28^{th} TJIMO

Alexandria, Virginia

Round: **Individual**

Problem 1. Compute $2^{0+1\cdot 6} - 2^0 \cdot 1 + 7$.

Answer. $\boxed{70}$

Solution. We follow the order of operations to obtain

$$\begin{aligned} 2^{0+1\cdot 6} - 2^0 \cdot 1 + 7 &= 2^{0+6} - 1 \cdot 1 + 7 \\ &= 2^6 - 1 + 7 \\ &= 64 - 1 + 7 = \boxed{70}. \end{aligned}$$

Problem 2. The sum of the digits of the year 2016 is $2 + 0 + 1 + 6 = 9$. What is the next year for which the sum of the digits is also 9?

Answer. $\boxed{2025}$

Solution. Note that for $2017, 2018$, and 2019 have sum of digits greater than 9. The sum of digits of 2020 is 4, so we need $9 - 4 = 5$ more years. Hence $\boxed{2025}$ is the next year with sum of digits equal to 9.

Problem 3. Compute $1 + 2 + 3 + \cdots + 63$.

Answer. $\boxed{2016}$

Solution. Note that $1 + 63 = 64$, $2 + 62 = 64$, $3 + 61 = 64$, and so forth. Therefore $(1 + 63) + (2 + 62) + (3+61)+\cdots(63+1) = 63\cdot 64$is twice the desired sum. Thus the desired sum is $\frac{1}{2}\cdot 63\cdot 64 = 63\cdot 32 = \boxed{2016}$.

Problem 4. A certain circle has an area whose value is twice the value of its circumference. Compute the diameter of the circle.

Answer. $\boxed{8}$

Solution. The area of a circle with radius r is πr^2, and its circumference is $2\pi r$. We have that $\pi r^2 = 2\cdot 2\pi r$. Dividing by πr on both sides leaves us with $r = 4$, so the diameter is $\boxed{8}$.

Problem 5. NAME has 5 shirts, each of a different color: red, green, blue, yellow, and black. He also has 4 pairs of pants of different colors: red, orange, yellow, green. If NAME does not want to wear pants of the same color as his shirt, compute the number of ways he can choose a shirt and a pair of pants.

Answer. $\boxed{17}$

Solution. Without the color restriction, there are simply $5 \cdot 4 = 20$ ways to choose a shirt and a pair of pants because there are 4 pairs of pants from which to choose for each of 5 shirts. There are 3 invalid sets: red shirt/pants, yellow shirt/pants, and green shirt/pants. Hence there are $20 - 3 = \boxed{17}$ valid combinations.

Time limit: 60 minutes.

Problem 6. NAME normally takes 30 minutes to walk to school at 3 miles per hour. One day, he left home 10 minutes later than usual. Compute the speed, in miles per hour, at which he must travel to still get to school on time.

Answer. $\boxed{4.5}$ (miles per hour)
Solution. The distance from NAME's home to school is $\frac{1}{2}\text{hour} \cdot 3\text{mph} = 1.5$ miles. If he leaves home 10 minutes late, then he needs travel the same distance in 20 minutes, or $\frac{1}{3}$ hour. Therefore he must travel at $\frac{1.5\text{miles}}{\frac{1}{3}\text{hour}} = \boxed{4.5}$ mph.

Problem 7. Compute the number of positive divisors of 2016.

Answer. $\boxed{36}$
Solution. The prime factorization of 2016 is $2^5 \cdot 3^2 \cdot 7$. A divisor of 2016 will have $0, 1, 2, 3, 4,$ or 5 factors of 2, for 6 choices. Similarly, there are 3 choices for the number of factors of 3, and 2 for factors of 7. Therefore there are $6 \cdot 3 \cdot 2 = \boxed{36}$ positive divisors of 2016.

Problem 8. A regular n-sided polygon has 2015 diagonals. Compute the number of diagonals in a regular $n+1$-sided polygon.

Answer. $\boxed{2079}$ (diagonals)
Solution. insert

Problem 9. logic/puzzle question

Answer. answer
Solution. solution

Problem 10. geometry

Answer. ans
Solution. soln

Problem 11. At a vending machine, 3 bags of chips and 2 bottles of water cost \$2.35, and 2 bags of chips and 3 bottles of water cost \$2.65. Compute the cost of 1 bag of chips and 1 bottle of water.

Answer. $\boxed{\$1}$
Solution. Let c be the cost of one bag of chips, and w be the cost of one bottle of water. We have:

$$\begin{cases} 3c + 2w = 2.35 \\ 2c + 3w = 2.65 \end{cases}$$

Adding these equations together gives $5c + 5w = 5$, so $c + w = \boxed{1}$. (Although not necessary, we can also solve for w and c explicitly to give $w = 0.35$ and $c = 0.65$.)

