B.Tech II year

Engineering Mathematics-IV (KAS-402)

Module-2 (Applications of partial differential equations)

Classification of linear partial differential equation of second order, Method of separation of variables, Solution of wave and heat conduction equation up to two dimension, Laplace equation in two dimensions, Equations of Transmission lines.

Contents

Course Outcome: To understand the classification of second order partial differentia equations and by using the method of separation of variables to evaluate the general solution of Heat, Wave, Laplace equations and Transmission lines.

Applications

- Classification of partial differential equations helps us to study it's characteristics of and choose the best effective numerical method to get the solution.
- The method of separation variables is a powerful tool to solve boundary value problems to get the solution in separation form of each variable function.
- The Heat, Wave and Laplace equations are well known boundary value problems in engineering and quantum mechanics to get the spread form of function and also determines the heat conduction, wave distribution and steady state temperature distribution respectively in of the concerned problems
- Transmission lines are used to transmit high frequency signals over long or short distances with minimum power loss and solution of Radio, Telegraph equation provides an optimum voltage and current distribution in a wire along with negligible potential and current drop.

2.1 Classification of PDE

Consider the following 2nd order linear partial differential equation (p.d.e)

$Rr + Ss + Tt + f(p, q, x, y, z) = 0$

p, q, r, s, t have their usual meaning.

Then

$S^2 - 4RT = 0 \Rightarrow$ Partial differential equation is parabolic

$S^2 - 4RT > 0 \Rightarrow$ Partial differential equation is hyperbolic and

$S^2 - 4RT < 0 \Rightarrow$ Partial differential equation is elliptic.

Example 1. Classify the following p.d.es

(i) $$2\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} + 3\frac{\partial u}{\partial y} - 6\frac{\partial u}{\partial x} = \cos x$$

Here $R = 2, S = -4, T = 2$

Therefore $S^2 - 4RT = 0 \Rightarrow$ equation is parabolic.

(ii) $$x\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + 2xy = 0$$

Here $R = x, S = -1, T = y$

Therefore $S^2 - 4RT = 1 - 4xy$ and we have following cases:

If $1 - 4xy > 0$, p.d.e is hyperbolic

If $1 - 4xy = 0$, p.d.e is parabolic and

If $1 - 4xy < 0$, p.d.e is elliptic.

(iii) $2xr - 3s - 6xy + p - 2q = 0$

Here $R = 2x, S = -3, T = 0$

And $S^2 - 4RT = 9 > 0$ implies the p.d.e is hyperbolic

(iv) $3u_{xx} + 4u_{yy} = x + e^{xy} + 5$

$R = 3, S = 0, T = 4$

And $S^2 - 4RT = -12$ which implies the p.d.e is elliptic.

2.2 Method of Separation of variables

Suppose the given partial differential equation involves n independent variable $x_1, x_2 ... x_n$ and the dependant variable u then we assume that it's solution is written in the form

$u(x_1, x_2 ... x_n) = X_1(x_1) X_2(x_2) ... X_n(x_n).$

Where X_i is a function of x_i only. We find each derivative involved get n ordinary differential equations using some constant and after that solve each equation at last find the complete solution using principle of superposition.

Example 2: solve $\frac{\partial u}{\partial x} = 3\frac{\partial u}{\partial y}$; $u(0, y) = 17e^{-3y}$.

Take $u = XY$ where X, Y are functions of x, y respectively

So that equation becomes $X'Y = 3XY'$, $X' \equiv \frac{d}{dx}, Y' \equiv \frac{d}{dy}$

Choosing constant k such that $\frac{X'}{3X} = \frac{Y'}{Y} = k$ we get differential equations

$X' - 3kX = 0$ and $Y' - kY = 0$

In first equation $\frac{dX}{dx} = 3kX$ which gives $X = Ae^{3kx}$ on solving

Similarly from second equation we obtain $Y = Be^{ky}$ and therefore

$u = XY = \lambda e^{k(3x+y)}$; $\lambda = AB$

Applying boundary condition $u(0, y) = 17e^{-3y} = \lambda e^{ky}$, we get $\lambda = 17, k = -3$

Hence $u = 17e^{-3(3x+y)}$ is the desired solution.

Example 3: Solve $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u; u(0, y) = 4e^{-y} - e^{-5y}$.

Consider $u = X(x)Y(y) \Rightarrow 4X'Y + XY' = 3XY$

Or $\quad 4\frac{X'}{X} + \frac{Y'}{Y} = 3$

Take a constant k such that $4\frac{X'}{X} = 3 - \frac{Y'}{Y} = k$

Now, $4\frac{X'}{X} = k \Rightarrow 4\frac{dX}{dx} = kX \Rightarrow X = c_1 e^{\frac{k}{4}x}$ and

$3 - \frac{Y'}{Y} = k \Rightarrow \frac{dY}{dy} + (k-3)Y = 0 \Rightarrow Y = c_1 e^{(3-k)y}$.

Therefore $u(x, y) = XY = c_1 e^{\frac{k}{4}x} . c_2 e^{(3-k)y} = Ae^{\frac{k}{4}x + (3-k)y}$

Consider the general solution $u(x, y) = \sum_{n=1}^{\infty} A_n e^{\frac{k_n}{4}x + (3-k_n)y}$

Now for n = 1, 2 only $\quad u(0, y) = 4e^{-y} - e^{-5y} . = A_1 e^{\frac{k_1}{4}x + (3-k_1)y} + A_2 e^{\frac{k_2}{4}x + (3-k_2)y}$

Comparing the constants we get $A_1 = 4, A_2 = -1, k_1 = 4, k_2 = 8$

Hence $u(x, y) = 4e^{x-y} - e^{2x-5y}$.

Example 4: Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$ given that $u = 0$ when t =0

and $\frac{\partial u}{\partial t} = 0$ when x =0.

Here we consider the solution $u(x, t) = X(x)T(t)$

So that $\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial X}{dx} \frac{\partial T}{\partial t} = e^{-t} \cos x$ which implies, for a constant $-p^2$

$$e^t \frac{dT}{dt} = \frac{\cos x}{\frac{dX}{dx}} = -p^2$$

Now, $e^t \dfrac{dT}{dt} = -p^2 \Rightarrow T = p^2 e^{-t} + c_1$ and

$$e^t \frac{dT}{dt} = -p^2 \Rightarrow T = \frac{-1}{p^2}\sin x + c_2$$

Therefore, $u(x,t) = X.T == (p^2 e^{-t} + c_1)\left(\dfrac{-1}{p^2}\sin x + c_2\right)$

$$u(x,0) = 0 \Rightarrow X(c_1 + p^2) = 0 \Rightarrow c_1 = -p^2$$

and $\dfrac{\partial u}{\partial t} = \left(-\dfrac{1}{p^2}\sin x + c_2\right)(-p^2 e^{-t})$

at x= 0 $\dfrac{\partial u}{\partial t} = 0$ implies $c_2(-p^2 e^{-t}) = 0 \Rightarrow c_2 = 0$

Hence $u(x,t) = \sin x(1 - e^{-t})$.

