Set 1

Problem 1. How many distinct arrangements of the word "MATH" are there?

Answer. 24
Solution. $4! = 24$.

Problem 2. Find the area of a square with perimeter 16.

Answer. 16
Solution. The area of an equilateral triangle with side length s is s^2, so $s = 4$. Then the perimeter is $4s = 16$.

Problem 3. Compute the smallest positive integer that leaves a remainder of 3 when divided by 4 and a remainder of 4 when divided by 5.

Answer. 19
Solution. The number is one less than a multiple of 4 and one less than a multiple of 5, so it must be one less than a multiple of 20. The smallest such positive integer is $20 - 1 = 19$.

Problem 4. If the sum of five consecutive integers is 25, what is the largest of the five integers?

Answer. 7
Solution. Let the five consecutive integers be $n - 2, n - 1, n, n + 1$, and $n + 2$. $(n - 2) + (n - 1) + (n) + (n + 1) + (n + 2) = 5n = 25$, so $n = 5$. The largest number is $n + 2 = 7$.

Time limit: 40 minutes.

Set 2

Problem 5. Compute $3 + 5 + 7 + \ldots + 25$.

Answer. 168

Solution. The sum $1 + 3 + 5 + \ldots + n + \ldots + (2n-1) = n^2$. Therefore, the sum is $13^2 - 1 = 168$.

Problem 6. If Kao has 3 fair, standard dice, what is the probability that the 3 numbers she rolls sum to 18?

Answer. $\dfrac{1}{216}$

Solution. The only way to get a sum of 18 with 3 rolls is if she rolls a 6 every time. Therefore, the probability is $\dfrac{1}{6 \cdot 6 \cdot 6} = \dfrac{1}{216}$.

Problem 7. Compute $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{9 \cdot 10}$.

Answer. $\dfrac{9}{10}$

Solution. This expression can be rewritten as a telescoping sum. Each term in the form of $\dfrac{1}{n(n+1)}$ can be rewritten as $\dfrac{1}{n} - \dfrac{1}{n+1}$. Using this, our original expression can be written as $\dfrac{1}{1} - \dfrac{1}{2} + \dfrac{1}{2} - \dfrac{1}{3} + \cdots + \dfrac{1}{9} - \dfrac{1}{10} = 1 - \dfrac{1}{10} = \dfrac{9}{10}$.

Problem 8. Costco only sells pencils in packs of 3 and 5. What is the largest integer number of pencils that Akshaj cannot purchase?

Answer. 7

Solution. By the Chicken McNugget Theorem, since 3 and 5 are relatively prime, the largest number of pencils that Akshaj cannot purchase is $(3)(5) - 3 - 5 = 7$ pencils. Alternatively, note that 7 is not attainable, $8 = 3 + 5$, $9 = 3 + 3 + 3$, and $10 = 5 + 5$, so any number 8 or above is attainable.

Time limit: 40 minutes.

Set 3

Problem 9. Jeffery has a very big lawn. If Akshaj can mow Jeffery's lawn in 8 hours and Michael can mow Jeffery's lawn in 12 hours, how many hours would it take Akshaj and Michael to mow Jeffery's lawn (working together)?

Answer. $\dfrac{24}{5}$

Solution. Akshaj can mow $\dfrac{1}{8}$ of Jeffery's lawn per hour, while Michael can mow $\dfrac{1}{12}$ of the lawn in an hour. Combined, those two can mow $\dfrac{5}{24}$ of Jeffery's enormous lawn in an hour, thus it will take them $\dfrac{24}{5}$ hours to mow Jeffery's lawn.

Problem 10. The Cookie Monster enjoys eating cookies every day. He starts out with 80 cookies and never adds any cookies to the jar. On the 1st day he eats 1 cookie, on the 2nd day he eats 2 cookies, and continues doing so such that on the nth day, he eats n cookies, unless there are less than n cookies remaining, in which case he simply eats all the remaining cookies. After how many days will the Cookie Monster's jar be empty?

Answer. 13

Solution. This problem uses consecutive sums: $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$. Let n be the number of days it will take Cookie Monster to finish his jar of cookies, so $\dfrac{n(n+1)}{2} = 80$. Since $n = 12$ is too small (just under 80), it will take Cookie Monster 13 days to finish his cookie jar.

Problem 11. What is the least possible integer for which 35% of that number is greater than 3?

Answer. 9

Solution. Let n be the least possible integer that satisfies the problem statement.

$0.35n > 3$ so $n > \frac{3}{0.35} \approx 8.57$. Therefore, the least possible integer is 9.

Problem 12. Harry has a standard deck of cards (52 cards, 13 in each suit). He deals out a hand with 13 cards in it. He then separates his hand into piles based on suit. The pile with the most cards in it has Q cards. What is the smallest value Q can assume?

Answer. 4

Solution. We are essentially distributing 13 cards into 4 piles, or in the terms of the Pigeonhole Principle, trying to fit 13 pigeons into 4 holes. We can see that no matter how we do the fitting, there is always going to be a pile with 4 or more cards.

Set 4

Problem 13. How many digits are in the numerical value of the product $4^4 \cdot 5^5$?

