

# Amenability and computability

Karol Duda

with an appendix by Karol Duda and Aleksander Ivanov

## Abstract

In this paper we extend the approach of M. Cavaleri to effective amenability to the class of computably enumerable groups, i.e. in particular we do not assume that groups are finitely generated. The main results of the paper concern some new directions in this approach. In the case of computable groups we study decidability of amenability of finitely generated subgroups, complexity of the set of all effective Følner sequences and effective paradoxical decomposition.

In the appendix we attach a version of the paper "On decidability of amenability in computable groups" by K. Duda and A. Ivanov which has been already published.

## 1 Introduction

M. Cavaleri has shown in [10] that every amenable finitely generated recursively presented group has computable Reiter functions and subrecursive Følner functions. Moreover, for a finitely generated recursively presented group with solvable Word Problem, amenability is equivalent to these conditions and in fact it is equivalent to so called effective amenability. The latter means existence of an algorithm which finds  $n$ -Følner sets for all  $n$ .

Since being finitely generated is not necessary for amenability, the question arises what happens if we consider the case of recursively presented groups without the assumption of finite generation. According to the approach of computable algebra the question concerns the class of computably enumerable groups and the subclass of computable groups, which corresponds to decidability of the Word Problem.

The following Theorem generalizes some results of Cavaleri to a case of computably enumerable groups:

**Theorem 1.** *Let  $G$  be a computably enumerable group. The following conditions are equivalent:*

- (i)  $G$  is amenable;
- (ii)  $G$  has computable Reiter functions;
- (iii)  $G$  has subrecursive Følner function.
- (iv)  $G$  is  $\Sigma$ -amenable (see Definition 2.9).

Moreover, computable amenability of  $G$  implies computability of it.

A paradoxical decomposition of a group is a triple  $(K, (A_k)_{k \in K}, (B_k)_{k \in K})$  consisting families  $A$  and  $B$  of subsets of  $G$  indexed by elements of a finite set  $K \subset G$  such that:

$$G = \left( \bigsqcup_{k \in K} kA_k \right) \bigsqcup \left( \bigsqcup_{k \in K} kB_k \right) = \left( \bigsqcup_{k \in K} A_k \right) = \left( \bigsqcup_{k \in K} B_k \right).$$

Here we use the definition given in [11] where some members  $A_k$  or  $B_k$  can be empty. It is equivalent to the traditional one. Thus the existence of such a paradoxical decomposition is opposite to amenability. By demanding families  $(A_k)$  and  $(B_k)$  to consist of computable sets, we introduce an effective paradoxical decomposition. Using an effective version of the Hall's Harem Theorem we prove the following theorem.

**Theorem 2.** *Let  $G$  be a computable group. There is an effective procedure which given  $K_0 \subset G$  such that for some natural  $n$  there is no  $n$ -Følner set with respect to  $K_0$ , finds a finite  $K$  with an effective paradoxical decomposition of  $G$  as above.*

We call such a set  $K_0$  a **witness** of the Banach-Tarski paradox. The question arises, how complex is the family (denoted by  $\mathfrak{W}_{BT}$ ) of such subsets of a computable group? We prove the following theorem.

**Theorem 3.** *For any computable group the family  $\mathfrak{W}_{BT}$  belongs to the class  $\Sigma_2^0$ . In the case of the fully residually free groups the family  $\mathfrak{W}_{BT}$  is computable.*

After this theorem a principal question arises if there is a computable group for which the family  $\mathfrak{W}_{BT}$  is not computable. Moreover it is worth mentioning that the latter condition is equivalent to undecidability of the problem if a finite subset generates an amenable subgroup. The appendix of this paper gives a required example. Using it we also build a finitely presented group with decidable word problem where the family  $\mathfrak{W}_{BT}$  is not computable. This shows that the statement of Theorem 3 cannot be extended to finitely presented groups with decidable word problem.

The paper is organized as follows. Section 2 contains some basic definitions and preliminary observations. In Sections 3 - 4 we generalize Cavaleri's characterizations (using very similar arguments) of some versions of effective amenability to the case of computably enumerable groups. In these sections we prove Theorem 1. In Section 5 we study complexity of the set of effective Følner sequences for computable groups. Sections 6 - 8 are dedicated to the effectiveness of a paradoxical decomposition. In Section 6 we introduce and prove an effective version of the Hall's Harem Theorem. We use it to prove Theorem 2 in Section 7. In Section 8 we introduce a notion of witnesses of the Banach-Tarski paradox and study the complexity of the set of witnesses (Theorem 3).

We mention papers of I. Bilanovic, J. Chubb and S. Roven [7] and of A. Darbinyan [17] as other papers in the field (applied to other group-theoretic properties).

The material of this paper is based on the master thesis of the author, written under supervision of Aleksander Ivanov. The author is grateful to him for support. The author is grateful to M. Cavaleri and T. Ceccherini-Silberstein for reading the paper and helpful remarks. In particular, the idea of Proposition 8.4 belongs to M. Cavaleri.

## 2 Preliminaries

From now on we identify each finite set  $F \subset \mathbb{N}$  with its Gödel number. For any sets  $X$  and  $Y$  we will write  $X \subset\subset Y$  to denote that  $X$  is a finite subset of  $Y$ . For any  $i \in \mathbb{N}$ , we denote the set  $\{1, 2, \dots, i\}$  by  $[i]$ . Throughout this paper,  $G$  is a countable group without any presumption about its generating set.

### 2.1 Computability

We use standard material from the computability theory (see [31]). A function is **subrecursive** if it admits a computable total upper bound. Sequence  $(n_i)_{i \in \mathbb{N}}$  of natural numbers is called **effective**, if the function  $k \rightarrow n_k$  is recursive.

Let  $G$  be a countable group generated by some  $X \subseteq G$ . The group  $G$  is called **recursively presented** (see Section IV.3 in [23]) if  $X$  can be identified with  $\mathbb{N}$  (or with some  $\{0, \dots, n\}$ ) so that  $G$  has a recursively enumerable set of relators in  $X$ . Below we give an equivalent definition, see Definition 2.3. It is justified by a possibility identification of the whole  $G$  with  $\mathbb{N}$ . We follow the approach of [21].

**Definition 2.1.** Let  $G$  be a group and  $\nu : \mathbb{N} \rightarrow G$  be a surjective function. We call the pair  $(G, \nu)$  a **numbered group**. The function  $\nu$  is called a **numbering** of  $G$ . If  $g \in G$  and  $\nu(n) = g$ , then  $n$  is called a number of  $g$ .

**Definition 2.2.** A numbered group  $(G, \nu)$  has a **computable presentation** if  $\nu$  is a bijection and the set

$$\text{MultT} := \{(i, j, k) : \nu(i)\nu(j) = \nu(k)\}$$

is computable (= decidable).

Any finitely generated group with decidable word problem obviously has a computable presentation. This also holds in the case of the free group  $\mathbb{F}_\omega$  with the free basis  $\{x_0, \dots, x_i, \dots\}$ . If we fix a computable presentation  $(\mathbb{F}_\omega, \nu_F)$  then for every recursively presented group  $G = \langle X \rangle$  and a natural homomorphism  $\rho : \mathbb{F}_\omega \rightarrow G$  (taking  $\omega$  onto  $X$ ) we obtain a numbering  $\nu = \rho \circ \nu_F$  which satisfies the following definition.

**Definition 2.3.** A numbered group  $(G, \nu)$  is **computably enumerable** if the set

$$\text{MultT} := \{(i, j, k) : \nu(i)\nu(j) = \nu(k)\}$$

is computably enumerable.

*Remark 2.4.* Let  $(G, \nu)$  be a computably enumerable group.

- (i) There exists a computable function  $\star : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $x, y \in \mathbb{N}$  the equality  $\nu(x)\nu(y) = \nu(x \star y)$  holds.
- (ii) There is a computable function which for every  $x \in \mathbb{N}$  finds  $y \in \mathbb{N}$  with  $\nu(x)\nu(y) = 1$ , i.e. a number  $(\nu(x))^{-1}$ . We denote it by  $x^*$ .
- (iii) The sets  $\{n : \nu(n) = 1\}$  and  $\{(n_1, n_2) : \nu(n_1) = \nu(n_2)\}$  are computably enumerable.

*Remark 2.5.* If  $(G, \nu)$  is a numbered group and the set **MultT** from Definition 2.3 is computable, then

- (i)  $G$  has a computable presentation (possibly under another numbering). Indeed, in this case the set of the smallest numbers of the elements of  $G$  is computable. Enumerating this set by natural numbers we obtain a required 1-1-enumeration.
- (ii) In this case we also have that the set  $\{(n_1, n_2) : \nu(n_1) = \nu(n_2)\}$  is computable.

Groups  $(G, \nu)$  as in this remark are called **computable** groups. They correspond to groups with solvable word problem. In this case the numbering  $\nu$  is called a **constructivization**.

## 2.2 Amenability

Let  $G$  be a group, and  $D \subset\subset G$ . Given  $n \in \mathbb{N}$ , we say that a subset  $F \subset\subset G$  is an  **$n$ -Følner set** with respect to  $D$  if

$$\forall x \in D \quad \frac{|F \setminus xF|}{|F|} \leq \frac{1}{n} \quad (1)$$

We denote by  $\mathfrak{Fol}_{G,D}(n)$  the set of all  $n$ -Følner sets with respect to  $D$ . Moreover, we say that a sequence  $(F_j)_{j \in \mathbb{N}}$  of non-empty finite subsets of  $G$  is a **Følner sequence** if for every  $g \in G$  the following condition holds:

$$\lim_{j \rightarrow \infty} \frac{|F_j \setminus gF_j|}{|F_j|} = 0. \quad (2)$$

We call the binary function:

$$Fol_G(n, D) = \min\{|F| : F \subseteq G \text{ such that } F \in \mathfrak{Fol}_{G,D}(n)\}, \quad (3)$$

where the variable  $D$  corresponds to finite sets, the **Følner function of  $G$** , [33].

It is easy to see that existence of Følner sets for every  $D$  and all  $n$  is equivalent to existence of a Følner sequence, i.e.  $G$  admits a Følner sequence if and only if  $Fol_G(n, D) < \infty$  for all finite  $D \subset G$  and  $n \in \mathbb{N}$ . In fact this is the **Følner condition of amenability**.

**Definition 2.6.** A summable non-zero function  $h : G \rightarrow \mathbb{R}_+$ ,  $\|h\|_{1,G} < \infty$ , is  **$n$ -invariant** with respect to  $D$ , if

$$\forall x \in D \quad \frac{\|h - {}_x h\|_{1,G}}{\|h\|_{1,G}} < \frac{1}{n}, \quad (4)$$

where  ${}_x h(g) := h(x^{-1}g)$ .

We denote by  $\mathfrak{Reit}_{G,D}(n)$  the set of all summable non-zero functions from  $G$  to  $\mathbb{R}_+$ , which are  $n$ -invariant with respect to  $D$ .

The following facts are well known and/or easy to prove.

**Lemma 2.7.** Let  $F, D \subset\subset G$ .

- (i)  $F \in \mathfrak{Fol}_{G,D}(n) \implies \forall g \in G \quad Fg \in \mathfrak{Fol}_{G,D}(n)$
- (ii)  $F \in \mathfrak{Fol}_{G,D}(n) \iff \forall x \in D \quad \frac{|F \cap xF|}{|F|} > 1 - \frac{1}{n}$
- (iii)  $F \in \mathfrak{Fol}_{G,D}(2n) \iff \chi_F \in \mathfrak{Reit}_{G,D}(n)$
- (iv) If  $h \in \mathfrak{Reit}_{G,D}(n)$  has a finite support then there exists  $F \subset \text{Supp}(h)$  such that for all  $x \in D$  following holds:

$$\frac{|F \setminus xF|}{|F|} < \frac{|D|}{2n}.$$

## 2.3 Effective amenability

In this section  $(G, \nu)$  is a numbered group.

**Definition 2.8.** We say that  $(G, \nu)$  has **computable Reiter functions**, if there exists an algorithm which, for every  $n \in \mathbb{N}$  and any finite set  $D \subset \mathbb{N}$  finds  $f : \mathbb{N} \rightarrow \mathbb{Q}_+$ , such that  $|Supp(f)| < \infty$  and

$$\forall x \in D, \quad \frac{\|\nu_{G*}(f) - \nu_{(x)} \nu_{G*}(f)\|_{1,G}}{\|\nu_{G*}(f)\|_{1,G}} < \frac{1}{n},$$

where  $\nu_{G*}(f)(g) := \sum_{i \in \nu^{-1}(g)} f(i)$ .

In the case of the Følner condition of amenability, we consider three types of effectiveness.

**Definition 2.9.** The group  $(G, \nu)$  is  **$\Sigma$ -amenable** if there exists an algorithm which for all pairs  $(n, D)$ , where  $n \in \mathbb{N}$  and  $D \subset \mathbb{N}$ , finds a set  $F \subset \mathbb{N}$  containing a subset  $F'$ , such that  $\nu(F') \in \mathfrak{Fol}_{G, \nu(D)}(n)$ .

**Definition 2.10.** We say that  $(G, \nu)$  has **computable Følner sets** if there exists an algorithm which, for all pairs  $(n, D)$ , where  $n \in \mathbb{N}$  and  $D \subset \mathbb{N}$ , finds a finite set  $F \subset \mathbb{N}$  such that  $\nu(F) \in \mathfrak{Fol}_{G, \nu(D)}(n)$ .