Example 5 : Solve $\dfrac{\partial\ z}{\partial x} - 3z = 2\dfrac{\partial\ z}{\partial y}$; $\quad z(x,0) = \theta_0 e^{4x}$

Solution: Consider $z = X(x)Y(y)$ so that equation becomes

$X'Y - 3XY = 2XY'$ or

$\dfrac{X'}{X} - 3 = 2\dfrac{Y'}{Y} = \omega$, where ω is a constant

We get differential equations $X' = (\omega + 3)X$ and $2Y' = \omega Y$

From first equation x $\dfrac{dX}{dx} = (\omega + 3)X$, solving we get $X = Ae^{(\omega+3)x}$ and

From second equation $2\dfrac{dY}{dy} = \omega Y$, solving we get $Y = Be^{\frac{\omega y}{2}}$

Therefore, $z = \xi e^{\left[(\omega+3)x + \frac{\omega y}{2}\right]}$ $(\xi = AB)$ and

$$z(x,0) = \theta_0 e^{4x} \Rightarrow \xi = \theta_0, \omega = 1$$

Hence $z = \theta_0 e^{\left[4x + \frac{y}{2}\right]}$

Example 6: solve the partial differential equation $y^3 \frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0$ using method of separation of variables.

Consider $u(x,y) = X(x)Y(y)$ so that $\frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$

Therefore $y^3 \frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial y} = 0 \Rightarrow y^3 X'Y + x^2 XY' = 0$

Or $\frac{y^3 Y}{Y'} = -\frac{x^2 X}{X'} = k$

Now $\frac{y^3 Y}{Y'} = k \Rightarrow \frac{Y'}{Y} = \frac{y^3}{k} \Rightarrow Y = c_1 e^{\frac{y^4}{4k}}$

And $-\frac{x^2 X}{X'} = k \Rightarrow \frac{X'}{X} = -\frac{x^2}{k} \Rightarrow X = c_2 e^{-\frac{x^3}{3k}}$

Hence $u(x,y) = c_1 c_2 e^{\frac{y^4}{4k}} e^{-\frac{x^3}{3k}} = Ae^{\frac{1}{k}\left(\frac{y^4}{4} - \frac{x^3}{3}\right)}$

Questions for practice

1. Classify the p.d.es (i) $tu_{tt} + 3u_{xt} + xu_{xx} + 15u_x = 1$

   (ii) $5v_{xx} - 9v_{xt} + 4u_{tt} = 0$

   Ans (i) hyperbolic if $xt < 9/4$, parabolic if $xt = 9/4$ and elliptic $xt > 9/4$
   (ii) hyperbolic.

2. Solve $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u; u(x,0) = 10e^{-x} - 6e^{-4x}$ Ans: $u(x,t) = 10e^{-(x+3t)} - 6e^{-2(2x+3t)}$

3. Use method of separation of variables to solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$

   $(A\cos kx + B\sin kx)e^{-(p^2+2)y}$

4. Solve $u_{xx} = u_y + 2u; u(0,y) = 0, \frac{\partial}{\partial x}u(0,y) = 1 + e^{-3y}$.

   :

   $u(x,y) = \frac{1}{\sqrt{2}}\sinh\sqrt{2}x + e^{-3y}\sin x$

2.3 One dimensional Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ where } c^2 = \frac{T}{m}$$

With the following conditions

$$u(0,t) = u(l,t) = 0 \quad \text{(boundary condition)}$$

$$u(x,0) = f(x), \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \quad \text{(initial conditions)}$$

Consider the solution $u(x,t) = X(x)T(t)$

So that $\frac{\partial^2 u}{\partial t^2} = XT''$ and $\frac{\partial^2 u}{\partial x^2} = TX''$

$$XT''' = c^2 TX'' \Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = k \text{, for any constant k}$$

Now, $\frac{X''}{X} = k \Rightarrow \frac{d^2 X}{dx^2} = kX$ and $\frac{T''}{c^2 T} = k \Rightarrow \frac{d^2 T}{dt^2} = c^2 kT$

Consider the following cases:

(i) when $k = 0$, $\frac{d^2 X}{dx^2} = 0 \Rightarrow X = c_1 x + c_2$ and

$$\frac{d^2 T}{dt^2} = 0 \Rightarrow T = c_3 t + c_4$$

$$u(x,t) = XT = (c_1 x + c_2)(c_3 t + c_4).$$

(ii) when $k = \lambda^2$, $\frac{d^2 X}{dx^2} = \lambda^2 X \Rightarrow X = (c_5 e^{\lambda x} + c_6 e^{-\lambda x})$ and

$$\frac{d^2 T}{dt^2} = c^2 \lambda^2 T \Rightarrow T = (c_7 e^{\lambda ct} + c_8 e^{-\lambda ct})$$

$$u(x,t) = XT = (c_5 e^{\lambda x} + c_6 e^{-\lambda x})(c_7 e^{\lambda ct} + c_8 e^{-\lambda ct})$$

(iii) when $k = -\lambda^2$,

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \Rightarrow X = (c_9 \sin \lambda x + c_{10} \cos \lambda x) \text{ and}$$

$$\frac{d^2 T}{dt^2} = -c^2 \lambda^2 T \Rightarrow T = (c_{11} \sin \lambda ct + c_{12} \cos \lambda ct)$$

Therefore,

$$u(x,t) = XT = (c_9 \sin\lambda x + c_{10}\cos\lambda x)(c_{11}\sin\lambda ct + c_{12}\cos\lambda ct)$$

In above all cases we get three different solutions out of which we have to choose that one which is consistent with the physical nature of the problem and provides a non trivial solution. So we will take (iii) as the most suitable

solution that is

$$u(x,t) = XT = (c_9 \sin\lambda x + c_{10}\cos\lambda x)(c_{11}\sin\lambda ct + c_{12}\cos\lambda ct)$$

Now we apply the boundary conditions

$$u(0,t) = 0 \Rightarrow (c_{10}\cos 0)T = 0 \Rightarrow c_{10} = 0 \text{ .}$$

$$u(l,t) = 0 \Rightarrow c_9 \sin\lambda l = 0 \Rightarrow \lambda = \frac{n\pi}{l}.$$

Also

$$\frac{\partial u}{\partial t} = (c_9 \sin\lambda x)(c_{11}\cos\lambda ct - c_{12}\sin\lambda ct)(\lambda c) = 0$$

and

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = (c_9 \sin\lambda x)(c_{11}\cos 0)(\lambda c) = 0 \Rightarrow c_{11} = 0$$

Therefore, $u(x,t) = (c_9 \sin\lambda x)(c_{12}\cos\lambda ct) = A\sin\lambda x\cos\lambda ct$

Where

$$A = c_9 c_{12}$$

Consider the general solution $u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$

Applying the last condition $u(x,0) = f(x)$ which gives

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \Rightarrow A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

using Fourier half range series.

Example 7: A tightly stretched string with fixed end points x = 0 and x = l is initially at rest and having zero initial velocity. If $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. Find the displacement of the string at any instant of time.

For string problem we consider the following one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with the following conditions

$$y(0,t) = y(l,t) = 0 = \left(\frac{\partial y}{\partial t}\right)_{t=0} \quad y(x,0) = y_0 \sin^3 \frac{\pi x}{l}.$$

Consider the solution $y(x,t) = X(x)T(t)$

So that $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = TX''$

$XT''' = c^2 TX'' \Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = k$, for any constant k

Now, $\frac{X''}{X} = k \Rightarrow \frac{d^2 X}{dx^2} = kX$ and $\frac{T''}{c^2 T} = k \Rightarrow \frac{d^2 T}{dt^2} = c^2 kT$

Consider the following cases:

(i) when $k = 0$ then $\frac{d^2 X}{dx^2} = 0 \Rightarrow X = c_1 x + c_2$ and

$\frac{d^2 T}{dt^2} = 0 \Rightarrow T = c_3 t + c_4$

$y(x,t) = XT = (c_1 x + c_2)(c_3 t + c_4).$

(ii) when $k = \lambda^2$ then $\frac{d^2 X}{dx^2} = \lambda^2 X \Rightarrow X = (c_5 e^{\lambda x} + c_6 e^{-\lambda x})$ and

$\frac{d^2 T}{dt^2} = c^2 \lambda^2 T \Rightarrow T = (c_7 e^{\lambda ct} + c_8 e^{-\lambda ct})$

$y(x,t) = XT = (c_5 e^{\lambda x} + c_6 e^{-\lambda x})(c_7 e^{\lambda ct} + c_8 e^{-\lambda ct})$

(iii) when $k = -\lambda^2$ then $\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \Rightarrow X = (c_9 \sin \lambda x + c_{10} \cos \lambda x)$

and $\frac{d^2 T}{dt^2} = -c^2 \lambda^2 T \Rightarrow T = (c_{11} \sin \lambda ct + c_{12} \cos \lambda ct)$

$$y(x,t) = XT = (c_9 \sin \lambda x + c_{10} \cos \lambda x)(c_{11} \sin \lambda ct + c_{12} \cos \lambda ct)$$

When we apply the provided conditions in solution (i) and solution (ii) we get not admissible solution

$u(x,t) = 0$
.

