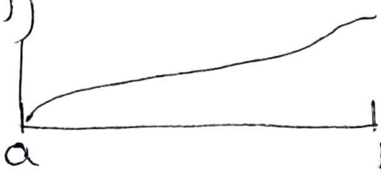
Boundary Value Problem

$$\frac{dy}{dx} = f(x, y) \quad , y(x_0) = y_0 \longrightarrow IVP$$

1^{st} order need one initial condition

Let us have $\frac{d^2y}{dx^2} = f(x, y(x), y'(x))$

and we have $y(a) = r_1$
$y(b) = r_2$



$y(x)$ is unknown function. The values of y at the boundary $x = a$, $x = b$ are given

This is called two point Boundary Value Problem (BVP)

Aim is to solve 2 point BVP

Two methods one can utilize to solve

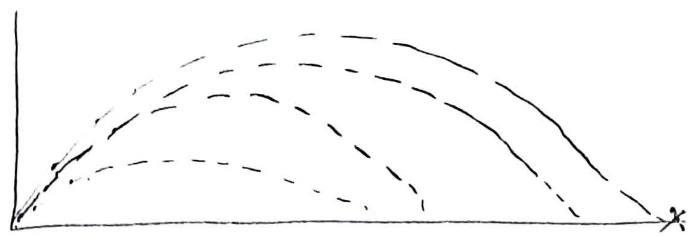
→ Shooting Method

→ Finite Difference Method.

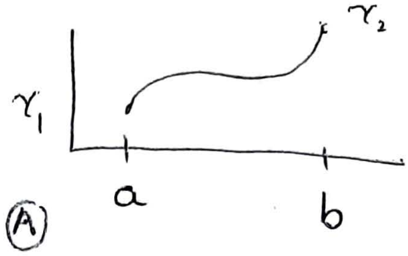
Shooting Method

One can guess the missing values, This will result in some solution, which may or may not satisfy boundary condition at the end. One can then inspect the discrepancy and estimate the needed change to make the in the guess to get proper solution satisfying the boundary condition at the end

It is more like shooting at a target



We have



$$\left.\begin{aligned} &y'' + p(x)y' + q(x)y = r(x) \\ &y(a) = r_1 \quad , \; y(b) = r_2 \end{aligned}\right) - (A)$$

$a < x < b$

We know how to solve IVP

If we convert (A) into the IVP $=$ ns.

One can supply an initial condition on $y'(a)$

Then we have

$$\boxed{\begin{aligned} &y'' + p(x)y' + q(x)y = r(x) \\ &y(a) = r_1 \\ &y'(a) = \cancel{\beta}\,\alpha \end{aligned}} - (B)$$

Let α be some value

(B) is the IVP $= n$

Using them one can easily get solution at $y(b)$

Based on slope one gets the solution.

Let if we use

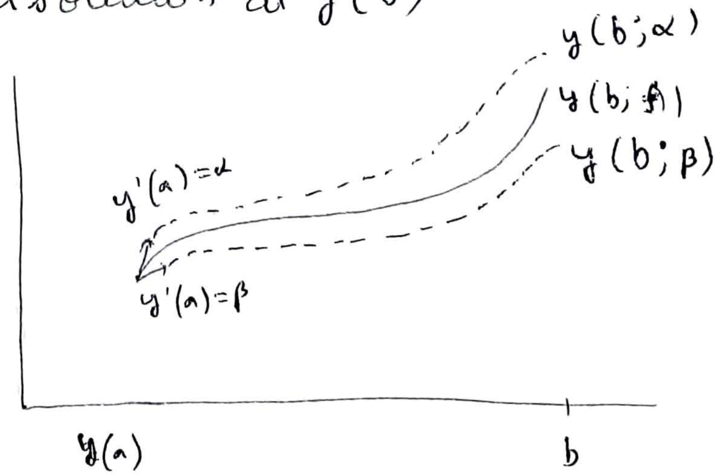
$y'(a) = \alpha$

we end up with $y(b;\alpha)$

If we use

$y'(b) = \beta$

end up with $\cancel{y(\beta}$ $y(b;\beta)$



One can simply adjust the slope such that we end up with suitable solution at $y(b)$

Linear Shooting

General form is written as

$$\frac{d^2y}{dx^2} = p(x)\frac{dy(x)}{dx} + q(x)y(x) + r(x) \qquad (1)$$

$$y(a) = \gamma_1 \quad , y(b) = \gamma_2$$

Lets construct IVP.

