28^{th} TJIMO

Alexandria, Virginia

Round: Practice Team

Problem 1. If the prime factorization of 60^6 is $2^a \cdot 3^b \cdot 5^c$, where a, b, c are positive integers, find $a+b+c$.

Problem 2. A bag contains marbles of three colors: 10 red marbles, p green marbles, and 55 blue marbles. The probability of randomly selecting a green marble from the bag is $\frac{p}{90}$. Find the probability of selecting a blue marble.

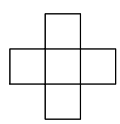
Problem 3. Joe lists all the three digit positive integers on a sheet of paper, starting with 100 and ending with 999. However, he skips all the integers that are divisible by three and all the integers that are divisible by five. How many positive integers does he count?

Problem 4. Given that x, y are positive real numbers satisfying $x + \frac{1}{y} = 5$ and $y + \frac{1}{x} = 7$, find $xy + \frac{1}{xy}$.

Problem 5. For how many positive integers $1 \leq x \leq 143$ is $x^2 + x^3$ the square of an integer?

Problem 6. Let triangle ABC have $AB = 7$, $BC = 24$, and $CA = 25$. Let X be the midpoint of AB, let Y be the midpoint of BC, and let Z be the midpoint of CA. Find the area of XYZ.

Problem 7. How many distinct ways are there to place the positive integers $1, 2, 3, 4, 5$ into the squares below such that every integer is used exactly once, each square contains exactly one integer, the horizontal row of three integers is in either increasing or decreasing order, and the vertical column of three integers is in either increasing or decreasing order?



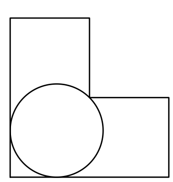
Problem 8. If x, y are distinct real numbers such that

$$x^2 + 4y = 25$$

$$y^2 + 4x = 25$$

find $x + y$.

Problem 9. Suppose the area of this L-shaped figure is 300. Find the radius of the largest circle that can fit inside this figure. Express your answer in the form $a - b\sqrt{2}$.



Problem 10. In a regular tetrahedron $ABCD$ with volume 400, we inscribe a sphere tangent to all four sides. Let R be the center of the sphere. Find the volume of $[RABC]$.