28^{th} TJIMO

Alexandria, Virginia

Round: **Individual**

Problem 1. Compute $2^{0+1\cdot 6} - 2^0 \cdot 1 + 7$.

Answer. $\boxed{70}$

Solution. We follow the order of operations to obtain

$$\begin{aligned} 2^{0+1\cdot 6} - 2^0 \cdot 1 + 7 &= 2^{0+6} - 1 \cdot 1 + 7 \\ &= 2^6 - 1 + 7 \\ &= 64 - 1 + 7 = \boxed{70}. \end{aligned}$$

Problem 2. The sum of the digits of the year 2016 is $2 + 0 + 1 + 6 = 9$. What is the next year for which the sum of the digits is also 9?

Answer. $\boxed{2025}$

Solution. Note that for $2017, 2018$, and 2019 have sum of digits greater than 9. The sum of digits of 2020 is 4, so we need $9 - 4 = 5$ more years. Hence $\boxed{2025}$ is the next year with sum of digits equal to 9.

Problem 3. Compute $1 + 2 + 3 + \cdots + 63$.

Answer. $\boxed{2016}$

Solution. Note that $1 + 63 = 64$, $2 + 62 = 64$, $3 + 61 = 64$, and so forth. Therefore $(1 + 63) + (2 + 62) + (3+61)+\cdots(63+1) = 63\cdot 64$is twice the desired sum. Thus the desired sum is $\frac{1}{2}\cdot 63\cdot 64 = 63\cdot 32 = \boxed{2016}$.

Problem 4. A certain circle has an area whose value is twice the value of its circumference. Compute the diameter of the circle.

Answer. $\boxed{8}$

Solution. The area of a circle with radius r is πr^2, and its circumference is $2\pi r$. We have that $\pi r^2 = 2\cdot 2\pi r$. Dividing by πr on both sides leaves us with $r = 4$, so the diameter is $\boxed{8}$.

Problem 5. NAME has 5 shirts, each of a different color: red, green, blue, yellow, and black. He also has 4 pairs of pants of different colors: red, orange, yellow, green. If NAME does not want to wear pants of the same color as his shirt, compute the number of ways he can choose a shirt and a pair of pants.

Answer. $\boxed{17}$

Solution. Without the color restriction, there are simply $5 \cdot 4 = 20$ ways to choose a shirt and a pair of pants because there are 4 pairs of pants from which to choose for each of 5 shirts. There are 3 invalid sets: red shirt/pants, yellow shirt/pants, and green shirt/pants. Hence there are $20 - 3 = \boxed{17}$ valid combinations.

Time limit: 60 minutes.

Problem 6. NAME normally takes 30 minutes to walk to school at 3 miles per hour. One day, he left home 10 minutes later than usual. Compute the speed, in miles per hour, at which he must travel to still get to school on time.

Answer. $\boxed{4.5}$ (miles per hour)
Solution. The distance from NAME's home to school is $\frac{1}{2}\text{hour} \cdot 3\text{mph} = 1.5$ miles. If he leaves home 10 minutes late, then he needs travel the same distance in 20 minutes, or $\frac{1}{3}$ hour. Therefore he must travel at $\frac{1.5\text{miles}}{\frac{1}{3}\text{hour}} = \boxed{4.5}$ mph.

Problem 7. Compute the number of positive divisors of 2016.

Answer. $\boxed{36}$
Solution. The prime factorization of 2016 is $2^5 \cdot 3^2 \cdot 7$. A divisor of 2016 will have $0, 1, 2, 3, 4$, or 5 factors of 2, for 6 choices. Similarly, there are 3 choices for the number of factors of 3, and 2 for factors of 7. Therefore there are $6 \cdot 3 \cdot 2 = \boxed{36}$ positive divisors of 2016.

Problem 8. A regular n-sided polygon has 2015 diagonals. Compute the number of diagonals in a regular $n + 1$-sided polygon.

