Objectivity in Continuum Mechanics

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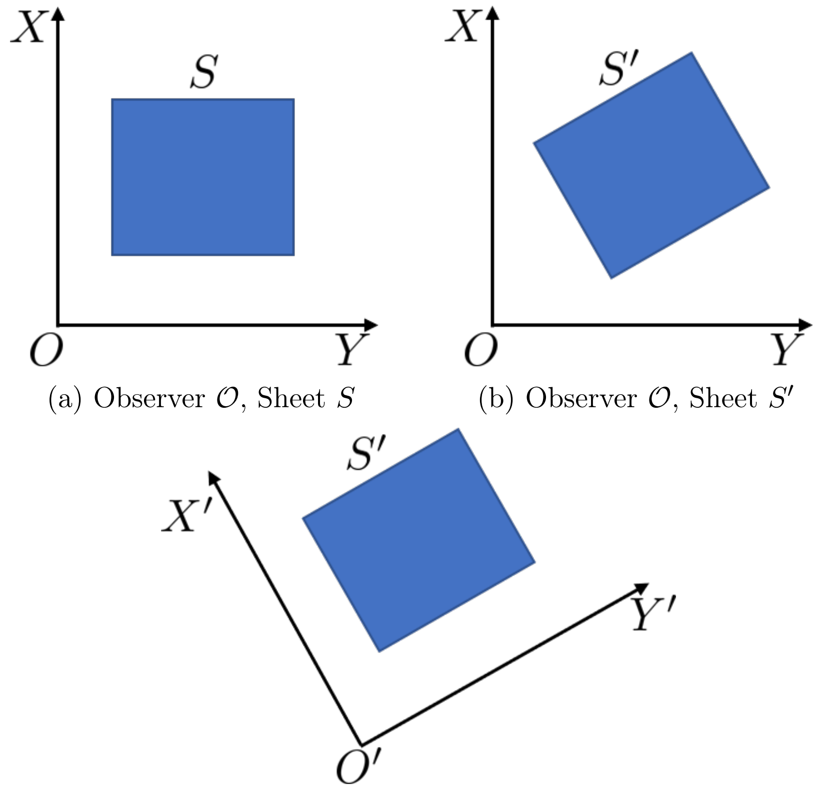
These notes are based on my understanding of the material presented in the text by Gurtin and collaborators[GFA10], and the lecture notes by Rohan Abeyaratne[Abe12]. Both these references are, in my opinion, excellent texts to understand and develop a foundation of continuum mechanics(particularly for solids) from a more mathematical perspective. I have thoroughly enjoyed reading these and any mistakes or inconsistencies that might appear reflect the gaping holes in my understanding and interpretation alone.

1. Objectivity - The basic idea

The underlying idea of objectivity is based on some form of invariance. What this invariance corresponds to, is illustrated here.
Say there is an scientist, identified by $\mathcal{O}$ who places a blue rectangular sheet of paper S on a rectangular glass table, with the hopes of sketching a scenery. Before beginning the sketch, $\mathcal{O}$ aligns that piece of paper on the table so that the edges of the sheet are parallel to edges of the table. Additionally, for ease of sketching $\mathcal{O}$ places this sheet a little away from the corner of the table[1]. Since $\mathcal{O}$ also happens to be a huge fan of coordinate-geometry, they recognize that the points on the sheet of paper can be uniquely pinpointed by choosing the corner of the table, O, as the origin, and the edges of the table, X and Y, as the basis of a coordinate system(Fig. 1a). $\mathcal{O}$ suddenly realizes that an integral part of the sketching process missing here - a good cup of coffee. While $\mathcal{O}$ goes off to prepare the coffee, a gust moves this piece of paper around on the table.
When $\mathcal{O}$ returns with the coffee they are disappointed to notice that the sheet of paper is no longer aligned with the edges of the table, and more so, has displaced of elsewhere (Fig. 1b). Instead of messing around with the displaced sheet S', what they choose to do is to lift this piece of paper, and align the table in such a way that the sheet S' is positioned on the table just as it was before the gust blew. As you would have guessed, this can be done by first moving the table and rotating it in exactly the same fashion as the gust moved and rotated the sheet of paper(Fig. 1c).
While the above exercise seemed pretty obvious, two important aspects must be noted :

1. Even after the gust blew and displaced the sheet S, the new piece of paper S' was the exact same blue rectangular piece of paper. In other words, the color did not change, and neither did the geometry nor the side lengths of the rectangle.

[1]Since $\mathcal{O}$ is right-handed while sketching some place is required to place the left hand for support



(a) Observer $\mathcal{O}$, Sheet S (b) Observer $\mathcal{O}$, Sheet S'

(c) Observer $\mathcal{O}'$, Sheet S'

Figure 1: Explaining objectivity

2. $\mathcal{O}$ was able to figure out that displacing and rotating the table to a new origin O' and new basis vectors X' and Y', in the same manner as the sheet of paper(S' relative to S), resulted in the same perspective as was before the gust blew.

The fact that the gust blew in a specific fashion to preserve some properties of the piece of paper(its color, geometry and dimensions) and that the original perspective could be achieved by $\mathcal{O}$ by moving the table and visualizing this entire situation as a new observer $\mathcal{O}'$, form the notion that is objectivity. The color of the rectangle and it's side lengths which remain unchanged, are objective quantities. For that matter, the relative distance between any two points on the sheet is an objective quantity.

2. Objectivity - The Formulation

Having given an intuitive idea of objectivity, we delve a bit into formulating the idea in mathematical terms. Since we primarily work with scalars, vectors and second-order tensors we shall focus on the objectivity of quantities defined using such mathematical constructs, although the idea can be extended in a straightforward manner to higher-order tensors. The mechanics of continua will be the setting in which the concept of objectivity will be developed further, and the physical process of interest is centered around kinetics, kinematics and thermodynamics. Central to this discussion is the idea of two deformations related by a rigid transformation - a uniform rotation followed by a uniform translation. Let $\boldsymbol{Y} : \mathcal{C}_0 \times [0, \infty) \to \mathbb{R}^3$ denote the deformation mapping which associates with every material point coordinate $p \in \mathcal{C}_0$ and time t, the position of the material point in the deformed configuration of the body $\mathcal{B}_t$. Let $\boldsymbol{Y}^*$ denote the deformation mapping generated by superposing a time-dependent rigid motion on the deformation mapping $\boldsymbol{Y}$. More explicitly

3. Objectivity of Kinematic Quantities

Since the principle of objectivity is purely geometric, kinematic quantities which are defined based on geometric relations can be immediately verified as being objective or note. Below we test the objectivity of some of the relevant or more commonly used kinematic quantities.

1. Velocity - $\boldsymbol{v}$

   The velocity is defined as the material time derivative of the deformation map. The velocity computed by observer $\mathcal{O}^*$ is

$$\begin{aligned}\boldsymbol{v}^*(\boldsymbol{X},t^*) &= \frac{\partial \boldsymbol{Y}^*(\boldsymbol{X},t)}{\partial t^*} = \frac{\partial \boldsymbol{Y}^*(\boldsymbol{X},t)}{\partial t} = \frac{\partial}{\partial t}\left(\boldsymbol{Q}(t)\boldsymbol{Y}(\boldsymbol{X},t)+\boldsymbol{c}(t)\right)\\ &= \boldsymbol{Q}(t)\frac{\partial \boldsymbol{Y}(\boldsymbol{X},t)}{\partial t} + \dot{\boldsymbol{Q}}(t)\boldsymbol{Y}(\boldsymbol{X},t)+\dot{\boldsymbol{c}}(t)\\ &= \underbrace{\boldsymbol{Q}(t)\boldsymbol{v}(\boldsymbol{X},t)}_{\text{Objective part}} + \dot{\boldsymbol{Q}}(t)\boldsymbol{Y}(\boldsymbol{X},t)+\dot{\boldsymbol{c}}(t)\end{aligned} \tag{1}$$

   The velocity is clearly **NOT OBJECTIVE** since the remaining contributions apart from the objective part in Eqn. 1 are in general non-zero. This happens precisely when one observer $\mathcal{O}^*$ has a non-zero velocity relative to observer $\mathcal{O}$.

