WUCT121

Discrete Mathematics

Logic

Tutorial Exercises Solutions

1. **Logic**
2. **Predicate Logic**
3. **Proofs**
4. **Set Theory**
5. **Relations and Functions**

Section 1: Logic

Question1

(i) If $x = 3$, then $x < 2$.

(a) Statement

(b) False

(c) $x = 3 \Rightarrow x < 2$

(ii) If $x = 0$ or $x = 1$, then $x^2 = x$.

(a) Statement

(b) True

(c) $(x = 0 \vee x = 1) \Rightarrow x^2 = x$

(iii) There exists a natural number x for which $x^2 = -2x$.

(a) Statement

(b) False

(iv) If $x \in \mathbb{N}$ and $x > 0$, then if $\sqrt{x} > 1$ then $x > 1$..

(a) Statement

(b) True

(c) $(x \in \mathbb{N} \wedge x > 0) \Rightarrow (\sqrt{x} > 1 \Rightarrow x > 1)$

(v) $xy = 5$ implies that either $x = 1$ and $y = 5$ or $x = 5$ and $y = 1$.

(a) Statement

(b) False. Consider $x = -1$ and $y = -5$ or $x = -5$ and $y = -1$.

(c) $xy = 5 \Rightarrow ((x = 1 \wedge y = 5) \vee (x = 5 \wedge y = 1))$

(vi) $xy = 0$ implies $x = 0$ or $y = 0$.

(a) Statement

(b) True

(c) $xy = 0 \Rightarrow x = 0 \vee y = 0$

(vii) $xy = yx$.

(a) Statement

(b) True

(viii) There is a unique even prime number.

(a) Statement

(b) True, $x = 2$.

Question2

(a) If x is odd and y is odd then $x+y$ is even.

p: x is odd. q: y is odd. r: $x+y$ is even.

Form: $p \wedge q \Rightarrow r$.

(b) It is not both raining and hot.

p: It is raining. q: It is hot

Form: $\sim(p \wedge q)$, alternatively $\sim p \vee \sim q$

(c) It is neither raining nor hot.

p: It is raining. q: It is hot

Form: $\sim p \wedge \sim q$, alternatively $\sim(p \vee q)$.

(d) It is raining but it is hot.

p: It is raining. q: It is hot.

Form: $p \wedge q$.

(e) $-1 \leq x \leq 2$.

$p: -1 < x, q: -1 = x, r: x < 2, s: x = 2$.

Form: $(p \vee q) \wedge (r \vee s)$.

Question3 :

(a) $P \vee Q$: Mathematics is easy or I do not need to study.

(b) $P \wedge Q$: Mathematics is easy and I do not need to study

(c) $\sim Q$: I need to study.

(d) $\sim\sim Q$: I do not need to study.

(e) $\sim P$: Mathematics is not easy.

(f) $\sim P \wedge Q$: Mathematics is not easy and I do not need to study.

(g) $P \Rightarrow Q$: If Mathematics is easy, then I do not need to study

Question4

(a) The truth tables for $(\sim p \vee q) \wedge q$ and $(\sim p \wedge q) \vee q$.

p	q	$(\sim p$	$\vee$	$q)$	$\wedge$	q	$(\sim p$	$\wedge$	$q)$	$\vee$	q
T	T	F	T		**T**		F	F		**T**	
T	F	F	F		**F**		F	F		**F**	
F	T	T	T		**T**		T	T		**T**	
F	F	T	T		**F**		T	F		**F**	
Step:		1	2		**3***		1	2		**3***	

The tables are the same

(b) The truth tables for $(\sim p \vee q) \wedge p$ and $(\sim p \wedge q) \vee p$.

p	q	$(\sim p$	$\vee$	$q)$	$\wedge$	p	$(\sim p$	$\wedge$	$q)$	$\vee$	p
T	T	F	T		**T**		F	F		**T**	
T	F	F	F		**F**		F	F		**T**	
F	T	T	T		**F**		T	T		**T**	
F	F	T	T		**F**		T	F		**F**	
Step:		1	2		**3***		1	2		**3***	

The tables are not the same. The student's guess is false

Question5

(a) The truth tables for $p \vee \sim p$ and $p \wedge \sim p$.

p	p	$\vee$	$\sim p$	p	$\wedge$	$\sim p$
T		**T**	F		**F**	T
F		**T**	T		**F**	F
		2*	1		**2***	1

(b) $p \vee \sim p$ is a tautology i.e. always true; $p \wedge \sim p$ is a contradiction, i.e. always false

(c) Use truth tables.

p	q	$(p$	$\vee$	$\sim p)$	$\vee$	q	$(p$	$\wedge$	$\sim p)$	$\wedge$	q
T	T		T	F	**T**			F	F	**F**	
T	F		T	F	**T**			F	F	**F**	
F	T		T	T	**T**			F	T	**F**	
F	F		T	T	**T**			F	T	**F**	
Step:			2	1	**3***			2	1	**3***	

Notice that "true $\vee$ anything" is true and "false $\wedge$ anything" is false

Conclusion: If you have a compound statement *R* of the form "$\mathrm{T} \vee P$", where T stands for a tautology (and *P* is any compound statement), then *R* is also a tautology. Similarly, if you have a compound statement, *S*, of the form "$\mathrm{F} \wedge P$", where F stands for a contradiction, then *S* is also a contradiction.

Question6

(a) The truth tables for the statements $(p \vee \sim p) \wedge (q \vee r)$ and $q \vee r$.

p	q	r	$(p$	$\vee$	$\sim p)$	$\wedge$	$(q$	$\vee$	$r)$	q	$\vee$	r
T	T	T		T	F	**T**		T			**T**	
T	T	F		T	F	**T**		T			**T**	
T	F	T		T	F	**T**		T			**T**	
T	F	F		T	F	**F**		F			**F**	
F	T	T		T	T	**T**		T			**T**	
F	T	F		T	T	**T**		T			**T**	
F	F	T		T	T	**T**		T			**T**	
F	F	F		T	T	**F**		F			**F**	
Step:				2	1	**4***		3			**1***	

Notice that the two statements are logically equivalent.
In fact, the truth value of the first is dependent entirely on the second

(b) The truth tables for the statements $(p \wedge \sim p) \vee (q \wedge r)$ and $q \wedge r$.

p	q	r	$(p$	$\wedge$	$\sim p)$	$\vee$	$(q$	$\wedge$	$r)$	q	$\wedge$	r
T	T	T		F	F	**T**		T			**T**	
T	T	F		F	F	**F**		F			**F**	
T	F	T		F	F	**F**		F			**F**	
T	F	F		F	F	**F**		F			**F**	
F	T	T		F	T	**T**		T			**T**	
F	T	F		F	T	**F**		F			**F**	
F	F	T		F	T	**F**		F			**F**	
F	F	F		F	T	**F**		F			**F**	
Step:				2	1	**4***		3			**1***	

Notice that the two statements are logically equivalent.
In fact, the truth value of the first is again dependent entirely on the second.

Conclusion: If you have a compound statement *R* of the form "$\mathrm{T} \wedge P$", where T stands for a tautology (and *P* is any compound statement), then the truth-value of *R* depends entirely on the truth-value of *P*. Similarly, if you have a compound statement, *S*, of the form "$\mathrm{F} \vee P$", where F stands for a contradiction, then the truth-value of *S* depends entirely on the truth-value of *P*.

Question7

(a) $(p \Rightarrow q) \vee (p \Rightarrow \sim q)$

	$(p$	$\Rightarrow$	$q)$	$\vee$	$(p$	$\Rightarrow$	$\sim$	$q)$
Step		1		4*		3	2	
Place F under main connective				F				
$\Rightarrow$ must be F		F				F		
1st $\Rightarrow$, p must be T and q must be F. 2nd $\Rightarrow$, p must be T and $\sim q$ must be F	T		**F**		T		F	
q must be T								**T**
q cannot be both T and F, thus $(p \Rightarrow q) \vee (p \Rightarrow \sim q)$ can only ever be true and is a tautology								

(b) $\sim (p \Rightarrow q) \vee (q \Rightarrow p)$

	$\sim($	p	$\Rightarrow$	$q)$	$\vee$	$(q$	$\Rightarrow$	$p)$
Step	2		1		4*		3	
Place F under main connective					F			
$\sim$must be F and $\Rightarrow$ must be F	F						F	
1st $\Rightarrow$ must be T. 2nd $\Rightarrow$, q must be T and p must be F			T			T		F
1st $\Rightarrow$ p can be F and q can be T, no conflict		F		T				
There is no contradiction, thus the statement is not a tautology								

(c) $(p \wedge q) \Rightarrow (\sim r \vee (p \Rightarrow q))$

	$(p$	$\wedge$	$q)$	$\Rightarrow$	$(\sim r$	$\vee$	$(p$	$\Rightarrow$	$q)$
Step		1		5*	2	4		3	
Place F under main connective				F					
$\wedge$ must be T and $\vee$ must be F		T				F			
$\wedge$ p must be T and q must be T $\vee$ $\sim r$ must be F and $\Rightarrow$ must be F	T		**T**		F			F	
$\Rightarrow$ p must be T and q must be F							T		**F**
q cannot be both T and F , thus $(p \wedge q) \Rightarrow (\sim r \vee (p \Rightarrow q))$ can only ever be true and is a tautology									

Question8

(a)

$(p \wedge q) \Rightarrow r \equiv \sim (p \wedge q) \vee r$ Implication Law

$\equiv (\sim p \vee \sim q) \vee r$ De Morgan's Law

$\equiv \sim p \vee \sim q \vee r$ Associativity

(b)

$p \Rightarrow (p \vee q) \equiv \sim p \vee (p \vee q)$ Implication Law

$\equiv \sim p \vee p \vee q$ Associativity

$\equiv T \vee q$ Negation Law

$\equiv T$ Dominance Law

Question9

(a)

LHS $= \sim (p \Rightarrow q)$

$\equiv \sim (\sim p \vee q)$ Implication Law

$\equiv \sim\sim p \wedge \sim q$ De Morgan's

$\equiv p \wedge \sim q$ Double Negation

$=$ RHS

(b)

LHS $= (p \wedge \sim q) \Rightarrow r$

$\equiv \sim (p \wedge \sim q) \vee r$ Implication Law

$\equiv (\sim p \vee \sim\sim q) \vee r$ De Morgan's

$\equiv (\sim p \vee q) \vee r$ Double Negation

$\equiv \sim p \vee (q \vee r)$ Associativity

$\equiv p \Rightarrow (q \vee r)$ Implication

$=$ RHS

Question10

(a) If x is a positive integer and $x^2 \le 3$ then $x = 1$.
The proposition is True.
If x is a positive integer, then $x^2 \le 3 \Rightarrow x \le \sqrt{3}$.
Now $\sqrt{3} \approx 1.7$ and so $x = 1$.

(b) $\left(\sim(x > 1) \vee \sim(y \le 0)\right) \Leftrightarrow \sim\left((x \le 1) \wedge (y > 0)\right)$.
The proposition is false. (You should have tried proving it using De Morgan's Laws and failed.)
Now find values of x and y that make the statement false.
Let $x = 0$ and $y = 1$.
$\sim(x > 1) \vee \sim(y \le 0)$ is True
$(x \le 1) \wedge (y > 0)$ is also True
Thus, $\sim\left((x \le 1) \wedge (y > 0)\right)$ is False
and the proposition is False.

Question11

(a)

$\sim(x > 1) \Rightarrow \sim(y \le 0)$	
$\equiv \sim(\sim(x > 1)) \vee \sim(y \le 0)$	Implication Law
$\equiv (x > 1) \vee \sim(y \le 0)$	Double Negation
$\equiv (x > 1) \vee (y > 0)$	Negation of $\le$

(b)

$(y \le 0) \Rightarrow (x > 1)$	
$\equiv \sim(y \le 0) \vee (x > 1)$	Implication Law .
$\equiv (y > 0) \vee (x > 1)$	Negation of $\le$

Question12

$\sim\left(\sim(p \vee q) \wedge \sim q\right)$	
$\equiv \sim\sim(p \vee q) \vee \sim\sim q$	De Morgan's
$\equiv (p \vee q) \vee q$	Double Negation
$\equiv p \vee q \vee q$	Associativity
$\equiv p \vee q$	Idempotent Law

Section 2 :Predicate Logic

Question1

(a) Every real number that is not zero is either positive or negative.
The statement is true.

(b) The square root of every natural number is also a natural number.
The statement is false (consider $n = 2$).

(c) Every student in WUCT121 can correctly solve at least one assigned problem.
Lecturers are yet to work out if this is true or false!

Question2

(a) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (xy = 0 \Rightarrow (x = 0 \wedge y = 0))$
The statement is false (consider $x = 1$ and $y = 0$).

(b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, x \leq y$
The statement is true.

(c) $\exists$ student *s* in WUCT121, $\forall$ lecturer's jokes *j*, *s* hasn't laughed at *j*.
True or false ??

Question3 Let *H* be the set of all people (human beings).

(a) $P: \exists p \in H, \forall q \in H, p \text{ loves } q.$

$\sim P: \sim (\exists p \in H, \forall q \in H, p \text{ loves } q)$

$\equiv \forall p \in H, \sim (\forall q \in H, p \text{ loves } q)$

$\equiv \forall p \in H, \exists q \in H, p \text{ doesn't love } q$

In a nice world, *P* is true!.