Assume $\bar{y}$ be the soln.

$$\bar{y}''(x) = p(x)\,\bar{y}'(x) + q(x)\bar{y}(x) + r(x) \qquad (2)$$

$$\bar{y}(a) = \gamma_1 \quad , \quad \bar{y}'(a) = \alpha$$

Assume $\tilde{y}$ be the soln.

$$\tilde{y}''(x) = p(x)\,\tilde{y}'(x) + q(x)\,\tilde{y}(x) + r(x) \qquad (3)$$

$$\tilde{y}(a) = \cancel{\gamma_2}\ \boxed{\gamma_1} \quad , \quad \tilde{y}'(a) = \beta$$

Solve (2) and (3) once obtain $(\bar{y}(x)\ \tilde{y}(x)$ on $a \leq x \leq b$

Now let

$$y(x) = \lambda\,\bar{y}(x) + (1-\lambda)\,\tilde{y}(x) \qquad — (4)$$

λ is a constant to be determined

One can write $y''(x)$ based on (4)

$$y''(x) = \lambda\,\bar{y}''(x) + (1-\lambda)\,\tilde{y}''(x)$$

Use (2) + (3)

$$y''(x) = \lambda\left(p(x)\bar{y}'(x) + q(x)\bar{y}(x) + r(x)\right) + (1-\lambda)\left(p(x)\tilde{y}'(x) + q(x)\tilde{y}(x) + r(x)\right)$$

Let write $p(x) \to p$. Same for y, q, r.

$$y'' = p(\lambda\bar{y}' + (1-\lambda)\tilde{y}') + q(\lambda\bar{y} + (1-\lambda\tilde{y}) + r(\lambda + 1 - \lambda)$$

$$= py' + qy + r$$

Suggest y always solve $= n$, no matter what λ once chooses [hold for all λ]

Lets check boundary condition

$\Rightarrow$ $y(a) = \lambda\bar{y}(a) + (1-\lambda)\tilde{y}(a)$

By construct $\tilde{y}(a) = \bar{y}(a) = \gamma_1$

$$y(a) = \lambda\gamma_1 + (1-\lambda)\gamma_1 = \gamma_1$$

$\Rightarrow$ $y(b) = \lambda\bar{y}(b) + (1-\lambda)\tilde{y}(b)$

$\bar{y}(b)$ + $\tilde{y}(b)$ depend upon the choice of

$\bar{y}'(a) = \alpha$ or $\tilde{y}'(a) = \beta$

As $y(b) = \gamma_2$

$$\lambda\bar{y}(b) + (1-\lambda)\tilde{y}(b) = \gamma_2$$

$$\lambda[\bar{y}(b) - \tilde{y}(b)] = \gamma_2 - \tilde{y}(b)$$

$$\boxed{\lambda = \frac{\gamma_2 - \tilde{y}(b)}{\bar{y}(b) - \tilde{y}(b)}} \quad \text{— } (5)$$

Superposition

$$y = \lambda \cdot \bar{y} + (1-\lambda)\tilde{y}$$

y given in (4) with λ in (5) is the solution of (1).

Based on $\bar{y}$ & $\tilde{y}$ & λ (linear combination).

Only condition $\lambda + (1-\lambda)$ should add to be 1

$\lambda \rightarrow$ fraction / weight

One can also solve for $\bar{y}$ & $\tilde{y}$ as initial value problem.