So we will take (iii) as

$$y(x,t) = XT = (c_9 \sin \lambda x + c_{10} \cos \lambda x)(c_{11} \sin \lambda ct + c_{12} \cos \lambda ct)$$

Now we apply the boundary conditions

$$y(0,t) = 0 \Rightarrow (c_{10} \cos 0)T = 0 \Rightarrow c_{10} = 0 \ .$$

$$u(l,t) = 0 \Rightarrow c_9 \sin \lambda l = 0 \Rightarrow \lambda = \frac{n\pi}{l}.$$

Also

$$\frac{\partial y}{\partial t} = (c_9 \sin \lambda x)(c_{11} \cos \lambda ct - c_{12} \sin \lambda ct)(\lambda c) = 0$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = (c_9 \sin \lambda x)(c_{11} \cos 0)(\lambda c) = 0 \Rightarrow c_{11} = 0$$

Therefore

$$y(x,t) = (c_9 \sin \lambda x)(c_{12} \cos \lambda ct) = A \sin \lambda x \cos \lambda ct$$

where $A = c_9 c_{12}$

consider the general solution $y(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$

applying the last condition $y \ (x,0) = y_0 \sin^3 \frac{\pi x}{l}$ which gives

$$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \Rightarrow \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = \frac{y_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right)$$

Comparing the like terms we get $n = 1, A_1 = \frac{3}{4} y_0, n = 3.A_3 = -\frac{y_0}{4}$

Hence the solution is $y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}$.

Example 8: A tightly stretched string of length 20 cm fastened at both ends and displaced from the position of equilibrium by imparting to each of its points. An initial velocity is given by

$$u_t(x,0) = \begin{cases} x, 0 < x < 10 \\ 20 - x, 10 < x < 20 \end{cases}$$

Consider the one dimensional Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

With the following conditions

$$u(0,t) = u(20,t) = 0\,, u(x,0) = 0, u_t(x,0) = \begin{cases} x, 0 < x < 10 \\ 20 - x, 10 < x < 20 \end{cases}$$

Consider the solution $u(x,t) = X(x)T(t)$

So that $\frac{\partial^2 u}{\partial t^2} = XT''$ and $\frac{\partial^2 u}{\partial x^2} = TX''$

$XT''' = c^2 TX'' \Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = k$, for any constant k

Now, $\frac{X''}{X} = k \Rightarrow \frac{d^2 X}{dx^2} = kX$ and $\frac{T''}{c^2 T} = k \Rightarrow \frac{d^2 T}{dt^2} = c^2 kT$

Consider the following cases:

(i) when $k = 0$ then $\frac{d^2 X}{dx^2} = 0 \Rightarrow X = c_1 x + c_2$ and

$$\frac{d^2 T}{dt^2} = 0 \Rightarrow T = c_3 t + c_4$$

$$\Rightarrow u(x,t) = XT = (c_1 x + c_2)(c_3 t + c_4).$$

(ii) when $k = \lambda^2$ then $\frac{d^2 X}{dx^2} = \lambda^2 X \Rightarrow X = (c_5 e^{\lambda x} + c_6 e^{-\lambda x})$ and

$$\frac{d^2 T}{dt^2} = c^2 \lambda^2 T \Rightarrow T = (c_7 e^{\lambda ct} + c_8 e^{-\lambda ct})$$

$$u(x,t) = XT = (c_5 e^{\lambda x} + c_6 e^{-\lambda x})(c_7 e^{\lambda ct} + c_8 e^{-\lambda ct})$$

(iii) when $k=-\lambda^2$ then $\frac{d^2X}{dx^2}+\lambda^2X=0 \Rightarrow X=(c_9\sin\lambda x+c_{10}\cos\lambda x)$ and

$$\frac{d^2T}{dt^2}=-c^2\lambda^2T \Rightarrow T=(c_{11}\sin\lambda ct+c_{12}\cos\lambda ct)$$

$$u(x,t)=XT=(c_9\sin\lambda x+c_{10}\cos\lambda x)(c_{11}\sin\lambda ct+c_{12}\cos\lambda ct)$$

we will take (iii) as the most suitable solution that is

$$u(x,t)=XT=(c_9\sin\lambda x+c_{10}\cos\lambda x)(c_{11}\sin\lambda ct+c_{12}\cos\lambda ct)$$

Now we apply the boundary conditions

$$u(0,t)=0 \Rightarrow (c_{10}\cos 0)T=0 \Rightarrow c_{10}=0.$$

$$u(20,t)=0 \Rightarrow c_9\sin\lambda 20=0 \Rightarrow \lambda=\frac{n\pi}{20}.$$

and

$$u(x,0)=0 \Rightarrow (c_9\sin\lambda x+c_{10}\cos\lambda x)(c_{12}\cos 0) \Rightarrow c_{12}=0$$

Also

$$\frac{\partial u}{\partial t}=(c_9\sin\lambda x)(c_{11}\cos\lambda ct)(\lambda c)=A\lambda c\sin\lambda x\cos\lambda ct$$

where

$$A=c_9c_{11}$$

So that $u_t(x,t)=\sum_{n=1}^{\infty}\frac{n\pi c}{20}A_n\sin\frac{n\pi x}{20}\cos\frac{n\pi ct}{20}$

Now,

$$u_t(x,0)=\begin{cases} x, 0<x<10 \\ 20-x, 10<x<20 \end{cases}=\sum_{n=1}^{\infty}\frac{n\pi c}{20}A_n\sin\frac{n\pi x}{20}$$

$\Rightarrow A_n=\frac{2}{20}\int_0^{20}\frac{n\pi c}{20}A_n\sin\frac{n\pi x}{20}dx$ using Fourier half range series

which gives

$$A_n=\frac{800}{n^3\pi^3}[1-(-1)^n]$$

Therefore

$$u(x,t) = \frac{1600}{c\pi^3}\left[\sin\frac{\pi x}{20}\sin\frac{\pi ct}{20} - \frac{1}{27}\sin\frac{3\pi ct}{20} + \ldots\right]$$

Example 9: A wire is stretched between two points x = 0 and x= L. At the time t = 0 the wire follows the function $0.032\sin\frac{2\pi x}{L}$. Find the displacement of the wire at any time t.

Proceeding exactly as example 7 we find the general solution of one dimensional wave equation as

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

Now $y(x,0) = 0.032\sin\frac{2\pi x}{L} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$

Comparing the coefficients $n = 2, A_2 = 0.032$

Hence $y(x,t) = 0.032\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L}$

Questions for practice

1. A tightly stretched violin string of length l fixed at it's ends is plucked at x = l/3 and assumes initially the shape of a triangle of height a. Find the displacement y at any distance x and any time t after the string is released from rest.