(b) $P: \forall p \in H, \forall q \in H, p \text{ loves } q.$

$\sim P: \sim (\forall p \in H, \forall q \in H, p \text{ loves } q)$

$\equiv \exists p \in H, \sim (\forall q \in H, p \text{ loves } q)$

$\equiv \exists p \in H, \exists q \in H, p \text{ doesn't love } q$

In a perfect world, *P* is true!

(c) $P: \exists p \in H, \exists q \in H, p \text{ loves } q.$

$\sim P: \sim (\exists p \in H, \exists q \in H, p \text{ loves } q)$

$\equiv \forall p \in H, \sim (\exists q \in H, p \text{ loves } q)$

$\equiv \forall p \in H, \forall q \in H, p \text{ doesn't love } q$

P is definitely true!

(d) $P: \forall p \in H, \exists q \in H, p \text{ loves } q.$

$\sim P: \sim (\forall p \in H, \exists q \in H, p \text{ loves } q)$

$\equiv \exists p \in H, \sim (\exists q \in H, p \text{ loves } q)$

$\equiv \exists p \in H, \forall q \in H, p \text{ doesn't love } q$

In our world, *P* is probably true!

(e) $P: \forall x \in \mathbb{Q}, x \in \mathbb{Z}$.

$\sim P: \sim (\forall x \in \mathbb{Q}, x \in \mathbb{Z})$

$\equiv \exists x \in \mathbb{Q}, x \notin \mathbb{Z}$

$\sim P$ is true.

(f)

$P: \sim (\forall n \in \mathbb{N}, \exists p \in \mathbb{N}, n = 2p)$

$\equiv \exists n \in \mathbb{N}, \sim (\exists p \in \mathbb{N}, n = 2p)$

$\equiv \exists n \in \mathbb{N}, \forall p \in \mathbb{N}, n \neq 2p$

$\sim P: \sim \sim (\forall n \in \mathbb{N}, \exists p \in \mathbb{N}, n = 2p)$

$\equiv \forall n \in \mathbb{N}, \exists p \in \mathbb{N}, n = 2p$

P is true.

(g) $P: \exists n \in \mathbb{N}, n \text{ is not prime}$.

$\sim P: \sim (\exists n \in \mathbb{N}, n \text{ is not prime})$

$\equiv \forall n \in \mathbb{N}, n \text{ is prime}$

P is true.

(h) $P: \forall \text{ triangle } T, T \text{ is a right triangle}$.

$\sim P: \sim (\forall \text{ triangle } T, T \text{ is a right triangle})$

$\equiv \exists \text{ triangle } T, T \text{ is not a right triangle}$

$\sim P$ is true.

<u>Question4</u>

(a) $\forall x \in \mathbb{R}, (x > 1 \Rightarrow x > 0)$

This statement is true. Clearly, $0 < 1 < x$, so $x > 0$

(b) $\forall x \in \mathbb{R}, (x > 1 \Rightarrow x > 2)$

This statement is false. Let $x = 1.5$. Then $x > 1$ but $x < 2$.

(c) $\exists x \in \mathbb{R}, (x > 1 \Rightarrow x^2 > x)$

This statement is true. Let $x = 2$. Then $x > 1$ and $x^2 = 4 > 2 = x$.

(d) $\exists x \in \mathbb{R}, \left(x > 1 \Rightarrow \frac{x}{x^2 + 1} < \frac{1}{3}\right)$

This statement is true. Let $x = 3$. Then $x > 1$ and $\frac{x}{x^2 + 1} = \frac{3}{10} < \frac{1}{3}$.

(e) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 = 9$

This statement is false. Let $x = 1$ and $y = 1$. Then $x^2 + y^2 = 2 \neq 9$.

(f) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 < y + 1$

This statement is true. For $x \in \mathbb{R}$, let $y = x^2$. Then clearly $x^2 < y + 1$.

(g) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 \geq 0$

This statement is true. Let $x = 0$. For each $y \in \mathbb{R}, y^2 \geq 0$, and we have $x^2 + y^2 = y^2 \geq 0$.

(h) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \left(x < y \Rightarrow x^2 < y^2\right)$

This statement is true. Let $x = 0$ and $y = 1$. Then $x < y$ and $x^2 = 0 < 1 = y^2$.

Question5 For each of the following statements,

(i) $\sim\left(\forall \xi > 0, \exists x \neq 0, |x| < \xi\right)$

$\equiv \exists \xi > 0, \sim\left(\exists x \neq 0, |x| < \xi\right)$

$\equiv \exists \xi > 0, \forall x \neq 0, |x| \geq \xi$

The negation of the statement is false.

For any $\xi > 0$, we can take $x = \frac{\xi}{2}$ and we have $x \neq 0$ but $|x| < \xi$.

(ii) $\sim\left(\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, y < x^2\right)$

$\equiv \forall y \in \mathbb{R}, \sim\left(\forall x \in \mathbb{R}, y < x^2\right)$

$\equiv \forall y \in \mathbb{R}, \exists x \in \mathbb{R}, y \geq x^2$

The negation of the statement is false.

Let $y = -1$. We know $x^2 \geq 0$ for all $x \in \mathbb{R}$, i.e. $x^2 > y$.

(iii) $\sim\left(\forall y \in \mathbb{R}, \forall x \in \mathbb{R}, \left(x < y \Rightarrow x < \frac{x+y}{2} < y\right)\right)$

$\equiv \exists y \in \mathbb{R}, \sim\left(\forall x \in \mathbb{R}, \left(x < y \Rightarrow x < \frac{x+y}{2} < y\right)\right)$

$\equiv \exists y \in \mathbb{R}, \exists x \in \mathbb{R}, \sim\left(x < y \Rightarrow x < \frac{x+y}{2} < y\right)$

$\equiv \exists y \in \mathbb{R}, \exists x \in \mathbb{R}, \left(x < y \wedge \left(\frac{x+y}{2} \leq x \vee \frac{x+y}{2} \geq y\right)\right)$

$\equiv \exists y \in \mathbb{R}, \exists x \in \mathbb{R}, \left(x < y \wedge \left(y \leq x \vee x \geq y\right)\right)$

The negation of the statement is false.

Clearly, $x < y \wedge \left(y \leq x \vee x \geq y\right)$ is equivalent to $x < y \wedge x \geq y$, which is impossible.

Question6

(a)

$\sim P :\sim (\exists x \in \mathbb{Q}, x^2 = 2)$

$\equiv \forall x \in \mathbb{Q}, \sim (x^2 = 2)$

$\equiv \forall x \in \mathbb{Q}, x^2 \neq 2$

(b)

$\sim Q :\sim (\forall x \in \mathbb{R}, x^2 + 1 \geq 2x)$

$\equiv \exists x \in \mathbb{R}, \sim (x^2 + 1 \geq 2x)$

$\equiv \exists x \in \mathbb{R}, x^2 + 1 < 2x$

Question7

(a)

$P : (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y < x)$

$\sim P :\sim (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y < x)$

$\equiv (\exists x \in \mathbb{R}, \sim (\exists y \in \mathbb{R}, y < x))$

$\equiv (\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \sim (y < x))$

$\equiv (\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y \geq x)$

The true statement is P because for a real number x, $x - 1$ is a smaller real number.

(b)

$Q : (\forall x \in \mathbb{R}, (x < 0 \vee x > 0))$

$\sim Q :\sim (\forall x \in \mathbb{R}, (x < 0 \vee x > 0))$

$\equiv (\exists x \in \mathbb{R}, \sim (x < 0 \vee x > 0))$

$\equiv \exists x \in \mathbb{R}, \sim (x < 0) \wedge \sim (x > 0)$

$\equiv \exists x \in \mathbb{R}, x \geq 0 \wedge x \leq 0$

$\equiv \exists x \in \mathbb{R}, x = 0$

The true statement is $\sim Q$ because $x = 0$ is neither positive nor negative.

Question8

(a)

$$\begin{aligned} \sim (\forall x \in \mathbb{R},\ x \geq 0) &\equiv \exists x \in \mathbb{R},\ \sim (x \geq 0) \\ &\equiv \exists x \in \mathbb{R},\ x < 0 \end{aligned}$$

The negation is true.

(b)

$\sim (\exists z \in \mathbb{Z}, (z \text{ is odd} \vee z \text{ is even}))$

$\equiv \forall z \in \mathbb{Z}, \sim (z \text{ is odd} \vee z \text{ is even})$

$\equiv \forall z \in \mathbb{Z}, (z \text{ is not odd} \wedge z \text{ is not even})$ (De Morgan's)

The original statement is true

(c)

$\sim (\exists n \in \mathbb{N}, (n \text{ is even} \wedge \sqrt{n} \text{ is prime}))$

$\equiv \forall n \in \mathbb{N}, \sim (n \text{ is even} \wedge \sqrt{n} \text{ is prime})$

$\equiv \forall n \in \mathbb{N}, (n \text{ is odd} \vee \sqrt{n} \text{ is not prime})$ (De Morgan's)

The original statement is true.

(d)

$$\sim\left(\forall y \in \mathbb{R}, \left(y \neq 0 \Rightarrow \frac{y+1}{y} < 1\right)\right)$$

$$\equiv \exists y \in \mathbb{R}, \sim\left(y \neq 0 \Rightarrow \frac{y+1}{y} < 1\right)$$

$$\equiv \exists y \in \mathbb{R}, \left(y \neq 0 \wedge \sim\left(\frac{y+1}{y} < 1\right)\right)$$

$$\equiv \exists y \in \mathbb{R}, \left(y \neq 0 \wedge \frac{y+1}{y} \geq 1\right)$$

The negation is true.

(e)

$\sim (\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy = 1)$
$\equiv \forall x \in \mathbb{R}, \sim (\forall y \in \mathbb{R}, xy = 1)$
$\equiv \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \sim (xy = 1)$
$\equiv \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy \neq 1$

The negation is true.

(f)

$\sim (\forall n \in \mathbb{N}, \exists p \in \mathbb{N}, n = 2p)$
$\equiv \exists n \in \mathbb{N}, \sim (\exists p \in \mathbb{N}, n = 2p)$
$\equiv \exists n \in \mathbb{N}, \forall p \in \mathbb{N}, \sim (n = 2p)$
$\equiv \exists n \in \mathbb{N}, \forall p \in \mathbb{N}, n \neq 2p$

The negation is true.

(g)

$\sim (\forall \varepsilon \in \mathbb{R}, \forall x \in \mathbb{Z}, \exists y \in \mathbb{Q}, (\varepsilon > 0 \Rightarrow |x - y| < \varepsilon))$
$\equiv \exists \varepsilon \in \mathbb{R}, \sim (\forall x \in \mathbb{Z}, \exists y \in \mathbb{Q}, (\varepsilon > 0 \Rightarrow |x - y| < \varepsilon))$
$\equiv \exists \varepsilon \in \mathbb{R}, \exists x \in \mathbb{Z}, \sim (\exists y \in \mathbb{Q}, (\varepsilon > 0 \Rightarrow |x - y| < \varepsilon))$
$\equiv \exists \varepsilon \in \mathbb{R}, \exists x \in \mathbb{Z}, \forall y \in \mathbb{Q}, \sim (\varepsilon > 0 \Rightarrow |x - y| < \varepsilon)$
$\equiv \exists \varepsilon \in \mathbb{R}, \exists x \in \mathbb{Z}, \forall y \in \mathbb{Q}, (\varepsilon > 0 \wedge \sim (|x - y| < \varepsilon))$ (Thm.1.4.2 pt 6)
$\equiv \exists \varepsilon \in \mathbb{R}, \exists x \in \mathbb{Z}, \forall y \in \mathbb{Q}, (\varepsilon > 0 \wedge |x - y| \geq \varepsilon)$

The statement is true.

<u>Question9</u>

(a) $\sim (\forall y \in \mathbb{R}, (y > -1 \Rightarrow y^2 > 1))$
$\equiv \exists y \in \mathbb{R}, \sim (y > -1 \Rightarrow y^2 > 1)$
$\equiv \exists y \in \mathbb{R}, (y > -1 \wedge \sim (y^2 > 1))$
$\equiv \exists y \in \mathbb{R}, (y > -1 \wedge y^2 \leq 1)$

The original statement is false. Take $y = 0$, then $y = 0 > -1 \Rightarrow y^2 = 0 < 1$)

(b)

$\sim\left(\exists x \in \mathbb{R}, x^2+1=0\right)$

$\equiv \forall x \in \mathbb{R}, \sim\left(x^2+1=0\right)$

$\equiv \forall x \in \mathbb{R},\ x^2+1 \neq 0$

The original statement is false. For any real number, x, $x^2 \geq 0$, so $x^2+1 \geq 1$. Thus, $x^2+1 \neq 0$.

(c)

$\sim(\forall x, y, z \in \mathbb{R}, x-(y-z) \neq (x-y)-z)$

$\equiv \exists x, y, z \in \mathbb{R}, \sim(x-(y-z) \neq (x-y)-z)$

$\equiv \exists x, y, z \in \mathbb{R}, x-(y-z)=(x-y)-z$

The original statement is false. Let $x=y=1$ and $z=0$.
Then $1-(1-0)=1-1=0$ and $(1-1)-0=0-0=0$.

(d)

$\sim(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x+y=0)$

$\equiv \exists x \in \mathbb{R}, \sim(\exists y \in \mathbb{R}, x+y=0)$

$\equiv \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \sim(x+y=0)$

$\equiv \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x+y \neq 0$

The negation is false. For any real number x, $x-x=0$, so let $y=-x$.

