27/11/18

PROBABILITY.

Random experiment:

An experiment that can result in different outcomes even though it is repeated in the same manner.

Sample Space:

Set of all possible outcomes.

Ex: When a coin is tossed 2 times:

$S = \{HH, HT, TH, TT\}$

Sample space can be discrete or continuous.

Discrete: finite or countable infinite set of outcomes

Continuous: interval (finite or infinite) of real nos.

Discrete ex: $\{1, 2, 3, 4, 5, 6\}$ (Dice tossed).

Continuous : $\{x : 1.0 \leq x \leq 20 \ \forall x \in \mathbb{R}\}$

Event:

Subset of sample space in an experiment.

Ex: When a coin is tossed twice

$S = \{HH, HT, TH, TT\}$

∴ Let the event E be, there is exactly one head.

$E = \{HT, TH\}$

Mutually Exclusive:

Two events are defined to be mutually exclusive if their intersection is null. i.e. they have no common outputs.

i.e. $E_1 \cap E_2 = \phi$

$\cup$ - union $A \cup B = B \cup A$

$\cap$ - intersection $A \cap B = B \cap A$

$-$ - difference $A - B \neq B - A$ also denoted as $\backslash$ $(A \backslash B \neq B \backslash A)$.

$\rightarrow (E')' = E$

$\rightarrow$ Distributive law: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ & $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\rightarrow$ De-Morgan law: $(A \cup B)' = A' \cap B'$ & $(A \cap B)' = A' \cup B'$

$\rightarrow$ Commutative law: $A \cup B = B \cup A$ & $A \cap B = B \cap A$

Probability:

defined in 3 ways:

(i) axiomatic defn

(ii) relative-frequency defn.

(iii) classical defn

- $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A) + \ldots$ where $A_1, A_2, A_3 \ldots$ are mutually exclusive.

Relative-frequency defn:

A random experiment performed n times, event A occurs n_A times,

$$\Rightarrow P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$

n_A/n - relative frequency of A.

Classical definition:

$$P(A) = \frac{\text{no. of fav. outcomes}}{\text{total no. of outcomes}} = \frac{N_A}{N}$$

(i) Axiomatic:

P(sure event) = 1

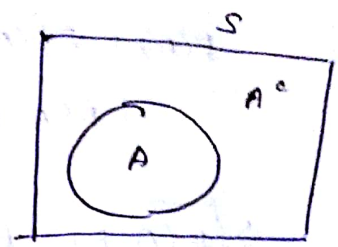
P(something not happening) = 0

(i) $0 \leq P(A) \leq 1$

(ii) $P(S) = 1$

(iii) Mutually exclusive

Some properties of probability



1. $P(A') = 1 - P(A)$

Proof:

A & A' is mutually exclusive. $\boxed{A' \text{ or } A^c}$

$S = A \cup A^c$

$\Rightarrow P(S) = P(A \cup A')$

$= P(A) + P(A')$ [$\because$ A & A' are mutually exclusive]

$1 = P(A) + P(A')$

$\Rightarrow \boxed{P(A') = 1 - P(A)}$

Hence proved.

2. $P(\phi) = 0$

ϕ: impossible event.
(Ex: 7 in rolling a dice)

Proof:

$\phi = S'$

$P(A) = P(S')$

$= 1 - P(S)$ [$\because P(A') = 1 - P(A)$]

$= 1 - 1$ [$\because P(S) = 1$]

$= 0$//.

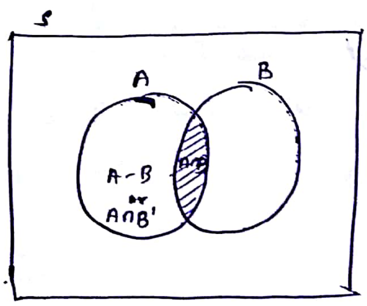
Hence proved//.

3. $P(A-B) = P(A) - P(A \cap B)$

(or)

$P(A) = \cancel{P(A-B)}\ P(A \cap B) + P(A \cap B')$

Proof



A-B or A∩B' and A∩B are mutually exclusive

$$A = (A-B) \cup (A \cap B)$$

$$P(A) = P[(A-B) \cup (A \cap B)]$$

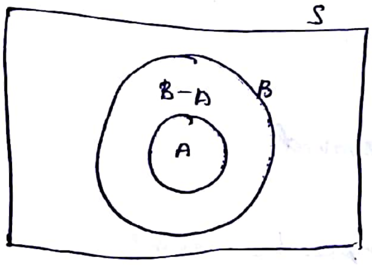
$$= P(A-B) + P(A \cap B)$$

$$\Rightarrow \boxed{P(A-B) = P(A) - P(A \cap B)}$$

Hence proved//

4. If $A \subset B$ then $P(A) \leq P(B)$

Proof:



$$B = A \cup (B-A)$$

$$P(B) = P[A \cup (B-A)]$$

$$= P(A) + P(B-A) \Rightarrow P(A) = P(B) - P(B-A)$$

If $P(B-A) = 0$,

$P(B) = P(A)$

Since $P(B-A) \geq 0$, we have

$P(A) \leq P(B)$

else if $P(B-A) > 0$,

$P(B) > P(A)$

$\Rightarrow P(B) \geq P(A)$

or

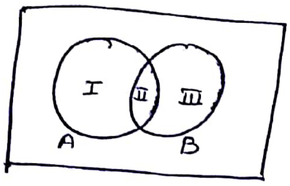
$P(A) \leq P(B)$

Hence proved.

5. For any 2 events, A & B

$$P(A\cup B) = P(A)+P(B) - P(A\cap B)$$

Proof:



I : $A-B$

II : $A\cap B$

III : $B-A$

I, II & III are mutually exclusive.

$$A\cup B = (A-B)\cup(A\cap B)\cup(B-A)$$

$$\begin{aligned} P(A\cup B) &= P(A-B)+P(A\cap B)+P(B-A) \\ &= P(A)-P(A\cap B)+P(A\cap B)+P(B)-P(B\cap A) \\ &= P(A)+P(B)-P(B\cap A) \\ &= P(A)+P(B)-P(A\cap B) \end{aligned}$$

Hence proved.

$A = I\cup II$

$P(A) = P(I) + P(II)$

$P(I) = P(A) - P(II)$ —— (1)

$B = II\cup III$

$P(B) = P(II)+P(III)$

$\Rightarrow P(III) = P(B) - P(II)$ —— (2)

$P(A\cup B) = P(I)+P(II)+P(III)$ —— (3)

Subs (1), (2) in (3),

$$\begin{aligned} P(A\cup B) &= P(A)-P(II)+P(II)+P(B)-P(II) \\ &= P(A)+P(B)-P(II) \\ &= P(A)+P(B)-P(A\cap B) \end{aligned}$$ //.

NOTE: If A, B & C are three events in S then,

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A) + P(A\cap B\cap C)$$

6. $P[A\cup(B\cup C)] = P(A)+P(B\cup C) - P(A\cap(B\cup C))$

Q). If A, B and C are any 3 events such that
$P(A) = P(B) = P(C) = 1/4$, $P(A\cap B) = P(B\cap C) = 0$ & $P(C\cap A) = 1/8$
Find the probability that atleast of the events occur.

PTO

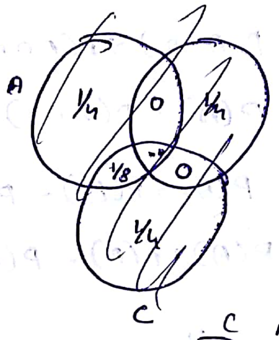
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

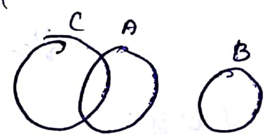
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - 0 - \frac{1}{8} - 0 + 0$$

$$= \frac{6-1}{8} = \frac{5}{8}$$

A, B & B, C are mutually exclusive.

$\therefore P(A\cap B \cap C) = 0$ $\left[\because P(A\cap B) = 0 \ \& \ P(B \cap C) = 0\right]$





Conditional Probability:

The conditional probability of an event A assuming that the event B has happened, is denoted by $P(A/B)$ and defined as

$$\boxed{P(A/B) = \frac{P(A\cap B)}{P(B)}} \quad \text{provided } P(B) \neq 0$$

Rolling 2 dice.

$P(A$ given $B)$ (A/B),

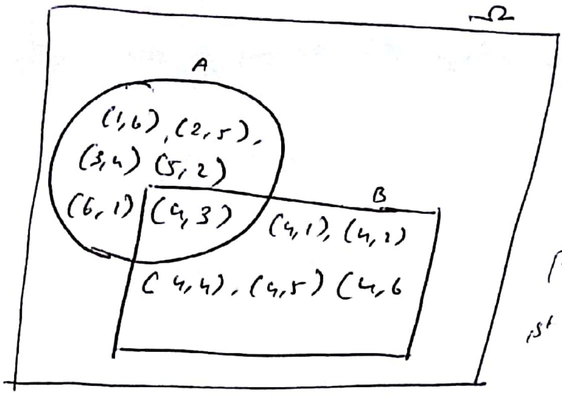
$S = \{(1,1), (1,2), \dots (6,6)\}$

$B = \{(2,1), (2,2) \dots (2,6)$
$(4,1) \dots (4,6), (6,1) \dots (6,6)\}$

Two dice are rolled.

A : sum of the two dice equal to 7

B : The first ~~sum~~ die resulted in 4.



Let A denote the event that the sum of 2 dice is 7.

A - {(1,6) ... }

B be the event that the 1st die is 4.

B - {(4,1), (4,2) ... }

$P(B) = \frac{n(B)}{n(S)} = \frac{1}{36}$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{1/36}{6/36}$$

$$= \frac{1}{6}$$

A bag contains 8 red balls, 4 green & 8 yellow balls. A ball is drawn at random from the bag & it is found ~~that~~ not to be one of the red balls. What is the probability that it is a green ball?

B: not red

A: green ball.

$P(A \cap B) = P(A) = 4/20 = 1/5$

$P(B) = 12/20$

$\therefore P(A/B) = 4/12 = 1/3$

Consider 2 rolls of 4 sided die. Given that in the first roll, the event occured with $\min(x,y) = 2$. What is the probability for an event to occur such that $\max(x,y) = 3$?

A : { (2,2), (2,3)(2,4) ~~(4,2) (2,6)~~ (3,2)(4,2) }

B : { (1,3) (2,3) (3,3) (3,2) (3,1) }

$$P(B/A) = \frac{2/16}{5/16} = \frac{2}{5}$$

Conditional probability obeys the axioms of the ordinary probabilities

(i) $0 \le P(A/B) \le 1$

Proof:

$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

$\because P(A\cap B) \le P(B)$, $P(A\cap B) \ge 0$ & $P(B) > 0$

$0 \le P(A/B) \le 1$

A: $1/4$

B:

(ii) $P(\Omega/B) = 1$

Proof:

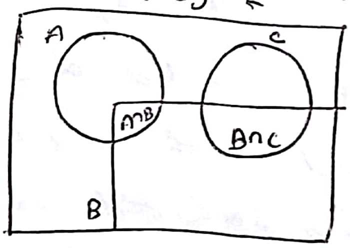
$$P(\Omega/B) = \frac{P(\Omega\cap B)}{P(B)}$$

$$= \frac{P(B)}{P(B)} = 1$$

H.P

(iii) If $A\cap C = \phi$ then $P[(A\cup C)/B] = P(A/B) + P(C/B)$

$\cancel{P\left[\frac{A\cup C}{B}\right]} = P$



Proof:

$$P[(A\cup C)/B] = \frac{P[(A\cup C)\cap B]}{P(B)}$$

$$= \frac{P[(A\cap B)\cup(C\cap B)]}{P(B)}$$

$$= \frac{P(A\cap B) + P(C\cap B)}{P(B)} =$$

$$= \frac{P(A\cap B)}{P(B)} + \frac{P(C\cap B)}{P(B)}$$

$$= P(A/B) + P(C/B)$$

H.P

3/12/18

NOTE: (Multiplication Rule) (Product Theorem):

By rewriting the conditional probability, we get

$P(A \cap B) = P(B) \cdot P(A/B)$ $\left[\because P(A/B) = \frac{P(A \cap B)}{P(B)}\right]$

This is sometimes referred to as product theorem of probability.

Independent Events:

I.

A: Getting atleast two 'O' grade
B: Got 'O' grade in PRP
} dependent $P(A/B) = \frac{P(A \cap B)}{P(B)}$

II

A: Getting 'O' grade in PRP
B: Got 'O' grade in CLP
} independent. $P(A/B) = P(A)$.

The events A & B are said to be independent if,

$P(A/B) = P(A)$ ——— (1)

$\because$ by multiplication rule,

$P(A \cap B) = P(A/B) \cdot P(B)$

(1) $\Rightarrow$ $\boxed{P(A \cap B) = P(A) \cdot P(B).}$

1. A red die and a blue die are rolled together. What is the probability that we obtain 4 on red & 2 on blue die?

Soln:

Let A: event ~~that~~, 4 on red die

B: ~~event~~, event, 2 on blue die

$P(A) = 1/6$; $P(B) = 1/6$

$\therefore$ $P(A \cap B) = P(A) \cdot P(B)$

$= \frac{1}{6} \cdot \frac{1}{6} = 1/36$

Q. Two coins are tossed. Let A : event that atmost one head on the 2 tosses & let B : event that one head & one tail in both tosses. Are the events independent.

Sample space, $\Omega = \{HH, HT, TH, TT\}$

$A = \{HT, TH, TT\}$ $P(A) = \frac{3}{4}$

$B = \{HT, TH\}$ $P(B) = \frac{2}{4}$

$P(A) = 3/4$

$A \cap B = B$

$P(B) = \frac{2}{4}$

$\Rightarrow$ $P(A \cap B) \neq P(A) \cdot P(B)$

$\Rightarrow$ the event is not independent.

Proposition:

If A & B are independents, then ~~A, B~~ A & B^c, A^c & B and ~~A~~ & ~~B~~ & A^c, B^c are also independent.

Proof:

if (A, B) are independent

$$P(A \cap B) = P(A) \cdot P(B) \quad \text{— (1)}$$

Now $A = (A \cap B) \cup (A \cap B^c)$

$$P(A) = P[(A \cap B) \cup (A \cap B^c)]$$

$$= P(A \cap B) + P(A \cap B^c)$$

$$= P(A) \cdot P(B) + P(A \cap B^c)$$

$$P(A \cap B') = P(A) P(B) - P(A)$$

$$= P(A) \cdot (1 - P(B))$$

$$= P(A) \cdot P(B^c)$$

NOTE: To prove A^c & B are independent, use

$$B = \cancel{A \cap B} (A \cap B) \cup (A^c \cap B)$$

(ii) To prove A^c & B^c are independent, use

$$A^c = (A^c \cap B) \cup (A^c \cap B^c).$$

3. A & B are 2 independent events defined in the sample space. They have the following probabilities $P(A) = x$ & $P(B) = y$. Find the probabilities of the following in terms of x & y.

(a) neither A nor B occurs

(b) Even A occurs but not B

(c) A or B

Soln:

(a) $P(A^c \cap B^c)$:

If A, B are independent,

A^c, B^c are also "

$$\therefore P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$$

$$= (1-x)(1-y) = (1-y) - x(1-y)$$

$$= 1 - y - x + xy$$

$$= 1 + xy - y - x$$

$A' \cup B' = (A' \cap B)'$

(ii) $P(A \cap B^c) = P(A) \cdot P(B^c)$

$$= x(1-y)$$

(iii) $P(A \cup B^c) = 1 - P(A^c \cap B)$

$$= 1 - P(A^c) \cdot P(B)$$

$$= 1 - (1-x) y = 1 - y + xy \text{ //.}$$

4.12

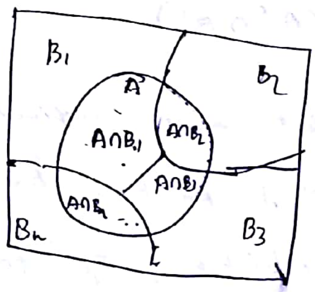
Total Probability :

$B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n = \Omega$

Let $\{B_1, B_2, \ldots, B_n\}$ be a partition of Ω. Each one of the events $B_1, B_2, \ldots B_n$ has non-zero probability. Let A be any event defined in Ω.

Then,

$$P(A) = P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2) + \ldots + P(A/B_n) \cdot P(B_n)$$



Proof :

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n)$$

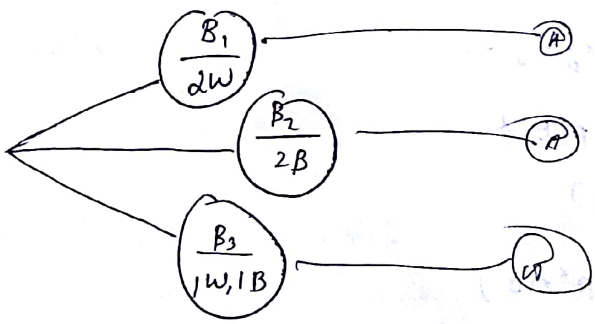
$$\because P(A \cap B_i) = P(A/B_i)\, P(B_i), \quad i = 1, 2, \ldots, n$$

$$P(A) = P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2) + \ldots + P(A/B_n) \cdot P(B_n)$$

Q).

	White	Black
Urn 1	10	3
Urn 2	3	5

2 balls are drawn at random from the first urn & placed in the 2nd urn. And then 1 ball is taken at random from the second urn. What is the probability that it is a white ball



P(A) =

$$P(B_1) = \frac{{}^{10}C_2 \cdot {}^{3}C_0}{{}^{13}C_2} = \frac{15}{26}$$

$$P(B_2) = \frac{{}^{10}C_0 \cdot {}^{13}C_2}{{}^{13}C_2} = \frac{1}{26}$$

$$P(B_3) = \frac{{}^{0}C_1 \cdot {}^{3}C_1}{{}^{13}C_2} = \frac{10}{26}$$

$$P(A/B_1) = \frac{5}{10} = \frac{1}{2}.$$

$$P(A/B_2) = \frac{3}{10}$$

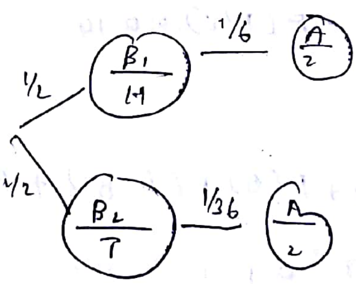
$$P(A/B_3) = \frac{4}{10}$$

$$\therefore P(A) = \frac{15}{26} \cdot \frac{5}{10} + \ldots$$

$$= \frac{1}{10} \cdot \frac{1}{26} \left[5 \cdot 15 + 1 \cdot 3 + 10 \cdot 4\right]$$

$$= \frac{\cancel{138}\ 118}{260} = \frac{59}{130}$$

In a coin tossing exp. if the coin shows head, one dice is thrown & the num is recorded but if the coin shows tail two dice are thrown & their sum is recorded. What is the probability no. is 2?



Baye's theorem

The conditional probality that event B_k occured given that A occured is given by

$$P(B_k/A) = \frac{P(A \cap B_k)}{P(A)}$$

$$P(B_k/A) = \frac{P(A/B_k).P(B_k)}{P(A)}$$

$$P(B_k/A) = \frac{P(A/B_k) \cdot P(B_k)}{\sum_{i=1}^{n} P(A/B_i)P(B_i)}$$

Q). Student buy

Supplier	No. of chips	P(chips to be defective)
A	1000	0.05
B	2000	0.1
C	3000	0.1

What is the probat iff one of the clip is selected,
(i) defective
(ii) from suppler A

$P(A) = 1/6$ $\quad P(D/A) = 0.05$
$P(B) = 2/6$ $\quad P(D/B) = 0.10$
$P(C) = 3/6$ $\quad P(D/C) = 0.10$

(i) $P(D) = P(A).P(D/A) + P(B)\,P(D/B) + P(C).P(D/C)$

$= \frac{1}{6} \cdot 0.05 + \frac{2}{6} \cdot 0.1 + \frac{3}{6} \cdot 0.1 = 0.09167$

$= \frac{1}{600}(5+20+30) = \frac{\cancel{55}^{11}}{\cancel{600}_{120}} = \frac{11}{120}$

(ii) $P(A/D) = \frac{P(A \cap D)}{P(D)} = \frac{P(D/A)P(A)}{P(D)} = \frac{1/6 \cdot 0.05}{0.9167} =$

7.12.'18

Q. A bag contains 5 balls & it is not ~~more~~ known, How many of them are white. 2 balls are drawn at random from the bag & they are noted to be white. What is the chance that all balls in the bag are white.

There might be

		Events
2 white balls		: B_1
3 " "		: B_2
4 " "		: B_3
5 " "		: B_4

All the events are mutually exclusive and ~~co~~ collectively exhaustive (totally there can be only 5 balls).
All the events are equally likely.

~~B~~ There are 4 possible events

$\therefore P(B_1) = P(B_2) = P(B_3) = P(B_4) = 1/4.$

Let A : drawing 2 balls at random.

$$P(A/B_1) = \frac{{}^2C_2 \cdot {}^3C_0}{{}^5C_2} = \frac{1}{10}$$

$$P(A/B_2) = \frac{{}^3C_2 \cdot {}^2C_0}{{}^5C_2} = \frac{3}{10}$$

$$P(A/B_3) = \frac{{}^4C_2 \cdot {}^1C_0}{{}^5C_2} = \frac{6}{10}$$

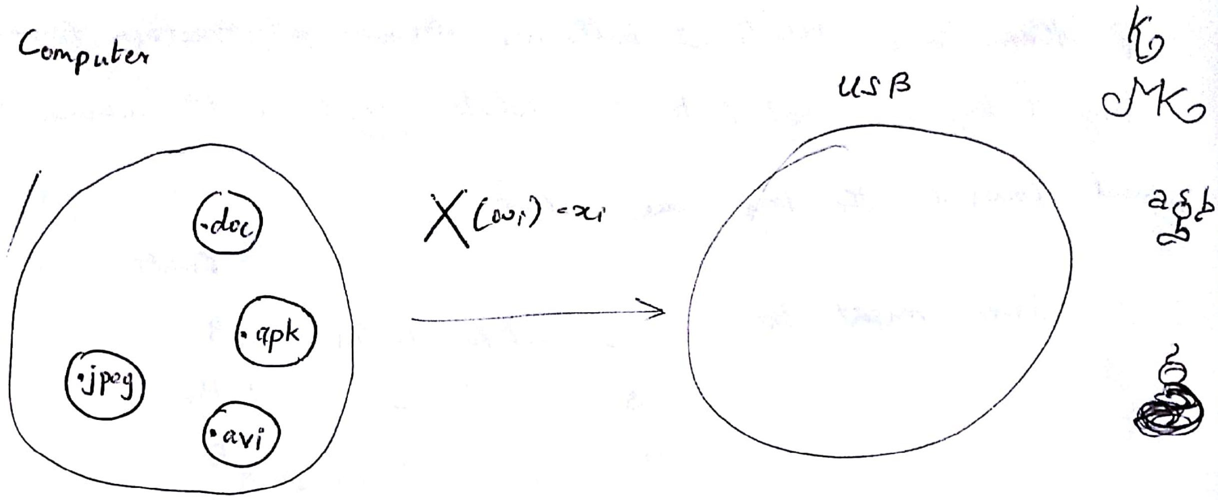
$$P(A/B_4) = \frac{{}^5C_2}{{}^5C_2} = 1$$

$$P(B_4/A) = \frac{P(B_4) \cap P(A))}{P(A)}$$

$$= \frac{P(B_4) \cdot P(A/B_4)}{\sum_{i=1}^{4} P(B_i) \cdot P(A/B_i)} = \frac{\cancel{1/4} \cdot 1}{\cancel{1/4}\left(\frac{1}{10} + \frac{6}{10} + \frac{3}{10} + \frac{10}{10}\right)}$$

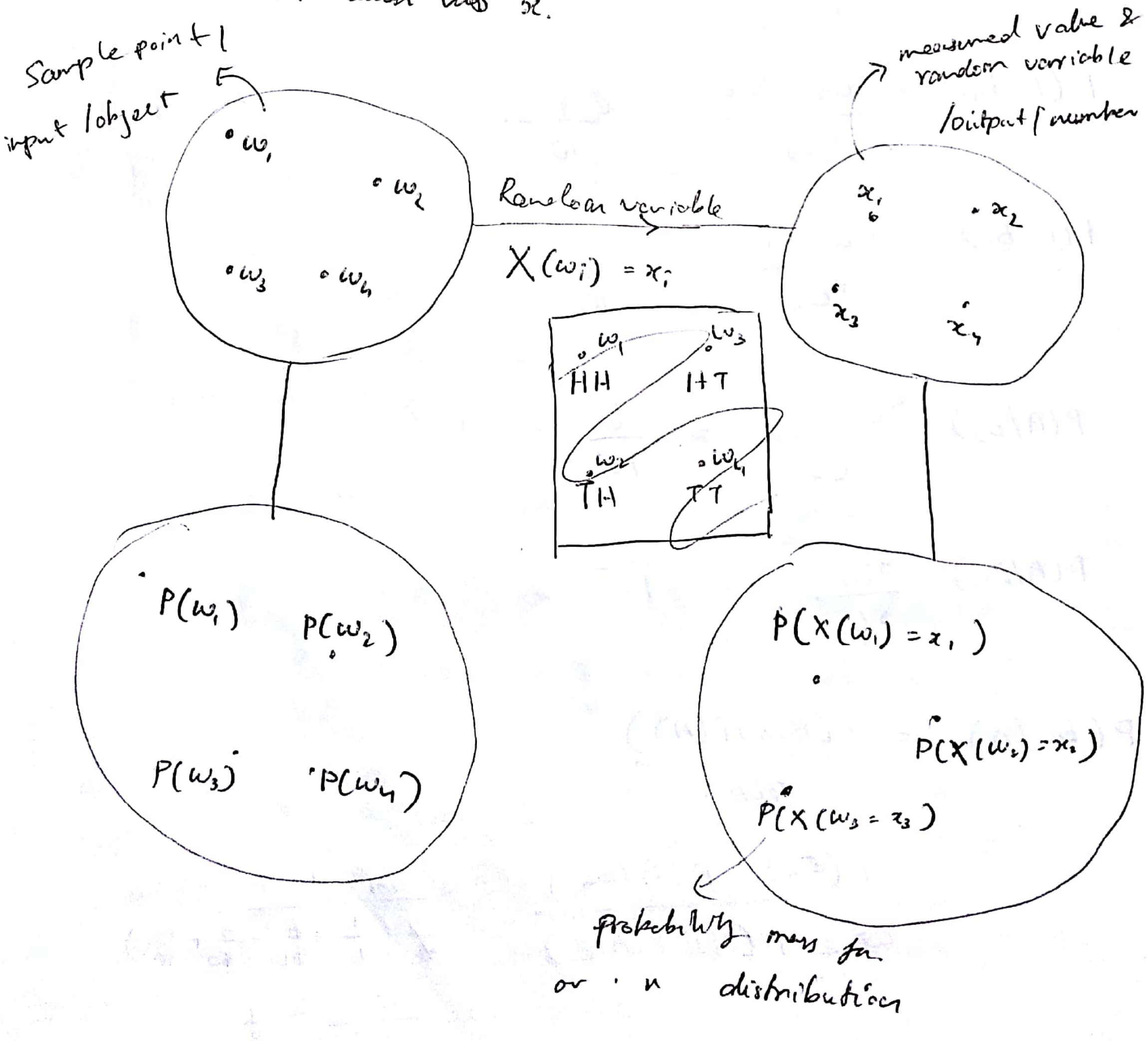
$$= \frac{10}{20} = \frac{1}{2}.$$

Transfer to USB

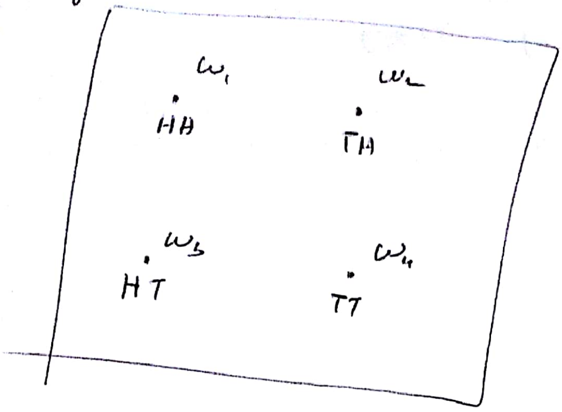


Random Variable

A random variable is a fn. that assigns a real no. to each outcome in the sample space of a random experiment. A random variable is denoted by uppercase letter such as X. The measured value of the random variable is denoted by lower case letter such as x.



tossing 2 coins



X =: no of heads in outcome

X can take values 0, 1, 2.

If we fix some "x", $X = x$ is an event.

Suppose $x = 1$, then $X = 1$

i.e. $\{\omega : X(\omega) = 1\} = \{HT, TH\}$ is an event.

$P[X = 1] = P(HT \cup TH)$

$= P(HT) + P(TH)$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ //.

Probability Mass Function (PMF)

$$P_X(x) = P(X(\omega) = x) = P[\{\omega \in \Omega : X(\omega) = x\}]$$

Now,

$P(X=0) = P(TT) = 1/4 \quad (f(x_1))$

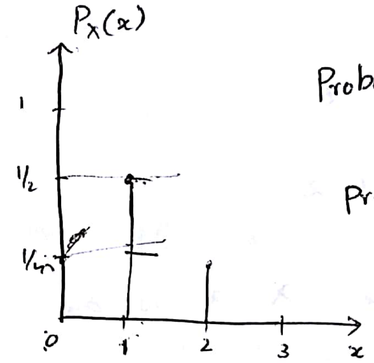
$P(X=1) = P(TH \cup HT) = 1/2 \quad f(x_2)$

$P(X=2) = P(HH) = 1/4 \quad f(x_3)$

$\overline{\quad 1 \quad}$

$$\boxed{\sum_{i=1}^{n} f(x_i) = 1}$$

x	$P[X(w_i) = x_i] = f(x_i)$
0	1/4
1	1/2
2	1/4



Probability mass function
or
Probability distribution

For a discrete variable X with possible values $x_1, x_2, \ldots, x_n$. A probability mass function is a function such that the following properties hold.