**Definition 2.11.** The group  $(G, \nu)$  is **computably amenable** if there exists an algorithm which for all pairs  $(n, D)$ , where  $n \in \mathbb{N}$  and  $D \subset \mathbb{N}$ , finds a set  $F \subset \mathbb{N}$  such that  $\nu(F) \in \mathfrak{Fol}_{G, \nu(D)}(n)$  and  $|F| = |\nu(F)|$ .

## 3 Effective amenability of computably enumerable groups

The main result of this section, Theorem 3.2, is a natural generalization of a theorem of M. Cavaleri from [10] (Theorem 3.1) to the case of groups which are not finitely generated. In fact we use the same arguments (appropriately adapted to our case). Throughout this section we assume that  $(G, \nu)$  is a computably enumerable group.

We start with some preliminary material concerning Reiter functions and partitions. Let  $X$  be a nonempty set. The family of sets  $\Pi$  is a **partition of a set**  $X$ , if and only if all of the following conditions hold:

1.  $\emptyset \notin \Pi$ ;
2.  $\bigcup_{A \in \Pi} A = X$ ;
3.  $\forall A, B \in \Pi, A \neq B \implies A \cap B = \emptyset$ .

Partition  $\Pi'$  is finer than partition  $\Pi$  (denoted by  $\Pi' \leq \Pi$ ), if for all  $A' \in \Pi'$  there exists  $A \in \Pi$ , such that  $A' \subset A$ .

Let  $f : \mathbb{N} \rightarrow \mathbb{Q}_+$  be a function with finite support  $F$ . Let  $D$  be a finite subset of  $\mathbb{N}$ . With every partition  $\Pi$  of the set  $F$  and every  $x \in D$  we associate the positive rational number:

$$M_{\Pi}^x(f) := \frac{\sum_{V \in \Pi} |\sum_{v \in V} (f(v) - f(x^* \star v))|}{\sum_{v \in F} f(v)},$$

where functions  $\star$  and  $*$  are taken from Remark 2.4. We denote by  $P$  the canonical partition of the set  $F$ , i.e. the partition into sets  $\{\nu^{-1}(\nu(k)), k \in F\} \cap F$ . Then for every  $x \in D$  we have

$$M_P^x(f) = \frac{\|\nu_{G*}(f) - \nu_{(x)} \nu_{G*}(f)\|_{1,G}}{\|\nu_{G*}(f)\|_{1,G}}. \quad (5)$$

By the triangle inequality for any two partitions  $\Pi, \Pi'$  of set  $F$ ,  $\Pi \leq \Pi'$  implies  $M_{\Pi}^x(f) \geq M_{\Pi'}^x(f)$ . In particular, for any partition  $\Pi \leq P$  and any  $x \in D$  the following inequality holds:

$$M_{\Pi}^x(f) \geq M_P^x(f). \quad (6)$$

**Lemma 3.1.** *Let  $(G, \nu)$  be a computably enumerable group. There exists a computable enumeration of the set of all triples  $(n, D, f)$ , where  $D \subset \mathbb{N}$  and  $f : \mathbb{N} \rightarrow \mathbb{Q}_+$  is a finitely supported function, such that  $\nu_{G*}(f) \in \mathfrak{Reit}_{G, \nu(D)}(n)$ .*

*Proof.* We apply the method of Theorem 3.1((i)  $\rightarrow$  (iv)) of [10]. Let us fix an enumeration of functions  $f_i$  with finite support and the corresponding enumeration of all triples of the form  $(n_i, D_j, f_k)$ . The following procedure, denoted below by  $\kappa(n, D, f)$ , determines triples satisfying the condition of the lemma.

We define the algorithm  $\kappa(n, D, f)$  as follows. For an input  $f$  let  $F = \text{supp} f$  and  $P_0 := \{\{x\} : x \in F\}$ , i.e. the finest partition of  $F$ . Let us fix an enumeration of the set  $\{(n_1, n_2) : \nu(n_1) = \nu(n_2)\}$ . Then on the  $m$ -th step of this enumeration we are trying to merge elements of the partition  $P_{m-1}$  obtained at step  $m-1$ . We do so when we meet  $(n_1, n_2)$ , such that  $|V_i \cap \{n_1, n_2\}| = |V_j \cap \{n_1, n_2\}| = 1$  for some pair  $V_i, V_j \in P_{m-1}$ . In this case we just merge this pair. We see that  $P_m \leq P$ . Then we verify if  $M_{P_m}^x(f) \leq \frac{1}{n}$  for all  $x \in D$ . We stop when these inequalities hold or when  $P_m = P$ . In the former case by (5) and (6) the function  $\nu_{G*}(f)$  is  $n$ -invariant. If there exist  $x$ , such that  $M_{P_m}^x(f) > \frac{1}{n}$  and  $P_m = P$ , then the function  $\nu_{G*}(f)$  is not  $n$ -invariant.  $\square$

The following theorem is a part of Theorem 1 from the introduction.

**Theorem 3.2.** *Let  $(G, \nu)$  be a computably enumerable group. Then the following conditions are equivalent:*

- (i)  $(G, \nu)$  is amenable;
- (ii)  $(G, \nu)$  has a subrecursive Følner function;
- (iii)  $(G, \nu)$  is  $\Sigma$ -amenable;
- (iv)  $(G, \nu)$  has computable Reiter functions.

*Proof.* It is clear that (iii)  $\implies$  (ii)  $\implies$  (i).

(iv)  $\implies$  (iii). By Definition 2.8 for all  $n \in \mathbb{N}$  and every  $D \subset \subset \mathbb{N}$  we find a function  $f : \mathbb{N} \rightarrow \mathbb{Q}^+$ ,  $|\text{supp}(f)| < \infty$ , such that  $\nu_{G*}(f) \in \mathfrak{Reit}_{G,D}$ . Denote  $F := \text{supp}(f)$ . By Lemma 2.7 (iv), there exists  $\epsilon \in \mathbb{R}^+$  such that  $\{g \in G : \nu_{G*}(f)(g) > \epsilon\}$  contains a subset that belongs to  $\mathfrak{Fol}_{G, \nu(D)}(n)$ . Since  $\{g \in G : \nu_{G*}(f)(g) > \epsilon\} \subset \nu(F)$ , then there exists  $F' \subseteq F$  such that  $\nu(F')$  satisfies the Følner condition.

To prove (i)  $\implies$  (iv) let us assume that the group  $G$  is amenable. Therefore for any  $n$  and  $D$  there exists  $F \subset \subset \mathbb{N}$  such that  $\nu(F) \in \mathfrak{Fol}_{G, \nu(D)}(2n)$  and  $|F| = |\nu(F)|$ . Since  $\nu$  is injective on  $F$ ,  $\nu_{G*}(\chi_F) = \chi_{\nu(F)} \in \mathfrak{Reit}_{G,D}(n)$ . We fix an enumeration of finite subsets of  $\mathbb{N} : F_1, F_2, \dots$  and we start the algorithms  $\kappa(n, D, \chi_{F_1}), \kappa(n, D, \chi_{F_2}), \dots$  constructed in Lemma 3.1, until one of them stops giving us a Reiter function for  $\nu(D)$ .  $\square$

## 4 Effective amenability of computable groups

The main results of this section, correspond to Theorem 4.1 and Corollary 4.2 of M. Cavaleri from [10]. In the proof we will use functions  $\star$  and  $*$  from Remark 2.4.

**Theorem 4.1.** *Let  $(G, \nu)$  be a computably enumerable group. The following conditions are equivalent:*

- (i)  $(G, \nu)$  is amenable and computable;
- (ii)  $(G, \nu)$  is computably amenable (Definition 2.11).

*Proof.* (i)  $\implies$  (ii). Suppose that  $(G, \nu)$  is amenable and computable. Let  $D \subset \subset \mathbb{N}$ . According the enumeration of all finite sets for every  $F \subset \subset \mathbb{N}$  we verify if the conditions of (ii) are satisfied. Verifying all equalities of the form  $\nu(f_i)\nu(d_k) = \nu(f_j)$ , where  $f_i, f_j \in F$  and  $d_k \in D$ , we can algorithmically check if  $\nu(F) \in \mathfrak{Fol}_{G, \nu(D)}(n)$ . Verifying all equalities of the form  $\nu(f_k) = \nu(f_l)$ , where  $f_k, f_l \in F$ , we can check if  $|F| = |\nu(F)|$ . Since  $(G, \nu)$  is amenable we eventually find the required  $F$ .

(ii)  $\implies$  (i). Our proof is a modification of the construction of Theorem 4.1 from [10]. It is clear that the existence of an algorithm for (ii) implies amenability of  $(G, \nu)$ . Therefore we only need to show that  $(G, \nu)$  is computable. It is sufficient to show that for any  $n_1, n_2, n_3 \in \mathbb{N}$  we can check if  $\nu(n_1)\nu(n_2) = \nu(n_3)$ .

Fix  $n_1, n_2, n_3$ . Let  $D$  be the set  $\{n_1, n_2, n_3\}$ . We use the algorithm for (ii) to find a set  $F$  corresponding to 4 and  $D$ , i.e.  $\nu(F) \in \mathfrak{Fol}_{G, \nu(D)}(4)$  and  $|F| = |\nu(F)|$ . Let  $F = \{f_1, f_2, \dots, f_k\}$ .

We fix an enumeration of the set of triples **MultT**. Using it we will enumerate the (directed) graph of the action by multiplication of  $\nu(n_1), \nu(n_2)$  and  $\nu(n_3)$  on the set  $\nu(F)$ . We start by setting  $\Sigma_1^0 = \Sigma_2^0 = \Sigma_3^0 = \emptyset$ .

At the  $m$ -th step of the construction we verify if the  $m$ -th triple of **MultT** is a triple of the form  $n_l \star f_i = f_j$  for  $l = 1, 2, 3$ . In this case we extend the corresponding  $\Sigma_l^{m-1}$  by the pair  $(i, j)$ . The graphs after step  $m$  are denoted by  $\Sigma_l^m$ ,  $l = 1, 2, 3$ .

Next we verify  $\min_l |\Sigma_l^m| > \frac{3k}{4}$ . If the inequality holds we stop the construction with  $\Sigma_l := \Sigma_l^m$ .

Since

$$\frac{|\{(i, j) : \nu(n_l \star f_i \star f_j^*) = 1\}|}{k} \geq \frac{|\nu(F) \cap \nu(n_l)\nu(F)|}{|\nu(F)|} > \frac{3}{4}$$

the procedure stops at some step  $m$ .

Let

$$\Sigma = \{i \in [k] : \exists j_1, j_2 \in [k], (i, j_1) \in \Sigma_1, (j_1, j_2) \in \Sigma_2, (i, j_2) \in \Sigma_3\}.$$

If  $\nu(n_1)\nu(n_2) = \nu(n_3)$  then for all  $i \in [k]$ ,  $\nu(n_1)\nu(n_2)\nu(f_i) = \nu(n_3)\nu(f_i)$ . Since each of the partial permutations  $\Sigma_1, \Sigma_2, \Sigma_3$  can be undefined for at most  $\frac{1}{4}|F|$  elements from  $|F|$ , then  $\nu(n_1)\nu(n_2) = \nu(n_3)$  implies  $|\Sigma| \geq \frac{1}{4}|F|$ . If equality does not hold, then  $\Sigma$  is an empty set. When we see which possibility holds we decide if  $\nu(n_1)\nu(n_2) = \nu(n_3)$ . □

The proof of Theorem 4.1 gives the following interesting observation.

**Corollary 4.2.** *Let  $(G, \nu)$  be a computably enumerable, amenable group. If for some  $n \geq 4$  there exists an algorithm, which for every  $D \subset \mathbb{N}$  finds a set  $F \subset \mathbb{N}$  such that  $\nu(F) \in \text{Føl}_{G, \nu(D)}(n)$  and  $|F| = |\nu(F)|$ , then  $G$  is computable.*

Using Theorem 4.1 we deduce a version of Theorem 3.2 for computable groups. This finishes the proof of Theorem 1.

**Theorem 4.3.** *Let  $(G, \nu)$  be a computable group. Then the following conditions are equivalent:*

- (i)  $(G, \nu)$  is amenable;
- (ii)  $(G, \nu)$  is computably amenable;
- (iii)  $(G, \nu)$  has computable Følner sets;
- (iv)  $(G, \nu)$  has computable Reiter functions;
- (v)  $(G, \nu)$  has subrecursive Følner function.

*Proof.* By Theorem 4.1 we have (i)  $\Rightarrow$  (ii) and by Lemma 2.7(iv) we have (iv)  $\Rightarrow$  (iii). Both (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (i) and (ii)  $\Rightarrow$  (v)  $\Rightarrow$  (i) are easy to see.

It follows that we only need to show that (ii)  $\Rightarrow$  (iv). We start with a finite set  $D$  and use an algorithm of (ii) to find a set  $F$  corresponding to  $2n$ . Then the characteristic function  $\chi_F$  can be taken as  $f$  from Definition 2.8. Indeed since the function  $\nu$  is injective on  $F$  then  $\nu_{G*}(\chi_F)$  is the characteristic function of  $\nu(F)$ , which is  $n$ -invariant by Lemma 2.7(iii). □

## 5 Effective Følner sequence

Let  $(G, \nu)$  be a computable group. Since in the case of computable groups we can assume that function  $\nu$  is 1-1, we identify the set  $G$  with  $\mathbb{N}$  and subsets  $F$  of  $N$  with  $\nu(F) \subset G$ .