Problem 12. Problem 12

Answer.
Solution. asfd

Problem 13. Two standard six-sided dice are rolled. Compute the probability that the product of the numbers rolled is divisible by 6.

Answer. $\boxed{\dfrac{8}{9}}$
Solution. We will use complementary counting. That is, we compute the probability that the product of the numbers rolled is not divisible by 6 and subtract that from 1. In order to not be divisible by 6, each die can either show a 1 or 5, for a $\frac{2}{6} = \frac{1}{3}$ chance each. For two independent dice, the probability is $\frac{1}{3} \cdot \frac{1}{3}$. Therefore the probability that the product is divisible by 6 is $1 - \frac{1}{9} = \boxed{\dfrac{8}{9}}$.

Problem 14. Rectangle $CDEF$ is inscribed in right triangle ABC with right angle C such that D is on AC, E is on AB, and F is on BC. If $AD = 3, CD = 4$, and $DE = 5$, find the area of $\triangle ABC$.

Answer. $\boxed{\dfrac{245}{6}}$
Solution. Since $\triangle AED \sim \triangle ABC$, we see that $BC = \frac{ED}{AD} \cdot AC = \frac{5}{3} \cdot 7 = \frac{35}{3}$. Thus, the area is $\dfrac{7 \cdot \frac{35}{3}}{2} = \boxed{\dfrac{245}{6}}$

Problem 15. algebra

Answer.
Solution. asfd

Problem 16. NAMES!!! A, B, and C are in a race. If A always beats C, but otherwise ties are allowed, find the number of possible results. For example, B and A at a tie for first and C second is one such result.

Answer. $\boxed{7}$
Solution. Insert solutoin

Problem 17. NAME's class can normally be split into even groups of 7 when everyone is present. When 2 people are absent and they try to split into groups of 9, there is 1 person left out. Find the minimum number of students in the entire class.

Answer. $\boxed{28}$ (students)

Solution. If there are n students in the class, then we know that n is divisible by 7 and $n-1$ is divisible by 9, or n is 1 more than a multiple of 9. Simply listing out multiples of 7, we have $7, 14, 21, 28, \ldots$. We see that $\boxed{28}$ is the first multiple of 7 that is 1 more than a multiple of 9, so that is our answer.

Problem 18. Two tanks are used to collect water to drain a full swimming pool. When only tank A is opened, it takes 2 hours for the pool to drain completely. When only tank B is opened, it takes 3 hours for the pool to drain completely. How many **minutes** will it take to drain the pool if both tanks are opened from the start?

Answer. $\boxed{72}$ (minutes)
Solution. We describe tanks A and B in terms of their individual drainage rates. Tank A has a rate of 1 swimming pool/2 hours. Tank B has a rate of 1 swimming pool/3 hours. Thus, the combined rate of tanks A and B working together is $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, which can be rewritten in terms of our full pool: $\frac{5}{6} = 1$ swimming pool/$\frac{6}{5}$ hours. hence it takes $\frac{6}{5}$ hours, or $\boxed{72}$ minutes, for both tanks to drain the pool.

Problem 19. problem 19

Answer.
Solution. solutgn

Problem 20. NAME1 and NAME2 are in the same SUBJECT class. They are both on track to get either an A, A-, or B+. The probability that NAME1 gets an A is 0.4, the probability that NAME2 gets an A is 0.5, and the probability that neither get an $A-$ but at least one get an A is 0.6. What is the probability that at least one of them gets an A but neither gets a B+?

Answer. $\boxed{0.3}$
Solution. Let $P(x, y)$ denote the probability that NAME1 gets grade x and NAME2 gets grade y. From the given information we have:

$$\begin{cases} P(A, A) + P(A, A-) + P(A, B+) = 0.4 \\ P(A, A) + P(A-, A) + P(B+, A) = 0.5 \\ P(A, A) + P(A, B+) + P(B+, A) = 0.6 \end{cases}$$

Adding the first two equations and then subtracting the third gives $P(A, A) + P(A-, A) + P(A, A-) = \boxed{0.3}$.

Problem 21. Compute the smallest positive integer value of n such that $n!$ (n factorial) ends in 16 consecutive zeros, where $n! = 1 \cdot 2 \cdot \ldots \cdot n$.

Answer. $\boxed{70}$
Solution. The number of zeros at the end of a number n is equal to teh largets power k such that 10^k is a factor of n. We solve this by incrementing n until n acquires all 16 zeros. We will always multiply by amultiple of 2 more frequently than a multiple of 5, so every time we multiply our current factorial by

a multiple of 5^p, we increase the nmber of zeros by p. The numbers $5, 10, 15$, and 20 each add one zero. Next, 25 adds two zeros, so 25! has 6 terminal zeros. We add another 6 zeros to the end by the time we reach 50!, with 12 terminal zeros. We add four more zeros to reach 16 with the numbers $55, 60, 65$, and 70. Hence, 70! is the smallest factorial ending in 16 zeros, so $n = \boxed{70}$.