(i) $f(x_i) \geq 0$

(ii) $\sum_{i=1}^{n} f(x_i) = 1$

(iii) $f(x_i) = P(X = x_i)$

Q). A shipment of 6 television sets contains two defective sets. A hotel makes a random purchase of three of the sets. If X is the no. of defective sets purchased by the hotel. Find the probability distribution of X.

Soln: X can take the values 0, 1 and 2.

$P(X=r)$: P(choosing r defective sets)
: P(choosing '$3-r$' good sets)

$\therefore \; P(X=r) = \dfrac{{}^2C_r \cdot {}^4C_{3-r}}{{}^6C_3}, \; r = 0, 1, 2$

$$P(x=0) = \frac{{}^2C_0 \cdot {}^4C_3}{{}^6C_3} = \frac{4}{20} = \frac{1}{5}$$

$$P(x=1) = \frac{{}^2C_1 \cdot {}^4C_2}{{}^6C_3} = \frac{2 \cdot 12}{20} = \frac{3}{5}$$

$$P(x=2) = \frac{{}^2C_2 \cdot {}^4C_1}{{}^6C_3} = \frac{4}{20} = \frac{1}{5}$$

$X = r$	$P(X=r)$
0	$1/5$
1	$3/5$
2	$1/5$
Total	$\sum_{i=0}^{2} f(x_i) = 1$

2. A random variable X has the following probability distribution:

x	-2	-1	0	1	2	3
$P(X=x)=f(x)$	0.1	k	0.2	2k	0.3	3k

(i) Find k

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$6k + 0.6 = 1$$

$$6k = \cancel{0}\,0.4$$

$$k = \frac{0.4}{6} = \frac{.2}{3} = \frac{2}{30} = \frac{1}{15}$$

(ii) $P(X<2) = \frac{0.1}{10} + \frac{1}{15} + \frac{2}{10} + \frac{2}{15}$

$$= \frac{3+2+6+4}{30} = \frac{1}{2}$$

(iii) $P(-2 < X < 2) = \frac{1}{2} - \frac{1}{10} = 0.5 - 0.1 = 0.4 = \frac{2}{5}$

Cummulative distribution function:

$F(x)$

or

$F_x(x) = P(X \leq x) = \sum_{x \leq x_i} F(x_i)$

The CDF of a discrete random variable X is denoted as $F(x)$ or $F_x(x)$ & is defined as

For a discrete random variable X, $F_x(x)$ satisfies the following properties:

(i) $F_x(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$

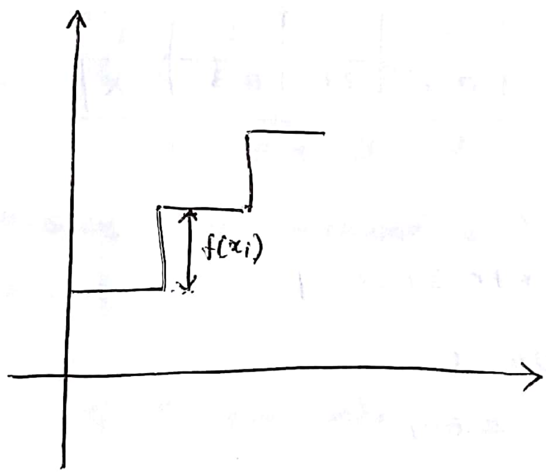
(ii) $0 \leq F_x(x) \leq 1$

(iii) If $x \leq y$, then $F_x(x) \leq F_x(y)$

(iv) $P(a < X \leq b) = F_x(b) - F_x(a)$

(v) $P(X > a) = 1 - P(X \leq a) = 1 - F_x(a)$

NOTE: The CDF $F_x(x)$ of a discrete random variable X is a step function. i.e. the value of $F_x(x)$ is constant in the interv $[x_{i-1}, x_i)$ and the takes a step or jump of size $f(x_i)$



Q. Let the random variable X denote the no. of head in three toss of a fair coin. What the PMF (X). Sketch the CDF of X

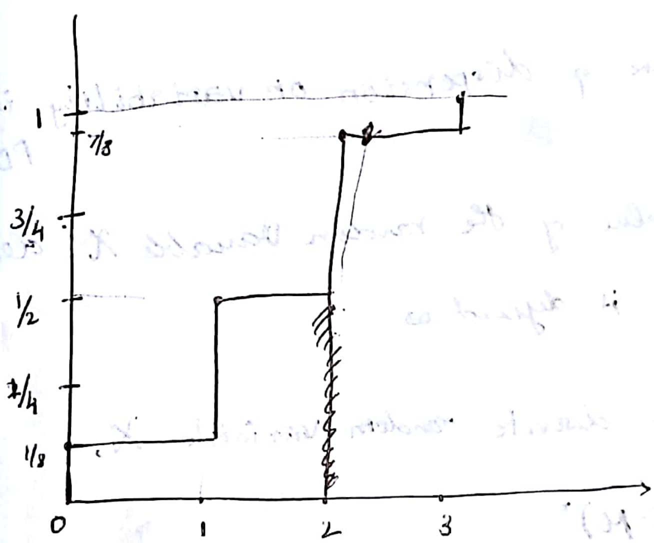
A: $X = 0, 1, 2, 3$

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

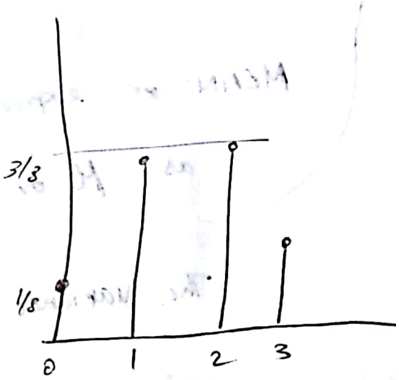
$P(X=0)$ x $P(X=x_i)=f(x_i)$

x	$P(X=x_i)=f(x_i)$
0	1/8
1	3/8
2	3/8.
3	1/8.

$F_x(0)=\frac{1}{8}$ $F_x(1)=\frac{4}{8}$ $F_x(2)=\frac{7}{8}$ $F_x(3)=\frac{8}{8}$



CDF



PMF

12.12.18

Obtaining the PMF from CDF

$$F_x(x)=\begin{cases} 0, & x<0 \\ 1/6, & 0\le x<2 \\ 1/2 \;\; (1/3) & 2\le x<4 \\ 5/8 \;\; (1/8) & 4\le x<6 \\ 1 \;\; (3/8) & x\ge 6 \end{cases}$$

Find PMF of X whose CDF is given by $F_x(x)$.

$3/6-1/6$ $1-\frac{5}{8}$

$\frac{1}{2}+x=\frac{5}{8}$

$X=0,2,4,6.$

PTO

$$P(X=x_i) = f(x_i) = \begin{cases} 1/6 , & x=0 \\ 1/3 , & x=2 \\ 1/8 , & x=4 \\ 3/8 , & x=6 \\ 0 & \text{otherwise} \end{cases}$$

$\frac{3}{6} + \frac{4}{8}$

$= 1$

Mean and Variance of a discrete Random variable.

Mean : $\mu = E(X) = \sum x_i . f(x_i)$

Mean is the measure of centre/middle of PD.

Variance is the measure of dispersion or variability in PD

MEAN: or expected value of the random variable X denoted as μ or $E(X)$ is defined as

The variance of the discrete random variable X,

$$\sigma^2 = V(X) = E(X-\mu)^2$$

$$= \sum_x (x-\mu)^2 . f(x)$$

The standard deviation; $\sigma = \sqrt{\text{variance}} = \sqrt{\sigma^2}$

NOTE: $V(x) = \sum_x (x-\mu)^2 . f(x)$

$$= \sum_x (x^2 - 2\mu x + \mu^2) f(x)$$

$$= \sum_x x^2 f(x) - 2\mu . \underbrace{\sum_x x f(x)}_{=\mu} + \mu^2 \underbrace{\sum_x f(x)}_{=1}$$

$$= \sum_x x^2 f(x) - 2\mu^2 + \mu^2$$

$$= \sum_x x^2 f(x) - \mu^2$$

$$\Rightarrow V(x) = E(X^2) - [E(X)]^2$$

Find the expected value of the discrete Random variable X with the following PMF

$$f(x) = P(X=x) = \begin{cases} 1/3, & x=0 \\ 2/3, & x=2 \end{cases}$$

$\boxed{P(X=x) \text{ or } P_X(x) \text{ or } P_i}$

$$E(X) = \sum_x x \cdot f(x)$$

$$= 0 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{4}{3}$$

Find expected value of the random variable K with the following PMF

$$f(x_k) = P(K=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, 3 \ldots$$

~~E(x)~~ $k \cdot f(x_k)$

$$E(K) = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \cancel{\sum_{k=0}^{\infty} \frac{\lambda^k}{(k-1)!} \cdot e^{-\lambda}}$$

$$= e^{-\lambda} \cancel{\sum_{k=0}^{\infty} \frac{\lambda^k}{(k-1)!}}$$

$$= e^{-\lambda}\left[0 + \frac{\lambda}{1!} + \frac{2\lambda^2}{2!} + \ldots\right]$$

$$= e^{-\lambda}\left[\lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \frac{\lambda^4}{3!} + \ldots\right]$$

$$= \lambda \cdot e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right]$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda}$$

$$= \lambda$$

Q). Find $E(x)$ & σ^2 of Random variable X if X represents the outcome when a fair die is thrown

$$E(X) = \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{21}{6} = \frac{7}{2}$$

21
21
21
42

$$\sigma^2 = -\left(\frac{21}{6}\right)^2 + \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2)$$

$\frac{n(n+1)\cdot 2(n+1)}{6}$

$$= -\frac{441}{36} + \frac{1}{6} 91$$

$\frac{6\cdot 7\cdot 13}{6}$

2
13
$\times 7$
91
$\times 6$
576
441
3) 135
12
1

$$= \frac{576-441}{36} = \frac{\cancel{135}\ 15}{\cancel{36}\ 4} = \frac{15}{4}$$

$$= \frac{35}{12}$$

$$\sigma^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

14/12/18

Continuous random variable:

A continuous random variable, is a random variable with an interval of real nos for its range.

Ex: Electric current, length, pressure, temp, time, weight, etc.

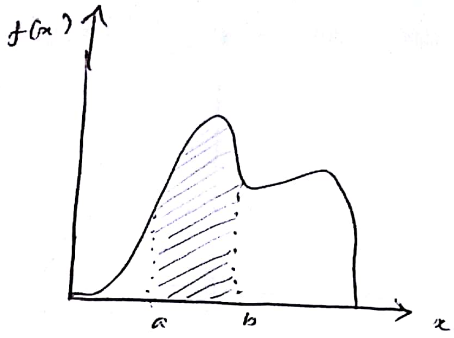
Probability density function: or Probability distribution

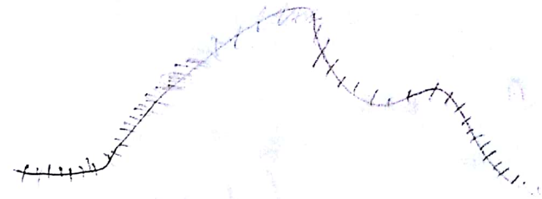
For a continuous random variable X, a probability density function is a function such that.

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ = area under $f(x)$ from a to b





NOTE:

→ Probabity ~~of rand~~ that a continuos random variable assumed at any fixed value is zero.

$$P(X=a) = \int_a^a f(x)\,dx = 0$$

→ If X is a continuous random variable, then for any x_1 & x_2

$$P(x_1 \le X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X \le x_2) = P(x_1 < X < x_2)$$

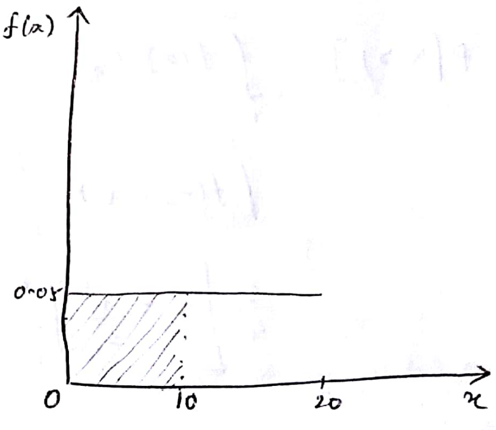
Q). Let the continuos random variable X denote the current measured in a thin copper wire in mA. Assume that, the range of X is ~~the~~ ~~clos~~ [0, 20] mA and assume that the probability density function of X is $f(x) = 0.05$ for the interval [0, 20]. What is the probability that a current measurement is less than 10 mA?

$$P[X<10] = \int_0^{10} f(x)\,dx$$

$$= \int_0^{10} (0.05)\,.dx$$

$$= 0.05\,[10-0]$$

$$= 0.5 = 1/2$$

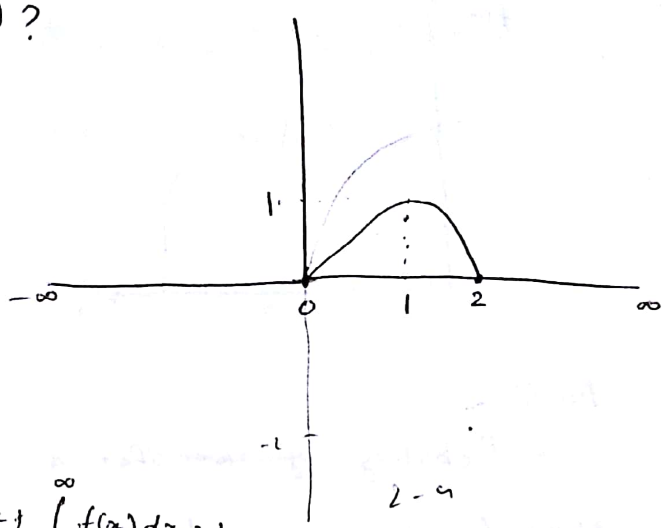


2. Assume that X is a continuos random variable with the following probability density function.

$$f(x) = \begin{cases} A(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(i) What is the value of A?

(ii) Find $P(x>1)$



A:

WKT,

(i)

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{2} f(x)\,dx + 0 = 1$$

$$\int_{0}^{2} A\cdot(2x-x^2)\,dx = 1$$

$$A\cdot\left[\frac{x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^2 = 1$$

$$A\left[4 - \frac{8}{3}\right] = 1$$

$$A = 3/4.$$

(ii) $$P[x>1] = \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_1^2 + 0$$

$$= \frac{3}{4}\left[4 - \frac{8}{3} - 1 + \frac{1}{3}\right]$$

$$= \frac{3}{4}\cdot\left[3 - \frac{7}{3}\right]$$

$$= \frac{\cancel{3}}{4}\cdot\frac{2}{\cancel{3}} = \frac{1}{2}//.$$

Cummulative Distribution function for cont. random Variable:

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

or $F(x)$

The CDF a continuous random variable X is defined as,

NOTE:

$$f(x) = \frac{d}{dx}[F_X(x)]$$

The probability density function $f(x)$ of a cont. random variable X can be determined from the cummulative distribution fn. $F_X(x)$ by differentiating $F_X(x)$.

Q). Consider the pdf of X given by $f(x) = \begin{cases} C & a \le x \le b \\ 0 & \text{, otherwise} \end{cases}$

(i) For what value of x is $f(x)$ pdf

(ii) Find the cummulative df of the cont. random variable X with the above PDF

A: (i) $f(x)$ will be PDF iff $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} c \cdot dx = 1$$

$$c\left[\int_{-\infty}^{a} dx + \int_{a}^{b} dx + \int_{b}^{\infty} dx\right] = 1$$

$$c \cdot [b-a] = 1$$

$$\Rightarrow c = \frac{1}{b-a}$$ //.

(ii) If $x < a$,

$\Rightarrow f(x) = 0$

$\Rightarrow F(x) = 0$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

For $a \le x < b$

$f(x) = c$

$$\therefore F(x) = 0 \cdot \int_{-\infty}^{a} c\,dx + \int_{a}^{x} f(x)\,dx$$

$$= \int_{a}^{x} c \cdot dx = \frac{1}{b-a} \cdot x-a = \frac{x-a}{b-a}$$ //.

For $x \geq b$

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$= \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)dx + \int_{b}^{x} f(x)dx$$

$$= 0 + \int_{a}^{b} c \cdot dx + 0$$

$$= \frac{1}{b-a}(b-a) = 1 //.$$

$$F(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \leq x < b \\ 1 & , x \geq b \end{cases}$$

17.12.18

Q. Mean & variance of a cont. random variable, ~~Suppos~~ X with pdf $f(x)$. the mean are expected value of X denoted as μ or $E(X)$ is defined as:

* $$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

* Variance, $$Var(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx - \mu^2$$

* Standard deviation,

$$\sigma = \sqrt{Var(X)}$$

Q. Let X be a cont. random variable with the pdf

$$f(x) = \begin{cases} 1/4 & , 2 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Find mean & variance of X.

52
3

208

36
6
216
8
52
~~208~~
~~72~~
3
52
48

$$\mu = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx$$

$$= \int_{-\infty}^{2} x \cdot f(x)\, dx + \int_{2}^{6} x \cdot f(x)\, dx + \int_{6}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{2}^{6} x \cdot 1/4 \, dx$$

$$= \frac{1}{8} [36 - 4]$$

$$= 4 //$$

$$\sigma^2 = \int_{2}^{6} x^2 \cdot 1/4 \, dx - 16$$

$$= \frac{1}{12} [216 - 8] - 16 \quad = \frac{52 - 48}{3}$$

$$= 4/3$$

Q. The CDF of cont. random variable is gvn by

$$F(x) = \begin{cases} 1 - (1+x)e^{-x} & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find pdf, mean & variance.

$$\text{pdf} = \frac{d}{dx}\begin{cases} 1 - (1+x)e^{-x} \\ 0 & \text{other} \end{cases}$$

$$f(x) = \begin{cases} 0 - [-(1+x) \cdot e^{-x}] + e^{-x} \\ 0 \end{cases}$$

$$\cancel{= e^{-x} + x + 1}$$

$$= \begin{cases} x e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = 1 - (1+x)e^{-x}$$

$$f(x) = 0 - [(1+x)e^{-x} + e^{-x})$$

$$= -e^{-x}(-1 - x + 1)$$

$$= x e^{-x}$$

$$\mu(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{\infty} x^2 e^{-x} dx$$

$$= \cancel{\int_{-\infty}^{\infty} x^2 e^{-x}} = \int_{0}^{\infty} x^2 e^{-x} dx$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

ILATE

$u = x^2$

$dv = e^{-x}$

$v = -e^{-x}$

$$M(x) = \int_0^\infty x^2 \cdot e^{-x}\,dx$$

$$= \int_0^\infty -x^2 e^{-x} + \int e^{-x} \cdot 2x\,dx$$

$$= -x^2 e^{-x} + 2$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$$

u', u'', u''' – successive differentiation

v_1, v_2, v_3 – " integration

$u = x^2$ | $v = e^{-x}$

$u' = 2x$ | $v_1 = -e^{-x}$

$u'' = 2$ | $v_2 = e^{-x}$

$u''' = 0$ | $v_3 = -e^{-x}$

$v_4 = e^{-x}$

$u = x^3$ | $v = e^{-x}$

$u' = 3x^2$

$u'' = 6x$

$u''' = 6$

$u'''' = 0$

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} - 2xe^{-x} + 2(-e^{-x})$$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x}$$

$$M = \int_0^\infty \left[x^2(e^{-x}) - 2x(e^{-x}) + 2(e^{-x}) \right]_0^\infty$$

$$= \cancel{\ } + 2 \cdot 4$$

$$\sigma^2 = \int_0^\infty x^2 \cdot f(x)\,dx - M^2$$

$$= \int_0^\infty x^3 e^{-x}\,dx - 4$$

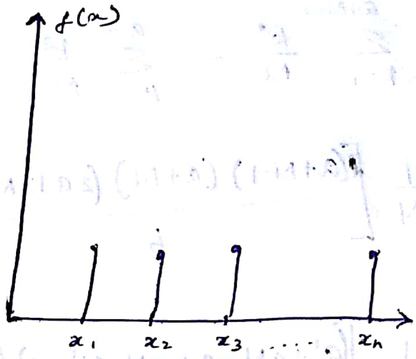
$$= \left[-x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} \right]_0^\infty - 4$$

$$= 6 - 4 = 24$$

SPECIAL PROBABILITY DISTRIBUTION

Discrete Uniform Distribution:

A random variable X has a discrete uniform distribution if for each of the n values in its range, say $x_1, x_2, \dots, x_n$ has equal probability $f(x_i) = \frac{1}{n}$



NOTE: Mean & Variance of Discrete uniform distribution

Suppose X is discrete random variable on the consecutive integers, $X = x = a, a+1, a+2, \dots, a+n-1$ and its PMF $f(x_i) = \begin{cases} \frac{1}{N}, & x = a, a+1, \dots, a+n-1 \\ 0, & \text{otherwise.} \end{cases}$

$$\therefore E(x) = a\left(\frac{1}{N}\right) + (a+1)\frac{1}{N} + (a+2)\frac{1}{N} + \dots + (a+N-1)\frac{1}{N}$$

$$= \sum_{k=a}^{a+n-1} \frac{k}{N}$$

$$= \sum_{k=1}^{a+N-1} \frac{k}{N} - \sum_{k=1}^{a-1} \frac{k}{N}$$

$$= \frac{1}{N}\frac{(a+N-1)(a+N)}{2} - \frac{1}{N}\frac{(a-1)(a)}{2}$$

$$= \frac{1}{2N}\left[\cancel{a^2} + aN + aN + N^2 - \cancel{a} - N - \cancel{a^2} + \cancel{a}\right]$$

$$= \frac{N^2 + 2aN - N}{2N}$$

$$= \frac{N + 2a - 1}{2}$$

$$= \frac{a + N - 1 + a}{2}$$

$$\boxed{E(x) = \frac{(a+N-1) + a}{2}}$$

[the sum of first, last element, by 2]

Thus the expected value is the arithmetic avg. of the lowest & highest values of the random variable X

Var (x)

$$E(x^2) = \sum_{k=a}^{a+N-1} \frac{k^2}{N}$$

$$= \sum_{k=1}^{a+N-1} \frac{k^2}{N} - \sum_{k=1}^{a-1} \frac{k^2}{N}$$

$$= \frac{1}{N}\left[\frac{(a+N-1)(a+N)(2a+2N-2+1)}{6} - \frac{(a-1)(a)(2a-2+1)}{6}\right]$$

$$= \frac{1}{6N}\left[(a^2+aN-a+aN+N^2-N)(2a+2N-1) - (a^2-a)(2a-1)\right]$$

$$= \frac{1}{6N}\left[2a^3+4a^2N-4a^3+2aN^2-2aN+\dots\right]$$

$$= \frac{2N^2+6aN+6a^2-6a-3N+1}{6}$$

$$\therefore \text{Var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{2N^2+6aN+6a^2-6a-3N+1}{6} - \frac{(a+N-1+a)^2}{2^2}$$

$$\boxed{\sigma^2 = \frac{N^2-1}{12}}$$

DISCRETE UNIFORM DISTRIBUTION

$f(x_i) = \frac{1}{n}$, $x_i = 1, 2, 3, \dots, n$

$$\boxed{\mu(x) = E(X) = \frac{(a+N-1)+a}{2}}$$

Variance $= V(X) = \frac{N^2-1}{12}$

1. Let X be the random variable that denotes the outcome of the roll of a fair die. Find mean & variance of X.

Soln: X can take 1, 2, 3, 4, 5, 6

PMF of X,

$$f(x_i) = \frac{1}{6}$$

The given distribution is discrete & uniform.

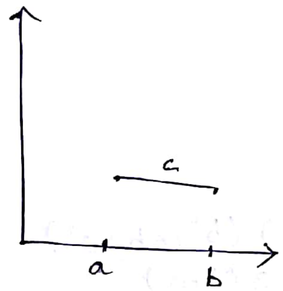
$$\therefore \mu(x) = \frac{a+N-1+a}{2}$$

$$= \frac{1+6-1+1}{2}$$

$$= \frac{7}{2} = 3.5$$

$$\sigma^2 = \frac{N^2-1}{12} = \frac{6^2-1}{12} = \frac{35}{12}$$

CONTINUOUS UNIFORM DISTRIBUTION



$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_a^b f(x)\,dx = 1$$

$$\int_a^b c \cdot dx = 1$$

$$c\,[x]_a^b = 1$$

$$c(b-a) = 1 \Rightarrow c = 1/b-a$$

$$\boxed{f(x_i) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & , \text{otherwise.} \end{cases}}$$

A cont. random variable X is ~~set~~ said to have a uniform distribution over ~~the~~ $[a, b]$. If its probability density fn. is given by:

* CDF of X is given by,

$$\boxed{F(x) = P(X \le x) \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & ; a \le x \le b \\ 1 & ; x > b \end{cases}}$$

* The expected value of X or mean of X

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_a^b x \cdot \cancel{f(x)} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2} //.$$

$$\mu(x) \quad \boxed{\mu = \frac{b+a}{2}}$$

Variance:

$$\int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}.$$

$$= \frac{a^2+ab+b^2}{3}$$

$$\sigma^2 = \left\{E(X^2) - [E(X)]^2\right.$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4a^2+4ab+4b^3-3a^2-6ab-3b^2}{12} = \frac{a^2-2ab+b^2}{12}$$

$$\boxed{\sigma^2 = \frac{(b-a)^2}{12}}$$

Q). If X is uniformly distributed over $(0,10)$. Calculate the probability that

(i) $X < 3$

(ii) $X > 6$

(iii) $3 \leq X < 8$

Soln:

(i) $f(x) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} \frac{1}{10} & , x \in (0, 10) \\ 0 & , \text{otherwise} \end{cases}$

(ii) ~~CDF~~ $= \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x \ge b \end{cases}$

$= \begin{cases} 0 & x < 0 \\ 3/10 & x < 3 \\ 1 & x \end{cases}$

$P(X<3) = \int_0^3 f(x)\, dx$

$= \int_0^3 \frac{1}{10}\, dx$

$= \frac{3}{10}$ //.

(ii) $P(X>6) = \int_6^{10} f(x)\, dx$

$= \frac{1}{10}[10-6] = \frac{4}{10} = \frac{2}{5}$

(iii) $P(3 \le X < 8) = \int_3^8 f(x)\, dx$

$= \frac{1}{10}[8-3] = \frac{8-3}{10} = \frac{5}{10}$

$= 1/2$ //.

2. The time that the teaching assistant takes to grade a paper is uniformly distributed. b/w 5 mins & 10 mins. Find the mean & variance of the time that the teaching assistant takes to grade a paper.

PTO

Let X be a random variable that denotes the time taken by the teaching assistant to grade a paper.

$\because X$ is uniformly distributed. The mean or expected value of X is $\frac{b+a}{2}$

$$f(x_i) = \begin{cases} \frac{1}{5} & , 5 \leq X \leq 10 \\ 0 & , \text{otherwise} \end{cases}$$

$$\mu = \frac{b+a}{2}$$

$$= \frac{15}{2} = 7.5$$

$$\sigma^2 : V = \frac{(10-5)^2}{12}$$

$$= \frac{25}{12} = 2.08$$

THE BERNOULLI TRIAL & BERNOULLI DISTRIBUTION

A Bernoulli trial is an experiment that results in two outcomes namely success & failure

In Benoulli trial we define the probability of success and probability of failure as

$P[\text{success}] = P \qquad 0 \leq P \leq 1$

$P[\text{failure}] : 1-P$

The Random Variable associated with Bernoulli Trial

Let us associate the events of Bernoullis Trial with a random variable X such that when the outcome of a trial is ~~~~ success. $X = 1$ and is failure then $X = 0$.

, PMF:

$$f(x) = \begin{cases} P & , x = 1 \\ 1-P & , x = 0 \end{cases}$$

An alternative way to define the PMF of X is

$$\boxed{f(x) = p^x (1-p)^{1-x} \quad , x = 0 \text{ or } 1}$$

* The CDF is,

$$\boxed{F(x) = \begin{cases} 0 & , x < 0 \\ 1-p & , 0 \leq x < 1 \\ 1 & , x \geq 1 \end{cases}}$$

* Mean of X

$$E(X) = 0(1-p) + 1 \cdot p$$

$$\boxed{\mu = E(X) = p}$$

* Variance of X.

$$E[x^2] = 0^2(1-p) + 1^2 \cdot p = p$$

Now, $$Var(X) = E[x^2] - (E[x])^2$$
$$= p - p^2$$

$$\boxed{\sigma^2 = p(1-p)}$$

19.12.18

Bernoulli Distribution:

$$P(S) = p$$
$$P(F) = 1-p$$

PMF: $$f(x) = p^x (1-p)^{n-x}$$

Binomial distribution for parameters n, p.

n – no. of trials

p – probability of 1 ~~event~~ trial.