The **effective Følner sequence** of the group  $(G, \nu)$ , is an effective sequence  $(n_j)_{j \in \mathbb{N}}$  such that for each  $j$ ,  $n_j$  is a Gödel number of the set  $F_j$ , with  $(F_j)_{j \in \mathbb{N}}$  being a Følner sequence.

In the previous section we have shown that amenability of  $(G, \nu)$  is equivalent to computable amenability. Note that this is also equivalent to existence of effective Følner sequences. Indeed, given  $j$  we use the algorithm for computable amenability and compute the Gödel number  $n_j$  of some  $F_j \in \mathfrak{Føl}_{G, [j]}(j)$ . Clearly, the sequence  $(F_j)_{j \in \mathbb{N}}$  is a Følner sequence and a sequence  $(n_j)_{j \in \mathbb{N}}$  is an effective Følner sequence.

The following Theorem classifies the set of all effective Følner sequences of the group  $(G, \nu)$  in the Arithmetical Hierarchy. The idea of it belongs to Aleksander Ivanov.

**Theorem 5.1.** *Let  $(G, \nu)$  be a computable group. The set of all effective Følner sequences of  $(G, \nu)$  belongs to the class  $\Pi_3^0$ . Moreover, for  $G = \bigoplus_{n \in \omega} \mathbb{Z}$  it is a  $\Pi_3^0$ -complete set.*

*Proof.* Let  $\varphi(x, y)$  be a universal recursive function, and  $\varphi_x(y) = \varphi(x, y)$  be a recursive function with a number  $x$ . We identify effective Følner sequences with numbers of recursive functions which produce these sequences. The set of these numbers is denoted by  $\mathfrak{F}_{seq}(G)$ . Then  $m$  is a number of an effective Følner sequence if and only if the following formula holds:

$$(\phi(m, y) \text{ is a total function}) \wedge (\forall g \in G)(\forall n)(\exists l)(\forall k) \left( k > l \wedge (\phi(m, k) = f) \right. \\ \left. \wedge (f \text{ is a Gödel number of } F_j) \rightarrow \frac{|F_j \setminus gF_j|}{|F_j|} < \frac{1}{n} \right). \quad (7)$$

Given number  $f$  the inequality  $\frac{|F_j \setminus gF_j|}{|F_j|} < \frac{1}{n}$  can be verified effectively. Since the set of numbers of all total functions belongs to the class  $\Sigma_2^0$  it is easy to see that the set of all  $m$  which satisfy (7) is a  $\Pi_3^0$  set. This proves the first part of the theorem.

We remind the reader that  $W_x = \text{Dom} \varphi_x$  is the computably enumerable set with a number  $x$ . The set  $\overline{Cof} = \{e : \forall n \ W_{\varphi_e(n)} \text{ is finite}\}$ , is known to be a  $\Pi_3^0$ -complete set ([31], p. 87). To prove the second part of the theorem, assume that  $G = \bigoplus_{n \in \omega} \mathbb{Z}$ . Let us show that the set  $\overline{Cof}$  is reducible to  $\mathfrak{F}_{seq}(G)$ . For each  $e$  let us fix a computable enumeration of the set  $\{(n, x) : x \in W_{\varphi_e(n)}\}$ . We can assume that this enumeration is without repetitions.

We present  $\bigoplus_{n \in \omega} \mathbb{Z}$  as  $\bigoplus_{n \in \omega} \langle g_n \rangle$ . We shall construct a sequence  $\{F_s^e\}$  such that  $e \in \overline{Cof}$  iff  $\{F_s^e\}$  is a Følner sequence.

For a given  $s$ , we use the enumeration of the set  $\{(n, x) : x \in W_{\varphi_e(n)}\}$  to find the element  $(n_s, x)$  with the number  $s$ . For each  $i = 1, \dots, s$  such that  $i \neq n_s$  let  $F_{s,i} = \{g_i, g_i^2 \dots g_i^s\}$ . For  $i = n_s$  we put  $F_{s,i} = \{g_i\}$ . Let  $F_s^e = \bigoplus_1^s F_{s,i}$ . Then in the former case  $F_s^e$  is an  $s$ -Følner set with respect to  $g_i$  and in the latter case  $F_s^e$  is not a 2-Følner set with respect to  $g_i$ . This ends the construction.

*Case 1.  $e \notin \overline{Cof}$ .* There exists  $n'$  such that  $W_{\varphi_e(n')}$  is an infinite set. Therefore there exist an increasing sequence  $\{s_i\}$  and the number  $i'$  such that for all  $i > i'$ ,  $F_{s_i}^e$  is not a 2-Følner set with respect to  $g_{n'}$ . Clearly the number of a sequence  $\{F_s^e\}$  does not belong to the set of numbers of a Følner sequences.

*Case 2.  $e \in \overline{Cof}$ .* For all  $n$ ,  $W_{\varphi_e(n)}$  is a finite set. Therefore for all  $n$ , there exists the number  $s'$  such that for all  $s > s'$ ,  $F_s^e$  is an  $s$ -Følner set with respect to  $g_n$ . This sequence is a Følner sequence.

Since for every  $e$  the number of the algorithm producing  $\{F_s^e\}$  can be effectively found it follows that the set  $\overline{Cof}$  is reducible to  $\mathfrak{F}_{seq}(G)$ , which completes the proof.  $\square$

## 6 An effective version of Hall's Harem Theorem

In this section we generalize the work of Kierstead [22] concerning an effective version of the Hall's Theorem. These results will be applied in the next section to effective paradoxical decompositions. Below we follow the presentation of [22].

A graph  $\Gamma = (V, E)$  is called a **bipartite graph** if the set of vertices  $V$  is partitioned into sets  $A$  and  $B$  in such way, that the set of edges  $E$  is a subset of  $A \times B$ . We denote such a bipartite graph by  $\Gamma = (A, B, E)$ . The set  $A$  (resp.  $B$ ) is called the set of **left** (resp. **right**) **vertices**.

From now on we concentrate on bipartite graphs. Although our definitions concern this case they usually have obvious extensions to all ordinary graphs. Let  $\Gamma = (A, B, E)$ . We will say that an edge  $(a, b)$  is **adjacent** to vertices  $a$  and  $b$ . In this case we say that  $a$  and  $b$  are adjacent. We also say that two edges  $(a, b), (a', b') \in E$  are **adjacent** if they have a common adjacent vertex.

Given a vertex  $x \in A \cup B$  the **neighbourhood** of  $x$  is a set

$$N_\Gamma(x) = \{y \in A \cup B : (x, y) \in E\}.$$

For subsets  $X \subset A$  and  $Y \subset B$ , we define the neighbourhood  $N_\Gamma(X)$  of  $X$  and the neighbourhood  $N_\Gamma(Y)$  of  $Y$  by

$$N_\Gamma(X) = \bigcup_{x \in X} N_\Gamma(x) \text{ and } N_\Gamma(Y) = \bigcup_{y \in Y} N_\Gamma(y).$$

We drop the subscript  $\Gamma$  if it is clear from the context.

The subset  $X$  of  $A$  (resp.  $Y$  of  $B$ ) is called **connected** if for all  $x, x' \in X$  (resp.  $y, y' \in Y$ ) there exist a path  $x = p_0, p_1, \dots, p_k = x'$  in  $\Gamma$  such that for all  $i$   $p_i \in X \cup N_\Gamma(X)$ .

We say that  $\Gamma$  is **locally finite** if the set  $N(x)$  is finite for all  $x \in A \cup B$ . If  $\Gamma$  is locally finite then the sets  $N(X)$  and  $N(Y)$  are finite for all finite subsets  $X \subset A$  and  $Y \subset B$ .

For a given vertex  $v$  a **star** of  $v$  is a subgraph  $S = (V', E')$  of  $\Gamma$ , with  $V' = \{v\} \cup N_\Gamma(v)$  and  $E' = \{(v, v') \in E\}$ .

A **matching** (a **(1,1)-matching**) from  $A$  to  $B$  is a subset  $M \subset E$  of pairwise nonadjacent edges. A matching  $M$  is called **left-perfect** (resp. **right-perfect**) if for all  $a \in A$  (resp.  $b \in B$ ) there exists exactly one  $b \in B$  (resp.  $a \in A$ ) with  $(a, b) \in M$ . The matching  $M$  is called **perfect** if it is both right and left-perfect.

We now introduce perfect  $(1, k)$ -matchings from  $A$  to  $B$  without defining  $(1, k)$ -matchings. We will use only perfect ones.

**Definition 6.1.** A **perfect (1, k)-matching** from  $A$  to  $B$  is a set  $M \subset E$  satisfying following conditions:

- (1) for all  $a \in A$  there exists exactly  $k$  vertices  $b_1, \dots, b_k \in B$  such that  $(a, b_1), \dots, (a, b_k) \in M$ ;
- (2) for all  $b \in B$  there is an unique vertex  $a \in A$  such that  $(a, b) \in M$ .

The following Theorem is known as **the Hall's Harem Theorem**, and the first of equivalent conditions is known as **Hall's k-harem condition**.

**Theorem 6.2.** Let  $\Gamma = (A, B, E)$  be a locally finite graph and let  $k \in \mathbb{N}$ ,  $k \geq 1$ . The following conditions are equivalent:

- (i) For all finite subsets  $X \subset A$ ,  $Y \subset B$  following inequalities holds  $|N(X)| \geq k|X|$ ,  $|N(Y)| \geq \frac{1}{k}|Y|$ .
- (ii)  $\Gamma$  has a perfect  $(1, k)$ -matching.

Given a  $(1, k)$ -matching  $M$  and a vertex  $a \in A$  an  $M$ -star of  $a$  is a graph consisting of the set of all vertices and edge adjacent to  $a$  in  $M$ .

**Definition 6.3.** A graph  $\Gamma$  is **computable** if there exists a bijective function  $\nu : \mathbb{N} \rightarrow V$  such that the set

$$R := \{(i, j) : (\nu(i), \nu(j)) \in E\}$$

is computable. A locally finite graph  $\Gamma$  is called **highly computable** if additionally there is a recursive function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $f(n) = |N_\Gamma(\nu(n))|$  for all  $n \in \mathbb{N}$ . This definition and the three definitions below are due to Kierstead [22].

**Definition 6.4.** A bipartite graph  $\Gamma = (A, B, E)$  is **computably bipartite** if  $\Gamma$  is computable and the set of  $\nu$ -numbers of  $A$  is computable.

Below we will identify the elements of  $\Gamma$  with numbers.

**Definition 6.5.** Let  $\Gamma = (A, B, E)$  be a computably bipartite graph. A perfect  $(1, k)$ -matching  $M$  from  $A$  to  $B$  is called a **computable perfect (1, k)-matching** if there is an algorithm which

- for each  $i$  with  $\nu(i) \in A$ , finds the tuple  $(i_1, i_2, \dots, i_k)$  such that  $(\nu(i), \nu(i_j)) \in M$ , for all  $j = 1, 2, \dots, k$
- when  $\nu(i) \notin A$  it finds  $i'$  such that  $(\nu(i'), \nu(i)) \in M$ .

The remainder of this section will be devoted to a proof that the following condition implies the existence of the computable perfect  $(1, k)$ -matching.

**Definition 6.6.** A bipartite graph  $\Gamma = (A, B, E)$  satisfies the **computable expanding Hall's harem condition with respect to  $k$**  (denoted *c.e.H.h.c.(k)*), if and only if there is a recursive function  $h : \mathbb{N} \rightarrow \mathbb{N}$  such that:

- $h(0) = 0$
- for all finite sets  $X \subset A$ , the inequality  $h(n) \leq |X|$  implies  $n \leq |N(X)| - k|X|$
- for all finite sets  $Y \subset B$ , the inequality  $h(n) \leq |Y|$  implies  $n \leq |N(Y)| - \frac{1}{k}|Y|$ .



Clearly, if the graph  $\Gamma$  satisfies the *c.e.H.h.c.(k)*, then it satisfies the Hall's  $k$ -harem condition.

**Theorem 6.7.** *If  $\Gamma = (A, B, E)$  is a highly computable bipartite graph satisfying the *c.e.H.h.c.(k)*, then  $\Gamma$  has a computable perfect  $(1, k)$ -matching.*

*Proof.* We extend the proof of Theorem 3 of the Kierstead's paper [22]. We fix a computable enumeration of  $A$  and  $B$ . Let  $h$  witness the *c.e.H.h.c.(k)* for  $\Gamma$ . We begin by setting  $M = \emptyset$ . At step  $s$  we update already constructed  $M$  in the following way. For a vertex  $x_s \in A \cup B$  we construct some subgraph  $\Gamma_s$  and a matching  $M_s$  in  $\Gamma_s$ . The matching  $M$  is updated by those elements of  $M_s$  which contain  $x_s$ . The subgraph  $\Gamma_s$  is constructed so that after removal of the  $M_s$ -star of  $x_s$  from  $\Gamma$ , we still have a highly computable bipartite graph satisfying the *c.e.H.h.c.(k)*.

At the first step of the algorithm we choose  $a_0$ , the first element of the set  $A$ . We construct the induced subgraph  $\Gamma_0 = (A_0, B_0, E_0)$  so that  $A_0 \cup B_0$  is the set of vertices with distance of at most  $\max\{2h(k) + 1, 3\}$  from  $a_0$ . Since the graph  $\Gamma$  is highly computable the graph  $\Gamma_0$  is finite and can be found effectively. It is clear that for all vertices  $v$  from  $A_0$ ,  $N_{\Gamma_0}(v) = N_{\Gamma}(v)$ . Therefore, for all  $X \subset A_0$  the inequality  $h(n) \leq |X|$  implies  $n \leq |N_{\Gamma_0}(X)| - k|X|$ .

