Other Numerical Methods for Parabolic PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

forward time

at (x_i, t_j) by average of values at j and $j+1$

$$\underbrace{\frac{u_{i,j+1} - u_{i,j}}{k}}_{\boxed{j+1} - \boxed{j}} = \frac{1}{2}\left\{\underbrace{\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}}_{\boxed{j+1}} + \underbrace{\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}}_{\boxed{j}}\right\}$$

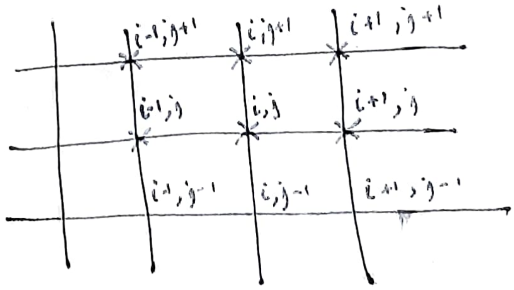
Lets simplify by taking $\lambda = k/h^2$

One can discretize it as :-

$$-\lambda u_{i-1,j+1} + 2(1+\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1-\lambda)u_{i,j} + \lambda u_{i+1,j} \quad \text{— (A)}$$

Called "Crank - Nicolson Implicit Scheme"

In $(j+1)$, we have more than one terms



Example

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$u(0,t) = u(1,t) = 0 \;\forall\; t \geq 0$

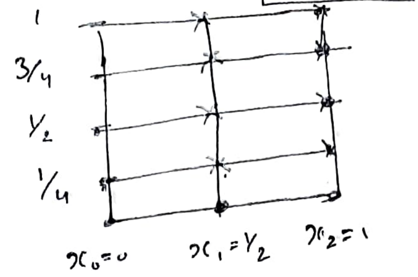
$u(x,0) = x - x^2, \quad 0 \leq x \leq 1$

Parameter $h = 1/2$, $k = 1/4 \Rightarrow \lambda = 1$

I.C. $u(x,0) = x - x^2$

$u_{i,0} = x_i - x_i^2$

$u_{0,0} = 0$; $u_{1,0} = \frac{1}{4}$; $u_{2,0} = 1 - 1 = 0$



$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

$j=0$

$i=1 \Rightarrow -\underset{0}{u_{0,1}} + 4u_{1,1} - \underset{0}{u_{2,1}} = \underset{1}{u_{0,0}} + \underset{0}{u_{2,0}}$

$u_{1,1} = 0$

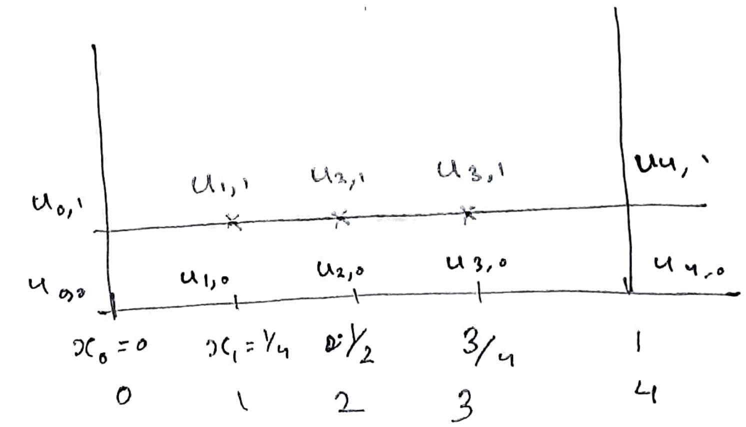
$j=1$ $\quad -\underset{0}{u_{0,2}} + 4u_{1,2} - \underset{0}{u_{2,2}} = \underset{0}{u_{0,1}} + u_{2,1}$

$u_{1,2} = 0$

Finer grid!

choose $h = 1/4$;

$\lambda = 1$



$i=1, j=0$

$u_{1,0} = \frac{1}{4}(\frac{1}{4} - 1)$

$u_{2,0} = x(x-1) = \frac{1}{2}(\frac{1}{2} - 1)$

$u_{3,0} = \frac{3}{4}(\frac{3}{4} - 1)$

$\cancel{4u_{1,1} - u_{2,1} = u_{2,0}}$

$$-u_{0,1} + 4u_{1,1} - u_{2,1} = u_{0,0} + u_{2,0}$$

$$4u_{1,1} - u_{2,1} = -1/4$$

✓ known

= calculate

$i=2, j=0$

$$-u_{1,1} + 4u_{2,1} - u_{3,1} = u_{1,0} + u_{3,0}$$

$$-u_{1,1} + 4u_{2,1} - u_{3,1} = u_{1,0} + u_{3,0}$$

$i=3, j=0$

$$-u_{2,1} + 4u_{3,1} - u_{4,1} = u_{2,0} + u_{4,0}$$

3 unknown ($u_{1,1}$, $u_{2,1}$, $u_{3,1}$) and 3 equations. One can solve.