Question10 Write the following statements using quantifiers. Find their negations and determine in each case whether the statement or its negation is false, giving brief reason where possible.

(a) $P: \forall n \in \mathbb{N}, \exists m \in \mathbb{N}, n>m$

$\sim P: \sim(\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, n>m)$

$\equiv \exists n \in \mathbb{N}, \sim(\exists m \in \mathbb{N}, n>m)$

$\equiv \exists n \in \mathbb{N}, \forall m \in \mathbb{N}, \sim(n>m)$

$\equiv \exists n \in \mathbb{N}, \forall m \in \mathbb{N}, n \leq m$

The statement *P* is false. Let $n=1$. All natural numbers *m* are greater than *n*.

(b) $P: \forall x \in \mathbb{R}, x^2 \geq 0$

$\sim P: \sim\left(\forall x \in \mathbb{R}, x^2 \geq 0\right)$

$\equiv \exists x \in \mathbb{R}, \sim\left(x^2 \geq 0\right)$

$\equiv \exists x \in \mathbb{R}, x^2<0$

The statement ~*P* is false. For any real number x, x^2 is not less than 0.

(c) Let *D* be the set of all dogs.

$P: \exists d \in D, d$ is vegetarian..

$\sim P: \sim(\exists d \in D, d \text{ is vegetarian})$

$\equiv \forall d \in D, \sim(d \text{ is vegetarian})$

$\equiv \forall d \in D, d$ is not vegetarian

The statement ~*P* is probably false.

(d) $P : \exists x \in \mathbb{R},\ x \text{ is rational}$.

$\sim P :\sim \left(\exists x \in \mathbb{R},\ x \text{ is rational}\right)$

$\equiv \forall x \in \mathbb{R}, \sim \left(x \text{ is rational}\right)$

$\equiv \forall x \in \mathbb{R},\ x \text{ is not rational}$

The statement $\sim P$ is false. The number 2 is real and rational.

(e) Let S be the set of all students and let M be the set of all mathematics subjects.

$P : \forall s \in S, \exists m \in M,\ s \text{ likes } m$.

$\sim P :\sim \left(\forall s \in S, \exists m \in M,\ s \text{ likes } m\right)$

$\equiv \exists s \in S, \sim \left(\exists m \in M,\ s \text{ likes } m\right)$

$\equiv \exists s \in S, \forall m \in M, \sim \left(s \text{ likes } m\right)$

$\equiv \exists s \in S, \forall m \in M,\ s \text{ dislikes } m$

Unfortunately, P is more likely to be false.

Section 3: Proofs

Question1

(a) The statement is of the form: $(P(x) \Rightarrow Q(x)) \wedge P(a)$, thus the conclusion is $Q(a)$. So, applying the universal rule of Modus Ponens, we conclude that Peter phones John.

(b) The statement is of the form: $(P(x) \Rightarrow Q(x)) \wedge (Q(x) \Rightarrow R(x))$, thus the conclusion is $P(x) \Rightarrow R(x)$ So, applying the Law of syllogism, we know the final conclusion is as follows: Therefore, if $x^2 - 3x + 2 = 0$, then $x = 2$ or $x = 1$.

(c) The statement is of the form: $(P(x) \Rightarrow Q(x)) \wedge \sim Q(a)$, thus the conclusion is $\sim P(a)$. So, applying the universal rule of Modus Tollens, we conclude that $y = \sqrt{-1}$ is not real.

Question2 Prove or disprove the following statements

(a) Statement is of the form $\forall x \in D, P(x)$, so must prove with general proof, or disprove with counterexample.

Disprove: Let $n = 29$. Then

$$\begin{aligned} n^2 + n + 29 &= 29^2 + 29 + 29 \\ &= 29(29 + 1 + 1) \\ &= 29 \times 31 \end{aligned}$$

In this case, $n^2 + n + 29$ is not prime, and thus we have a counterexample.

Therefore, it is false to say "$\forall n \in \mathbb{N}, n^2 + n + 29$ is prime".

(b) Statement is of the form $\exists x \in D, \forall y \in D, P(x, y)$. So, to prove, must find one $x \in D$ that for all $y \in D$, $P(x, y)$ is true.

Prove: Let $x = 0$, and let $y \in \mathbb{Q}$. Then $xy = 0 \neq 1$.

Thus, the statement is true.

(c) Statement is of the form $\forall x \in D, \forall y \in D, P(x, y)$, so must prove with general proof, or disprove with counterexample.

Disprove: Let $a = b = 1$. Then,

$(a + b)^2 = (1 + 1)^2 = 2^2 = 4$ and $a^2 + b^2 = 1^2 + 1^2 = 2 \neq (a + b)^2$.

Thus we have a counterexample.

Therefore, it is false to say that $\forall a, b \in \mathbb{R}, (a + b)^2 = a^2 + b^2$

(d) Statement is of the form $\forall x \in D, \forall y \in D, P(x, y)$, so must prove with general proof, or disprove with counterexample.

Disprove: Let $n = 1$ and $m = 3$, both of which are odd. Then the average is $\frac{n+m}{2} = \frac{1+3}{2} = 2$, which is not odd.

Thus we have a counterexample.

Therefore, it is false to say that the average of any two odd integers is odd.

<u>**Question3**</u> Find the mistakes in the following "proofs".

(a) Statement is of the form $\forall x \in D, P(x)$, that is a universal statement, so requires proof with general proof, or disprove with counterexample.

(b) The mistake is in the use of the definitions of odd and even numbers.

When using an existential statement on two separate occasions, you should not use the same variable; that is, if we use *k* for defining *n* as an odd integer ($n = 2k+1$ for some $k \in \mathbb{Z}$), then we must use a different letter for defining *m* as an even integer (e.g. $m = 2q$ for some $q \in \mathbb{Z}$).

<u>**Question4**</u>

(a) Statement is of the form $\forall x \in D, Q(x)$, where $Q(x)$ is "$x^2 + 1 \geq 2x$".

Thus we must find a $P(x)$ to give the form $\forall x \in D, P(x) \Rightarrow Q(x)$

We know that for all $x \in \mathbb{R}, x^2 \geq 0$, so let $P(x)$ be "$x^2 \geq 0$".

$$x^2 \geq 0 \Rightarrow (x-1)^2 \geq 0$$
$$\Rightarrow x^2 - 2x + 1 \geq 0$$
$$\Rightarrow x^2 + 1 \geq 2x$$

Therefore for $x \in \mathbb{R}$, $x^2 + 1 \geq 2x$.

(b) Statement is of the form $\forall n \in D, P(n) \Rightarrow Q(n)$, where $P(n)$ is "*n* is odd" and $Q(n)$ is "n^2 is odd"

$$n \text{ is odd} \Rightarrow n = 2p+1 \qquad p \in \mathbb{N},$$
$$\Rightarrow n^2 = 4p^2 + 4p + 1$$
$$\Rightarrow n^2 = 2\left(2p^2 + 2p\right) + 1$$
$$\Rightarrow n^2 = 2q + 1 \qquad \text{where } q = 2p^2 + 2p \in \mathbb{N}$$
$$\Rightarrow n^2 \text{ is odd}$$

Therefore, For $n \in \mathbb{N}$, if *n* is odd, n^2 is odd.

(c) Statement is of the form $\forall x \in D, \forall y \in D, P(x,y) \Rightarrow Q(x,y)$, where $P(x,y)$ is "any two odd integers" and $Q(x,y)$ is "sum is even".

Let *x*, *y* be any two odd integers.

x is odd $\Rightarrow x = 2p+1 \qquad p \in \mathbb{Z}$

y is odd $\Rightarrow y = 2q+1 \; q \in \mathbb{Z}$

$$\begin{aligned} x+y &= (2p+1)+(2q+1) \\ &= 2p+2q+2 \\ &= 2(p+q+1) \\ &= 2r \qquad r = p+q+1 \in \mathbb{Z} \end{aligned}$$

Therefore, the sum of any two odd integers is even.

(d) Statement is of the form $\forall x \in D, P(x) \Rightarrow Q(x)$.

Let *ABC* be a triangle, with angles *A*, *B* and *C*.

We are given that the sum of two angles is equal to the third angle, i.e.

$A + B = C \ldots (1)$.

We know that $A+B+C = 180°$, since the angle sum of a triangle is $180°$.

$$\begin{aligned} A+B+C = 180° &\Rightarrow C + C = 180° \qquad \text{by (1)} \\ &\Rightarrow 2C = 180° \\ &\Rightarrow C = 90° \\ &\Rightarrow ABC \text{ is a right angled triangle} \end{aligned}$$

Therefore f the sum of two angles of a triangle is equal to the third angle, then the triangle is a right angled triangle

Question5 Statement is of the form $\forall x \in D, P(x) \Rightarrow Q(x)$, where $P(x)$ is "*x* is negative real number", and $Q(x)$ is "$(x-2)^2 > 4$".

We know that for all $x \in \mathbb{R}, x < 0$

$$\begin{aligned} x < 0 &\Rightarrow x - 4 < 0 \\ &\Rightarrow x(x-4) > 0 \\ &\Rightarrow x^2 - 4x > 0 \\ &\Rightarrow x^2 - 4x + 4 > 4 \\ &\Rightarrow (x-2)^2 > 4 \end{aligned}$$

Therefore if *x* is a negative real number, then $(x-2)^2 > 4$..

Question6 Statement is of the form $\exists x \in D, P(x)$, so to prove, must show one $x \in D$, which makes $P(x)$ true.

Let $n = 7$. $2^7 - 1 = 128 - 1 = 127$, which is prime.

Therefore, there is an integer $n > 5$ such that $2^n - 1$ is prime

Question7 Statement is of the form $\forall x \in D, P(x)$, where D is finite. So to prove, must show for all $x \in D$, $P(x)$ is true.

Using the method of exhaustion:

$n=1$: $n^2 - n + 41 = 1 - 1 + 41 = 41$ is prime

$n=2$: $n^2 - n + 41 = 4 - 2 + 41 = 43$ is prime

$n=3$: $n^2 - n + 41 = 9 - 3 + 41 = 47$ is prime

$n=4$: $n^2 - n + 41 = 16 - 4 + 41 = 53$ is prime

$n=5$: $n^2 - n + 41 = 25 - 5 + 41 = 61$ is prime

$n=6$: $n^2 - n + 41 = 36 - 6 + 41 = 71$ is prime

$n=7$: $n^2 - n + 41 = 49 - 7 + 41 = 83$ is prime

$n=8$: $n^2 - n + 41 = 64 - 8 + 41 = 97$ is prime

$n=9$: $n^2 - n + 41 = 81 - 9 + 41 = 113$ is prime

$n=10$: $n^2 - n + 41 = 100 - 10 + 41 = 131$ is prime

Therefore, for each integer n such that $1 \le n \le 10$, $n^2 - n + 41$ is a prime number.

Question8 Statement is of the form $\forall n \in D, P(n) \Rightarrow Q(n)$, where $P(n)$ is "n is an odd number", and $Q(n)$ is "$(-1)^n = -1$".

$\forall n \in \mathbb{Z}$ n is odd $\Rightarrow n = 2p+1$ $p \in \mathbb{Z}$

$$\begin{aligned}
(-1)^n &= (-1)^{2p+1} \\
&= (-1)^{2p}(-1) \\
&= (1)^p(-1) \\
&= 1 \times (-1) \\
&= -1
\end{aligned}$$

Therefore, if n is an odd integer, then $(-1)^n = -1$.

Question9 Statement is of the form: $\forall n \in D, P(n) \Rightarrow Q(n)$, where $P(n)$ is "n^2 is even", and $Q(n)$ is "n is even".

To prove by contraposition we must show $\forall n \in D, \sim Q(n) \Rightarrow \sim P(n)$. $\sim Q(n)$ is "n is not even", i.e. "n is odd", and $\sim P(n)$ is "n^2 is not even", i.e. "n^2 is odd".

n is odd $\Rightarrow n = 2p+1 \qquad p \in \mathbb{Z},$

$\Rightarrow n^2 = 4p^2 + 4p + 1$

$\Rightarrow n^2 = 2\left(2p^2 + 2p\right) + 1$

$\Rightarrow n^2 = 2q + 1 \qquad$ where $q = 2p^2 + 2p \in \mathbb{Z}$

$\Rightarrow n^2$ is odd

Therefore, if n is odd, n^2 is odd, and so by proof by contraposition, if n^2 is even, then n is even

<u>Question10</u> Statement is of the form: $\forall m \in D, P(m) \Rightarrow Q(m)$, where $P(m)$ is "m is an integer", and $Q(m)$ is "$m^2 + m + 1$ is always odd". Now if m is an integer, then m is even or m is odd, thus $P(m) \equiv R(m) \vee S(m)$, where $R(m)$ is "m is even", and $S(m)$ is "m is odd".