If x denotes the no. of success,

then,

PMF,

$$P(X = x) = f(x_i) = \binom{n}{x} p^x (1-p)^{n-x}$$

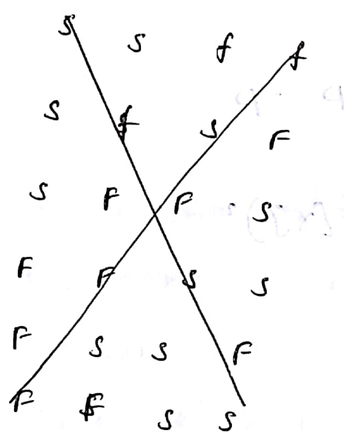
Suppose we conduct n independent Bernoulli trials & we represent the no. of success in those n trials by random variable

X or X(n), then X(n) is defined as A binomial random variable with parameters (n,p). The pmf of a random variable X(n) with parameteres (n,p) is given by:

$$\boxed{f(x) = P(X=x) = {}^{n}C_x \cdot p^x \cdot (1-p)^{n-x} \qquad x = 0,1,2,\dots,n}$$

The binomial coefficient $\binom{n}{x}$ represents the no. of ways of arranging x successes and $n-x$ failures. For instance,

If $n=4$, $x=2$ then there are $\binom{4}{2} = 6$ ways in which the 4 trials can result in 2 successes, namely



S	S	F	F	$p^2(1-p^2)$ +
S	F	S	F	" +
S	F	F	S	" +
F	S	S	F	" +
F	S	F	S	" +
F	F	S	S	" +

$$= 6 \cdot p^2(1-p^2) = \binom{4}{2} p^2 (1-p^2)$$

NOTE:

$$\sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} {}^{n}C_x \cdot p^x (1-p)^{n-x}$$

$$= (p+1-p)^n$$

$$= 1^n$$

$$= 1$$

Binomial expansion

$$(a+b)^n = \sum_{k=0}^{n} {}^{n}C_k \cdot a^k \cdot b^{n-k}$$

* The mean of X(n)

$$E[X(n)] = \sum_{x=0}^{n} x \cdot f(x)$$

$$= \sum_{x=0}^{n} \left(x \cdot {}^{n}C_x \, p^x \cdot (1-p)^{n-x} \right)$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!}\, p^{x}\, (1-p)^{x}$$

$$= \sum_{x=1}^{n} \frac{x \cdot n!}{x(x-1)!\,(n-x)!}\, p^{x} (1-p)^{x}$$

$$= \sum_{x=1}^{n}$$

$$= \sum_{x=}^{n}$$

$$= n \cdot p \sum_{x=1}^{n} \left[\frac{(n-1)!}{(x-1)!\,(n-x)!} \cdot p^{x-1} \cdot (1-p)^{n-x} \right]$$

$= np$

Let $\hat{x} = x-1$, when $x=1$, $\hat{x}=0$

$x=n$, $\hat{x} = n-1$

$$E[X(n)] = np \sum_{\hat{x}=0}^{n-1} \frac{(n-1)!}{\hat{x}!\,(n-\hat{x}-1)!} \cdot p^{\hat{x}} \cdot (1-p)^{n-\hat{x}-1}$$

$$= np \sum_{\hat{x}=0}^{n-1} \binom{n-1}{\hat{x}} p^{\hat{x}} \cdot (1-p)^{n-\hat{x}-1}$$

$$= np\, [p + (1-p)]^{n-1}$$

$$= np\, [1]^{n-1} = np$$

$$\boxed{E[X(n)] = np}$$

* The variance of $X(n)$:

$$E[X^2] = E[X(X-1)] + E[X]$$

$$= \sum_{x=0}^{n} x(x-1) f(x) + \sum_{x=0}^{n} x \cdot f(x)$$

$$= \sum_{x=0}^{n} x^2 f(x) - \sum_{x=0}^{n} x f(x) + \sum_{x=0}^{n} x \cdot f(x)$$

$$= \sum_{x=0}^{n} x^2 f(x)$$

$$E[X^2] = E[X(X-1)] + E[X]$$

$$= \sum_{x=0}^{n} x(x-1)f(x) + E[x]$$

$$= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x} + np$$

$$= \sum_{x=2}^{n} \frac{x(x-1)\cdot n(n-1)\cdot(n-2)!}{x\cdot(x-1)(x-2)!\,(n-x)!} p^2 p^{x-2} (1-p)^{n-x} + np$$

$$= \cancel{\sum_{x=2}^{n}}$$

$$= p^2 \cdot n\cdot(n-1) \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} (1-p)^{n-x} + np$$

Let $\hat{x} = x-2$, when $x=2$, $\hat{x}=0$

when $x=n$, $\hat{x}=n-2$

$$E[x^2] = p^2 n(n-1) \sum_{\hat{x}=0}^{n-2} \frac{(n-2)!}{\hat{x}!\,(n-2-\hat{x})!} p^{\hat{x}} \cdot (1-p)^{n-\hat{x}-2} + np$$

$$\cancel{= p^2 n(n-1)}$$

$$= p^2 n(n-1) \sum_{\hat{x}=0}^{n-2} \binom{n-2}{\hat{x}} p^{\hat{x}} \cdot (1-p)^{n-2-\hat{x}} + np$$

$$= p^2 n(n-1)\,[p+1-p]^{n-2} + np$$

$$= p^2 n(n-1) + np$$

$$= np\,[p(n-1)+1]$$

$$= np\,[np-p+1]$$

$$\boxed{Var(X) = np(1-p)}$$

$$E[X^2] = p^2 n(n-1) + np$$

$$\therefore\ Var(X) = E[X^2] - (E[X])^2$$

$$= p^2 n(n-1) + np - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= -np^2 + np$$

$$= np(1-p)$$

* The CDF of X is given by..

$$\boxed{F(x) = P(X \leq x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}, \quad x = 0, 1, \ldots, n}$$

$$P(X = x) = f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

21·12·2018

Binomial Distribution

PMF

$$f(x) = P(X = x) = \cancel{p^x (1-p)^{n-x}}\ {}^nC_x \cdot p^x (1-p)^{n-x}$$

$x = 0, 1, 2, \ldots, n$

CDF:

$$F(x) = P(X < x) = \sum_{k=0}^{x} {}^nC_k \cdot p^k \cdot (1-p)^{n-k}$$

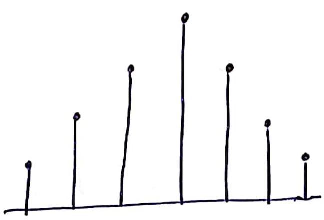
Mean

$$\mu = \text{Mean} = E(X) = np$$

Variance

$$Var(X) = \sigma^2 = np \cdot (1-p)$$

Binomial distribution follow symmetry about some point. needn't be always in midpt.



Q.) Five fair coins are flipped. If X denotes the number of heads appears in five tosses. Find pmf, mean & variance of X

Soln: X can take values 0,1,2,3,4,5

PMF: $f(x)$ ${}^nC_x$ Since X is a binomial random variable with parameters, $(n, p) = (5, 1/2)$

PMF:

$$f(x) = P(X = x) = {}^nC_x \cdot p^x \cdot (1-p)^{n-x} \quad x = 0, 1, 2, 3, 4, 5$$

$\cancel{\text{}}$

(i) $P(X=0) = {}^5C_0 \cdot \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^5$

$= \frac{1}{2^5} = \frac{1}{32}$

(ii) $P(X=1) = {}^5C_1 \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^4$

$= \frac{5}{32}$

(iii) $P(X=2) = {}^5C_2 \left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^3 = \frac{10}{32}$

(iv) $P(X=3) = {}^5C_3 \left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^2 = \frac{10}{32}$

(v) $P(X=4) = {}^5C_4 \left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right) = \frac{5}{32}$

(vi) $P(X=5) = {}^5C_5 \left(\tfrac{1}{2}\right)^5\left(\tfrac{1}{2}\right)^0 = \frac{1}{32}$

x	0	1	2	3	4	5
$P(X=x)$	1/32	5/32	10/32	10/32	5/32	1/32

Mean $= \mu = np$

$= \frac{5}{2}$ //

81
6
486

Variance $= np(1-p)$

$= 5 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{4}$ //

Q.) The chance that a bit transmitted through a digital transmission channel is received in error is 0.1. Assume that the transmission trials are independent. Let X denote the no. of bytes in error in the next 4 bits transmitted. Find the $P(X=2)$, mean & var. of X.

$P(X=2) = {}^4C_2 \cdot \left(\tfrac{1}{10}\right)^2 \left(\tfrac{9}{10}\right)^2$

$= 6 \cdot \frac{1}{100} \cdot \frac{81}{100}$

$= \frac{486}{10000} = 0.0486$ //

Mean = np

$= 4 \times 0.1 = 0.4$

Variance $= np(1-p) = 4 \times 0.1 \times 0.9$

$= \frac{36}{100} = 0.36\%$

Q.) The mean of a binomial distribution is 3 & variance is 9/4. Find the following.

(i) Value of n

(ii) $P(X \geq 7)$

(iii) $P(1 \leq X < 6)$

(i)

$np = 3$

$np(1-p) = 9/4$

$1 - p = 3/4 \Rightarrow p = 1/4$

$np = 3$

$\Rightarrow n = 12$

(ii) $P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10) + P(X=11) + P(X=12)$

$$= {}^{12}C_7 \left(\tfrac{1}{4}\right)^7 \left(\tfrac{3}{4}\right)^5 + {}^{12}C_8 \left(\tfrac{1}{4}\right)^8 \left(\tfrac{3}{4}\right)^4 + {}^{12}C_9 \left(\tfrac{1}{4}\right)^9 \left(\tfrac{3}{4}\right)^3 + \binom{12}{11} \left(\tfrac{1}{4}\right)^{11} \left(\tfrac{3}{4}\right)^1$$

$$+ {}^{12}C_{10} \left(\tfrac{1}{4}\right)^{10} \left(\tfrac{3}{4}\right)^2 + {}^{12}C_{12} \left(\tfrac{1}{4}\right)^{12} \left(\tfrac{3}{4}\right)^0$$

$$= \frac{1}{4^{12}} \left(\cancel{122487}\ 239122\right) = 239122$$

POISSON DISTRIBUTION

A discrete random variable X is called a Poisson random variable with parameter $\boxed{\lambda = np}$ where $\lambda > 0$, if its pmf is given by:

$$\cancel{f(x) = P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots}$$

$$\boxed{f(x) = P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots}$$

Poisson Approximation of Binomial Distribution

The Poisson random variable is an approximation for a binomial random variable with parameters (n,p). When n is large & p is small enough so that np is a moderate size.

Now, the PMF of binomial distribution is

$$f(x) = P(X=x) = {}^nC_x \cdot p^x \cdot (1-p)^{n-x}$$

$$P(X=x) = \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \qquad [\because \lambda = np]$$

$$= \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(\frac{n-\lambda}{n}\right)^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} \cdot \frac{\lambda^x}{n^x} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$$

$$= \frac{n(n-1)(n-2)\ldots(n-x)!}{x!\,(n-x)!} \frac{\lambda^x}{n^x} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$$

$$= \frac{n(n-1)(n-2)\ldots(n-(x-1))}{x!\; n^x} \lambda^x \frac{(1-\lambda/n)^n}{(1-\lambda/n)^x}$$

For very large 'n' & moderate λ,

$$\left(1-\frac{\lambda}{n}\right)^n \approx e^{-\lambda}, \quad \frac{n(n-1)(n-2),\ldots(n-(x-1))}{n^x} = 1$$

and $$\left(1-\frac{\lambda}{n}\right)^x \approx 1$$

$\therefore$ For large 'n' & moderate λ,

$$P[X=x] = \frac{e^{-\lambda}\cdot\lambda^x}{x!}$$

Ex:
~~NOTE~~: The no. of customers entering a post office on a given day
The no. of rung telephone calls that are dialled in a day
The no. of α-particles dicharged in a fixed period of time from some radio-active material. The no. of mis-prints on the page of a book.

22.12.15

Poisson Distribution

PMF:

$$f(x) = P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!} \quad , x = 0, 1, 2, \ldots$$

CDF:

$$F(x) = P(X \leq x] = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}$$

Q). Suppose that the no. of typographical errors on a single page of a book has a poisson distribution with parameter $\lambda = 1/2$. Calculate the probability that there is atleast one error on the page

Soln: Let X denote the no. of errors on a page. Since X has the Poisson distribution, thus its PMF is given by

$$f(x) = P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{where } \lambda = 1/2$$

$$\Rightarrow f(x) = \frac{e^{-1/2}(1/2)^x}{x!} \quad ; x = 0, 1, 2$$

$$P[X \geq 1] = 1 - P[X<1]$$

$$= 1 - P[X=0]$$

$$= 1 - \frac{e^{-1/2} \cdot (1/2)^0}{0!}$$

$$= 1 - e^{-1/2}$$

$$\approx 0.393$$

Q). Suppose that the probability that an item produced by a certain machine will be defective is 0.1. Find the probability that a sample of 10 items will contain atmost 1 defective item

~~Using~~ Let X denote the no. of defective items

(1) Using Binomial distribution $(n,p)=(10, 0.1)$

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= {}^{10}C_0 \left(\frac{1}{10}\right)^0 \cdot \left(\frac{9}{10}\right)^{10} + {}^{10}C_1 \left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^9$$

$$= \cancel{0.3486}\ 0.7361$$

(2) Poisson Distribution : $\lambda = np$

$$= 10 \times 0.1$$

$$= 1$$

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \frac{e^{-1} \cdot 1^0}{0!} + \frac{e^{-1} \, 1^1}{1!}$$

$$= 2 \times e^{-1}$$

$$\simeq 0.7358$$

NOTE :

~~* The expectation for~~

* Mean:

$$E[X] = \sum_{x=0}^{\infty} x f(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \cdot \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \lambda \sum_{\hat{x}=0}^{\infty} \frac{\lambda^{\hat{x}}}{\hat{x}!} \qquad \text{where } \hat{x} = x-1$$

$$= e^{-\lambda} \cdot \lambda \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots \right]$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda}$$

$$= \lambda$$

$$\therefore \boxed{E[X] = \lambda}$$

* Variance of X:

$$Var(X) = \sigma^2 = E[X^2] - (E[X])^2$$

$$E[X^2] = \sum_{x=0}^{\infty} x^2 f(x)$$

$$= \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$\cancel{= e^{-\lambda} \cdot \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}}$$

$$\cancel{= e^{-\lambda} \cdot \lambda^2 \sum_{\hat{x}=0}^{\infty} \frac{\lambda^{\hat{x}}}{\hat{x}!} \quad ; \hat{x} = x-2}$$

$$= \cancel{\lambda^2}$$

$\sigma^2 =$

$$= e^{-\lambda} \cdot \lambda \sum_{x=01}^{\infty} x \cdot \frac{\lambda^{x-1}}{(x-1)!} \qquad \begin{matrix} x-1 = \hat{x} \\ x = \hat{x}+1 \end{matrix}$$

$$= \lambda \cdot e^{-\lambda} \sum_{\hat{x}=0}^{\infty} (\hat{x}+1) \cdot \frac{\lambda^{\hat{x}}}{\hat{x}!}$$

$$= \lambda \cdot e^{-\lambda} \cdot (\lambda+1) \cdot e^{\lambda}$$

$$= \lambda^2 + \lambda$$

$$\therefore \sigma^2 = \lambda^2 + \lambda - \lambda^2$$

$$= \lambda //.$$

$$\therefore \boxed{\sigma^2 = \lambda}$$

24.12.18

POISSON DISTRIBUTION

μ = Mean = $E[X]$ = λ

σ^2 = Variance = $Var[X]$ = λ

1. Consider an experiment that consists of counting of the no. of α-particles given off in 1s by 1 g of radioactive material. If we know from past experience that on the average 3.2 such α-particles are given off. What is a good approximation to the probability that no more than two α-particles will appear?

Soln: If we think of the gram of radio-active material as consisting of a large number n of atoms then the no. of α-particles given-off will be a poisson random variable with parameter $\lambda = 3.2$

$\therefore$ The PMF of X is

$$P[X=x] = \frac{e^{-\lambda} \cdot \lambda^x}{x!} \quad , x = 0, 1, 2, \ldots$$

$$P[X \le 2] = P[X=0] + P[X=1] + P[X=2]$$

$$= e^{-3.2} + \frac{e^{-3.2} \cdot 3.2}{1} + \frac{e^{-3.2} \cdot (3.2)^2}{2}$$

$$= e^{-3.2}[1 + 3.2 + 5.12]$$

$$= 9.32 \times e^{-3.2}$$

$$= 0.3799$$

$$\simeq 0.38$$

2.) Message arrived at a switch board in a poisson manner at an average rate of 6 per hour. Find the probability for each of the following events.

(i) Exactly 2 messages arrive within 1 hour

(ii) No message arrives in one hr.

(iii) Atleast 3 msgs arrive within 1 hour.

Given. Let X denote the no. of messages

$\lambda = 6$

(ii) $P[X=0] = \dfrac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-6} = 2.478 \times 10^{-3}$

$= 0.002478$

(i) $P[X=2] = \dfrac{e^{-6} \cdot 6^2}{2!} = e^{-6} \cdot 18 = 0.0446$

(iii) $P[X \geq 3] = 1 - P[X \lesssim 3]$

$= 1 - P[X=0] - P[X=1] - P[X=2]$

$= 1 - 2.478 \times 10^{-3} - 0.01487$

$= 1 - 0.0619$

$= 0.938$

3. Suppose that the no. of flaws follows a poisson distribution, with a mean of 2.3 flaws per mm on a Cu wire. Determine the probability of 10 flaws in five mm of wire

Soln: Let X denote the no. of flaws in 5 mm. of wire

The X has a poisson distribution with mean,

$\therefore \lambda = 5 \times 2.3$

$= 11.5$

$\therefore P[X=10] = \dfrac{e^{-11.5} \times (11.5)^{10}}{10!} = 0.1129$ //.

Note : If X is a Poisson with parameter λ, then

$$\frac{P[X=x+1]}{P[X=x]} = \frac{e^{-\lambda} \cdot \lambda^{x+1}/(x+1)!}{e^{-\lambda} \cdot \lambda^{x}/x!}$$

$$= \frac{\lambda^{x}}{x!} \cdot \frac{\lambda}{x+1}$$

$$\Rightarrow \boxed{P[X=x+1] = \frac{\lambda}{x+1} \cdot P[X=x]}$$

Note : $P[X=x] = \dfrac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

$P[X=0] = e^{-\lambda}$ $\qquad\qquad e^{-\lambda} =$

Q). Suppose that the no. of customers that enter a bank in an hour is a poisson random variable and suppose that $P[X=0] = 0.05$. Determine the mean, variance

$e^{-\lambda} = 0.05$

$-\lambda = \ln(0.05)$

$\Rightarrow \lambda = 2.996$

$\lambda \simeq 3$

$\therefore \mu = 3$

& $\sigma^2 = 3$

$\therefore P[X=1] = \dfrac{\lambda}{0+1} \cdot P[X=0]$

$= \cancel{3}\ 2.9957 \times 0.05$

$= 0.149786613$

Special Probability distribution on cont. Random Variable.

(i) Cont. Uniform Distribution

(ii) Exponential distribution

26.12.18

Exponential distribution

* Let the random variable $N(t)$ denotes the no. of flaws in "x" mm of wire. If the mean no. of flaws is λ per mm, $N(t)$ has a Poisson distribution with mean "λx". The PMF of "N" is

$$\boxed{f(x) = \frac{e^{-\lambda x} (\lambda x)^t}{t!}, \quad t = 0, 1, 2, \ldots}$$

* The distance between flaws is another random variable that is often of interest

* Let the random variable X denote the length from any starting point on the wire until a flaw is detected

* The distribution of X can be obtained from knowledge of the distribution of the number of flaws

* The distance to the 1st flaw exceeds x mm iff there are no flaws with a length of x mm.

i.e.

$$P[X > x] = P[N = 0]$$

$$= \frac{e^{-\lambda x} \cdot (\lambda x)^0}{0!}$$

$$= e^{-\lambda x} \quad //$$

$$\Rightarrow \boxed{P[X > x] = e^{-\lambda x}}$$

Now, The CDF of X is,

$$F(x) = P[X \leq x]$$

$$= 1 - P[X > x]$$

$$\Rightarrow \boxed{F(x) = 1 - e^{-\lambda x}}$$

The random variable X that equals the distance b/w the successive counts of a Poisson process with mean $\lambda > 0$ is an exponential random variable with parameter λ. The PDF of X is $\lambda \cdot e^{-\lambda x}$, for $0 \le x < \infty$

WKT, $f(x) = \frac{d}{dx} F(x)$

$$\boxed{f(x) = \lambda \cdot e^{-\lambda x}} \quad x \ge 0$$

The PDF of X is

$f(x) = \lambda e^{-\lambda x}$; for $0 \le x < \infty$

* The expected value of X is given by

$$E[X] = \int_0^\infty x \cdot f(x)\, dx$$

$$= \lambda \left[x \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 1 \cdot \left(\frac{e^{-\lambda x}}{\lambda^2} \right) \right]_0^\infty$$

$$\boxed{E[X] = \frac{1}{\lambda}}$$

* The variance of X:

$$V(x) = E[x^2] - (E[x])^2$$

$$E[x^2] = \int_0^\infty x^2 f(x)\, dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x^2 e^{-\lambda x}\, dx = \lambda \left[x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{\lambda^3} \right) \right]_0$$

$$E[x^2] = \frac{2}{\lambda^2}$$

$$\therefore \ Var(x) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

$$\boxed{\sigma^2 = 1/\lambda^2}$$

Q). Suppose that the length of a phone call in mins is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth find the probability that you will have to wait:

(1) more than 10 mins

(2) b/w 10 & 20 mins

Soln: Let X denotes the length of the phone call in mins

The PDF of X is

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

where $\lambda = 1/10$

$$\therefore f(x) = \frac{1}{10} e^{-1/10 \, x} \quad ; x \geq 0$$

(1) $P[X > 10] = \int_{10}^{\infty} f(x) \, dx$

$$= \int_{10}^{\infty} \frac{1}{10} e^{-x/10} \, dx$$

$$= \frac{1}{10} \left[\frac{e^{-x/10}}{-1/10} \right]_{10}^{\infty}$$

$$= -\left[0 - e^{-1} \right]$$

$$= e^{-1} = 0.368$$

(or)

$P[X \geq 10] = 1 ~~- P[X<x]~~$

$= e^{-\lambda x}$

(ii) $P[10 < x < 20] = \int_{10}^{20} \frac{1}{10} e^{-x/10} \, dx$

$$= \cancel{\frac{1}{10}} - \left[e^{-x/\cancel{10}} \right]_{10}^{20}$$

$$= e - e^{-2} + e^{-1}$$

$$= 0.2325$$

Q). In a large corporate comp. n/w, user log ons to the system can be modelled as a Poisson process with a mean of 1/25 log ons per hour. (i) Probability that there are no log on in an interval of 6 mins. (ii) Probability that the time until the next logon is b/w 2 & 3 mins (iii) Determine the time interval such that the probability that no log ons occurs in the interval is 0.90.

Soln: Let X denote the time in hour from the first logon

The PDF of X is

$$f(x) = \lambda e^{-\lambda x} \quad , 0 \leq x < \infty$$

Mean $= \frac{1}{25}$

$\Rightarrow \frac{1}{\lambda} = \frac{1}{25}$

$\Rightarrow \lambda = 25$

$$\therefore f(x) = 25 e^{-25x} \quad , 0 \leq x < \infty$$

(i) $P[x > 6 \text{ min}] = P[X > 0.1 \text{ hour}]$

$$= e^{-\lambda x}$$

$$= e^{-25 \times 0.1}$$

$$= 0.08211$$

(ii) $P[2m < X < 3m] = P\left[\frac{2}{60} < x < \frac{3}{60}\right]$

$$= \int_{2/60}^{3/60} e^{-25x} \, dx \quad = 5.9 \times 10^{-3}$$

$$= 0.152$$

(iii) $P[X > x] = 0.9$

$$e^{-25x} = 0.9$$

$$-25x = 0.9$$

$$x = 0.0963$$

(iii) $P[X > x] = 0.9$

$$1 - P[X \le x] = 0.9$$

$$1 - F(x) = 0.9$$

$$0.9 = 1 - (1 - e^{-25x})$$

$$\Rightarrow e^{-25} = 0.9$$

$$-25x = \ln(0.90)$$

$$x = 0.1054 \text{ hr}$$

$$= 0.213 \text{ min}$$

28-12-18

Mean and Variance of distribution fns. using moment generation fn.

* $E[X], E[X^2], E[X^3], \ldots$ are called the moments about the origin or raw moments
* $E[X-\mu], E[X-\mu]^2, E[X-\mu]^3, \ldots$ are called the moments about the mean or central moments

Moment Generating Function (MGF):

$$M_X(t) = E[e^{tx}] = \begin{cases} \sum_{x=-\infty}^{\infty} e^{tx} f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx & \text{if } X \text{ is continuous.} \end{cases}$$

If X is a random variable, the it's moment Generating fn. is defined as above.

Now,

$$M_x(t) = E\left[e^{tx}\right] = E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots\right]$$

$$= 1 + t \cdot E[x] + \frac{t^2}{2!} \cdot E[x^2] + \dots + \frac{t^r}{r!} \cdot E[x^r] + \dots$$

The coefficient of,

$$\frac{t^r}{r!} = \boxed{E[x^r] = \left\{ \frac{d^r}{dt^r} M_x(t) \right\}_{t=0}}$$

Property 1:

$$M_{cx}(t) = M_x(ct)$$

Property 2: $M_{x_1+x_2}(t) = M_{x_1}(t) \cdot M_{x_2}(t)$

(ii) Binomial Random Variable & its distribution.

The PMF of x is

$$f(x) = {}^nC_x \cdot p^x \cdot (1-p)^{n-x} \quad ; x = 0, 1, 2, \dots$$

MGF :

$$M_x(t) = E\left[e^{tx}\right] = \sum_{x=-\infty}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{\infty} \binom{n}{x} (e^t p)^x (1-p)^{n-x}$$

$$\boxed{M_x(t) = (e^t p + 1 - p)^n}$$

* Mean of x:

$$\mu = E[x] = \left\{ \frac{d}{dt} M_x(t) \right\}_{t=0}$$

$$= \left[\frac{d}{dt}\left(e^t p + 1 - p\right)^n\right]_{t=0}$$

$$= \left[n\left[e^t p + 1 - p\right)^{n-1} \cdot p e^t\right]_{t=0}$$

$$= \left[n \cdot 1 \cdot p \cdot 1\right]$$

$$\boxed{E[x] = np}$$

* Variance of X :

$$E[x^2] = \left[\frac{d^2}{dt^2}\{M_x(t)\right]_{t=0}$$

$$\cancel{\frac{d}{dt}\left[\right.}$$

$$= \left[\frac{d^2}{dt^2}\left(e^t p + 1 - p\right)^n\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{n\left(e^t p + 1 - p\right)^{n-1} \cdot p e^t\right\}\right]_{t=0}$$

$$= \left[np \cdot \left\{\left(e^t p + 1 - p\right)^{n-1} \cdot e^t + e^t \cdot (n-1)\left(e^t p + 1 - p\right)^{n-2} \cdot e^t p\right\}\right]_{t=0}$$

$$= \left[np\left[1 \cdot 1 + 1 \cdot (n-1) \cdot p\right]\right]$$

$$= np\left[1 + (n-1)p\right]$$

$$\therefore Var(x) = \sigma^2 = E[x^2] - (E[x])^2$$

$$= np\left[1 + (n-1)p\right] - (np)^2$$

$$= np\left[1 + np - p - np\right]$$

$$\boxed{\sigma^2 = np(1-p)}$$

Poisson Random Variable & its distribution

The PMF of X is,

$$f(x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} \qquad x = 0, 1, \dots$$

MGF

$$M_X(t) = E[e^{tx}] = \sum_{x=-\infty}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \cdot (\lambda \cdot e^{t})^{x}}{x!}$$

$$= e^{-\lambda} \left[1 + \frac{e^{t}\lambda}{1!} + \frac{(e^{t}\lambda)^2}{2!} + \dots \right]$$

$$= e^{-\lambda} \cdot e^{\lambda e^{t}}$$

$$= e^{\lambda e^{t} - \lambda}$$

$$\boxed{M_X(t) = e^{\lambda(e^{t}-1)}}$$

* Mean of X :

$$E[X] = \left[\frac{d}{dt} \cdot M_X(t) \right]_{t=0}$$

$$= \left[\frac{d}{dt} e^{\lambda(e^{t}-1)} \right]_{t=0}$$

$$= \left[\lambda e^{t} \cdot e^{\lambda(e^{t}-1)} \right]_{t=0}$$

$$= \lambda$$

$$\therefore \boxed{E[X] = \lambda}$$

* Variance of X:

$$E[x^2] = \left[\frac{d}{dt}\left(\lambda e^t \cdot e^{\lambda(e^t-1)}\right)\right]$$

$$= \left[\lambda e^t\left[\lambda e^t \cdot e^{\lambda(e^t-1)}\right] + e^{\lambda(e^t-1)}\lambda e^t\right]_{t=0}$$

$$= \left[\lambda\left[\lambda \cdot 1\right] + \lambda\right]$$

$$= \lambda(\lambda+1)$$

$$= \lambda^2 + \lambda$$

$$Var(X) = \sigma^2 = E[x^2] - (E[x])^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

$$\boxed{\sigma^2 = \lambda}$$

Continuous Uniform Distribution:

The ~~pro~~ PDF of x is

$$f(x) = \begin{cases} 1/b-a & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Mean ~~usin~~ and Variance using MGF:

$$M_x(x) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{e^{bt} - e^{at}}{t}\right] = \frac{1}{(b-a)t}\left(\left[1 + \frac{bt}{1!} + \frac{(bt)^2}{2!} + \dots\right] - \left[1 + \frac{at}{1!} + \frac{(at)^2}{2!} \dots\right]\right)$$

$$= \frac{1}{(b-a)t}\left[\frac{(b-a)t}{1!} + \frac{(b^2-a^2)}{2!}t^2 + \frac{(b^3-a^3)t^3}{3!} + \dots\right]$$

$$\frac{1}{(b-a)t}$$

$$= \frac{t}{t}\left[1 + \frac{(b+a)t}{2!} + \frac{b^2+ab+a^2}{3!} + \dots\right]$$

Mean $= E[x]$

$$= \frac{d}{dt}\left[1 + \left(\frac{b+a}{2!}\right)t + \dots\right]\Bigg|_{t=0}$$

$$= \frac{b+a}{2}$$ //.

