain

Not in syllabus

If you have a function of time $f(t)$, then Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

But, in this case, we have the Fourier transform.

Inversion theorem: If we know Laplace transform, $\int_0^\infty \mathcal{L}\{f(t)\} e^{st} ds$ gives us $f(t)$.

If PSD uses the Fourier transform, which is:

For a function $f(t)$, which could be a signal,

then

$$F\{f(t)\}(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t}\, dt = \hat{f}(\omega)$$

omega

Here, the inversion theorem is:

If we have a Fourier transform $\hat{f}(\omega)$, then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$$

PSD is a Fourier transform of a some specific function.

autocorrelation function, to be specific

when $\int_{-\infty}^{\infty} |g(t)| < \infty$, is integrable,

then we can get a Fourier transform.

Now,

Let $X(t)$ be a random process.

Let $R_{XX}(\tau)$ be an autocorrelation function of $X(t)$.

Then PSD of $X(t)$ is defined by

omega

$$F(R_{XX}(\tau))(\omega)$$

$$= S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega t}\, d\tau$$

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(9)

eg. $R_{xx}(\tau) = e^{-a|\tau|}$ $\quad a > 0$

Let us check if PSD exists.

PSD is : $S_{xx}(\omega) = \int_{-\infty}^{\infty} e^{-a|\tau|} e^{-i\omega\tau} \, d\tau$

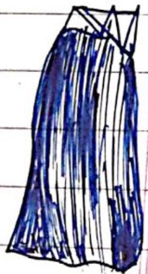
$|\tau|$ is τ for $\tau > 0$

$-\tau$ for $\tau < 0$

Splitting over intervals,

$$S_{xx}(\omega) = \int_{-\infty}^{0} e^{-a \cdot -\tau} e^{-i\omega\tau} \, d\tau + \int_{0}^{\infty} e^{-a\tau} e^{-i\omega\tau} \, d\tau$$

$$= \int_{-\infty}^{0} e^{a\tau - i\omega\tau} \, d\tau + \int_{0}^{\infty} e^{-a\tau - i\omega\tau} \, d\tau$$



$$= \left[\frac{e^{(a-i\omega)\tau}}{a-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-\tau(a+i\omega)}}{-(a+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{a-i\omega} - 0 + 0 - \left(-\frac{1}{a+i\omega}\right)$$

$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{a+i\omega+a-i\omega}{a^2 - i^2\omega^2}$$

$$= \frac{2a}{a^2+\omega^2}$$

$$\begin{array}{r} 10.00 \\ 9.71 \\ \hline 0.29 \end{array}$$

SRISHTI

$x(\tau)$ | $F\{x(\tau)\}$

① $e^{-a|\tau|}$ $a>0$ — $\frac{2a}{a^2+\omega^2}$

② $e^{-a\tau}$ $a>0, \tau>0$ — $1/a+i\omega$ not PSD

③ $e^{b\tau}$ $b>0, \tau<0$ — $1/b-i\omega$ not PSD

④ $\tau e^{-a\tau}$ $a>0, \tau<=0$ — $1/(a+i\omega)^2$ not PSD

⑤ 1 — $2\pi\,\delta(\omega-\omega_0)$

↳ Dirac-Delta function

Note: $\delta(x) = \infty \quad x = 0$
$\qquad\quad\; 0 \quad x \neq 0$

⑥ $\delta(\tau)$ — 1

⑦ $e^{i\omega_0\tau}$ — $2\pi\,\delta(\omega-\omega_0)$

⑧ $\cos(\omega_0\tau)$ — $\pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$

⑨ $\sin(\omega_0\tau)$ — $-i\pi[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$

↳ not PSD

Properties: (PSD)

① $S_{xx}(\omega) \geq 0$ (Wiener - Khintchine theorem)

② $S_{xx}(-\omega) = S_{xx}(\omega)$

③ $S_{xx}(\omega)$ is always real.

④ Average power (or) mean square value

$$E[X^2(t)] = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega$$

$$\because S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i\omega\tau}\,d\tau$$

(11)

By inversion theorem,

$$R_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{+i\omega\tau}\, d\omega$$

$$\text{So, } R_{xx}(0) = E[x^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega$$

(5) $S_{xx}(\omega) = S_{xx}^{*}(\omega)$ (or) $\overline{S_{xx}(\omega)}$

(i.e. if $z = x+iy$, then $\bar{z} = x-iy$)

(6) If $\int_{-\infty}^{\infty} |R_{xx}(\tau)|\, d\tau < \infty$ | eg. $\int_{-\infty}^{\infty} |x|\, dx = \infty$ ↳ not integrable since not finite

then $S_{xx}(\omega)$ is continuous.