Hence $P(m) \Rightarrow Q(m) \equiv (R(m) \vee S(m)) \Rightarrow Q(m)$

$\equiv (R(m) \Rightarrow Q(m)) \wedge (S(m) \Rightarrow Q(m))$

Case 1: Prove: $R(m) \Rightarrow Q(m)$, i.e. If m is even, then $m^2 + m + 1$ is always odd

n is even $\Rightarrow n = 2p \qquad p \in \mathbb{Z},$

$\Rightarrow m^2 + m + 1 = 4p^2 + 2p + 1$

$\Rightarrow m^2 + m + 1 = 2\left(2p^2 + p\right) + 1$

$\Rightarrow m^2 + m + 1 = 2q + 1 \qquad$ where $q = 2p^2 + p \in \mathbb{Z}$

$\Rightarrow m^2 + m + 1$ is odd

Therefore if m is even, then $m^2 + m + 1$ is always odd

Case 2: Prove: $S(m) \Rightarrow Q(m)$, i.e. If m is odd, then $m^2 + m + 1$ is always odd

m is odd $\Rightarrow m = 2k + 1 \qquad k \in \mathbb{Z},$

$\Rightarrow m^2 + m + 1 = 4k^2 + 4k + 1 + 2k + 1 + 1$

$\Rightarrow m^2 + m + 1 = 2\left(2k^2 + 3k + 1\right) + 1$

$\Rightarrow m^2 + m + 1 = 2l + 1 \qquad$ where $l = 2k^2 + 3k + 1 \in \mathbb{Z}$

$\Rightarrow m^2 + m + 1$ is odd

Therefore if m is odd, then $m^2 + m + 1$ is always odd.

Thus if m is even or m is odd, then $m^2 + m + 1$ is always odd, and so if m is an integer, then $m^2 + m + 1$ is always odd.

Question11 Disprove the statement: $\forall a,b \in \mathbb{Z},\ a \neq 0, b \neq 0,\ \frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$. Are there any values for *a*, *b* that make the statement true? Explain.

Statement is of the form $\forall x \in D, P(x)$, that is a universal statement, so requires disproof with counterexample

Let $a = 1$ and $b = 2$.

Then $\frac{1}{a+b} = \frac{1}{1+2} = \frac{1}{3}$.

But, $\frac{1}{a} + \frac{1}{b} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \neq \frac{1}{a+b}$.

Thus by counterexample the statement $\forall a,b \in \mathbb{Z},\ a \neq 0, b \neq 0,\ \frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$ is false

There are no real values that make the statement true.

If you try to solve for *a* and *b*, you come across a quadratic with only complex solutions

Question12 Prove or disprove this statement: For all integers, *a*, *b* if $a < b$, then $a^2 < b^2$.

Statement is of the form $\forall x \in D, \forall y \in D, P(x,y) \Rightarrow Q(x,y)$, so requires general proof or disproof with a counterexample.

Counterexample: Let $a = -5$ and let $b = 2$.

$a < b$ but $a^2 = 25 > 4 = b^2$

Thus by counterexample the statement $\forall x \in D, \forall y \in D, P(x,y) \Rightarrow Q(x,y)$ is false.

Question13 Prove if n^2 is odd, then *n* is odd.

Statement is of the form: $\forall n \in D, P(n) \Rightarrow Q(n)$, where $P(n)$ is "n^2 is odd", and $Q(n)$ is "n is odd". Direct proof is not possible, thus use proof by contraposition.

To prove by contraposition we must show $\forall n \in D, \sim Q(n) \Rightarrow \sim P(n)$. $\sim Q(n)$ is "n is not odd", i.e. "n is even", and $\sim P(n)$ is "n^2 is not odd", i.e. "n^2 is even".

$$
\begin{aligned}
n \text{ is even} &\Rightarrow n = 2p \qquad p \in \mathbb{Z}, \\
&\Rightarrow n^2 = 4p^2 \\
&\Rightarrow n^2 = 2 \times 2p^2 \\
&\Rightarrow n^2 = 2q \qquad \text{where } q = 2p^2 \in \mathbb{Z} \\
&\Rightarrow n^2 \text{ is even}
\end{aligned}
$$

Therefore, if *n* is even, n^2 is even, and so by proof by contraposition, if n^2 is odd, then *n* is odd

Question14 Prove there is no smallest positive real number.

Statement is of the form $\forall x \in D, P(x)$. Where $P(x)$ is "there is no smallest positive real number" So to prove, must show for all $x \in D,\ \ P(x)$ is true. Prove by contradiction.

Assume $\sim P(x)$, that is assume there is a smallest positive real number, $n \in \mathbb{R}$. Then $n-1 \in \mathbb{R},\ n-1 < n$. This contradicts our assumption, thus $\sim P(x)$ is false and the original statement "there is no smallest positive real number" is true.

Question15 Prove each of the following using proof by cases

(a) If $x = 4, 5, \text{ or } 6,$ then $x^2 - 3x + 21 \neq x.$

Statement is of the form $[R(x) \vee S(x) \vee T(x)] \Rightarrow Q(x)$, where $R(x)$ is $x = 4$, $S(x)$ is $x = 5$, $T(x)$ is $x = 6$ and $Q(x)$ is $x^2 - 3x + 21 \neq x.$

Case 1: Prove: $R(x) \Rightarrow Q(x)$, i.e. If $x = 4$, then $x^2 - 3x + 21 \neq x.$

$$\begin{aligned} &4^2 - 3\times 4 + 21 \\ &= 25 \\ &\neq 4 \end{aligned}$$

Therefore If $x = 4$, then $x^2 - 3x + 21 \neq x.$

Case 2: Prove: $S(x) \Rightarrow Q(x)$, i.e. If $x = 5$, then $x^2 - 3x + 21 \neq x.$

$$\begin{aligned} &5^2 - 3\times 5 + 21 \\ &= 31 \\ &\neq 5 \end{aligned}$$

Therefore If $x = 5$, then $x^2 - 3x + 21 \neq x..$

Case 3: Prove: $T(x) \Rightarrow Q(x)$, i.e. If $x = 6$, then $x^2 - 3x + 21 \neq x.$

$$\begin{aligned} &6^2 - 3\times 6 + 21 \\ &= 39 \\ &\neq 6 \end{aligned}$$

Therefore If $x = 6$, then $x^2 - 3x + 21 \neq x..$

Thus If $x = 4, 5, \text{ or } 6,$ then $x^2 - 3x + 21 \neq x.$

(b) $\forall x \in \mathbb{Z},\ \ x \neq 0 \Rightarrow 2^x + 3 \neq 4$

Question16 Prove there is a perfect square that can be written as the sum of two other perfect squares. (Note an integer *n* is a perfect square if and only if $\exists k \in \mathbb{Z}, n = k^2$)

Statement is of the form $\exists n \in D, P(n)$, so we must show one example.

$\exists n \in \mathbb{Z}$, (n is a perfect square $\wedge \exists k, l \in \mathbb{Z}, n = k^2 + l^2$).

Let $n = 25 = 5^2$. n is a perfect square and $n = 4^2 + 3^2$.

Therefore, there is a perfect square that can be written as a sum of two other perfect squares.

Question17 Prove that the product of two odd integers is also an odd integer.

Statement is of the form $\forall x \in D, \forall y \in D, P(x, y) \Rightarrow Q(x, y)$, where *D* is the integers, $P(x, y)$ can be written as "*x* is odd and *y* is odd", $Q(x, y)$ can be written as "$x \times y$ is odd".

x is odd $\Rightarrow x = 2k + 1$ $\quad k \in \mathbb{Z}$,

y is odd $\Rightarrow y = 2l + 1$ $\quad l \in \mathbb{Z}$,

$$\begin{aligned} x \times y &= (2k+1)(2l+1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1 \\ &= 2n + 1 \qquad \text{where } n = 2kl + k + l \in \mathbb{Z} \end{aligned}$$

$\therefore x \times y$ is odd

Therefore the product of two odd integers is also an odd integer

Question18 Prove or disprove the following statements:

(a) The difference between any two odd integers is also an odd integer.

Statement is of the form $\forall x \in D, \forall y \in D, P(x, y) \Rightarrow Q(x, y)$, where *D* is the integers, $P(x, y)$ can be written as "*x* is odd and *y* is odd", $Q(x, y)$ can be written as "$x - y$ is odd". Disprove with counterexample or prove with general proof.

Counterexample: Let $x = 5$, $y = 3$, $x - y = 5 - 3 = 2$, which is even. Hence by counterexample the statement "The difference between any two odd integers is also an odd integer" is false.

(b) For any integer n, $3 \mid n(6n+3)$.

Statement is of the form $\forall n \in D, P(n)$, where D is the integers, $P(n)$ can be written as "$n(6n+3) = 3k, k \in \mathbb{Z}$". Disprove with counterexample or prove with general proof.

$$\begin{aligned} n(6n+3) &= 3n(2n+1) \\ &= 3(2n^2+n) \\ &= 3k, k = 2n^2+n \in \mathbb{Z} \\ &\therefore 3 \mid n(6n+3) \end{aligned}$$

Therefore for any integer n, $3 \mid n(6n+3)$.

(c) The cube of any odd integer is an odd integer.

Statement is of the form $\forall x \in D, P(x) \Rightarrow Q(x)$, where D is the integers, $P(x)$ can be written as "x is odd", $Q(x)$ can be written as "x^3 is odd".

x is odd $\Rightarrow x = 2k+1 \qquad k \in \mathbb{Z}$,

$$\begin{aligned} x^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \\ &= 2l + 1 \qquad \text{where } l = 4k^3 + 6k^2 + 3k \in \mathbb{Z} \\ &\therefore x^3 \text{ is odd} \end{aligned}$$

Therefore the cube of any odd integer is an odd integer

(d) For any integers a, b, c, if $a \mid c$, then $ab \mid c$.

Disprove by counterexample: Let $a = 2, b = 3, c = 4$.

$c = 4 = 2 \times 2 = 2a \quad \therefore a \mid c$ However $ab = 6 \nmid 4$, $ab \nmid c$

Thus by counterexample "For any integers a, b, c, if $a \mid c$, then $ab \mid c$" is false.

(e) There is no largest even integer.

Proof by contradiction.

Assume that there is a largest even integer, n, say.

Then, $\exists k \in \mathbb{Z}, n = 2k$.

Consider the number $m = n + 2 = 2k + 2 = 2(k+1)$.

Let $l = k + 1 \in \mathbb{Z}$. Then $m = 2l$.

Therefore, by definition, m is an even integer. Also, we have $m > n$.

However, we said that n was the largest even integer. Thus we have a contradiction.

Therefore, our assumption must be wrong.

Therefore, there must be no largest even integer

(f) For all integers *a*, *b*, *c*, if $a \nmid bc$, then $a \nmid b$.

Statement form is $P(a,b,c) \Rightarrow Q(a,b,c)$, where $P(a,b,c): a \nmid bc$ and $Q(a,b,c): a \nmid b$

Proof by contraposition, i.e. prove $\sim Q(a,b,c) \Rightarrow \sim P(a,b,c)$. Where $\sim P(a,b,c): a \mid bc$, and $\sim Q(a,b,c): a \mid b$

$a \mid b \Rightarrow b = ak \quad k \in \mathbb{Z}$

$\Rightarrow bc = akc$

$\Rightarrow bc = al \qquad l = kc \in \mathbb{Z}$

$\therefore a \mid bc$

Therefore For all integers *a*, *b*, *c*, if $a \mid b$, then $a \mid bc$, and so by contraposition for all integers *a*, *b*, *c*, if $a \nmid bc$, then $a \nmid b$.

(g) For all integers *n*, $4(n^2+n+1)-3n^2$ is a perfect square.

$$\begin{aligned} 4(n^2+n+1)-3n^2 &= 4n^2+4n+4-3n^2 \\ &= n^2+4n+4 \\ &= (n+2)^2 \\ &= k^2 \qquad k = n+2 \in \mathbb{Z} \end{aligned}$$

Therefore for all integers *n*, $4(n^2+n+1)-3n^2$ is a perfect square.

(h) For any integers *a*, *b*, if $a \mid b$ then $a^2 \mid b^2$.

$a \mid b \Rightarrow b = ak \qquad k \in \mathbb{Z}$

$\Rightarrow b^2 = (ak)^2$

$\Rightarrow bc = a^2 l \qquad l = k^2 \in \mathbb{Z}$

$\therefore a^2 \mid b^2$

Therefore for any integers *a*, *b*, if $a \mid b$ then $a^2 \mid b^2$.

(i) For all integers *n*, $n^2 - n + 41$ is prime.

Disprove by counterexample. Let $n = 41$.

Then $n^2 - n + 41 = (41)^2 - 41 + 41 = (41)^2$, which is clearly not prime.

(j) For all integers, *n* and *m*, if $n-m$ is even, then n^3-m^3 is even.
Statement is of the form $\forall n \in D, \forall m \in D, P(n,m) \Rightarrow Q(n,m)$, where *D* is the integers, $P(n,m)$ is "$n-m$ is even", $Q(n,m)$ is "n^3-m^3 is even".

$n-m$ is even $\Rightarrow n-m=2k \qquad k \in \mathbb{Z}$,

$$\begin{aligned} n^3-m^3 &= (n-m)(n^2+nm+m^2) \\ &= 2k(n^2+nm+m^2) \\ &= 2(kn^2+knm+km^2) \\ &= 2l \qquad \text{where } l = kn^2+knm+km^2 \in \mathbb{Z} \\ &\therefore n^3-m^3 \text{ is even} \end{aligned}$$

Therefore for all integers, *n* and *m*, if $n-m$ is even, then n^3-m^3 is even

<u>Question19</u> Prove that the product of any four consecutive numbers, increased by one, is a perfect square?

$\forall n \in D, P(n)$, where *D* is the integers, $P(n)$ is "product of any four consecutive numbers, increased by one, is a perfect square".

Let $n, n+1, n+2, n+3$ be four consecutive integers.

$$\begin{aligned} n(n+1)(n+2)(n+3)+1 &= n^4+6n^3+11n^2+6n+1 \\ &= (n^2+3n+1)^2 \\ &= k^2 \qquad k=(n^2+3n+1) \in Z \end{aligned}$$

Hence $n(n+1)(n+2)(n+3)+1$ is a perfect square

Thus the product of any four consecutive numbers, increased by one, is a perfect square.