~~take (A)~~ Lets take (A) L10 - ② a

$$-\lambda u_{i-1,j+1} + 2(1+\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} =$$
$$\lambda u_{i-1,j} + 2(1-\lambda)u_{i,j} + \lambda u_{i+1,j}$$

$j=0$ ~~level~~ give $j=1$ level

$$-\lambda u_{i-1,1} + 2 + 2(1+\lambda)u_{i,1} - \lambda u_{i+1,1} = \lambda u_{i-1,0} + 2(1-\lambda)u_{i,0} + \lambda u_{i+1,}$$

$i=1$

$$-\lambda u_{0,1} + 2(1+\lambda)u_{1,1} - \lambda u_{2,1} = \lambda u_{0,0} + 2(1-\lambda)u_{1,0} + \lambda u_{2,0}$$
$$2(1+\lambda)\underline{u_{1,1}} - \lambda \underline{u_{2,1}} = \lambda u_{0,0} + 2(1-\lambda)u_{1,0} + \lambda u_{2,0} + \lambda u_{0,1}$$

$i=2$

$$-\lambda \underline{u_{1,1}} + 2(1+\lambda)\underline{u_{2,1}} - \lambda \underline{u_{3,1}} = \lambda u_{1,0} + 2(1-\lambda)u_{2,0} + \lambda u_{3,0}$$

$i=3$

$$-\lambda \underline{u_{2,1}} + 2(1+\lambda)\underline{u_{3,1}} - \lambda \underline{u_{4,1}} = \lambda u_{2,0} + 2(1-\lambda)u_{3,0} + \lambda u_{4,0}$$

$\vdots$

$i=N-1$

$$-\lambda u_{N-2,1} + 2(1+\lambda)u_{N-1,1} - \lambda u_{N,1} = \lambda u_{N-2,0} + 2(1-\lambda)u_{N-1,0} + \lambda u_{N,0}$$
$$-\lambda \underline{u_{N-2,1}} + 2(1+\lambda)\underline{u_{N-1,1}} = \lambda u_{N-2,0} + 2(1-\lambda)u_{N-1,0} + \lambda u_{N,0} + \lambda u_{N,1}$$

$$\begin{pmatrix} 2(1+\lambda) & -\lambda & 0 & 0 & \cdots & & \\ -\lambda & 2(1+\lambda) & -\lambda & 0 & & & \\ 0 & -\lambda & 2(1+\lambda) & -\lambda & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & -\lambda & 2(1+\lambda) & -\lambda \\ & & & & 0 & -\lambda & 2(1+\lambda) \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ \vdots \\ \vdots \\ u_{N-2,1} \\ u_{N-1,1} \end{pmatrix} =$$

$$(d_{1,1}\ \ d_{2,1}\ \ d_{3,1}\ \ \dots\ \ d\ \ \ )^T$$

Crank-Nicolson give (N-1)-dimensional linear system for each time step.

As seen the rh is diagonal system. (Tridiagonal)

One can also use iteration technique like Jacobi or Gauss-Seidel

$\boxed{\lambda = k/h^2}$

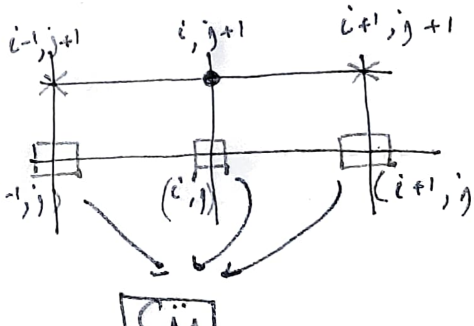
$$-\lambda u_{i-1,j+1} + 2(1+\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1-\lambda)u_{i,j} + \lambda u_{i+1,j}$$

Rearranging

$$\underbrace{(1+\lambda)u_{i,j+1}}_{j+1\text{ level}} = \frac{\lambda}{2}\Big[\underbrace{u_{i-1,j+1} + u_{i+1,j+1}}_{(j+1)\text{ level}}\Big] + \underbrace{u_{i,j} + \frac{\lambda}{2}\big[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\big]}_{j^{th}\text{ level}}$$

$$u_{i,j+1} = \frac{\lambda}{2(1+\lambda)}\big[u_{i-1,j+1} + u_{i+1,j+1}\big] + \frac{C_{i,j}}{(1+\lambda)}$$

here $C_{i,j} = u_{i,j} + \frac{\lambda}{2}\big[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\big)$



This can be put as a system.