Ans: $y(x,t) = \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$

2. Solve $y_{tt} = 4y_{xx}; y(0,t) = 0 = y(5,t), y(x,0) = 0, y_t(x,0) = f(x)$

Ans: $y(x,t) = \frac{5}{2\pi}\sin\pi x\sin 2\pi t.$

3. Find the deflection of the vibrating string of length π and ends are fixed corresponding to zero initial velocity and initial deflection $f(x) = k(\sin x - \sin 2x), c^2 = 1$

Ans: $f(x) = k(\sin x\cos t - \sin 2x\cos 2t).$

4. A tightly stretched flexible string has it's ends fixed at x = 0 and x = l. At t = 0 the string is given a shape defined by $\mu x(l - x)$ and then released. Find the displacement of the string.

Ans: $\frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}.$

2.4 One dimensional Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

With the following conditions

$u(0,t) = u(l,t) = 0$ (boundary conditions)

$u(x,0) = f(x)$ (initial condition)

Consider the solution $u(x,t) = X(x)T(t)$

So that $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = TX''$

$XT' = c^2 TX'' \Rightarrow \frac{T'}{c^2 T} = \frac{X''}{X} = k$, for any constant k

Now, $\frac{X''}{X} = k \Rightarrow \frac{d^2 X}{dx^2} = kX$ and $\frac{T'}{c^2 T} = k \Rightarrow \frac{dT}{dt} = c^2 kT$

Consider the following cases:

(i) when $k = 0$ then $\frac{d^2 X}{dx^2} = 0 \Rightarrow X = c_1 x + c_2$ and

$$\frac{dT}{dt} = 0 \Rightarrow T = c_3$$

$$u(x,t) = XT = (c_1 x + c_2)(c_3)$$

(ii) when $k = \lambda^2$ then $\frac{d^2 X}{dx^2} = \lambda^2 X \Rightarrow X = (c_5 e^{\lambda x} + c_4 e^{-\lambda x})$ and

$$\frac{dT}{dt} = c^2 \lambda^2 T \Rightarrow T = (c_6 e^{\lambda^2 c^2 t})$$

$$\Rightarrow u(x,t) = XT = (c_5 e^{\lambda x} + c_4 e^{-\lambda x})(c_6 e^{\lambda^2 c^2 t})$$

(iii) when $k = -\lambda^2$ then $\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \Rightarrow X = (c_7 \sin \lambda x + c_8 \cos \lambda x)$

$$\frac{d\ T}{dt} = c^2\lambda^2 T \Rightarrow T = (c_9 e^{-\lambda^2 c^2 t})$$

$$\Rightarrow u(x,t) = XT = (c_7 \sin\lambda x + c_8 \cos\lambda x)(c_9 e^{-\lambda^2 c^2 t})$$

In above all cases we get three different solutions out of which we have to chose that one which is consistent with the physical nature of the problem and provides a non trivial solution. So we will take (iii) as the most suitable solution that is

$$u(x,t) = XT = (c_7 \sin\lambda x + c_8 \cos\lambda x)(c_9 e^{-\lambda^2 c^2 t})$$

Now we apply the boundary conditions

$$u(0,t) = 0 \Rightarrow (c_8 \cos 0)T = 0 \Rightarrow c_8 = 0\ .$$

$$u(l,t) = 0 \Rightarrow c_7 \sin\lambda l = 0 \Rightarrow \lambda = \frac{n\pi}{l}.$$

Therefore

$$u(x,t) = (c_7 \sin\lambda x)(c_9 e^{\lambda^2 c^2 t}) = A \sin\lambda x e^{-\lambda^2 c^2 t}$$

Where

$$A = c_9 c_7$$

Consider the general solution $u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} e^{-\frac{n^2 c^2 \pi^2 t}{l^2}}$

Applying the last condition $u(x,0) = f(x)$ which gives

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \Rightarrow A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

using Fourier half range series.

Example 10: Determine the solution of one dimensional heat equation

$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial^2 x^2}$ where the boundary condition are $u(0,t) = u(l,t) = 0$ and initial condition $u(x,0) = x, l$ being the length of the bar.

Consider the solution $u(x,t) = X(x)T(t)$

So that $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = TX''$

$XT' = c^2 TX'' \Rightarrow \frac{T'}{c^2 T} = \frac{X''}{X} = k$, for any constant k

Now, $\frac{X''}{X} = k \Rightarrow \frac{d^2 X}{dx^2} = kX$ and $\frac{T''}{c^2 T} = k \Rightarrow \frac{d\ T}{dt} = c^2 kT$

Consider the following cases:

(i) when $k = 0$ then $\frac{d^2 X}{dx^2} = 0 \Rightarrow X = c_1 x + c_2$ and

$$\frac{d\ T}{dt} = 0 \Rightarrow T = c_3$$

$$u(x,t) = XT = (c_1 x + c_2)(c_3)$$

(ii) when $k = \lambda^2$ then $\frac{d^2 X}{dx^2} = \lambda^2 X \Rightarrow X = (c_5 e^{\lambda x} + c_4 e^{-\lambda x})$ and

$$\frac{d\ T}{dt} = c^2 \lambda^2 T \Rightarrow T = (c_6 e^{\lambda^2 c^2 t})$$

$$u(x,t) = XT = (c_5 e^{\lambda x} + c_4 e^{-\lambda x})(c_6 e^{\lambda^2 c^2 t})$$

(iii) when $k = -\lambda^2$ then $\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \Rightarrow X = (c_7 \sin \lambda x + c_8 \cos \lambda x)$

and $\frac{d\ T}{dt} = c^2 \lambda^2 T \Rightarrow T = (c_9 e^{-\lambda^2 c^2 t})$

$$u(x,t) = XT = (c_7 \sin \lambda x + c_8 \cos \lambda x)(c_9 e^{-\lambda^2 c^2 t})$$

The most suitable solution is

$$u(x,t) = XT = (c_7 \sin \lambda x + c_8 \cos \lambda x)(c_9 e^{-\lambda^2 c^2 t})$$

Now we apply the boundary conditions

$$u(0,t) = 0 \Rightarrow (c_8 \cos 0)T = 0 \Rightarrow c_8 = 0\ .$$

$$u(l,t) = 0 \Rightarrow c_7 \sin \lambda l = 0 \Rightarrow \lambda = \frac{n\pi}{l}.$$

Therefore

$$u(x,t) = (c_7 \sin \lambda x)(c_9 e^{\lambda^2 c^2 t}) = A \sin \lambda x e^{-\lambda^2 c^2 t}$$

where

$A = c_9 c_7$

Consider the general solution $u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} e^{-\frac{n^2 c^2 \pi^2 t}{l^2}}$

Applying the last condition $u(x,0) = x$ which gives

$$x = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \Rightarrow A_n = \frac{2}{l}\int_0^l x \sin \frac{n\pi x}{l} dx = \frac{2l}{n\pi}(-1)^{n+1}$$

using Fourier half range series

Hence

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2l}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{l} e^{-\frac{n^2 c^2 \pi^2 t}{l^2}}$$

Example 11: The temperature of a bar 10 cm long with insulated sides has ends A and B maintained at temperature 50C and 100C respectively until steady state condition prevail. The temperature at A is suddenly raised to 90C and at the same time at B is lowered to 60C. Find the temperature distribution in the bar at time t.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial^2 x^2}$$

The steady state condition

$$\frac{\partial u}{\partial t} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = Ax + B$$