Answer. $\boxed{2079}$ (diagonals)
Solution. insert

Problem 9. logic/puzzle question

Answer. answer
Solution. solution

Problem 10. geometry

Answer. ans
Solution. sol

Problem 11. Two standard six-sided dice are rolled. Compute the probability that the product of the numbers rolled is divisible by 6.

Answer. $\boxed{\dfrac{8}{9}}$
Solution. We will use complementary counting. That is, we compute the probability that the product of the numbers rolled is not divisible by 6 and subtract that from 1. In order to not be divisible by 6, each die can either show a 1 or 5, for a $\frac{2}{6} = \frac{1}{3}$ chance each. For two independent dice, the probability is $\frac{1}{3} \cdot \frac{1}{3}$. Therefore the probability that the product is divisible by 6 is $1 - \frac{1}{9} = \boxed{\dfrac{8}{9}}$.

Time limit: 60 minutes.

Problem 12. Two circles with equal radii are placed inside a 16×18 rectangular box as shown in the diagram. Each circle touches two walls of the box. The two circles are also touching each other at one point inside the box. What is the radius of both circles?

Answer. $\boxed{5}$
Solution. The point of tangency of the circles is the center of the rectangle. Therefore, we can compress the original rectangle by a factor of $\frac{1}{2}$ to make one of its corners lie on the tangency point of the two circles, as shown. Then we add in line segments representing the radii of one of our circles, each radius directed toward a different point of tangency.

Now we consider the shaded right triangle in the diagram. If we let r be the radius of the circle, the right triangle has legs of length $8-r$ and $9-r$ and hypotenuse of length r. Thus, by the Pythagorean Theorem, we have $(8-r)^2+(9-r)^2=r^2$, which reduces to $r^2-34r+145=0$. Factoring this gives us $(r-5)(r-29)=0$. The radius of the circle cannot be 29 (the circle needs to fit inside a 16×18 box), so our only solution is $r = \boxed{5}$.

Problem 13. If $\frac{1}{x}+\frac{1}{y}=5$ and $x+y=10$, what is x^2+y^2?

Answer. $\boxed{96}$
Solution. Multiplying the two equations, we get $\frac{x}{y}+\frac{y}{x}+2=50$. Subtracting by 2 and multiplying by xy, we see that $x^2+y^2=48xy$. Also, note that from squaring the second equation, $x^2+y^2+2xy=100$. We now have a system of equations with xy and x^2+y^2. Solving, we get our answer of $\boxed{96}$.

Problem 14. There are 2016 mathematicians at the Annual Mathematics Conference, and each mathematician brought along a nonmathematician partner. NAME1, a mathematician, brought NAME2, a journalist, who asked each of the 4031 people besides himself how many people they knew besides themselves and their partner. (Knowing is mutual, so if person A knows person B then person B knows person A.) NAME2 received a different answer from each person. Compute the number of people that NAME1 knows.

Answer. $\boxed{2016}$ (people)
Solution. insert solution

Problem 15. Quadrilateral $ABCD$ has sides $AB = 1008$, $BC = 2016$, and $CD = 2016$. If $\angle BAD$ is a right angle and $m\angle ADC = \frac{1}{2}m\angle BCD$, compute $m\angle BCD$ in degrees.

Answer. $\boxed{108}$ (degrees)

Solution. Reflect the quadrilateral about side AD, and let B' be the reflection of B and C' that of C. Then pentagon $BCDC'D'$ is equilateral. Furthermore, $\angle BCD \cong \angle CDC' \cong DC'B'$, this construction is uniquely defined, and our pentagon is in fact a regular pentagon. Therefore $\angle BCD$ is an angle in a regular pentagon, so it measures $\boxed{108^\circ}$.

Problem 16. Compute the probability that two randomly chosen positive integers are relatively prime. You may find the fact $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ useful.

Answer. $\boxed{\dfrac{6}{\pi^2}}$

Solution. asdf