2. Acceleration - $\boldsymbol{a}$

   The acceleration is defined as the material time derivative of the velocity. The acceleration computed by observer $\mathcal{O}^*$ is

$$\begin{aligned}\boldsymbol{a}^*(\boldsymbol{X},t^*) &= \frac{\partial \boldsymbol{v}^*(\boldsymbol{X},t)}{\partial t^*} = \frac{\partial \boldsymbol{v}^*(\boldsymbol{X},t)}{\partial t} = \frac{\partial}{\partial t}\Big(\boldsymbol{Q}(t)\boldsymbol{v}(\boldsymbol{X},t)+\dot{\boldsymbol{Q}}(t)\boldsymbol{Y}(\boldsymbol{X},t)+\dot{\boldsymbol{c}}(t)\Big)\\ &= \boldsymbol{Q}(t)\frac{\partial \boldsymbol{v}(\boldsymbol{X},t)}{\partial t} + 2\dot{\boldsymbol{Q}}(t)\boldsymbol{v}(\boldsymbol{X},t)+\ddot{\boldsymbol{Q}}(t)\boldsymbol{Y}(\boldsymbol{X},t)+\ddot{\boldsymbol{c}}(t)\\ &= \underbrace{\boldsymbol{Q}(t)\boldsymbol{a}(\boldsymbol{X},t)}_{\text{Objective part}} + 2\dot{\boldsymbol{Q}}(t)\boldsymbol{v}(\boldsymbol{X},t)+\ddot{\boldsymbol{Q}}(t)\boldsymbol{Y}(\boldsymbol{X},t)+\ddot{\boldsymbol{c}}(t)\end{aligned} \tag{2}$$

   The acceleration is clearly **NOT OBJECTIVE** since the remaining contributions apart from the objective part in Eqn. 2 are in general non-zero. This will happens precisely when one observer $\mathcal{O}^*$ has a non-zero acceleration relative to observer $\mathcal{O}$.

3. Deformation gradient tensor - $\boldsymbol{F}$

   The deformation gradient is defined as the gradient of the deformation map relative to the position vector of the material points in the reference configuration . The deformation gradient computed by observer $\mathcal{O}^*$ is

$$\boldsymbol{F}^*(\boldsymbol{X},t^*) = \frac{\partial \boldsymbol{Y}^*(\boldsymbol{X},t)}{\partial \boldsymbol{X}} = \frac{\partial}{\partial \boldsymbol{X}}\left(\boldsymbol{Q}(t)\boldsymbol{Y}(\boldsymbol{X},t)+\boldsymbol{c}(t)\right) = \boldsymbol{Q}(t)\frac{\partial \boldsymbol{Y}(\boldsymbol{X},t)}{\partial \boldsymbol{X}} \tag{3}$$

   Hence the deformation gradient is **NOT OBJECTIVE** since it is missing a multiplication with $\boldsymbol{Q}(t)^T$ to the right.

4. Right stretch tensor - $\boldsymbol{U}$

$$\begin{aligned}\boldsymbol{U}^{*}\left(\boldsymbol{X}, t^{*}\right) &= \sqrt{\boldsymbol{F}^{*}\left(\boldsymbol{X}, t^{*}\right)^{T} \boldsymbol{F}^{*}\left(\boldsymbol{X}, t^{*}\right)} = \sqrt{\boldsymbol{F}\left(\boldsymbol{X}, t\right)^{T} \boldsymbol{Q}(t)^{T} \boldsymbol{Q}(t) \boldsymbol{F}\left(\boldsymbol{X}, t\right)} \\ &= \sqrt{\boldsymbol{F}\left(\boldsymbol{X}, t\right)^{T} \boldsymbol{F}\left(\boldsymbol{X}, t\right)}\end{aligned} \tag{4}$$

Hence the right stretch tensor is **NOT OBJECTIVE** since it is invariant under the prescribed observer transformations. This result is expected since based on the polar decomposition of the deformation gradient, the right stretch tensor encodes all of the stretching and any rigid body rotations imposed thereafter, enter in the form a left multiplication by a proper orthogonal tensor, leaving the right stretch tensor untouched. On this note the polar decomposition of $\boldsymbol{F}^{*}$ yields

$$\boldsymbol{F}^{*} = \boldsymbol{Q}\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{R}\boldsymbol{U} = \left(\boldsymbol{Q}\boldsymbol{R}\right)\boldsymbol{U} = \boldsymbol{R}^{*}\boldsymbol{U}^{*}$$

From uniqueness of the polar decomposition we obtain

$$\boldsymbol{R}^{*} = \boldsymbol{Q}\boldsymbol{R}\ ;\ \boldsymbol{U}^{*} = \boldsymbol{U} \tag{5}$$

From Eqn. 5 we also note that the rotation tensor identified by $\boldsymbol{R}$(computed from the polar decomposition of $\boldsymbol{F}$) is **NOT OBJECTIVE**.

5. Left stretch tensor - $\boldsymbol{V}$

$$\begin{aligned}\boldsymbol{V}^{*}\left(\boldsymbol{X}, t^{*}\right) &= \boldsymbol{F}^{*}\left(\boldsymbol{X}, t^{*}\right) \boldsymbol{R}^{*}\left(\boldsymbol{X}, t^{*}\right)^{T} = \boldsymbol{Q}(t)\boldsymbol{F}\left(\boldsymbol{X}, t\right) \boldsymbol{R}^{*}\left(\boldsymbol{X}, t^{*}\right)^{T} \\ &= \boldsymbol{Q}(t)\boldsymbol{F}\left(\boldsymbol{X}, t\right) \boldsymbol{R}\left(\boldsymbol{X}, t\right)^{T} \boldsymbol{Q}(t)^{T} = \boldsymbol{Q}(t)\boldsymbol{V}\left(\boldsymbol{X}, t\right) \boldsymbol{Q}(t)^{T}\end{aligned} \tag{6}$$

Hence the left stretch tensor is **OBJECTIVE**.

6. Velocity gradient tensor - $\boldsymbol{L}$

$$\begin{aligned}\boldsymbol{L}^{*}\left(\boldsymbol{X}, t^{*}\right) &= \dot{\boldsymbol{F}}^{*}\left(\boldsymbol{X}, t^{*}\right) \boldsymbol{F}^{*}\left(\boldsymbol{X}, t^{*}\right)^{-1} = \frac{\partial}{\partial t}\left(\boldsymbol{Q}(t)\boldsymbol{F}\left(\boldsymbol{X}, t\right)\right) \boldsymbol{F}\left(\boldsymbol{X}, t\right)^{-1} \boldsymbol{Q}(t)^{T} \\ &= \left(\dot{\boldsymbol{Q}}(t)\boldsymbol{F}\left(\boldsymbol{X}, t\right) + \boldsymbol{Q}(t)\dot{\boldsymbol{F}}\left(\boldsymbol{X}, t\right)\right) \boldsymbol{F}\left(\boldsymbol{X}, t\right)^{-1} \boldsymbol{Q}(t)^{T} \\ &= \dot{\boldsymbol{Q}}(t)\boldsymbol{Q}(t)^{T} + \underbrace{\boldsymbol{Q}(t)\boldsymbol{L}\left(\boldsymbol{X}, t\right) \boldsymbol{Q}(t)^{T}}_{\text{Objective part}}\end{aligned} \tag{7}$$

Hence the velocity gradient tensor is **NOT OBJECTIVE** and the contribution arising apart from the objective part in Eqn. 7 is present when observer $\mathcal{O}^{*}$ spins relative to observer $\mathcal{O}$. We note additionally that this contribution is a skew-symmetric tensor which can be shown from the proper orthogonal property of $\boldsymbol{Q}(t)$ as follows :

$$\begin{aligned}&\boldsymbol{Q}(t)\boldsymbol{Q}(t)^{T} = \boldsymbol{I} \\ \implies &\overline{\boldsymbol{Q}(t)\boldsymbol{Q}(t)^{T}}^{\boldsymbol{\cdot}} = \boldsymbol{0} \\ \implies &\dot{\boldsymbol{Q}}(t)\boldsymbol{Q}(t)^{T} + \boldsymbol{Q}(t)\dot{\boldsymbol{Q}}(t)^{T} = \boldsymbol{0} \\ &\dot{\boldsymbol{Q}}(t)\boldsymbol{Q}(t)^{T} + \left(\dot{\boldsymbol{Q}}(t)\boldsymbol{Q}(t)^{T}\right)^{T} = \boldsymbol{0}\end{aligned} \tag{8}$$

7. Stretching tensor - $\boldsymbol{D}$

$$\begin{aligned}
\boldsymbol{D}^*\left(\boldsymbol{X},t^*\right) &= \frac{1}{2}\left(\boldsymbol{L}^*\left(\boldsymbol{X},t^*\right)+\boldsymbol{L}^*\left(\boldsymbol{X},t^*\right)^T\right)\\
&= \frac{1}{2}\left(\boldsymbol{Q}(t)\boldsymbol{L}\left(\boldsymbol{X},t\right)\boldsymbol{Q}(t)^T+\boldsymbol{Q}(t)\boldsymbol{L}\left(\boldsymbol{X},t\right)^T\boldsymbol{Q}(t)^T\right)\\
&= \boldsymbol{Q}(t)\frac{1}{2}\left(\boldsymbol{L}\left(\boldsymbol{X},t\right)+\boldsymbol{L}\left(\boldsymbol{X},t\right)^T\right)\boldsymbol{Q}(t)^T\\
&= \boldsymbol{Q}(t)\boldsymbol{D}\left(\boldsymbol{X},t\right)\boldsymbol{Q}(t)^T
\end{aligned} \tag{9}$$

Hence the stretching tensor is **OBJECTIVE**.