Solve using any iterative method

Jacobi iteration

$$u^{(n)}_{i,j+1} = \frac{\lambda}{2(1+\lambda)}\Big[u^{(n-1)}_{i-1,j+1} + u^{(n-1)}_{i+1,j+1}\Big] + \frac{C_{i,j}}{(1+\lambda)}$$

from previous time level.

(n) → iteration

Gauss-Seidel

$$u^{(n+1)}_{i,j+1} = \frac{\lambda}{2(1+\lambda)}\Big[u^{(n+1)}_{i-1,j+1} + u^{(n+1)}_{i+1,j+1}\Big] + \frac{C_{i,j}}{(1+\lambda)}$$

Always use new component value as soon as they become available → r..dm Convergence !

updating of unknowns must be done successively; in contrast to Jacobi method [where unknowns can be updated in any order or even simultaneously]

Successive-over-Relaxation [SOR]

$$u_{i,j+1}^{(n+1)} = u_{(i,j)}^{(n)} + \omega\left\{\left[\frac{\lambda}{2(1+\lambda)}\left(u_{i-1,j+1}^{(n)} + u_{i+1,j+1}^{(n)}\right)\right] - u_{(i,j)}^{(n)} + \frac{c_{ij}}{(1+\lambda)}\right\}$$

ω — relaxation parameter.

$\omega > 1$ gives over-relaxation, $\omega < 1$ gives under-relaxation

Currently the methods we see were two level.
One can also have higher level method

Three level difference method

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$\frac{\partial u}{\partial t}$ — Central, $\frac{\partial^2 u}{\partial x^2}$ — Central

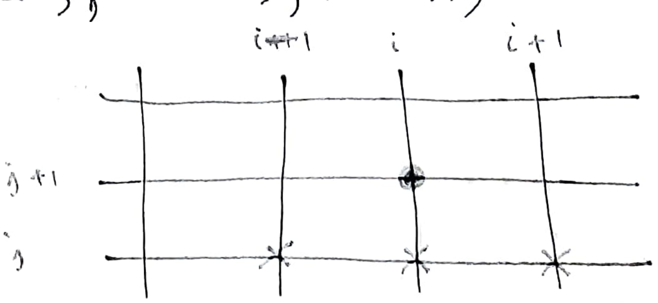
$$\boxed{\lambda = k/h^2}$$

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(k^2 + h^2)$$

$$u_{i,j+1} = \overbrace{u_{i,j-1}}^{\text{One more time level}} + 2\lambda\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

RM1

Richardson's method



DuFort - Frankel Method

$\lambda = k/h^2$

Using RM1

$$u_{i,j+1} = u_{i,j-1} + 2\lambda\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

Replace $u_{i,j} \simeq \frac{1}{2}\left[u_{i,j+1} + u_{i,j-1}\right]$ (Average)

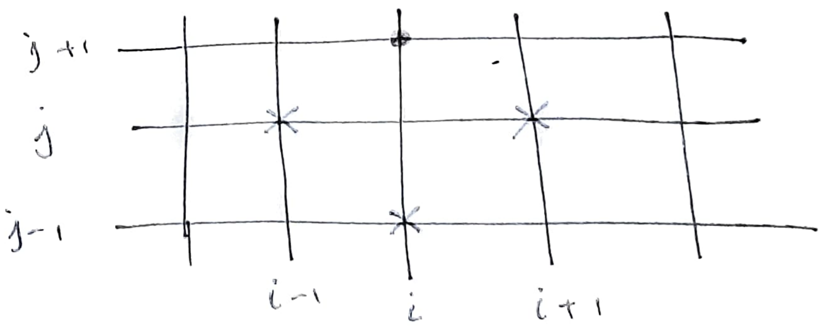
$$u_{i,j+1} = u_{i,j-1} + 2\lambda\left[u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + u_{i+1,j}\right)$$

Rearranging

$$u_{i,j+1} + 2\lambda\, u_{i,j+1} = (1-2\lambda)\, u_{i,j-1} + 2\lambda\left[u_{i-1,j} + u_{i+1,j}\right]$$

$$u_{i,j+1} = \frac{(1-2\lambda)}{(1+2\lambda)}\, u_{i,j-1} + \frac{2\lambda}{(1+2\lambda)}\left[u_{i-1,j} + u_{i+1,j}\right]$$

Explicit but demand 2 previous level.