The initial temperature distribution in the bar is

$$u_1 = 50 + \left(\frac{100-50}{10}\right)x = 50 + 5x$$

And final tempreture distribution is given by

$$u_2 = 90 + \left(\frac{60-90}{10}\right)x = 90 - 3x$$

At any intermediate period

$u(x,t) = u_2 + u'$

Where u' satisfies one dimensional heat equation such that

$u(0,t)=90, u(10,t)=60$

Now,

$$u(x,t)=90-3x+\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{10} e^{-\frac{n^2\pi^2c^2t}{100}}$$

$$u(x,0)=50+5x \Rightarrow 8x-40=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{10}$$

Therefore $A_n = \frac{2}{10}\int_0^{10}(8x-40)\sin\frac{n\pi x}{10}\,dx = -\frac{160}{n\pi}$

$$u(x,t)=90-3x-\frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{5} e^{-\frac{n^2\pi^2c^2t}{25}}.$$

Example 12: The temperature distribution in a metal rod of length π is governed by the partial differential equation $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial^2 x^2}$

It is given that both the ends of the rod are perfectly insulated. Determine the temperature distribution in the rod at any instant t. The initial temperature distribution

$u(x,0)=3\cos 5x$

Since the solution of one dimensional heat equation

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial^2 x^2}$$

is

$$u(x,t)=c_1e^{-\lambda^2 t}(c_2\cos\lambda x+c_3\sin\lambda x)$$

Since both the ends of the rod are insulated so boundary conditions

$\frac{\partial u}{\partial x}=0$ at $x=0,\pi$

Now $\frac{\partial u}{\partial x}=c_1e^{-\lambda^2 t}\lambda(-c_2\sin\lambda x+c_3\cos\lambda x)$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0}=0\Rightarrow c_3=0$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=\pi}=0\Rightarrow c_1e^{-\lambda^2 t}\lambda(-c_2\sin\lambda\pi)=0\Rightarrow\lambda=n$$

Therefore,

$$u(x,t)=Ae^{-\lambda^2 t}\cos\lambda x$$

and general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \cos nx$$

$$u(x,0) = 3\cos 5x \Rightarrow \sum_{n=0}^{\infty} A_n \cos nx = 3\cos 5x \Rightarrow n = 5, A_5 = 3$$

$$u(x,t) = 3e^{-25t} \cos 5x.$$

Questions for practice

1. Find the temperature in a homogeneous bar of heat conducting material of length L cm with its ends kept at zero temperature and initial temperature given by $\dfrac{x(L-x)}{L^2}$

Ans: $\dfrac{8L}{\pi^3} \sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \sin \dfrac{(2n-1)\pi x}{L} e^{-\frac{(2n-1)^2 \pi^2 c^2 t}{L^2}}$

2. The temperature of a bar 50 cm long with insulated sides is kept at 0C and 100C respectively until steady state condition prevail. The two ends are hidden suddenly insulated so that the temperature gradient is zero at each and thereafter. Find the temperature distribution.

Ans: $\dfrac{200}{\pi} \sum_{n-1}^{\infty} \dfrac{(-1)^{n+1}}{n} \sin \dfrac{n\pi x}{50} e^{-\frac{n^2\pi^2 kt}{2500}}$

3. Solve $u_t = a^2 u_{xx}$ under the conditions $u_x(0,t) = u_x(\pi,t) = 0$ and $u(x,0) = x^2$

Ans: $\dfrac{\pi^3}{3} + 4\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2} \cos nx e^{-a^2 n^2 t}$

4. Find the temperature in a slab whose whose ends $x = 0, x = L$ are kept at zero Temperature and initial temperature is given by $f(x) = \begin{cases} k, 0 < x < L/2 \\ 0, L/2 < x < L \end{cases}$

Ans: $\dfrac{4k}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{n} \sin^2 \dfrac{n\pi}{4} \sin \dfrac{n\pi x}{L} e^{-\frac{c^2\pi^2 n^2 t}{L^2}}$

2.5 Two dimensional Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

with the following conditions

$u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0$ (boundary conditions)

$u(x,y,0)=f(x,y), \left(\dfrac{\partial u}{\partial t}\right)_{t=0}=0$ (initial conditions)

Consider the solution $u(x,t)=X(x)Y(y)T(t)$

So that $\dfrac{\partial^2 u}{\partial t^2}=XYT''$ $\dfrac{\partial^2 u}{\partial x^2}=TX''Y$ and $\dfrac{\partial^2 u}{\partial y^2}=TXY''$

$XYT'''=c^2(TYX''+TXY'') \Rightarrow \dfrac{T''}{c^2T}=\dfrac{X''}{X}+\dfrac{Y''}{Y}=-k^2$, for any constant k

Now, $\dfrac{T''}{c^2T}=-k^2 \Rightarrow \dfrac{d^2T}{dt^2}=-c^2k^2T \Rightarrow T=(c_1\cos kct+c_2\sin kct)$

Consider

$k^2=\lambda^2+\mu^2$

$\dfrac{X''}{X}=-\lambda^2 \Rightarrow \dfrac{d^2X}{dx^2}+\lambda^2X=0 \Rightarrow X=(c_3\cos\lambda x+c_4\sin\lambda x)$

$\dfrac{Y''}{Y}=-\mu^2 \Rightarrow \dfrac{d^2Y}{dy^2}+\lambda^2Y=0 \Rightarrow Y=(c_5\cos\mu y+c_6\sin\mu y)$

So that

$$u(x,t)=XYT=(c_1\cos kct+c_2\sin kct)(c_3\cos\lambda x+c_4\sin\lambda x)$$

$$(c_5\cos\mu y+c_6\sin\mu y)$$

Now we apply the boundary conditions

$u(0,y,t)=0 \Rightarrow c_3=0$.

$u(a,y,t)=0 \Rightarrow c_4\sin\lambda a=0 \Rightarrow \lambda=\dfrac{n\pi}{a}$.

$u(x,0,t)=0 \Rightarrow c_5=0$.

$u(x,b,t)=0 \Rightarrow c_6\sin\mu b=0 \Rightarrow \mu=\dfrac{m\pi}{b}$.

Also

$$\frac{\partial u}{\partial t} = (c_4 \sin \lambda x)(c_6 \sin \mu y)(-c_1 \sin kct + c_2 \cos kct)(kc)$$

and

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = (c_4 \sin \lambda x)(c_6 \sin \mu y)(c_{21} \cos 0)(kc) = 0 \Rightarrow c_2 = 0$$

Therefore

$$u(x,t) = (c_4 \sin \lambda x)(c_6 \sin \mu y)(c_1 \cos kct) = A \sin \lambda x \sin \mu y \cos kct$$

Consider the general solution $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \cos kct$

where

$$k = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

Applying the last condition $u(x,y,0) = f(x,y)$ which gives

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

$$\Rightarrow A_{mn} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b f(x,y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}\, dydx$$

using double Fourier half range series

Example 13: Find the deflection of the square membrane with $a = b = 2$ if the initial velocity is zero and the initial deflection is $u(x,y,0) = \psi \sin \frac{\pi x}{2} \sin \frac{3\pi y}{2}$

The problem is to solve

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

With the following conditions

$u(0,y,t) = u(2,y,t) = u(x,0,t) = u(x,2,t) = 0$ (boundary conditions)

$$u(x,y,0)=f(x),\left(\frac{\partial u}{\partial t}\right)_{t=0}=0 \quad \text{(initial conditions)}$$