8. Spin tensor - $\boldsymbol{W}$

$$\begin{aligned}
\boldsymbol{W}^*\left(\boldsymbol{X},t^*\right) &= \frac{1}{2}\left(\boldsymbol{L}^*\left(\boldsymbol{X},t^*\right)-\boldsymbol{L}^*\left(\boldsymbol{X},t^*\right)^T\right)\\
&= \dot{\boldsymbol{Q}}(t)\boldsymbol{Q}(t)^T+\frac{1}{2}\left(\boldsymbol{Q}(t)\boldsymbol{L}\left(\boldsymbol{X},t\right)\boldsymbol{Q}(t)^T-\boldsymbol{Q}(t)\boldsymbol{L}\left(\boldsymbol{X},t\right)^T\boldsymbol{Q}(t)^T\right)\\
&= \dot{\boldsymbol{Q}}(t)\boldsymbol{Q}(t)^T+\boldsymbol{Q}(t)\frac{1}{2}\left(\boldsymbol{L}\left(\boldsymbol{X},t\right)-\boldsymbol{L}\left(\boldsymbol{X},t\right)^T\right)\boldsymbol{Q}(t)^T\\
&= \dot{\boldsymbol{Q}}(t)\boldsymbol{Q}(t)^T+\underbrace{\boldsymbol{Q}(t)\boldsymbol{W}\left(\boldsymbol{X},t\right)\boldsymbol{Q}(t)^T}_{\text{Objective part}}
\end{aligned} \tag{10}$$

Hence the spin tensor is **NOT OBJECTIVE** and the contribution apart from the objective part in Eqn. 10 is present when observer $\mathcal{O}^*$ spins relative to observer $\mathcal{O}$, akin to velocity gradient.

The results are summarized in Table. 3

Kinematic quantity	**Symbol**	**Type**	**Objectivity**
Velocity	$\boldsymbol{v}$	Vector	**NOT OBJECTIVE**
Acceleration	$\boldsymbol{a}$	Vector	**NOT OBJECTIVE**
Deformation gradient	$\boldsymbol{F}$	2^{nd} Order tensor	**NOT OBJECTIVE**
Rotation tensor	$\boldsymbol{R}$	2^{nd} Order tensor	**NOT OBJECTIVE**
Right Stretch	$\boldsymbol{U}$	2^{nd} Order tensor	**NOT OBJECTIVE**
Left Stretch	$\boldsymbol{V}$	2^{nd} Order tensor	**OBJECTIVE**
Velocity gradient	$\boldsymbol{L}$	2^{nd} Order tensor	**NOT OBJECTIVE**
Stretching tensor	$\boldsymbol{D}$	2^{nd} Order tensor	**OBJECTIVE**
Spin tensor	$\boldsymbol{W}$	2^{nd} Order tensor	**NOT OBJECTIVE**

4. Convecting and Co-rotating Bases

On a slightly different note we delve a bit into the topic of convecting and co-rotating bases and establish their significance in the context of constructing objective rates of objective quantities. The idea of convecting bases is linked to the following question(s) - Suppose we have an objective spatial field , is its material time derivative also objective? If not, can we construct an appropriate time derivative that is objective?

A spatial vector field $\boldsymbol{f}(\boldsymbol{x},t)$ is a convecting vector field if there exists a corresponding time invariant material vector field $\boldsymbol{g}(\boldsymbol{X})$ such that

$$
\boldsymbol{f}(\boldsymbol{x},t) = \boldsymbol{F}(\boldsymbol{X},t)\boldsymbol{g}(\boldsymbol{X}) \tag{11}
$$

$$
\begin{aligned}
\implies \dot{\boldsymbol{f}}(\boldsymbol{x},t) &= \dot{\boldsymbol{F}}(\boldsymbol{X},t)\boldsymbol{g}(\boldsymbol{X}) = \boldsymbol{L}(\boldsymbol{x},t)\boldsymbol{F}(\boldsymbol{X},t)\boldsymbol{g}(\boldsymbol{X}) \\
&= \boldsymbol{L}(\boldsymbol{x},t)\boldsymbol{f}(\boldsymbol{x},t)
\end{aligned} \tag{12}
$$

The physical interpretation of Eqn. 11 is that the vector field $\boldsymbol{f}(\boldsymbol{X},t)$ follows the material fiber aligned along $\boldsymbol{g}(\boldsymbol{X})$ all throughout the deformation. Let $\boldsymbol{d}\tilde{\boldsymbol{X}}$ denote a material fiber tangential to $\boldsymbol{g}(\boldsymbol{X})$, and $\boldsymbol{d}\tilde{\boldsymbol{x}}$ its spatial counterpart after deformation. Then

$$
\begin{aligned}
\boldsymbol{f}(\boldsymbol{x},t) \times \boldsymbol{d}\tilde{\boldsymbol{x}} &= (\boldsymbol{F}(\boldsymbol{X},t)\boldsymbol{g}(\boldsymbol{X})) \times \left(\boldsymbol{F}(\boldsymbol{X},t)\boldsymbol{d}\tilde{\boldsymbol{X}}\right) \\
&= \det\left(\boldsymbol{F}(\boldsymbol{X},t)\right)\boldsymbol{F}(\boldsymbol{X},t)^{-T}\left(\boldsymbol{g}(\boldsymbol{X}) \times \boldsymbol{d}\tilde{\boldsymbol{X}}\right) = \boldsymbol{0}
\end{aligned}
$$

implying that the convecting spatial vector field follows a specific material fiber characterized by its corresponding material vector field. For the rest of the discussion we dispense with the arguments of the associated vector and tensor field for a succint exposition.

Consider a set of three convecting spatial vector fields - $\mathcal{B} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ which form a linearly independent set at time $t = t_0$. This ensures that they remain linearly independent for all time since

$$
\begin{aligned}
[\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3] &= [\boldsymbol{F}\boldsymbol{g}_1, \boldsymbol{F}\boldsymbol{g}_2, \boldsymbol{F}\boldsymbol{g}_3] = \det(\boldsymbol{F})\,[\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3] \\
&= \det(\boldsymbol{F})\left(\det(\boldsymbol{F}_{t_0})^{-1}\right)\left[\boldsymbol{f}_{1,t_0}, \boldsymbol{f}_{2,t_0}, \boldsymbol{f}_{3,t_0}\right] \neq 0\ \forall t
\end{aligned} \tag{13}
$$

where $[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ denotes the box product of the three vectors (taken in that order), $\boldsymbol{F}_{t_0} := \boldsymbol{F}(\boldsymbol{X}, t_0)$ and $\boldsymbol{f}_{i,t_0} := \boldsymbol{f}_i(\boldsymbol{x}, t_0)\ \forall i \in \{1,2,3\}$. Then the set $\mathcal{B}$, which consists of convecting spatial vector fields which form a basis, is referred to as a convecting basis .

Now consider an arbitrary spatial vector field $\boldsymbol{a}(\boldsymbol{x},t)$. The projection of $\boldsymbol{a}$ on the convecting spatial vector field $\boldsymbol{f}_i$ is

$$
a_i = \boldsymbol{a} \cdot \boldsymbol{f}_i\ \forall i \in \{1,2,3\} \tag{14}
$$

so that the material time derivative of this projection yields

$$
\overset{\triangle}{a}_i = \dot{\overline{\boldsymbol{a} \cdot \boldsymbol{f}_i}} = \dot{\boldsymbol{a}} \cdot \boldsymbol{f}_i + \boldsymbol{a} \cdot \dot{\boldsymbol{f}}_i = \dot{\boldsymbol{a}} \cdot \boldsymbol{f}_i + \boldsymbol{a} \cdot (\boldsymbol{L}\boldsymbol{f}_i) = \left(\dot{\boldsymbol{a}} + \boldsymbol{L}^T\boldsymbol{a}\right) \cdot \boldsymbol{f}_i \tag{15}
$$

after which we define the convected time derivative of the spatial vector field $\boldsymbol{a}$ as follows

$$
\overset{\triangle}{\boldsymbol{a}} = \dot{\boldsymbol{a}} + \boldsymbol{L}^T\boldsymbol{a} \tag{16}
$$

It is important to note that $\overset{\triangle}{\boldsymbol{a}}$ is different from $\dot{\boldsymbol{a}}$. While components of $\dot{\boldsymbol{a}}$ denote the material time derivative of each component obtained from a projection of $\boldsymbol{a}$ on a time invariant basis, the components of $\overset{\triangle}{\boldsymbol{a}}$ denote the material time derivative of each component of $\boldsymbol{a}$ relative to a basis moving or 'convecting' with the material fibers. As we shall see in a bit, the material time derivative of objective quantities are generally not objective. Their convected time derivatives however turn out to be objective.