Problem:

Determine the autocorrelation function of the random process with PSD given by

$$S_{xx}(\omega) = \begin{cases} S_0 & |\omega| < \omega_0 \\ 0 & \text{otherwise} \end{cases}$$

Applying the formula,

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{+i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} S_0\, e^{i\omega\tau}\, d\omega = \frac{1}{2\pi} \times S_0 \left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{S_0}{2\pi i\tau} \times e^{i\tau(\omega_0+\omega_0)} = \frac{S_0}{2\pi i\tau}\, e^{2\omega_0 i\tau}$$

(or), $\frac{S_0}{2\pi i \tau}\left[e^{-i\omega_0\tau} - e^{-i\omega_0\tau}\right]$

$= \frac{S_0}{2\pi i\tau}\sin(\omega_0\tau)$ ↳ By Euler's formula,

Refer Oliver Ibe's Book for more problems.

Find $E[X^2(t)]$ for $S_{XX}(\omega) = \frac{24}{\omega^2+16}$ Ans: 3

22/3/19 Exercise Problems: 8.31 of Oliver Ibe

① $S_{YY}(\omega) = \frac{9}{\omega^2+64}$ Find a) average power
b) autocorrelation

a) Average power $E[X^2(t)] = R_{XX}(0)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, \underbrace{e^{i\omega t}}_{=1 \text{ at } t=0} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{9}{\omega^2+64}d\omega$$

$$= \frac{9}{2\pi}\times\frac{1}{8}\left(\tan^{-1}\frac{\omega}{8}\right)_{-\infty}^{\infty} = \frac{9}{2\cancel{\pi}}\times\frac{1}{8}\times\cancel{\pi}$$

$$= \frac{9}{16}$$

$\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

(b) If you consult the table, and apply Fourier inverse transform we see that:

$$e^{-a|\tau|} \longrightarrow \frac{2a}{\omega^2+a^2}$$

$$S(\omega) = \frac{9}{\omega^2+8^2}\times\frac{2\times 8}{2\times 8} \Rightarrow \underbrace{\frac{9}{2\times 8}}_{\text{constant}}\times\frac{2\times 8}{\omega^2+8^2}$$

$$= \frac{9}{16}e^{-8|\tau|}$$

$S_{XX}(\omega)\,d\omega$

$\int_0^\infty |x|\,dx = \infty$ ↳ not integrable since not finite (or) use this method:

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{9}{\omega^2+8^2}\, e^{i\omega\tau}\, d\omega$$ should evaluate to the result given.

8.30

$$S_{xx}(\omega) = \begin{cases} 4 - \dfrac{\omega^2}{9} & |\omega| \le 6 \\ 0 & \text{otherwise} \end{cases}$$

We have to use direct method here.

Average power: $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega$

$$= \frac{1}{2\pi}\int_{-6}^{6} 4 - \frac{\omega^2}{9}\, d\omega$$

$$= \frac{1}{2\pi}\left(4\left[\omega\right]_{-6}^{6} - \left(\frac{\omega^3}{3\times 9}\right)_{-6}^{6}\right)$$

$$= \frac{1}{2\pi}\left(4\times 12 - \left(\frac{216 + 216}{27}\right)\right)$$

$$= \frac{1}{2\pi}\left(48 - \frac{432}{27}\right)$$

$$= \frac{1}{2\pi} \times \frac{1296 - 432}{27} = \frac{\cancel{864}^{\,432}}{27} \times \frac{1}{\cancel{2}\pi}$$

$$= \frac{1}{\pi} \times \frac{\cancel{432}^{\,144}}{\cancel{27}\,9} = \frac{144}{9\pi}$$

3
36
6
216

216
216
432

27
48
216
108
1296

1296 −
432
864

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$\int u\,dv = uv - u'v_1 + u''v_2 - \ldots \quad u\,dv = uv - \int v\,du$

↳ Leibnitz Rule

b) $R_{xx}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-6}^{6}\left(4 - \frac{\omega^2}{9}\right) e^{i\omega\tau}\,d\tau$

u   dv

$$= \frac{1}{2\pi}\int_{-6}^{6}\left[\left(4 - \frac{\omega^2}{9}\right)\left(\frac{e^{i\omega\tau}}{i\tau}\right) - \int e^{i\omega\tau} \cdot -\frac{2\omega}{9}\,d\tau\right.$$

$$= \frac{1}{2\pi}\left[\left(4 - \frac{\omega^2}{9}\right)\left(\frac{e^{i\omega\tau}}{i\tau}\right) - \left(-\frac{2\omega}{9}\right)\left(\frac{e^{i\omega\tau}}{-\tau^2}\right)\right.$$

$$\left. + \left(-\frac{2}{9}\right)\left(\frac{e^{\omega\tau}}{-i\tau^3}\right)\right]_{-6}^{6}$$