Answer. 6
Solution. We can see that the product is equivalent to $2^8 \cdot 5^5$. By regrouping the product, we get $2^3(2 \cdot 5)^5$ which equals $8 \cdot 10^5 = 800000$. We can easily see that the final product has 6 digits.

Problem 14. There are 100 light bulbs lined up in a row in a long room. Each bulb has its own switch and is currently switched off. The room has an entry door and an exit door. There are 100 people lined up outside the entry door. Each bulb is numbered consecutively from 1 to 100 and so is each person. Person 1 enters the room, switches on every bulb, and exits. Person 2 enters and flips the switch on every second bulb (turning off bulbs 2, 4, 6...). Person 3 enters and flips the switch on every third bulb (changing the state on bulbs 3, 6, 9...). This continues until all 100 people have passed through the room. How many light bulbs are on at the very end?

Answer. 90
Solution. We observe that every light bulb n is flipped k times, where k is the number of positive divisors of n. We can see that every nonsquare number has an even number of factors, so any light bulb of a nonsquare number is flipped an even number of times. If a light is flipped an even number of times, it will go back to its original state, which was off. We can see that the number of factors of all perfect squares is odd. There are 10 squares up to 100, so our answer is $100 - 10 = 90$.

Problem 15. The TJ Varsity Math team went out and ordered 9 eight-slice pizzas. Frankie came and ate $\frac{1}{6}$ of all the pizza slices. Oxahaj was next and he ate $\frac{2}{5}$ of all the pizza slices left. Samuel then ate $\frac{1}{2}$ of all the slices remaining. How many slices of pizza are left?

Answer. 18
Solution. There are a total of 72 slices of pizza. Frankie ate $\frac{1}{6} \cdot 72 = 12$ slices, leaving 60 slices. Oxhaj ate $\frac{2}{5} \cdot 60 = 24$ of the remaining pizza, leaving Samuel with 36 slices. Samuel ate $\frac{1}{2} \cdot 36 = 18$ slices, which means there are 18 slices of pizzas remaining.

Problem 16. Sam buys some tacos from Taco Bell. Six of the tacos have beef, eight have chicken, eleven have pork, three have beef and pork, two have beef and chicken, five have chicken and pork, and two have beef, chicken, and pork. How many tacos did Sam buy?

Answer. 17
Solution. By the Principle of Inclusion-Exclusion (PIE), we have $|B \cup C \cup P| = |B| + |C| + |P| - |B \cap C| - |B \cap P| - |C \cap P| + |B \cap C \cap P|$ where B, C, P, represent the set of tacos that include beef, chicken, and pork, respectively. Therefore, we have $6 + 8 + 11 - 3 - 2 - 5 + 2 = 17$.

Time limit: 40 minutes.

Set 5

Problem 17. Harry lives on a coordinate plane. Starting from point A, he walks 4 units north, 3 units east, 8 units north, and then 6 more units east, in this order, ending at point B. What is the straight-line distance from A to B?

Answer. 15

Solution. In total, Harry walks $4 + 8 = 12$ units up and $3 + 6 = 9$ units to the right. These two lengths are the legs of a right triangle. The length of the hypotenuse is the shortest distance from A to B, so applying the Pythagorean Theorem, we have: $\sqrt{9^2 + 12^2} = 15$.

Problem 18. What is the probability that three standard six-sided dice show exactly two distinct face values when rolled?

Answer. $\frac{5}{12}$

Solution. One way to solve this is with complementary counting. There are 6 out of 216 ways to roll three equal face values (one distinct face value), so this outcome has probability $\frac{1}{36}$. There are $_6P_3 = 6\cdot5\cdot4 = 120$ out of 216 ways to choose and permute three distinct face values across the three dice, so this outcome has probability $\frac{20}{36}$. Hence, the probability of rolling exactly two distinct face values is $1 - \frac{1}{36} - \frac{20}{36} = \frac{5}{12}$.

Problem 19. If $\sqrt{a} + \sqrt{b} = \sqrt{13 + 2\sqrt{42}}$ for some positive integers a and b, what is $a^2 + b^2$?

Answer. 85

Solution. Squaring both sides of the equation gives $a + b + 2\sqrt{ab} = 13 + 2\sqrt{42}$. Thus we have $a + b = 13$ and $ab = 42$, so $a = 7$ and $b = 6$. Then $a^2 + b^2 = 49 + 36 = 85$.

Problem 20. If the greatest common divisor of two positive integers is 5 and their least common multiple is 30, what is their product?

Answer. 150

Solution. It is well known that for any integers a and b, we have $\gcd(a, b) \cdot \mathrm{lcm}(a, b) = ab$. Thus $ab = 5 \cdot 30 = 150$. One can also guess and check to find that $a = 10$ and $b = 15$ works.

Time limit: 40 minutes.

Set 6

Problem 21. Points A, B, C, and D lie on a circle. Chords AC and BD intersect at a right angle at point E inside the circle. If $AE = 6$, $BE = 8$, and $CE = 4$, compute the length of CD.