We carry out the previous steps in a similar manner for a spatial tensor field $\boldsymbol{A}(\boldsymbol{x}, t)$ whose projection onto the pair of basis elements $\left(\boldsymbol{f}_i, \boldsymbol{f}_j\right)$ and subsequent material time derivative may be defined and computed as

$$
\begin{aligned}
& A_{ij} = \boldsymbol{f}_i \cdot \left(\boldsymbol{A}\boldsymbol{f}_j\right) \\
\implies & \overset{\triangle}{A}_{ij} = \overline{\boldsymbol{f}_i \cdot \left(\boldsymbol{A}\boldsymbol{f}_j\right)}^{\boldsymbol{\cdot}} = \dot{\boldsymbol{f}}_i \cdot \left(\boldsymbol{A}\boldsymbol{f}_j\right) + \boldsymbol{f}_i \cdot \left(\dot{\boldsymbol{A}}\boldsymbol{f}_j\right) + \boldsymbol{f}_i \cdot \left(\boldsymbol{A}\dot{\boldsymbol{f}}_j\right) \\
& \quad = \boldsymbol{L}\boldsymbol{f}_i \cdot \left(\boldsymbol{A}\boldsymbol{f}_j\right) + \boldsymbol{f}_i \cdot \left(\dot{\boldsymbol{A}}\boldsymbol{f}_j\right) + \boldsymbol{f}_i \cdot \left(\boldsymbol{A}\boldsymbol{L}\boldsymbol{f}_j\right) \\
& \quad = \boldsymbol{f}_i \cdot \left(\boldsymbol{L}^T\boldsymbol{A}\boldsymbol{f}_j\right) + \boldsymbol{f}_i \cdot \left(\dot{\boldsymbol{A}}\boldsymbol{f}_j\right) + \boldsymbol{f}_i \cdot \left(\boldsymbol{A}\boldsymbol{L}\boldsymbol{f}_j\right) \\
& \quad = \boldsymbol{f}_i \left(\left(\boldsymbol{L}^T\boldsymbol{A} + \dot{\boldsymbol{A}} + \boldsymbol{A}\boldsymbol{L}\right)\boldsymbol{f}_j\right)
\end{aligned} \tag{17}
$$

which motivates the notation and definition of the convected time derivative of the spatial tensor field $\boldsymbol{A}$ as

$$
\overset{\triangle}{\boldsymbol{A}} = \boldsymbol{L}^T\boldsymbol{A} + \dot{\boldsymbol{A}} + \boldsymbol{A}\boldsymbol{L} \tag{18}
$$

In a similar vein in which a convecting spatial vector field was defined, a co-rotating spatial vector field $\boldsymbol{f}(\boldsymbol{x}, t)$ is one which satisfies

$$
\dot{\boldsymbol{f}} = \boldsymbol{W}\boldsymbol{f} \tag{19}
$$

where $\boldsymbol{W}$ is the spin tensor. Consider a set of three co-rotating spatial vector fields - $\mathcal{B} = \{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ which form a linearly independent set at time $t = t_0$. This ensures that they remain linearly independent for all time which can be shown as follows

$$
\begin{aligned}
& \overline{[\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3]}^{\boldsymbol{\cdot}} = \left[\dot{\boldsymbol{f}}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\right] + \left[\boldsymbol{f}_1, \dot{\boldsymbol{f}}_2, \boldsymbol{f}_3\right] + \left[\boldsymbol{f}_1, \boldsymbol{f}_2, \dot{\boldsymbol{f}}_3\right] \\
& \quad = [\boldsymbol{W}\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3] + [\boldsymbol{f}_1, \boldsymbol{W}\boldsymbol{f}_2, \boldsymbol{f}_3] + [\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{W}\boldsymbol{f}_3] \\
& \quad = [\boldsymbol{\omega} \times \boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3] + [\boldsymbol{f}_1, \boldsymbol{\omega} \times \boldsymbol{f}_2, \boldsymbol{f}_3] + [\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{\omega} \times \boldsymbol{f}_3] \\
& \quad = (\boldsymbol{\omega} \times \boldsymbol{f}_1) \cdot (\boldsymbol{f}_2 \times \boldsymbol{f}_3) + (\boldsymbol{\omega} \times \boldsymbol{f}_2) \cdot (\boldsymbol{f}_3 \times \boldsymbol{f}_1) + (\boldsymbol{\omega} \times \boldsymbol{f}_3) \cdot (\boldsymbol{f}_1 \times \boldsymbol{f}_2) \\
& \quad = (\boldsymbol{\omega} \cdot \boldsymbol{f}_2)(\boldsymbol{f}_1 \cdot \boldsymbol{f}_3) - (\boldsymbol{\omega} \cdot \boldsymbol{f}_3)(\boldsymbol{f}_1 \cdot \boldsymbol{f}_2) + (\boldsymbol{\omega} \cdot \boldsymbol{f}_3)(\boldsymbol{f}_1 \cdot \boldsymbol{f}_2) \\
& \quad - (\boldsymbol{\omega} \cdot \boldsymbol{f}_1)(\boldsymbol{f}_2 \cdot \boldsymbol{f}_3) + (\boldsymbol{\omega} \cdot \boldsymbol{f}_1)(\boldsymbol{f}_2 \cdot \boldsymbol{f}_3) - (\boldsymbol{\omega} \cdot \boldsymbol{f}_2)(\boldsymbol{f}_1 \cdot \boldsymbol{f}_3) \\
& \quad = 0 \\
\implies & [\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3] = C = \left[\boldsymbol{f}_{1,t_0}, \boldsymbol{f}_{2,t_0}, \boldsymbol{f}_{3,t_0}\right] \neq 0
\end{aligned}
$$

where ω is the unique axial vector corresponding to the skew-symmetric spin tensor $\boldsymbol{W}$. Then repeating the earlier procedure to define co-rotated derivatives of the spatial vector and tensor fields, $\boldsymbol{a}$ and $\boldsymbol{A}$ respectively, we obtain

$$
\overset{\circ}{\boldsymbol{a}} = \dot{\boldsymbol{a}} - \boldsymbol{W}\boldsymbol{a} \tag{20}
$$

$$
\overset{\circ}{\boldsymbol{A}} = \dot{\boldsymbol{A}} + \boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A} \tag{21}
$$

5. Objective Rates

In this section we are particularly interested in constructing objective rates of objective quantities. We address the questions posed at the start of the previous section - Is the material time derivative of an objective quantity objective?
Consider first an objective spatial scalar field $a(\boldsymbol{x}, t)$. When measured by two observers $\mathcal{O}$ and $\mathcal{O}^*$ we have

$$\begin{aligned} & a^*(\boldsymbol{x}^*, t^*) = a(\boldsymbol{x}, t^*) \\ \implies & \frac{\partial}{\partial t^*}\left(a^*(\boldsymbol{x}^*, t^*)\right) = \frac{\partial}{\partial t}\left(a^*(\boldsymbol{x}^*, t^*)\right) = \frac{\partial}{\partial t}\left(a(\boldsymbol{x}, t)\right) \end{aligned} \tag{22}$$

which implies that the material time derivative of an objective scalar spatial field is objective. Note that the material time derivative as computed by observer $\mathcal{O}^*$ is computed by a partial derivative relative to t^*.
Now consider an objective spatial vector field $\boldsymbol{a}(\boldsymbol{x}, t)$. The material time derivative of its counterpart as measured by observer $\mathcal{O}^*$ yields

$$\begin{aligned} \frac{\partial}{\partial t^*}\left(\boldsymbol{a}^*(\boldsymbol{x}^*, t^*)\right) = \frac{\partial}{\partial t}\left(\boldsymbol{a}^*(\boldsymbol{x}^*, t^*)\right) &= \frac{\partial}{\partial t}\left(\boldsymbol{Q}(t)\boldsymbol{a}(\boldsymbol{x}, t)\right) \\ &= \dot{\boldsymbol{Q}}(t)\boldsymbol{a}(\boldsymbol{x}, t) + \underbrace{\boldsymbol{Q}(t)\dot{\boldsymbol{a}}(\boldsymbol{x}, t)}_{\text{Objective part}} \end{aligned}$$