Let  $B_{S_0}$  denote the set of vertices  $v \in B_0$  of the distance  $\max\{2h(k) + 1, 3\}$  from  $a_0$ . It is clear that  $N_{\Gamma_0}(B_0 \setminus B_{S_0}) = N_{\Gamma}(B_0 \setminus B_{S_0}) = A_0$ . On the other hand since it can happen that  $N_{\Gamma}(B_{S_0})$  is not contained in  $A_0$ , it is possible that there exist  $Y \subset B_{S_0}$ , such that  $|N_{\Gamma_0}(Y)| \leq \frac{1}{k}|Y|$ .

Since  $\Gamma$  contains a perfect  $(1, k)$ -matching, there exists a  $(1, k)$ -matching in  $\Gamma_0$ , that satisfies the conditions of perfect  $(1, k)$ -matchings for all  $a \in A_0$ ,  $b \in B_0 \setminus B_{S_0}$ . We denote it by  $M_0$ . Since  $\Gamma_0$  is finite, the matching  $M_0$  can be obtained effectively. Let  $\{b_{0,1}, \dots, b_{0,k}\}$  be all elements that  $(a_0, b_{0,i})$  belongs to  $M_0$ . We define  $M$  to be the set of all these pairs.

Let  $\Gamma'$  be a subgraph obtained from  $\Gamma$  through removal of the  $M_0$ -star of  $a_0$ . Since the sets  $A \cup B$ ,  $A$  and  $E$  are computable, and the matching  $M_0$  is found effectively, hence the sets  $A' \cup B'$ ,  $A'$  and  $E'$  are also computable. Therefore  $\Gamma'$  is a computably bipartite graph. Since  $\Gamma'$  is locally finite and we can compute the neighbourhood of every vertex,  $\Gamma'$  is highly computable. To finish this step it suffices to show that  $\Gamma'$  satisfies *c.e.H.h.c.(k)*.

Let

$$h'(n) = \begin{cases} 0, & \text{if } n = 0, \\ h(n+k), & \text{if } n > 0. \end{cases}$$

We claim that  $h'$  works for  $\Gamma'$ . We start with the case when  $X \subset A'$  and  $n > 0$ . Since  $|N_{\Gamma'}(X)| \geq |N_{\Gamma}(X)| - k$ , then for  $n \geq 1$  the inequality  $|X| > h'(n)$  implies  $|N_{\Gamma'}(X)| - k|X| \geq |N_{\Gamma}(X)| - k|X| - k \geq n$ .

Let us consider the case when  $n = 0$  and  $X$  is still a subset of  $A'$ . If  $X$  is not connected, then its neighbourhood would be the union of neighbourhoods of its connected subsets. Therefore without the loss of the generality, we can assume that  $X$  is connected. If  $X \subset A_0$ , then  $|N_{\Gamma'}(X)| - k|X| \geq 0$ , since  $M_0$  was a  $(1, k)$ -matching from  $A_0$  to  $B_0$  that was perfect for subsets of  $A_0$ .

Now, let us assume that there exists  $a' \in X \setminus A_0$ . If  $b_{0,1}, \dots, b_{0,k} \notin N_{\Gamma'}(X)$ , then  $|N_{\Gamma'}(X)| = |N_{\Gamma}(X)|$ , so  $|N_{\Gamma'}(X)| - k|X| \geq 0$ . Assume that for some  $i \leq k$  and some  $a \in X$ , there exists  $(a, b_{0,i}) \in E$ . Since the distance between  $a$  and  $a'$  is at least  $2h(k)$  we have  $|X| \geq h(k) + 1$ . Thus  $|N_{\Gamma}(X)| - k|X| \geq k$  and it follows that  $|N_{\Gamma'}(X)| - k|X| \geq 0$ . We conclude that the case of finite subsets of  $A'$  is verified.

Now we need to show that  $\Gamma'$  satisfies *c.e.H.h.c.(k)* for sets  $Y \subset B'$ . We have to show that for all finite sets  $Y \subset B$ , the inequality  $h'(n) \leq |Y|$  implies  $n \leq |N_{\Gamma'}(Y)| - \frac{1}{k}|Y|$ . Note  $Y \subset B' = B \setminus \{b_{0,1}, \dots, b_{0,k}\}$  and  $|N_{\Gamma'}(Y)| \geq |N_{\Gamma}(Y)| - 1$ .

In the case  $n > 0$  the inequality  $|Y| > h'(n)$  implies  $|N_{\Gamma'}(Y)| - \frac{1}{k}|Y| \geq |N_{\Gamma}(Y)| - \frac{1}{k}|Y| - 1 \geq n + k - 1 \geq n$ .

Let us consider the case  $n = 0$ . As before, we can assume that  $Y$  is connected. If  $Y \subset B_0 \setminus B_{S_0}$ , then  $|N_{\Gamma'}(Y)| - \frac{1}{k}|Y| \geq 0$ , since  $M_0$  satisfied the conditions of a perfect  $(1, k)$ -matching for elements of  $B_0 \setminus B_{S_0}$ .

Let us assume that there exists  $b' \in Y \setminus (B_0 \setminus B_{S_0})$ . If  $a_0 \notin N_{\Gamma'}(Y)$ , then  $N_{\Gamma'}(Y) = N_{\Gamma}(Y)$  and  $|N_{\Gamma'}(Y)| - \frac{1}{k}|Y| \geq 0$ .

Assume that for some  $b \in Y$  there exists the edge  $(a_0, b) \in E$ . Since the distance between  $b$  and  $b'$  is at least  $2h(k)$  we have  $|Y| \geq h(k) + 1$ . It follows that  $|N_{\Gamma}(Y)| - \frac{1}{k}|Y| \geq k$  and  $|N_{\Gamma'}(Y)| - \frac{1}{k}|Y| \geq k - 1 \geq 0$ .

As a result we have that the graph  $\Gamma'$  satisfies *c.e.H.h.c.(k)*. To force the matching  $M$  to be a perfect  $(1, k)$ -matching we use back and forth. Therefore we start the next step of an algorithm by choosing an element  $b_{1,1}$  of  $B'$ .

We construct the induced subgraph  $\Gamma_1 = (A_1, B_1, E_1)$  so that  $A_1 \cup B_1$  is a set of vertices of  $\Gamma'$  with distance of at most  $\max\{2h'(k) + 2, 4\}$  from  $b_{1,1}$ . Let  $B_{S_1}$  denote the set of vertices of the distance  $\max\{2h'(k) + 2, 4\}$  from  $b_{1,1}$ . Since  $\Gamma'$  contains a perfect  $(1, k)$ -matching, there exist a  $(1, k)$ -matching in  $\Gamma_1$  that satisfies the conditions of a perfect  $(1, k)$ -matching for all  $a \in A_1$  and  $b \in B_1 \setminus B_{S_1}$ . We denote

it by  $M_1$ . We choose  $a_1$  with  $(a_1, b_{1,1}) \in M_1$ . Let  $\{b_{1,2}, \dots, b_{1,k}\}$  be all remaining elements that  $(a_1, b_{1,i})$  belongs to  $M_1$ . We update  $M$  by all edges adjacent to  $a_1$  in  $M_1$ .

Let  $\Gamma''$  be a subgraph obtained from  $\Gamma'$  through removal of the  $M_1$ -star of  $a_1$ . Then  $\Gamma''$  is also highly computable bipartite graph. We need to show that  $\Gamma''$  satisfies *c.e.H.h.c.(k)*.

Let

$$h''(n) = \begin{cases} 0, & \text{if } n = 0, \\ h'(n+k), & \text{if } n > 0. \end{cases}$$

To prove that  $h''(n)$  works for  $\Gamma''$  we use the same method as in the case  $h'(n)$  and  $\Gamma'$ .

We continue iteration by taking the elements of  $A$  at even steps and the elements of  $B$  at odd steps. At every step  $n$ , the graph  $\Gamma^{(n)}$  satisfies the conditions for existence of perfect  $(1, k)$ -matchings and we update  $M$  by  $k$  edges adjacent to  $a_n$ . Every vertex  $v$  will be added to  $M$  at some step of the algorithm. It follows that  $M$  is a perfect  $(1, k)$ -matching of the graph  $\Gamma$ . Effectiveness of our back and forth construction guarantees that we have an algorithm satisfying Definition 6.5.  $\square$

## 7 Effective paradoxical decomposition

Throughout this section,  $(G, \nu)$  is a computable group. For simplicity of notation we identify the set  $G$  with  $\mathbb{N}$  and subsets  $F$  of  $\mathbb{N}$  with  $\nu(F) \subset G$ . As before by  $x^* \in \mathbb{N}$  we denote a number with  $\nu(x^*)\nu(x) = 1$ .

**Definition 7.1.** The group  $G$  has an **effective paradoxical decomposition**, if there exists a finite set  $K \subset G$  and two families of computable sets  $(A_k)_{k \in K}, (B_k)_{k \in K}$ , such that:

$$G = \left( \bigsqcup_{k \in K} kA_k \right) \bigsqcup \left( \bigsqcup_{k \in K} kB_k \right) = \left( \bigsqcup_{k \in K} A_k \right) = \left( \bigsqcup_{k \in K} B_k \right).$$

We call  $(K, (A_k)_{k \in K}, (B_k)_{k \in K})$  a paradoxical decomposition of  $G$ .

**Theorem 7.2.** *There is an effective procedure which for any finite subset  $K_0 \subset G$  satisfying the condition:*

*there is a natural number  $n$  such that for any finite subset  $F \subset G$ , there exists  $k \in K_0$  such that  $\frac{|F \setminus kF|}{|F|} \geq \frac{1}{n}$ ,*

*finds a finite subset  $K \subset G$  which defines an effective paradoxical decomposition as in Definition 7.1.*

*Proof.* The proof is an adaptation of the proof of Theorem 4.9.2 from [11]. Consider the set  $K_1 = K_0 \cup \{1\}$ . For any  $F \subset G$  we have:

$$K_1 F \supset F \text{ and } K_1 F \setminus F = K_0 F \setminus F.$$

Thus there is  $k \in K_0$  so that

$$|K_1 F| - |F| = |K_1 F \setminus F| = |K_0 F \setminus F| \geq |kF \setminus F| \geq \frac{|F|}{n}.$$

It follows that

$$|K_1 F| \geq \left(1 + \frac{1}{n}\right)|F|.$$

Choose  $n_1 \in \mathbb{N}$  such that  $(1 + \frac{1}{n})^{n_1} \geq 3$  and set  $K = K_1^{n_1}$ . We see that  $K$  is found effectively by  $K_0$ . Note that for any  $F \subset G$  we have  $|KF| \geq 3|F|$ .

To find the corresponding effective paradoxical decomposition consider the bipartite graph  $\Gamma_K(G) = (\mathbb{N}, \mathbb{N}, E)$ , where the set  $E \subset \mathbb{N} \times \mathbb{N}$  consists of all pairs  $(g, h)$  with  $h \in Kg$ , where  $g, h$  are viewed as elements of  $G$ . Since  $G$  is computable and  $K$  is finite, the graph  $\Gamma_K(G)$  is computably bipartite. Since the degree of every vertex is equal to  $|K|$ , the graph is highly computable.

Let  $F$  be a finite subset of the first copy of  $G$ . Then  $|N_\Gamma(F)| = |KF| \geq 3|F|$ . It follows that:

$$|N_\Gamma(F)| - 2|F| \geq 3|F| - 2|F| = |F|.$$

Therefore for any  $n \in \mathbb{N}$  the inequality  $n \leq |F|$  implies that  $n \leq |N_\Gamma(F)| - 2|F|$ .

On the other hand, if we consider a finite set  $F$  in the second copy of  $G$ , then any  $k \in K$  satisfies  $N_\Gamma(F) \supset k^*F$ . Consequently:

$$|N_\Gamma(F)| \geq |k^*F| = |F| \geq \frac{1}{2}|F|.$$

Since the function  $h(n) = 2n$  is recursive, the graph  $\Gamma_K(G)$  satisfies *c.e.H.h.c.*(2) with respect to  $h$ . By virtue of the Effective Hall Harem Theorem, we deduce the existence of a computable perfect  $(1, 2)$ -matching  $M$  in  $\Gamma_K(G)$ . In other words, there is a computable surjective  $(2 \rightarrow 1)$ -map  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $n\phi(n)^* \in K$  for all  $n \in \mathbb{N}$ .

We now define functions  $\psi_1, \psi_2$  as follows:

$$\begin{cases} \psi_1(n) = \min(n_1, n_2) \\ \psi_2(n) = \max(n_1, n_2) \end{cases}, \text{ where } \phi(n_1) = n = \phi(n_2), n_1 \neq n_2.$$

Since the function  $\phi$  realizes a computable perfect  $(1, 2)$ -matching, both  $\psi_1$  and  $\psi_2$  are recursive.

Define  $\theta_1(n) := \psi_1(n)n^*$ ,  $\theta_2(n) := \psi_2(n)n^*$ . Observe that  $\theta_1, \theta_2$  are recursive and  $\theta_1(n), \theta_2(n) \in K$  for all  $n \in \mathbb{N}$ .

For each  $k \in K$  define sets  $A_k$  and  $B_k$  in the following way:

$$A_k = \{n \in \mathbb{N} : \theta_1(n) = k\}, B_k = \{n \in \mathbb{N} : \theta_2(n) = k\}.$$

It is clear that these sets are computable and

$$G = \bigsqcup_{k \in K} A_k = \bigsqcup_{k \in K} B_k.$$

For each  $n \in A_k$ , the value  $\psi_1(n)$  is  $k \cdot n$  under the group multiplication. Thus  $\psi_1(\mathbb{N}) = \bigsqcup_{k \in K} kA_k$ .

