28$^{\text{th}}$ TJIMO

Alexandria, Virginia

Round: **Team**

Problem 1. If the prime factorization of 60^6 is $2^a \cdot 3^b \cdot 5^c$, where a, b, c are positive integers, find $a+b+c$.

Answer. $\boxed{24}$

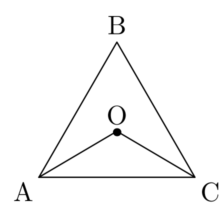
Solution. Notice $60^6 = (2^2 \cdot 3 \cdot 5)^6 = 2^{12} \cdot 3^6 \cdot 5^6$. Thus our answer is $12+6+6 = \boxed{24}$.

Problem 2. A bag contains marbles of three colors: 10 red marbles, p green marbles, and 55 blue marbles. The probability of randomly selecting a green marble from the bag is $\frac{p}{90}$. Find the probability of selecting a blue marble.

Answer. $\boxed{\frac{11}{18}}$

Solution. Notice the probability of selecting a green marble from the bag is $\frac{p}{55+10+p} = \frac{p}{65+p}$. If this is equal to $\frac{p}{90}$, then $65+p = 90$ and $p = 25$, so the probability of selecting a blue marble is $\frac{55}{10+25+55} = \frac{55}{90} = \boxed{\frac{11}{18}}$.

Problem 3. The area of equilateral triangle ABC (AB = BC = CA) is 300. O is the center of the triangle, as shown below.



Find the area of triangle AOC.

Answer. $\boxed{100}$

Solution. Notice by symmetry $[AOB] = [BOC] = [COA]$ and they sum to 300 so our answer is $\boxed{100}$.

Problem 4. Given that x, y are positive real numbers satisfying $x + \frac{1}{y} = 5$ and $y + \frac{1}{x} = 7$, find $\frac{x}{y}$.

Answer. $\boxed{\frac{5}{7}}$

Solution. We can clear denominators in both equations to obtain $xy+1 = 5y$ and $xy+1 = 7x$. Thus, $5y = 7x$ and we see $\frac{x}{y} = \boxed{\frac{5}{7}}$.

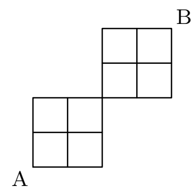
Problem 5. Let right triangle ABC have $AB = 6$, $BC = 8$, and $CA = 10$. Let X be the midpoint of AB, let Y be the midpoint of BC, and let Z be the midpoint of CA. Find the area of XYZ.

Time limit: 30 minutes

Answer. $\boxed{6}$

Solution. The area of the middle triangle is $\frac{1}{4}$ the area of the outer triangle, so our answer is $\frac{1}{4} \cdot \frac{1}{2} \cdot 6 \cdot 8 = \boxed{6}$.

Problem 6. Alex the ant is at point A on the below grid.



If he can only move up or to the right on the grid, find the number of distinct paths Alex can take to go from point A to B.

Answer. $\boxed{36}$ (paths)

Solution. Label the middle point as C. We can count 6 paths to go from A to C. (Alternatively, we notice that to go from A to C, we must move up twice and move to the right twice. Then there are $\binom{4}{2}$ ways to choose the order in which to make the moves) Then similarly there are 6 paths to go from C to B. Thus, there are $6 \cdot 6 = \boxed{36}$ total paths to go from A to B.

Problem 7. If x, y are distinct positive integers such that

$$x^2 + 2xy = 40$$

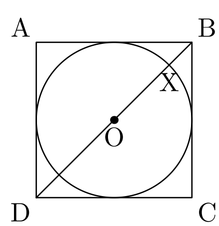
$$y^2 + 2xy = 33$$

find $x + y$.

Answer. $\boxed{7}$

Solution. Subtracting the equations gives $x^2 - y^2 = 7$ or $(x - y)(x + y) = 7$. We see $x + y = \boxed{7}$.

Problem 8. Suppose square $ABCD$ has side length of 10 and circle O is inscribed in the square, as shown below.



Segment BO intersects the circle again at X. Find the length of BX.

Answer. $\boxed{5\sqrt{2}-5}$
Solution. Let the other intersection of BD and circle O be Y. Notice by symmetry, $BX = DY$. Also, $BX + XY + YD = BD = 10\sqrt{2}$. We know XY is the diameter of circle O, so $XY = 10$. Thus $BX + YD = 10\sqrt{2} - 10$, so $BX = \boxed{5\sqrt{2}-5}$.

Problem 9. Joe lists all the three digit positive integers on a sheet of paper. However, he skips all the integers that are divisible by three and all the integers that are divisible by five. Determine how many integers Joe counts.

Answer. $\boxed{480}$ (integers)
Solution. We use the Principle of Inclusion-Exclusion (PIE). There are 900 total three-digit positive integers. We subtract off the 300 multiples of three and 180 multiples of five, and add the 60 multiples of 15 to obtain $900 - 300 - 180 + 60 = \boxed{480}$.

Problem 10. Compute the number of positive integers $1 \le x \le 143$ such that $x^2 + x^3$ is a perfect square.

Answer. $\boxed{11}$ (integers)
Solution. If $x^2 + x^3 = x^2(x+1)$ is the square of an integer, then $(x+1)$ must be the square of an integer. We also know $2 \le x + 1 \le 144$. Thus $x + 1 \in \{2^2, 3^2, \cdots, 12^2\}$, so there are $\boxed{11}$ possible values of x.