which is not objective. In a similar fashion consider an objective spatial tensor field $\boldsymbol{A}(\boldsymbol{x}, t)$ for which the material time derivative computed by observer $\mathcal{O}^*$ would be

$$\begin{aligned} \frac{\partial}{\partial t^*}\left(\boldsymbol{A}^*(\boldsymbol{x}^*, t^*)\right) = \frac{\partial}{\partial t}\left(\boldsymbol{A}^*(\boldsymbol{x}^*, t^*)\right) &= \frac{\partial}{\partial t}\left(\boldsymbol{Q}(t)\boldsymbol{A}(\boldsymbol{x}, t)\boldsymbol{Q}(t)^T\right) \\ &= \dot{\boldsymbol{Q}}(t)\boldsymbol{A}(\boldsymbol{x}, t)\boldsymbol{Q}(t)^T + \underbrace{\boldsymbol{Q}(t)\dot{\boldsymbol{A}}(\boldsymbol{x}, t)\boldsymbol{Q}(t)^T}_{\text{Objective part}} + \boldsymbol{Q}(t)\boldsymbol{A}(\boldsymbol{x}, t)\dot{\boldsymbol{Q}}(t)^T \end{aligned}$$

which is not objective.
Now we test the same for the convected and co-rotated time derivatives of the spatial vector and tensor fields.

5.1. Objectivity of the convected time derivative

For the objective spatial vector field $\boldsymbol{a}(\boldsymbol{x}, t)$ the convected time derivative as computed by observer $\mathcal{O}^*$ is

$$\begin{aligned} \overset{\triangle}{\boldsymbol{a}}{}^* = \dot{\boldsymbol{a}}^* + \boldsymbol{L}^{*T}\boldsymbol{a}^* &= \dot{\boldsymbol{Q}}\boldsymbol{a} + \boldsymbol{Q}\dot{\boldsymbol{a}} + \left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T\right)^T \boldsymbol{Q}\boldsymbol{a} \\ &= \dot{\boldsymbol{Q}}\boldsymbol{a} + \boldsymbol{Q}\dot{\boldsymbol{a}} + \boldsymbol{Q}\dot{\boldsymbol{Q}}^T\boldsymbol{Q}\boldsymbol{a} + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{a} \\ &= \dot{\boldsymbol{Q}}\boldsymbol{a} + \boldsymbol{Q}\dot{\boldsymbol{a}} - \dot{\boldsymbol{Q}}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{a} + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{a} \\ &= \dot{\boldsymbol{Q}}\boldsymbol{a} + \boldsymbol{Q}\dot{\boldsymbol{a}} - \dot{\boldsymbol{Q}}\boldsymbol{a} + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{a} \\ &= \boldsymbol{Q}\left(\dot{\boldsymbol{a}} + \boldsymbol{L}^T\boldsymbol{a}\right) \\ &= \boldsymbol{Q}\overset{\triangle}{\boldsymbol{a}} \end{aligned} \tag{23}$$

The above derivation was a mechanical one somewhat devoid of intuition. One can arrive at the same result by using the fact that the convecting bases vectors are objective (Appendix A), which after employing simple arguments and concise computation confirms the objectivity for the convected time derivative of an objective spatial vector field. Following the same procedure for an objective spatial tensor field we obtain

$$
\begin{aligned}
\overset{\triangle}{\boldsymbol{A}}{}^{*} &= \dot{\boldsymbol{A}}^{*} + \boldsymbol{L}^{*T}\boldsymbol{A}^{*} + \boldsymbol{A}^{*}\boldsymbol{L}^{*} \\
&= \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^{T} + \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^{T} \\
&+ \left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{T}\right)^{T}\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{T}\left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{T}\right) \\
&= \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^{T} + \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^{T} \\
&- \dot{\boldsymbol{Q}}\boldsymbol{Q}^{T}\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{L}^{T}\boldsymbol{Q}^{T}\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{T} - \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{T}\boldsymbol{Q}\dot{\boldsymbol{Q}}^{T} + \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{T}\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^{T} \\
&= \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^{T} + \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^{T} \\
&- \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{L}^{T}\boldsymbol{A}\boldsymbol{Q}^{T} - \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^{T} + \boldsymbol{Q}\boldsymbol{A}\boldsymbol{L}\boldsymbol{Q}^{T} \\
&= \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{L}^{T}\boldsymbol{A}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{A}\boldsymbol{L}\boldsymbol{Q}^{T} \\
&= \boldsymbol{Q}\left(\dot{\boldsymbol{A}} + \boldsymbol{L}^{T}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{L}\right)\boldsymbol{Q}^{T} \\
&= \boldsymbol{Q}\overset{\triangle}{\boldsymbol{A}}\boldsymbol{Q}^{T}
\end{aligned}
\tag{24}
$$

The same conclusion can be achieved from much less tedious computation if we work with a convecting basis set and use the property that the convecting basis vectors are objective (Appendix A).
The exact same procedure can be extended to show that the co-rotated time derivative of an objective spatial fields is also objective. Basically, all instances of $\boldsymbol{L}$ and $\boldsymbol{L}^{*}$ can just be replaced by $\boldsymbol{W}$ and $\boldsymbol{W}^{*}$ respectively.

6. Solved Excercises

6.1. Objectivity of the deformed unit normal

Let $\boldsymbol{n}(\boldsymbol{x}, t)$ denote the oriented unit normal in the deformed configuration. Verifying it's objectivity we have

$$
\begin{aligned}
\boldsymbol{n}^{*} &= \frac{\det(\boldsymbol{F}^{*})\boldsymbol{F}^{*-T}\boldsymbol{N}}{|\det(\boldsymbol{F}^{*})\boldsymbol{F}^{*-T}\boldsymbol{N}|} = \frac{\det(\boldsymbol{Q}\boldsymbol{F})\,(\boldsymbol{Q}\boldsymbol{F})^{-T}\,\boldsymbol{N}}{|\det(\boldsymbol{Q}\boldsymbol{F})\,(\boldsymbol{Q}\boldsymbol{F})^{-T}\,\boldsymbol{N}|} \\
&= \frac{\det(\boldsymbol{Q})\det(\boldsymbol{F})\boldsymbol{Q}\boldsymbol{F}^{-T}\boldsymbol{N}}{|\det(\boldsymbol{Q})\det(\boldsymbol{F})\boldsymbol{Q}\boldsymbol{F}^{-T}\boldsymbol{N}|} = \frac{\det(\boldsymbol{F})\boldsymbol{Q}\boldsymbol{F}^{-T}\boldsymbol{N}}{|\det(\boldsymbol{F})\boldsymbol{Q}\boldsymbol{F}^{-T}\boldsymbol{N}|} \\
&= \frac{\det(\boldsymbol{F})\boldsymbol{Q}\boldsymbol{F}^{-T}\boldsymbol{N}}{|\det(\boldsymbol{F})\boldsymbol{F}^{-T}\boldsymbol{N}|} = \boldsymbol{Q}\frac{\det(\boldsymbol{F})\boldsymbol{F}^{-T}\boldsymbol{N}}{|\det(\boldsymbol{F})\boldsymbol{F}^{-T}\boldsymbol{N}|} \\
&= \boldsymbol{Q}\boldsymbol{n}
\end{aligned}
$$

so that the oriented unit normal vector in the deformed configuration is objective.

6.2. Objectivity of some other rates

Let $\boldsymbol{A}(\boldsymbol{x}, t)$ denote an objective spatial tensor field. We would like to verify the objectivity of the following time derivatives.