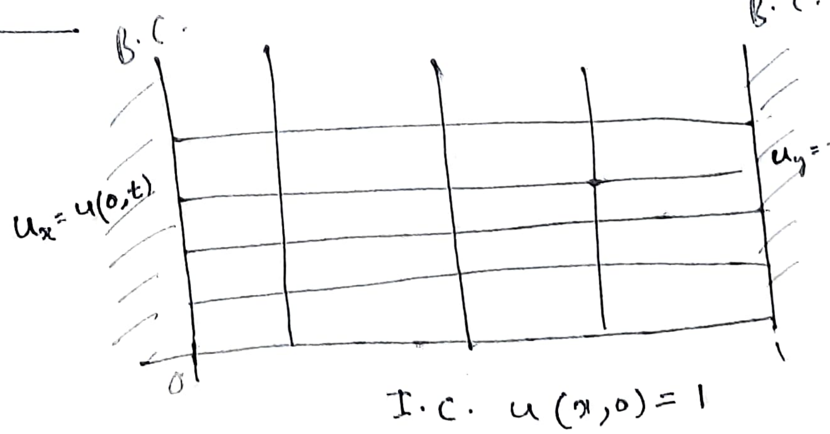
Derivative Boundary Condition

Let consider $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

I.C. $u(x,0) = 1$

B.C. $\dfrac{\partial u}{\partial x}(0,t) = u(0,t)$

$\dfrac{\partial u}{\partial x}(1,t) = -u(1,t)$



I.C. $u(x,0) = 1$

$0 \le x \le 1$ $h = 0.2$, $\lambda = 1/4$

Discretize with either explicit or implicit scheme.
Then discretize the boundary condition

Let consider explicit scheme

$$u_{i,j+1} = u_{i,j} + \lambda\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) \quad -(1)$$

I.C. $u(x,0) = 1 \Rightarrow \boxed{u_{i,0} = 1} \quad -(2)$

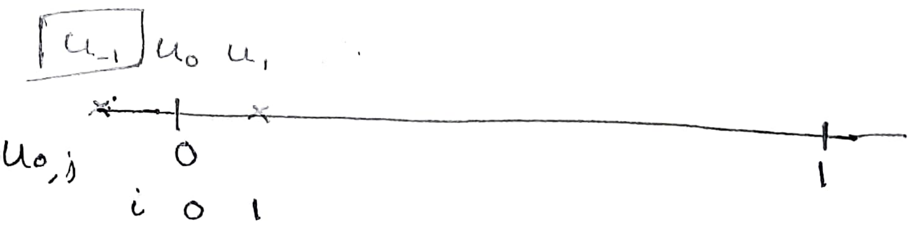
Now consider the B.C.

$$\frac{\partial u}{\partial x} = u \text{ at } x = 0 \quad \text{for all } t$$

One can use forward, backward or central.

$$\frac{u_{i+1,j} - u_{i-1,j}}{2h} = u_{i,j} \Rightarrow u_{i+1,j} - u_{i-1,j} = 2h\,u_{i,j}$$

at $x = 0$; $i = 0$



$\therefore\; u_{1,j} - u_{-1,j} = 2h\,u_{0,j}$

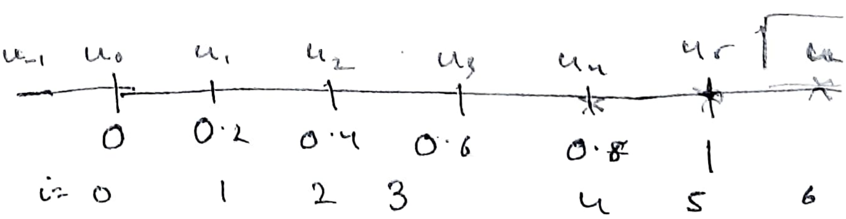
u_{-1} outside the domain $\rightarrow$ fictitious Value.

$$\boxed{u_{-1,j} = u_{1,j} - 2h\,u_{0,j}}$$

Let use $h = 0.2$
$k = 0.01$

Now at $x = 1$, $\frac{\partial u}{\partial x} = -u$ for all t

$$\frac{u_{i+1,j} - u_{i-1,j}}{2h} = -u_{i,j}$$



at $x=1$, $i = 5$

$$\frac{u_{6,j} - u_{4,j}}{2h} = -u_{5,j} \Rightarrow \boxed{u_{6,j} = u_{4,j} - 2h\,u_{5,j}}$$