Similarly we can show that  $\psi_2(\mathbb{N}) = \bigsqcup_{k \in K} kB_k$ . Since  $\mathbb{N} = \psi_1(\mathbb{N}) \sqcup \psi_2(\mathbb{N})$ , we have

$$G = \left( \bigsqcup_{k \in K} kA_k \right) \sqcup \left( \bigsqcup_{k \in K} kB_k \right).$$

Therefore  $(K, (A_k)_{k \in K}, (B_k)_{k \in K})$  is an effective paradoxical decomposition of the group  $G$ .  $\square$

## 8 Complexity of paradoxical decompositions

We preserve the assumption of Section 7.

**Definition 8.1.** Let

$$\mathfrak{W}_{BT} = \left\{ K : (K \subset\subset G) \wedge \exists n \in \mathbb{N} (\forall F \subset\subset G) (\exists k \in K) \left( \frac{|F \setminus kF|}{|F|} \geq \frac{1}{n} \right) \right\}.$$

We call this family **witnesses of the Banach-Tarski paradox**.

This term is justified by Theorem 7.2 where  $\mathfrak{W}_{BT}$  appears in the formulation.

**Proposition 8.2.** *For any computable group the family  $\mathfrak{W}_{BT}$  belongs to the class  $\Sigma_2^0$ .*

*Proof.* Since group  $G$  is computable, for any finite subsets  $K, F$  of  $G$ , and any  $n \in \mathbb{N}$ , we can effectively check if the inequality  $\frac{|F \setminus kF|}{|F|} < \frac{1}{n}$  holds for all  $k \in K$ . Therefore, the set of triples  $(n, K, F)$  such that  $\frac{|F \setminus kF|}{|F|} < \frac{1}{n}$  holds for all  $k \in K$  is computably enumerable.

Since the projection of this set to the first two coordinates is also computably enumerable, the set

$$\mathfrak{W}'_{BT} = \{(K, n) : (\forall F \subset\subset \Gamma) (\exists k \in K) \left( \frac{|F \setminus kF|}{|F|} \geq \frac{1}{n} \right)\}$$

belongs to the class  $\Pi_1^0$ . The set  $\mathfrak{W}_{BT}$  consists of  $K$  such that there exists  $n \in \mathbb{N}$  with  $(K, n) \in \mathfrak{W}'_{BT}$ . Thus  $\mathfrak{W}_{BT}$  belongs to the class  $\Sigma_2^0$ .  $\square$

The following question has become principal for us.

- Are there natural examples with computable/non-computable  $\mathfrak{W}_{BT}$ ?

In the appendix we give an example of a finitely presented group with decidable word problem and non-computable  $\mathfrak{W}_{BT}$ . In the present section we give positive examples. The most natural ones are provided by the following theorem.

**Theorem 8.3.** *The family  $\mathfrak{W}_{BT}$  is computable for any finitely generated free group.*

The proof of this theorem is based on some reformulation of witnessing. It belongs to M. Cavaleri. It simplifies our original argument.

**Proposition 8.4.** *Let  $G$  be a group and  $K \subset\subset G$ . Then  $K \in \mathfrak{W}_{BT}$  if and only if  $\langle K \rangle$  is a non-amenable subgroup of  $G$ .*

*Proof.* The necessity is obvious. Assume that  $K \notin \mathfrak{W}_{BT}$ . It follows that for every  $n$  there exists set  $F_n$  such that  $F_n \in \mathfrak{Fol}_{G,K}(n)$ . Set  $n \in \mathbb{N}$ . Let  $m = n|K|$ . We now follow a proof of Proposition 9.2.13 from [14] to show that there exists  $t_0 \in G$  such that the set  $F_m t_0^{-1} \cap \langle K \rangle = \{k \in \langle K \rangle : kt_0 \in F_m\}$  is an  $n$ -Følner for  $K$ . Let  $T \subset G$  be a complete set of representatives of the right cosets of  $\langle K \rangle$  in  $G$ . Clearly, every  $g \in G$  can be uniquely written in the form  $g = ht$  with  $h \in \langle K \rangle$  and  $t \in T$ . We then have:

$$|F_m| = \sum_{t \in T} |F_m t^{-1} \cap \langle K \rangle| \quad (8)$$

For every  $x \in K$ , we have  $x F_m = \bigsqcup_{t \in T} (x F_m t^{-1} \cap \langle K \rangle) t$ , hence:

$$x F_m \setminus F_m = \bigsqcup_{t \in T} ((x F_m t^{-1} \cap \langle K \rangle) \setminus (F_m t^{-1} \cap \langle K \rangle)) t.$$

This gives us:

$$|x F_m \setminus F_m| = \sum_{t \in T} |(x F_m t^{-1} \cap \langle K \rangle) \setminus (F_m t^{-1} \cap \langle K \rangle)|. \quad (9)$$

Since for all  $x \in K$ ,

$$|x F_m \setminus F_m| \leq \frac{|F_m|}{m},$$

using (8) and (9), we get

$$\sum_{t \in T} |(K F_m t^{-1} \cap \langle K \rangle) \setminus (F_m t^{-1} \cap \langle K \rangle)| = \sum_{t \in T} \left| \bigcup_{x \in K} ((x F_m t^{-1} \cap \langle K \rangle) \setminus (F_m t^{-1} \cap \langle K \rangle)) \right| \leq \frac{|K|}{m} \sum_{t \in T} |F_m t^{-1} \cap \langle K \rangle|$$

By the pigeonhole principle, there exists  $t_0 \in T$  such that the set  $|(K F_m t_0^{-1} \cap \langle K \rangle) \setminus (F_m t_0^{-1} \cap \langle K \rangle)| \leq \frac{1}{n} |F_m t_0^{-1} \cap \langle K \rangle|$ . Clearly  $F_m t_0^{-1} \cap \langle K \rangle$  is an  $n$ -Følner set with respect to  $K$ . Since  $n$  was arbitrary,  $\langle K \rangle$  is amenable, a contradiction.  $\square$

*Proof. (Theorem 8.3).* Let  $\mathbb{F}$  be a finitely generated free group. Since it is computable, the equation  $xy = yx$  can be effectively verified for every  $x, y \in \mathbb{F}$ . We will show that  $K \in \mathfrak{W}_{BT}$  if and only if there exist  $x, y \in K$  such that  $xy \neq yx$ . This will give the result.

( $\Rightarrow$ ) Let us assume that  $xy = yx$  for every  $x, y \in K$ . Since  $\mathbb{F}$  is a free group, there exists  $z \in \mathbb{F}$  such that all words from  $K$  are powers of  $z$ . Since the subgroup  $\langle z \rangle$  is cyclic, the subgroup  $\langle K \rangle$  is amenable and for every  $n$  there is a finite set  $F$ , which is an  $n$ -Følner with respect to  $K$ . Clearly  $K \notin \mathfrak{W}_{BT}$ .

( $\Leftarrow$ ) Let us assume that there exist  $x, y \in K$  with  $xy \neq yx$ . Then  $x, y$  generate a free subgroup of  $\mathbb{F}$  of rank 2. By Proposition 8.4 there is a natural number  $n$  such that  $\mathfrak{Fol}_{\mathbb{F}, \{x, y\}}(n) = \emptyset$ . Thus  $\mathfrak{Fol}_{\mathbb{F}, K}(n)$  is also empty.  $\square$

We remind the reader that a group  $G$  is called **fully residually free** if for any finite collection of nontrivial elements  $g_1, \dots, g_n \in G \setminus \{1\}$  there exists a homomorphism  $\phi : G \rightarrow \mathbb{F}$  onto a free group  $\mathbb{F}$  such that  $\phi(g_1) \neq 1, \dots, \phi(g_n) \neq 1$ , [19]. The class of fully residually free groups as well as residually free groups has deserved a lot of attention mainly in connection with algorithmic and model-theoretic investigations in group theory, see for example [20] and [34].

**Theorem 8.5.** *The family  $\mathfrak{W}_{BT}$  is computable for any computable fully residually free group.*

*Proof.* Let  $(G, \nu)$  be a computable fully residually free group. Since  $(G, \nu)$  is computable, it suffices to show that  $K \in \mathfrak{W}_{BT}$  if and only if there exist  $x, y \in K$  such that  $[x, y] \neq 1$ .

( $\Rightarrow$ ) Let us assume that  $[x, y] = 1$  for all  $x, y \in K$ . Therefore subgroup  $\langle K \rangle$  is a finitely generated abelian group. Thus it is amenable and  $K \notin \mathfrak{W}_{BT}$ .

( $\Leftarrow$ ) Let us assume that there exist  $x, y \in K$  with  $[x, y] \neq 1$ . Since  $x, y, [x, y]$  are nontrivial elements of  $G$  we have  $\phi : G \rightarrow F_2$  such that  $\phi(x) \neq \phi(y) \neq \phi([x, y]) \neq 1$ . Clearly,  $\langle \phi(x), \phi(y) \rangle$  is a free group of rank 2. Thus  $\langle x, y \rangle$  is also a free subgroup of rank 2. It remains to apply Proposition 8.4 exactly as in the proof of Theorem 8.3.

□

# Appendix: On decidability of amenability in computable groups\*

KAROL DUDA

Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2  
Wrocław, 50-384, Poland  
Karol.Duda@math.uni.wroc.pl

ALEKSANDER IVANOV

Department of Applied Mathematics, Silesian University of Technology, ul. Kaszubska  
23  
Gliwice, 44-100, Poland  
Aleksander.Iwanow@polsl.pl

## Abstract

The main result of the paper states that there is a finitely presented group  $G$  with decidable word problem where detection of finite subsets of  $G$  which generate amenable subgroups is not decidable.

## 1 Introduction

In this paper we discuss the following problem. Given algebraic property of groups  $\mathcal{P}$  is there a finitely presented group  $G_{\mathcal{P}}$  with decidable word problem such that there is no algorithm to determine whether or not the subgroup generated by an arbitrary finite set of words in the given generators of  $G_{\mathcal{P}}$  has the property  $\mathcal{P}$ ? If one removes the requirement of decidability of the word problem, this task becomes easier. For example Baumslag, Boon and Neumann have proved in [5] a general theorem which gives a positive answer for a long list of properties (see also Section 3 in [24]). If we insist on decidability of the word problem the question becomes hard. Although the methods of contemporary algorithmic geometric group theory have become very advanced (for example see [27] - [29]) we do not know any example where they work for typical properties  $\mathcal{P}$ . Usually they are applied to properties of different kinds: the conjugacy problem [28], the power problem [29] or the problem of solvability of exponential equations [8].

The main result of our paper is a theorem which for some natural properties  $\mathcal{P}$  states existence of a finitely presented group  $G_{\mathcal{P}}$  with decidable word problem where detection of finite subsets of  $G_{\mathcal{P}}$  which generate subgroups satisfying  $\mathcal{P}$  is not decidable. We emphasize the case when  $\mathcal{P}$  is the property of **amenability**. The reason will be explained in the second part of the introduction.

The proof uses an idea which is completely new in geometric group theory. It consists of two steps. Firstly by application of the techniques of **intrinsically computable relations** of Ash, Knight and Nerode from computable model theory we build a computable group with undecidable amenability problem for subgroups. In Section 3 using some kind of Higman's embedding we transform it into a group which is also finitely presented. The construction of the first step is quite general and can be applied to other properties studied in group theory. Therefore we present it in a very general form.

For deeper discussion of the results we now introduce the main definitions of the paper.

A first-order structure  $M$  with domain  $\mathbf{N}$  is called **computable**, if all relations and operations of  $M$  are computable and this condition is satisfied uniformly. In group theory the terminology is slightly different. Let  $G = \langle X \mid \mathcal{R} \rangle$  be a countable group generated by  $X \subseteq G$ . The group  $G$  is called **recursively presented** (see Section IV.3 in [23]) if  $X$  can be identified with  $\mathbf{N}$  (or with some  $\{0, \dots, n\}$ ) so that the set of relators  $\mathcal{R} \subset (X \cup X^{-1})^*$  is computably enumerable. When  $G$  is recursively presented and has decidable word problem, then there is an algorithm computing a 1-1 function  $f : \mathbf{N} \rightarrow (X \cup X^{-1})^*$  such

---

\*Archive for Mathematical Logic, <https://doi.org/10.1007/s00153-022-00819-5>

that each element of  $G$  is uniquely presented by a word from  $f(\mathbf{N})$  (we neglect the easy case when  $G$  is finite). The group operations of  $G$  induce a group with domain  $\mathbf{N}$  which is isomorphic to  $G$ . Since each computable group can be obviously viewed as a recursively presented one, we see that recursively presented groups with decidable word problem exactly correspond to computable groups.

Amenability is one of the basic concepts of mathematics. It was created by Banach, von Neumann and Tarski in the beginning of the 20-th century and nowadays it has become fundamental in dynamical systems, group theory, logic, measure theory, etc., see [32]. Very recently Cavaleri and Moriakov have begun investigations of algorithmic aspects of this topic, see [9], [10] and [25].

We now give two equivalent definitions of amenability. Let  $G$  be a discrete group and  $D$  be a finite subset of  $G$ . Given  $n \in \mathbf{N}$ , we say that a finite  $F \subset G$  is an  $\frac{1}{n}$ -**Følner set** with respect to  $D$  if

$$\forall x \in D \quad \frac{|F \setminus xF|}{|F|} \leq \frac{1}{n}.$$

The group  $G$  is called **amenable** if for every  $n \in \mathbf{N}$  and every finite  $D \subset G$  there is a finite  $F \subset G$  which is  $\frac{1}{n}$ -Følner with respect to  $D$  (this is so called Følner's condition).