1. $\overset{\triangledown}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{L}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{L}^T$

$$
\begin{aligned}
\overset{\triangle}{\boldsymbol{A}}{}^* &= \dot{\boldsymbol{A}}^* - \boldsymbol{L}^*\boldsymbol{A}^* - \boldsymbol{A}^*\boldsymbol{L}^{*T} \\
&= \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^T \\
&\quad - \left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T\right)\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T\left(\dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T\right)^T \\
&= \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^T \\
&\quad - \dot{\boldsymbol{Q}}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T\boldsymbol{Q}\dot{\boldsymbol{Q}}^T - \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{L}^T\boldsymbol{Q}^T \\
&= \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^T \\
&\quad - \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{L}\boldsymbol{A}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^T - \boldsymbol{Q}\boldsymbol{A}\boldsymbol{L}^T\boldsymbol{Q}^T \\
&= \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{L}\boldsymbol{A}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{A}\boldsymbol{L}^T\boldsymbol{Q}^T \\
&= \boldsymbol{Q}\left(\dot{\boldsymbol{A}} - \boldsymbol{L}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{L}^T\right)\boldsymbol{Q}^T \\
&= \boldsymbol{Q}\overset{\triangledown}{\boldsymbol{A}}\boldsymbol{Q}^T
\end{aligned}
$$

implying that $\overset{\triangledown}{\boldsymbol{A}}$ is objective. Since $\overset{\triangle}{\boldsymbol{A}}$ is also objective, so is the following rate

$$
\overset{\square}{\boldsymbol{A}} = \frac{1}{2}\left(\overset{\triangle}{\boldsymbol{A}} - \overset{\triangledown}{\boldsymbol{A}}\right) = \boldsymbol{D}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{D}
$$

Note that the previous rate may be proved objective also from the fact that $\boldsymbol{D}$ and $\boldsymbol{A}$ are both objective so that their multiplication is too. This can be quite simply shown as follows. Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be objective spatial tensor fields with their product $\tilde{\boldsymbol{A}} = \boldsymbol{A}_1\boldsymbol{A}_2$. Then

$$
\tilde{\boldsymbol{A}}^* = \boldsymbol{A}_1^*\boldsymbol{A}_2^* = \left(\boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{Q}^T\right)\left(\boldsymbol{Q}\boldsymbol{A}_2\boldsymbol{Q}^T\right) = \boldsymbol{Q}\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{Q}^T = \boldsymbol{Q}\tilde{\boldsymbol{A}}\boldsymbol{Q}^T
$$

Using the principle of mathematical induction one can show that an arbitrary product of objective tensor fields is also objective.

2. What is the form of objective spatial vector and tensor fields such that their material time derivative is also objective? For this to happen, the non-objective parts should evaluate to zero. Carrying this out for an objective spatial vector field $\boldsymbol{a}(\boldsymbol{x}, t)$ we obtain

$$
\begin{aligned}
\dot{\boldsymbol{a}}^* - \boldsymbol{Q}\dot{\boldsymbol{a}} = \dot{\boldsymbol{Q}}\boldsymbol{a} = \boldsymbol{0}\ ;\ \forall\dot{\boldsymbol{Q}} \\
\implies \boldsymbol{a} = \boldsymbol{0}
\end{aligned}
$$

Similarly for an objective spatial tensor field $\boldsymbol{A}(\boldsymbol{x},t)$ we obtain

$$
\begin{aligned}
\dot{\boldsymbol{A}}^* - \boldsymbol{Q}\dot{\boldsymbol{A}}\boldsymbol{Q}^T = \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{A}\dot{\boldsymbol{Q}}^T &= \boldsymbol{0}\ ;\ \forall \boldsymbol{Q},\dot{\boldsymbol{Q}} \\
\implies \dot{\boldsymbol{Q}}\boldsymbol{A}\boldsymbol{Q}^T - \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T\dot{\boldsymbol{Q}}\boldsymbol{Q}^T &= \boldsymbol{0}\ ;\ \forall \boldsymbol{Q},\dot{\boldsymbol{Q}} \\
\implies \dot{\boldsymbol{Q}}\boldsymbol{Q}^T\boldsymbol{A}^* - \boldsymbol{A}^*\dot{\boldsymbol{Q}}\boldsymbol{Q}^T &= \boldsymbol{0}\ ;\ \forall \boldsymbol{Q},\dot{\boldsymbol{Q}} \\
\implies \tilde{\boldsymbol{S}}\boldsymbol{A}^* - \boldsymbol{A}^*\tilde{\boldsymbol{S}} &= \boldsymbol{0}\ ;\ \forall \tilde{\boldsymbol{S}} \in \mathrm{skw}(3) \\
\implies \tilde{\boldsymbol{S}}\boldsymbol{A}^*\boldsymbol{v} - \boldsymbol{A}^*\tilde{\boldsymbol{S}}\boldsymbol{v} &= \boldsymbol{0}\ ;\ \forall \tilde{\boldsymbol{S}} \in \mathrm{skw}(3), \boldsymbol{v} \in \mathbb{R}^3 \\
\implies \boldsymbol{\omega} \times (\boldsymbol{A}^*\boldsymbol{v}) - \boldsymbol{A}^*(\boldsymbol{\omega} \times \boldsymbol{v}^*) &= \boldsymbol{0}\ ;\ \forall \boldsymbol{\omega}, \boldsymbol{v} \in \mathbb{R}^3 \\
\implies \boldsymbol{\omega} \times (\boldsymbol{A}^*\boldsymbol{\omega}) - \boldsymbol{A}^*(\boldsymbol{\omega} \times \boldsymbol{\omega}^*) &= \boldsymbol{0}\ ;\ \forall \boldsymbol{\omega} \in \mathbb{R}^3 \\
\implies \boldsymbol{\omega} \times (\boldsymbol{A}^*\boldsymbol{\omega}) &= \boldsymbol{0}\ ;\ \forall \boldsymbol{\omega} \in \mathbb{R}^3 \\
\implies \boldsymbol{A}^*\boldsymbol{\omega} &= \lambda\boldsymbol{\omega}\ ;\ \forall \boldsymbol{\omega} \in \mathbb{R}^3 \\
\implies \boldsymbol{A}^* &= \beta(\boldsymbol{x},t)\boldsymbol{I}
\end{aligned}
$$

6.3. The Rivlin-Ericksen tensor

Let $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ and $\overset{(n)}{\boldsymbol{C}}$ denote its n^{th} material time derivative. Define the n^{th} Rivlin-Ericksen tensor $\boldsymbol{A}_n$ as follows

$$\boldsymbol{A}_n = \boldsymbol{F}^{-T}\overset{(n)}{\boldsymbol{C}}\boldsymbol{F}^{-1}$$

We now verify the objectivity of this tensor. Note first that $\boldsymbol{C}$ is invariant and so is its material time derivative. As a result, it's n^{th} derivative is invariant too. Then the n^{th} Rivlin-Ericksen tensor as computed by the observer $\mathcal{O}^*$ is

$$
\begin{aligned}
\boldsymbol{A}_n^* &= \boldsymbol{F}^{*-T}\overset{(n)}{\boldsymbol{C}^*}\boldsymbol{F}^{*-1} = \boldsymbol{Q}\boldsymbol{F}^{-T}\overset{(n)}{\boldsymbol{C}}\boldsymbol{F}^{-1}\boldsymbol{Q}^T \\
&= \boldsymbol{Q}\boldsymbol{A}_n\boldsymbol{Q}^T
\end{aligned} \tag{25}
$$

so that the n^{th} Rivlin-Ericksen tensor is objective. In a similar fashion, if $\boldsymbol{C}$ is an invariant tensor field, its material time derivative is also invariant and the tensors $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ defined as

$$\boldsymbol{C}_1 = \boldsymbol{F}^{-T}\boldsymbol{C}\boldsymbol{F}^{-1}\ ,\ \boldsymbol{C}_2 = \boldsymbol{F}\boldsymbol{C}\boldsymbol{F}^T$$

are objective. Then given an objective tensor spatial field $\boldsymbol{B}$ define its pullback relative to the reference configuration via $\boldsymbol{F}$ as

$$\boldsymbol{B}_{PB} := \boldsymbol{F}^{-1}\boldsymbol{B}\boldsymbol{F}^{-T}$$

which is clearly invariant since

$$\boldsymbol{B}^*_{PB} = \boldsymbol{F}^{*-1}\boldsymbol{B}^*\boldsymbol{F}^{*-T} = \boldsymbol{F}^{-1}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{F}^{-T} = \boldsymbol{F}^{-1}\boldsymbol{B}\boldsymbol{F}^{-T} = \boldsymbol{B}_{PB}$$

Then the material time derivative of $\boldsymbol{B}_{PB}$ is invariant so that it's push forward to the deformed configuration defined by

$$\overset{\times}{\boldsymbol{B}}{}^*{}_{PF} = \boldsymbol{F}\dot{\boldsymbol{B}}_{PB}\boldsymbol{F}^T = \boldsymbol{F}\overline{\boldsymbol{F}^{-1}\dot{\boldsymbol{B}}\boldsymbol{F}^{-T}}\boldsymbol{F}^T$$

is objective. Hence considering an objective tensor field, computing it's pull back followed by the material time derivative, and then the push forward results in an objective rate of that tensor field[Gar14].