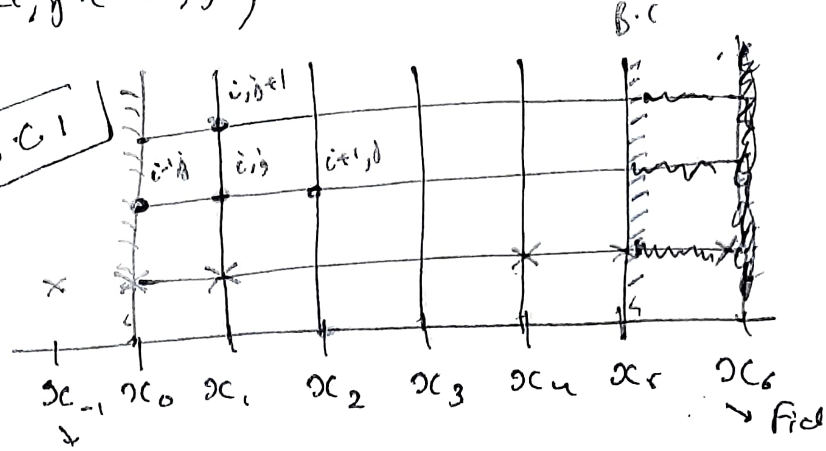
One gets two fictitious values at boundary condition

$u_{i,j+1} = u_{i,j} + \lambda(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$

B.C. $u_{-1,j} = u_{1,j} + 2h\,u_{0,j}$

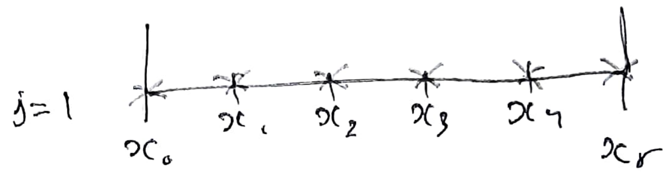
$u_{6,j} = u_{4,j} - 2h\,u_{5,j}$ $\boxed{B.C\,1}$

I.C. $u_{i,0} = 1$



Now we have four eqns and 6 unknown for one timestep as we don't know B.C. B. Value.

However if we try to get the value at B. Value



$i=0$ $u_{0,j+1} = u_{0,j} + \lambda(\underline{u_{-1,j}} - 2u_{0,j} + u_{1,j})$ we get fictitious values.

$i=5$ $u_{5,j+1} = u_{5,j} + \lambda(u_{4,j} - 2u_{5,j} + \underline{u_{6,j}})$

One can then plug B.C. eqn $\boxed{B.C.1}$ to eliminate fictitious values

$i=0$

$$u_{0,j+1} = u_{0,j} + \lambda(u_{1,j} + 2h\,u_{0,j} - 2u_{0,j} + u_{1,j})$$

$$= u_{0,j} + \lambda[u_{1,j} + 2(h-1)u_{0,j} + u_{1,j}]$$

$i=5$

$$u_{5,j+1} = u_{5,j} + \lambda[u_{4,j} - 2u_{5,j} + u_{4,j} - 2h\,u_{5,j})$$

$$= u_{5,j} + \lambda[u_{4,j} - \cancel{2u_{5,j}} - 2(1+h)u_{5,j} + u_{4,j})$$

Now we have 6 equations & 6 unknowns.

Compute the unknowns

Example: Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ using Crank-Nicolson Scheme.