A theorem of A. Tarski states that the group  $G$  is not amenable if and only if  $G$  has a **paradoxical decomposition**. The latter means that there exists a finite set  $K \subset G$  and two families of subsets of  $G$ :  $(A_k)_{k \in K}$  and  $(B_k)_{k \in K}$  such that

$$G = \left( \bigsqcup_{k \in K} k \cdot A_k \right) \bigsqcup \left( \bigsqcup_{k \in K} k \cdot B_k \right) = \left( \bigsqcup_{k \in K} A_k \right) = \left( \bigsqcup_{k \in K} B_k \right).$$

Here we use a version of the definition given in [12], where some members  $A_k$  or  $B_k$  can be empty. The first author has shown in his master thesis that given computable non-amenable  $G$  there is an effective procedure which assigns to any  $D$  without  $\frac{1}{n}$ -Følner sets a paradoxical decomposition consisting of computable pieces. Moreover the family of such  $D$  is computable if and only if there is an algorithm which distinguishes all finite subsets of  $G$  generating amenable subgroups. This observation explains our original motivation.

As we have already mentioned above standard methods of algorithmic group theory do not seem helpful for producing examples of the form  $G_{\mathcal{P}}$  as in the beginning of the section. It seems to us that Proposition 1.1 below describes basic obstacles in the field. It roughly says that when we build a computable group with undecidable amenability it should not resemble free groups. Note that standard finitely presented examples in group theory are usually amenable or “free-like”. For example they often satisfy the Tits alternative: it is virtually soluble (i.e. amenable) or contains a non-abelian free group. Thus the proposition suggests that in order to build non-amenable groups of the form  $G_{\mathcal{P}}$  we probably need new tools. In particular, computability theory techniques may become essential in this respect.

To present Proposition 1.1 we remind the reader that a group  $G$  is called **fully residually free** if for any finite collection of nontrivial elements  $g_1, \dots, g_n \in G \setminus \{1\}$  there exists a homomorphism  $\phi : G \rightarrow \mathbf{F}$  onto a free group  $\mathbf{F}$  such that  $\phi(g_1) \neq 1, \dots, \phi(g_n) \neq 1$ , [19]. The class of fully residually free groups as well as residually free groups has received a lot of attention mainly in connection with algorithmic and model-theoretic investigations in group theory, see for example [20] and [34].

**Proposition 1.1.** *If  $G$  is a computable fully residually free group then the problem if a finite subset of  $G$  generates an amenable subgroup is decidable.*

*Proof.* We use the well-known fact that abelian groups are amenable contrary to groups containing non-abelian free subgroups.

Let  $G$  be a computable fully residually free group. It suffices to show that a finite  $K \subset G$  generates a non-amenable subgroup if and only if there exist  $x, y \in K$  such that  $[x, y] \neq 1$ .

( $\Rightarrow$ ) Let us assume that  $[x, y] = 1$  for all  $x, y \in K$ . Therefore subgroup  $\langle K \rangle$  is a finitely generated abelian group, i.e. it is amenable.

( $\Leftarrow$ ) Let us assume that there exist  $x, y \in K$  with  $[x, y] \neq 1$ . Since  $x, y, [x, y]$  are nontrivial elements of  $G$  we have a homomorphism  $\phi$  from  $G$  into  $F_2$  such that the set  $\{\phi(x), \phi(y), \phi([x, y]), 1\}$  consists of pairwise distinct elements. Clearly,  $\langle \phi(x), \phi(y) \rangle$  is a free group of rank 2. Thus  $\langle x, y \rangle$  is also a free subgroup of rank 2. In particular  $\langle K \rangle$  is not amenable.  $\square$

## 1.1 Technical preliminaries

We do not assume that the reader has any special knowledge in recursion theory. We give all necessary definitions making the material available for group theorists. From now on we identify each finite set

$F \subset \mathbf{N}$  with its Gödel number. The following notion is convenient when one builds a computable copy of an abstract group.

**Definition 1.2.** Let  $G$  be a group and  $\nu : \mathbf{N} \rightarrow G$  be a surjective function. We call the pair  $(G, \nu)$  a **numbered group**. The function  $\nu$  is called a **numbering** of  $G$ . If  $g \in G$  and  $\nu(n) = g$ , then  $n$  is called a number of  $g$ .

Note that when the numbering  $\nu$  is a bijection and the set

$$\text{MultT} := \{(i, j, k) : \nu(i)\nu(j) = \nu(k)\}$$

is computable, the map  $\nu$  is an isomorphism of  $G$  with a computable group, and the latter is called a **computable copy** of  $G$ . On the other hand if a numbering  $\nu$  of a group  $G$  is not necessary injective but the set  $\text{MultT}$  above is still computable, then

- $G$  has a computable copy (possibly under another numbering). Indeed, in this case the set of the smallest numbers of the elements of  $G$  is computable. Enumerating this set by natural numbers we obtain a required 1-1 enumeration.
- In this case we also have that the set  $\{(n_1, n_2) : \nu(n_1) = \nu(n_2)\}$  is computable.

We apply the theory of **intrinsically computable relations**, [2], [3], [4]. In fact we use the advanced version of it from [3]. According Section 2 of that paper the binary relation  $\bar{a} \leq_0 \bar{b}$  on tuples of the same length  $n$  from a computable structure  $M$  is defined to be the property that  $M \models \phi(\bar{a})$  implies  $M \models \phi(\bar{b})$  where  $\phi(x_{i_1}, \dots, x_{i_l})$  is atomic or the negation of an atomic formula with the Gödel number  $< n$  and variables among  $x_1, \dots, x_n$ .

When  $\alpha$  is an ordinal  $> 0$ , then the relation  $\bar{a} \leq_\alpha \bar{b}$  is defined that for any  $\gamma < \alpha$  and any  $\bar{d} \in M$  there exists  $\bar{c}$  such that  $\bar{b}\bar{d} \leq_\gamma \bar{a}\bar{c}$ . For tuples with  $|\bar{a}| = n \leq |\bar{b}|$  this relation means that  $\bar{a} \leq_\alpha \bar{b}|_n$ .

The structure  $M$  is  $\alpha$ -**friendly** if the relations  $\leq_\beta$  are uniformly c.e. for all  $\beta < \alpha$ . The following statement is Theorem 2.1 from [3].

Let  $M$  be an  $\alpha$ -friendly computable structure and a relation  $R$  is computable on  $M$ . Suppose that for any  $\bar{c}$  there is a tuple  $\bar{a} \notin R$  such that for any tuple  $\bar{a}_1$  and  $\beta < \alpha$  there exist  $\bar{a}' \in R$  and  $\bar{a}'_1$  such that  $\bar{c}\bar{a}\bar{a}_1 \leq_\beta \bar{c}\bar{a}'\bar{a}'_1$  in  $M$ . Then for every  $\Sigma_\alpha^0$  set  $C$  there is an isomorphism  $f$  from  $M$  onto a computable structure  $M'$  such that  $C$  and  $f(R)$  are of the same Turing degree modulo  $\Delta_\alpha^0$ .

In our construction below we use the formulation of Theorem 2.1 in the case when  $\alpha = 1$ . Thus it is worth noting here that 1-**friendliness** of a structure just means that the relation  $\leq_0$  is computably enumerable on the set of tuples of arbitrary length. This condition is always satisfied in a computable structure. In particular we will omit it below.

## 2 A computable group with undecidable amenability

### 2.1 The construction

Let  $(G_i, g_{i,1}, \dots, g_{i,l}), i \in 2\mathbf{N}+1$  be a sequence of groups (indexed by odd numbers) such that the following conditions are satisfied.

- For every odd  $i$  the group  $G_i$  is generated by the tuple  $g_{i,1}, \dots, g_{i,l}$ .
- For every pair  $i \leq j$  the (directed and coloured) Cayley graph of  $G_j$  with respect to the generators  $g_{j,1}, \dots, g_{j,l}$  has the same  $i$ -balls of 1 with the Cayley graph of  $(G_i, g_{i,1}, \dots, g_{i,l})$ .

In terms of [11] these conditions exactly mean that the sequence of marked groups  $(G_i, g_{i,1}, \dots, g_{i,l})$  (i.e. a group with a distinguished tuple of generators),  $i \in 2\mathbf{N}+1$ , is convergent in the Grigorchuk's topology. By compactness of the latter (see [11], Section 2.2) there is a marked group  $(G_\infty, g_{\infty,1}, \dots, g_{\infty,l})$  which is the limit of the sequence, i.e. for every  $i \in 2\mathbf{N}+1$  the Cayley graph of  $G_i$  with respect to the generators  $g_{i,1}, \dots, g_{i,l}$  has the same  $i$ -balls of 1 with the Cayley graph of  $(G_\infty, g_{\infty,1}, \dots, g_{\infty,l})$ .

A standard example of such a situation (which will be used below) is as follows. Let  $F_2$  be a 2-generated free group with the basis  $\{a, b\}$ . Since  $F_2$  is residually finite, for every  $i \in 2\mathbf{N}+1$  there is a finite group  $G_i$  and a surjective homomorphism  $\rho_i : F_2 \rightarrow G_i$  such that for the generators  $a_i = \rho_i(a)$  and  $b_i = \rho_i(b)$  the Cayley graph of  $(G_i, a_i, b_i)$  has the same  $i$ -ball of 1 with the Cayley graph of  $(F_2, a, b)$ .

Let  $s$  be a positive natural number. We will assume below that for every  $i \in (2\mathbf{N}+1) \cup \{\infty\}$



- the group  $G_i$  is a computable group with domain  $\mathbf{N}$  or some  $\{1, \dots, n\}$  (if the group is finite) and
- a computable  $s$ -ary relation  $R_i$  is fixed on  $G_i$  which is  $\text{Aut}(G_i)$ -invariant and contains the  $s$ -tuple of units  $\bar{1}$ .

Furthermore, we assume that the multiplication in  $G_i$  and the characteristic function of  $R_i$  are given by a uniform algorithm on  $i$ .

Let  $G = \langle g_1, \dots, g_l \mid \mathcal{R} \rangle$  be a recursively presented group with decidable word problem and let  $(G, g_1, \dots, g_l)$  be the limit of a sequence of finite marked groups. Then there is a sequence  $(G_i, g_{i,1}, \dots, g_{i,l})$ ,  $i \in 2\mathbf{N} + 1$  which together with  $(G, g_1, \dots, g_l)$  (viewed as  $(G_\infty, g_{\infty,1}, \dots, g_{\infty,l})$ ) satisfies the assumptions given above. Indeed, fix a computable copy of  $G$  and view it as  $G$ . Take any computable enumeration of multiplication tables of finite groups. For every  $i \in 2\mathbf{N} + 1$  determine the  $i$ -ball of the unit in  $(G, g_1, \dots, g_l)$ . Then let  $(G_i, g_{i,1}, \dots, g_{i,l})$  be the finite group with the same  $i$ -ball such that its multiplication table has the minimal number.

We define the sequence  $(G_i, \bar{g}_i)$ ,  $i \in \omega$ , of marked groups so that for even  $i$  the group  $(G_i, \bar{g}_i)$  coincides with  $(G_\infty, g_{\infty,1}, \dots, g_{\infty,l})$ . For odd  $i$  we assume that  $G_i$  is as before. Let  $G = \sum_{i \in \omega} G_i$ . Let

- $\text{Odd}(x)$  to be the unary predicate on  $G$  for the subgroup  $\sum_{i \in 2\mathbf{N}+1} G_i$ ;
- $\text{Pr}_i(x, y)$  be the relation from  $G \times G$  that  $y$  is the  $G_i$ -projection of  $x$ ,  $i \in \omega$ ;
- $R \subset G^s$  be the relation consisting of all  $s$ -tuples  $\bar{g}$  such that the projection of  $\bar{g}$  to every  $G_i$  satisfies  $R_i$ ,  $i \in \omega$ .

We define a numbering  $\nu : \mathbf{N} \rightarrow G$  as follows. We fix a 1-1 numbering of all finite subsets of  $\mathbf{N} \times \mathbf{N}$  with pairwise distinct first coordinates. Given  $k$  let  $\{(k_1, l_1), \dots, (k_s, l_s)\}$  be the subset with the number  $k$  where  $k_1 < \dots < k_s$ . Let  $\nu(k)$  be the element of  $\sum_{i \in \omega} G_i$  represented by the sequence where the only non-unit elements are  $l_j$  at the corresponding places of  $\{(k_1, l_1), \dots, (k_s, l_s)\}$ .

If some  $G_i$  are finite we modify  $\nu$  so that when  $\{(k_1, l_1), \dots, (k_s, l_s)\}$  is a subset with the number  $k$  then each  $l_i$  is strictly less than  $|G_i|$ .

**Lemma 2.1.** *The numbered structure  $(G, \cdot, 1, R, \text{Odd}, \{\text{Pr}_i\}_{i \in \omega}, \nu)$  is computable, i.e. the group operations of  $G$  and the relations  $R$ ,  $\text{Odd}$  and  $\text{Pr}_i$  are computable with respect to  $\nu$ .*

*Proof.* The structure  $(G, \cdot, 1, R, \text{Odd}, \{\text{Pr}_i\}_{i \in \omega}, \nu)$  is computable by the definition of  $\nu$  and the observation that an equation is satisfied in  $G$  if and only if it is satisfied in all  $G_i$ .  $\square$

Having this lemma we see that the structure  $(G, \cdot, 1, R, \{\text{Pr}_i\}_{i \in \omega})$  has a computable copy (for example by an argument given in Section 1.1). From now on we identify  $G$  with  $\mathbf{N}$ . The following theorem is the crucial technical statement.