6.4. The relative deformation gradient tensor

Let $\mathcal{C}_\tau$ and $\mathcal{C}_t$ be configurations of the body at times τ and t respectively. Then we have the deformation gradients of the deformation mapping at the two times as $\boldsymbol{F}(\boldsymbol{X},\tau)$ and $\boldsymbol{F}(\boldsymbol{X},t)$ respectively. Define the relative deformation gradient $\boldsymbol{F}_t(\boldsymbol{X},\tau)$ as the deformation gradient of the configuration at time τ relative to time t. Then

$$\boldsymbol{F}_t(\boldsymbol{X},\tau) = \boldsymbol{F}(\boldsymbol{X},\tau)\boldsymbol{F}(\boldsymbol{X},t)^{-1}$$

This metric as measured by the observer $\mathcal{O}^*$ is

$$\begin{aligned}\boldsymbol{F}^*_{t^*}(\boldsymbol{X},\tau^*) &= \boldsymbol{F}^*(\boldsymbol{X},\tau^*)\boldsymbol{F}^*(\boldsymbol{X},t^*)^{-1}\\ &= \boldsymbol{Q}(\tau)\boldsymbol{F}(\boldsymbol{X},\tau)\boldsymbol{F}(\boldsymbol{X},t)^{-1}\boldsymbol{Q}(t)^T\\ &= \boldsymbol{Q}(\tau)\boldsymbol{F}_t(\boldsymbol{X},\tau)\boldsymbol{Q}(t)^T\end{aligned}$$

which is not objective in general since $\boldsymbol{Q}(\tau)$ is not necessarily equal to $\boldsymbol{Q}(t)$.

7. Objectivity of non-kinematic quantities

While objectivity can be plainly verified for purely kinematic quantities, that is not the case for those which have some non-geometric component in them. These include energetic and kinetic quantities like the spatial Helmholtz free energy density , the spatial traction vector, the spatial specific entropy, etc. Objectivity in such cases needs to be enforced and one must do so in a consistent manner. Since enforcing objectivity in such cases takes some motivation from a physical scenario, the quantities on which objectivity is forced are spatial counterparts since they are really parametrized relative to the deformed coordinates and hence, are the quantities measured in experiments. The objectivity of the corresponding referential quantities can then be verified since they are related to the spatial quantities through geometry alone.

In the development of the balance laws and thermodynamic inequality the following non-kinematic fields were introduced : density, traction vector, body force density vector, heat flux, heat supply, specific internal energy, thermodynamic temperature and specific entropy. It is now postulated that all these fields be objective.

Objectivity on non-kinematic fields $\begin{cases} \rho(\boldsymbol{x},t) = \rho^*(\boldsymbol{x}^*,t^*) & \text{Density} \\ \boldsymbol{t}(\boldsymbol{x},t) = \boldsymbol{t}^*(\boldsymbol{x}^*,t^*) & \text{Traction vector} \\ \boldsymbol{b}(\boldsymbol{x},t) = \boldsymbol{b}^*(\boldsymbol{x}^*,t^*) & \text{Body force density vector} \\ h(\boldsymbol{x},t,\boldsymbol{n}) = h^*(\boldsymbol{x}^*,t^*,\boldsymbol{n}^*) & \text{Heat flux} \\ r(\boldsymbol{x},t) = r^*(\boldsymbol{x}^*,t^*) & \text{Heat supply} \\ \epsilon(\boldsymbol{x},t) = \epsilon^*(\boldsymbol{x}^*,t^*) & \text{Specific internal energy} \\ \theta(\boldsymbol{x},t) = \theta^*(\boldsymbol{x}^*,t^*) & \text{Thermodynamic temperature} \\ \eta(\boldsymbol{x},t) = \eta^*(\boldsymbol{x}^*,t^*) & \text{Specific entropy} \end{cases}$

We can now evaluate the objectivity of the Cauchy stress using Cauchy's relation

$$\begin{aligned} & \boldsymbol{T}\boldsymbol{n} = \boldsymbol{t} \ , \ \boldsymbol{T}^*\boldsymbol{n}^* = \boldsymbol{t}^* \\ \Longrightarrow\ & \boldsymbol{T}^*\boldsymbol{Q}\boldsymbol{n} = \boldsymbol{Q}\boldsymbol{t} \implies \left(\boldsymbol{Q}^T\boldsymbol{T}^*\boldsymbol{Q}\right)\boldsymbol{n} = \boldsymbol{t} \\ \Longrightarrow\ & \left(\boldsymbol{Q}^T\boldsymbol{T}^*\boldsymbol{Q} - \boldsymbol{T}\right)\boldsymbol{n} = \boldsymbol{0} \ ; \ \forall \boldsymbol{n} \in \mathbb{S}^3 \\ \Longrightarrow\ & \boldsymbol{T}^* = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T \end{aligned}$$

which shows that the Cauchy stress is objective. Now we verify the objectivity of the stress power[2] defined as

$$\begin{aligned} & p = \boldsymbol{T}\cdot\boldsymbol{D} \ , \ p^* = \boldsymbol{T}^*\cdot\boldsymbol{D}^* \\ \Longrightarrow\ & p^* = \left(\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T\right)\cdot\left(\boldsymbol{Q}\boldsymbol{D}\boldsymbol{Q}^T\right) \\ \Longrightarrow\ & p^* = \left(\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T\right)\cdot\left(\boldsymbol{D}\boldsymbol{Q}^T\right) \\ \Longrightarrow\ & p^* = \left(\boldsymbol{T}\boldsymbol{Q}^T\right)\cdot\left(\boldsymbol{D}\boldsymbol{Q}^T\right) \\ \Longrightarrow\ & p^* = \left(\boldsymbol{T}\boldsymbol{Q}^T\boldsymbol{Q}\right)\cdot\boldsymbol{D} \\ \Longrightarrow\ & p^* = \boldsymbol{T}\cdot\boldsymbol{D} \\ \Longrightarrow\ & p^* = p \end{aligned}$$

In a similar fashion as was done for the Cauchy stress we can show that the spatial heat flux vector is objective

$$\begin{aligned} & \boldsymbol{q}\cdot\boldsymbol{n} = h \ , \ \boldsymbol{q}^*\cdot\boldsymbol{n}^* = h^* \\ \Longrightarrow\ & \boldsymbol{q}^*\cdot(\boldsymbol{Q}\boldsymbol{n}) = h \implies \left(\boldsymbol{Q}^T\boldsymbol{q}^*\right)\cdot\boldsymbol{n} = h \\ \Longrightarrow\ & \left(\boldsymbol{Q}^T\boldsymbol{q}^* - \boldsymbol{q}\right)\cdot\boldsymbol{n} = 0 \ ; \ \forall \boldsymbol{n} \in \mathbb{S}^3 \\ \Longrightarrow\ & \boldsymbol{q}^* = \boldsymbol{Q}\boldsymbol{q} \end{aligned}$$

Now we can turn to verify the objectivity of some of the referential quantities.

[2] This is the part of the externally supplied power which does not translate to kinetic power and is the stored internal mechanical power.

1. The first Piola-Kirchoff stress is not objective.

$$\begin{aligned}
\boldsymbol{S}^* &= \det\left(\boldsymbol{F}^*\right)\boldsymbol{T}^*\boldsymbol{F}^{*-T} = \det\left(\boldsymbol{F}\right)\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{F}^{-T} \\
&= \boldsymbol{Q}\det\left(\boldsymbol{F}\right)\boldsymbol{T}\boldsymbol{F}^{-T} \\
&= \boldsymbol{Q}\boldsymbol{S}
\end{aligned}$$

2. The referential heat flux vector is not objective. Moreover it is invariant.

$$\begin{aligned}
\boldsymbol{q}_0^* &= \det\left(\boldsymbol{F}^*\right)\boldsymbol{F}^{*-1}\boldsymbol{q}^* \\
&= \det\left(\boldsymbol{F}\right)\boldsymbol{F}^{-1}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{q} \\
&= \det\left(\boldsymbol{F}\right)\boldsymbol{F}^{-1}\boldsymbol{q} \\
&= \boldsymbol{q}_0
\end{aligned}$$

3. The referential traction is objective

$$\begin{aligned}
\boldsymbol{t}_0^* &= \boldsymbol{S}^*\boldsymbol{N}^* = \boldsymbol{S}^*\boldsymbol{N} \\
&= \boldsymbol{Q}\boldsymbol{S}\boldsymbol{N}^* \\
&= \boldsymbol{Q}\boldsymbol{t}_0
\end{aligned}$$

4. The referential heat flux is objective.

$$\begin{aligned}
h^* &= \boldsymbol{q}_0^* \cdot \boldsymbol{N}^* \\
&= \boldsymbol{q}_0 \cdot \boldsymbol{N} \\
&= h
\end{aligned}$$

5. The second Piola-Kirchoff stress is not objective. Moreover, it is invariant.

$$\begin{aligned}
\tilde{\boldsymbol{S}}^* &= \boldsymbol{F}^{*-1}\boldsymbol{S}^* \\
&= \boldsymbol{F}^{-1}\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{S} \\
&= \boldsymbol{F}^{-1}\boldsymbol{S} \\
&= \tilde{\boldsymbol{S}}
\end{aligned}$$