$\frac{1}{i} = -i$

$$= \frac{1}{2\pi}\left[\frac{e^{i\omega\tau}}{\tau}\left(-i\left(4 - \frac{\omega^2}{9}\right) - \frac{2\omega}{9\tau} - i\frac{2}{9\tau^2}\right]_{-6}^{6}\right.$$

$$= \frac{1}{2\pi}\left[\frac{e^{6i\tau}}{\tau}\left(-i\left(\underbrace{4 - \frac{36}{9}}_{0}\right) - \frac{12}{9\tau} - i\cdot\frac{2}{9\tau^2}\right.\right.$$

$$\left. - \left(\frac{e^{-6i\tau}}{\tau}\left(-i\left(\underbrace{4 - \frac{36}{9}}_{0}\right) + \frac{12}{9\tau} - i\frac{2}{9\tau^2}\right)\right)\right]$$

$$= \frac{1}{2\pi}\left[\cancel{\frac{e^{6i\tau}}{\tau}} - \frac{12}{9\tau} - \frac{12}{9\tau} - \frac{2i}{9\tau^2} + \frac{2i}{9\tau^2}\right]$$

$$= -\frac{24}{9\tau} \times \frac{1}{2\pi} = \frac{-\cancel{12}^{4}}{\cancel{9}3\pi} = -\frac{4}{3\pi\tau}$$

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$\ln 0 = -\infty$

Lavender Border

8.33: Which of these can be power spectral density of a W.S.S process?

a) $S_{XX}(\omega) = \frac{\sin\omega}{\omega}$

c) $S_{XX}(\omega) = \frac{8}{\omega^2+16}$

b) $S_{XX}(\omega) = \frac{\cos\omega}{\omega}$ → odd func.

d) $S_{XX}(\omega) = \frac{5\omega^2}{1+3\omega^2+4\omega^4}$

e) $S_{XX}(\omega) = \frac{5\omega}{1+5\omega^2+4\omega^2}$

$f(\omega)$ is a PSD if $\int_{-\infty}^{\infty} |f(\omega)|\, d\omega$ is finite

For (i), $\int_{-\infty}^{\infty} \frac{|\sin\omega|}{|\omega|} d\omega \le \int_{0}^{\infty} \frac{1}{\omega} d\omega$

$= \ln \omega \Big|_0^{\infty} = \infty$

So (i) cannot be a PSD

Note:

$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$

8.29 Find PSD of $R_{XX}(\tau) = 2e^{-|\tau|} + 4e^{-4|\tau|}$

PSD: $S_{XX}(\omega)$

$= 2F(e^{-|\tau|}) + 4F(e^{-4|\tau|})$

$= 2 \times \frac{1}{1^2+\omega^2} + 4 \times \frac{2}{4^2+\omega^2}$

$= \frac{2}{\omega^2+1} + \frac{8}{\omega^2+16}$

Backyardigans

Definition

Let $R_{XY}(\tau)$ be cross correlation(τ) function

then $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-\omega\tau} d\tau$ is

the cross power spectral density.

Note:

Let $X(n)$ be a discrete time process

Then, PSD of $X(n)$ is defined by

$$S_{XX}(\Omega) = \sum_{m=-\infty}^{\infty} R_{XX}(m) e^{-i\Omega m}$$

$\Omega \in (-\pi, \pi)$

~~summation should become a function of~~ since the function is periodic.

Given $S_{XX}(\Omega)$,

$$R_{XX}(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{XX}(\Omega) e^{i\Omega m} d\Omega$$

Properties (discrete)

① $S_{XX}(\Omega) \geq 0$

② $S_{XX}(-\Omega) = S_{XX}(\Omega)$

③ $S_{XX}(\Omega + 2\pi) = S_{XX}(\Omega)$ → periodic with period π

Problems:

① Find PSD of $R(\tau) = e^{-a\tau^2}$

Ans: $S_{XX}(\omega) = \sqrt{\pi/a}\, e^{-\omega^2/4a}$

② $S_{XX}(\omega) = \begin{cases} 1 & \text{when } |\omega| < a \\ 0 & \text{otherwise} \end{cases}$ Find $R_{XX}(\tau)$

Ans: $\frac{\sin a\tau}{\pi\tau}$

③ If $S_{XX}(\Omega) = \begin{cases} b/a\,(a - |\omega|) & |\omega| \leq a \\ 0 & |\omega| > a \end{cases}$

Show that $R_{XX}(\tau) = \frac{ab}{\pi}\left(\frac{\sin a\tau/2}{a\tau/2}\right)^2$

23/3/19 Markov Process

Example & Definition:

Let (X_n) be a discrete process

A process in which the future values do not depend on the past, but do depend on the current value, can be termed as Markov Process.

Let X represent tossing a coin. We can do a mapping as follows:

$X(\text{head}) = 1$

$X(\text{tail}) = 0$

So, let (X_n) be a process which follows the above X.

X_n can be represented as a binary sequence. You can have multiple observations.