I.C. $u(x,0) = 2x$

B.C. $u(0,t) = 0$

$u(20,t) = 40$

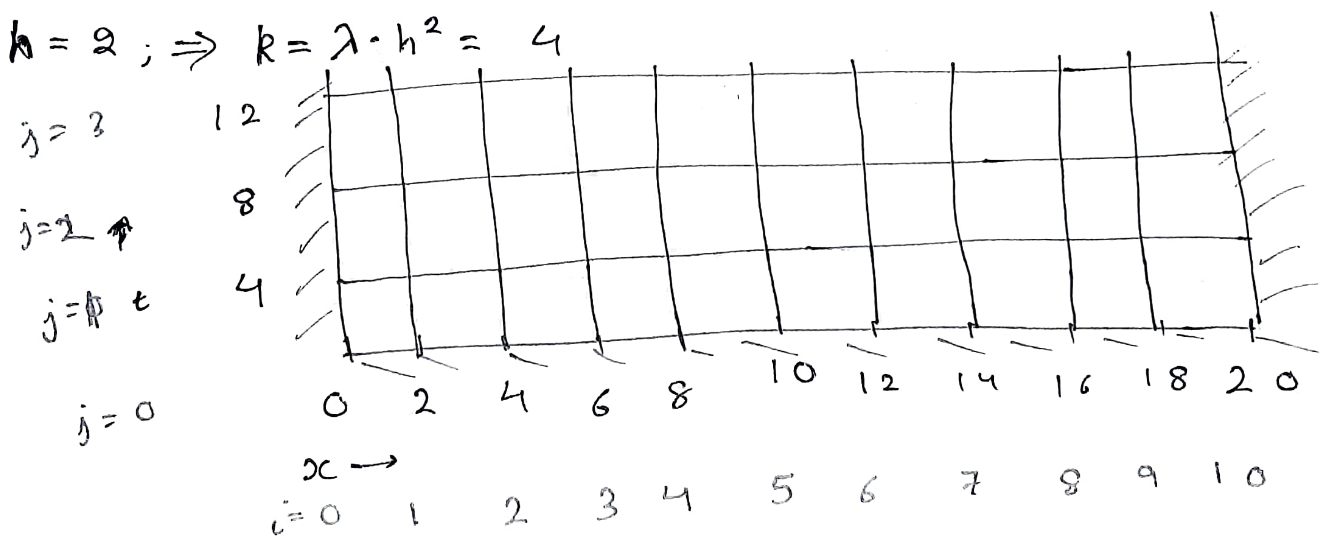
Crank Nicolson Scheme

$$-\lambda u_{i-1,j+1} + 2(1+\lambda)\, u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + 2(1-\lambda) u_{i,j} + \lambda u_{i+1,j}$$

Let $\lambda = 1$

$$-u_{i-1,j+1} + 4u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

Let $h = 2$; $\Rightarrow k = \lambda \cdot h^2 = 4$



I.C. give ~~u(~~

~~$u_{i,j}$~~ $u_{i,0} = 2x_i$

$u(0,0) = 0$; $u(1,0) = 4$; $u(2,0) = 8$; $u(4,0) = 16$; $u(6,0) = \cancel{12}\ 24$; $u(8,0) = \cancel{16}\ 32$;

$u(3,0) = \cancel{6}\ 12$; $u(5,0) = 20$; $u(7,0) = 28$; $u(9,0) = 36$; $u(10,0) = 40$

$u(0,1) = u(0,2) = u(0,3) = \ldots = 0$

$u(20,0) = u(20,1) = u(20,2) = u(20,3) = \ldots = 0$

$j = 0$

$$-u_{i-1,1} + 4u_{i,1} - u_{i+1,1} = u_{i-1,0} + u_{i+1,0}$$

$j=0$

$$-u_{i-1,1}+4u_{i,1}-u_{i+1,1}=u_{i-1,0}+u_{i+1,0}$$

$i=1$

$$\underset{B.C.}{-u_{0,1}}+4u_{1,1}-u_{2,1}=u_{0,0}+u_{2,0}$$

$i=2$

$$-u_{1,1}+4u_{2,1}-u_{3,1}=u_{1,0}+u_{3,0}$$

$i=3$

$$-u_{2,1}+4u_{3,1}-u_{4,1}=u_{2,0}+u_{4,0}$$

$i=4$

$$-u_{3,1}+4u_{4,1}-u_{5,1}=u_{3,0}+u_{4,0}$$

$i=5$

$$-u_{4,1}+4u_{5,1}-u_{6,1}=u_{4,0}+u_{6,0}$$

$i=6$

$$-u_{5,1}+4u_{6,1}-u_{7,1}=u_{5,0}+u_{7,0}$$

$i=7$

$$-u_{6,1}+4u_{7,1}-u_{8,1}=u_{6,0}+u_{8,0}$$

$i=8$

$$-u_{7,1}+4u_{8,1}-u_{9,1}=u_{7,0}+u_{9,0}$$

$i=9$

$$-u_{8,1}+4u_{9,1}-\underset{B.C.}{u_{10,1}}=\underset{I.C.}{u_{8,0}}+\underset{I.C.}{u_{10,0}}$$

$u_{1,1}$
$u_{2,1}$
$u_{3,1}$
$u_{4,1}$
$u_{5,1}$
$u_{6,1}$
$u_{7,1}$
$u_{8,1}$
$u_{9,1}$

9 unknown
9 equations

Tridiagonal system

$$\begin{pmatrix}
4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
-1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & \\
0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 &
\end{pmatrix}$$