**Theorem 2.2.** *Assume that  $G_\infty^s \setminus R_\infty$  is non-trivial, but for all odd  $i$  the relation  $R_i$  coincides with  $G_i^s$ . Then the group  $G$  has a computable copy so that  $R$  is not computable.*

*Proof.* Let us verify the condition of Theorem 2.1 from [3] which is stated in Section 1.1. Let  $\bar{c}$  be a tuple from  $G$  and  $t_1 < t_2 < \dots < t_r$  be the support of  $\bar{c}$ , i.e. the indices of those  $G_i$  where elements from  $\bar{c}$  have non-trivial projections. These numbers can be found algorithmically by Lemma 2.1.

Let  $t_{r+1}$  be the first even index greater than  $t_r$ . We define  $\bar{a}$  to be a tuple of  $G_{t_{r+1}}$ , say  $(a_{t_{r+1},1}, \dots, a_{t_{r+1},s})$  which is not in  $R_{t_{r+1}}$ . We consider it as a tuple of elements of  $G$  (assuming to be  $\bar{1}$  for other coordinates).

Let  $\bar{a}_1$  be any tuple from  $G$ , and let  $n$  be the length of  $\bar{c}\bar{a}\bar{a}_1$ . We want to find  $\bar{a}' \in R$  and  $\bar{a}'_1$  as in the formulation (given in Section 1.1). In particular verifying  $\bar{c}\bar{a}\bar{a}_1 \leq_0 \bar{c}\bar{a}'\bar{a}'_1$  we only consider formulas of Gödel numbers  $< n$ . We may suppose that these formulas are as follows:

$$\{w_i(\bar{z}, \bar{x}, \bar{x}_1) = 1 : i \in I_1\} \cup \{v_i(\bar{z}, \bar{x}, \bar{x}_1) \neq 1 : i \in I_2\},$$

where  $w_i$  and  $v_i$  are group words. For a word  $w(\bar{z}, \bar{x}, \bar{x}_1)$  let  $w(\bar{c}, \bar{a}, \bar{a}_1)(t)$  be the word written in the generators  $g_{t,1}, \dots, g_{t,l}$  which is obtained by the substitution of the  $G_t$ -projections of elements from  $\bar{c}\bar{a}\bar{a}_1$  into  $w(\bar{z}, \bar{x}, \bar{x}_1)$  (before reductions). Let

$$n_0 = \max \bigcup \{ \{ |w_i(\bar{c}, \bar{a}, \bar{a}_1)(t)| : i \in I_1 \} \cup \{ |v_i(\bar{c}, \bar{a}, \bar{a}_1)(t)| : i \in I_2 \} : t \in \text{supp}(\bar{c}\bar{a}\bar{a}_1) \} + 1.$$

Let  $\hat{t}$  be the first odd index greater than  $\max(\text{supp}(\bar{c}\bar{a}\bar{a}_1))$ , such that the  $n_0$ -ball of 1 in the Cayley graph of  $G_{t_{r+1}}$  is isomorphic to the  $n_0$ -ball of 1 in the Cayley graph of  $G_{\hat{t}}$ . Let us define  $\bar{a}'\bar{a}'_1$  as follows:

$\bar{\mathbf{a}}'\bar{\mathbf{a}}'_1(t) = \bar{\mathbf{a}}\bar{\mathbf{a}}_1(t)$  when  $t \notin \{t_{r+1}, \hat{t}\}$ ,  $\bar{\mathbf{a}}'\bar{\mathbf{a}}'_1(t_{r+1}) = \bar{1}$ ,  
and the words of the sequence  $\bar{\mathbf{a}}'\bar{\mathbf{a}}'_1(\hat{t})$  coincide with ones of the sequence  
 $\bar{\mathbf{a}}\bar{\mathbf{a}}_1(t_{r+1})$  under the correspondence  $(g_{\hat{t},1}, \dots, g_{\hat{t},l}) \leftrightarrow (g_{t_{r+1},1}, \dots, g_{t_{r+1},l})$ .

It is clear that  $\bar{\mathbf{a}}' \in R$ . Since the sequences  $\bar{\mathbf{c}}\bar{\mathbf{a}}\bar{\mathbf{a}}_1$  and  $\bar{\mathbf{c}}\bar{\mathbf{a}}'\bar{\mathbf{a}}'_1$  coincide on the sets of indices

$$\text{supp}(\bar{\mathbf{c}}\bar{\mathbf{a}}\bar{\mathbf{a}}_1) \setminus \{t_{r+1}, \hat{t}\} = \text{supp}(\bar{\mathbf{c}}\bar{\mathbf{a}}'\bar{\mathbf{a}}'_1) \setminus \{t_{r+1}, \hat{t}\},$$

their realizations on the formulas of Gödel numbers  $< n$  are equivalent on this part of the support. Note that  $t_{r+1} \notin \text{supp}(\bar{\mathbf{c}}\bar{\mathbf{a}}'\bar{\mathbf{a}}'_1)$  and  $\hat{t} \notin \text{supp}(\bar{\mathbf{c}}\bar{\mathbf{a}}\bar{\mathbf{a}}_1)$ . Thus to obtain the result it suffices to note that for any word  $w$  appearing in the formula of the Gödel number  $< n$  the equality  $w(\bar{\mathbf{c}}, \bar{\mathbf{a}}, \bar{\mathbf{a}}_1)(t_{r+1}) = 1$  is equivalent to  $w(\bar{\mathbf{c}}, \bar{\mathbf{a}}', \bar{\mathbf{a}}'_1)(\hat{t}) = 1$ . The latter follows from the choice of  $n_0$  and  $\hat{t}$ .

Since the conditions of Theorem 2.1 from [3] (see Section 1.1 above) are satisfied, applying it to a non-computable  $C$  we have that there is an 1-1 numbering  $\nu'$  of the group  $G$  such that  $(G, \nu')$  is a computable group where the relation  $R$  is not computable with respect to  $\nu'$ .  $\square$

In the following corollary we will use some standard facts concerning the Haagerup property and property **(T)**. They can be found in [6] and [13]. To keep the paper in a compact form we only give definitions of these important notions adapted to the case of countable discrete groups. In this case **(T)** is equivalent to so called FH. The latter states that any affine isometric action of  $G$  on a real Hilbert space has a fixed point (Sections 2.1 and 2.12 in [6]). The Haagerup property is a strong negation of FH. It states that there is an affine isometric action of  $G$  on a real Hilbert space  $H$  which is metrically proper, i.e. for all bounded subsets  $B \subset H$  the set  $\{g \in G : gB \cap B \neq \emptyset\}$  is finite (Section 1.1.1 in [13]).

**Corollary 2.3.** *There is a countable group  $G$  which has a computable copy  $G_{nA}$  such that the following problems are undecidable in  $G_{nA}$ :*

- (i) *when does a finite subset of  $G_{nA}$  generate an amenable subgroup?*
- (ii) *when does a finite subset of  $G_{nA}$  generate a group without free non-abelian subgroups?*

Furthermore, there is a computable copy  $G_{nT} \cong G$  such that the following problems are undecidable in  $G_{nT}$ :

- (iii) *when does a finite subset of  $G_{nT}$  generate a finite subgroup?*
- (iv) *when does a finite subset of  $G_{nT}$  generate a subgroup having property **(T)**?*

The group  $G$  has the Haagerup property.

*Proof.* Let  $(G_i, a_i, b_i)$ ,  $i \in 2\mathbb{N} + 1$ , be the sequence defined in Remark 2.1. We consider the free group  $(F_2, a, b)$  as  $(G_\infty, g_{\infty,1}, \dots, g_{\infty,l})$  and define  $(G_i, a_i, b_i)$ ,  $i \in \omega$ , taking even members to be equal to it. Let  $G$  be the computable group as in the construction and let

$$R_i = \{(g_1, g_2) \mid \langle g_1, g_2 \rangle \text{ is not a free non-abelian subgroup of } G_i\}.$$

For an even  $i$  we have

$$(g_1, g_2) \in R_i \Leftrightarrow g_1 g_2 = g_2 g_1.$$

Thus  $R_i$  is computable. Moreover it is easy to see that in  $G$  a pair  $(g_1, g_2)$  satisfies  $R$  if and only if the subgroup  $\langle g_1, g_2 \rangle$  is not isomorphic to  $F_2$  (i.e. its even projections are cyclic). The latter is equivalent to the property that  $\langle g_1, g_2 \rangle$  is amenable. Applying Theorem 2.2 we obtain  $G_{nA}$  satisfying statement (i) of the corollary.

It is worth noting that in the group  $G$  the condition to be amenable subgroup coincides with the condition to be without non-abelian free subgroup. Thus conditions (i) and (ii) are equivalent.

Now let us consider the unary relation *Odd* on  $G$ . Keeping  $G_\infty = F_2$  we see that its odd projections are the corresponding  $G_i$ , but its even projections are trivial. Applying Theorem 2.2 to  $G$  and *Odd* we have  $G_{nT}$  satisfying statement (iii) of the corollary.

It is known that any free group does not have property **(T)** but finite groups have this property. Moreover property **(T)** is preserved under homomorphisms. Now it is easy to see that a finite subset of  $G$  generates a subgroup with property **(T)** if and only if it generates a finite subgroup. Thus for singletons such subsets are represented by *Odd*. This proves (iv).

The last statement of the corollary follows from the facts that free and finite groups have the Haagerup property and the latter is preserved under direct sums (according chapt. 6 of [13] it is preserved under finite sums and holds in the group when all finitely generated subgroups satisfy it).  $\square$

## 2.2 Comments

(1) Is it possible to realize undecidability of conditions (i), (iii), (iv) of Corollary 2.3 (even separately) in computable torsion groups (or of bounded exponent)? By the positive solution of the restricted Burnside problem [35] if the group is of finite exponent then the size of a finite subgroup is bounded by a function depending on the number of the generators. Thus question (iii) is decidable in computable groups of finite exponent.

If  $G$  is  $B(m, n)$ , the free Burnside group, and  $n$  is odd and sufficiently large, then by a theorem of S.V. Ivanov [18] any infinite subgroup of  $G$  contains a copy of  $B(m, n)$ . Since  $B(m, n)$  are not amenable for sufficiently large odd  $n$ , [1], question (i) is decidable in such  $G$ .

This means that when  $n$  is large and odd and  $(B(m, n), a_1, \dots, a_m)$  (marked by the free generators) is a limit of finite marked groups  $(G_i, \bar{g}_i)$ ,  $i \in 2\mathbb{N} + 1$ , then Theorem 2.2 gives a computable torsion group with undecidable question (i) of Corollary 2.3. It is an open problem of geometric group theory if such a situation can be realized. V. Pestov notes in [30] (after G. Arzhantseva) that this question has positive solution provided that all hyperbolic groups are residually finite. The latter is a famous open problem of geometric group theory.

(2) In order to investigate more advanced properties of the group  $G$  from Corollary 2.3 it is interesting to know for which  $\alpha$  the relation  $\bar{a} \leq_\alpha \bar{b}$  from [3] is decidable in the group  $G$ . In the case  $\alpha = 1$  this relation exactly means that every existential formula satisfied by  $\bar{b}$  is satisfied by  $\bar{a}$ . The authors do not even know if this relation is computably enumerable (=decidable) in free groups. We only know that there is an algorithm which decides if  $\bar{a}$  and  $\bar{b}$  have the same existential type in  $F_2$ . Indeed by the Whitehead algorithm it is possible to determine if  $\bar{a}$  and  $\bar{b}$  belong to the same orbit. By [26] the latter is equivalent to the equality of existential types. It is worth noting that even if  $\leq_1$  is decidable in  $F_2$  in order to apply it to  $G$  we need a stronger result: decidability of the corresponding problem for direct sums of free groups. This problem seems to be more difficult.

(3) A relation  $R$  on a computable structure  $M$  is called **intrinsically computable** if it is computable in any computable copy of  $M$ . The following statement is Theorem 3.1 from [2] (originally proved in [4]). It is a slightly simplified version of Theorem 2.1 from [3], which we used above.

Let  $(M, R)$  be a computable structure whose existential diagram (i.e. the set of existential formulas with parameters which hold in  $(M, R)$ ) is computable. Then  $R$  is intrinsically computable on  $M$  if and only if both  $R$  and its complement are formally computably enumerable on  $M$ .

In this formulation formally c.e. means that  $R$  is equivalent to a c.e. disjunction (possibly infinite)  $\bigvee \phi_n(\bar{x}, \bar{c})$  of existential formulas over a tuple  $\bar{c}$ . The following proposition is very close to the proof of Theorem 2.2.