At any n^{th} state/trial, the outcomes are not affected by previous events (n-1th trial, for example) and such. (Independent)

So you will not consider this experiment a Markov process.

Using this process, we can consider n^{th} partial sums:

Let $S_n = X_1 + X_2 + \dots X_n$

where $S_1 = X_1$

$S_2 = X_1 + X_2$ & so on.

S_n is thus a collection of random variables & is hence a random process.

We should show that (S_n) is a Markov Process.

S_1 can be either 0 or 1
S_2 can have a value of max 2 $(0, 1, 2)$
. . . .

$\therefore$ S_n can take values from $0, 1, 2 \ldots n$

We know the range of S_n, but we don't know what its state is.

future, present

So, $\boxed{S_{n+1} = X_{n+1} + S_n}$

$\Rightarrow$ If $S_n = j$, then $0 \leq j \leq n$
then $S_{n+1} = j$ or $j+1$

(S_n) is a Markov process.

$\therefore$ The definition of a Markov Process is:

A random process $\{X(t) \,/\, t \in T\}$ is called a first-order Markov process if

$$P[X(t_n) \leq x_n \mid X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \ldots, X(t_0) = t_0]$$

$$= P[X(t_n) \leq x_n \mid X(t_{n-1}) = x_{n-1}]$$

where $t_0 < t_1 < t_2 \ldots < t_n$

$\hookrightarrow$ conditional cumulative distribution function

i.e. probability value of future state given all past states & present state equals probability value of future state given only the present state.

(19)

If time is discrete, => Discrete-time Markov process
Same for continuous.

Markov Chain:

If the states are discrete variables (i.e. $X(t)$ values) given then this discrete state Markov process is called Markov chain.

Time \ State Space	Discrete	Continuous
Discrete	Discrete time Markov chain	Discrete time Markov Process
Continuous	Continuous time Markov chain	Cont-time Markov Process

Discrete Time Markov Chains

A discrete process
$$\{X_K \mid K = 0, 1, 2, 3 \ldots\}$$
$\hookrightarrow$ K represents time

is called a Markov Chain if

$$P[X_K = j \mid X_{K-1} = i, X_{K-2} = n, \ldots X_0 = m]$$

$$= P[X_K = j \mid X_{K-1} = i]$$

$$= \underbrace{P_{ijK}}_{\text{state values}}$$

If probability is independent of time variable, the process is called homogeneous Markov chain.

Homogeneous Markov Chain:

$$P_{ijK} = P_{ij}$$

P_{ij} = Homogeneous State transition probability

The conditions it satisfies are:

1. $0 \le p_{ij} \le 1$
2. $\sum_{j} p_{ij} = 1 \qquad i = 1, 2, \ldots n$

(adding up elements in a row, transition matrix → matrix form $\begin{bmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix}$

(or)

State transition probability matrix

Problem:

Determine the missing elements denoted by x in the following transition probability matrix.

$$\begin{bmatrix} x & 1/3 & 1/3 & 1/3 \\ 2/5 & x & 1/5 & 2/5 \\ x & x & x & 1 \\ 3/5 & 2/5 & x & x \end{bmatrix}$$

$x = 0$, ~~always~~

$1 - \frac{1}{10} = \frac{9}{10}$

$\frac{3 \times 4\,1}{5 \times 2\,10}$

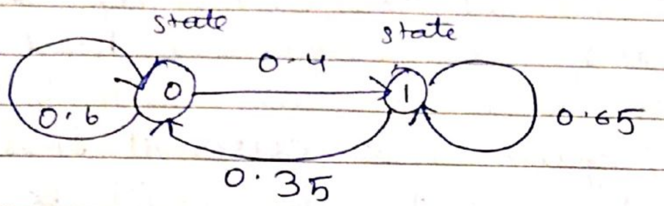
Every Markov ~~process~~ chain is completely specified by the transition matrix.

The transition matrix can be specified by a state transition diagram. (graph)

No. of states = no. of nodes

Let $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix}$

No. of nodes = 2 (2 states exist i.e. in an $n \times n$ matrix there are n states.)



3/19 The n-step state transition probability

Let $X^{(n)}$, $n = 0, 1, 2 \dots$ be a discrete Markov chain

$P_{ij}(n)$ (or) $P_{ij}^{(n)}$ is called the n-step state transition probability.

Note

> the probability when the current state of process, i, transforms into state j after n transitions.

(or)

probability that the system will be in state j after exactly n transitions, given that it is presently in state i.

$$\Rightarrow P_{ij}(n) = Pr\left\{ X_{m+n} = j \,/\, X_m = i \right\}$$

Note:

$$P_{ij}(0) = \begin{cases} P_{ij} & i \neq j \\ 1 & i = j \end{cases}$$

→ sure event (state also doesn't change)

$P_{ij}(1) = P_{ij}$

Note: $P_{ij} = Pr\{X_k = j / X_{k-1} = i\} = P_{ij}$

Consider $P_{ij}(2)$ (or) the 2-step transition probability.