Section 4: Set Theory

Question1

(a) $A \cup B = (0, 1]$
$= \{x \in \mathbb{R} : 0 < x \leq 1\}$

(b) $A \cap B = \emptyset$

(c) $B \cap C = B$

(d) $A \cup C = C$

(e) $A \cap C = A$

(f) $\overline{A} = \{x \in \mathbb{R} : x \neq 1\}$

(g) $\overline{C} = (-\infty, 0) \cup (1, \infty)$
$= \{x \in \mathbb{R} : x < 0 \vee x > 1\}$

(h) $C - A = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}$

(i) $C - B = \{0, 1\}$

(j) $A - C = \emptyset$

The sets *A* and *B* are disjoint.

Question2

(a) $A \cup B = \mathbb{N}$

(b) $A \cap B = \emptyset$

(c) $B \cap P = \{2\}$

(d) $A \cup P = A \cup \{2\}$
$= \{1, 2, 3, 5, 7, 9, 11, 13, \ldots\}$

(e) $A \cap P = P - \{2\}$
$= \{3, 5, 7, 11, 13, \ldots\}$

(f) $\overline{A} = B$

(g) $\overline{P} = \{x \in \mathbb{N} : x \text{ is not prime}\}$
$= \{x \in \mathbb{N} : x \text{ is composite } \vee x = 1\}$

(h) $P - A = \{2\}$

(i) $B - P = B - \{2\}$

(j) $A - B = A$

A and *B* are disjoint as $A \cap B = \emptyset$.

P is not a subset of *A*, since $2 \in P$ but $2 \notin A$.

Question3 Let $X = \{1, 2, 3, 4\}$.

(a) $\mathcal{P}(X) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\},$
$\{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\} \}$

(b) $\mathcal{P}(X)$ has $2^4 = 16$ elements.

(c) Yes, $\emptyset \in \mathcal{P}(X)$ is true.

(d) Yes, $\{\emptyset\} \subseteq \mathcal{P}(X)$ is true.

<u>Question4</u> $\mathcal{P}(\emptyset) = \{\emptyset\}$. $\mathcal{P}(\emptyset)$ has $2^0 = 1$ element.

<u>Question5</u> $\mathcal{P}(X)$ has 2^n elements.

<u>Question6</u>

(a) False.
Let $B = \{2\} \in \mathcal{P}(X)$ and. $C = \{1\} \in \mathcal{P}(X)$. Then
$2 \in B, 2 \notin C \therefore B \not\subseteq C$ also $1 \in C, 1 \notin B \therefore C \not\subseteq B$

(b) True. Let $B = \emptyset$..

(c) True. Let $B = X$.

(d) True. All subsets but X are proper subsets

<u>Question7</u> Since $\mathcal{P}(X)$ has four elements, $\mathcal{P}(\mathcal{P}(X))$ will have $2^4 = 16$ elements.

$$\begin{aligned}\mathcal{P}(\mathcal{P}(X)) = \mathcal{P}(\{\emptyset, \{1\}, \{2\}, \{1,2\}\})\\ = \{ \ &\emptyset, \{\emptyset\}, \{\{1\}\}, \{\{2\}\}, \{\{1,2\}\}, \{\emptyset, \{1\}\}, \{\emptyset, \{2\}\},\\ &\{\emptyset, \{1,2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{1,2\}\}, \{\{2\}, \{1,2\}\},\\ &\{\emptyset, \{1\}, \{2\}\}, \{\emptyset, \{1\}, \{1,2\}\}, \{\{1\}, \{2\}, \{1,2\}\},\\ &\{\emptyset, \{2\}, \{1,2\}\}, \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \ \}\end{aligned}$$

Since Y has three elements, $\mathcal{P}(\mathcal{P}(Y))$ will have $2^{2^3} = 256$ elements.

$\emptyset, \{\{1\}\}$ and $\{\{2\}\}$ belong to $\mathcal{P}(\mathcal{P}(Y))$.

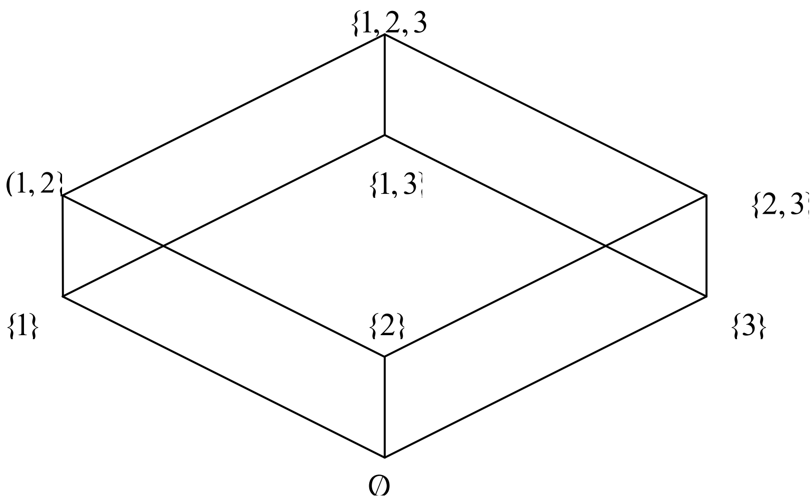
<u>Question8</u> $\emptyset, [-1,1], \{1\}, (0,1)$, etc.

The elements of $\mathcal{P}(\mathbb{R})$ cannot be listed. (There are too many of them!)

The set $\mathcal{P}(\mathbb{R})$ has an infinite number of elements.

$[-1,1] = \{x \mid x \in \mathbb{R} \wedge -1 \le x \le 1\} \in \mathcal{P}(\mathbb{R})$ is true.

Question9



Question10 Omitted

Question11 Let Claim(n) be "If $X = \{1, 2, \ldots, n\}$, then $\mathcal{P}(X)$ has 2^n elements."

Step 1: Claim(1) is "If $X = \{1\}$, then $\mathcal{P}(X)$ has $2^1 = 2$ elements."

$\mathcal{P}(X) = (\emptyset, \{1\}\}$. $\mathcal{P}(X)$ has 2 elements, so, Claim(1) is true.

Step 2: Assume that Claim(k) is true for some $k \in \mathbb{N}$; that is, "If $X = \{1, 2, \ldots, k\}$, then $\mathcal{P}(X)$ has 2^k elements." …(1)

Prove Claim($k+1$) is true; that is, prove that "If $X = \{1, 2, \ldots, k, k+1\}$, then $\mathcal{P}(X)$ has 2^{k+1} elements."

We know that the set $\{1, 2, \ldots, k\}$ has 2^k subsets which contain the elements 1, 2, 3, …, k.

These subsets will also be subsets of $X = \{1, 2, \ldots, k, k+1\}$.

So, we already have 2^k subsets of X.

How do we take into account the element $k+1$? Each of these original 2^k subsets will determine a "new" subset when the element $k+1$ is included in the original subset and all subsets containing $k+1$ will be so determined.

Thus, we have the subsets of $\{1, 2, 3, \ldots, k\}$ and the "new" subsets.

So the total number of subsets of $X = \{1, 2, 3, \ldots, k, k+1\}$ is $2^k + 2^k = 2\left(2^k\right) = 2^{k+1}$.

So Claim($k+1$) is true.

Thus, by Mathematical Induction, Claim(n) is true for all $n \in \mathbb{N}$.

Question12

(a) Let $X = \{1\}$ and $Y = \{2\}$.

Then $\mathcal{P}(X) = \{\emptyset, \{1\}\}$, $\mathcal{P}(Y) = \{\emptyset, \{2\}\}$ and $X \cup Y = \{1, 2\}$.

$\mathcal{P}(X) \cup \mathcal{P}(Y) = \{\emptyset, \{1\}, \{2\}\}$, $\mathcal{P}(X \cup Y) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Clearly, $\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$ but $\mathcal{P}(X) \cup \mathcal{P}(Y) \neq \mathcal{P}(X \cup Y)$.

(b) Let $X = \{1, 2\}$ and $Y = \{2, 3\}$.

Then $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $\mathcal{P}(Y) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$ and $X \cap Y = \{2\}$.

$\mathcal{P}(X) \cap \mathcal{P}(Y) = \{\emptyset, \{2\}\}$, $\mathcal{P}(X \cap Y) = \{\emptyset, \{2\}\}$

Clearly, $\mathcal{P}(X) \cap \mathcal{P}(Y) = \mathcal{P}(X \cap Y)$.

Question13

(a) Prove $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$

KNOW: $A \subseteq B$, that is, $x \in A \Rightarrow x \in B \ldots (1)$

PROVE: $A \cup C \subseteq B \cup C$, that is, $x \in A \cup C \Rightarrow x \in B \cup C$.

PROOF: Let $x \in A \cup C$.

$$\begin{aligned} x \in A \cup C &\Rightarrow x \in A \vee x \in C \\ &\Rightarrow x \in B \vee x \in C \text{ by (1)} \\ &\Rightarrow x \in B \cup C \end{aligned}$$

Therefore, $A \cup C \subseteq B \cup C$.

(b) To prove $(A \cup B) \cap B = B$, we must prove two things:

1. $(A \cup B) \cap B \subseteq B$, that is, $x \in (A \cup B) \cap B \Rightarrow x \in B$

2. $B \subseteq (A \cup B) \cap B$, that is, $x \in B \Rightarrow x \in (A \cup B) \cap B$

Proof of 1:

$$\begin{aligned} x \in (A \cup B) \cap B &\Rightarrow x \in (A \cup B) \wedge x \in B \\ &\Rightarrow x \in B \\ \therefore (A \cup B) \cap B &\subseteq B \end{aligned}$$

Proof of 2:

$$\begin{aligned}
x \in B &\Rightarrow x \in B \wedge x \in B \\
&\Rightarrow (x \in B \vee x \in A) \wedge x \in B \quad (\vee\text{-introduction}) \\
&\Rightarrow x \in (B \cup A) \wedge x \in B \\
&\Rightarrow x \in (A \cup B) \cap B \\
&\therefore B \subseteq (A \cup B) \cap B
\end{aligned}$$

Thus, $(A \cup B) \cap B = B$

<u>Question14</u> Let U be the universal set and let A, B and C be subsets of U.

Using properties of union, intersection and complement and known set laws, simplify the following:

(a)

$$\begin{aligned}
\overline{(A \cap \overline{B})} \cap A &= (\overline{A} \cup B) \cap A \\
&= (\overline{A} \cap A) \cup (B \cap A) \\
&= \emptyset \cup (B \cap A) \\
&= (B \cap A)
\end{aligned}$$

(b)

$$\begin{aligned}
(C \cup B) \cup \overline{C} &= C \cup B \cup \overline{C} \\
&= C \cup \overline{C} \cup B \\
&= U \cup B \\
&= U
\end{aligned}$$

(c)

$$\begin{aligned}
(A \cap \emptyset) \cap U &= \emptyset \cap U \\
&= \emptyset
\end{aligned}$$

(d)

$$\begin{aligned}
(A \cap U) \cup \overline{A} &= A \cup \overline{A} \\
&= U
\end{aligned}$$

<u>Question15</u>

Let $A = \{0, 1\}$, $B = \left\{ n \in \mathbb{Z} : \exists k \in \mathbb{Z}, \left(n = \frac{1+(-1)^k}{2} \right) \right\}$

Step 1: Prove $A \subseteq B$.

Let $x \in A$. Then $x = 0$ or $x = 1$. Proof by cases.

Case 1: $x = 0 \Rightarrow x = \frac{1-1}{2} = \frac{1+(-1)^1}{2}$.

Therefore, $\exists k \in \mathbb{Z}, \left(x = \dfrac{1+(-1)^k}{2} \right)$.

Case 2: $x = 1 \Rightarrow x = \dfrac{1+1}{2} = \dfrac{1+(-1)^2}{2}$.

Therefore, $\exists k \in \mathbb{Z}, \left(x = \dfrac{1+(-1)^k}{2} \right)$.

Therefore, $A \subseteq B$.

Step 2: Prove $B \subseteq A$.

Let $y \in B$. Then $\exists k \in \mathbb{Z}, \left(y = \dfrac{1+(-1)^k}{2} \right)$.

k can be an odd integer or an even integer.

Let *k* be an odd integer.

Then $y = \dfrac{1+(-1)^k}{2} = \dfrac{1+(-1)}{2} = \dfrac{0}{2} = 0$.

Let *k* be an even integer.

Then $y = \dfrac{1+(-1)^k}{2} = \dfrac{1+1}{2} = \dfrac{2}{2} = 1$.

Therefore, $y = 0$ or $y = 1$.

Thus, $y \in A$.

Therefore, $B \subseteq A$.

Therefore, by Step 1 and Step 2, $A = B$.

<u>Question16</u> $A = \{1, 3, 5, 9, \ldots\}$ $B = \{2, 5, 8, 11, \ldots\}$.

$$
\begin{aligned}
t \in A \cap B &\Rightarrow t \in A \wedge t \in B \\
&\Rightarrow \exists k \in \mathbb{Z}\ (t = 2k - 1) \wedge \exists w \in \mathbb{Z}\ (t = 3w + 2) \\
&\Rightarrow 2k - 1 = 3w + 2 \\
&\Rightarrow 2k = 3w + 3 = 3(w+1). \text{ But } 2k \text{ is even so } w+1 \text{ must be even.} \\
&\Rightarrow w \text{ is an odd number}
\end{aligned}
$$

Therefore, there is an odd integer $w \in \mathbb{Z}$ such that $t = 3w + 2$.