$j=1$

$$-u_{i-1,2}+4u_{i,2}-u_{i+1,2}=u_{i-1,1}+u_{i+1,1}$$

$i=1$

$$-u_{0,2}+4u_{1,2}-u_{2,2}=u_{0,1}+u_{2,1}$$

$i=2$

$$-u_{1,2}+4u_{2,2}-u_{3,2}=u_{1,1}+u_{3,1}$$

$\vdots$

$$\begin{pmatrix}
4 & -1 & 0 & 0 & \\
-1 & 4 & -1 & 0 & 0 \\
0 & -1 & 4 & -1 & \\
0 & & & &
\end{pmatrix}$$

soon

sparse mt

mf is matrix [I use this notation] | L10 - 6 a

Sparse mf is useful. One can also save space → when storing only the non-zero entries

$$A = \begin{pmatrix} 0 & a & b & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & e & f & 0 & 0 \\ g & 0 & 0 & 0 & h \\ i & 0 & 0 & 0 & 0 \end{pmatrix}$$

3 vectors Z, c, r

Nonzero vector

$Z = [a, b, c, d, e, f, g, h, i]$

Row vector
how many non-zero element in a row

$r = [0, 2, 4, 6, 8, 9]$

row1 = $r[1] - r[0] = 2 - 0 = 2$
row2 = $r[2] - r[1] = 4 - 2 = 2$
row3 = $r[3] - r[2] = 6 - 4 = 2$

Column index

$c = [2, 3, 1, 2, 2, 3, 1, 5, 1]$

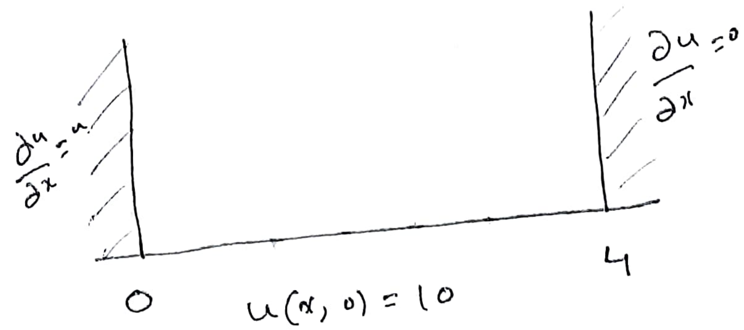
Efficient way of storing. Might be helpful when dealing with a very large mf [due to finer grid point]

Derivative Boundary condition.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad ; \quad u(x,0) = 10 \quad [\text{I.C.}]. \qquad 0 < x < 4$$

B.C. $\dfrac{\partial u}{\partial x}(0,t) = u(0,t)$

$\dfrac{\partial u}{\partial x}(4,t) = 0$



Explicit scheme — 1

$$u_{i,j+1} = u_{i,j} + \lambda\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

I.C. $u(x,0) = 10$

$u_{i,0} = 10$

B.C. $\frac{\partial u}{\partial x} = u$ at $x=0$

Put $\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$

$\Rightarrow \frac{u_{i+1,j} - u_{i-1,j}}{2h} = u_{i,j}$

$i=0$

$$u_{1,j} - \boxed{u_{-1,j}} = 2h\,u_{0,j}$$

$$\left.\frac{\partial u}{\partial x}\right|_{i=N} = 0 \Rightarrow \boxed{u_{n+1,j}} - u_{n-1,j} = 0$$

① → $i=0$

fictitious

$$u_{0,j+1} = u_{0,j} + \lambda(\underline{u_{-1,j}} - 2u_{0,j} + u_{1,j})$$

Use - $u_{-1,j} = u_{1,j} - 2h\,u_{0,j}$

$$\Rightarrow u_{0,j+1} = u_{0,j} + \lambda(u_{1,j} - 2h\,u_{0,j} - 2u_{0,j} + u_{1,j})$$

$$u_{0,j+1} = (1 - 2h\lambda - 2\lambda)u_{0,j} + 2\lambda\,u_{1,j}$$

$i=1$ ① become

$$u_{1,j+1} = u_{1,j} + \lambda(u_{0,j} - 2u_{1,j} + u_{2,j})$$

$$= \lambda u_{0,j} + (1-2\lambda)u_{1,j} + \lambda u_{2,j}$$

$i=2$

$$u_{2,j+1} = \lambda u_{1,j} + (1-2\lambda)u_{2,j} + \lambda u_{3,j}$$

fictitious

$i=N$ $u_{N,j+1} = u_{N,j} + \lambda(u_{N-1,j} - 2u_{N,j} + \underline{u_{N+1,j}})$

$$u_{N,j+1} = u_{N,j} + \lambda\left(u_{N+1,j} - 2u_{N,j} + u_{N-1,j}\right)$$
$$= \cancel{u_{N,j}}\ \cancel{\lambda}\ 2\lambda u_{N-1,j} + (1-2\lambda)u_{N,j}$$

N+1 equations having N+1 unknown.