$100 \cdot \frac{t}{3} = \frac{100}{3} \cdot \frac{2}{3}$

$= \frac{200}{9}$

$np < np(1-p)$

$1 < 1-p \; q$

$p < 0$

$$E[x^2] = \frac{d^2}{dt^2}\left[M_x(e^{tx})\right]\Bigg|_{t=0}$$

$$= \frac{d^2}{dt^2}\left[a + \left(\frac{b+a}{2}\right)t + \frac{b^2+ab+a^2}{3!}\cdot 2t + \dots\right]\Bigg|_{t=0}$$

$$= \frac{b^2+ab+a^2}{3}$$

Variance, $V(x) = E[x^2] - (E[x])^2$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+ab^2}{4}$$

$$= \frac{4b^2+4ab+2a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$ //.

Exponential Distribution function.

The PDF of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & 0 < x < \infty, \ \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

Mean and Variance using MGF:

$$E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tx} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} e^{-(\lambda - t)x}\, dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \lambda \cdot \frac{1}{\lambda - t}$$

$$= (1 - t/\lambda)^{-1}$$

Mean, $E[X] = \frac{d}{dt} M_x(e^{tx}) \Big|_{t=0}$

$$= \left\{ \frac{d}{dt} (1 - t/\lambda)^{-1} \right\}_{t=0}$$

$$= -1 \cdot (1 - t/\lambda)^{-2} \cdot -1/\lambda$$

$$= 1/\lambda //.$$

$$E[X^2] = \frac{d^2}{dt^2} \left[M_x(e^{tx}) \right]_{t=0}$$

$$= \left\{ \frac{d}{dt} \left[\frac{1}{\lambda} \cdot (1 - t/\lambda)^{-2} \right] \right\}_{t=0}$$

$$= \left[\frac{1}{\lambda} \cdot -2 \cdot \left(1 - \frac{t}{\lambda}\right)^{-3} \cdot -\frac{1}{\lambda} \right]_{t=0}$$

$$= \frac{2}{\lambda^2}$$

$$\therefore \text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2} //$$

DISCREET

A random variable X has a discrete distribution $P[X=x] = \frac{1}{k} \cdot x$, $x = 1, 2, \ldots, k$. Find MGF of $E[x]$.

Given,

$$P[X=x] = \sum_{x=1}^{k} \frac{1}{k}$$

MGF,

$$E[e^{tx}] = \sum_{x=-\infty}^{\infty} e^{tx} \cdot f(x)$$

$$= \sum_{x=1}^{k} e^{tx} \frac{1}{k}$$

$$= \frac{1}{k} \sum_{x=1}^{k} e^{tx}$$

$$= \frac{1}{k} \left[e^{t} + e^{2t} + e^{3t} + \ldots + e^{kt} \right]$$

$$\cancel{= \frac{e^{t}}{k} \left[1 + e^{t} + e^{2t} + \ldots + e^{(k-1)t} \right]}$$

$$= \frac{1}{k} \left[\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots\right) + \left(1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \ldots\right) + \ldots \right]$$

t^2

$2t$

$$E[x] = \frac{d}{dt} M_x(t) \Big]_{t=0}$$

$$= \frac{1}{k}(1+2+3+\ldots+k)$$

$$= \frac{k(k+1)}{2k}$$

$$= \frac{k+1}{2}$$

$$\cancel{E[x^2] = \left[\frac{d^2}{dt^2}\{M_x(t)\}\right]_{t=0}}$$

$$\cancel{= 1+}$$

a b

a b)

b . c

c d

$$E[x^2] = \left[\frac{d^2}{dt^2}\{M_x(t)\}\right]_{t=0}$$

$$= \left[\frac{d^2}{dt^2}\left(e^t + e^{2t} + e^{3t} + \ldots\right)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(e^t + 2e^{2t} + 3e^{3t} + \ldots\right)\right]_{t=0}$$

$$= \frac{1}{k}(1+4+9+\ldots k^2)$$

$$= \frac{1}{\cancel{k}} \cdot \frac{\cancel{k}(k+1)(2k+1)}{6}$$

$$= \frac{2k^2+k+2k+1}{6} = \frac{2k^2+3k+1}{6}$$

Variance, $V(x) = E[x^2] - (E[x])^2$

$$= \frac{2k^2+3k+1}{6} - \frac{k^2+2k+1}{4}$$

$$= \frac{4k^2+6k+2-3k^2-6k-3}{12}$$

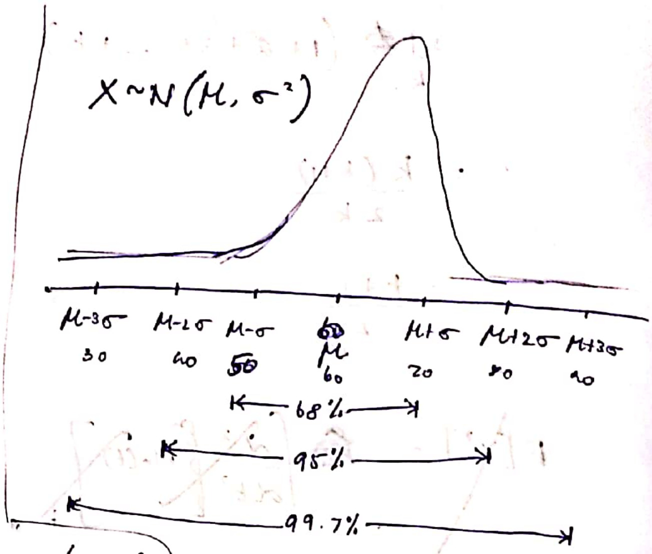
$$= \frac{k^2-1}{12}$$

21.1.19

NORMAL OR GAUSSIAN DISTRIBUTION

$f(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\int_{-\infty}^{\infty} f(x) = \sigma\sqrt{2\pi}$



But for $f(x)$ to be PDF,
$\int_{-\infty}^{\infty} f(x)\,dx$ should be 1.

$\therefore$ we divide $f(x)$ by $\sigma\sqrt{2\pi}$ as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

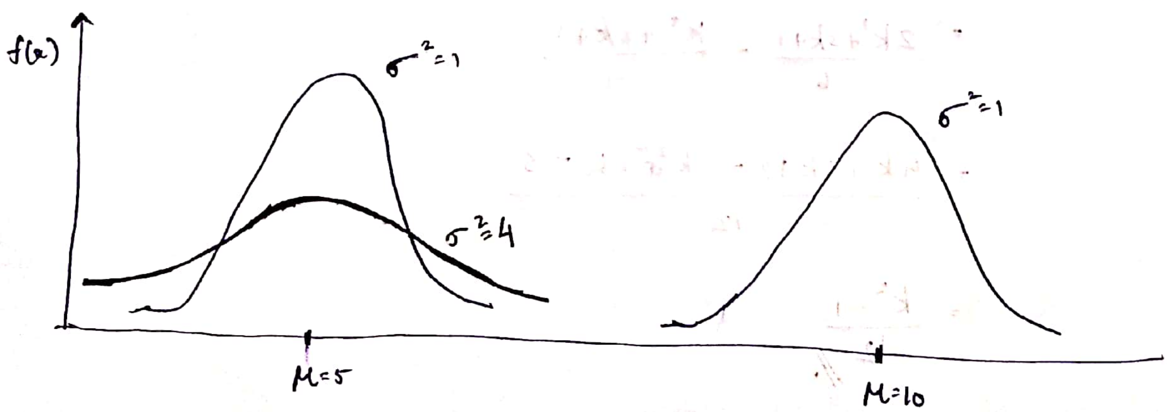
Width : 6σ

The normal curve will be symmetric about the mean.
$\therefore P(X>\mu) = P(X<\mu) = 0.5$

A random variable X with PDF $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in (-\infty, \infty)$ is a normal random variable with parameter μ where $\sigma > 0$. Also $E[X] = \mu$ & variance of $X = \sigma^2$ and the notation $N(\mu, \sigma^2)$ is used to denote the distribution.

NOTE:

* The value of $E[X] = \mu$ determines the centre of the PDF (Probability Density Function) and the value of variance of $X = \sigma^2$ determines the width



* For any normal random variable, by symmetry of $f(x)$

(i) $P(X>\mu) = P(X<\mu) = 0.5$

* For any normal random variable,

(i) $P(\mu-\sigma < X < \mu+\sigma) = 0.6827$

(ii) $P(\mu-2\sigma < X < \mu+2\sigma) = 0.9545$

(iii) $P(\mu-3\sigma < X < \mu+3\sigma) = 0.9973$

STANDARD NORMAL RANDOM VARIABLE:

A normal random variable with $\mu=0$ and $\sigma^2=1$ is called a standard normal random variable and is denoted as Z

$$f(x) = e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}}$$

i.e. $$f(z) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$P(Z \leq z) = \int_{-\infty}^{z} f(u)\,du$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du$$

The CDF of Standard NRV, is denoted as $\Phi(z) = P(Z \leq z)$

NOTE: Standardizing a random variable X

* If X is a normal random variable with $E[X]=\mu$ and variance of $X = \sigma^2$ then the random variable

$$Z = \frac{X-\mu}{\sigma}$$ is a normal random variable

with $E[Z]=0$ and $Var(Z)=1$ i.e. Z is a standard normal random variable. ~~Suppose~~

* Suppose X is a normal random variable with mean μ and variance σ^2, then $P(X \leq x)$ will be equal to $P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$ $P\left(Z \leq \frac{x-\mu}{z}\right)$

$= P[Z \leq z]$

Where X is a standard normal variable and $z = \frac{x-\mu}{\sigma}$ is the Z value obtained by standardizing X.

Q). Assume Z is a std. normal random variable, find

(i) $P(Z > 1.26)$ (ii) $P(Z < -0.86)$ (iii) $P(Z > -1.37)$ (iv) $P(-1.25 < Z < 0.37)$

Soln:

(i) $P(Z > 1.26) = 1 - P(Z \leq 1.26)$

$= 1 - 0.89616$

$= 0.10384$

(ii) $P(Z < -0.86) = 0.19490$

(iii) $P(Z > -1.37) = 1 - P(Z \leq -1.37)$

$= P(Z \leq 1.37)$

(iv) $P(-1.25 < Z < 0.37) = P(Z \leq 0.37) - P(Z \leq -1.25)$

22-1-19

If X is normally distributed with $\mu = 3$, $V(x) = 9$.

$X \sim N(3, 9)$ which means that X is a normal random variable with mean, $\mu = 3$ and variance, $\sigma^2 = 9$. Find the probability that X lies b/w 2 and 5

Soln: Gn// $\mu = 3$ & $\sigma^2 = 9$

$$P(2 < X < 5) = P\left[\frac{2-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right]$$

$$= P\left[\frac{2-3}{3} < Z < \frac{5-3}{3}\right] \quad \because Z = \frac{X-\mu}{\sigma}$$

$= P[-1/3 < Z < 2/3]$

$= P[-0.33 < Z < 0.67]$

~~$= P(0.67) - P(-0.33)$~~

$= P(Z \leq 0.67) - P(Z \leq -0.33)$

$= 0.7486 - 0.37070$

$= 0.37791.$

Q7. Suppose that the weights of 800 students are normally distributed with mean, $\mu = 140$ pounds & std. dev. 10 pounds. find the no. of students whose weights are b/w 138 & 148 (ii) more than 152

$$P(138 \leq X \leq 148) = P\left(\frac{138-140}{10} < \frac{X-\mu}{\sigma} < \frac{148-140}{10}\right)$$

$$= P\left(\frac{-2}{10} < Z < \frac{8}{10}\right)$$

$$= P(-0.2 < Z < 0.8)$$

$$= P(Z \leq 0.8) - P(Z \leq -0.2)$$

$$= 0.788145 - \cancel{0.385908}\ 0.420740$$

$$= \cancel{0.402237}\ 0.367405$$

No. of students $= 800 \times 0.367405$

$= 294$

(ii) $P(X \geq 152) = 1 - P(X \leq 152)$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{152-140}{10}\right)$$

$$= 1 - P(X \leq 1.2)$$

$$= 1 - 0.884930$$

$$= 0.11507$$

No. of students $= 800 \times 0.11507$

$= 92$ students.

Q) Let X be a normal random variable $X \sim N(\mu, \sigma)$ mean μ, std. dev $= \sigma$. If Z is the std. normal random variable such that $Z = -0.8$ when $X = 26$ &

$Z = 2$ when $X = 40$

$$Z = \frac{X-\mu}{\sigma}$$

$$2 = \frac{40-\mu}{\sigma} \qquad -0.8 = \frac{26-\mu}{\sigma}$$

$$2\sigma = 40-\mu$$

$$\mu + 2\sigma = 40 \quad -(1) \qquad \& \quad \mu - 0.8\sigma = 26 \quad -(2)$$

$\Rightarrow (1) - (2) \Rightarrow$

$$2.8\sigma = 14$$

$$\sigma = \frac{14}{2.8} = \frac{1}{0.2} = 5 \qquad \boxed{\sigma = 5}$$

$$\cancel{2\sigma}\ 10 = 40 - \mu \Rightarrow \boxed{\mu = 30}$$

$$P[X > 45] = 1 - P[X \leq 45]$$

$$= 1 - P\left[\frac{X-\mu}{\sigma} < \frac{45-30}{5}\right]$$

$$= 1 - P[Z < 3]$$

$$= 1 - 0.99865$$

$$= 0.00135$$

$$P[|X-30| > 5] = 1 - P[-5 < X-30 < 5]$$

$$= 1 - P[25 < X < 35]$$

$$= 1 - P[-1 < X < 1] = 1 - [P(X \leq 1) - P(X \leq -1)]$$

$$= 1 - [0.84135 - 0.15865]$$

$$= 1 - 0.68269 = 0.31731$$

Q). If X is a normal random variable with mean 12 and ~~std. dev~~ var = 16. Find value of x such that $P(X > x) = 0.24$

$$P(X > x) = 0.24$$

$$0.24 = 1 - P(X \leq x)$$

$$= 1 - P\left(Z \leq \frac{x-\mu}{\sigma}\right)$$

$$\Rightarrow P\left(Z \leq \frac{x-12}{4}\right) = 0.76$$

$$\frac{x-12}{4} = 0.71$$

$$x = 14.84$$

23.1.19

Derivation of Mean and Variance using MGF.

WKT,

$$M_X(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

~~Let Z =~~

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\, dx$$

Let $z = \frac{x-\mu}{\sigma}$, $x = -\infty \Rightarrow z = -\infty$, $x = \infty \Rightarrow z = \infty$

$dx = \sigma\, dz$

$\because x = \sigma z + \mu$

$$\therefore M_X(t) = \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2}\, \sigma\, dz$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\mu t} \int_{-\infty}^{\infty} e^{-(z^2/2 - \sigma t)}\, \sigma\, dz$$

$$= \frac{e^{\mu t} \cdot \sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2\sigma t + \sigma^2 t^2 - \sigma^2 t^2)}\, dz$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

$$= \cancel{\frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}}}$$

Let $u = z - \sigma t$

$du = dz$

$$M_x(t) = \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du$$

$$= \frac{e^{\mu t + \frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \cdot \sqrt{2\pi}$$

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Mean:

$$E[X] = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[e^{\mu t + \frac{\sigma^2 t^2}{2}} (\mu + \sigma^2 t)\right]_{t=0}$$

$$= \mu$$

Variance:

$$E[X^2] = \left[\frac{d}{dt}\left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right)(\mu + \sigma^2 t)\right]_{t=0}$$

$$= \left[\sigma^2 \cdot e^{\mu t + \frac{\sigma^2 t^2}{2}} + (\mu + \sigma^2 t)^2 e^{\mu t + \frac{\sigma^2 t^2}{2}}\right]_{t=0}$$

$$= \sigma^2 + \mu^2$$

$$\therefore \operatorname{Var}(X) = E[X^2] - (E[X])^2$$

$$= \sigma^2 + \mu^2 - \mu^2$$

$$= \sigma^2$$

Q) In a digital comm channel assume that the no. of bits received in error can be modeled by a binomial random variable & assume that the P(a bit is received in error) = 10^{-5}, If 16 million bits are transmitted, P(X > 150 errors)

$$\lambda = 16{,}000{,}000 \times 10^{-5} = 160$$

P(X > 150 - λ)

NORMAL APPROXIMATION TO BINOMIAL & POISSON DISTRIBUTION.

If X is a binomial random variable,

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately a standard normal random variable.

NOTE: The approx. is good if $np > 5$ & $n(1-p) > 5$

Binomial:

$$P(X > 150) = 1 - P[X \leq 150]$$

$$= 1 - \sum_{x=0}^{150} \binom{16000000}{150} (10^{-5})^{x} (1-10^{-5})^{16\times10^6 - x}$$

$$Z = \frac{X - 160}{\sqrt{160(1-10^{-5})}} = 6.25 \times 10^{-3} (X - 160)$$

$$\therefore P(Z > \overset{-0.79}{\cancel{100}}) = 1 - P(Z \leq 150)$$

$= P(Z < 0.79)$

$= 0.7852366$

$$= 1 - \int_{0}^{-0.79} P(\cancel{6.25\times10^{-3}})$$

P

$$= 1 - P(-0.79)$$

$$= 1 - 0.214764$$

$$= 0.785\cancel{9}2364$$

NORMAL APPROX. TO POISSON DISTRIBUTION:

If X is a poisson random variable,

$$Z = \frac{X-\lambda}{\sqrt{\lambda}} \qquad \boxed{\begin{array}{l}\mu = \lambda \\ \sigma^2 = \lambda\end{array}}$$

is approximately a std. normal random variable

NOTE: Approximation is good for $\lambda > 5$

Q). Assume that the no. of asbestos particles in a sq. m of dust on a surface follows a poisson distribution with a mean of 1000. If a sq. m of dust is analysed, what is the P($\leq$950 particles are found).

$$P(X \leq 950) = P\left(Z \leq \frac{950-1000}{\sqrt{1000}}\right)$$

$$= P(Z \leq -1.58)$$

$$\cancel{\approx 0.063522}$$

$$= 0.057053$$

25.1.19

CHEBYSHEV'S INEQUALITY

PMF/PDF of X - unknown

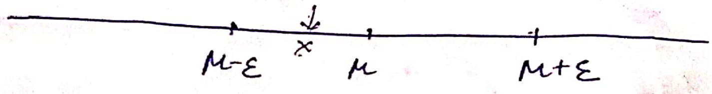
μ, σ^2 - known

If X is a random variable (discrete or continuos) with mean μ & variance σ^2, then

$$P[|X-\mu| \geq \varepsilon] \leq \frac{\sigma^2}{\varepsilon^2}$$ (Upper bound for probability),

(or) , where ε (epsilon) is non-negative

$$P[|X-\mu| \leq \varepsilon] \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$ (Lower bound)



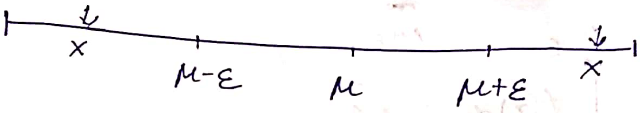
$$\Rightarrow \quad \mu - \varepsilon < X \le \mu + \varepsilon$$

$$-\varepsilon < X - \mu \le \varepsilon$$

$$\Rightarrow \quad |X - \mu| \le \varepsilon$$

$$\therefore \; P[\mu - \varepsilon \le X \le \mu + \varepsilon] = P[|X - \mu| \le \varepsilon]$$

Case (1)



$$X \le \mu - \varepsilon \qquad\qquad x \ge \mu + \varepsilon$$

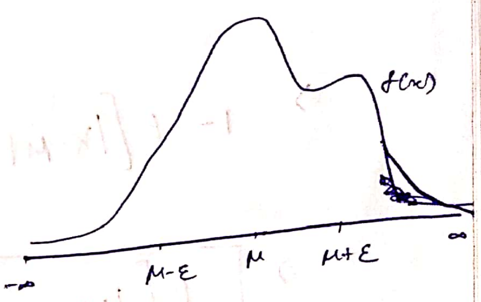
$$X - \mu \le -\varepsilon \qquad\qquad x - \mu \ge \varepsilon$$

$$\Rightarrow \quad |X - \mu| \ge \varepsilon$$

<u>Proof</u> (Continuous)

Let X be a cont. random variable with PDF $f(x)$.

Then by defn.,



$$\sigma^2 = V[X] = E[(X-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\mu-\varepsilon} (x-\mu)^2 f(x)\,dx + \int_{\mu-\varepsilon}^{\mu+\varepsilon} (x-\mu)^2 f(x)\,dx + \int_{\mu+\varepsilon}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\Rightarrow \quad \sigma^2 \ge \int_{-\infty}^{\mu-\mu} (x-\mu)^2 f(x)\,dx + \int_{\mu+\varepsilon}^{\infty} (x-\mu)^2 f(x)\,dx$$

← Eliminating the middle term in RHS, we have [∵ it is nonnegative]

In "I"st integral

'x' varies from $-\infty$ to $\mu-\varepsilon$

$x \le \mu - \varepsilon$

$x - \mu \le -\varepsilon$

$-(x-\mu) \ge \varepsilon$

$(x-\mu)^2 \ge \varepsilon^2$

in IInd integral

'x' varies from $\mu+\varepsilon$ to ∞

$x \ge \mu + \varepsilon$

$x - \mu \ge \varepsilon$

$(x-\mu)^2 \ge \varepsilon^2$

$$\sigma^2 \ge \int_{-\infty}^{\mu-\varepsilon} \varepsilon^2 f(x)\,dx + \int_{\mu+\varepsilon}^{\infty} \varepsilon^2 f(x)\,dx$$

$$\ge \varepsilon^2 \int_{-\infty}^{\mu-\varepsilon} f(x)\,dx + \varepsilon^2 \int_{\mu+\varepsilon}^{\infty} f(x)\,dx$$

$$\ge \varepsilon^2 \left[P(X \le \mu-\varepsilon) + P(X \ge \mu+\varepsilon) \right]$$

$$\sigma^2 \ge \varepsilon^2 . P\left[|X-\mu| \ge \varepsilon \right]$$

$$\Rightarrow \boxed{P\left[|X-\mu| \ge \varepsilon \right] \le \frac{\sigma^2}{\varepsilon^2}}$$

$$\Rightarrow 1 - P\left[|X-\mu| \le \varepsilon \right] \le \frac{\sigma^2}{\varepsilon^2}$$

$$\Rightarrow \boxed{P\left[|X-\mu| \le \varepsilon \right] \ge 1 - \frac{\sigma^2}{\varepsilon^2}}$$

Q). A random variable X has a mean of 4 and variance of 2. Use Chebbyshev's inequality to obtain an upper bound for

(i) $P(|X-4| \geq 3)$

Soln: $P(|X-4| \geq 3) \leq \frac{\sigma^2}{\varepsilon^2} \leq \frac{2}{9}$ //.

Q). A random variable X has mean 12 & variance 9. Using Chebbyshev's inequality estimate the lower bound for

(i) $P(6 < X < 18)$ (ii) $P[3 < X \leq 21]$

Soln:

(i) $P(6 < X < 18) = P(-6 < X - 12 < 6)$

$= P(|X-12| \leq 6)$

$\geq 1 - \frac{9}{36} = \frac{27}{36}$

$\geq \frac{27}{36}$ // $\geq \frac{3}{4}$

Upper bound = 1

(ii) $P(3 < X < 21) = P(-9 < X-12 < 9)$

$= P(|X-12| < 9)$

$\geq 1 - \frac{9}{81} = \frac{72}{81} \frac{8}{9}$

$\geq \frac{8}{9}$ //.

Upper bound = 1.

Q). Compute $P[\mu - 2\sigma < X < \mu + 2\sigma]$, where X has density fn.

$f(x) = 6x(1-x), \quad 0 < x < 1$

& compare with Chebyshev's lower bound.

Soln: $\mu = \int_0^1 x f(x)\, dx$

$= \int_0^1 (6x^2 - 6x^3)\, dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$ //.

Variance;

$$E[X^2] = \int_0^1 x^2\, 6x\cdot(1-x)\, dx$$

$$= 6\int_0^1 x^3 - x^4 \, dx$$

$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{6}{20} = \frac{3}{10}$$

$$\sigma^2 = \frac{3}{10} - \frac{1}{4} = \frac{6-5}{20} = \frac{1}{20} \,//.$$

$2\sigma = \frac{2}{\sqrt{2}}$ $\quad \mu - 2\sigma = \frac{1}{2} - \frac{\sqrt{2}}{\cancel{\sqrt{2}}}$

$$\therefore P(\mu-2\sigma < X < \mu+2\sigma)$$

$$= P\left(\frac{1}{2} - \sqrt{2} < X < \frac{1}{2} + \sqrt{2}\right)$$

$$= \int_{\frac{1}{2}-\sqrt{2}}^{\frac{1}{2}+\sqrt{2}} 6x(1-x)\, dx$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{\frac{1}{2}-\sqrt{2}}^{\frac{1}{2}+\sqrt{2}}$$

$$= 6\left[\frac{\left(\frac{1}{2}+\sqrt{2}\right)^2}{2} - \frac{\left(\frac{1}{2}+\sqrt{2}\right)^3}{3} - \frac{\left(\frac{1}{2}-\sqrt{2}\right)^2}{2} + \frac{\left(\frac{1}{2}-\sqrt{2}\right)^3}{3}\right]$$

$$= 0.9849 \,//$$

By Chebyshev's inequality,

$$P(\mu-2\sigma < X < \mu+2\sigma) \geq 0.75 \,//.$$

28.1.19

JOINT PROBABILITY DISTRIBUTION

Two discrete random variables:

The Joint PMF of discrete random variables X, Y denoted as $f_{XY}(X, Y)$ satisfies:

(i) $f_{XY}(X, Y) \geq 0$

(ii) $\sum_x \sum_y f_{xy}(x, y) = 1$

(iii) $f_{XY}(x, y) = P(X = x, Y = y)$

Marginal PMF:

If X & Y are discrete random variables with joint pmf, $f_{xy}(x, y)$ then the marginal pmf of X & Y are defined as

$$f_X(x) = P[X = x] = \sum_y f_{xy}(x, y)$$

and $$f_Y(y) = P[Y = y] = \sum_x f_{XY}(x, y)$$

Independent Random Variables:

If X & Y are independent random variables, then

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

Q). Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white & 5 blue balls. If we let X & Y denote respectively the no. of red & white balls chosen, then find the joint pmf of X, Y.

					Row sum $P[Y=y]$
3	4/220				4/220
2	30/220	18/220			48/220
1	40/220	60/220	12/220		112/220
0	10/220	30/220	15/220	1/220	56/220
y↑ x→	0	1	2	3	
Column sum $P[X=x]$	84/220	108/220	27/220	1/220	

(i) $f_{XY}(0,0) = \frac{{}^5C_3}{{}^{12}C_3} = \frac{10}{220} = \frac{1}{22}$

(ii) $f_{XY}(0,1) = \frac{{}^3C_0 \cdot {}^4C_1 \cdot {}^5C_2}{{}^{12}C_3} = \frac{40}{220}$

(iii) $f_{XY}(0,2) = \frac{{}^3C_0 \cdot {}^4C_2 \cdot {}^5C_1}{{}^{12}C_3} = \frac{30}{220}$

(iv) $f_{XY}(0,3) = \frac{{}^4C_3}{{}^{12}C_3} = \frac{4}{220}$

(v) $f_{XY}(1,0) = \frac{{}^3C_1 \cdot {}^5C_2}{{}^{12}C_3} = \frac{30}{220}$

(vi) $f_{XY}(1,1) = \frac{{}^3C_1 \cdot {}^4C_1 \cdot {}^5C_1}{{}^{12}C_3} = \frac{60}{220}$

(vii) $f_{XY}(1,2) = \frac{{}^3C_1 \cdot {}^4C_2}{{}^{12}C_3} = \frac{18}{220}$

~~(viii) $f_{XY}(1,3)$~~

(viii) $f_{XY}(2,0) = \frac{{}^3C_2 \cdot {}^5C_1}{{}^{12}C_3} = \frac{15}{220}$

(ix) $f_{XY}(2,1) = \frac{{}^3C_2 \cdot {}^4C_1}{{}^{12}C_3} = \frac{12}{220}$

(x) $f_{XY}(3,0) = \frac{{}^3C_3}{{}^{12}C_3} = \frac{1}{220}$

(b) Marginal PMF of X:

$P[X=x] = \sum_y f_{XY}(x,y)$

$\cancel{P[X=0] = \frac{10+40+30+4}{220} = \frac{84}{220}}$

$P[X=x] = \sum_y f_{XY}(x,y)$

Q). $f_{XY}(x,y) = \begin{cases} k(2x+y) & x=1,2;\ y=1,2 \\ 0 & \text{otherwise} \end{cases}$

where 'k' is a constant,

(i) What is the value of 'k'?

(ii) Find the marginal PMF's of X & Y.

(iii) Are X & Y independent.?