Answer. 5

Solution. By Power of a Point, $AE \cdot CE = BE \cdot DE$, or $6 \cdot 4 = 8 \cdot DE$, so $DE = 3$. Therefore, by the Pythagorean Theorem, $CD = \sqrt{CE^2 + DE^2} = \sqrt{3^2 + 4^2} = 5$.

Problem 22. What is the area enclosed by the x-axis, $y = 3x + 6$, and $y = -3x + 12$?

Answer. 27

Solution. The equations create a triangular area with a base of 6 and a height of 9. The area is $\left(\dfrac{1}{2}\right)(6)(9) = 27$.

Problem 23. The Thorne Miniature room in the Chicago Art Institute consists of 64 tiny replicas of real-world rooms. If a 5 feet long table is represented by a 6 inch replica, and the scales are constant within the exhibition, what is the real life size area, in square inches, of a 25 square inch carpet replica?

Answer. 2500 square inches

Solution. The scaling ratio is $\dfrac{6 \text{ inch}}{5 \text{ feet}} = \frac{6}{60} = \frac{1}{10}$, so the ratio of the area of the real carpet to that of its replica is $10^2 = 100$, so the area is $25 \cdot 100 = 2500$.

Problem 24. How many positive integers less than or equal to 50 are relatively prime to 50?

Answer. 20

Solution. Euler's totient function, $\phi(n)$, finds the number of numbers less than or equal to n that are relatively prime to n, where n is a positive integer. $\phi(50) = 50\left(1 - \dfrac{1}{2}\right)\left(1 - \dfrac{1}{5}\right) = 20$.

Alternatively, note that any such number must be odd, eliminating the 25 even numbers. Of the remaining, 5 are multiples of 5 (5, 15, 25, 35, and 45), leaving a total of 20 integers relatively prime to 50.

Set 7

Problem 25. Suppose that the following are true:

- Hao always laughs in math class.

- Math class is on Wednesday and Friday.

- If Hao laughs in a class, he will laugh for the rest of the day.

How many of the following are always true?

- Hao only laughs two days a week.

- Hao will laugh on Friday.

- Hao will not laugh on Monday.

- Hao will laugh in Physics, which he has after math.

Answer. 2

Solution. The second and fourth statements are always true, but there is no guarantee that any other statement is always true.

Problem 26. In how many ways can 1128 be written as a product of two positive integers if the order doesn't matter?

Answer. 8

Solution. 1128 can be prime factored into $2^3 \cdot 3 \cdot 47$. We can see that 1128 has $4 \cdot 2 \cdot 2 = 16$ positive factors and each product pair has two prime factors, thus there are $16 \div 2 = 8$ ways 1128 can be written as a product of two positive integers.

Problem 27. A circle is inscribed inside a square with perimeter 40, and a square is inscribed inside the circle. Compute the area of the smaller square.

Answer. 50

Solution. The side length of the big square is 10, which is a diameter of the circle, which is a diagonal of the smaller square. Therefore the area of the smaller square is $\frac{1}{2} \cdot 10 \cdot 10 = 50$.

Problem 28. Ray started with a positive integer x, divided it by 4, discarded any remainder, and obtained an integer y. If $y = 12$, what is the sum of all possible values of x?

Answer. 198

Solution. Working backwards, we can get the possible values of x: $4y$, $4y+1$, $4y+2$, $4y+3$, which would be 48, 49, 50, and 51, for a sum of 198.

Time limit: 40 minutes.

Set 8

Problem 29. Big Ben is a huge philanthropist. He loves to give back to the community and sometimes even gives people things they didn't ask for. If he donated 100 dollars to charity on his first year of donation and increased the amount by 100% every year, in which year will the sum of his donations surpass $10,000?

Answer. 7th year

Solution. The second year, he will donate $200, for a total of $300. The third year, he will donate $400, for a total of $700. In general, the nth year, he will donate $100 \cdot 2^{n-1}$, and his total will be $100 \cdot (2^n - 1)$. The first value of n for which this is greater than 10000 is $n = 7$.

Problem 30. Determine the smallest positive integer that ends in 7 and is divisible by exactly 3 primes, all of which are distinct.

Answer. 357

Solution. Note that the primes must be odd and cannot be 5. Simple experimentation yields $3*7*17 = 357$

Problem 31. Harry decides to paint the entire outside of a $10 \times 12 \times 8$ rectangular prism. He then decides to cut the rectangular prism into $1 \times 1 \times 1$ cubes and picks one of them. What is the probability that the cube picked has at least 2 sides painted?

Answer. $\frac{13}{120}$

Solution. There are a total of $10 \cdot 12 \cdot 8 = 960$ cubes. Note that the cubes with at least two sides painted lie on the edges of the box. Adding up the cubes on the edges, we get that there are 104 cubes with at least two sides painted. Therefore, the probability is $\dfrac{104}{960} = \dfrac{13}{120}$.

Problem 32. Estimate the total word count in the solutions to this Guts Round. Your answer will be considered correct if it is within 100 of the correct answer.

Answer. 2617

Solution. Trivial. Just count. Or write a program.

Time limit: 40 minutes.