Example
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} \qquad 0 \le x \le 1$$

I.C. $u(x,0) = 1 - x^2$ $\quad t > 0$

B.C. $\frac{\partial u}{\partial x} = 0$ at $x = 0$

$u = 0$ at $x = 1$

At $x = 0$, we have a problem.

$$\frac{2}{x}\frac{\partial u}{\partial x} = \frac{0}{0}; \quad \lim_{x\to 0} \frac{2}{x}\frac{\partial u}{\partial x} = \frac{0}{0}$$

One can get $\lim_{x\to 0} \frac{2}{x}\frac{\partial u}{\partial x} \to 2\frac{\partial^2 u}{\partial x^2}$

$\therefore$ at $x = 0$ $\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}$

~~where~~ So:
$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} \quad ; \text{ at } x = 0$$
$$= \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} \quad \text{at } x \neq 0$$

At $i = 0$
$$\underbrace{\frac{u_{0,j+1} - u_{0,j}}{k}}_{\text{forward time}} = \underbrace{\frac{3\left(u_{-1,j} - 2u_{0,j} + u_{1,j}\right)}{h^2}}_{\text{central space}} \quad (1)$$

As at $x=0$ $\frac{\partial u}{\partial x}=0$ [B.C.

$$\frac{u_{-1,j} - u_{1,j}}{h} = 0 \Rightarrow \boxed{u_{-1,j} = u_{1,j}} \text{ B.C.}$$

(1) becomes

$$u_{0,j+1} - u_{0,j} = 3(u_{-1,j} - 2u_{0,j} + u_{1,j})\frac{k}{h^2} \qquad \lambda = k/h^2$$

$$u_{0,j+1} - u_{0,j} = 3\lambda(2u_{1,j} - 2u_{0,j})$$

$$u_{0,j+1} = (1-6\lambda)u_{0,j} + 6\lambda u_{1,j}$$

at $x \neq 0$
$i \geq 1$

Let $i=1$ $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x}$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{2}{(ih)}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2h}\right)$$

x

$$u_{i,j+1} = \lambda\left(1-\frac{1}{i}\right)u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda\left(1+\frac{1}{i}\right)u_{i+1,j}$$

$$u_{i,0} = 1 - x_i^2$$

$i=1$

$$u_{1,j+1} = \lambda\left(1-\frac{1}{1}\right)u_{0,j} + (1-2\lambda)u_{1,j} + \lambda\left(1+\frac{1}{1}\right)u_{2,j}$$
$$= (1-2\lambda)u_{1,j} + 2\lambda u_{2,j}$$

$i=2$

$$u_{2,j+1} = \lambda\left(1-\frac{1}{2}\right)u_{1,j} + (1-2\lambda)u_{2,j} + \lambda\left(1+\frac{1}{2}\right)u_{3,j}$$

$$= \frac{\lambda}{2}u_{1,j} + (1-2\lambda)u_{2,j} + \frac{3\lambda}{2}u_{3,j}$$

L10 -(8)a

$i = p$;

$$u_{p,j+1} = \lambda\left(1-\frac{1}{p}\right)u_{p-1,j} + (1-2\lambda)u_{p,j} + \lambda\left(1+\frac{1}{p}\right)u_{p+1,j}$$

As $i = n-1$

$$u_{n,j+1} = \lambda\left(1-\frac{1}{n-1}\right)u_{n-2,j} + (1-2\lambda)u_{n-1,j} + \lambda\left(1+\frac{1}{n-1}\right)\underbrace{u_{n,j}}_{0}$$

$u = 0$ at $x = 1$ (B.C. $u_{n,j} = 0$

(n)

$$\begin{pmatrix} (1-6\lambda) & 6\lambda & 0 & 0 & & & \\ 0 & (1-2\lambda) & 2\lambda & 0 & & & \\ 0 & \frac{\lambda}{2} & (1-2\lambda) & \frac{3\lambda}{2} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \lambda\left(1-\frac{1}{p}\right) & (1-2\lambda) & \lambda\left(1+\frac{1}{p}\right) & \\ & & & & \ddots & \ddots & \\ & & & & & \lambda\left(1-\frac{1}{n-1}\right) & (1-2\lambda) \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix}$$

$$\bar{u}_{j+1} = \bar{A}\,\bar{u}_j$$