28^{th} TJIMO

Alexandria, Virginia

Round: **Practice Guts**

Set 1

Problem 1. Compute $1 - 2 + 3 - 4 + 5 - 6 + \cdots + 19 - 20$.

Answer. $\boxed{-10}$

Solution. Addition is associative, so by grouping every two terms, we have $(1-2)+(3-4)+(5-6)+\cdots+(19-20) = \underbrace{-1+(-1)+(-1)+\cdots+(-1)}_{\text{10 -1s}} = \boxed{-10}$.

Problem 2. How many diagonals are in a hexagon?

Answer. $\boxed{9}$ (diagonals)

Solution. Since each vertex cannot form a diagonal with itself or its two adjacent vertices, it can only form $6-3$ diagonals. As there are 6 vertices, and a diagonal has 2 endpoints, we see that there are a total of $\frac{(6)(6-3)}{2} = \boxed{9}$ diagonals.

Problem 3. Using only 3 straight cuts, what is the maximum number of pieces into which a cube can be cut?

Answer. $\boxed{8}$

Solution. Imagine the vertices of the cube are at $(\pm1, \pm1, \pm1)$. Then, the three cuts are the planes $x = 0, y = 0$, and $z = 0$. Each "quadrant" will become a piece.

Problem 4. Find all two-digit numbers $\underline{AB}$ that satisfy the equation $\sqrt{A + \sqrt{\underline{AB}}} = A$, where A is the tens digit, B is the units digit, and $\underline{AB}$ is a two-digit number.

Answer. $\boxed{36}$

Solution. Squaring both sides of the equation gives $\sqrt{\underline{AB}} = A^2 - A$. Since $A^2 - A$ is a positive integer, $\underline{AB}$ must be the square of a positive integer. A quick check of the two-digit perfect squares yields $\boxed{36}$ as the only solution.

Set 2

Problem 5. Akshaj is struggling in APUSH. His test scores were $47, 51, 43, 64$, and 35. If he gets a 42 on his next test, what will be the average of all his test scores?

Answer. $\boxed{47}$
Solution. The average is $\frac{47+51+43+64+35+42}{6} = \boxed{47}$.

Problem 6. There are 30 people in one of the math-team trailers. 18 of them like geometry, 21 of them like algebra, and 20 of them like number theory. If 10 of them like algebra and geometry, 12 of them like algebra and number theory, and 8 of them like geometry and number theory, how many of them like all 3 subjects?

Answer. $\boxed{1}$ (person)
Solution. If we add the number of people who like one subject together, $18+21+20 = 59$, we overcount the students who like two or more subjects. If we subtract the number of people who like 2 subjects from the previous sum, $59-10-12-8 = 29$, we oversubtract the students who like all 3, so 29 plus the number of students who like all 3 should equal the number of total students, or 30, so there is $\boxed{1}$ person who likes all three subjects.

Problem 7. What is the remainder when 2^{2016} is divided by 7?

Answer. $\boxed{1}$
Solution. Note that $2^3 = 8 \equiv 1 \pmod 7$ is 1 more than a multiple of 7, and $2^{2016} = (2^3)^{672} \equiv 1^{672} = 1 \pmod 7$, so the remainder is $\boxed{1}$.

Problem 8. In quadrilateral $ABCD$, $\angle DAC = 75^\circ$, $\angle ACB = 40^\circ$, $\angle DBC = 75^\circ$, and $\angle BDC = 25^\circ$. Find the measure of angle $\angle DCA$.

Answer. $\boxed{40^\circ}$
Solution. We observe that $\angle DCA = 180^\circ - \angle DBC - \angle ACB - \angle BDC = 40^\circ$.

Set 3

Problem 9. Lilian does Alex's evil bidding. If Lilian averages 30 tasks per hour between 7 pm and 9 pm and does 51 tasks between 7 pm and 8:30 pm, how many tasks does she do between 8:30 pm and 9 pm?

Answer. $\boxed{9}$ (tasks)
Solution. Since Lilian averages 30 tasks per hour, in the two hours between 7 pm and 9 pm, she will do 60 tasks. As she has completed 51 tasks between 7 pm and 8:30 pm, she will do $60 - 51 = \boxed{9}$ tasks between 8:30 pm and 9 pm.

Problem 10. Michael has a playlist with 10 songs on it. The lengths of the songs form an arithmetic sequence with common difference 6 seconds and sum 30 minutes. What is the length of the shortest song in seconds?

Answer. $\boxed{153}$ (seconds)
Solution. The average length is $\frac{30}{10} = 3$ minutes or 180 seconds. The average length is halfway between the fifth and sixth shortest songs, so we have to subtract 4.5 times the common difference from the average to get the shortest song, which is $180 - 4.5 \cdot 6 = 180 - 27 = \boxed{153}$ seconds.

Problem 11. How many distinct, non-congruent rectangles with positive integer side lengths have an area that is 52 more than their perimeter?

Answer. $\boxed{4}$ (rectangles)
Solution. Let x and y be the side lengths of the rectangle. The area is xy, and the perimeter is $2(x+y)$. We are given that $xy = 52 + 2(x+y)$, so $xy - 2x - 2y = 52$. The trick here is to add 4 to both sides of the equation, "forcing" the left side to factor: $(x-2)(y-2) = xy - 2x - 2y + 4 = 52 + 4 = 56$. Since x and y must be positive integers, our task is now to count the number of distinct factor pairs of 56. These pairs are 1 and 56, 2 and 28, 4 and 14, and 7 and 8. Thus, there are $\boxed{4}$ ordered pairs (x, y) that serve as the side lengths of four distinct rectangles.

Problem 12. There is a 70% chance that it is cloudy, and a 60% chance that it will rain. It is twice as likely to rain when it is cloudy than when it is not. What is the probability that it will rain, given that it is not cloudy?

Answer. $\boxed{\dfrac{6}{17}}$
Solution. Let p be the probability it rains when it is not cloudy. Then $2p$ is the probability it will rain when it is cloudy. We are given $\frac{7}{10} \cdot 2p + \frac{3}{10} \cdot p = \frac{6}{10}$, so solving gives $p = \boxed{\dfrac{6}{17}}$.