Soln:

(i) Marginal PMF of X:

$$P[X=x] = \sum_y f_{XY}(x,y)$$

$y \uparrow$ / $x \rightarrow$	1	2	
2	4k	6k	10k
1	3k	5k	8k
	7k	11k	

Total probability = 1

$\sum_x \sum_y f_{XY}(x,y) = 1$ $\qquad 18k = 1$

$3k + 4k + 5k + 6k = 1$ $\qquad \therefore k = \frac{1}{18}$

(ii) Marginal PMF of X:

X	1	2
$P[X=x]$	7/18	11/18

Y	1	2
$P[Y=y]$	8/18	10/18

$x = -3/4$

$(x+3/4)(x+3)$

$f_x(x) = \sum_y f_{XY}(x,y)$

$= \sum_{y=1,2} \frac{1}{18}(2x+y)$

$= \frac{1}{18}(4x+3)$

$f_x(y) = \sum_x f_{XY}(x,y)$

$= \sum_{x=1}^{2} \frac{1}{18}(2x+y)$

$= \frac{1}{18}(2y+6)$

(iii) $f_x(x) \cdot f_x(y) = \frac{1}{18 \cdot 18}(2y+6)(4x+3)$

$= \frac{8x^2+6x+24x+18}{18 \cdot 18} = \frac{8x^2+30x+18}{18 \cdot 18} = \cancel{\frac{(x+3/4)(x+3)}{18 \cdot 18}} = \cancel{\frac{(4x+3)(x+3)}{4 \cdot 18 \cdot 18}}$

$\neq f_{XY}(x,y)$

$\therefore X, Y$ are dependent.

Conditional Probability Distribution:

Conditional PMF:

Giv discrete random variable X, Y with joint PMF $f_{XY}(x,y)$, the conditional PMF of Y given X = x is

$$f_{Y/X}(y) = \frac{f_{XY}(x,y)}{f_X(x)} \quad \text{for } f_X(x) > 0$$

IIIly

$$f_{X/Y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad \text{for } f_Y(y) > 0$$

Independency:

$$f_{XY}(x,y) = \begin{cases} \frac{1}{18}(2x+y) & ; x=1,2;\ y=1,2 \\ 0 & ; \text{otherwise} \end{cases}$$

(i) Find conditional ~~probabil~~ PMF of X given Y

(ii) " " " " Y given X

Soln:

$f_X(x) = \frac{1}{18}(4x+3)$

$f_Y(y) = \frac{1}{18}(2y+6)$

(i) $f_{X/Y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{1}{18}(2x+y)}{\frac{1}{18}(2y+6)} = \frac{2x+y}{2y+6}$ //

(ii) $f_{Y/X}(y) = \frac{2x+y}{4x+3}$ //

1/2/19

Mean and Variance from joint Distribution:

$$E[X] = \mu_x = \sum_x x f(x)$$

$$= \sum_x x \left(\sum_y f_{XY}(x,y) \right)$$

$$= \sum_x \sum_y x \cdot f_{XY}(x,y)$$

$$\boxed{E[X] = \sum_x \sum_y x \, f_{XY}(x,y)}$$

$$V[X] = \sum_x (x-\mu_x)^2 f_X(x)$$

$$= \sum_x (x-\mu_x)^2 \sum_y f_{XY}(x,y)$$

$$\boxed{V[X] = \sum_x \sum_y (x-\mu_x)^2 f_{XY}(x,y)}$$

III ly.

$$\boxed{\begin{aligned} E[Y] &= \sum_x \sum_y y \, f_{XY}(x,y) \\ V[Y] &= \sum_x \sum_y (y-\mu_y)^2 f_{XY}(x,y) \end{aligned}}$$

1. The joint PMF of X & Y is given by,

$y \uparrow$ / $x \rightarrow$	0	1	2	3	
3	0.0375	0	0	0	0.0375
2	0.0875	0.1125	0	0	0.2
1	0.10	0.175	0.1125	0	0.3875
0	0.15	0.10	0.0875	0.0375	0.375
	0.375	0.3875	0.2	0.0375	

(i) Find Marginal PMF of X & Y. & mean, variance of X & Y

(ii)

Soln: Marginal PMF of X is

$$f_X(x) = P[X=x] = \sum_y f_{XY}(x,y)$$

(a) $f_X(0) = P[X=0] = \sum_{0}^{3} f_{XY}(0,y)$

$$= f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(0,2) + f_{XY}(0,3)$$

$$= 0.375$$

$|||^{ly}$ $f_X(1) = 0.3875$

$f_X(2) = 0.2$

$f_X(3) = 0.0375$

x	0	1	2	3
$P[X=x]$	0.375	0.3875	0.2	0.0375

& $f_Y(0) = 0.375$

$f_Y(1) = 0.3875$

$f_Y(2) = 0.2$

$f_Y(3) = 0.0375$ //.

y	0	1	2	3
$P[Y=y]$	0.375	0.3875	0.2	0.037

X:

Mean:

$E[X] = \cancel{\sum_x \sum_y x f_{XY}(x,y)}$

$E[X] = 0.375 \times 0 + 0.3875 \times 1 + 0.2 \times 2 + 0.0375 \times 3$

$= 0.9$ //.

$E[X] = \sum_x \sum_y x\, f_{XY}(x,y)$

$= \sum_0^3 \sum_0^3 x\, f_{XY}(x,y)$

$= 0[f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(0,2) + f_{XY}(0,3)] + 1[f_{XY}(1,0) + \ldots f_{XY}(1,3)] + \ldots$

$+ 3[f_{XY}(3,0) + f_{XY}(3,1) + f_{XY}(3,2) + f_{XY}(3,3)]$

$Var(X) = \sum_x \sum_y (x-\mu_x)^2 f_{XY}(x,y)$

$= (1 \times 0.3875 + 4 \times 0.2 + 9 \times 0.0375) - (0.9)^2$

$= 1.525 - 0.81$

$= \cancel{0.751}$ //. 0.715

$= (0-0.9)^2[f_{XY}(0,0) + \ldots + f_{XY}(0,3)] + (1-9)^2[f_{XY}(1,0) + \ldots + f_{XY}(1,$

$(2-0.9)^2[f_{XY}(2,0) + \ldots + f_{XY}(2,3)] + (3-0.9)^2[\quad " \quad]$

Conditional mean and Variance

Defn: The conditional mean of Y given X = x denoted as $E[Y/x]$ or $\mu_{Y/x}$ is defined as,

$$\boxed{E[Y/x] = \sum_{y} y \, f_{Y/x}(y)}$$

and the conditional variance of Y given X = x denoted as $V[Y/x]$ or $\sigma^2_{Y/x}$ is defined as :

$$\boxed{V[Y/x] = \sum_{y} (y - \mu_{Y/x})^2 \, f_{Y/x}(y)}$$

IIIly conditional mean & variance of X given Y = y,

$$E[X/y] = \sum_{x} x \, f_{X/y}(x)$$

$$V[X/y] = \sum_{x} (x - \mu_{X/y})^2 \, f_{X/y}(x)$$

Q). The joint PMF of X & Y is given by,

↑y / x→	0	1	2	3	4
4	4.1×10^{-5}				
3	4.1×10^{-5}	1.84×10^{-3}			
2	1.54×10^{-5}	1.38×10^{-3}	3.11×10^{-2}		
1	2.56×10^{-6}	3.46×10^{-4}	1.56×10^{-2}	0.2333	
0	1.6×10^{-7}	2.88×10^{-5}	1.94×10^{-3}	5.83×10^{-2}	0.6561
$f_X(X=x)$	0.0001	0.0036	0.0486	0.2916	0.6561

(i) Find conditional probability of Y given X = x

(ii) " " mean & variance of Y given X = 2.

PTO

(i) WKT,

$$f_{Y/x}(y) = \frac{f_{XY}(x,y)}{f_X(x)} \quad , \quad f_X(x) > 0$$

$f_{Y/x}$:

4	0.410	0	0	0	0
3	0.410	0.511	0	0	0
2	0.154	0.383	0.640	0	0
1	2.56×10^{-2}	0.096	0.320	0.8	0
0	1.6×10^{-3}	0.008	0.040	0.2	0.1
$x \rightarrow$ \ $\uparrow y$	0	1	2	3	4

(ii)

$$\mu_{Y/x} = \sum_y y\, f_{Y/x}(y)$$

$$\mu_{Y/2} = \sum_{y=0}^{4} y\, f_{Y/2}(y)$$

$$= \cancel{0 \cdot f_{Y/2}(2,0) + 1 \cdot f_{Y/2}}$$

$$= 0(0.04) + 1(0.320) + 2(0.640) + 3(0) + 4(0)$$

$$= 1.6$$

$$\cancel{\sigma^2_{Y/2} = 0^2 \cdot 0.4 + 1^2 \cdot 0.32 + 2^2 \cdot 0.64}$$

$$\cancel{= 2.88}$$

$$\sum_y$$

$$\sigma^2_{Y/x} = \sum_y (y - \mu_{Y/x})^2 f_{Y/x}(y)$$

$$\sigma^2_{Y/2} = \sum_{y=0}^{4} (y-1.6)^2 f_{Y/2}(x)$$

$$= (0-1.6)^2(0.04) + (1-1.6)^2(0.32) + (2-1.6)^2 \cdot (0.64)$$

$$= 0.32$$

2-2-19

Determining Probabilities from a joint CDF

The joint CDF of 2-d discrete random variables X & Y is defined as:

$$F_{XY}(x,y) = P[X \le x, Y \le y] = \sum_{X\le x}\sum_{Y \le y} f_{XY}(x,y)$$

Properties:

(i) $F_{XY}(-\infty,-\infty) = 0$, $F_{XY}(-\infty, y) = 0$ & $F_{XY}(x, -\infty) = 0$

(ii) $F_{XY}(\infty, \infty) = 1$

(iii) $0 \le F_{XY}(x,y) \le 1$

Q). The joint CDF of 2 discrete random variable, X & Y is given as follows:

$$F_{XY}(x,y) = \begin{cases} 1/8 & ; x=1, y=1 \\ 5/8 & ; x=1, y=2 \\ 1/4 & ; x=2, y=1 \\ 1 & ; x=2, y=2 \end{cases}$$

Determine (i) Joint PMF of X & Y.
(ii) Marginal PMF of X & Y.

Soln: X can take values: 1, 2

Y can take values: 1, 2

↑ y \ x →	1	2	
2	4/8	1/4	6/8
1	1/8	1/8	2/8
	5/8	3/8	

WKT,

$$F_{XY}(x,y) = \sum_{X\le x}\sum_{Y\le y} f_{XY}(x,y)$$

(a) $F_{XY}(1,1) = 1/8 = f_{XY}(1,1)$

$$F_{XY}(1,2) = \sum_{X\le 1}\sum_{Y\le 2} f_{XY}(x,y)$$

$$\frac{5}{8} = f(1,1) + f(1,2)$$

$$= \frac{1}{8} + f(1,2) \Rightarrow f(1,2) = \frac{4}{8}$$

$F_{XY}(2,1) = \sum_{X\le 2}\sum_{Y\le 1} f_{XY}(x,y) = f(1,1) + \cancel{f(\ldots)}\ f(2,1)$

$\frac{2}{8} = \frac{1}{8} + f(2,1) \Rightarrow f(2,1) = \frac{1}{8}$

$f(2,2) = 1 - \frac{1}{8} - \frac{1}{8} - \frac{4}{8} = \frac{2}{8} = \frac{1}{4}$

Two dimensional Continuos Random Variables

Joint Probability Distribution:

A joint probability density function for the continuous random variables X & Y denoted as $f_{XY}(x,y)$, satisfies the following properties:

(i) $f_{XY}(x,y) \geq 0 \quad \forall\, x,y$

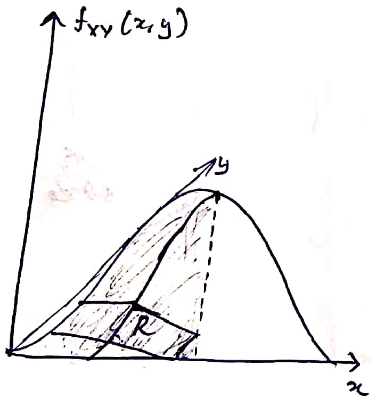
(ii) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dy\, dx = 1$

(iii) For any region R of 2-dimensional space,

$$P[(x,y) \in R] = \iint_R f_{XY}(x,y)\, dx\, dy$$

i.e. $\displaystyle P[a \leq x \leq b,\ c \leq y \leq d] = \int_c^d \int_a^b f_{XY}(x,y)\, dx\, dy$

NOTE: Probability that (X,Y) is in the region R is determined by the volume of $f_{XY}(x,y)$ over R.



NOTE:

$$F_{XY}(x,y) = P[X \in (-\infty, x],\ Y \in (-\infty, y]]$$

$$= \cancel{\int_{-\infty}^{x}\int_{-\infty}^{y} f_{xy}\, dx\, dy} = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\, du\, dv$$

It follows that

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x\, \partial y} F_{XY}(x,y)$$

Marginal PDF:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

NOTE:

$$P[a<x<b] = P[a<X<b, -\infty<Y<\infty]$$

$$= \int_a^b \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$$

$$= \int_a^b f_X(x)\,dx \,//.$$

Mean and Variance from joint distribution:

$$E[X] = \mu_x = \int_{-\infty}^{\infty} x\, f_x(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$$

$$\boxed{\mu_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{XY}(x,y)\,dy\,dx \,//.}$$

$$V[X] = \sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_x(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\mu_x)^2 \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx$$

$$\boxed{V[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)^2 \cdot f_{XY}(x,y)\,dy\,dx}$$

IIIly

$$\mu_Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{XY}(x,y)\,dx\,dy$$

$$\sigma_Y^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\mu_Y)^2 f_{XY}(x,y)\,dx\,dy \,//.$$

Q). The joint density fn. of X & Y is given by

$$f_{XY}(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & x, y \in (0,\infty) \\ 0, & \text{otherwise} \end{cases}$$

Compute:

(i) $P[X>1, Y<1]$

(ii) $P[X<Y]$

(iii) $P[X<a]$, $(a>0)$

Soln:

(i) $$P[X>1, Y<1] = \int_1^\infty \int_0^1 f_{XY}(x,y)\, dy\, dx$$

$$= \int_1^\infty \int_0^1 2e^{-x}e^{-2y}\, dy\, dx$$

$$= 2\int_1^\infty e^{-x}\left[\frac{e^{-2y}}{-2}\right]_0^1 dx$$

$$= -\int_1^\infty e^{-x}(e^{-2}-1)\, dx$$

$$= (1-e^{-2})\int_1^\infty e^{-x}\, dx$$

$$= (1-e^{-2})\left[\frac{e^{-x}}{-1}\right]_1^\infty$$

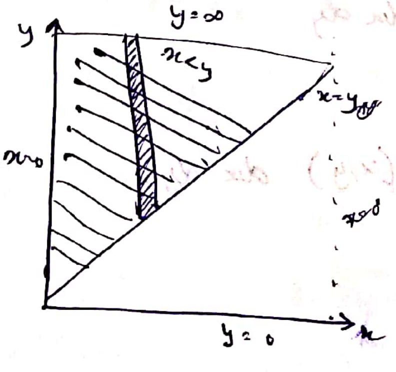
$$= (e^{-2}-1)(0-1)$$

$$= (1-e^{-2})\, e^{-1}$$

$$= \cancel{0.86476}$$

$$= 0.3181$$

(ii) $P[X<Y]$:



$$\therefore P[X<Y] = \int_{0}^{\infty}\int_{x}^{\infty} f_{XY}(x,y)\,dy\,dx = \int_{0}^{\infty}\int_{0}^{y} f_{XY}(x,y)\,dx\,dy$$

$$= \int_{0}^{\infty}\int_{x}^{\infty} 2e^{-x}e^{-2y}\,dy\,dx$$

$$= 2\int_{0}^{\infty} e^{-x}\left[\frac{e^{-2y}}{-2}\right]_{x}^{\infty} dx$$

$$= -\int_{0}^{\infty} e^{-x}\left[0-e^{-2x}\right] dx$$

$$= \int_{0}^{\infty} e^{-3x}\,dx$$

$$= -\frac{1}{3}\left[e^{-3x}\right]_{0}^{\infty}$$

$$= -\frac{1}{3}[0-1] = \frac{1}{3}$$ //.

(iii) . $$P[X<a] = \int_{0}^{a}\int_{0}^{\infty} f_{XY}(x,y)\,dy\,dx$$

$$= \int_{0}^{a}\int_{0}^{\infty} 2e^{-x}e^{-2y}\,dy\,dx$$

$$= 2\int_{0}^{a} e^{-x}\int_{0}^{\infty} e^{-2y}\,dy\,dx$$

$$= -\int_{0}^{a} e^{-x}\left[e^{-2y}\right]_{0}^{\infty} dx$$

$$= \int_{0}^{a} e^{-x}\,dx$$

$$= -\left[e^{-x}\right]_{0}^{a}$$

$$= -(e^{-a}-1)$$

$$= 1-e^{-a}$$ //.

4.2.19

Q). Assume that the random variable X & Y have the joint PDF

$$f_{XY}(x,y) = \frac{1}{2}x^3y \qquad 0 \le x \le 2,\ 0 \le y \le 1$$

Are X & Y independent?

Soln:

If X & Y are independent,

then, $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \int_0^1 \frac{1}{2}x^3y\, dy$$

$$= \frac{1}{4}x^3\left[y^2\right]_0^1$$

$$= \frac{1}{4}x^3 \qquad , 0 \le x \le 2$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

$$= \int_0^2 \frac{1}{2}x^3y\, dx$$

$$= \frac{1}{8}y\left[x^4\right]_0^2$$

$$= 2y \qquad 0 \le y \le 1.$$

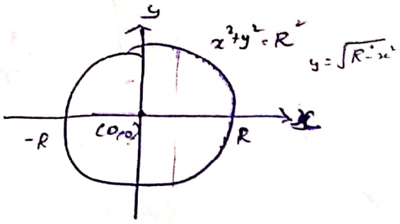
$$f_X(x) \cdot f_Y(y) = \frac{1}{4}x^3 \cdot 2y$$

$$= \frac{1}{2}x^3y$$

$$= f_{XY}(x,y).$$

$\therefore$ X & Y are independent.

2. A gun is aimed at a certain pt. (origin of the coordinate system). because of the random factors the actual hit point can be any point (X, Y) in a circle of radius R about the origin. Assume that the joint density of X & Y is given by,



$$f_{XY}(x,y) = \begin{cases} c & x^2+y^2 \le R^2 \\ 0 & \text{otherwise} \end{cases}$$

(1) Compute c.

(2) Show that $f_X(x) = \begin{cases} \frac{2}{\pi R}\sqrt{1-(x/R)^2} & -R \le x \le R \\ 0 & \text{otherwise.} \end{cases}$

Soln:

(1) WKT,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dy\, dx = 1$$

$$\Rightarrow 1 \Rightarrow \int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} c\, dy\, dx$$

$$1 = \int_{-R}^{R} c\,[y]_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\, dx$$

$$= \int_{-R}^{R} 2c\left[\sqrt{R^2-x^2}\right] dx$$

$$= 2c\int_{-R}(R^2-x^2)^{1/2}\, dx$$

$$= 2c\cdot\left[\frac{x}{2}\sqrt{R^2-x^2} + \frac{R^2}{2}\sin^{-1}\left(\frac{x}{R}\right)\right]_{-R}^{R}$$

$$= 2c\left[\frac{R^2}{2}\cdot\frac{\pi}{2} + \frac{R^2}{2}\cdot\frac{\pi}{2}\right]$$

$$= c\cdot R^2\,\pi$$

$$\Rightarrow c = \frac{1}{\pi R^2}$$

$R^2 - x^2 = z^2$

$-2x\, dx = 2z\, dz$

$x = \sqrt{R^2 - z^2}$

$R^2 - x^2 = z^2$

$-2x\, dx = 2z\, dz$

$-x\, dx = z\, dz$

NOTE:

$$\int\sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

(ii) $f_X(x,y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

$$= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\, dy$$

$$= \frac{1}{\pi R^2} \left[y \right]_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}$$

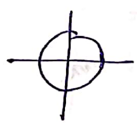
$$= \frac{2}{\pi R^2} \sqrt{R^2-x^2}$$

$$= \frac{2}{\pi R} \sqrt{1-(x/R)^2} \quad //$$

H.P

Aliter,

Changing to polar coordinates.



$x = r\cos\theta$ r varies from 0 to R

$y = r\sin\theta$ θ varies from 0 to 2π

$$dx\,dy = r\,dr\,d\theta$$

$$\therefore \int_0^{2\pi}\int_0^{R} c \cdot r\, dr\, d\theta = 1 \Rightarrow \int_0^{2\pi} \frac{c}{2}\left[R^2\right] d\theta = \frac{cR^2}{2}\left[2\pi\right] = \pi R^2 c$$

$$\Rightarrow c = \frac{1}{\pi R^2}$$

Q. If the joint PDF of (X, Y) is gvn by,

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left\{ \frac{-(x^2+y^2)}{2\sigma^2} \right\}, \quad -\infty < x, y < \infty$$

Find $P\left[X^2+Y^2 \le a^2\right]$

Hint: Transform into ~~for~~ polar coordinates.

S.ln :

$$P(x^2+y^2 \le a^2) = \int_0^{2\pi}\int_0^{a} \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{r^2}{2\sigma^2}} \cdot r\,dr\,d\theta$$

$$= \frac{1}{2\pi\sigma^2} \int_0^{2\pi}\int_0^{a} r\, e^{-\frac{r^2}{2\sigma^2}}\, dr\, d\theta$$

$\frac{r^2}{2\sigma^2} = z$

$2r\,dr = 2\sigma^2 dz$

$r=0,\ z=0$

$r=a\quad z=\frac{a^2}{2\sigma^2}$

$$= \frac{1}{2\pi} \int_0^{2\pi}\int_0^{a^2/2\sigma^2} e^{-z}\, dz\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(1 - e^{-a^2/2\sigma^2}\right) d\theta$$

$$= \frac{1}{2\pi}\left(1 - e^{-\frac{a^2}{2\sigma^2}}\right) \cdot 2\pi$$

$$= 1 - e^{-a^2/2\sigma^2} \quad //.$$

~~Ans: 1 -~~ e

Conditional Probability Distributions

$$f_{Y/x}(y) = \frac{f_{XY}(x,y)}{f_X(x)}, \quad f_X(x) > 0$$

$$f_{X/y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)}, \quad f_Y(y) > 0$$

Note:

(i) $f_{Y/x}(y) \ge 0$

(ii) $\int_{-\infty}^{\infty} f_{Y/x}\, dy = 1$

(iii) $P[Y \in [a,b]/X=x] = \int_a^b f_{Y/x}(y)\, dy$

All these properties m.b. for X given Y=y.

Conditional Mean & Variance

$$\mu_{Y/x} = E[Y/x] = \int_{-\infty}^{\infty} y \cdot f_{Y/x}(y)\, dy$$

and $$\sigma^2_{Y/x} = \int_{-\infty}^{\infty} (y-\mu_{Y/x})^2 f_{Y/x}(y)\, dy = E[Y^2/x] - (E[Y/x])^2$$

PTO

m^{ly},

$$\mu_{x/y} = E[X/y] = \int_{-\infty}^{\infty} x \, f_{x/y}(x)\,dx$$

$$\sigma^2_{x/y} = \int_{-\infty}^{\infty} (x-\mu_{x/y})^2 f_{x/y}(x)\,dx = \{ E[x^2/y] - (E[x/y])^2 \text{ //.}$$

Q). Two random variables X & Y have the following Joint PDF.

$$f_{XY}(x,y) = \begin{cases} x e^{-x(y+1)} & , 0 \le x < \infty,\ 0 \le y < \infty \\ 0 & , \text{otherwide} \end{cases}$$

Determine $f_{x/y}(x)$ & $f_{y/x}(x)$.

<u>Soln:</u>

WKT//

$$f_{x/y}(x) = \frac{f_{xy}(x,y)}{f_Y(y)} \quad \& \quad f_{y/x}(y) = \frac{f_{xY}(x,y)}{f_x(y)}$$

$$f_x(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^{\infty} x e^{-x(y+1)} \, dy$$

$$= -e^{-x}\left[e^{-xy}\right]_0^{\infty}$$

$$= e^{-x} \text{ //.}$$

$u = x \qquad v = e^{-x(y+1)}$

$u' = 1 \qquad v_1 = \frac{e^{-x(y+1)}}{-(y+1)}$

$u'' = 0 \qquad v_2 = \frac{e^{-x(y+1)}}{(y+1)^2}$

$$f_Y(y) = \int_0^{\infty} x e^{-x(y+1)} \, dx$$

$$= \left[\frac{x.e^{-x(y+1)}}{-(y+1)} - \frac{e^{-x(y+1)}}{(y+1)^2}\right]_0^{\infty}$$

$$= \cancel{\frac{e^{-x(y+1)}}{y+1}}$$

$$= \left[0 - \left(\frac{0.e^0}{-(y+1)} - \frac{e^0}{(y+1)^2}\right)\right]$$

$$= \frac{1}{(y+1)^2}$$ //.

$$\therefore f_{X/y}(x) = \frac{x e^{-x(y+1)}}{e^{-x}} = x e^{-xy}$$

$$f_{Y/x}(y) = x e^{-x(y+1)} \cdot (y+1)^2 = x(y+1)^2 e^{-x(y+1)}$$ //.

Q). Compute the conditional mean of X give $Y=y$, If the Joint PDF of X & Y is gvn by,

$$f_{XY}(x,y) = \begin{cases} \frac{e^{-(x/y)} e^{-y}}{y} & , 0 \le x, y \le \infty \\ 0 & , \text{otherwise} \end{cases}$$

~~Soln:~~

Ans:

$$f_Y(y) = e^{-y} \quad , 0 \le y < \infty$$

$$f_{X/y}(x) = \frac{e^{-(x/y)}}{y} \quad \& \quad E[X/y] = y.$$

Soln:

$$f_Y(y) = \int_0^\infty \frac{e^{-x/y} \cdot e^{-y}}{y} dx$$

$$= \frac{e^{-y}}{y} \cdot y \left[e^{-x/y} \right]_0^\infty$$

$$= e^{-y}$$ //.

$$f_{X/y} = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{e^{-x/y}}{y}$$

$u = x \quad v = e^{-x/y}$

$u' = 1 \quad v_1 = -y e^{-x/y}$

$u'' = 0 \quad v_2 = \frac{y^2 e^{-x/y}}{}$

$$E[X/y] = \int_0^\infty x \cdot \frac{e^{-x/y}}{y} dx$$

$$= \frac{1}{y} \int_0^\infty x e^{-x/y} dx$$

$$= \frac{1}{y} \left[-xy e^{-x/y} - y^2 e^{-x/y} \right]_0^\infty$$

$$= \frac{1}{y}(y^2) = y$$ //.

5/2/19

$$f_{XY}(x,y) = 6\times10^{-6} \exp(-0.001x - 0.002y) \quad \text{for } x<y$$

Assume that the joint PDF for X & Y is given above.

