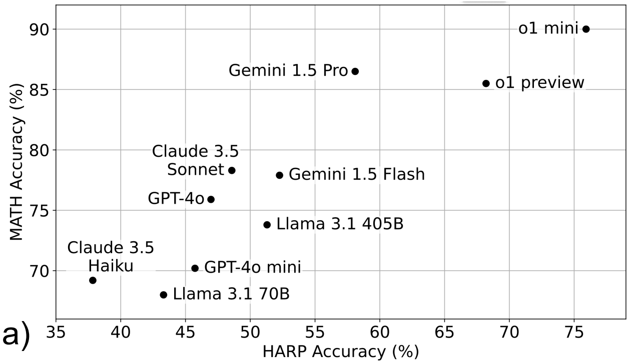

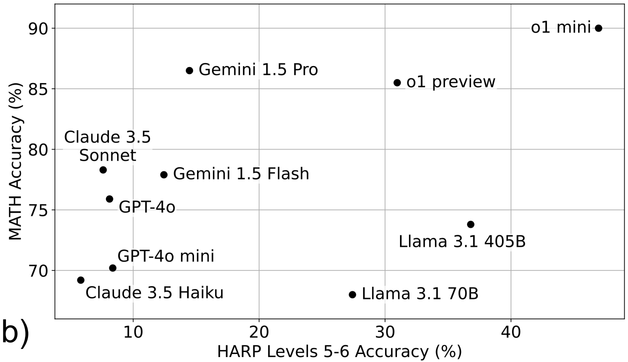

c)

Problem:

When n standard six-sided dice are rolled, the product of the numbers rolled can be any of 936 possible values. What is n?

Metadata:

Subject: Number Theory
Difficulty: Level 4
Year/Contest/#: 2023/AMC 12/23
Requires choices: No

Choices:

A. 11 B. 6
C. 8 D. 10
E. 9

Human-written Solutions:

We start by trying to prove a function of n, and then we can apply the function and equate it to 936 to find the value of n. It is helpful to think of this problem in the format $(1+2+3+4+5+6)\cdot(1+2+3+4+5+6)\ldots$. Note that if we represent the scenario in this manner, we can think of picking a 1 for one factor and then a 5 for another factor to form their product - this is similar thinking to when we have the factorized form of a polynomial. Unfortunately this is not quite accurate to the problem because we can reach the same product in many ways: for example for $n = 2$, 4 can be reached by picking 1 and 4 or 2 and 2. However, this form gives us insights that will be useful later in the problem. Note that there are only 3 primes in the set $\{1, 2, 3, 4, 5, 6\}$: 2, 3, and 5. Thus if we're forming the product of possible values of a dice roll, the product has to be written in the form $2^h \cdot 3^i \cdot 5^j$ (the choice of variables will become clear later), for integer nonnegative values h, i, j. So now the problem boils down to how many distinct triplets (h, i, j) can be formed by taking the product of n dice values. We start our work on representing j: the powers of 5, because it is the simplest in this scenario because there is only one factor of 5 in the

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The product can be written as

$$2^a 3^b 4^c 5^d 6^e = 2^{a+2c+e} 3^{b+e} 5^d.$$

Therefore, we need to find the number of ordered tuples $(a+2c+e, b+e, d)$ where a, b, c, d, e are non-negative integers satisfying $a+b+c+d+e \leq n$. We denote this number as $f(n)$. Denote by $g(k)$ the number of ordered tuples $(a+2c+e, b+e)$ where $(a,b,c,e) \in \Delta_k$ with $\Delta_k \triangleq \{(a,b,c,e) \in \mathbb{Z}_+^4 : a+b+c+e \leq k\}$. Thus,

$$\begin{aligned} f(n) &= \sum_{d=0}^{n} g(n-d) \\ &= \sum_{k=0}^{n} g(k). \end{aligned}$$

Next, we compute $g(k)$. Denote $i = b+e$. Thus, for each given i, the range of $a+2c+e$ is from 0 to $2k-i$. Thus, the number of $(a+2c+e, b+e)$ is

$$\begin{aligned} g(k) &= \sum_{i=0}^{k} (2k-i+1) \\ &= \frac{1}{2}(k+1)(3k+2). \end{aligned}$$

Therefore,

$$f(n) = \sum_{k=0}^{n} g(k)$$

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