28th TJIMO

Alexandria, Virginia

Round: **Practice Team**

Problem 1. Determine the value of $(3^{3^2} - 3^{2^3} + 3^{2^2} - 2^{3^3})(3+3-3+2)^3(3^2-2^3-1)(3^3+2^2+1^1)$.

Answer. $\boxed{0}$
Solution. Since one of the factors is $3^2 - 2^3 - 1 = 9 - 8 - 1 = 0$, the entire product is simply $\boxed{0}$.

Problem 2. Michael has a playlist with 6 songs in it, but 1 of the songs is a repeat of another. If he presses shuffle, how many possible orders are there for his playlist?

Answer. $\boxed{360}$ (orders)
Solution. For 6 songs, the number of ways to order them is $6! = 720$. However, since 2 of them are the same, we overcounted by a factor of $2! = 2$, so $\frac{6!}{2!} = \frac{720}{2} = \boxed{360}$.

Problem 3. Edwin is swimming in a circular lake. He swims at 2π meters per minute. If it takes him 2400 seconds to swim around the edge of the lake, what is the radius of the lake?

Answer. $\boxed{40}$ (meters)
Solution. Since 2400 seconds is $\frac{2400}{60} = 40$ minutes, the circumference of the lake is $40 \cdot 2\pi = 80\pi$ meters. If the radius is r, then the circumference is given by $2\pi r$, so the radius is $r = \boxed{40}$ meters.

Problem 4. If $x + \frac{1}{x} = 4$, what is $x^2 + \frac{1}{x^2}$?

Answer. $\boxed{14}$
Solution. Observe that $\left(x + \frac{1}{x}\right)^2 = x^2 + 2x \cdot \frac{1}{x} + \frac{1}{x^2} = x^2 + \frac{1}{x^2} + 2$. Since $x + \frac{1}{x} = 4$, $\left(x + \frac{1}{x}\right)^2 = 16$, so $x^2 + \frac{1}{x^2} = 16 - 2 = \boxed{14}$.

Problem 5. What is the area of a square with all four of its vertices on a circle of radius 10?

Answer. $\boxed{200}$ (square units)
Solution. The four vertices must be evenly spaced around the circle, so the diagonals of the square are diameters of the circle, which have length 20. A square is also a rhombus, so its area is $\frac{1}{2} \cdot 20 \cdot 20 = \boxed{200}$.

Problem 6. Timmy rolls 4 standard, fair, six-sided die. What is the probability that at least one of the number he rolls is prime?

Answer. $\boxed{\frac{15}{16}}$
Solution. The possible prime numbers that can be rolled are $2, 3$, and 5, so there is a $\frac{3}{6} = \frac{1}{2}$ chance for each roll to be prime. Then there is a $1 - \frac{1}{2} = \frac{1}{2}$ probability for each roll to not be prime. Therefore, the probability that none of the 4 dice show a prime number is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, so the probability that at least one of them is prime is the complement, which is $1 - \frac{1}{16} = \boxed{\frac{15}{16}}$.

Problem 7. Let a, b, and c be real numbers such that $a + b + 2c = 2015$, $a + 2b + c = 2016$, and $2a + b + c = 2017$. What is the value of $a + b + c$?

Answer. $\boxed{1512}$
Solution. Adding the three equations together gives us $4a + 4b + 4c = 2015 + 2016 + 2017 = 3 \cdot 2016$, so $a + b + c = 3 \cdot 504 = \boxed{1512}$.

Problem 8. Aaditya is downloading an Android app. Every second, his phone has an equal chance to either download 20% of the app or do nothing. What is the probability that after 8 seconds, the app will have finished downloading?

Answer. $\boxed{\frac{7}{32}}$
Solution. There are $2^8 = 256$ possible sequences of events in 8 seconds. In order for the app to have finished downloading after 8 seconds, 5 seconds must have been spent downloading 20% of the app each, and 3 seconds must have been spent doing nothing. There are $\binom{8}{5} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 56$ different combinations of 5 out of 8 seconds for the phone to have spent downloading. Therefore the probability is $\frac{56}{256} = \boxed{\frac{7}{32}}$.

Problem 9. A triangle with integer side lengths has a perimeter of 5. What is its area?

Answer. $\boxed{\frac{\sqrt{15}}{4}}$ (square units)
Solution. By the triangle inequality, the only possible triangle has side lengths 2, 2, and 1. This is an isosceles triangle with base 1 and legs of length 2. Its altitude is $\sqrt{2^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{15}}{2}$, so its area is $\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{15}}{2} = \boxed{\frac{\sqrt{15}}{4}}$.

Problem 10. What is the largest prime factor of $17^3 + 1$?

Answer. $\boxed{13}$
Solution. Recall the sum of cubes factorization $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. Thus we have $17^3 + 1 = (17 + 1)(17^2 - 17 \cdot 1 + 1^2) = 18(273) = (2 \cdot 3^2)(3 \cdot 7 \cdot 13)$, so its largest prime factor is $\boxed{13}$.