We have already obtain the general solution

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{n\pi x}{2}\sin\frac{m\pi y}{2}\cos kct$$

where

$$k=\pi\sqrt{\frac{n^2}{2^2}+\frac{m^2}{2^2}}$$

Applying the last condition $u(x,y,0)=\psi\sin\frac{\pi x}{2}\sin\frac{3\pi y}{2}$ which gives

$$\psi\sin\frac{\pi x}{2}\sin\frac{3\pi y}{2}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{n\pi x}{2}\sin\frac{m\pi y}{2}$$

$$m=3, n=1, A_{31}=\psi$$

Hence $u(x,y,t)=\psi\sin\frac{\pi x}{2}\sin\frac{3\pi y}{2}\cos kct$

where

$$k=\pi\sqrt{\frac{1^2}{2^2}+\frac{3^2}{2^2}}=\frac{\sqrt{10}}{2}\pi$$

2.6 Two dimensional Heat equation

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

With the following conditions

$$u(0,y,t)=u(a,y,t)=u(x,0,t)=u(x,b,t)=0 \quad \text{(boundary conditions)}$$

$$u(x,y,0)=f(x,y) \quad \text{(initial condition)}$$

Consider the solution $u(x,t)=X(x)Y(y)T(t)$

So that $\frac{\partial^2 u}{\partial t^2} = XYT'$ $\frac{\partial^2 u}{\partial x^2} = TX''Y$ and $\frac{\partial^2 u}{\partial y^2} = TXY''$

$XYT' = c^2(TYX'' + TXY'') \Rightarrow \frac{T'}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = -k^2$, for any constant k

Now, $\frac{T'}{c^2 T} = -k^2 \Rightarrow \frac{d\ T}{dt} = -c^2 k^2 T \Rightarrow T = c_1 e^{-c^2 k^2 t}$

Consider

$k^2 = \lambda^2 + \mu^2$

and

$$\frac{X''}{X} = -\lambda^2 \Rightarrow \frac{d^2 X}{dx^2} + \lambda^2 X = 0 \Rightarrow X = (c_3 \cos \lambda x + c_4 \sin \lambda x)$$

$$\frac{Y''}{Y} = -\mu^2 \Rightarrow \frac{d^2 Y}{dy^2} + \lambda^2 Y = 0 \Rightarrow Y = (c_5 \cos \mu y + c_6 \sin \mu y)$$

So that

$$u(x,t) = XYT = c_1 e^{-c^2 k^2 t}(c_3 \cos \lambda x + c_4 \sin \lambda x)(c_5 \cos \mu y + c_6 \sin \mu y)$$

Now we apply the boundary conditions

$u(0, y, t) = 0 \Rightarrow c_3 = 0.$

$u(a, y, t) = 0 \Rightarrow c_4 \sin \lambda a = 0 \Rightarrow \lambda = \frac{n\pi}{a}.$

$u(x, 0, t) = 0 \Rightarrow c_5 = 0$.

$u(x, b, t) = 0 \Rightarrow c_6 \sin \mu b = 0 \Rightarrow \mu = \frac{m\pi}{b}.$

Therefore

$$u(x,t) = (c_4 \sin \lambda x)(c_6 \sin \mu y) c_1 e^{-c^2 k^2 t} = A \sin \lambda x \sin \mu y e^{-c^2 k^2 t}$$

Consider the general solution $u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} c_1 e^{-c^2 k^2 t}$

where

$$k = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

Applying the last condition $u(x, y, 0) = f(x, y)$ which gives

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

$$\Rightarrow A_{mn} = \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b f(x, y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dydx$$

using double Fourier half range series

Example 13: Find the temperature distribution in a rectangular metal plate having dimensions $a = 2, b = 3$. And initial temperature is given $u(x, y, 0) = xy$.

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

With the following conditions

$u(0, y, t) = u(2, y, t) = u(x, 0, t) = u(x, 3, t) = 0$ (boundary conditions)

$u(x, y, 0) = f(x)$ (initial condition)

Consider the general solution $u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{2}\sin\frac{m\pi y}{3}c_1 e^{-c^2k^2t}$

Where

$$k = \pi\sqrt{\frac{n^2}{2^2} + \frac{m^2}{3^2}}$$

Applying the last condition $u(x, y, 0) = xy$ which gives

$$xy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{2}\sin\frac{m\pi y}{3}$$

$$\Rightarrow A_{mn} = \frac{2}{2}\frac{2}{3}\int_0^2\int_0^3 xy\sin\frac{n\pi x}{2}\sin\frac{m\pi y}{3}\,dydx = \frac{4}{mn\pi^2}[1-(-1)^n][1-(-1)^m]$$

Hence

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4}{mn\pi^2}[1-(-1)^n][1-(-1)^m]\sin\frac{n\pi x}{2}\sin\frac{m\pi y}{3}c_1e^{-c^2k^2t}$$

where

$$k=\pi\sqrt{\frac{n^2}{2^2}+\frac{m^2}{3^2}}$$

Questions for practice

1. A tightly stretched unit square membrane starts vibrating from rest and it's initial displacement is $k\sin 2\pi x\sin\pi y$. Show that the deflection at any instant

$k\sin 2\pi x\sin\pi y\cos(\sqrt{5}\pi ct)$

2. A thin rectangular plate whose surface is impervious to heat flow has at t =0 an arbitrary distribution of temperature $f(x,y)$. Its four edges $x=0, y=0, x=a, y=b$ are kept at zero temperature Determine the temperature at a point of a plate as t increases.

Ans: $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}e^{-c^2p\ t}; A_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}dxdy$

where

$$p^2=\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)$$

2.7 Laplace equation

When we impose the steady state condition $\frac{\partial u}{\partial t}=0$ in two dimensional heat equation, it becomes following Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Example 14: Consider the following problem $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$$u(0,y)=u(l,y)=u(x,0)=0, u(x,a)=\mu\sin\frac{3\pi x}{l}$$

Here first we consider $u(x,y)=X(x)Y(y)$

So that $\frac{\partial^2 u}{\partial x^2}=X''Y$ and $\frac{\partial^2 u}{\partial y^2}=XY''$

$YX''+XY''=0 \Rightarrow = \frac{X''}{X} = -\frac{Y''}{Y} = -k^2$, for any constant k

Now,

$$\frac{Y''}{Y} = k^2 \Rightarrow \frac{d^2Y}{dy^2} - k^2 Y = 0 \Rightarrow Y = (c_1 e^{ky} + c_2 e^{-ky})$$

And

$$\frac{X''}{X} = -k^2 \Rightarrow \frac{d^2X}{dx^2} + k^2 X = 0 \Rightarrow X = (c_3 \cos kx + c_4 \sin kx)$$

So that

$$u(x,y) = XY = (c_1 e^{ky} + c_2 e^{-ky})(c_3 \cos \lambda x + c_4 \sin \lambda x)$$

Now we apply the boundary conditions

$$u(0,y) = 0 \Rightarrow c_3 = 0 \ .$$

$$u(l,y) = 0 \Rightarrow c_4 \sin kl = 0 \Rightarrow k = \frac{n\pi}{l}.$$

$$u(x,0) = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

So we may write

$$u(x,y) = (c_1 e^{ky} - c_1 e^{-ky})(c_4 \sin kx)$$

$$= 2c_1 c_4 \sin kx \sinh ky = A \sin kx \sinh ky$$

General solution

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l} x \sinh \frac{n\pi}{l} y$$

Now

$$u(x,a) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l} x \sinh \frac{n\pi a}{l} = \mu \sin \frac{3\pi x}{l}$$

which implies

$$n = 3, A_3 = \frac{\mu}{\sinh \frac{3\pi a}{l}}$$

Hence

$$u(x, y) = \frac{\mu \sin \frac{3\pi x}{l} \sinh \frac{3\pi y}{l}}{\sinh \frac{3\pi a}{l}}$$