**Proposition 2.4.** *Let  $G$  and  $R$  be the group and the relation defined in Theorem 2.2. Then  $\neg R$  is not formally computably enumerable on  $G$ .*

*Proof.* Let  $\bar{\mathbf{c}} \in G$  and let  $\bigvee \phi_n(x_1, \dots, x_s, \bar{\mathbf{c}})$  be a disjunction of existential formulas of the group theory language. If  $\neg R$  is defined by this disjunction then each  $\phi_n(x_1, \dots, x_s, \bar{\mathbf{c}})$  implies  $\neg R(x_1, \dots, x_s)$ . Moreover if  $t$  is an even index outside the support of  $\bar{\mathbf{c}}$  and  $G_t \models \neg R(a_{t,1}, \dots, a_{t,s})$ , then the tuple  $\mathbf{a}_1, \dots, \mathbf{a}_s, \bar{\mathbf{c}}$  with

$$\mathbf{a}_j(t) = a_{t,j} \text{ for } 1 \leq j \leq s, \text{ and } \mathbf{a}_j(i) = 1 \text{ for } 1 \leq j \leq s \text{ and } i \neq t,$$

realizes some  $\phi_n(x_1, \dots, x_s, \bar{\mathbf{y}})$ . Fixing this  $n$  assume that

$$\phi_n(x_1, \dots, x_s, \bar{\mathbf{y}}) = \exists \bar{z} \phi'(x_1, \dots, x_s, \bar{\mathbf{y}}, \bar{z}),$$

where  $\phi'$  is quantifier free. Let  $m$  be a number which is greater than the sum of the lengths of the words appearing in  $\phi'(x_1, \dots, x_s, \bar{\mathbf{y}}, \bar{z})$ . Take  $\bar{\mathbf{d}}$  realizing  $\phi'(\mathbf{a}_1, \dots, \mathbf{a}_s, \bar{\mathbf{c}}, \bar{z})$  in  $G$ . Let  $\bar{d}$  be the projection of  $\bar{\mathbf{d}}$  to  $G_t$ . By the choice of  $G_i$  with odd  $i$  there is a sufficiently large odd index  $\hat{t}$  outside the support of  $\bar{\mathbf{c}}\bar{\mathbf{d}}$  and a tuple  $a'_{\hat{t},1}, \dots, a'_{\hat{t},s}, \bar{1}, \bar{d}' \in G_{\hat{t}}$  such that the map

$$w(a_{t,1}, \dots, a_{t,s}, \bar{1}, \bar{d}) \rightarrow w(a'_{\hat{t},1}, \dots, a'_{\hat{t},s}, \bar{1}, \bar{d}') \text{ where } |w(x_1, \dots, x_s, \bar{\mathbf{y}}, \bar{z})| \leq m,$$

induces an isomorphism from the substructure of the Cayley graph of  $G_t$  with respect to the generators  $a_{t,1}, \dots, a_{t,s}, \bar{d}$  to the corresponding substructure of  $(G_{\hat{t}}, a'_{\hat{t},1}, \dots, a'_{\hat{t},s}, \bar{d}')$ . Since  $G_t \models \phi'(a_{t,1}, \dots, a_{t,s}, \bar{1}, \bar{d})$ , we have  $G_{\hat{t}} \models \phi'(a'_{\hat{t},1}, \dots, a'_{\hat{t},s}, \bar{1}, \bar{d}')$ . Let us define

$$\mathbf{a}'_j(\hat{t}) = a'_{\hat{t},j} \text{ for } 1 \leq j \leq s, \text{ and } \mathbf{a}'_j(i) = 1 \text{ for } 1 \leq j \leq s \text{ and } i \neq \hat{t}, \text{ and}$$

$$\bar{\mathbf{d}}'(\hat{t}) = \bar{d}' , \bar{\mathbf{d}}'(t) = \bar{1} , \bar{\mathbf{d}}'(i) = \bar{\mathbf{d}}(i) \text{ for } i \notin \{\hat{t}, t\}.$$

Then the tuple  $\mathbf{a}'_1, \dots, \mathbf{a}'_s, \bar{\mathbf{c}}, \bar{\mathbf{d}}'$  satisfies  $\phi'(x_1, \dots, x_s, \bar{y}, \bar{z})$  and the tuple  $\mathbf{a}'_1, \dots, \mathbf{a}'_s, \bar{\mathbf{c}}$  satisfies  $\phi_n(x_1, \dots, x_s, \bar{y})$  in  $G$ . Since  $\mathbf{a}'_1, \dots, \mathbf{a}'_s$  satisfies  $R$ , we obtain a contradiction.  $\square$

This proposition suggests that the group  $G$  and the relation  $R$  from the proof of Corollary 2.3(i) also satisfy the conditions of Theorem 3.1 from [2] (they are stronger than ones used in the proof of Corollary 2.3). Furthermore, according [4] to get the conclusion of Theorem 3.1 from [2] one can weaken decidability of the the existential diagram of the structure  $(G, \cdot, 1, R)$  (this is the only remaining task) by decidability of the set of existential formulas of the diagram which contain only one  $R$ -literal. However the authors were not able to prove either of these conditions. The attempts which were made lead us to the following problem.

Is there a family of finite two-generated groups

$$\mathcal{G} = \{G_l = \langle a_l, b_l \rangle : l \in \omega\}$$

such that the universal (or elementary) theory of  $\mathcal{G}$  is decidable and  $(F_2, a, b)$  is a limit group of this family in the Grigorchuk's topology?

It is worth noting that the elementary theory of  $(F_2, a, b)$  is decidable, [20].

### 3 A finitely presented group with undecidable amenability

The following theorem is based on Corollary 2.3. We will also use Theorem 1 of [16] (see also [17]).

**Theorem 3.1.** *There is a finitely presented group  $H_{nA}$  with decidable word problem such that detection of finite subsets of  $H_{nA}$  which generate amenable subgroups is not decidable. Detection of finite subsets of  $H_{nA}$  which generate subgroups which do not embed free non-abelian groups is also not decidable. Moreover, the corresponding finitely presented example  $H_{nT}$  can be found for properties (iii) and (iv) of Corollary 2.3.*

*Proof.* Let  $G_{nA}$  be a computable group given by Corollary 2.3. Our construction starts with the following statement of A. Darbinyan.

- For any computable group  $G$  there is a two-generated group  $G_1 = \langle c, s \rangle$  with decidable word problem and a computable embedding  $\phi_1 : G \rightarrow G_1$ .

This is a part of the statement of Theorem 1 of [16] and the definition of  $\phi_1$  at line 19 on p.4927 of that paper. We only add that applying this theorem

- we consider our  $G$  with respect to the generators given by any enumeration of  $G = \{a^{(1)}, a^{(2)}, \dots\}$ ,
- $\phi_1$  is denoted by  $\phi$  in [16], and
- the definition of  $\phi$  in [16] should be corrected by  $a^{(i)} \rightarrow [c, c^{s^{2^i-1}}]$ .

We apply this theorem to  $G = G_{nA}$ . Then it is clear that detection of finite subsets of  $G_1$  which generate amenable subgroups is not decidable. Theorem 6 of [15] states that

- a finitely generated group  $G_1$  with decidable word problem can be embedded into a finitely presented group with decidable word problem.

Applying this theorem we obtain the group  $H_{nA}$  as in the formulation. Indeed the embedding  $\phi_2 : G_1 \rightarrow H_{nA}$  is obviously defined by the restriction of  $\phi_2$  to the finite set of generators of  $G_1$ . Thus  $\phi_2$  is computable. Having this we see that the problem if a finite subset of  $H_{nA}$  generates an amenable subgroup is not decidable.

The same argument can be applied to properties (iii) - (iv) of Corollary 2.3.  $\square$

## Acknowledgements

The authors are grateful to M. Cavaleri, T. Ceccherini-Silberstein and L. Kołodziejczyk for reading the paper and helpful remarks. We also thank M. Sapir for right advice concerning Higman's embedding. The authors are grateful to the referee for remarks which substantially improved the exposition.

This research was partially supported by (Polish) Narodowe Centrum Nauki, grant UMO-2018/30/M/ST1/00668.

## References

- [1] Adyan, S.: Random walks on free periodic groups, *Mathematics of the USSR - Izvestiya* 21(3), 425 – 434 (1983)
- [2] Ash, C. J.: Isomorphic recursive structures. in: Ershov, Y.L., Goncharov, S.S., Nerode, A., Remmel, J.B., Mark, V.W. (eds.) *Handbook of recursive mathematics*, Vol. 1, pp. 167 - 181, *Stud. Logic Found. Math.*, 138, North-Holland, Amsterdam (1998)
- [3] Ash, C. J. and Knight, J. F.: Possible degrees in recursive copies. II. *Ann. Pure Appl. Logic* 87, 151 - 165. (1997)
- [4] Ash, C. J. and Nerode, A.: Intrinsically recursive relations, in: Crossley, J.N. (ed) *Aspects of effective algebra* (Clayton, 1979), pp. 26 - 41, Upside Down A Book Co., Yarra Glen, Vic. (1981)
- [5] Baumslag, G., Boon, W.W., Neumann, B.H.: Some unsolvable problems about elements and subgroups of groups, *Math. Scand.* 7, 191 – 201 (1959)
- [6] Bekka, B., de la Harpe P., Valette, A.: *Kazhdan's Property (T)*, *New Mathematical Monographs*, 11, Cambridge University Press, Cambridge (2008)
- [7] Bilanovic, I., Chubb J. and Roven, S.: Detecting properties from presentations of groups. *Arch. Math. Log.* 59 293 – 312 (2020)
- [8] Bogopolski, O., Ivanov, A.: Notes about decidability of exponential equations, [arXiv:2105.06842](https://arxiv.org/abs/2105.06842)
- [9] Cavaleri, M.: Computability of Følner sets, *International Journal of Algebra and Computation*, vol. 27, 819-830 (2017)
- [10] Cavaleri, M.: Følner functions and the generic Word Problem for finitely generated amenable groups, *Journal of Algebra*, vol. 511, 2018, Pages 388-404 (2018)
- [11] Ceccherini-Silberstein, T. and Coornaert, M.: *Cellular automata and groups*, *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, (2010)
- [12] Champetier, Ch., Guirardel, V.: Limit groups as limits of free groups: compactifying the set of free groups, *Israel J. Math.* 146, 1 – 75 (2005)
- [13] Cherix, P.-A., Cowling, M., Jolissaint, P., Julg, P., and Valette, A.: *Groups with the Haagerup Property*. Birkhäuser, *Progress in Mathematics* 197 (2001)
- [14] Coornaert, M.: *Topological dimension and dynamical systems*, Universitext, Springer International Publishing Switzerland, (2015)
- [15] Clapham, C.R.J.: An embedding theorem for finitely generated groups, *Proc. London Math. Soc.* 17, 419 – 430 (1967)
- [16] Darbinyan, A.: Group embeddings with algorithmic properties, *Comm. Algebra* 43, no. 11, 4923 – 4935 (2015)
- [17] Darbinyan, A.: Computability, orders, and solvable groups. *The Journal of Symbolic Logic*, 85(4), 1588-1598 (2020)
- [18] Ivanov, S.V.: On subgroups of free Burnside groups of large odd exponent, *Illinois Journ. Math.* 47, 299 – 304 (2003)
- [19] Kapovich, I.: Subgroup properties of fully residually free groups, *Transactions of the American Mathematical Society* 354. 335-362 (2001)

- [20] Kharlampovich, O. and Myasnikov, A.: Elementary theory of free non-abelian groups, *Journal of Algebra* 302, no. 2, 451 - 552 (2006)
- [21] Khoushainov, B. and Myasnikov, A.: Finitely presented expansions of groups, semigroups, and algebras, *Transactions of the American Mathematical Society*. 366. (2014)
- [22] Kierstead, H.: An effective version of Hall's Theorem, *Proc. Amer. Math. Sur.*, vol. 88(1983) 124-128. doi:10.2307/2045123
- [23] Lyndon, R.C.: and Schupp, P.E.: *Combinatorial Group Theory*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* 89, Springer-Verlag, Berlin, Heidelberg, New York (1977)
- [24] Miller III, C. F.: Decision Problems for Groups – Survey and Reflections, in: *Algorithms and Classification in Combinatorial Group Theory* (ed. G. Baumslag et al), pp 1 – 59 , Springer - Verlag, New York (1992)
- [25] Moriakov, N.: On effective Birkhoff's ergodic theorem for computable actions of amenable groups. *Theory Comput. Syst.* 62, 1269 – 1287 (2018)
- [26] Nies, A.: Aspects of free groups, *J. Algebra* 263, 119–125 (2003)
- [27] Olshanskii, A.Yu., Sapir, M.V.: The conjugacy problem and Higman embeddings, *Memoirs of the AMS* 170, no. 804 (2004)
- [28] Olshanskii, A.Yu., Sapir, M.V.: Subgroups of finitely presented groups with solvable conjugacy problem, *Int. J. Algebra and Comput.*, **15**, 1-10 (2005)
- [29] Olshanskii, A.Yu., Sapir, M.V.: Algorithmic problems in groups with quadratic Dehn function, *Arxiv*: 2012.10417
- [30] Pestov, V.: Hyperlinear and sofic groups: a brief guide, *Bul. Symb. Logic* 14, 449–480 (2008)
- [31] Soare, R. I.: *Turing Computability, Theory and Applications*, Springer-Verlag Berlin Heidelberg (2016)
- [32] Tomkowicz G., Wagon S.: *The Banach-Tarski Paradox (Encyclopedia of Mathematics and its Applications)* 2nd Edition Cambridge University Press (2019)
- [33] Vershik, A.: Amenability and approximation of infinite groups, *Selecta Math. Soviet* 2, no. 4, 311-330. (1982) Selected translations.
- [34] Sela, Z.: Diophantine geometry over groups I: Makanin-Razborov diagrams , *Publications Mathematiques de l'IHES* 93, 31 – 105 (2001)
- [35] Zelmanov, E.: Solution of the restricted Burnside problem for groups of odd exponent, *Mathematics of The USSR-izvestiya* 36, 41 – 60 (1991)