(i) Determine the conditional PDF of Y gn X = x

(ii) " " " mean for Y gn that x = 1500

Soln:

(i) $$f_{Y/x}(y) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \int_{x}^{\infty} 6\times10^{-6} \exp(-0.001x - 0.002y)\, dy$$

$$= 6\times10^{-6} \int_{x}^{\infty} e^{-(0.001x + 0.002y)}\, dy$$

$$= 6\times10^{-6}\, e^{-0.001x} \int_{x}^{\infty} e^{-0.002y}\, dy$$

$$= 6\times10^{-6} \frac{e^{-0.001x}}{-2\times10^{-3}} \left[e^{-2\times10^{-3}\times y}\right]_{x}^{\infty}$$

$$= -3\times10^{-3}\, e^{-0.001x} \cdot (0 - e^{-0.002x})$$

$$= 3\times10^{-3}\, e^{-0.003x}$$

$$\therefore f_{Y/x}(y) = \frac{6\times10^{-6} \times e^{-0.001x} \cdot e^{-0.002y}}{3\times10^{-3}\, e^{-0.001x} \cdot e^{-0.002x}}$$

$$= 2\times10^{-3}\, e^{-0.002(y-x)}$$

$$= 2\times10^{-3} \cdot e^{-0.002(y-x)}$$

$$\begin{array}{r} 1500 \\ 2 \\ \hline 3000 \end{array}$$

(ii) $f_{Y/x=1500}(y) =$

$$\mu_{Y/x} = \int_{-\infty}^{\infty} y \cdot f_{Y/x}(y)\, dy$$

$$= \int_{1500}^{\infty} y \cdot 2\times10^{-3} \cdot e^{-0.002(y-1500)} \, dy$$

$$= 2\times10^{-3} \cdot e^{3} \int_{1500}^{\infty} y \, e^{-0.002y} \, dy$$

$u = y$ $\quad v = e^{-0.002y}$

$u' = 1$ $\quad v_1 = \frac{e^{-0.002y}}{-0.002}$

$u'' = 0$ $\quad v_2 = \frac{e^{-0.002y}}{4\times10^{-6}}$

$$= 2\times10^{-3} \cdot e^{3} \left[-y \frac{e^{-0.002y}}{0.002} - \frac{e^{-0.002y}}{4\times10^{-6}} \right]_{1500}^{\infty}$$

$$= 2\times10^{-3} \cdot e^{3} - \left[\frac{-1500 \cdot e^{-3}}{0.002} - \frac{e^{-3}}{4\times10^{-6}} \right]$$

$$= \frac{\cancel{2\times10^{-3}} \, e^{3}}{2\times10^{-3} \, \cancel{e^{3}}} - \left[-1500 - \frac{1}{2\times10^{-3}} \right]$$

$$= 2000$$

Q). The joint PDF of 2-D random variable (X, Y) is given by.

$$f_{XY}(x,y) = xy^2 + \frac{x^2}{8} \quad , 0 \le x \le 2, \quad 0 \le y \le 1$$

Compute (i) $P[X>1 / Y<1/2]$

(ii) $P[Y<1/2 / X>1]$

Soln:

WKT

(i) $$P[X>1 / Y<1/2] = \frac{f_{XY}(X>1, Y<1/2)}{f_Y(Y<1/2)}$$

$$f_{XY}(X>1, Y<1/2) = \int_{1}^{2}\int_{0}^{1/2} xy^2 + \frac{x^2}{8} \, dy \, dx$$

$$= \int_{1}^{2} \left[\frac{xy^3}{3} + \frac{yx^2}{8} \right]_{0}^{1/2} dx$$

$$= \int_{1}^{2} \left[\frac{x}{3} \cdot \frac{1}{8} + \frac{x^2}{2} \right] dx$$

$$= \left[\frac{x^2}{48} + \frac{x^3}{8} \right]_{1}^{2}$$

$$= \left[\frac{4}{48} + \frac{8}{8} - \frac{1}{48} - \frac{1}{8} \right] = \frac{3}{48} + \frac{7}{8} = \frac{1}{16} + \frac{14}{16} = \frac{9}{16}$$

$\frac{1}{16}$ $\frac{3}{48}$

$= 5/24$

$$P[Y<1/2] = P[0<Y<1/2, -\infty<x<\infty]$$

$$= \int_0^{1/2}\int_0^2 f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^{1/2}\int_0^2 xy^2+\frac{x^2}{8}\,dx\,dy$$

$$= \int_0^{1/2}\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_0^2 dy$$

$$= \int_0^{1/2}\left(2y^2+\frac{1}{3}\right)dy$$

$$= \left[\frac{2y^3}{3}+\frac{y}{3}\right]_0^{1/2}$$

$$= \frac{\cancel{2}}{\cancel{8}\,4\cdot 3}+\frac{1}{6}$$

$$= \frac{1+2}{12} = \frac{3}{12} = \frac{1}{4}.$$

$$\therefore P[X>1/Y<1/2] = \frac{P[X>1, Y<1/2]}{P[Y<1/2]} = \frac{5/24}{1/4} = \frac{5}{6}.$$

(ii) $P[Y<1/2 / X>1]$:

$$P[X>1] = \int_0^1\int_1^2 xy^2+\frac{x^2}{8}\,dx\,dy$$

$$= \int_0^1\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_1^2 dy$$

$$= \int_0^1\left(2y^2+\frac{8}{24}-\frac{y^2}{2}-\frac{1}{24}\right)dy$$

$$= \int_0^1 \frac{3}{2}y^2+\frac{7}{24}\,dy$$

$$= \left[\frac{y^3}{2} + \frac{7y}{24}\right]_0^1$$

$$= \frac{1}{2} + \frac{7}{24}$$

$$= \frac{19}{24}$$

$$\therefore P[Y < 1/2 \,/\, X > 1] = \frac{5/24}{19/24} = \frac{5}{19}$$

Q). Let the random variable X & Y denote the lengths of a dimension of a machine part. Assume that X & Y are independent random variables & further assume that the distribution of X is normal with mean 10.5 mm and variance 0.0025 mm^2 & that the distribution of Y is normal with mean 3.2 mm & variance 0.0036 mm^2. Determine the probability that

$$P[10.4 < X < 10.6,\ 3.15 < Y < 3.25]$$

$= P[10.4$

Soln: $\because$ X & Y are independent rad.

$$P[10.4 < X < 10.6;\ 3.15 < Y < 3.25]$$

$$= P[10.4 < X < 10.6] \cdot P[3.15 < Y < 3.25]$$

$= P[10.4-$

$$P[10.4 < X < 10.6] = P\left[\frac{10.4-10.5}{0.05} < Z_1 < \frac{10.6-10.5}{0.05}\right]$$

$$= P[-2 < Z_1 < 2]$$

$$= P(2) - P(-2)$$

$$= 0.977250 - 0.022750$$

$$= 0.9545$$

$$= 0.955$$

$$P[3.15 < Y < 3.25] = P\left[\frac{3.2-3.15}{0.06} < Z_2 < \frac{3.25-3.2}{0.06}\right]$$

$$= P[-0.833 < Z_2 < 0.833]$$

$$= 0.796731 - 0.203260$$

$$= 0.593471$$

$$\therefore P[10.4 < X < 10.6,\ 3.15 < Y < 3.25] =$$

$$= 0.9545 \times 0.593471$$

$$= 0.56646$$

$$= 0.5665\%.$$

6.2.19

Covariance & Correlation

The covarionce b/w the random variables X, Y denoted as $COV(X, Y)$ or σ_{XY} is:

$$Cov(X, Y) = \sigma_{XY} = E[(X-\mu_x)(Y-\mu_y)]$$

Ex: When copying files from computer to USB,

X - no. of bits; Y - no. of files.

NOTE: $Cov(X, Y) = E[(X-\mu_x)(Y-\mu_y)]$

$$= E[XY - X\cdot\mu_y - Y\cdot\mu_x + \mu_x\cdot\mu_y]$$

(aX+b)

$$= E[XY] - E[\mu_y\, X] - E[\mu_x Y] + \mu_x\mu_y$$

$$= E[XY] - \mu_y E[X] - \mu_x E[Y] + \mu_x \mu_y$$

$$= E[XY] - \mu_y\cdot\mu_x - \mu_x\cdot\mu_y + \mu_x\mu_y$$

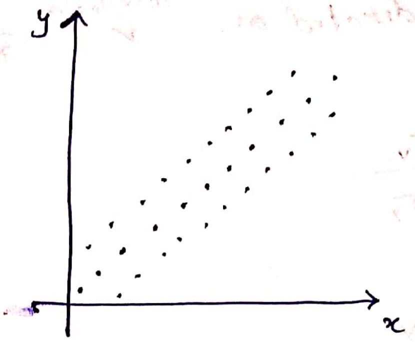
$$= E[XY] - \mu_x\cdot\mu_y$$

i.e. $\boxed{cov(X, Y) = \sigma_{XY} = E[XY] - E[X]\cdot E[Y]}$

NOTE: If X & Y are independent, then,

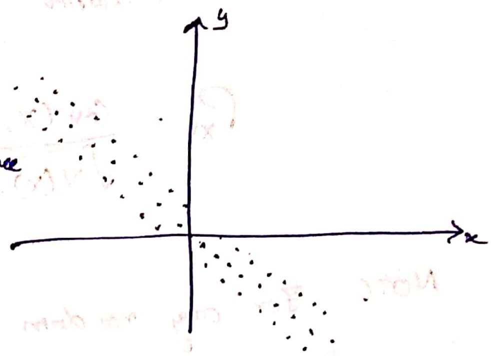
$$Cov(X,Y) = E[X]\cdot E[Y] - E[X]E[Y]$$
$$= 0 //$$

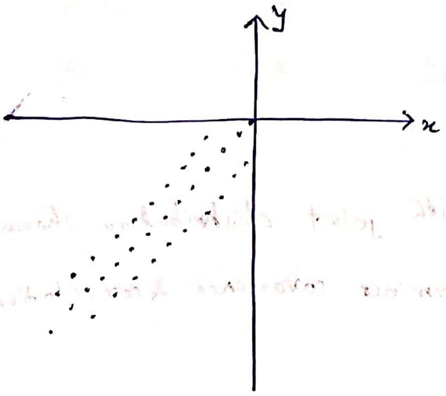
(I) Positive covariance



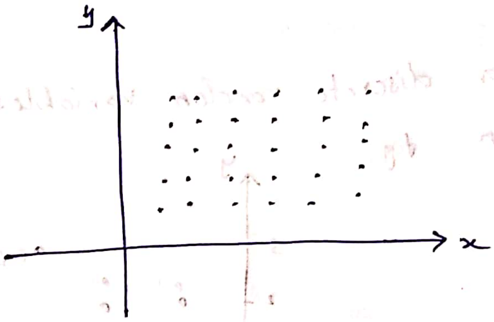
The joint PMF and the sign of covariance between X & Y

(II) Negative covariance





(III) Zero Covariance



X	1	2	3
$f(x)$	$1/4$	$1/4$	$1/2$

$$E[X] = \frac{1}{4} + \frac{2}{4} + \frac{6}{4}$$
$$= \frac{9}{4} = 2.25 //$$

f(x) is higher when X is higher

So, when X is higher & Y is higher or when X is lower & Y is lower then covariance is positive.

When X is higher & Y is lower or when X is lower & Y is higher the covariance is negative.

The covariance is zero, when X & Y are independent of each other

NOTE:

(I) $cov(X,X) = Var(X)$

(II) $cov(aX+b, Y) = a\, cov(X,Y)$

Proof: $cov(aX+b, Y) = E[(aX+b)Y] - E[aX+b]E[Y]$

$$= E[aXY + bY] - [aE[X]+b]E[Y]$$
$$= aE[XY] + bE[Y] - aE[X]E[Y] + bE[Y]$$
$$= a[E[XY] - E[X]E[Y]] = a \cdot cov(X,Y) //$$

(iii) $cov(X+a, Y+b) = cov(X, Y)$

Correlation:

- Dimensionless ~~pr~~ quantity (no units) : useful when Ex: height X ↑ ft., weight Y ↑ kg
covariance – ft·kg
correlation – $\cancel{\frac{ft\cdot kg}{ft\cdot kg}}$

~~The~~ The correlation b/w X & Y denoted as:

$$\rho_{XY} = \frac{Cov(X, Y)}{\sqrt{V(X)\cdot V(Y)}} = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}$$

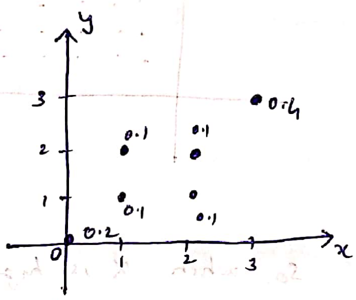
NOTE: For any random variables, X & Y,

$$-1 \le \rho_{XY} \le 1$$

Negative correlation   positive correlation

Q). For discrete random variables X, Y with joint distribution shown in fig.

(i) determine covariance & correlation



(i) $cov(X, Y) = E[XY] - E[X]E[Y]$

$$E[XY] = \sum_x \sum_y xy\, f_{XY}(x, y)$$

$$= 0\cdot 0\cdot 0.2 + 1\times1\times0.1 + 1\times2\times0.1 + 2\times1\times0.1 + 2\times2\times0.1 + 3\times3\times0.4$$

$$= 4.5$$

$$E[X] = (0.2)(0) + 1\cdot(0.2) + 2(0.2) + 3(0.4)$$

$$= 1.8$$

$$E[Y] = 0(0.2) + 1(0.2) + 2(0.2) + 3(0.4)$$

$$= 1.8$$

$$V[X] = (0-1.8)^2(0.2) + (1-1.8)^2(0.2) + (2-1.8)^2(0.2) + (3-1.8)^2(0.4)$$

$$= 1.36$$

$V[Y] = 1.36$

$cov(X,Y) = E[XY] - E[X]E[Y]$

$= 4.5 - 1.8 \times 1.8$

$= 1.26$ //.

$\therefore \rho_{XY} = \dfrac{\sigma_{XY}}{\sigma_x \cdot \sigma_y} = \dfrac{cov(X,Y)}{\sqrt{V(X)V(Y)}}$ //.

$= \dfrac{1.26}{\sqrt{1.36 \times 1.36}}$

$= \dfrac{1.26}{1.36} = 0.9265$

Q). Two refills for a pen are selected at random from a box that contains 3 blue, 2 red and 3 green refills. If X is the no. of blue refills & Y is the no. of red refills selected, Find

a) the joint PDF for X & Y (b) $P(X+Y \le 1)$ (c) The covariance of X & Y

11.2.19

NOTE :

(i) $E[X+Y] = E[X] + E[Y]$

(ii) $Var[X+Y] = Var[X] + Var[Y] + 2 \cdot cov(X,Y)$

$Var[X-Y] = Var[X] + Var[Y] - 2 \cdot cov(X,Y)$

Q). If the independent random variables X and Y have the variance 36 & 16 respectively. Then find the correlation coefficient b/w X+Y & X−Y.

Soln: Given X & Y are independent.

$\sqrt{2\cdot3\cdot2\cdot3+4\cdot4}$

$2\sqrt{3\cdot3+4}$

$2\sqrt{13}$

Let $u = X+Y$

and $v = X-Y$

$\rho_{uv} = \dfrac{cov(u,v)}{\sigma_u \cdot \sigma_v}$ —— (1)

$\sigma_u^2 = Var(u)$

$= Var(X+Y)$

$= Var(X) + Var(Y)$ [$\because$ X & Y are independent $\Rightarrow cov(X,Y) = 0$]

$\Rightarrow 2cov(X,Y) = 0$

$= 36 + 16$

$= 52$ //.

$\sigma_u = 2\sqrt{13}$

$\sigma_v^2 = Var(v)$

$= Var(X-Y)$

$= var(X) + var(Y) - 2\, cov(X,Y)$

$= 36+16 \qquad [\because cov(X,Y)=0$

$= 52$

$\sigma_v = 2\sqrt{13}$

$2\sqrt{13} \cdot 2\sqrt{13}$
$4 \cdot 13$
$= 52$

~~Cov(U,V)~~

$cov(u,v) = E[UV] - E[U] \cdot E[V]$

$= E[(X+Y)(X-Y)] - E[X+Y]\, E[X-Y]$

$= E[X^2-Y^2] - (E[X]+E[Y])(E[X]-E[Y])$

$= E[X^2] - E[Y^2] - (E[X])^2 + E[Y]^2$

$= E[X^2] - (E[X])^2 - (E[Y^2] - (E[Y])^2)$

$= Var.(X) - Var.(Y)$

$= 36-16$

$= 20$

$\therefore \rho_{uv} = \frac{20}{52} = \frac{10}{26} = \frac{5}{13}$

$\rho_{uv} = \frac{cov(u,v)}{\sigma_u \cdot \sigma_v}$

$= \frac{20}{2\sqrt{13} \cdot 2\sqrt{13}} = \frac{5}{13}$

Q) If X, Y & Z are uncorrelated random variables with 0 means & std. dev. 5, 12 & 9 respectively & if $U = X+Y$ & $V = Y+Z$ Find the correlation coefficient b/w U & V.

Soln: Given X, Y, Z are uncorrelated ⇒ X, Y & Z are independent.

$E[X] = E[Y] = E[Z] = 0$

$\sigma_x = 5 \quad \sigma_y = 12 \quad \sigma_z = 9$

$$Var[x] = E[x^2] - (E[x])^2$$

$$5^2 = E[x^2]$$

$$E[x^2] = 25$$

$$\text{lly } E[Y^2] = 144 \quad \& \quad E[Z^2] = 81$$

$$\sigma_u^2 = Var(X+Y)$$

$$= Var(X) + Var(Y) + 2\cancel{Var}\, cov(X,Y)$$

$$= 25 + 144$$

$$= 169$$

$$\sigma_u = 13$$

$$\sigma_v^2 = Var(Y+Z)$$

$$= Var(Y) + Var(Z) + 2cov(Y,Z)$$

$$= 144 + 81$$

$$= 225$$

$$\sigma_v = 15$$

$$cov(u,v) = E[uv] - E[u]E[v]$$

$$= E[(X+Y)(X+Z)] - E[X+Y]E[X+Z]$$

$$= E[XY + XZ + Y^2 + YZ] - (E[X] + E[Y])(E[Y] + E[Z])$$

$$= E[XY] + E[XZ] + E[Y^2] + E[YZ] - E[X]E[Y] - E[X]E[Z] - (E[Y])^2 + E[Y]E[Z]$$

WKT $cov(X,Y) = 0$

$$E[XY] - E[X]E[Y] = 0$$

$$\Rightarrow E[XY] = 0$$

$$\text{lly } E[YZ] = E[ZX] = 0$$

$$\therefore cov(u,v) = 0 + E[Y^2]$$

$$= 144 \text{//}$$

$$\therefore \rho_{uv} = \frac{cov(u,v)}{\sigma_u \cdot \sigma_v}$$

$$= \frac{\cancel{144}\, 12 \cdot \cancel{12}^{4}}{13 \cdot \cancel{15}_{5}} = \frac{48}{65}$$

Q. The Central limit Theorem:

Let $X_1, X_2, \ldots, X_n$ be a sequence of mutually independent & identically distributed random variables each of which has a finite mean, μ & variance, σ^2.

Let $S_n = X_1 + X_2 + \ldots + X_n$. The central limit theorem states that for large $n (n \geq 30)$, the distribution of S_n is approximately normal regardless of the distribution of X_k, $k = 1, 2, \ldots, n$

Now,

$$\mu_{S_N} = E[S_n]$$
$$= E[X_1 + X_2 + \ldots + X_n]$$
$$= E[X_1] + E[X_2] + \ldots + E[X_n]$$
$$= \mu + \mu + \ldots + \mu$$
$$= n\mu$$

$$\boxed{E[S_n] = \mu_{SN} = n\mu}$$

$$Var[S_N] = \sigma^2_{SN}$$
$$= E[S_N^2] - E[S_N]^2$$
$$= Var[X_1 + \cancel{X_2} + \ldots + X_n]$$
$$= Var[X_1] + Var[X_2] + \ldots + Var[X_n]$$
($\because X_1, X_2, \ldots X_n$ are independent)

$$= \sigma^2 + \sigma^2 + \ldots + \sigma^2$$
$$= n\sigma^2$$

$$\boxed{Var[S_N] = \sigma^2_{SN} = n\sigma^2}$$

Converting S_N to a standard Normal Random Variable

(i.e. with mean zero & variance one).

$$Z = \frac{S_N - \mu_{S_N}}{\sigma_{S_N}}$$

$$= \frac{S_N - n\mu}{\sqrt{n\sigma^2}}$$

$$\boxed{Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}}$$

Q). Assume that the random variable S_n is the sum of 48 independent experimental values of the random variable X whose PDF is given by,

$$f_X(x) = \begin{cases} 1/3 & \text{if } 1 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that S_n lies in the range $108 \le S_n \le 126$

Soln: X is a uniform random variable

$$\therefore E[X] = \frac{a+b}{2} = \frac{1+4}{2}$$

$$\boxed{E[X] = 2.5}$$

$$Var[X] = \frac{(b-a)^2}{12}$$

$$= \frac{(4-1)^2}{12} = \frac{3}{4}$$

$$\sigma^2 = \frac{3}{4}$$

Thus, the mean & variance of S_n are

$$\mu_{S_N} = E[M_S]$$

$$= \frac{5}{2} \cdot 48$$

$$= \cancel{1} \; 120 ''$$

$$\sigma_{S_N}^2 = n\sigma^2$$

$$= 48 \cdot \frac{3}{4} = 36 ''$$

$$\therefore P[108 \le S_n \le 126] = P\left[\frac{108-120}{6} \le \frac{S_n-120}{6} \le \frac{126-120}{6}\right]$$

$$= P[-2 \le Z \le 1]$$

$$= P(Z < 1) - P(Z \le -2)$$

$$= 0.8185$$

12.2.19

NOTE: If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population with mean μ & variance σ^2. and if $\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$ $\bar{X}$ is the sample mean then the limiting form of the distribution as $n \to \infty$ is the normal distribution with mean $\mu_{\bar{x}} = \frac{\mu + \mu + \ldots + \mu}{n}$

$\mu_{\bar{x}} = \mu$

~~With $\mu_{\bar{x}} = \mu$ and variance, $\sigma_{\bar{x}}^2 = \frac{\sigma^2 + \sigma^2 + \ldots + \sigma^2}{n}$ [$\because X_1, X_2, \ldots X_n$ are independent]~~

~~$= \frac{n\sigma^2}{n} = \sigma^2$~~

$$\mu_{\bar{x}} = \frac{\mu + \mu + \ldots + \mu}{n}$$

$$= \frac{n\mu}{n} = \mu$$

$$\boxed{\mu_{\bar{x}} = \mu}$$

Variance:

$$\sigma_{\bar{x}}^2 = Var\left[\frac{X_1 + X_2 + X_3 + \ldots + X_n}{n}\right]$$

$$= \frac{1}{n^2}(n\sigma^2) \quad [\because Var[aX] = a^2 Var[X]]$$

$$= \frac{\sigma^2}{n}$$

$$\boxed{\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}}$$

$$\boxed{\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}}$$

Now, $\bar{x}$ has normal distribution with mean μ & variance $\frac{\sigma^2}{n}$

$$\therefore \boxed{Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma}}$$

Q). The lifetime of a certain brand of an electric bulb may be considered as a random variable with mean 1200 hrs & std. dev 250 hrs. Use the Central limit thm to find the probability that the sand fill mean of the lifetime with a sample of size 36 is b/w 1100 hrs and 1300 hrs.

Soln: Let X be the lifetime of an electric bulb

Giv, $\mu_x = 1200$ & $\sigma_x = 250$

$n = 36.$

Let $\bar{x}$ denote the sample mean obtained from a sample size of $n = 36$,

then

$$\mu_{\bar{x}} = \mu_x$$

$$= 1200 \,//.$$

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}$$

$$= \frac{250}{6} = \frac{125}{3}$$

Then, the reqd. probability is,

$$P[1100 < X < 1300] = P\left[\frac{1100-1200}{125/3} < \frac{X-\mu}{\sigma} < \frac{1300-1200}{125/3}\right]$$

$$= P[-2.4 < Z < 2.4]$$

$$= P(Z<2.4) - P(Z<-2.4)$$

$$= 0.991802 - 0.008298$$

$$= 0.9835 \,//.$$

Q). A random variable of size 100 is taken from a population whose mean is 60 and variance 400. Using the central lt. thm, find what probability can be assessed that the mean of sample will not differ from mean, $\mu = 60$ by more than 4.

Soln: Gn // $\mu_x = 60$ $\sigma_x^2 = 400$

$$\Rightarrow \mu_{\bar{x}} = 60 \qquad \sigma_{\bar{x}} = \frac{20}{\sqrt{100}} = 2$$

IMf

Then, the reqd. probability is,

$$P[|\bar{X}-\mu| \leq 4] = P[-4 \leq \bar{x}-\mu \leq 4]$$

$$= P[-2 \leq z \leq 2]$$

$$= P(z \leq 2) - P(z \leq -2)$$

$$= 0.977250 - 0.022750$$

$$= 0.9545\%.$$

25/2/19

INTRODUCTION TO RANDOM PROCESSES

Random function:

is a fn $X(t,e)$ that is chosen randomly from a family of functions $\{X(t,e_i)\}, i = 1,2,\ldots$ where t is is usually a time ~~parti~~ parameter.

Ex: Assume that a coin is tossed 10 times at diff. pts of time $t_1, t_2, \ldots t_i, \ldots t_{10}$ in $(0,t)$.

Let $x(t,e)$ be random variable, where t in time parameter, then we have 10 such random variables $x(t_1,e), x(t_2,e), \ldots, x(t_{10},e)$.

Assume you gain ₹10 for heads & lose ₹5 for each tail multiplied by the n^{th} toss.

We can represent the random ~~var~~ function $X(t,\xi)$ as.

$$X(t,\xi) = \begin{cases} -5t & \text{if tail turns up (i.e } \xi = e_1 = T) \\ 10t & \text{if heads turns up (i.e. } \xi = e_2 = H) \end{cases}$$

Random Process: $\{X(t,\xi)\}$

defined as a collection of random functions together with a probability rule

		State space	Time parameter
$\{X(t,\xi)\}$	- Random process	Variable	Variable
$\{X(t,\xi)\}$	- Random fn.	Fixed ~~Variable~~	Variable ~~Random fn.~~
$X(t_1,\xi)$	- Random variable	Variable	Fixed
$X(t_1,\xi_1)$	- Numerical value.	Fixed	Fixed

Random process denoted by $\{X(t, \xi)\}$ is the collection of (the uncountably infinite if the state space ξ is cont. or countably infinite if the state space ξ is discrete) random fns. $X(t; \xi_1), X(t, \xi_2), \ldots$ with state space $\{\xi_1, \xi_2, \ldots \xi_n\}$

Collection of random fns is also called as ensemble.

41+49
= 90
100

Q-9×30

Classification of random variable

		State space (ξ)	
		Discrete	Cont.
Time parameter (t)	Discrete	Discrete random seq.	Discrete random seq
	Cont.	Discrete rand. process	Cont. random process

26.2.19

PROBABILITY DISTRIBUTIONS AND STATISTICAL AVERAGE

~~Gn a random variable pr~~

* PMF & PDF:

PMF: Denoted as $P[X(t) = x] = f_X(x)$ aka first order pmf

PDF: Denoted as $f_X(x, t)$ aka first order PDF

Second order PMF (Joint PMF): $P[X(t_1) = x_1, X(t_2) = x_2]$

Second order PDF: $f_{XX}(x_1, x_2, t_1, t_2)$ or $f(x_1, x_2; t_1, t_2)$ or

~~$f_{X(t_1)X(t_2)}(x_1, x_2)$~~ $f_{X(t_1)X(t_2)}(x_1, x_2)$,

27.2.19

Q) In an exp. of tossing a fair coin, the random process $\{X(t)\}$ is defined as

$$X(t) = \begin{cases} \sin(\pi t) & \text{, if head turns up} \\ 2t & \text{, if tail turns up} \end{cases}$$

(i) Find $E[X(t)]$ at $t = 1/4$

(ii) Find the cummulative distribution function $F(x, t)$ at $t = 1/4$

PTO

come on

Soln:

WKT

$$P(H) = P(T) = 1/2$$

$\Rightarrow P[X(t) = \sin\pi t] = \frac{1}{2}$ & $P[X(t) = 2t] = 1/2$

When $t = 1/4$,

$P[X(1/4) = \sin\frac{\pi}{4}]$

$= P[X(1/4) = 1/\sqrt{2}] = 1/2$

$P[X(1/4) = 2 \cdot 1/4] = 1/2$

$P[X(1/4) = 1/2] = 1/2$

x	$1/2$	$1/\sqrt{2}$
$P[X(1/4) = x]$	$1/2$	$1/2$

(i) $\therefore E[X(1/4)] = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)$

$= \frac{1}{2}\frac{(\sqrt{2}+2)}{2\sqrt{2}}$

$= \frac{\sqrt{2}}{2}\frac{(1+\sqrt{2})}{2\cancel{\sqrt{2}}}$

$= \frac{1}{4}(1+\sqrt{2})$

$E[X(t)] = \sum_x x \cdot P[X(t) = x]$

$= (\sin\pi t) \cdot P[X(t) = \sin\pi t] + 2t \cdot P[X(t) = 2t]$

at $t = 1/4$,

$E[X(t)] = \sin\frac{\pi}{4} \cdot P[X(1/4) = \sin\frac{\pi}{4}] + \frac{2}{4} \cdot P[X(1/4) = 2/4]$

$= \frac{1}{\sqrt{2}}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}(\sqrt{2}+1)$

≈ 0.6036

(ii) CDF of $X(t)$ at $t = 1/4$

$$F(x,t) = \begin{cases} 0 & , \text{ if } x < 1/2 \\ 1/2 & , \text{ if } 1/2 \le x < 1/\sqrt{2} \\ 1 & , \text{ if } x \le 1/\sqrt{2} \end{cases}$$

Q). Let $\{X(t)\}$ be a random process where

$$X(t) = Y|\cos(2\pi f t)|, \quad t \ge 0$$ where ω is a constant with exponential PDF given by

$$\boxed{f = \frac{\omega}{2\pi}}$$

$$f_Y(y) = \begin{cases} \frac{1}{10} e^{-y/10} & , y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Then obtain PDF of $\{X(t)\}$

Soln: CDF of $X(t)$ can be given by,

$$F_{X(t)}(x) = P[X(t) \le x]$$

$$= P[Y|\cos(2\pi f t)| \le x]$$

$$= P\left[Y \le \frac{x}{|\cos(2\pi f t)|}\right]$$ Let $z = \frac{x}{|\cos 2\pi f t|}$

$$= \int_{-0}^{z} f_Y(y)\, dy$$

$$= \int_{-0}^{z} \frac{1}{10} e^{-y/10}$$

$$= \frac{-10}{10}\left[e^{-y}\right]_0^{\frac{x}{|\cos 2\pi|} z}$$

$$= -\left[e^{-z} - 1\right]$$

$$= 1 - e^{-z}$$

$$F_{X(t)}(x) = 1 - e^{-x/|\cos(2\pi f t)| 10} = 1 - e^{-\frac{x}{10|\cos(2\pi f t)|}}$$

//.

