Lecture 3

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Euler method and fourth-order RK methods are called single-step methods, where to calculate y_{i+1} one only requires the knowledge of the y_i.

While Modified Euler method is a multi-step method since for computation of y_{i+1}, the knowledge of y_i is not enough.

→It is a predictor-corrector method, in which a predictor formula is used to predict the value y_{i+1} of y at x_{i+1} and then a corrector formula is used to improve the value of y_{i+1}.

Adams-Moulton Method

Also known as Adams-Bashorth method or Adams-Bashforth-Moulton method.

Also a predictor-corrector method.

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0$$

$$y_1 = y_0 + \int_{x_0}^{x_0+h} f(x, y)dx \qquad y_1 = y(x_0 + h)$$

Replacing *f(x,y)* by Newton's Backward Interpolation Formula

$$f(x_i + uh) = f(x_i) + u\nabla f(x_i) + \frac{u(u+1)}{2!}\nabla^2 f(x_i) + \frac{u(u+1)(u+2}{3!}\nabla^3 f(x_i) + \dots + \frac{u(u+1)(u+2)\dots(u+n-1)}{n!}\nabla^n f(x_i)$$

$$\nabla f(x_i) = f(x_i) - f(x_{i-1})$$

$$\nabla^2 f(x_i) = \nabla f(x_i) - \nabla f(x_{i-1})$$

$$\nabla^3 f(x_i) = \nabla^2 f(x_i) - \nabla^2 f(x_{i-1})$$

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
$(= x_0 + h)$		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
$(= x_0 + 2h)$		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
$(= x_0 + 3h)$		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
$(= x_0 + 4h)$		∇y_5				
x_5	y_5					
$(= x_0 + 5h)$						

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0 \qquad y_1 = y_0 + \int_{x_0}^{x_0+h} f(x,y)dx \qquad y_1 = y(x_0+h)$$

$$y_1 = y_0 + h\int_0^1 \left\{ f_0 + u\nabla f_0 + \frac{u(u+1)}{2}\nabla^2 f_0 + \frac{u(u+1)(u+2)}{6}\nabla^3 f_0 + \ldots \right\} du$$

$x = x_0 + hu \rightarrow dx = h\,du$ Limit of *u* are from 0 to 1

$$y_1 = y_0 + h\left(f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + \ldots \right)$$

Neglecting the fourth order and higher order differences

$$\nabla f_0 = f_0 - f_{-1}$$
$$\nabla^2 f_0 = f_0 - 2f_{-1} + f_{-2}$$
$$\nabla^3 f_0 = f_0 - 3f_{-1} + 3f_{-2} - f_{-3}$$

$x_i = x_0 + h$
$x_{-1} = x_0 - h$
$x_{-2} = x_0 - 2h$
$x_{-3} = x_0 – 3h$

$f_0 = f(x_0,y_0)$
$f_1 = f(x_1,y_1)$
$f_{-1} = f(x_{-1},y_{-1})$
$f_{-2} = f(x_{-2},y_{-2})$
$f_{-3} = f(x_{-3},y_{-3})$

One get the following as simplified

$$y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

This is known as Adams-Bashforth or Adams-Moulton-Predictor formula

$$y_{n+1} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

$$y_{n+1}^{P} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

Having found y_1, we find $f_1 = f(x_0+h, y_1)$

To find a better value of y_1, one derive a corrector formula by substituting Newton's backward interpolation formula at f_1 at place of $f(x.y)$

$$y_1 = y_0 + h\int_{-1}^{0}\left\{f_1 + u\nabla f_1 + \frac{u(u+1)}{2}\nabla^2 f_1 + \frac{u(u+1)(u+2)}{6}\nabla^3 f_1 + ...\right\}du$$

$$x = x_0 + hu \rightarrow dx = h\,du$$

$$y_1 = y_0 + h\left(f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_0 - \frac{1}{24}\nabla^3 f_0 + ...\right)$$

$$\nabla f_1 = f_1 - f_0$$

$$\nabla^2 f_1 = f_1 - 2f_0 + f_{-1}$$

$$\nabla^3 f_1 = f_1 - 3f_0 + 3f_{-1} - f_{-2}$$

One get the following as simplified

$$y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$y_{n+1}^{C} = y_n + \frac{h}{24}(9f_{n+1} - 19f_n - 5f_{n-1} + f_{n-2})$$

$$y_{n+1}^{P} = y_n + \frac{h}{24}(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3})$$

$$y_{n+1}^{C} = y_n + \frac{h}{24}(9f_{n+1} - 19f_n - 5f_{n-1} + f_{n-2})$$

$$y_{n+1}^{P} = y_n + \frac{h}{24}(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3})$$

$$y_{n+1}^{C} = y_n + \frac{h}{24}(9y'_{n+1} - 19y'_n - 5y'_{n-1} + y'_{n-2})$$

Known as Fourth-order Adams Method

Milne’s Method

Based on same. First one gets value from predictor and then correct.

As in Adams-Basforth Method, here also one get the formula after simplifying the Newton’s Forward Interpolation Formula

$$y_1 = y_0 + h \int \left\{ f_0 + u\Delta f_0 + \frac{u(u+1)}{2}\Delta^2 f_0 + \frac{u(u+1)(u+2)}{6}\Delta^3 f_0 + ... \right\} du$$

Simplifying, one will get

Predictor

$$y_n^P = y_{n-4} + \frac{4h}{3}(2y'_{n-3} - y'_{n-2} + 2y'_{n-1})$$

Corrector

$$y_n^C = y_{n-2} + \frac{h}{3}(2y'_{n-2} + 4y'_{n-1} + y'^P_n)$$

Forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
$(= x_0 + h)$		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
$(= x_0 + 2h)$		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
$= (x_0 + 3h)$		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
$= (x_0 + 4h)$		Δy_4				
x_5	y_5					
$= (x_0 + 5h)$						

Systems of equations

Most of the practical problems require the solution of system of simultaneous ODEs rather than a single equation

Such system represented by

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, ..., y_n)$$

$$\frac{dy_2}{dx} = f_1(x, y_1, y_2, ..., y_n)$$

$$\frac{dy_3}{dx} = f_1(x, y_1, y_2, ..., y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_1(x, y_1, y_2, ..., y_n)$$

Solution of such a system requires that n initial conditions be known at the starting value of *x*

Modifying the code of single ODE for system of equations

Take Euler for simplicity

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}), \text{ where } x_n = x_{n-1} + h$$

$$\frac{dy}{dx} = f_1(x, y, z, t) \qquad \frac{dz}{dx} = f_1(x, y, z, t) \qquad \frac{dt}{dx} = f_1(x, y, z, t)$$

Define N Number of equations .
Define the N Initial dependent variables
Compute the slope at each of the dependent variable

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), \text{ where } x_n = x_{n-1} + h$$

$$z_n = z_{n-1} + hf(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), \text{ where } x_n = x_{n-1} + h$$

$$t_n = t_{n-1} + hf(x_{n-1}, y_{n-1}, z_{n-1}, t_{n-1}), \text{ where } x_n = x_{n-1} + h$$