Thus, $t \in A \cap B \Rightarrow \exists w \in \mathbb{Z}\ (w \text{ is odd} \wedge t = 3w + 2)$.

Now, let t be an integer such that $\exists w \in \mathbb{Z}\ (w \text{ is odd} \wedge t = 3w + 2)$.

$t \in B$ by the definition of *B*. We must show that $t \in A$.

$t \in \mathbb{Z}$ such that $\exists w \in \mathbb{Z}\,(w \text{ is odd} \wedge t = 3w+2)$

$\Rightarrow p \in \mathbb{Z}\,(w = 2p+1 \wedge t = 3(2p+1)+2)$
$\Rightarrow p \in \mathbb{Z}\,(w = 2p+1 \wedge t = (6p+3)+2 = 2(3p+3)-1)$
$\Rightarrow t \in A$

Therefore, $t \in A \cap B$.

Thus, $\exists w \in \mathbb{Z}\,(w \text{ is odd} \wedge t = 3w+2) \Rightarrow t \in A \cap B$

Question17

(a) We must prove two things:

1. $\overline{(A \cup B)} \subseteq \overline{A} \cap \overline{B}$, that is, $x \in \overline{(A \cup B)} \Rightarrow x \in \overline{A} \cap \overline{B}$

2. $\overline{A} \cap \overline{B} \subseteq \overline{(A \cup B)}$, that is, $x \in \overline{A} \cap \overline{B} \Rightarrow x \in \overline{(A \cup B)}$

Proof of 1.:

$$
\begin{aligned}
& x \in \overline{(A \cup B)} \Rightarrow \sim (x \in A \cup B) \\
& \Rightarrow \sim (x \in A \vee x \in B) \\
& \Rightarrow \sim (x \in A) \wedge \sim (x \in B) \\
& \Rightarrow x \in \overline{A} \wedge x \in \overline{B} \\
& \Rightarrow x \in \overline{A} \cap \overline{B} \\
& \therefore \overline{(A \cup B)} \subseteq \overline{A} \cap \overline{B}
\end{aligned}
$$

Proof of 2: Reverse the steps for proof of 1. $\therefore \overline{A} \cap \overline{B} \subseteq \overline{(A \cup B)}$

Therefore, $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

(b) We must prove two things:

1. $A \cap (B - C) \subseteq (A \cap B) - C$, that is $x \in A \cap (B - C) \Rightarrow x \in (A \cap B) - C$

2. $(A \cap B) - C \subseteq A \cap (B - C)$, that is $x \in (A \cap B) - C \Rightarrow x \in A \cap (B - C)$

Proof of 1:

$$
\begin{aligned}
& x \in A \cap (B - C) \Rightarrow x \in A \wedge x \in B - C \\
& \Rightarrow x \in A \wedge (x \in B \wedge x \notin C) \\
& \Rightarrow (x \in A \wedge x \in B) \wedge x \notin C \\
& \Rightarrow x \in A \cap B \wedge x \notin C \\
& \Rightarrow x \in (A \cap B) - C \\
& \therefore A \cap (B - C) \subseteq (A \cap B) - C
\end{aligned}
$$

Proof of 2: Reverse the steps for proof of 1. $\therefore (A \cap B) - C \subseteq A \cap (B - C)$

Therefore, $A \cap (B - C) = (A \cap B) - C$.

Question18 Let U be the universal set and let A, B and C be subsets of U.

Using properties of union, intersection and complement and known set laws, simplify the following:

(a)
$$\begin{aligned}(C \cap U) \cup \overline{C} &= \left(C \cup \overline{C}\right) \cap \left(U \cup \overline{C}\right) \\ &= U \cap U \\ &= U\end{aligned}$$

(c)
$$\begin{aligned}\overline{\overline{(C \cup \emptyset)} \cup C} &= (C \cup \emptyset) \cap \overline{C} \\ &= C \cap \overline{C} \\ &= \emptyset\end{aligned}$$

(b)
$$\begin{aligned}\overline{(A \cap U)} \cup \overline{A} &= \left(\overline{A} \cup \emptyset\right) \cup \overline{A} \\ &= \overline{A} \cup \overline{A} \\ &= \overline{A}\end{aligned}$$

(d)
$$\begin{aligned}(A \cap B) \cap \overline{A} &= A \cap B \cap \overline{A} \\ &= \left(A \cap \overline{A}\right) \cap B \\ &= \emptyset \cap B \\ &= \emptyset\end{aligned}$$

Question19

(a) Let $n \in T$. Then n^2 is an odd integer.

Let's assume that n is an even integer.

Then, $\exists k \in \mathbb{Z}\ (n = 2k)$

$\Rightarrow n^2 = (2k)^2 = 4k^2 = 2\left(2k^2\right)$.

Therefore, n^2 is an even integer.

This leads us to a contradiction, as n^2 is an odd integer.

So our assumption must be wrong.

Therefore, n must be an odd integer $\Rightarrow n \in O$.

Thus, $T \subseteq O$.

(b) Let $m \in O$.

Then m is an odd integer $\Rightarrow \exists k \in \mathbb{Z}\ (m = 2k + 1)$

$$\begin{aligned}\Rightarrow m^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2\left(2k^2 + 2k\right) + 1, \\ &\quad \left(2k^2 + 2k\right) \in \mathbb{Z}\end{aligned}$$

Therefore, m^2 is an odd integer, so $m \in T$.

Thus, $O \subseteq T$.

(c) From Part (a) $T \subseteq O$ and from part (b) $O \subseteq T$. Therefore $T = O$.

Question20 Let $x = 1$ and $y = 4$. $x^2 + y^2 = 1 + 16 = 17$ and $17 \notin E$.

Thus, T is not a subset of E.

Question21

(a) Prove $A \cap B = A \Rightarrow A \subseteq B$.

KNOW: $A \cap B = A$., that is $x \in A \Rightarrow x \in A \cap B$ and $x \in A \cap B \Rightarrow x \in A \ldots (1)$

PROVE: $A \subseteq B$, that is, $x \in A \Rightarrow x \in B$.

$$\begin{aligned} x \in A &\Rightarrow x \in A \cap B \text{ (by 1)} \\ &\Rightarrow x \in A \wedge x \in B \\ &\Rightarrow x \in B \end{aligned}$$

Thus, $A \subseteq B$.

(b) Disprove the statement.

Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{2\}$.

Then $A \cap B = A \cap C = \{1\}$, but $B \neq C$.

Question22 Determine if the following statements are true or false:

(a) True

Prove $A \cap B = \emptyset \Rightarrow A \subseteq \overline{B}$.

KNOW: $A \cap B = \emptyset$.

PROVE: $A \subseteq \overline{B}$, that is, $x \in A \Rightarrow x \in \overline{B}$.

Let $x \in A$. Suppose $x \in B$.

Then $x \in A \cap B$, but $A \cap B = \emptyset$.

Therefore, we have a contradiction and $x \notin B$, that is, $x \in \overline{B}$.

(b) True

Prove $\left(A \subseteq \overline{B} \wedge \overline{A} \subseteq \overline{B}\right) \Rightarrow B = \emptyset$.

KNOW: $A \subseteq \overline{B}$ and $\overline{A} \subseteq \overline{B}$.

PROVE: $B = \emptyset$.

Let $B \neq \emptyset$, that is, there exists x such that $x \in B$.

Now, we have two cases.

Either $x \in A$ or $x \in \overline{A}$.

$x \in A \Rightarrow x \in \overline{B}$, which is a contradiction.

$x \in \overline{A} \Rightarrow x \in \overline{B}$, which is also a contradiction.

Therefore, x does not exist, so $B = \emptyset$.

(c) True

Prove A and $B - A$ are disjoint, that is $A \cap (B - A) = \emptyset$.

Suppose $A \cap (B - A) \neq \emptyset$, that is, there exists x such that $x \in A \cap (B - A)$

$$\Rightarrow x \in A \wedge x \in (B - A)$$
$$\Rightarrow x \in A \wedge (x \in B \wedge x \notin A)$$
$$\Rightarrow x \in A \wedge x \notin A \wedge x \in B$$

This statement is false.

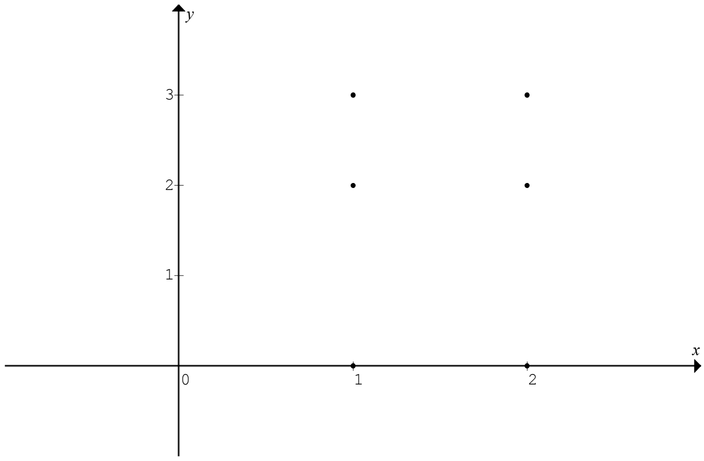
Therefore, $A \cap (B - A) = \emptyset$.

Section 5: Relations and Functions

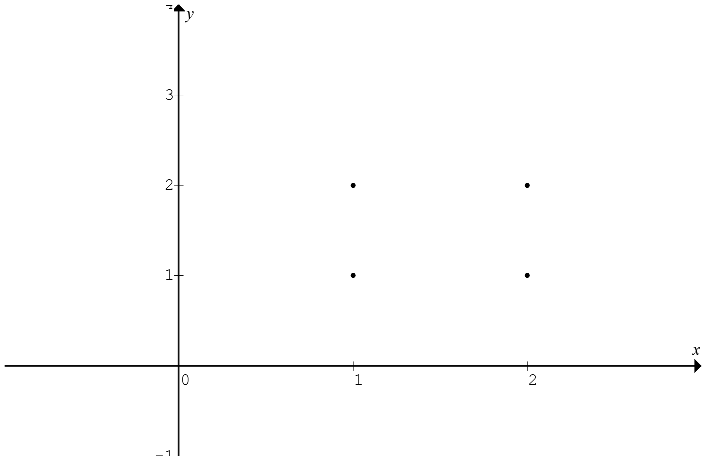
Question1

(a)

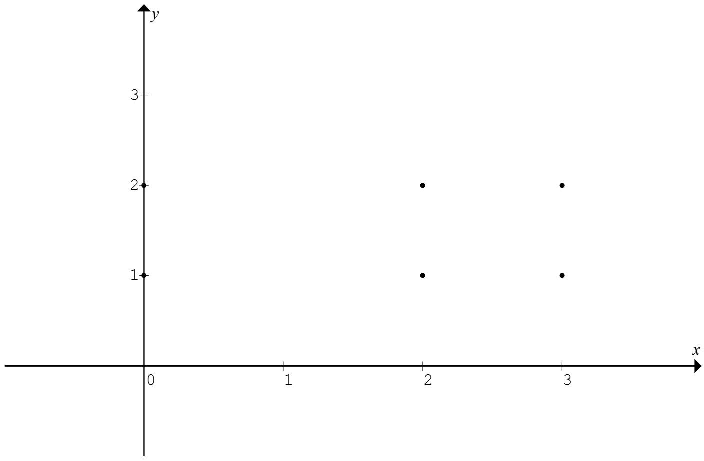
(i) $A \times B = \{(1,0),(1,2),(1,3),(2,0),(2,2),(2,3)\}$



(ii) $A \times A = \{(1,1),(1,2),(2,1),(2,2)\}$



(iii) $B \times A = \{(0,1),(0,2),(2,1),(2,2),(3,1),(3,2)\}$



(b) Is $A \times B \subseteq B \times B$ No

(c) $A \cup B = \{0,1,2,3\}$ $(A \cup B) \times C$

$(A \cup B) \times C = \{(0,a),(0,b),(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$

$(A \times C) = \{(1,a),(1,b),(2,a),(2,b)\}$.

$(B\times C)=\{(0,a),(0,b),(2,a),(2,b),(3,a),(3,b)$

$(A\times C)\cup(B\times C)=\{(0,a),(0,b),(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)$

What do you notice? $(A\cup B)\times C=(A\times C)\cup(B\times C)\}$

(d)

$(A\times B)\times C=\{(1,0,a),(1,0,b),(1,2,a),(1,2,b),(1,3,a),(1,3,b),$
$(2,0,a),(2,0,b),,(2,2,a),(2,2,b)(2,3,a),(2,3,b)\}$

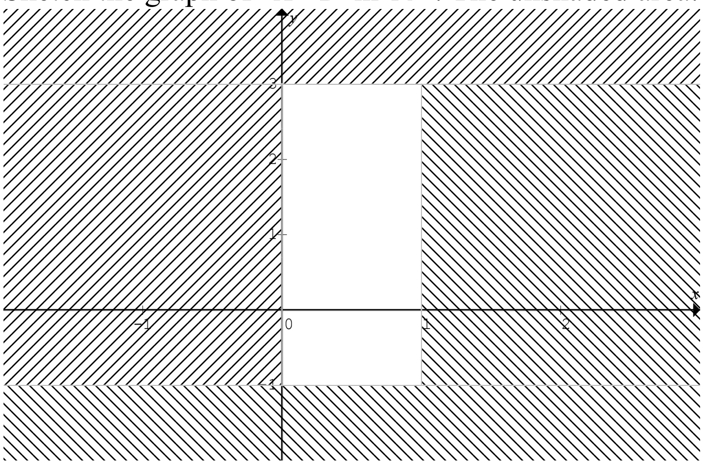
$C\times(A\times A)=\{(a,1,1),(a,1,2),(a,2,1),(a,2,2),$
$(b,1,1),(b,1,2),(b,2,1),(b,2,2)\}$.