Since the Cauchy stress is objective, different objective rates of the Cauchy stress can be constructed, a few of which are included below :

$$\begin{aligned}
&\overset{\triangle}{\boldsymbol{T}} = \dot{\boldsymbol{T}} + \boldsymbol{L}^T\boldsymbol{T} + \boldsymbol{T}\boldsymbol{L}\ ,\ \text{Convected Rate} \\
&\overset{\triangledown}{\boldsymbol{T}} = \dot{\boldsymbol{T}} - \boldsymbol{L}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{L}^T\ ,\ \text{Oldroyd Rate} \\
&\overset{\circ}{\boldsymbol{T}} = \dot{\boldsymbol{T}} - \boldsymbol{W}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{W}\ ,\ \text{Co-Rotational or Jaumann Rate} \\
&\overset{\square}{\boldsymbol{T}} = \frac{1}{2}\left(\overset{\triangle}{\boldsymbol{T}} - \overset{\triangledown}{\boldsymbol{T}}\right) = \boldsymbol{D}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{D} \\
&\overset{\otimes}{\boldsymbol{T}} = \dot{\boldsymbol{T}} - \boldsymbol{\Omega}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{\Omega}\ ,\ \text{Green-Naghdi Rate}
\end{aligned}$$

The Green-Naghdi rate can be shown to be objective in the following manner

$$
\begin{aligned}
\overset{\otimes}{\boldsymbol{T}}{}^{*} &= \dot{\boldsymbol{T}}^{*} - \boldsymbol{\Omega}^{*}\boldsymbol{T}^{*} + \boldsymbol{T}^{*}\boldsymbol{\Omega}^{*} \\
&= \dot{\boldsymbol{Q}}\boldsymbol{T}\boldsymbol{Q}^{T} + \boldsymbol{Q}\dot{\boldsymbol{T}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{T}\dot{\boldsymbol{Q}}^{T} - \dot{\overline{\boldsymbol{Q}\boldsymbol{R}}}\boldsymbol{R}^{T}\boldsymbol{T}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^{T}\dot{\overline{\boldsymbol{Q}\boldsymbol{R}}}\boldsymbol{Q}\boldsymbol{R}^{T} \\
&= \dot{\boldsymbol{Q}}\boldsymbol{T}\boldsymbol{Q}^{T} + \boldsymbol{Q}\dot{\boldsymbol{T}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{T}\dot{\boldsymbol{Q}}^{T} - \dot{\overline{\boldsymbol{Q}\boldsymbol{R}}}\boldsymbol{R}^{T}\boldsymbol{T}\boldsymbol{Q}^{T} - \boldsymbol{Q}\boldsymbol{T}\boldsymbol{R}\dot{\overline{\boldsymbol{Q}\boldsymbol{R}}}^{T} \\
&= \dot{\boldsymbol{Q}}\boldsymbol{T}\boldsymbol{Q}^{T} + \boldsymbol{Q}\dot{\boldsymbol{T}}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{T}\dot{\boldsymbol{Q}}^{T} - \dot{\boldsymbol{Q}}\boldsymbol{T}\boldsymbol{Q}^{T} - \boldsymbol{Q}\dot{\boldsymbol{R}}\boldsymbol{R}^{T}\boldsymbol{T}\boldsymbol{Q}^{T} \\
&- \boldsymbol{Q}\boldsymbol{T}\dot{\boldsymbol{Q}}^{T} - \boldsymbol{Q}\boldsymbol{T}\boldsymbol{R}\dot{\boldsymbol{R}}^{T}\boldsymbol{Q}^{T} \\
&= \boldsymbol{Q}\dot{\boldsymbol{T}}\boldsymbol{Q}^{T} - \boldsymbol{Q}\dot{\boldsymbol{R}}\boldsymbol{R}^{T}\boldsymbol{T}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{T}\dot{\boldsymbol{R}}\boldsymbol{R}^{T}\boldsymbol{Q}^{T} \\
&= \boldsymbol{Q}\dot{\boldsymbol{T}}\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{\Omega}\boldsymbol{T}\boldsymbol{Q}^{T} + \boldsymbol{Q}\boldsymbol{T}\boldsymbol{\Omega}\boldsymbol{Q}^{T} \\
&= \boldsymbol{Q}\left(\dot{\boldsymbol{T}} - \boldsymbol{\Omega}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{\Omega}\right)\boldsymbol{Q}^{T} \\
&= \boldsymbol{Q}\overset{\otimes}{\boldsymbol{T}}\boldsymbol{Q}^{T}
\end{aligned}
$$

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Appendices

A. Objective rates : A simpler derivation

Consider the convecting basis set $\mathcal{B} = \{\boldsymbol{f}_1(\boldsymbol{x},t), \boldsymbol{f}_2(\boldsymbol{x},t), \boldsymbol{f}_3(\boldsymbol{x},t)\}$ and an objective spatial vector field $\boldsymbol{a}(\boldsymbol{x},t)$. For the observer $\mathcal{O}^*$ the convecting basis set is $\mathcal{B}^* = \{\boldsymbol{Q}(t)\boldsymbol{f}_1(\boldsymbol{x},t), \boldsymbol{Q}(t)\boldsymbol{f}_2(\boldsymbol{x},t), \boldsymbol{Q}(t)\boldsymbol{f}_3(\boldsymbol{x},t)\}$ and the objective spatial vector field is $\boldsymbol{a}^*(\boldsymbol{x}^*,t^*) = \boldsymbol{Q}(t)\boldsymbol{a}(\boldsymbol{x},t)$ so that projection along the convecting basis is

$$
\begin{aligned}
a_i^* &= \boldsymbol{a}^*(\boldsymbol{x}^*,t^*) \cdot \boldsymbol{f}_i^*(\boldsymbol{x}^*,t^*) = (\boldsymbol{Q}(t)\boldsymbol{a}(\boldsymbol{x},t)) \cdot (\boldsymbol{Q}(t)\boldsymbol{f}_i(\boldsymbol{x},t)) \\
&= (\boldsymbol{Q}(t)^T\boldsymbol{Q}(t)\boldsymbol{a}(\boldsymbol{x},t)) \cdot \boldsymbol{f}_i(\boldsymbol{x},t) \\
&= \boldsymbol{a}(\boldsymbol{x},t) \cdot \boldsymbol{f}_i(\boldsymbol{x},t)
\end{aligned}
$$

so that the projection is an objective spatial scalar field. We know that the material time derivative of an objective spatial scalar field is also objective which yields

$$
\begin{aligned}
\overset{\triangle}{a_i^*} &= \left(\dot{\boldsymbol{a}^*} + \boldsymbol{L}^{*T}\boldsymbol{a}^*\right) \cdot \boldsymbol{f}_i^* = \left(\dot{\boldsymbol{a}} + \boldsymbol{L}^T\boldsymbol{a}\right) \cdot \boldsymbol{f}_i = \left(\dot{\boldsymbol{a}} + \boldsymbol{L}^T\boldsymbol{a}\right) \cdot \left(\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{f}_i\right) \\
&= \left(\boldsymbol{Q}\left(\dot{\boldsymbol{a}} + \boldsymbol{L}^T\boldsymbol{a}\right)\right) \cdot (\boldsymbol{Q}\boldsymbol{f}_i) \\
&= \left(\boldsymbol{Q}\left(\dot{\boldsymbol{a}} + \boldsymbol{L}^T\boldsymbol{a}\right)\right) \boldsymbol{f}_i^* \\
\implies & \left(\dot{\boldsymbol{a}^*} + \boldsymbol{L}^{*T}\boldsymbol{a}^*\right) \cdot \boldsymbol{f}_i^* = \left(\boldsymbol{Q}\left(\dot{\boldsymbol{a}} + \boldsymbol{L}^T\boldsymbol{a}\right)\right) \boldsymbol{f}_i^* \; ; \; \forall \boldsymbol{f}_i^* \\
\implies & \left(\dot{\boldsymbol{a}^*} + \boldsymbol{L}^{*T}\boldsymbol{a}^*\right) = \boldsymbol{Q}\left(\dot{\boldsymbol{a}} + \boldsymbol{L}^T\boldsymbol{a}\right) \\
\implies & \overset{\triangle}{\boldsymbol{a}^*} = \boldsymbol{Q}\overset{\triangle}{\boldsymbol{a}}
\end{aligned}
$$

The same idea of derivation can be extended to verify the objectivity of the convected time derivative of an objective spatial tensor field, and the co-rotated time derivative of objective spatial vector and tensor fields.
The present derivation while simpler assumes the existence of a convecting basis set signifying its property and importance. The earlier more tedious computation did not rely on a basis set since we just started with the construction of the convected time derivative and verified its objectivity.