Example 15: A long rectangular plate of width a cm with insulated surfaces has its temperature v equal to zero on both the long sides and one of the short sides so that

$$v(0, y) = v(a, y) = 0, \lim_{y \to \infty} v(x, y) = 0, v(x,0) = \theta x.$$

Then show that the steady state temperature within the plate is

$$v(x, y) = -\frac{2a\theta}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp\left(-\frac{n\pi y}{a} \right) \sin \frac{n\pi x}{a}$$

Here first we consider $v(x, y) = X(x)Y(y)$

So that $\frac{\partial^2 v}{\partial x^2} = X''Y$ and $\frac{\partial^2 v}{\partial y^2} = XY''$

$YX'' + XY'' = 0 \Rightarrow = \frac{X''}{X} = -\frac{Y''}{Y} = -k^2$, for any constant k

Now,

$$\frac{Y''}{Y} = k^2 \Rightarrow \frac{d^2 Y}{dy^2} - k^2 Y = 0 \Rightarrow Y = (c_1 e^{ky} + c_2 e^{-ky})$$

and

$$\frac{X''}{X} = -k^2 \Rightarrow \frac{d^2 X}{dx^2} + k^2 X = 0 \Rightarrow X = (c_3 \cos kx + c_4 \sin kx)$$

So that

$$v(x, y) = XY = (c_1 e^{ky} + c_2 e^{-ky})(c_3 \cos \lambda x + c_4 \sin \lambda x)$$

Now we apply the boundary conditions

$$v(0, y) = 0 \Rightarrow c_3 = 0.$$

$$v(a, y) = 0 \Rightarrow c_4 \sin kl = 0 \Rightarrow k = \frac{n\pi}{a}.$$

So we may write

$$\lim_{y \to \infty} v(x, y) = 0 \Rightarrow c_1 = 0$$

Therefore

$$v(x, y) = (c_2 e^{-ky})(c_4 \sin kx) = c_2 c_4 \sin kx e^{-ky} = A \sin kx e^{-ky}$$

And the general solution is $v(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} e^{-\frac{n\pi y}{a}}$

Now

$$v(x,0) = \theta x. = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a}$$

$$\Rightarrow A_n = \frac{2\theta}{a} \int_0^a x \sin \frac{n\pi x}{a} dx = \frac{2(-1)^{n+1} a}{n\pi} \theta$$

Hence $v(x, y) = -\frac{2a\theta}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp\left(-\frac{n\pi y}{a}\right) \sin \frac{n\pi x}{a}$

Example 16 : The temperature along the upper horizontal edge is given by

$$u(x,0) = x(L - x); 0 < x < L$$

While other three edges are kept at zero temperature. Find the steady state temperature in the plate. The dimensions of the plate are $0 < x < l, 0 < y < l$.

We know that the solution of Laplace equation is

$$u(x, y) = XY = (c_1 e^{ky} + c_2 e^{-ky})(c_3 \cos kx + c_4 \sin kx)$$

With boundary conditions

$$u(x, l) = u(0, y) = u(L, y) = 0, u(x,0) = x(L - x)$$

$u(0, \mathrm{y}) = 0 \Rightarrow c_3 = 0$

$u(\mathrm{L}, y) = 0 \Rightarrow c_4 \sin \mathrm{L}\, k = 0 \Rightarrow k = \dfrac{n\pi}{L}$

$u(x, l) = 0 \Rightarrow c_4 \sin kx(c_1 e^{kl} + c_2 e^{kl}) = 0 \Rightarrow c_1 e^{kl} + c_2 e^{kl} = 0$

$\Rightarrow c_1 e^{kl} = -c_2 e^{kl} = -\dfrac{1}{2} c$

$u(x, l) = \dfrac{1}{2}(c\ e^{(l-y)k} - c\ e^{-(l-y)k})c_4 \sin kx = A \sinh k(l - y) \sin kx$

Where

$A = cc_4$

General solution is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(l - y)}{L}$$

Now

$$u(x,0) = x(L - x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi l}{L}$$

$$A_n = \frac{2}{L \sinh \frac{n\pi l}{L}} \int_0^L (Lx - x^2) \sin \frac{n\pi x}{L} dx$$

$$= \frac{4L^2}{n^2\pi^2}[1 - (-1)^n] \operatorname{cosech} \frac{n\pi l}{L}$$

Hence $u(x, y) = \displaystyle\sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2}[1 - (-1)^n] \operatorname{cosech} \frac{n\pi l}{L} \sin \frac{n\pi x}{L}$

Questions for practice

1. Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in a rectangle

$u(0, y) = u(a, y) = u(x, b) = 0, u(x,0) = f(x)$ along x axis Ans:

$$u(x, y) = \sum_{n-1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi(b - y)}{a}; B_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

2. An infinitely long plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at temperature u_0 at all points and the other edges are kept at zero temperature. Determine the temperature at any point of the plate in the steady state. Ans: $\frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)xe^{-(2n-1)y}$

3. A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may bee considered infinite in length. If the temperature along one short edge y = 0 is $u(x,0) = 100\sin\frac{\pi x}{8}, 0 < x < 8$ while the two long edges $x = 0, x = 8$

As well as the other short edge are kept at 0C show that the steady state temperature at any point is given by $u(x,y) = 100e^{-\frac{\pi y}{8}}\sin\frac{\pi x}{8}$

4. Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in a square

$u(0,y) = u(\pi,y) = u(x,\pi) = 0, u(x,0) = \sin^2 x$

Ans: $u(x,y) = \frac{-8}{\pi}\sum_{n=1,3,5..}^{\infty}\frac{\sin nx \sinh n(\pi - y)}{n(n^2-4)\sinh n\pi}$

2.8 Transmission lines

Consider the flow of electricity in an insulated cable. Let V, i be the current and voltage respectively. Then potential drop and current drop are given by

$-\frac{\partial V}{\partial x} = Ri + L\frac{\partial i}{\partial t}$ and

$-\frac{\partial i}{\partial x} = GV + C\frac{\partial V}{\partial t}$ respectively.

Where the symbols G, R, C and L have their usual meaning.

Differentiating these two equations w. r. t. x and t and eliminating i and V, we get

Following Telephone equation

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC + LG)\frac{\partial V}{\partial t} + RGV \text{ and}$$

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (RC + LG)\frac{\partial i}{\partial t} + RGi$$

Case I : when $L = G = 0$ we obtain following **Telegraph** equations

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t} \quad \text{and} \quad \frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t}$$

Case II : when $R = G = 0$ we obtain following **Radio** equations

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2}$$

Case III: when $L = C = 0$ we obtain following Submarine cable equations

$$\frac{\partial^2 V}{\partial x^2} = RGV \quad \text{and} \quad \frac{\partial^2 i}{\partial x^2} = RGi$$

Case IV: when $R = G = 0$ we obtain following Transmission line equations

$$\frac{\partial\ V}{\partial x} = -L\frac{\partial i}{\partial t} \quad \text{and} \quad \frac{\partial\ i}{\partial x} = -C\frac{\partial V}{\partial t}$$

Example 17 : Neglecting R and G find the e.m.f V(x,t) in a line of length l, t sec after the ends were suddenly grounded. Given that $i(x,0) = i_0$ and

$$V(x,0) = 0.006\sin\frac{\pi x}{l} - 0.012\sin\frac{3\pi x}{l}$$

Consider the Radio equation $\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}$ with conditions $V(0,t) = V(l,t) = 0$

