28^{th} TJIMO
ALEXANDRIA, VIRGINIA

Round: **Individual**

Problem 1. Compute $2^{0+1\cdot 6} - 2^0 \cdot 1 + 7$.

Answer. $\boxed{70}$
Solution. We follow the order of operations to obtain

$$\begin{aligned} 2^{0+1\cdot 6} - 2^0 \cdot 1 + 7 &= 2^{0+6} - 1 \cdot 1 + 7 \\ &= 2^6 - 1 + 7 \\ &= 64 - 1 + 7 = \boxed{70}. \end{aligned}$$

Problem 2. A certain circle has an area whose value is twice the value of its circumference. Compute the diameter of the circle.

Answer. $\boxed{8}$
Solution. The area of a circle with radius r is πr^2, and its circumference is $2\pi r$. We have that $\pi r^2 = 2 \cdot 2\pi r$. Dividing by πr on both sides leaves us with $r = 4$, so the diameter is $\boxed{8}$.

Problem 3. The sum of the digits of the year 2016 is $2 + 0 + 1 + 6 = 9$. What is the next year for which the sum of the digits is also 9?

Answer. $\boxed{2025}$
Solution. Note that for $2017, 2018$, and 2019 have sum of digits greater than 9. The sum of digits of 2020 is 4, so we need $9 - 4 = 5$ more years. Hence $\boxed{2025}$ is the next year with sum of digits equal to 9.

Problem 4. Compute $1 + 2 + 3 + \cdots + 63$.

Answer. $\boxed{2016}$
Solution. Note that $1 + 63 = 64$, $2 + 62 = 64$, $3 + 61 = 64$, and so forth. Therefore $(1 + 63) + (2 + 62) + (3+61)+\cdots(63+1) = 63\cdot 64$is twice the desired sum. Thus the desired sum is $\frac{1}{2}\cdot 63\cdot 64 = 63\cdot 32 = \boxed{2016}$.

Problem 5. A regular n-sided polygon has 2015 diagonals. Compute the number of diagonals in a regular $n + 1$-sided polygon.

Answer. $\boxed{2079}$ (diagonals)
Solution. insert

Problem 6. Two circles with equal radii are placed inside a 16×18 rectangular box as shown in the diagram. Each circle touches two walls of the box. The two circles are also touching each other at one point inside the box. What is the radius of both circles?

Time limit: 60 minutes.

Answer. 5
Solution. The point of tangency of the circles is the center of the rectangle. Therefore, we can compress the original rectangle by a factor of $\frac{1}{2}$ to make one of its corners lie on the tangency point of the two circles, as shown. Then we add in line segments representing the radii of one of our circles, each radius directed toward a different point of tangency.

Now we consider the shaded right triangle in the diagram. If we let r be the radius of the circle, the right triangle has legs of length $8-r$ and $9-r$ and hypotenuse of length r. Thus, by the Pythagorean Theorem, we have $(8-r)^2+(9-r)^2=r^2$, which reduces to $r^2-34r+145=0$. Factoring this gives us $(r-5)(r-29)=0$. The radius of the circle cannot be 29 (the circle needs to fit inside a 16×18 box), so our only solution is $r=\boxed{5}$.

Problem 7. If $\frac{1}{x}+\frac{1}{y}=5$ and $x+y=10$, what is x^2+y^2?

Answer. 96
Solution. Multiplying the two equations, we get $\frac{x}{y}+\frac{y}{x}+2=50$. Subtracting by 2 and multiplying by xy, we see that $x^2+y^2=48xy$. Also, note that from squaring the second equation, $x^2+y^2+2xy=100$. We now have a system of equations with xy and x^2+y^2. Solving, we get our answer of $\boxed{96}$.

Problem 8. There are 2016 mathematicians at the Annual Mathematics Conference, and each mathematician brought along a nonmathematician partner. NAME, a mathematician, brought NAME, a journalist, who asked each of the 4031 people besides himself how many people they knew besides themselves and their partner. (Knowing is, so if person A knows person B then person B knows person A.) NAME received a different answer from each person. Compute the number of people that NAME knows.

Answer. $\boxed{2016}$ (people)
Solution. insert solution

Problem 9. Quadrilateral $ABCD$ has sides $AB=1008$, $BC=2016$, and $CD=2016$. If $\angle BAD$ is a right angle and $m\angle ADC=\frac{1}{2}m\angle BCD$, compute $m\angle BCD$ in degrees.

Answer. $\boxed{108}$ (degrees)
Solution. Reflect the quadrilateral about side AD, and let B' be the reflection of B and C' that of C. Then pentagon $BCDC'D'$ is equilateral. Furthermore, $\angle BCD\cong\angle CDC'\cong DC'B'$, this construction is uniquely defined, and our pentagon is in fact a regular pentagon. Therefore $\angle BCD$ is an angle in a regular pentagon, so it measures $\boxed{108^\circ}$.

Problem 10. What is the probability that two randomly chosen positive integers are relatively prime? You may find the fact $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$ useful.

Answer. $\boxed{\dfrac{6}{\pi^2}}$
Solution. asdf