Question2 $D\times A=\{(a,1),(b,1),(a,2,),(b,2)\}$.

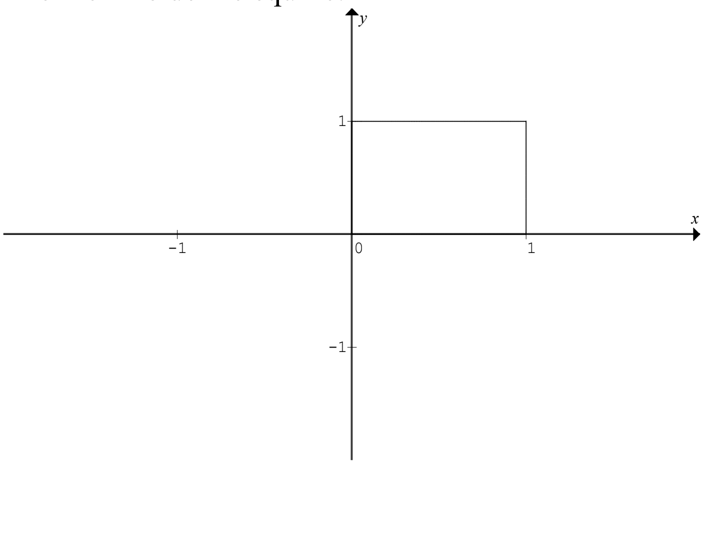
$A\times D=\{(1,a),(1,b),(2,a),(2,b)\}$, they are not equal

Question3 Let $A=\{x\in\mathbb{R}:0<x<1\}$, $B=\{x\in\mathbb{R}:-1<x<3\}$ and $C=\{x\in\mathbb{R}:0\le x\le1\}$.

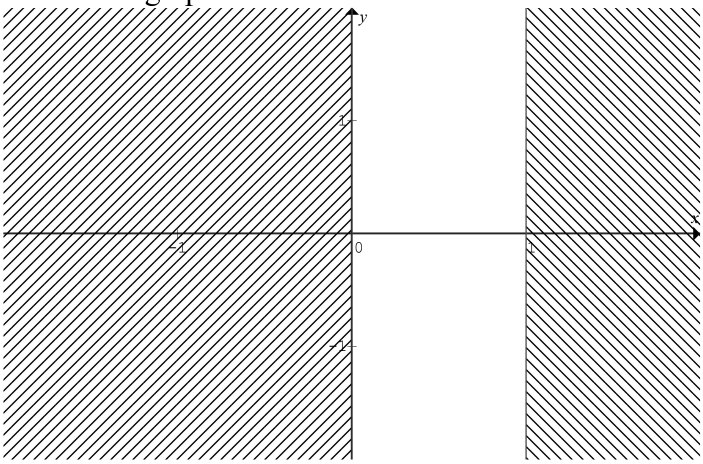
(a) Sketch the graph of $A\times B$ in $\mathbb{R}^2$. The unshaded area:



(b) Sketch the graph of $C\times C$ in $\mathbb{R}^2$. Note: $C\times C$ is called the until square in $\mathbb{R}^2$. The area inside the square:



(c) Sketch the graph of $C \times \mathbb{R}$ in $\mathbb{R}^2$. The unshaded area:



Question4 Let $A = \{a_1, a_2, \ldots a_n\}$ and $B = \{b_1, b_2, \ldots b_m\}$.

(a) There will be mn elements in $A \times B$.

(b)

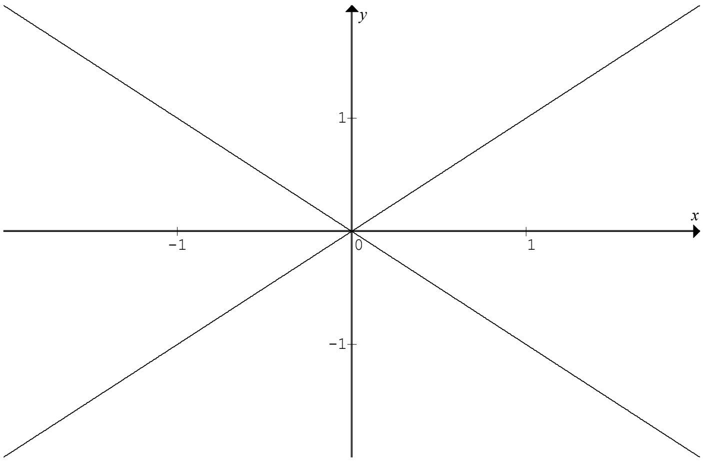
$$
\begin{aligned}
A \times B = \{&(a_1,b_1),(a_1,b_2),\ldots(a_1,b_m),\\
&(a_2,b_1),(a_2,b_2),\ldots(a_2,b_m),\ldots.\\
&(a_n,b_1),(a_n,b_2),\ldots(a_n,b_m)\}
\end{aligned}
$$

Question5

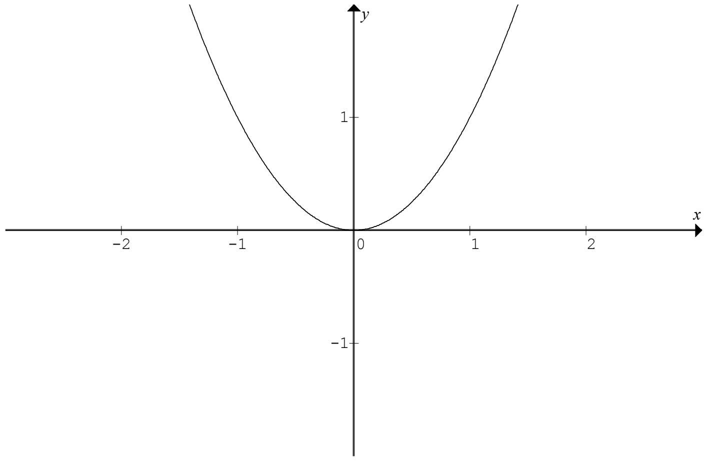
$$
\begin{aligned}
&(a,b) \in (A \cup B) \times C\\
&\quad \Leftrightarrow a \in (A \cup B) \wedge b \in C\\
&\quad \Leftrightarrow (a \in A \vee a \in B) \wedge b \in C\\
&\quad \Leftrightarrow (a \in A \wedge b \in C) \vee (a \in B) \wedge b \in C)\\
&\quad \Leftrightarrow (a,b) \in A \times C \vee (a,b) \in B \times C\\
&\quad \Leftrightarrow (a,b) \in (A \times C) \cup (B \times C)\\
&\quad \therefore (A \cup B) \times C = (A \times C) \cup (B \times C)
\end{aligned}
$$

Question6 Sketch the graphs of the following relations in $\mathbb{R}^2$.

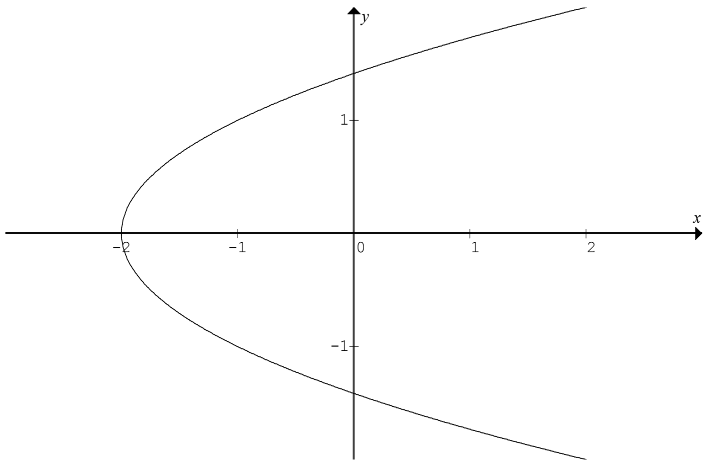
(a) $R_1 = \{(x,y) : x = y \wedge x = -y\}$.



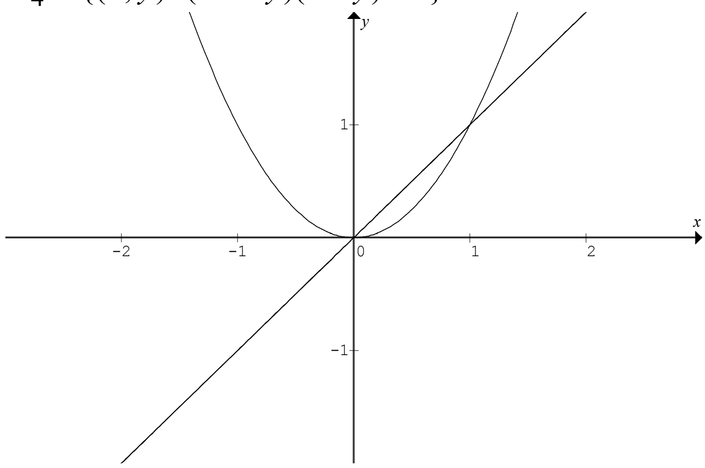
(b) $R_2 = \{(x, y) : x^2 - y = 0\}$.



(c) $R_3 = \{(x, y) : y^2 = 2 + x\}$.

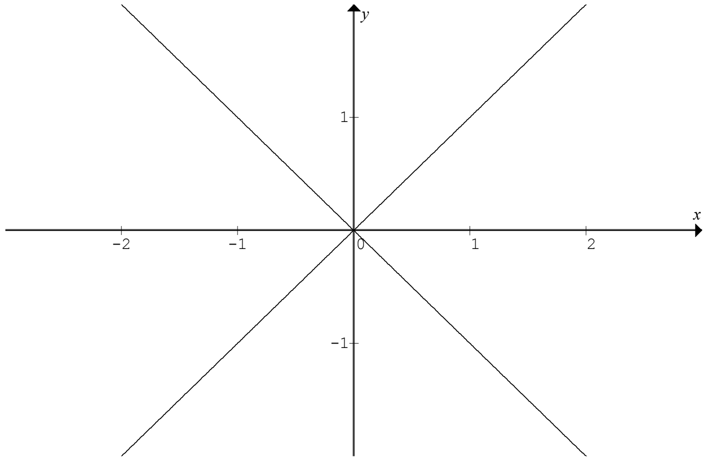


(d) $R_4 = \{(x, y) : (x^2 - y)(x - y) = 0\}$.

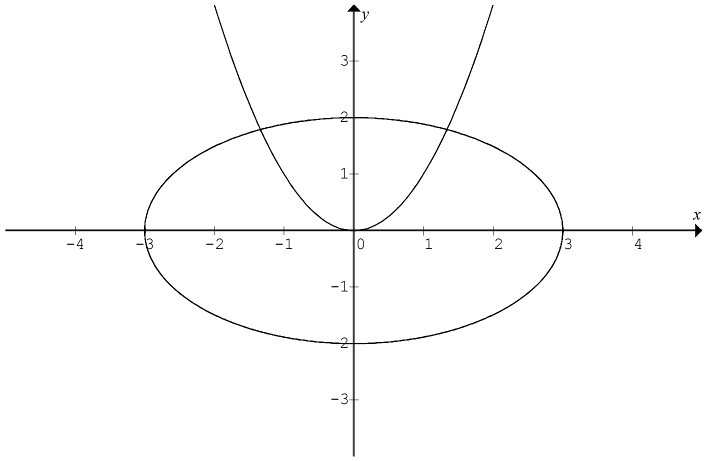


<u>Question7</u> Sketch the graphs of the following relations in $\mathbb{R}^2$.

(a) $R_1 = \{(x, y) : |x| = |y|\}$



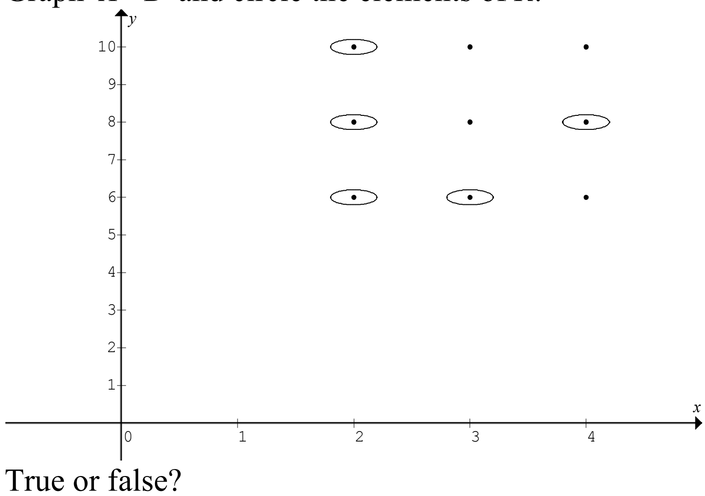
(b) $R_2 = \{(x, y) : (x^2 - y)(4x^2 + 9y^2 - 36) = 0\}$



Question8

(a) $R = \{(2, 6), (2, 8), (2, 10), (3, 6), (4, 8)\}$

(b) Graph $A \times B$ and circle the elements of *R*.