Now from case IV, $i(x,0) = i_0 \Rightarrow \left(\frac{\partial V}{\partial t}\right)_{t=0} = 0$

We assume that $V(x,t) = X(x)T(t)$

So that $\frac{\partial^2 V}{\partial t^2} = XT''$ and $\frac{\partial^2 V}{\partial x^2} = TX''$

$XT''' = c^2TX'' \Rightarrow \frac{T''}{c^2T} = \frac{X''}{X} = k$, for any constant k

Now, $\frac{X''}{X} = k \Rightarrow \frac{d^2X}{dx^2} = kX$ and $\frac{T''}{X} = \frac{1}{LC}k \Rightarrow \frac{d^2T}{dt^2} = \frac{1}{LC}kT$

$k = -\lambda^2$ then $\frac{d^2X}{dx^2} + \lambda^2 X = 0 \Rightarrow X = (c_1 \sin \lambda x + c_2 \cos \lambda x)$ and

$$\frac{d^2T}{dt^2} = -\frac{1}{LC}\lambda^2 T \Rightarrow T = (c_3 \sin \lambda \frac{1}{\sqrt{LC}} t + c_4 \cos \lambda \frac{1}{\sqrt{LC}} t)$$

$$V(x,t) = XT = (c_1 \sin \lambda x + c_2 \cos \lambda x)(c_3 \sin \lambda \frac{1}{\sqrt{LC}} t + c_4 \cos \lambda \frac{1}{\sqrt{LC}} t)$$

Applying the conditions

$$V(0,t) = 0 \Rightarrow (c_2 \cos 0)T = 0 \Rightarrow c_2 = 0.$$

$$V(l,t) = 0 \Rightarrow c_1 \sin \lambda l = 0 \Rightarrow \lambda = \frac{n\pi}{l}.$$

$$\left(\frac{\partial V}{\partial t}\right)_{t=0} = (c_1 \sin \lambda x)(c_3 \cos 0)(\lambda \frac{1}{\sqrt{LC}}) = 0 \Rightarrow c_3 = 0$$

Therefore

$$V(x,t) = (c_1 \sin \lambda x)(c_4 \cos \lambda \frac{1}{\sqrt{LC}} t) = A \sin \lambda x \cos \lambda \frac{1}{\sqrt{LC}} t$$

where

$$A = c_1 c_4$$

Consider the general solution $V(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$

Applying the last condition $V(x,0) = 0.006 \sin \frac{\pi x}{l} - 0.012 \sin \frac{3\pi x}{l}$ gives

$$0.006 \sin \frac{\pi x}{l} - 0.012 \sin \frac{3\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

which implies

$$n = 1, n = 3, A_1 = 0.006, A_3 = -0.012$$

Hence $V(x,t) = 0.006 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} - 0.012 \sin \frac{3\pi x}{l} \cos \frac{3\pi t}{l\sqrt{LC}}$

Example 18 The current distribution in a wire of length 20 cm is determined by the equation $\frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t}$ with conditions $i(0,t) = i(20,t) = 0, i(x,0) = \omega$. Find the current the wire at any time t.

Consider the solution $i(x,t) = X(x)T(t)$

So that $\frac{\partial\ i}{\partial t} = XT'$ and $\frac{\partial^2 i}{\partial x^2} = TX''$

$XT' = \frac{1}{RC}TX'' \Rightarrow \frac{RCT'}{T} = \frac{X''}{X} = k$, for any constant k

Now, $\frac{X''}{X} = k \Rightarrow \frac{d^2 X}{dx^2} = kX$ and $\frac{1}{RC}\frac{T'}{T} = k \Rightarrow \frac{d\ T}{dt} = \frac{1}{RC}kT$

If $k = -\lambda^2$ then $\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \Rightarrow X = (c_1 \sin \lambda x + c_2 \cos \lambda x)$ and

$\frac{d\ T}{dt} = \frac{1}{RC}\lambda^2 T \Rightarrow T = (c_3 e^{-\lambda^2 \frac{1}{RC}t})$

$i(x,t) = XT = (c_1 \sin \lambda x + c_2 \cos \lambda x)(c_3 e^{-\lambda^2 \frac{1}{RC}t})$

Now we apply the boundary conditions

$i(0,t) = 0 \Rightarrow (c_2 \cos 0)T = 0 \Rightarrow c_2 = 0.$

$i(20,t) = 0 \Rightarrow c_1 \sin 20\lambda = 0 \Rightarrow \lambda = \frac{n\pi}{20}.$

Therefore

$i(x,t) = (c_1 \sin \lambda x)(c_3 e^{-\lambda^2 \frac{t}{RC}}) = A \sin \lambda x e^{-\lambda^2 \frac{t}{RC}}$ where $A = c_1 c_3$

Consider the general solution $i(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{20} e^{-\frac{n^2\pi^2 t}{400RC}}$

Applying the last condition $u(x,0) = \omega$ which gives

$\omega = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \Rightarrow A_n = \frac{2}{20}\int_0^{20} \omega \sin\frac{n\pi x}{20} dx = \frac{2}{n\pi}[1-(-1)^n]$

Therefore, $i(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[1-(-1)^n]\sin\frac{n\pi x}{20} e^{-\frac{n^2\pi^2 t}{400RC}}$

Questions for practice

1. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time t = 0 the receiving end is grounded. Find the voltage and current t sec later. Neglected leakance and inductance.

Ans: $v(x,t) = \frac{20(l-x)}{l} + \frac{24}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{RCl^2}}$

$i(x,t) = \frac{20}{Rl} + \frac{24}{lR}\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi x}{l} e^{-\frac{n^2\pi^2 t}{RCl^2}}$

2. Solve $\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}$ assuming that the initial voltage is $K\sin\frac{2\pi x}{l}; V_t(x_0) = 0$ and V = 0 at the ends $x = 0, x = l$ for all t. Ans : $K\sin\frac{2\pi x}{l}\cos\frac{2\pi t}{l\sqrt{LC}}$

3. Find the current I and voltage V in a line of length l, t sec after the ends are suddenly grounded, given that $I(x,0) = I_0, V(x,0) = V_0 \sin\frac{\pi x}{l}$. R and G are negligible.

Ans: $V = V_0 \sin\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}}$ $\quad I = I_0 - V_0\sqrt{\frac{C}{L}}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{LC}}$

Links for e-contents

https://www.youtube.com/watch?v=BcwMJ-8-cvk

https://www.youtube.com/watch?v=CveZCpDq3Y4

(Method of separation of variables)

https://www.youtube.com/watch?v=6uMswn4les4&list=PL2bgr0gejLWM60EV8Njl2yyrQc4kCgGhT

https://www.youtube.com/watch?v=Do_q2ErNAoQ

https://www.youtube.com/watch?v=5AiGSzZ1QqU

https://www.youtube.com/watch?v=yTavjRbrp7A

(One dimensional Heat and Wave equation)

https://www.youtube.com/watch?v=S0Wg7k3FNJU

(Two dimensional Heat equation)

https://www.youtube.com/watch?v=8stWtUuMYKY

https://www.youtube.com/watch?v=KAS7JBztw8E

(Two dimensional Wave equation)

https://www.youtube.com/watch?v=h7IqoB1wGf0

https://www.youtube.com/watch?v=dHlBL6ImtIM

(Two dimensional Heat and Laplace equation)

https://www.youtube.com/watch?v=surDm-x5Uwo

https://www.youtube.com/watch?v=M2uiKA1qG-o

(Transmission lines)

https://www.youtube.com/watch?v=KxvujWiRIbE

https://www.youtube.com/watch?v=FPd9C03WAX4

(Applications of pde)