28^{th} TJIMO
ALEXANDRIA, VIRGINIA

Round: **Practice Power**

1 Introduction

Unlike the other rounds, just getting the answer right is not enough on the Power Round. Make sure you explain your answer and use words to describe how you arrived at your answer. In the words of middle school math teachers across the nation – no work, no credit!

This Practice Power Round (worth 25 points) is divided into two sections. In the first section, we will discuss the shoelace theorem. We will then start proving the shoelace theorem for triangles.

Feel free to use results from previous problems on this round to prove a later problem (that is, you can use Problem 2 to prove Problem 3, but not vice versa). You do not need to have solved the earlier problem to cite its result.

2 The Shoelace Theorem

The shoelace theorem can be used to calculate the area of polygons, given the Cartesian coordinates of the vertices.

Theorem 1 (The Shoelace Theorem)**.** *Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in counterclockwise order. Then the area of P is*

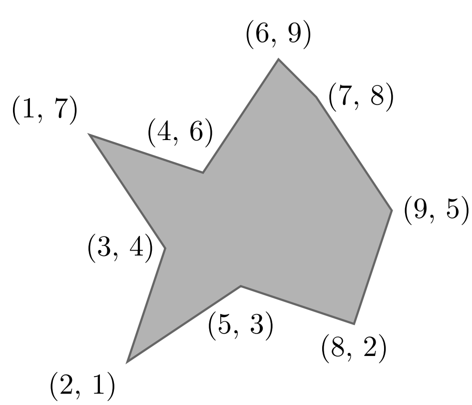
$$\frac{1}{2}|(a_1b_2 + a_2b_3 + \cdots + a_nb_1) - (b_1a_2 + b_2a_3 + \cdots + b_na_1)|$$

Definition 1.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{vmatrix} = \frac{1}{2}|(a_1b_2 + a_2b_3 + \cdots + a_nb_1) - (b_1a_2 + b_2a_3 + \cdots + b_na_1)|$$

Note that the shoelace theorem gets its name from the criss-crossing that results (in the first expression) when one marks the pairs of coordinates to be multiplied.

Problem 1 (2 points total)**.** Use shoelace to find, with proof, the area of the nonagon below:



Time limit: 30 minutes.

Problem 2 (4 points total)**.** Prove that the shoelace theorem works for a triangle with a vertex at $(0, 0)$ and a base on the x-axis. (Hint: think of a way to describe the vertices.)
Problem 3 (4 points total)**.** Prove that the shoelace theorem works for squares with two adjacent vertices on the positive x-axis and positive y-axis and the other two vertices in the first quadrant.

3 Areas of Triangles

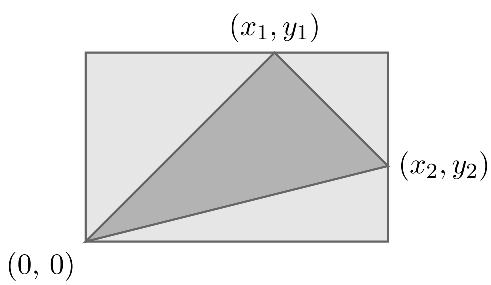
We will begin proving shoelace theorem for just triangles.

3.1 Conveniently Located Triangles

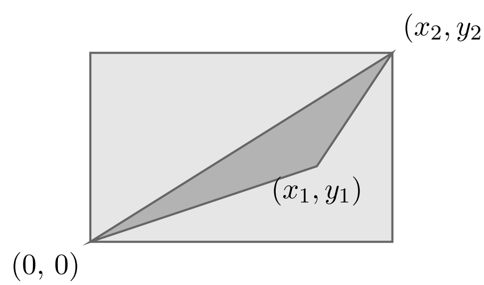
Let the three vertices of triangle ABC lie at $(0, 0), (x_1, y_1)$, and (x_2, y_2).

Problem 4 (5 points each)**.** Prove that the shoelace theorem applies to the following cases for $x_1, y_1, x_2, y_2 \geq 0$: (Hint: Draw a box around the triangle and subtract out unnecessary areas)

(a) $x_1 \leq x_2$ and $y_1 \geq y_2$



(b) $x_1 < x_2$ and $y_1 < y_2$



3.2 General Triangles

Problem 5 (5 points total)**.** Prove that the shoelace theorem is consistent when a triangle is rotated by $90°$ counterclockwise about the origin. (Hint: (x, y) becomes $(-y, x)$)

*******PRACTICE POWER ENDS HERE. YOU HAVE BEEN WARNED.*******

Time limit: 30 minutes.