Let $m = 0$

$$P_{ij}(2) = Pr\{X_2 = j / X_0 = i\}$$

$$= \sum_k P_r[X_2 = j, X_1 = k / X_0 = i]$$

$$= \sum_k Pr[X_2 = j / X_1 = k, X_0 = i] * Pr[X_1 = k / X_0 = i] \quad \text{— (1)}$$

$$\because \frac{P[X_2 = j, X_1 = k, X_0 = i]}{\cancel{P[X_1 = k, X_0 = i]}} * \frac{\cancel{P[X_1 = k, X_0 = i]}}{P[X_0 = i]}$$

$$= \frac{P[X_2 = j, X_1 = k, X_0 = i]}{P[X_0 = i]}$$

$$= P[X_2 = j, X_1 = k / X_0 = i]$$

By Markov property, X_2 (future state) depends only on present state X_1

$$\text{(1)} = \sum_k P[X_2 = j / X_1 = k]\, P[X_1 = k / X_0 = i]$$

$$= \sum_k P_{kj} P_{ik}$$

$$= \sum_k P_{ik} P_{kj}$$

Graph

∴ The equation known as Chapman-Kolmogorov equation:

Carousel Ferris-wheel

For all $0 < r < n$,

$$P_{ij}(n) = \sum_{k} P_{ik}(r) \, P_{kj}(n-r)$$

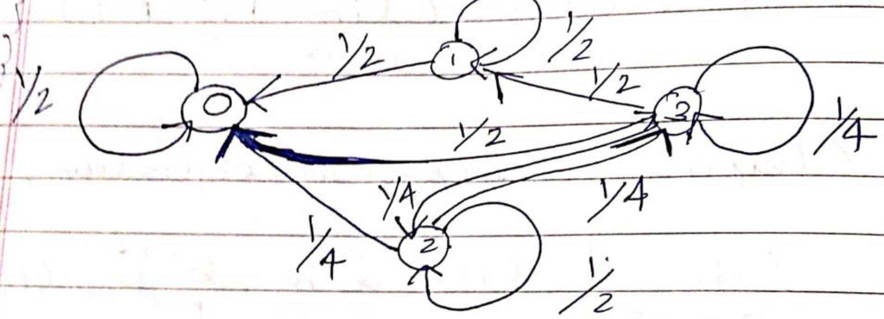
matrix product thing

Problems:

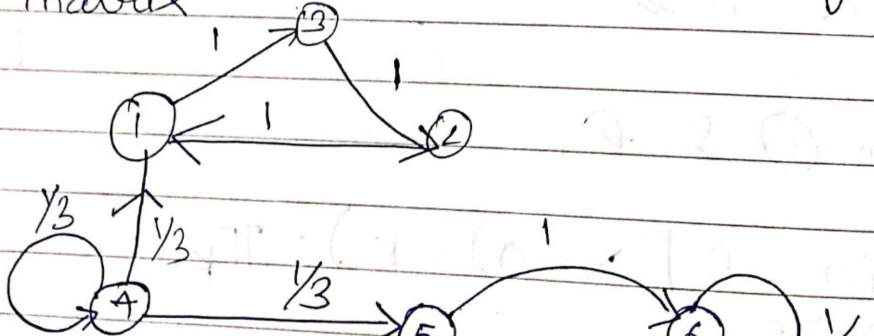
① Draw the state transition diagram.

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$$
$$\begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \end{bmatrix}$$

Aerial Blue



② Compute the transition probability matrix



$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{array}$$

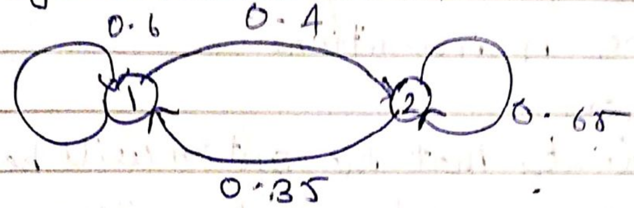
(5) Consider tosses of a biased coin that the outcome of next toss depends on the outcome of current toss.
Given that current toss comes up heads, the next toss will come up head with probability 0.6 & tails, with 0.4.
Similarly, given that current toss comes up tails, the next toss will be heads w/ probability 0.35 & tails with 0.65

i) Compute transition probability matrix

$P = [p_{ij}]$

$$= \begin{array}{c c} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix} \end{array}$$

ii) Diagram



iii) Compute n-step transition probability

$p_{12}(3)$

$P[1 \to 1 \to 1 \to 2] + P[1 \to 1 \to 2 \to 2]$

$+ P[1 \to 2 \to 1 \to 2] + P[1 \to 2 \to 2 \to 2]$

$= \cancel{0.6 \times 0.4 \times 0.65}\ 0.6 \times 0.6 \times 0.4$
$+ 0.6 \times 0.4 \times 0.65 + 0.4 \times 0.35 \times 0.4$
$+ 0.4 \times 0.65 \times 0.65$