Problem 22. Two cars are traveling towards each other, with car A traveling at 70 km/h and car B travling at 65 km/hr. Initially, they are 15 km apart, and a bird begins to fly from car A to car B. When the bird reaches car B, it immediately starts flying back to car A. The bird continues to do this until the cars collide. Given that the bird flies at a speed of 20 km/h, compute the total distance the bird will fly before stopping

Answer. $\boxed{\dfrac{20}{9}}$ (km)

Solution. The amount of time the bird flies for is the same amount of time it takes for the cars to collide. The cars will collide in $\frac{15}{70+65} = \frac{1}{9}$ h. Thus, the distance is $20 \cdot \frac{1}{9} = \boxed{\dfrac{20}{9}}$

Problem 23. Two circles with equal radii are placed inside a 16×18 rectangular box as shown in the diagram. Each circle touches two walls of the box. The two circles are also touching each other at one point inside the box. What is the radius of both circles?

Answer. $\boxed{5}$

Solution. The point of tangency of the circles is the center of the rectangle. Therefore, we can compress the original rectangle by a factor of $\frac{1}{2}$ to make one of its corners lie on the tangency point of the two circles, as shown. Then we add in line segments representing the radii of one of our circles, each radius directed toward a different point of tangency.

Now we consider the shaded right triangle in the diagram. If we let r be the radius of the circle, the right triangle has legs of length $8 - r$ and $9 - r$ and hypotenuse of length r. Thus, by the Pythagorean Theorem, we have $(8 - r)^2 + (9 - r)^2 = r^2$, which reduces to $r^2 - 34r + 145 = 0$. Factoring this gives us $(r - 5)(r - 29) = 0$. The radius of the circle cannot be 29 (the circle needs to fit inside a 16×18 box), so our only solution is $r = \boxed{5}$.

Problem 24. Find the least positive integer whose sum of positive divisors is 42.

Answer. $\boxed{20}$

Solution. For a general number whose prime factorization is $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, the sum of its divisors is $(1 + p_1 + p_1^2 \cdots + p_1^{e_1}) \cdots (1 + p_k + \cdots + p_k^{e_k})$. We have $42 = 2 \cdot 3 \cdot 7$, but 2 is not workable ($1 + 1$ is invalid as 1 is not prime), so the possible groupings are $6 \cdot 7 = (1 + 5)(1 + 2 + 4)$, $3 \cdot 14 = (1 + 2)(1 + 13) =$, and $42 = 1{+}41$. Of these, the least option is $(1{+}5)(1{+}2{+}4)$, which corresponds to the number $5{\cdot}4 = \boxed{20}$.

Problem 25. problem 25

Answer.
Solution. akfa;kfej

Problem 26. If $\frac{1}{x} + \frac{1}{y} = 5$ and $x + y = 10$, what is $x^2 + y^2$?

Answer. $\boxed{96}$
Solution. Multiplying the two equations, we get $\frac{x}{y} + \frac{y}{x} + 2 = 50$. Subtracting by 2 and multiplying by xy, we see that $x^2 + y^2 = 48xy$. Also, note that from squaring the second equation, $x^2 + y^2 + 2xy = 100$. We now have a system of equations with xy and $x^2 + y^2$. Solving, we get our answer of $\boxed{96}$.

Problem 27. probabilty with states (2 player knockout)

Answer.
Solution. asdf

Problem 28. There are 2016 mathematicians at the Annual Mathematics Conference, and each mathematician brought along a nonmathematician partner. NAME1, a mathematician, brought NAME2, a journalist, who asked each of the 4031 people besides himself how many people they knew besides themselves and their partner. (Knowing is mutual, so if person A knows person B then person B knows person A.) NAME2 received a different answer from each person. Compute the number of people that NAME1 knows.

Answer. $\boxed{2016}$ (people)
Solution. insert solution

Problem 29. Quadrilateral $ABCD$ has sides $AB = 1008$, $BC = 2016$, and $CD = 2016$. If $\angle BAD$ is a right angle and $m\angle ADC = \frac{1}{2}m\angle BCD$, compute $m\angle BCD$ in degrees.

Answer. $\boxed{108}$ (degrees)
Solution. Reflect the quadrilateral about side AD, and let B' be the reflection of B and C' that of C. Then pentagon $BCDC'D'$ is equilateral. Furthermore, $\angle BCD \cong \angle CDC' \cong DC'B'$, this construction is uniquely defined, and our pentagon is in fact a regular pentagon. Therefore $\angle BCD$ is an angle in a regular pentagon, so it measures $\boxed{108^\circ}$.

Problem 30. Compute the probability that two randomly chosen positive integers are relatively prime. You may find the fact $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ useful.

Time limit: 60 minutes.

Answer. $\boxed{\dfrac{6}{\pi^2}}$

Solution. asdf