(c) True or false?

(i) $4R6$ False, 4 is not a factor of 6

(ii) $4R8$ True, $8 = 4 \times 2$

(iii) $(3, 8) \in R$ False, 3 is not a factor of 8

(iv) $(2, 10) \in R$ True, $10 = 2 \times 5$

(v) $(4, 12) \in R$ False, $12 \notin B$

Question9

(a) $R \cup S = R$

(b) $R \cap S = S$

Question10 Write down the domain and range of the relation R on the given set A.

$A = \{h : h \text{ is a human being}\}$, $R = \{(h_1, h_2) : h_1 \text{ is the sister of } h_2\}$

Dom $R = \{f \in A : f \text{ is female} \wedge f \text{ has a sibling}\}$,

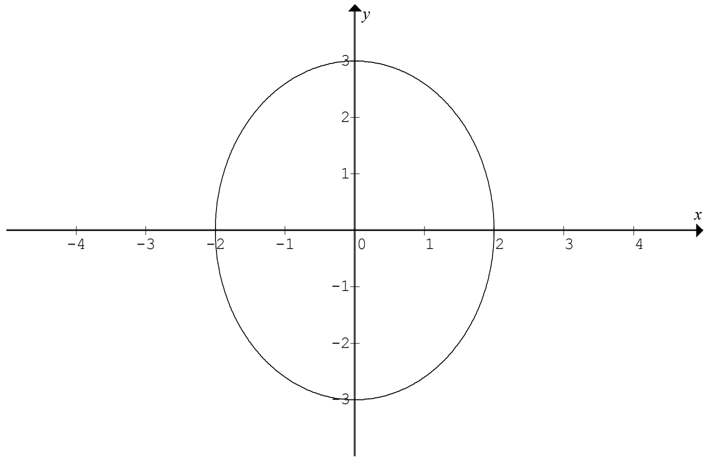
Range $R = \{p \in A : p \text{ has a sister}\}$

Question11 $R = \{(3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)\}$.

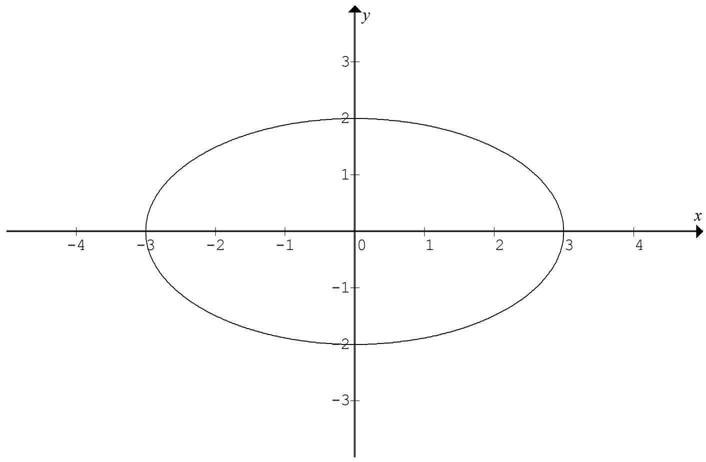
$R^{-1} = \{(4, 3), (5, 3), (6, 3), (5, 4), (6, 4), (6, 5)\}$

Question12 $T^{-1} = \{(x, y) : \dfrac{y^2}{4} + \dfrac{x^2}{9} = 1\}$.

Sketch of T:



Sketch of T^{-1}:



Question13 Determine whether or not the given relation is reflexive, symmetric or transitive. Give a counterexample in each case in which the relation does not satisfy the property.

(a) R_1 on the set $A = \{h : h \text{ is a human being}\}$ given by

$R_1 = \{(h_1, h_2) : h_1 \text{ is the sister of } h_2\}$

R_1 is not reflexive, symmetric or transitive. Consider a family with three siblings, Jane, Mary and John.

R_1 is not reflexive as Jane is not her own sister

R_1 is not symmetric as Jane is John's sister, however John is not Jane's sister

R_1 is not transitive as Jane is Mary's sister and Mary is Jane's sister, however, Jane is not her own sister.

(b) R_2 on the set $A = \{a, b, c, d\}$ given by

$R_2 = \{(a,a),(a,b),(b,a),(b,b),(b,c),(c,b),(c,c),(d,d)\}$

R_2 is reflexive on $A = \{a, b, c, d\}$:

$(a, a) \in \mathrm{R}_2$, $(b, b) \in \mathrm{R}_2$, $(c, c) \in \mathrm{R}_2$ and $(d, d) \in \mathrm{R}_2$.

R_2 is symmetric:

$(a, b) \in \mathrm{R}_2$ and $(b, a) \in \mathrm{R}_2$; $(b, c) \in \mathrm{R}_2$ and $(c, b) \in \mathrm{R}_2$.

All other elements in R_2 are of the form (x, x) so satisfy the symmetry property.

R_2 is not transitive:

$(a, b) \in \mathrm{R}_2$ and $(b, c) \in \mathrm{R}_2$. However, $(a, c) \notin \mathrm{R}_2$.

So transitivity fails.

<u>Question14</u> Determine whether or not the following relation is an equivalence relation.

R on $A = \{0,1,2,3\}$ given by $R = A \times A$.

$R = A \times A = \{(x, y) : x, y \in A\}$. That is, $\forall x, y \in A, (x, y) \in \mathrm{R}$.

R is Reflexive: $\forall x \in A, (x, x) \in A \times A \Rightarrow (x, x) \in \mathrm{R}$.

R is Symmetric:

$$\begin{aligned} &\forall x, y \in A, (x, y) \in A \times A \\ &\quad \Rightarrow y, x \in A \\ &\quad \Rightarrow (y, x) \in A \times A \end{aligned}$$

Thus, $\forall x, y \in A, (x, y) \in R \Rightarrow (y, x) \in R$

R is Transitive:

$\forall x, y, z \in A, (x, y) \in A \times A, (y, z) \in A \times A,$ and $(x, z) \in A \times A.$

Thus, $\forall x, y, z \in A, (x, y) \in \mathrm{R} \wedge (y, z) \in \mathrm{R} \Rightarrow (x, z) \in \mathrm{R}.$

Therefore, the relation R is an equivalence relation on A

<u>Question15</u> Show that the relation R on the set $A = \{0,1,2,3,4\}$ given by

$R = \{(0,0),(0,4),(1,1),(1,3),(2,2),(3,1),(3,3),(4,0),(4,4)\}$ is an equivalence relation.

Find all the classes of R.

R is Reflexive on $A = \{0,1,2,3,4\}$:

$(0, 0) \in R, (1, 1) \in R, (2, 2) \in R, (3, 3) \in R$ and $(4, 4) \in R.$

Thus, $\forall a \in A, (a, a) \in R$.

R is Symmetric:

$(0, 4) \in R$ and $(4, 0) \in R; (1, 3) \in R$ and $(3, 1) \in R$.

All other elements in R are of the form (x, x), so satisfy the symmetry property.

Thus, $\forall a, b \in A, (a, b) \in R \Rightarrow (b, a) \in R$.

R is Transitive:

$(0, 0), (0, 4) \in R$ and $(0, 4) \in R;$

$(0, 4), (4, 0) \in R$ and $(0, 0) \in R;$

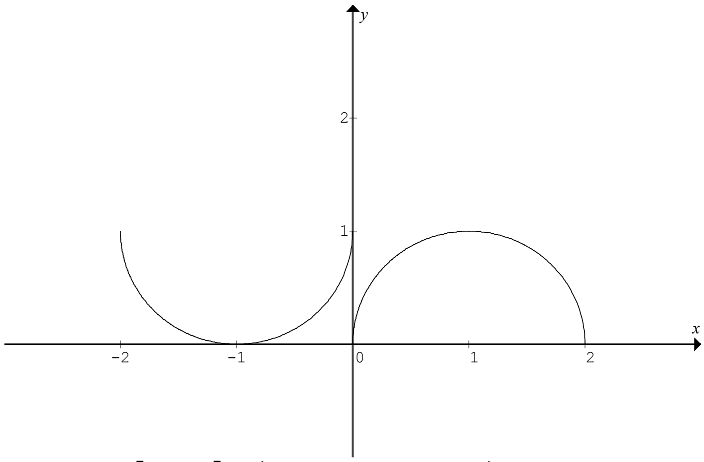
$(0, 4), (4, 4) \in R$ and $(0, 4) \in R;$

$(1, 1), (1, 3) \in R$ and $(1, 3) \in R;$

$(1, 3), (3, 1) \in R$ and $(1, 1) \in R;$

$(1, 3), (3, 3) \in R$ and $(1, 3) \in R$;
$(3, 1), (1, 1) \in R$ and $(3, 1) \in R$;
$(3, 1), (1, 3) \in R$ and $(3, 3) \in R$;
$(3, 3), (3, 1) \in R$ and $(3, 1) \in R$;
$(4, 0), (0, 0) \in R$ and $(4, 0) \in R$;
$(4, 0), (0, 4) \in R$ and $(4, 4) \in R$;
$(4, 4), (4, 0) \in R$ and $(4, 0) \in R$.
Elements of the form (x, x) also satisfy the transitive property.
Thus, $\forall a, b, c \in A, (a, b), (b, c) \in R \Rightarrow (a, c) \in R$.
Therefore, R is an equivalence relation.
$\text{class}(0) = \{0, 4\}$; $\text{class}(1) = \{1, 3\}$; $\text{class}(2) = \{2\}$; $\text{class}(3) = \{1, 3\} = \text{class}(1)$;
$\text{class}(4) = \{0, 4\} = \text{class}(0)$.

Question16 Is the following relation a function? Give brief reason.
R on $[-2, 2] = \{x \in \mathbb{R} : -2 \le x \le 2\}$, where
$R = \{(x, y) : y = \sqrt{1-(x-1)^2} \vee y = 1 - \sqrt{1-(x+1)^2}\}$.



Dom $R = [-2, 2] = \{x \in \mathbb{R} : -2 \le x \le 2\}$.
However, the relation doesn't satisfy the vertical line test as both $(0, 0)$ and $(0, 1)$ are elements of the relation

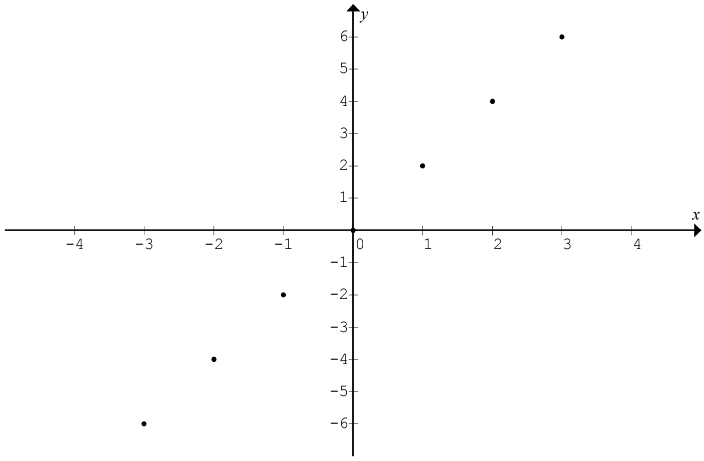
Question17

(i) Let $A = \{1, 5, 9\}$ and $B = \{3, 4, 7\}$. $F_1 \subseteq A \times B$ and $F_1 = \{(1, 7), (5, 3), (9, 4)\}$

(a) F_1 is one-to-one as each element in the range appears only once.

(b) F_1 is onto as range $F_1 = B$

(ii) F_2 on $\mathbb{Z}$ and $F_2 = \{(x, y) : y = 2x\}$



(a) The function satisfies the horizontal line test, thus F_2 is one-to-one

(b) Range $F_2 = \{2n : n \in \mathbb{Z}\} \neq \mathbb{Z}$, thus, F_2 is not onto

Question18 Let $A = \{4,5,6\}$ and $B = \{5,6,7\}$ and define the relations *S* and *T* from *A* to *B* as follows: $S = \{(x, y) : x - y \text{ is even}\}$ and $T = \{(4,6),(6,5),(6,7)\}$.

(a) S^{-1} from *B* to *A*, $S^{-1} = \{(5,5),(6,4),(6,6),(7,5)\}$.

and T^{-1} from *B* to *A*, $T^{-1} = \{(6,4),(5,6),(7,6)\}$

(b) $S = \{(4,6),(5,5),(5,7),(6,6)\}$, $(5,5) \in S$ and $(5,7) \in S$ thus *S* is not a function,

Dom $T = \{4,6\} \neq A$ and $(6,5) \in T$ and ,$(6,7) \in T$, thus *T* is not a function,

$(6,4) \in S^{-1}$ and $(6,6) \in S^{-1}$, thus S^{-1} is not a function

Dom $T^{-1} = \{5,6,7\} = B$ each element in the domain appears only once, thus T^{-1} is a function.

Question19 Simplify the following:

(a) $(1\ \ 3\ \ 4)(3\ \ 2\ \ 4) = (1\ \ 2\ \ 4)$

(b) $(1\ \ 3\ \ 4)^{-1} = (4\ \ 3\ \ 1)$

(c) $(2\ \ 5\ \ 4\ \ 1)^{-1} = (1\ \ 4\ \ 5\ \ 2)$

(d) $(3\ \ 2)(3\ \ 2\ \ 4)(3\ \ 1)(4\ \ 2) = (3\ \ 2\ \ 1\ \ 4)$