$= 0.525$

(or) multiply matrix n times & find i^{th}, j^{th} entry

$$\begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix}\begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.4375 & 0.5625 \end{bmatrix}\begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix}$$

$$P_{12} = 0.525 \qquad \begin{bmatrix} 0.475 & 0.525 \\ 0.459375 & 0.540625 \end{bmatrix}$$

26/3/19 Limiting state probabilities (or) steady state probabilities

Let $X_0, X_1, \ldots X_N$ be N-state random processes with transition matrix

$$P = (p_{ij})_{N\times N}$$

Now, $P^n = (p_{ij}(n))_{N\times N}$

If $n \to \infty$, limit of this matrix (at every entry, find limit) becomes constant.

– Proof: Let $P[X(0)=i]$ denote probability that process is in state i, before it makes first transition.

multilevel feedback

$$\therefore \sum_i P[X(0)=i] = 1$$

For a single random variable itself, all the states/results sum up to 1.

Let $P[X(n)=j]$ denote state probability i.e. At the end of the first n transitions, probability that process is in state j.

Then $$P[X(n)=j] = \sum_{i=1}^{N} P[X(0)=i] \times P_{ij}(n) \quad \text{— (1)}$$

To find

$$\lim_{n\to\infty} P[X(n)=j] = \pi_j$$ (assumed) value — ② ; where j varies from 1 to N possible states

$$\pi_j = \lim_{n\to\infty} P[X(n)=j]$$

$$\Rightarrow = \lim_{n\to\infty} \sum_{i=1}^{N} P[X(0)=i]\, P_{ij}(n)$$

Summation is independent of n, so we can interchange Σ & limit.

$$\sum_{i=1}^{N} \lim_{n\to\infty} P[X(0)=i]\, P_{ij}(n)$$

$$= \sum_{i=1}^{N} P[X(0)=i] \left(\lim_{n\to\infty} P_{ij}(n)\right)$$

By Chapman-Kolmogorov equation,

$$\lim_{n\to\infty} P_{ij}(n) = \lim_{n\to\infty} \sum_{K} [P_{iK}(n-1)]\, P_{Kj}$$

$$= \sum_{K} \lim_{n\to\infty} \underbrace{[P_{iK}(n-1)]}_{X(n)=K} P_{Kj} = \sum_{K} \pi_K P_{Kj}$$ — ③

From ① & ②,

$$\lim_{n\to\infty} P[X(n)=K) = \pi_K$$

$\Rightarrow$ ~~The equation~~ ~~...~~ $\pi_K = \lim_{n\to\infty} \sum_{i=1}^{n}$ ~~$\sum_K$~~ $P[X(0)=\cancel{K}]\, P_{\cancel{K}iK}(n)$

$$= \sum_{i\cancel{K}=1}^{n} P[X(0)=\cancel{K}] \times \lim_{n\to\infty} P_{\cancel{K}iK}(n)$$

27

Assuming the limit independent of K,

$$= \lim_{n\to\infty} P_{ij}(n) \times 1$$ summation

provided, $\lim_{n} P_{ik}(n-1)$

independent of initial condition $P[X(0)=i]$

From (3),

$$\pi_j = \sum_{K} \pi_K P_{Kj} \qquad j=1,2,\dots N$$

and $\sum \pi_j = 1$

The π-matrix thus formed can be evaluated as follows:

$$[\pi_1 \ \pi_2 \dots \pi_N] = [\pi_1 \ \pi_2 \dots \pi_N] \times \begin{bmatrix} P_{11} & \dots & P_{1N} \\ \vdots & \ddots & \vdots \\ P_{N1} & \dots & P_{NN} \end{bmatrix}$$

matrix multiplication

and $\pi_1 + \pi_2 \dots + \pi_N = 1$

Then $[\pi_1 \dots \pi_N]$ is called steady state probability.

Problem example:

Consider the biased coin problem.

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix}$$

Here, To find limiting probability.

Here, there are 2 states

$$[\pi_1 , \pi_2] = [\pi_1 \quad \pi_2] \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix}$$

and $\pi_1 + \pi_2 = 1$

$\pi_1 = 0.6\pi_1 + 0.35\pi_2$
$\pi_2 = 0.4\pi_1 + 0.65\pi_2$

Solving, we use this rule (Rule of Thumb)
take only first N-1 equations

$\pi_1 = 0.6\pi_1 + 0.35\pi_2$ (or) $0.4\pi_1 - 0.35\pi_2 = 0$

and $\underbrace{\pi_1 + \pi_2 = 1}_{\text{this one necessary}}$

$\cancel{0.6\pi_1 + 0.35\pi_2} +$ $\pi_1 = 0.46$
$\pi_2 = 0.53$

$[0.46 \quad 0.53]$ is the steady state probability.