WKT,

$$f_x(x) = \frac{d}{dx} F_{X(t)}(x)$$

$$= \left(1 - e^{\frac{-x}{10|\cos 2\pi f t|}}\right)'$$

$$= 0$$

$$= \begin{cases} \dfrac{1}{10|\cos 2\pi f t|} \, e^{-\frac{x}{10|\cos(2\pi f t)|}} , & x \geq 0 \\ 0 , & \text{otherwise} \end{cases}$$

$\left(\because Y \leq \dfrac{x}{|\cos(2\pi f t)|} \ \& \ Y \geq 0\right)$

Q). If $\{X(t)\}$ is a random process with $\mu(t) = 8$ and auto correlation. $R(t_1, t_2) = 64 + 10e^{-2|t_1 - t_2|}$, then find, mean, variance & covariance of the random variables $Z = X(6)$ and $W = X(9)$

Soln: Gn,

$$E[X(t)] = \mu_X(t) = 8$$

and $R(t_1, t_2) = 64 + 10e^{-2|t_1 - t_2|}$

$$R(t_1, t_2) = E[X(t_1)\, X(t_2)]$$

(i) Mean :

$$E[Z] = E[X(6)] = \mu(6) = 8$$

$$E[W] = E[X(9)] = \mu(9) = 8$$

(ii) Variance:

$$\begin{aligned} Var(Z) &= E[Z^2] - (E[Z])^2 \\ &= E[X(6)\,X(6)] - (E[X(6)])^2 \\ &= R(6,6) - (\mu(6))^2 \\ &= 64 + 10e^0 - 8^2 \\ &= 74 - 64 \\ &= 10 \end{aligned}$$

$$Var(W) = E[W^2] - (E[W])^2$$

$$= R(9,9) - 8^2$$

$$= 74 - 64$$

$$= 10$$

$$cov(Z, W) = E[ZW] - E[Z]E[W]$$

$$= E[X(6)X(9)] - 8 \cdot 8$$

$$= R(6,9) - 64$$

$$= 64 + 10e^{-6} - 64$$

$$= 10e^{-6}$$

$$= 0.02479$$ //

Q). If $\{Z(t)\}$ is a random process defined by

$Z(t) = Xt + Y$ where X & Y are a pair of random variables with means μ_x & μ_y, variances σ_x^2, σ_y^2 respectively & correlation coefficient ρ_{xy}.

Find (i) mean (ii) variance (iii) auto-correlation (iv) auto-covariance of $\{Z(t)\}$

(i) $E[Z(t)] = E[Xt + Y]$

$$= E[Xt] + E[Y]$$

$$= t \cdot E[X] + E[Y]$$

$$= \mu_x t + \mu_y$$

(ii) Variance of Z,

$$Var[Z(t)] = Var[Xt + Y]$$

$$= E[(Xt+Y)^2] - (E[Xt+Y])^2$$

$$= E[X^2t^2 + Y^2 + 2XYt] - (\mu_x t + \mu_y)^2$$

$$= t^2 \cdot E[X^2] + E[Y^2] + 2t\,E[XY] -$$

$$Var[aX+bY] = a^2 Var[X] + b^2 Var[Y] + 2ab\, cov(X,Y)$$

$$\therefore Var[Xt+Y] = t^2 \cdot Var[X] + Var[Y] + 2t\, cov(X,Y)$$

$$\cancel{cov(X,Y) = E[XY] - E[X]E[Y]}$$

$$\rho_{XY} = \frac{cov(X,Y)}{\sigma_x \sigma_y}$$

$$cov(X,Y) = \rho_{XY} \cdot \sigma_x \sigma_y$$

$$\Rightarrow Var[Z(t)] = t^2 \sigma_x^2 + \sigma_y^2 + 2t\, \rho_{xy} \sigma_x \sigma_y$$

(III) $R(t_1, t_2) = E[Z(t_1)\, Z(t_2)]$

$$= E[(Xt_1+Y)(Xt_2+Y)]$$

$$= E[t_2 t_1 X^2 + (t_1+t_2)(X+Y) + Y^2]$$

$$= t_1 t_2\, E[X^2] + (t_1+t_2)(E[X] + E[Y]) + E[Y^2]$$

$$= t_1 t_2\, E[X^2] + E[Y^2] + (t_1+t_2)(\mu_x + \mu_y)$$

Stationarity of Random Processes:

Stationary refers to time invariance of some, or all statistics of a random process, mean, variance, autocorrel...

2 Types of ~~Station~~ stationarity,

(i) strict sense stationary (SSS)

(ii) wide sense stationary (WSS)

STATIONARITY OF RANDOM PROCESSES

WSS (Wide Sense Stationary) Process:

- Mean is constant.
- Auto-correlation is time invariant i.e. not a fn. of t

SSS (Strict Sense Stationary) Process:

- ~~Atleast one of the moments should be time invariant.~~
- All the moments should be time invariant.

5.3.19

Q) $X(t) = Y\cos t + Z\sin t$

Y & Z are independent.

$Y=y$	-2	1	$Z=z$	-2	1
$P(Y=y)$	$1/3$	$2/3$	$P[Z=z]$	$1/3$	$2/3$

Show that $\{X(t)\}$ is WSS not SSS

Soln:

$E\{X(t)\} = E\{Y\cos t + Z\sin t\}$

$= \cos t \cdot E[Y] + \sin t\, E[Z]$

$= \cos t \cdot \sum_y y\, P[Y=y] + \sin t \sum_z P[Z=z]$

$= \cos t \cdot \left(-2\cdot\frac{1}{3} + 1\cdot 2/3\right) + \sin t\,(0)$

$= 0$ //.

$\therefore\ E\{X(t)\} = 0$ //

Now, $R(t_1, t_2) = E[X(t_1)\,X(t_2)]$

$= E\left[(Y\cos t_1 + Z\sin t_1)(Y\cos t_2 + Z\sin t_2)\right]$

$= E\left[Y^2\cos t_1\cos t_2 + YZ\cos t_1\sin t_2 + YZ\sin t_1\cos t_2 + Z^2\sin t_1\sin t_2\right]$

$= \cos t_1\cdot\cos t_2\, E[Y^2] + (\cos t_1\sin t_2 + \sin t_1\cos t_2)\,E[YZ] + \sin t_1\sin t_2\, E[Z^2]$

$E[y^2] = \sum_y y^2 f(y)$

$= 4\cdot\frac{1}{3} + \frac{2}{3} = 2$ Ily $E[Z^2] = 2$

$E[YZ] = E[Y]\cdot E[Z]$ ($\because$ Y & Z are independent)

$= 0\cdot 0$

$= 0$

$\therefore R(t_1, t_2) = 2(\cos t_1\cos t_2 + \sin t_1\sin t_2)$

$= 2\cos(t_2 - t_1)$

$= 2\cos\tau$, where $\tau = t_2 - t_1$

$\therefore \boxed{R(t_1, t_2) = 2\cos\tau}$

~~E[t_1, t_2]~~

$\because$ $E\{X(t)\} = 0$ is constant & auto correlation $R(t_1, t_2) = 2\cos\tau$ is a fn. of time difference thus $\{X(t)\}$ is WSS

$E\{X^2(t)\} = E\{(Y\cos t + Z\sin t)^2\}$

$= E[Y^2\cos^2 t + 2YZ\cos t\sin t + Z^2\sin^2 t]$

$= \cos^2 t\, E[Y^2] + 2\cos t\sin t\, E[YZ] + \sin^2 t\, E[Z^2]$

$= 2(\cos^2 t + \sin^2 t) + 0$

$= 2u$

$E\{X^3(t)\} = E\{(Y\cos t + Z\sin t)^3\}$

$= E[Y^3\cos^3 t + Z^3\sin^3 t + 3Y^2\cos^2 t\, Z\sin t + 3Y\cos t\, Z^2\sin^2 t]$

$= \cos^3 t\, E[Y^3] + \sin^3 t\, E[Z^3] + 3\cos^2 t\sin t\, E[Y^2 Z] + 3\cos t\sin t\, E[Y^2 Z]$

$= \cos^3 t\, E[Y^3] + \sin^3 t\, E[Z^3]$

$E[Y^3] = -\frac{8}{3} + \frac{2}{3} = -\frac{6}{3} = E[Z^3] = -2$

$= -2(\cos^3 t + \sin^3 t)$

which is not time invariant, as it depend on 't'

$\therefore \{X(t)\}$ is not SSS

Q) Consider a random process $\{X(t)\}$ such that

$X(t) = A\cos(\omega t + \theta)$ where A & ω are contant & θ is an uniform random variable distributed in the interval $(-\pi, \pi)$. Check whether $\{X(t)\}$ is a WSS

~~E[θ]~~.

Soln: θ is uniform in $(-\pi, \pi)$, we have

$$f(\theta) = \frac{1}{2\pi} \quad (\pi - (-\pi)) \quad , -\pi \le \theta \le \pi$$

Now,

$$E[\theta] = \int_{0} \theta f(\theta)$$

$$= \int_{-\pi}^{\pi} \theta \cdot \frac{1}{2\pi} \, d\theta$$

$$= \frac{1}{4\pi} \cdot \left[\theta^2\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi} (\pi^2 - \pi^2) = 0$$

$$E\{x(t)\} = E[A\cos(\omega t + \theta)]$$

$$= A\, E[\cos(\omega t + \theta)]$$

$$= A \cdot \int_{-\pi}^{\pi} \cos(\omega t + \theta) \, f(\theta) \, d\theta \qquad \left[E\{x(\theta)\} = \int_{-\infty}^{\infty} x(\theta) f(\theta) d\theta\right]$$

$$E\{x(t)\} = A \int_{-\pi}^{\pi} \cos(\omega t + \theta) \frac{1}{2\pi} \, d\theta$$

$$= \frac{A}{2\pi} \left[\sin(\omega t + \theta)\right]_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi} \left[\sin(\omega t + \pi) - \sin(\omega t - \pi)\right]$$

$$= 0$$

$$E\{x(t)\} = 0$$

$$R_{xx}(t_1, t_2) = E[X(t_1) X(t_2)]$$

$$= E[A\cos(\omega t_1 + \theta) \, A\cos(\omega t_2 + \theta)]$$

$$= \frac{A^2}{2} E[\cos(\omega t_1 + \theta - \omega t_2 - \theta) + \cos(\omega t_1 + \theta + \omega t_2 + \theta)]$$

$$= \frac{A^2}{2} E[\cos(\omega t_1 - \omega t_2) + \cos[\omega(t_1 + t_2) + 2\theta])$$

$$= \frac{A^2}{2} E[\cos\omega(t_1 - t_2) + \cos[\omega(t_1 + t_2) + 2\theta]]$$

$$= \frac{A^2}{2}\cos\omega(t_1-t_2) + \frac{A^2}{2} E\left[\cancel{\omega t_1}\cos(\omega(t_1+t_2)+2\theta)\right]$$

$$= \frac{A^2}{2}\cos\omega(t_1-t_2) + \frac{A^2}{2}\int_{-\pi}^{\pi}\cos\left[\omega(t_1+t_2)+2\theta\right] f(\theta)\, d\theta$$

$$= \frac{A^2}{2}\cos\omega(t_1-t_2) + \frac{A^2}{2}\cdot\frac{1}{2\pi}\left[\frac{\sin[\omega(t_1+t_2)+2\theta]}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{A^2}{2}\cos\omega(t_1-t_2) + \frac{A^2}{8\pi}\left[\sin[\omega(t_1+t_2)+2\pi] - [\sin\omega(t_1+t_2)-2\pi]\right]$$

$$= \frac{A^2}{2}\cos\omega(t_1-t_2) \qquad = 0$$

$$R_{xx}(t_1,t_2) = \frac{A^2}{2}\cos\omega(t_1-t_2)$$

$\therefore$ WSS

PROPERTIES OF AUTO-CORRELATION:

→ Auto-correlation of an stationary random process $\{X(t)\}$ is an even fn. i.e. $R_{xx}(\tau) = R_{xx}(-\tau)$

→ Auto-correlation of stationary random process is max. at $\tau = 0$.
i.e. $|R_{xx}(\tau)| \le R_{xx}(0)$; Also $R_{xx}(0) = E[x^2(t)]$

→ If $R_{xx}(\tau)$ is the auto correlation fn. of a stationary random process $\{x(t)\}$, then the mean of the random process can be obtained as

$$\mu_x = \sqrt{\lim_{\tau\to\infty} R_{xx}(\tau)}$$

→ The auto-correlation $R(\tau)$ of a stationary process $X(t)$ is periodic with period h with $h \neq 0$

$$R_{xx}(\tau+h) = R_{xx}(\tau)$$

$$E[x(t)\,x(t+\tau)] = R(\tau) \quad ; \quad E[x(t+\tau)\,x(t+\tau)] = R(0)$$

6.3.19

Q). A stationary random process $\{X(t)\}$ has an auto-correlation function $R_{XX}(\tau) = \frac{25\tau^2+36}{6.25\tau^2+4}$. Then find mean and variance of the process.

Soln:

(i) Mean:

$$\mu_X = \sqrt{\lim_{\tau\to\infty} R_{XX}(\tau)}$$

$$= \left(\underset{\tau\to\infty}{Lt} \frac{25\tau^2+36}{6.25\tau^2+9}\right)^{1/2}$$

$$= \left(\underset{\tau\to\infty}{Lt} \frac{25+36/\tau^2}{6.25+9/\tau^2}\right)^{1/2}$$

$$= \left(\frac{25}{6.25}\right)^{1/2}$$

$$= \left(\frac{\cancel{25}\times 100\ 4}{\cancel{25}\times\cancel{25}}\right)^{1/2} = 2 //.$$

(ii) Variance:

WKT,

$$E[X^2(t)] = R_{XX}(0)$$

$$\therefore \operatorname{Var}\{X(t)\} = R_{XX}(0) - (E\{X(t)\})^2$$

$$= \frac{36}{4} - 2^2$$

$$= 9-4$$

$$= 5 //.$$

Q). If $\{X(t)\}$ is a wide sense stationary process with auto-correlation function $R(\tau) = 4e^{-2|\tau|}$, then find $E\{[X(t+\tau) - X(t)]^2\}$ i.e. second moment of $X(t+\tau) - X(t)$

Soln:

$$E\{[X(t+\tau) - X(t)]^2\} = E\{[X^2(t+\tau) - 2X(t+\tau)X(t) + X^2(t)]$$

$$= E[X^2(t+\tau)] - 2E[X(t+\tau)X(t)] + E[X^2(t)]$$

$$= R(0) - 2R(\tau) + R(0)$$

$$= 4 - 8e^{-2|\tau|} + 4 = 8 - 8e^{-2|\tau|} //$$

Q) Let $\{X(t)\}$ be a random process and $X(t_1)$ and $X(t_2)$ are the random variables of the process at 2 time points t_1 and t_2 with autocorrelation function $R_{XX}(t_1, t_2)$. If $Y(t)$ is another random process such that $Y(t) = X(t_1) + X(t_2)$ with auto correlation function $R_{YY}(t_1, t_2)$ then show that $R_{YY}(t, t) = R_{XX}(t_1, t_1) + R_{XX}(t_2, t_2) + 2R_{XX}(t_1, t_2)$

Soln:

$$R_{YY}(t, t) = E[Y(t)\, Y(t)]$$

$$= E[(X(t_1) + X(t_2))(X(t_1) + X(t_2))]$$

$$= E[X(t_1)X(t_1) + 2X(t_1)X(t_2) + X(t_2)X(t_2)]$$

$$= E[X(t_1)X(t_1)] + 2E[X(t_1)X(t_2)] + E[X(t_2)X(t_2)]$$

$$R_{YY}(t, t) = R_{XX}(t_1, t_1) + 2R_{XX}(t_1, t_2) + R_{XX}(t_2, t_2)$$

NOTE: $R_{XX}(t_1, t_2)$ cannot be equal to $R_{XX}(0)$. $R_{XX}(t_1, t_1) = R_{XX}(0)$ only when $\{X(t)\}$ is a stationary random process.

PROPERTIES OF CROSS-CORRELATION:

* The cross-correlation fn. $R_{X_1X_2}(\tau)$ of two stationary random process $\{X_1(t)\}$ & $\{X_2(t)\}$ is an even function.

i.e. $R_{X_1X_2}(\tau) = R_{X_1X_2}(-\tau)$

* If $\{X_1(t)\}$ & $\{X_2(t)\}$ are a stationary random processes with auto-correlation functions $R_{X_1X_1}(\tau)$ & $R_{X_2X_2}(\tau)$ respectively and let $R_{X_1X_2}(\tau)$ be their cross correlation function, then

$$|R_{X_1X_2}(\tau)| \leq \sqrt{R_{X_1X_1}(0)\, R_{X_2X_2}(0)}$$

* If $\{X_1(t)\}$ & $\{X_2(t)\}$ are two stationary random process then, the cross relation:

$$|R_{X_1X_2}(\tau)| \leq \frac{1}{2}\{R_{X_1X_1}(0) + R_{X_2X_2}(0)\}$$

* If $\{X_1(t)\}$ and $\{X_2(t)\}$ are two independent stationary random processes with mean values $E\{X_1(t)\} = \mu_{x_1}$ and $E\{X_2(t)\} = \mu_{x_2}$, then

$$R_{x_1x_2}(\tau) = \mu_{x_1} \cdot \mu_{x_2}$$

Proof :

$$R_{x_1x_2}(\tau) = E[X_1(t)\, X_2(t+\tau)]$$

$$= E[X_1(t)] \cdot E[X_2(t+\tau)] \quad \{\because X_1 \,\&\, X_2 \text{ are independent}\}$$

$$= \mu_{x_1} \cdot \mu_{x_2}$$

NOTE: If the random process $\{X(t)\}$ is integrable in mean square sense, then

$$E\left\{\int_a^b X(t)\, dt\right\}^2 = \int_a^b \int_a^b E[X(t_1)\, X(t_2)]\, dt_1\, dt_2$$

$$= \int_a^b \int_a^b R(t_1, t_2)\, dt_1\, dt_2$$

Q). Let $\{X(t)\}$, and $\{Y(t)\}$ be two random stationary processes such that ~~$\{X(t)\}, \{Y(t)\}$~~

$X(t) = 3\cos(\omega t + \theta)$ $\quad Y(t) = 2\cos(\omega t + \theta - \pi/2)$, where θ is a random variable uniformly distributed in $(0, 2\pi)$, then prove that

$$\sqrt{R_{xx}(0)\, R_{yy}(0)} \geq |R_{xy}(\tau)|$$

Soln:

$$R_{xx}(\tau) = E[X(t)\, X(t+\tau)]$$

$$= E\{3\cos(\omega t + \theta) \cdot 3\cos(\omega(t+\tau) + \theta)\}$$

$$= \frac{9}{2} E\{\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\theta)\}$$

$$= \frac{9}{2} \cancel{E} \cos \omega\tau + \frac{9}{2} E[\cos(2\omega t + \omega\tau + 2\theta)]$$

$$= \frac{9}{2}\cos\omega\tau + \frac{9}{2}\int_0^{2\pi} \cos(2\omega t+\omega\tau+2\theta)\, f(\theta)\,d\theta$$

$$= \frac{9}{2}\cos\omega\tau + \frac{9}{2}\int_0^{2\pi} \cos(2\omega t+\omega\tau+2\theta)\cdot\frac{1}{2\pi}\,d\theta \quad \left[\because f(\theta)=\frac{1}{2\pi-0}=\frac{1}{2\pi}\right]$$

$$= \frac{9}{2}\cos\omega\tau + \frac{9}{4\pi}\,\frac{1}{2}\Big[\sin(2\omega t+\omega\tau+2\theta)\Big]_0^{2\pi}$$

$$= \frac{9}{2}\cos\omega\tau + \frac{9}{8\pi}\Big[\sin(2\omega t+\omega\tau+4\pi) - \sin(2\omega t+\omega\tau+0)\Big]$$

$$= \frac{9}{2}\cos\omega\tau + \frac{9}{8\pi}\Big[\sin(2\omega t+\omega\tau) - \sin(2\omega t+\omega\tau)\Big]$$

$R_{xx}(\tau) = \frac{9}{2}\cos\omega\tau$ // |||ly

$R_{xx}(0) = \frac{9}{2}$

$R_{yy}(\tau) = E\{Y(t)\,Y(t+\tau)\}$

$$= E\{2\cos(\omega t+\theta-\pi/2)\cdot 2\cos(\omega(t+\tau)+\theta-\pi/2)\}$$

$$= \frac{4}{2}\,E\Big[\cos(\omega\tau) + \cos(2\omega t+\omega\tau+2\theta-\pi)\}\Big]$$

$$= 2\cos\omega\tau + E\Big[\cos(2\omega t+\omega\tau+2\theta-\pi)\Big]$$

$R_{yy}(\tau) = 2\cos\omega\tau$

$R_{yy}(0) = 2\cdot 1 = 2.$

|||ly $R_{xy}(\tau) = 3\sin\omega\tau$

$R_{xx}(0)\cdot R_{yy}(0) = 2\cdot\frac{9}{2} = 9$

$\sqrt{R_{xx}(0)\cdot R_{yy}(0)} = 3$

$R_{xy}(\tau) = 3\sin\omega\tau$

$\Rightarrow R_{xy}(\tau) \le 3 \quad \because -1 \le \sin\omega\tau \le 1$

H.P //

11.3.19

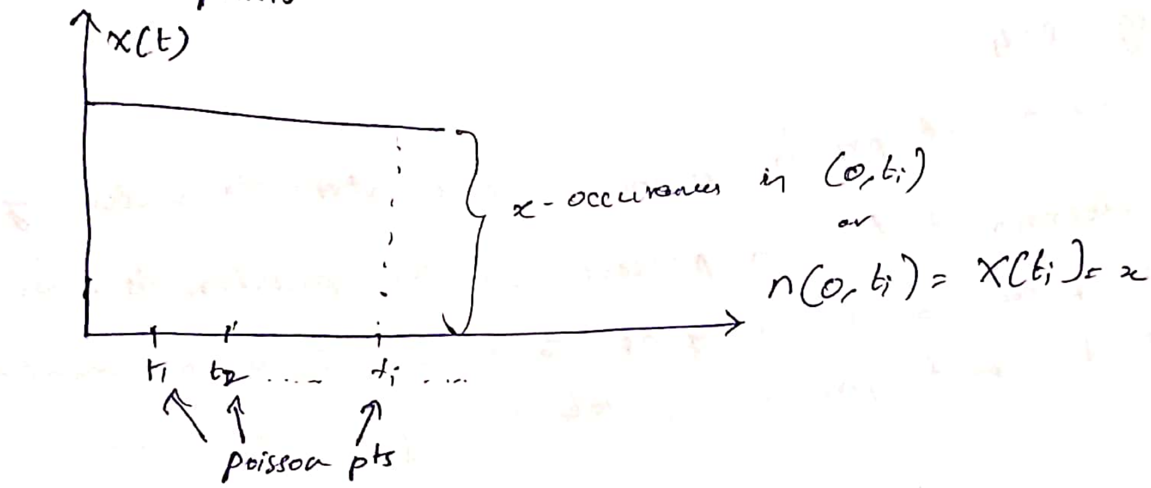
POISSON PROCESS

Poisson Distribution

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad , x=0,1,2\ldots$$

where parameter $\lambda \geq 0$ represents rate of occurance of events

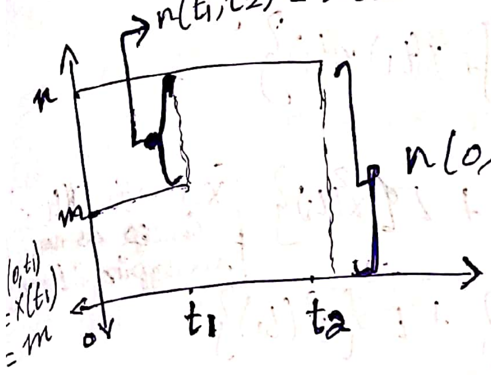
Poisson points



$\{x(t)\}$ $\quad x(t_1) = R.V$

here, $x(t_i)$ = ~~no.~~ occurrances of x in an interval of $(0, t_i)$.

$n(t_1, t_2) = X(t_2) - X(t_1) = n - m$



$n(0, t_2) = X(t_2) = n$

$(0, t_1) = X(t_1) = m$

Poisson Process:

The random process $\{X(t)\}$ is said to be a poisson process with $\lambda t > 0$, if the PMF of $X(t)$ is given by,

$$\boxed{P\{X(t) = x\} = \frac{e^{-(\lambda t)} (\lambda t)^x}{x!}, \quad x = 0, 1, 2 \ldots}$$

(OR) a counting process $\{X(t)\}$ is said to be Poisson process with parameter $\lambda t > 0$, if,

(i) $X(t) = 0$ when $t = 0$.

(ii) $\{X(t)\}$ has independent increments, i.e, if the intervals (t_1, t_2) and (t_2, t_3) are non-overlapping, then the rand. vars.

$n(t_1, t_2) = X(t_2) - X(t_1)$,

and $n(t_2, t_3) = X(t_3) - X(t_2)$

Mean:

$$E\{X(t)\} = \lambda t$$

Variance:

$$Var\{X(t)\} = \lambda t = E\{X^2(t)\} - (E\{X(t)\})^2$$

$$\therefore \boxed{E\{X^2(t)\} = \lambda^2 t^2 + \lambda t}$$

Autocorrelation:

$$R(t_1,t_2) = E\{X(t_1)\,X(t_2)\}$$

$$= E\{X(t_1)X(t_2) - X(t_1)X(t_1) + X(t_1)X(t_1)\}$$

$$= E\{X(t_1)[X(t_2) - X(t_1)] + X(t_1)X(t_1)\}$$

$$= E\{X(t_1)[X(t_2) - X(t_1)]\} + E\{X^2(t_1)\}$$ [$X(t_1)$, $X(t_2)-X(t_1)$ are indep as non overlapping interval]

$$= E\{X(t_1)\}\,E\{X(t_2) - X(t_1)\} + E\{X^2(t_1)\}$$

$$= (\lambda t_1)(\lambda(t_2-t_1)) + \lambda^2 t_1^2 + \lambda t_1$$

$$= \lambda^2 t_1 t_2 - \cancel{\lambda^2 t_1^2} + \cancel{\lambda^2 t_1^2} + \lambda t_1$$

$$\therefore \boxed{R(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda t_1} \quad (\text{if } t_2 > t_1)$$

$|||^{ly}$ $$\boxed{R(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda t_2} \quad (\text{if } t_1 > t_2)$$

$$\therefore \boxed{R(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1,t_2)}$$

Auto-covariance:

$$C_{xx}(t_1,t_2) = R_{xx}(t_1,t_2) - E\{X(t_1)\}E\{X(t_2)\}$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 - (\lambda t_1)(\lambda t_2) \quad \text{if } (t_2 > t_1)$$

$$\therefore Cov_{xx}(t_1,t_2) = \begin{cases} \lambda t_1, & \text{if } t_2 > t_1 \\ \lambda t_2, & \text{if } t_1 > t_2 \end{cases}$$

Note: If $t_1 = t_2 = t$ we have $C_{xx}(t_1,t_2) = \lambda t$, which is nothing but the variance of the process (poisson) $\{X(t)\}$.

* **Theorem:**

If $\{X_1(t)\}$ & $\{X_2(t)\}$ 2 indep. poisson processes with parameters $\lambda_1 t, \lambda_2 t$ resp., then the process $\{Y(t)\} = X_1(t) + X_2(t)$ is a poisson process with params $(\lambda_1+\lambda_2)t$, i.e, sum of 2 indep. poisson processess is also a **poisson process**.

Proof:

$$Y(t) = X_1(t) + X_2(t)$$

$$E\{Y(t)\} = E\{X_1(t)\} + E\{X_2(t)\}$$

$$= \lambda_1 t + \lambda_2 t$$

$$= (\lambda_1 + \lambda_2)t$$

$$V\{Y(t)\} = V\{X_1(t) + X_2(t)\}$$

$$= V\{X_1(t)\} + V\{X_2(t)\} + \underbrace{2\,Cov_{X_1X_2}(t)}_{\to\, 0 \text{ as indep.}}$$

$$= \lambda_1 t + \lambda_2 t$$

$$= (\lambda_1 + \lambda_2)t$$

$E\{Y(t)\} = V\{Y(t)\}$, ~~so they~~ $\{Y(t)\}$ is ~~also~~ a Poisson Process

13·3·19

POISSON PROCESS

$X(t_1) = n(0, t_1) = x$

$X(t_2) - X(t_1) = n(t_1, t_2) = n$

$E\{X(t)\} = \lambda t$

$Var\{X(t)\} = \lambda t$

NOTE: The poisson process is not a stationary process

$$P[X(t) = n(0,t) = x] = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \quad ; x = 0, 1, 2, \ldots$$

Q). At a service counter, customers arrive according to Poisson process with mean rate of 3 per min. Find the probabilities that during a time interval of 2 mins (i) exactly 4 customers arrive (ii) more than 4 costumers arrives

Given $\lambda = 3$

(i) $P[X(2) = 4] = \frac{e^{-3\cdot 2}(3\cdot 2)^4}{4!} = \frac{e^{-6}\cdot 6^{4}}{4\cdot 3\cdot 2} = \frac{e^{-6}\cdot 36\cdot 6}{4}$

$= 54e^{-6} = 0.1339$

(ii) $P[X(2) > 4] = 1 - P[X(2) \leq 4] = 1 - \sum_{x=0}^{4} \frac{e^{-6}\cdot 6^x}{x!}$

$P[X(2) = 0] = \frac{e^{-6}\cdot 6^0}{0!} = e^{-6}$

$\therefore P[X(2) \leq 4] = e^{-6}\left[1 + \frac{6}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!}\right]$

$\therefore P[X(2) > 4] = 0.7174$ //.

Q). If $\{X(t)\}$ is a Poisson process, such that $E\{X(9)\} = 6$ then find (i) mean & variance of $X(8)$ (ii) find $P\{X(4) \leq 5 / X(2) = 3\}$

Soln: Given

$E\{X(9)\} = 6$

$\lambda \cdot 9 = 6$

$\lambda = 2/3$

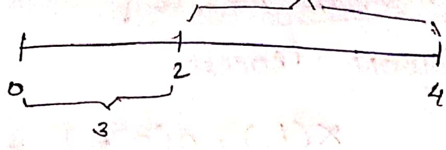
$\therefore E\{X(8)\} = V\{X(8)\}$

$= \lambda \cdot 8$

$= 16/3$ //.