Note:
From derivation,

$$\pi_j = \lim_{n \to \infty} p_{ij}(n) = \lim_{n \to \infty} P^n \quad j = 1 \ldots N$$

when $P = (p_{ij})$

$$\text{(or)} \quad \pi_j = \lim_{n \to \infty} (P[X(n) = j])$$

state probability

Note 2:

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.35 & 0.65 \end{bmatrix}$$

P^2 gives some value
P^3 and so on.

$$P^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.4375 & 0.5625 \end{bmatrix}$$

$$= \begin{bmatrix} \pi_1 & \pi_1 \\ \pi_2 & \pi_2 \end{bmatrix}$$

in limiting behaviour
when $n \to \infty$

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Problem:

① $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$

Obtain

i) $\lim_{n \to \infty} P^n$

ii) steady state probability distribution

② Initial conditions:

$P[X(0) = 1] = 1/2$

$P[X(0) = 2] = 1/2$

Ans: $[\pi_1 \ \ \pi_2] = [2/3 \ \ 1/3]$

missed notes

Stationary Processes

Stationary Processes are of 2 kinds
① SSS - Strict Sense Stationary
② WSS - Wide Sense Stationary

Let $\{X(t) \mid t \in (0,T)\}$ be a random process

$X(t) = X(t+c) \quad \forall c$
$F_X(x,t) = F_X(x,t+c) \ \forall c$
So $F_X(x_1, x_2, t_1, t_2) = F_X(x_1, x_2, t_1+c, t_2+c)$

For strict sense,

$f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1+c, t_2+c)$

and for wide sense,
$f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1-t_2, 0)$

$\hookrightarrow n=2$ where $c = -t_2$

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Strict Sense Stationary Process

A random process $X(t)$ is called SSS if its statistical properties are invariant with shift of origin.

i.e. $F_X(x_1, x_2, \ldots x_n, t_1, t_2 \ldots t_n)$
$= F_X(x_1, x_2 \ldots x_n, t_1+c, t_2+c \ldots t_n+c)$ $\forall c$

$f = \frac{\delta^n F_X}{\delta x_i^n}$ $\Rightarrow f(x_1, x_2, \ldots x_n, t_1+c, t_2+c \ldots t_n+c)$
$= f(x_1, x_2 \ldots x_n, t_1, t_2 \ldots t_n)$

Refer prev page for n=2 case. $\forall c$ & for all n.

Thus, $\mu_X(t) = \mu_X(0)$ $\forall t$, where $\mu_X(t)$ is a constant.

$R_{XX}(t, t+\tau) = R(0, \tau)$ $\tau = t_1 - t_2$
$C_{XX}(t, t+\tau) = C_{XX}(0, \tau)$ $\tau = t_1 - t_2$

Wide Sense Stationary Process

A random process $X(t)$ is called WSS if its
i) mean is constant
ii) Autocorrelation depends only on $t_1 - t_2$ (let's say τ)

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Eg:
A random process $X(t)$ is defined by
$X(t) = A\cos t + B\sin t$ $\quad -\infty < t < \infty$

A & B are independent random variables, each of which has a value -2 with probability $1/3$ and a value 1 with probability $2/3$. Show that $X(t)$ is WSS but not SSS.

Given $X(t) = A\cos t + B\sin t$

x	-2	1	same for
$P(x)$ $=A$	$1/3$	$2/3$	$x = B$

$$E[x(t)] = E[A\cos t + B\sin t]$$
$$= E[A\cos t] + E[B\sin t]$$
$$= -2\times\frac{1}{3} + 1\times\frac{2}{3} + -2\times\frac{1}{3} + 1\times\frac{2}{3}$$
$$= 0 + 0 = 0$$

$$R_{xx}(t,s) = E[X(t)X(s)]$$
$$= E[(A\cos t + B\sin t)(A\cos s + B\sin s)]$$
$$= E[A^2\cos t\cos s + AB\cos t\sin s + AB\cos s\sin t + B^2\sin t\sin s]$$

$$= E[A^2\cos t\cos s] + E[\underbrace{AB\,\sin(t+s)}_{\text{turns to 0}}] + E[B^2\sin t\sin s]$$

because A & B are independent

$$E[A^2] = \Sigma x^2 p(x)$$
$$= (-2)^2\times\frac{1}{3} + 1^2\times\frac{2}{3} = \frac{4}{3} + \frac{2}{3} = \frac{6}{3} = 2$$
$$= E[B^2]$$