(ii)

$P\{X(4) \le 5 / X(2) = 3\}$



$P\{X(4) \le 5 / X(2) = 3\}$

$$= \frac{P\{X(2) = 3 \cap X(4) \le 5\}}{P\{X(2) = 3\}}$$

$$= P\{n(2,4) \le 2\}$$

$$\frac{e^{-\lambda(t_2-t_1)}[\lambda(t_2-t_1)]^x}{x!}$$

$$= \sum_{x=0}^{2} \frac{e^{-\frac{2}{3}(4-2)}[2/3(2)]^x}{x!}$$

$$= 0.8494$$

Theorem:

The time X (waiting time or service time) b/w the occurances of events in a poisson process with parameter λx is an exponential

(or)

If there is an arrival at time pt. t_0 & the next arrival is at time pt. t_1, then the time b/w these 2 poisson pts is given by

$X = t_1 - t_0$, follows exponential distribution with PDF

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

Proof:

WKT

$$P\{n(t_0,t_1) = k\} = \frac{e^{-\lambda(t_1-t_0)}[\lambda(t_1-t_0)]^k}{k!} \quad k = 0,1,2,\ldots$$

$$= \frac{e^{-\lambda x}(\lambda x)^k}{k!} \quad x = t_1 - t_0$$

which implies $P\{n(t_0,t_1) = 0\} = e^{-\lambda x}$

$P\{X > x\} = P\{$no occurances in $(t_0,t_1)\}$

$$= P\{n(t_0,t_1) = 0\}$$

$$\boxed{P\{X > x\} = e^{-\lambda x} \; ; \; x = t_1 - t_0}$$

41

Now,

$$F(x) = P\{X \le x\}$$

$$= 1 - P\{X > x\}$$

$$= 1 - e^{-\lambda x}$$

$$f(x) = \frac{d}{dx} F(x)$$

$$\boxed{f(x) = \lambda e^{-\lambda x}} \quad x > 0$$

Q). If arrival of customers at a counter is in accordance with a poisson process with mean arrival rate of 2 per min. Then find the probability that interval b/w 2 conseutive arrival is (i) more than one min (ii) b/w 1 & 2 mins (iii) ≤ 4 mins

Soln: Let X is a random variable representing the time b/w arrivals then it follows exponential distribution fn. whose pdf is gvn by.

$$f(x) = \lambda e^{-\lambda x}, \quad x \ge 0$$

(i) $P[X > 1] = \int_1^\infty f(x)\, dx$

$$= \int_1^\infty \lambda e^{-\lambda x}\, dx$$

$$= -\frac{2}{2}\left[e^{-2x}\right]_1^\infty$$

$$= e^{-2} = \cancel{0.13}\ 0.1353$$

(ii) $P[1 < x < 2] = \int_1^2 f(x)\, dx$

$$= \int_1^2 2e^{-2x}\, dx$$

$$= 0.1170$$

(iii) $P[x > 4] = 1 - e^{-2\cdot 4} = 1 - e^{-8} = 0.9997$

$$= \int_0^4 2e^{-2x}\, dx$$

$$\int_0^4 2e^{-2x}\, dx$$

$$-\left[e^{-2x}\right]_0^4$$

$$-\{e^{-8} - 1\}$$

15.3.19

Time averages

$$\bar{x}_T = \frac{1}{2T}\int_{-T}^{T} x_T(t)\,dt$$ if time interval $(-T, T)$

$$\bar{x}_T = \frac{1}{T}\int_{0}^{T} x_T(t)\,dt$$ if time interval $(0, T)$

Ergodic Process

A stationary random process is ergodic if its ensemble avg (mean of random process) involving the process can be estimated using the time average of one of the sample functions (realisations) of the process.

Mean Ergodic Process

time interval $(-T, T)$; if time avg tends to constant ensemble avg as $T \to \infty$.

i.e. $E\{X(t)\} = \mu$ (constant)

$$\lim_{T\to\infty} \bar{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt = \mu$$

$$E\{X(t)\} = \lim_{T\to\infty} \bar{X}_T = \mu$$

Convergence in Probability:

Time average μ_T converges to ensemble average μ if

$$P[|\mu_T - \mu| \le \varepsilon] \to 1 \quad \text{as } T \to \infty$$

Mean Ergodic theorem:

$$\lim_{t\to\infty} V\{\bar{X}_T\} = 0$$

Proof:

$$\bar{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$

$$\Rightarrow E\{\bar{X}_T\} = \frac{1}{2T}\int_{-T}^{T} E\{X(t)\}\,dt = \mu \quad [\because E\{X(t)\} = \mu]$$

$\therefore E\{X(t)\} = \mu$

By Chebyshev's inequality,

$$P\left\{\left|\bar{X}_T - E(\bar{X}_T)\right| \le \varepsilon\right\} \ge 1 - \frac{V\{\bar{X}(t)\}}{\varepsilon^2}, \quad \varepsilon > 0$$

$$\Rightarrow P\left\{\left|\lim_{T\to\infty} \bar{X}_T - \mu\right| \le \varepsilon\right\} \ge 1 - \lim_{T\to\infty} \frac{V\{\bar{X}_T\}}{\varepsilon^2}$$

If $\lim V\{\bar{X}_T\} = 0$,

$$P\left\{\left|\lim_{T\to\infty} \bar{X}_T - \mu\right| \le \varepsilon\right\} = 1$$

By using convergence in probability, H.P

NOTE : If $C(\tau)$ is auto-covariance fn. of the stationary random process

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} C(\tau)\left(1 - \frac{|\tau|}{2T}\right) d\tau = 0$$

or

$$\lim_{T\to\infty} \frac{1}{T} \int_{0}^{2T} C(\tau)\left(1 - \frac{\tau}{2T}\right) d\tau = 0$$

or

$$\lim_{T\to\infty} \frac{2}{T} \int_{0}^{T} C(\tau)\left(1 - \frac{\tau}{T}\right) d\tau = 0$$

Q) A random process $\{X(t)\}$ has the sample fn of the form $X(t) = A\cos(\omega t + \theta)$ where ω is a constant, A is a random variable that has magnitude $+1$ & -1 with equal probabilities & θ is a random variable i.e uniformly distributed b/w 0 & 2π. Assume that A & θ are independent. Is $\{X(t)\}$ mean ergodic process?

Soln

(i) Ensemble avg: $E\{X(t)\} = \mu$ (constant)

(ii) Time average : $\lim_{T\to\infty} \bar{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt = \mu$

} for $\{X(t)\}$ to be mean ergodic.

Now

$$E\{X(t)\} = E\{A\cos(\omega t+\theta)\}$$

$$= E\{A\} \cdot E\{\cos(\omega t+\theta)\} \quad [\because A \,\&\, \theta \text{ are independent}]$$

$$= \left[-1\cdot\tfrac{1}{2} + 1\cdot\tfrac{1}{2}\right] E\{\cos(\omega t+\theta)\}$$

$$= 0 //.$$

$E\{\cos(\omega t+\theta)\}$

$\because \theta$ is uniformly distributed in $(0, 2\pi)$

$$f(\theta) = \frac{1}{2\pi}, \quad \theta \in [0, 2\pi]$$

$$E[\cos(\omega t+\theta)] = \int_0^{2\pi} \cos(\omega t+\theta)\frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi}\left[\sin(\omega t+\theta)\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\sin\omega t - \sin\omega t\right]$$

$$= 0 //.$$

(b) Now

$$\overline{X}_T = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$

$$= \frac{1}{2T}\int_{-T}^{T} A\cos(\omega t+\theta)\,dt$$

$$= \frac{A}{2T} \left[\frac{\sin(\omega t+\theta)}{\omega}\right]_{-T}^{T}$$

$$= \frac{A}{2\omega T}\left[\sin(\omega T+\theta) - \sin(-\omega t+\theta)\right]$$

$$= \frac{A}{2\omega T}\left[2\cos\theta \sin\omega T\right]$$

$$\lim_{T\to\infty} \overline{X}_T = \frac{\cancel{C}}{2\omega\cdot\infty} \; 0 //.$$

$$= A\cos\theta \Big(\underbrace{\lim_{T\to\infty} \frac{\sin\omega T}{\omega T}}_{=0}$$

$$= 0 //.$$

Time Averages

$$\overline{X_T} = \frac{1}{2T}\int_{-T}^{T} X(t)\,dt$$

Mean Ergodic Theorem:

If $\lim_{T\to\infty} V\{\overline{X_T}\} = 0$, then $\{X(t)\}$ is mean ergodic, where $\{X(t)\}$ is stationary.

① Show that R.P. with constant mean is mean ergodic if $\lim_{T\to\infty}\left\{\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)\,dt_1 dt_2\right\} = 0$

Solⁿ:

$$V(\overline{X_T}) = E(\overline{X_T}^2) - \{E(\overline{X_T})\}^2$$

$$= E\left(\frac{1}{(2T)^2}\int_{-T}^{T}\int_{-T}^{T} X(t_1)X(t_2)\,dt_1 dt_2\right) - E\left\{\left(\frac{1}{2T}\int_{-T}^{T} X(t_1)\,dt_1\right)\left(\frac{1}{2T}\int_{-T}^{T} X(t_2)\,dt_2\right)\right\}$$

$$V(\overline{X_T}) = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)]\,dt_1 dt_2 - \left(\int_{-T}^{T} E[X(t_1)]\,dt_1\right)\left(\int_{-T}^{T} E[X(t_2)]\,dt_2\right)$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}\left[E\{X(t_1)X(t_2)\} - E\{X(t_1)\}E\{X(t_2)\}\right] dt_1 dt_2$$

$$= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}\left[R(t_1,t_2) - E\{X(t_1)\}E\{X(t_2)\}\right] dt_1 dt_2$$

$$\cancel{\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)}$$

$$V(\overline{X_T}) = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} \mathrm{Cov}(t_1,t_2)\,dt_1 dt_2$$

If $\mu = 0$ of $\{X(t)\}$,

the $V(\overline{X_T}) = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} R(t_1,t_2)\,dt_1 dt_2$

Therefore, the ran. process $\{X(t)\}$ with constant mean is mean ergodic, if

$$\lim_{T\to\infty}\left\{\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)\,dt_1\,dt_2\right\} = 0$$

Note: If $\{X(t)\}$ is stationery then,

$$V(\overline{X}_T) = \frac{1}{T}\int_0^{2T} C(\tau)\left(1-\frac{\tau}{2T}\right)dt$$

Note: if Mean of $\{X(t)\} = 0$, then

$$V(\overline{X}_T) = \frac{1}{T}\int_0^{2T} R(\tau)\left(1-\frac{\tau}{2T}\right)dt$$

Note:

$$V(\overline{X}_T) = \frac{1}{T}\int_{-T}^{T} R(\tau)\left(1-\frac{|\tau|}{T}\right)d\tau. \text{ for interval } (-T,T)$$

Q If $E\{X(t)\} = 0$, with $\{X(t)\}$ as wide sense stationarity with $R_{XX}(\tau) = e^{-2|\tau|}$, show that $\{X(t)\}$ is mean ergodic

Sol^n: $R_{XX}(\tau) = e^{-2|\tau|}$

To prove,

$$\lim_{T\to\infty} V(\overline{X}_T) = \lim_{T\to\infty}\frac{1}{T}\int_0^{2T} C(\tau)\left(1-\frac{\tau}{2T}\right)d\tau$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^{2T} e^{-2|\tau|}\left(1-\frac{\tau}{2T}\right)d\tau \quad [\because E\{x(t)\}=0]$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^{2T} \underbrace{e^{-2\tau}}_{V}\underbrace{\left(1-\frac{\tau}{2T}\right)}_{u}d\tau$$

$$= \lim_{T\to\infty}\frac{1}{T}\left[\left(1-\frac{\tau}{2T}\right)\frac{e^{-2\tau}}{-2} - \left(\frac{-1}{2T}\right)\frac{e^{-2\tau}}{4}\right]_0^{2T}$$

$$= \lim_{T\to\infty}\left[\frac{1}{2} + \frac{e^{-4T}}{8T} - \frac{1}{8T}\right]\times\frac{1}{T}$$

$$\lim_{T\to\infty} V(\overline{X}_T) = \lim_{T\to\infty}\left(\frac{e^{-4T}}{8T^2}\right) + \lim_{T\to\infty}\frac{1}{2T} + \lim_{T\to\infty}\frac{1}{8T^2} = 0 \quad \text{[Apply L-Hospital Rule]}$$

$\therefore$ Mean Ergodic.

② A zero mean wss process $\{x(t)\}$ has a auto-corr. func. $R_{xx}(\tau) = e^{-2\alpha|\tau|}$, $\alpha > 0$. Show that $\{x(t)\}$ is mean ergodic.

Sol^n: $\lim_{T\to\infty} V(\bar{X}_T) = 0 = \lim_{T\to\infty} \frac{1}{T} \int_0^{2T} c(\tau)\left(1-\frac{\tau}{2T}\right)d\tau$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^{2T} e^{-2\alpha\tau}\left(1-\frac{\tau}{2T}\right)d\tau$$

$$= \lim_{T\to\infty} \frac{1}{T}\left[\left(1-\frac{\tau}{2T}\right)\frac{e^{-2\alpha\tau}}{-2\alpha} - \left(\frac{-1}{2T}\times\frac{e^{-2\alpha\tau}}{4\alpha^2}\right)\right]_0^{2T}$$

$$= \lim_{T\to\infty} \frac{1}{T}\left[\left(0+\frac{e^{-4\alpha T}}{8T\alpha^2}\right) - \left(\frac{-1}{2\alpha}+\frac{1}{8T\alpha^2}\right)\right]$$

$$= 0$$

So, Mean Ergodic.

---

③ A binary transm. process $\{x(t)\}$ has zero mean & auto-corr. func. $R(\tau) = 1 - \frac{|\tau|}{T}$. Find mean & variance of time average of the process $\{x(t)\}$ over the interval $(-T, T)$. Verify whether mean ergodic.

Sol^n: $E(\bar{X}_T) = \frac{1}{2T}\int_{-T}^{T} E\{x(t)\}\,dt$

$$= \frac{1}{2T}\int_{-T}^{T} 0\,dt \qquad [\because \text{given, mean} = 0]$$

$$= 0$$

$$V(\bar{X}_T) = \frac{1}{T}\int_{-T}^{T} c(\tau)\left(1-\frac{|\tau|}{T}\right)d\tau$$

$$= \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)\left(1-\frac{|\tau|}{T}\right)d\tau$$

As even function we rewrite it as,

$$Var(\overline{X_T}) = 2 \times \frac{1}{T} \int_0^T \left(1 - \frac{\tau}{T}\right)^2 d\tau$$

$$= 2 \times \frac{1}{T} \left[\frac{\left(1 - \frac{\tau}{T}\right)^3}{3} \times \frac{1}{-1/T} \right]_0^T$$

$$= -\frac{2}{T} \left[\frac{\left(1 - \frac{\tau}{T}\right)^3}{3} \times T \right]_0^T$$

$$= \frac{-2}{3T} \left[0 - (T) \right]$$

$$\therefore Var(\overline{X_T}) = 2/3$$

$$\lim_{T \to \infty} V(\overline{x_T}) = \lim_{T \to \infty} 2/3 \neq 0$$. So non-mean ergodic.

Power Spectral Density Functions:

Suppose,

		0	$\pi/2$	π	$3\pi/2$	2π
$X(t_1)$	$= \sin t$	0	1	0	-1	0
$X(t_2)$	$= \cos t$	1	0	-1	0	1

$$p(x,y) = 1/4 \ \forall x,y$$

$$E[X(t_1)X(t_2)] = \sum_{x,y} xy\, p(x,y).$$

$$= 0\times1\times 1/4 + 1\times0\times 1/4 + 0\times -1\times 1/4 + -1\times0\times 1/4 + 0\times1\times 1/4$$

$$= 0$$

$\boxed{\text{SDF } (\omega) \text{ is a func. of frequency}}$, used to describe how the power of time series is distributed with frequency.

(or) PSD refers the power per unit of freq. as a func. of freq.

$E\{X^2(t)\} = R_{xx}(t,t) = R_{xx}(0)$ gives the avg. power of $\{X(t)\}$

classmate

Date
Page

Fourier Transformation:

$$F[f(t)] = \hat{f}(\omega)$$

PSDF:

If $\{X(t)\}$ is a stationary process with auto-correlation $R_{XX}(\tau)$, the Fourier transformation of $R_{XX}(\tau)$ is called "power spectral density PSD" function of the process $\{X(t)\}$ and is given by

$$\boxed{S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau}$$ where $\omega = 2\pi f$

Inverse transformation

$$\boxed{R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{+i\omega\tau} d\omega}, \omega = 2\pi f$$

Cross Power Spectral Density Function:

$$\boxed{S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau}, \omega = 2\pi f$$

Inverse Cross PSD Function:

$$\boxed{R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(\omega) e^{+i\omega\tau} d\omega}$$

Properties:

1. PSD at '0' freq. = total area under $R_{XX}(\tau)$.
2. $E\{X^2(t)\} = R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\tau$, area under PSD curve

3. The PSD func $S_{xx}(\omega)$ is even function.

$S_{xx}(\omega) = S_{xx}(-\omega)$, $S_{xy}(\omega) = S_{xy}(-\omega)$.

Q. The auto-correlation function of the random signal process is given by,

$R(\tau) = a^2 e^{-2\gamma|\tau|}$. Determine power spectral density of the random signal.

Sol^n: WKT,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= a^2 \int_{-\infty}^{\infty} e^{-2\gamma|\tau|} \times e^{-i\omega\tau} dt$$

$$= a^2 \int_{-\infty}^{+\infty} e^{-2\gamma|\tau|} (\cos(\omega\tau) - i\sin(\omega\tau)) d\tau$$

$$= \underbrace{a^2 \int_{-\infty}^{\infty} e^{-2\gamma\tau} \cos(\omega\tau) d\tau}_{\text{Even Function}} - \underbrace{i a^2 \int_{-\infty}^{\infty} e^{-2\gamma|\tau|} \sin(\omega\tau) d\tau}_{\text{odd function}}$$

$$= 2a^2 \int_0^{\infty} e^{-2\gamma\tau} \cos(\omega\tau) d\tau + 0$$

Note:

$$\boxed{\int_0^{\infty} e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)}$$

$$= 2a^2 \left[\frac{e^{-2\gamma\tau}}{(-2\gamma)^2 + (\omega^2)} \{(-2\gamma)\cos(\omega x) + \omega \sin(\omega x)\} \right]_0^{\infty}$$

$$= 2a^2 \left[(0) - \left(\frac{1}{4\gamma^2 + \omega^2} (-2\gamma + 0) \right) \right]$$

$$\boxed{S(\omega) = \frac{4\gamma a^2}{4\gamma^2 + \omega^2}}$$ //

Q). Find the power spectral density function of the random process whose auto correlation function given by $R(\tau) = e^{-a\tau^2}$

Soln:

$$R_{xx}(\tau) = e^{-a\tau^2}$$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau^2} \cdot e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-(a\tau^2 + i\omega\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\left(\tau^2 + \frac{i\omega\tau}{a}\right)}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\left(\tau^2 + \frac{i\omega\tau}{a} + \left(\frac{i\omega}{2a}\right)^2 - \left(\frac{i\omega}{2a}\right)^2\right)}\, d\tau$$

$$= e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(\tau + \frac{i\omega}{2a}\right)^2}\, d\tau \qquad [i^2 = -1]$$

$\cancel{\left(\tau + \frac{i\omega}{2a}\right)^2 = x}$

$\cancel{2(d\tau = dx}$

e^{x^2}

Let $u = \sqrt{a}\left(\tau + \frac{i\omega}{2a}\right)$

If $\tau = -\infty \quad u = -\infty$

$\tau = \infty \quad u = \infty$

$$\frac{du}{d\tau} = \sqrt{a} \Rightarrow d\tau = \frac{du}{\sqrt{a}}$$

$$S_{xx}(\omega) = e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{a}}$$

$$= \frac{e^{-\frac{\omega^2}{4a}}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2}\, du$$

$$= \frac{2 \cdot e^{-\omega^2/4a}}{\sqrt{a}} \int_{0}^{\infty} e^{-u^2}\, du$$

Let $v = u^2$

$dv = 2u\, du$

$\Rightarrow \frac{dv}{du} = 2\sqrt{v}$.

$$\therefore S_{xx}(\omega) = \frac{1}{\sqrt{a}} e^{-\omega^2/4a} \cdot \cancel{2} \int_0^\infty e^{-v} \frac{dv}{\cancel{2}\sqrt{v}}$$

$$= \frac{1}{\sqrt{a}} e^{-\omega^2/4a} \int_0^\infty v^{-1/2} e^{-v}\, dv$$

$$= \frac{1}{\sqrt{a}} e^{-\omega^2/4a} \int_0^\infty e^{-v} v^{\frac{1}{2}-1}\, dv.$$

NOTE: Gamma function:

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\therefore S_{xx}(\omega) = \frac{1}{\sqrt{a}} e^{-\omega^2/4a} \cdot \Gamma\tfrac{1}{2} \qquad \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$= \frac{1}{\sqrt{2}} e^{-\omega^2/4a} \cdot \sqrt{\pi} //.$$

Q. If the power spectral density of a stationary ~~object~~ random process $\{X(t)\}$ is given by

$$S(\omega) = \begin{cases} \frac{b}{a}(a - |\omega|), & |\omega| \le a \\ 0, & |\omega| > a \end{cases} \qquad -a \le \omega \le a$$

then find the auto-correlation function of $\{X(t)\}$.

Soln:

$$R_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-a}^{a} \frac{b}{a}(a - |\omega|)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \cdot \frac{b}{a} \int_{-a}^{a} (a - |\omega|)\, e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \frac{b}{a} \int_{-a}^{a} (a - |\omega|)\left[\cos(\omega\tau) + i\sin(\omega\tau)\right] d\omega$$

$$= \frac{1}{2\pi} \int_{-a}^{a} \frac{b}{a}(a-|\omega|) \cos\omega\tau \, d\omega + \frac{1}{2\pi} i \int_{-a}^{a} (a-|\omega|) \sin\omega\tau \, d\omega$$

even fn. even fn. even fn odd fn

$$= \frac{1}{2\pi} \cdot 2 \int_{0}^{a} \frac{b}{a}(a-|\omega|) \cos\omega\tau \, d\omega$$

$$= \frac{b}{\pi a}\left[(a-\omega)\frac{\sin\omega\tau}{\tau} + \frac{-\cos(\omega\tau)}{\tau^2}\right]_0^a$$

$$= \frac{b}{\pi a}\left[\frac{-\cos a\tau}{\tau^2} - \left(0-\frac{1}{\tau^2}\right)\right]$$

$$= \frac{b}{\pi a}\left[\frac{1}{\tau^2} - \frac{\cos a\tau}{\tau^2}\right]$$

$$= \frac{b}{\pi a\tau^2}(1-\cos a\tau) // = \frac{b}{\pi a\tau^2} \cdot 2\sin^2(a\tau/2)$$

3. Find the power spectral density fn of the random process whose autocorrelation fn. is given by $R_{xx}(\tau) = e^{-a\tau^2} \cos b\tau$ where a & b are constants

Soln:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau^2} \cos b\tau \, e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \cos b\tau \, e^{-(a\tau^2 + i\omega\tau)} d\tau$$

$$= \int_{-\infty}^{\infty} \cos b\tau \cdot e^{-a\left(\tau^2 + \frac{i\omega\tau}{a}\right)} d\tau$$

$$= \int_{-\infty}^{\infty} \cos b\tau \cdot e^{-a\left(\tau^2 + \frac{i\omega\tau}{a} + \left(\frac{i\omega}{2a}\right)^2 - \left(\frac{i\omega}{2a}\right)^2\right)} d\tau$$

$$= e^{\frac{-\omega^2}{4a}} \int_{-\infty}^{\infty} \cos b\tau \cdot e^{-a\left(\tau + \frac{i\omega}{2a}\right)^2} d\tau$$

Let $\sqrt{a} \cdot \left(\tau + \frac{i\omega}{2a}\right) = u \Rightarrow \frac{du}{d\tau} = \sqrt{a}$

$$S(\omega) = e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} \left(\frac{e^{ib\tau} + e^{-ib\tau}}{2}\right) e^{-a\left(\tau + \frac{\omega i}{2a}\right)^2} d\tau$$

$$S(\omega) = \int_{-\infty}^{\infty} e^{-a\tau^2} \left(\frac{e^{ib\tau} + e^{-ib\tau}}{2}\right) e^{-i\omega\tau} d\tau$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{-[a\tau^2 + i(\omega-b)\tau]} + e^{-[a\tau^2 + i(\omega+b)\tau]}\right) d\tau$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-[a\tau^2 + i(\omega-b)\tau]} d\tau + \frac{1}{2}\int_{-\infty}^{\infty} e^{-a[a\tau^2 + i(\omega+b)\tau]} d\tau \quad \text{—— (1)}$$

~~Let~~

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-[a\tau^2 + i(\omega-b)\tau]} d\tau = \int_{-\infty}^{\infty} e^{-a[\tau^2 + i(\omega-b)\tau]^2} e^{-\frac{(\omega-b)^2}{4a}} d\tau$$

Let $u = \sqrt{a}\left(\tau + \frac{i(\omega-b)}{2a}\right)$

$$d\tau = \frac{du}{\sqrt{a}}$$

$$\Rightarrow \frac{1}{2}\int_{-\infty}^{\infty} e^{-[a\tau^2 + i(\omega-b)\tau]} d\tau = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{(\omega-b)^2}{4a}} \, //.$$

llly $\frac{1}{2}\int_{-\infty}^{\infty} e^{-[a\tau^2 + i(\omega+b)\tau]} d\tau = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{(\omega+b)^2}{4a}}$ //.

$$\therefore S_{xx}(\omega) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\left[e^{-\frac{(\omega-b)^2}{4a}} + e^{-\frac{(\omega+b)^2}{4a}}\right] //.$$

22.3.19 **MARKOV PROCESS**

$$P\{X(t_n) = \mathcal{E}_n / X(t_0) = \mathcal{E}_0, X(t_1) = \mathcal{E}_1, \ldots, X(t_{n-1}) = \mathcal{E}_{n-1}\} = \cancel{P\{X(t_n) = \mathcal{E}_n / X(t_{n-1}) = \mathcal{E}_{n-1}\}}$$

$$= P\{X(t_n) = \mathcal{E}_n / X(t_{n-1}) = \mathcal{E}_{n-1}\}$$

Cummulative probability

$$P\{X(t) \le \mathcal{E}_n / X(t_0) = \mathcal{E}_0, X(t_1) = \mathcal{E}_1, \ldots, X(t_{n-1}) = \mathcal{E}_{n-1}\} = P\{X(t_n) \le \mathcal{E}_n / X(t_{n-1}) = \mathcal{E}_{n-1}\}$$

MARKOVIAN PROPERTY: the state of a process depends only on the state of the process at immediate past time.

Markov ~~ian~~ chain:

A Markovian process is said to be markov chain if the state space E is discrete irrespective of whether the time parameter is discrete or continuous.

If time parameter is assumed to be discrete, it is always represented as steps.

Illy for discrete states of state space are represented by small cases a, b, c, d, etc.

$$P\{X_n = j / X_0 = a, X_1 = b, \ldots X_{n-1} = i\} = P\{X_n = j / X_{n-1} = i\}$$

Transition probability,

The probability that the process that was in state i in $(n-1)$th step moved to step j in nth step denoted by

$P\{X_n = j / X_{n-1}\ i\}$, $n = 1, 2, 3, \ldots$; $i, j = 1, 2, 3, \ldots, k$

One-step transition probability

Homogenous Markov Chain:

transition probabilities depend only on difference of steps but not on the actual steps.

Transition Probabity Matrix:

Assume that Markov chain is homogeneous if there are k no. of states then one-step probabilities can be obtained for $i, j = 1, 2, 3, \ldots, k$

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ k \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & \cdots & j & \cdots & k \end{array} \\
\begin{bmatrix}
p_{11}^{(1)} & P_{12}^{(1)} \ldots & P_{1j}^{(1)} & \ldots & P_{1j}^{(k)} \\
P_{21}^{(1)} & P_{22}^{(1)} \ldots & P_{2j}^{(2)} & \ldots & P_{2k}^{(1)} \\
\vdots & & & & \\
P_{i1}^{(1)} & P_{i2}^{(1)} \ldots & P_{ij}^{(1)} & \ldots & P_{ik}^{(1)} \\
\vdots & & & & \\
P_{k1}^{(1)} & P_{k2}^{(1)} & P_{k(j)}^{(1)} & & P_{kk}^{(1)}
\end{bmatrix}
\end{array}$$

(i) $0 \le P_{ij}^{(n)} \le 1 \quad \forall i, j$

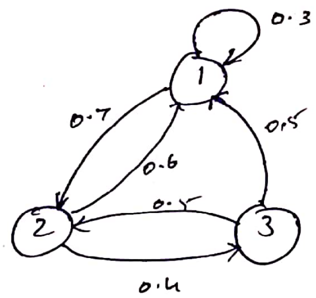
(ii) $\sum_{j=1}^{k} P_{ij}^{(n)} = 1$

(iii) If state j in not reachable from i, $P_{ij} = 0$

udhari

Transition Diagram:

Representation of the transitions among states of a Markov ~~[illegible]~~ chain



1-step:

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.6 & 0 & 0.4 \\ 0.5 & 0.5 & 0 \end{bmatrix} & \\ 3 & & & \end{array}$$

2-step transition probability

$P_{11}^{(2)} = (0.3)(0.3) + (0.7)(0.6) = 0.51$

$P_{12}^{(2)} = (0.3)(0.7) = 0.21$

$P_{13}^{(2)} = (0.7)(0.4) = 0.28$

⋮

$$P^{(2)} = \begin{bmatrix} 0.51 & 0.21 & 0.28 \\ 0.38 & 0.62 & 0 \\ 0.45 & 0.35 & 0.2 \end{bmatrix}$$