$\therefore$ $2\cos t\cos s + 2\sin t\sin s = 2\cos(t-s)$

$E[X^2(t)] = E[A^2\cos^2 t + B^2\sin^2 t + 2AB\cos t \sin t]$

$= 2\cos^2 t + 2\sin^2 t + \underbrace{2AB\cos t \sin t}_{0}$

$= 2$

$E[X^3(t)] = E[(A\cos t + B\sin t)^3]$

$= E[A^3\cos^3 t + B^3\sin^3 t + 3A^2B\cos t\sin t + 3AB^2\cos t\sin t]$

$E[A^3] = E[B^3] = (-2)^3 \times \frac{1}{3} + 1 \times \frac{2}{3}$

$= -\frac{8}{3} + \frac{2}{3} = -\frac{6}{3} = -2$

$E[X^3(t)] = -2(\cos^3 t + \sin^3 t)$

Not constant

$\Rightarrow X(t)$ is not SSS

Properties of autocorrelation function:

1. ~~Stellisca~~ For a WSS,
$R_{XX}(\tau) = R_{XX}(0, t-s) \quad \tau = t-s$
$= R_{XX}(t, t+\tau)$

2. $R_{XX}(\tau)$ is even always
i.e. $R_{XX}(\tau) = R_{XX}(-\tau)$

3. $R_{XX}(0) = E[X^2(t)]$ or average power or mean square value

4. $|R_{XX}(\tau)| \leq R_{XX}(0)$

5. $\mu_X^2(\tau) = \lim_{|\tau| \to \infty} R_{XX}(\tau)$

Page

Proofs:

② $R_{XX}(-\tau) = R_{XX}(0, -\tau) = R_{XX}(t, t-\tau)$

$= E[X(t)X(t-\tau)]$ where you say $t = t+\tau$ by property

$= E[X(t+\tau)X(t+\tau-\tau)]$

$= R_{XX}(t+\tau, t)$

$= R_{XX}(\tau)$

∴ even function

④ By Cauchy-Schwartz inequality,

$$\int fg \leq \int f \int g$$

$$|\langle f, g\rangle| \leq \|f\| \, \|g\|$$

$$\{E[XY]\}^2 \leq E[X^2]\,E[Y^2]$$

$$\{E[X(t)X(t+\tau)]\}^2 \leq E[X^2(t)]\,E[X^2(t+\tau)]$$

$$\leq E[X^2(t)]\,E[X^2(t)]$$

$$\leq (E[X^2(t)])^2 = [R_{XX}(0)]^2$$

$$\Rightarrow |R_{XX}(\tau)| \leq R_{XX}(0)$$

Compute variance of process $X(t)$ whose autocorrelation function is

$$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

$$\mu_X^2(t) = \lim_{\tau \to \infty} R_{XX}(\tau)$$

$$= 25$$

$E[x^2(t)] = R_{xx}(0) = 25+4 = 29$

$Var(x(t)) = 29 - 25 = 4$

Let $x(t)$ be a random process with

$$R_{xx}(\tau) = \frac{4\tau^2+6}{\tau^2+1}$$

Find mean square value, mean and variance

$$\mu_x^2(t) = \lim_{\tau \to \infty} \frac{4 + 6/\tau^2}{1 + 1/\tau^2} = \frac{4+0}{1+0} = 4$$

$\boxed{\mu_x = 2}$

$E[x^2(t)] = R_{xx}(0)$

$= 6$ = Average power

Variance $= 6 - \cancel{4} = 2$

Can the function $R_{xx}(\tau) = \frac{1+\tau^4}{1+\tau^6}$ serve as a valid autocorrelation function for a WSS process? Justify.

$|R_{xx}(\tau)| \le R_{xx}(0)$

$\hookrightarrow = 1$

$$\frac{1+\tau^4}{1+\tau^6} \le 1$$

$1+\tau^4 \le 1+\tau^6 \qquad \tau^4 \le \tau^6 \qquad \tau > 1$

Now, when $\tau = 1/2$

$$\frac{1+(1/2)^4}{1+(1/2)^6} = \frac{\frac{16+1}{16}}{\frac{64+1}{64}} = \frac{17}{65} \cdot 4 = 1.046 > 1$$

which shouldn't happen since $\tau > 1$ has been found.

$\Rightarrow R_{xx}(\tau)$ is not WSS

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Compute mean, variance of $X(t)$ whose autocorrelation function is

$$R_{XX}(\tau) = \frac{25\tau^2+36}{625\tau^2+4}$$

$$\mu_X^2(t) = \lim_{\tau \to \infty} R_{XX}(\tau)$$

$$= \frac{25+\frac{36}{\tau^2}}{625+4/\tau^2} = \frac{25}{625} = \frac{1}{25}$$

$$\mu_X = 1/5$$

$$E[X^2(t)] = R_{XX}(0) = \frac{36}{4} = 9$$

$$\text{Var} = 9 - \frac{1}{25} = \frac{225-1}{25} = \frac{224}{25}$$