$u_1 = \bar{y}$ $\qquad$ $u_3 = \tilde{y}$

$u_2 = \bar{y}'$ $\qquad$ $u_4 = \tilde{y}'$

$u_1' = \bar{y}' = u_2$

$u_2' = \bar{y}'' = p u_2 + q u_1 + r$ [from (2)]

$u_3' = \tilde{y}' = u_4$

$u_4' = \tilde{y}'' = p u_4 + q u_3 + r$ [from (3)]

I.C. $\quad u_1(a) = \gamma_1$

$u_2(a) = \alpha$

$u_3(a) = \gamma_1$

$u_4(a) = \beta$

This becomes system of first order ODE, which could be solved easily!

Thus is for linear problem.

What about the non-linear?

General non linear second order B-value ODE can be written as:

$$y'' = p(x, y)\, y' + q(x, y)\, y + r(x)$$

$$y(a) = \gamma_1 \quad \text{and} \quad y(b) = \gamma_2$$

where the coefficient $p(x, y)$ and $q(x, y)$ may be linear or non linear function of y

When solved by the shooting method, based on method for solving initial value ODEs, the non linear terms pose no special problems.

~~Example~~ <u>Iteration Method</u>

$$y'' + p y' + q y = r \qquad a < x < b$$

$$y(a) = \gamma_1, \quad y(b) = \gamma_2$$

Let

$$y' = z \qquad y(a) = \gamma_1$$

$$z' = r - pz - qy \qquad z(a) = y'(a) = ?$$

Let assume $z(a) = \alpha$ [which is $y'(a)$] then we have ~~BVP~~ converted this eqn into IVP.

$$BVP \longrightarrow IVP$$

Solved by <u>RK, Taylor, Euler</u>

However we don't know α <u>!</u>

We need two initial approximation α_0, α_1, 2 different slopes

2 initial slopes, α_0, α_1 define two IVPs

IVP (a)
$$y'' + py' + qy = r$$
$$y(a) = \gamma_1 \quad ; \quad y'(a) = \alpha_0$$

IVP (b)
$$y'' + py' + qy = r$$
$$y(a) = \gamma_1 \quad ; \quad y'(a) = \alpha_1$$

Solve IVP (a) and IVP (b) to get

$y(b; \alpha_0)$ $\qquad$ $y(b; \alpha_1)$

Then
$$\phi(\alpha_0) = y(b; \alpha_0) - \gamma_2 \quad \text{—} \quad (6)$$
$$\phi(\alpha_1) = y(b; \alpha_1) - \gamma_2 \quad \text{—} \quad (7)$$

One check how far (6) and (7) are.
And estimate α_2 using secant method

$$\alpha_2 = \alpha_1 - \frac{(\alpha_1 - \alpha_0)}{(\phi(\alpha_1) - \phi(\alpha_0))} \cdot \phi(\alpha_1)$$

Then solve IVP (c)
$$y'' + py' + qy = r$$
$$y(a) = \gamma_1 \quad ; \quad y'(a) = \alpha_2$$

Then get $\phi(\alpha_2) = y(b; \alpha_2) - \gamma_2 \quad - \quad (8)$

n	0	1	2	3	$\ldots$	n
x	$y(a)$ γ_1					$y(b) \to \gamma_2$
assume $y'(a)=\alpha$	γ_1	$y(x_1;\alpha)$	$y(x_2;\alpha)$	$\ldots$		$y(b;\alpha)$
$y'(a)=\beta$	γ_1	$y(x_1;\beta)$	$y(x_2;\beta)$	$\ldots$		$y(b;\beta)$

Our Target is $y(b)$ to be γ_2

Hence compare $y(b;\alpha)-\gamma_2$

if $|y(b;\alpha)-\gamma_2| < \epsilon$

ϵ is the pre assigned accuracy needed.

We can solve the BVP

So our aim is $\boxed{y(b;\alpha)-\gamma_2}$

Looking for α such that $y(b;\alpha)-\gamma_2 \simeq 0$

One can also see it as; we are looking for a solution of $\phi(\alpha) = y(b;\alpha)-\gamma_2 = 0$.

Treat it as non-linear eqn. Several methods to get the root of this eqn

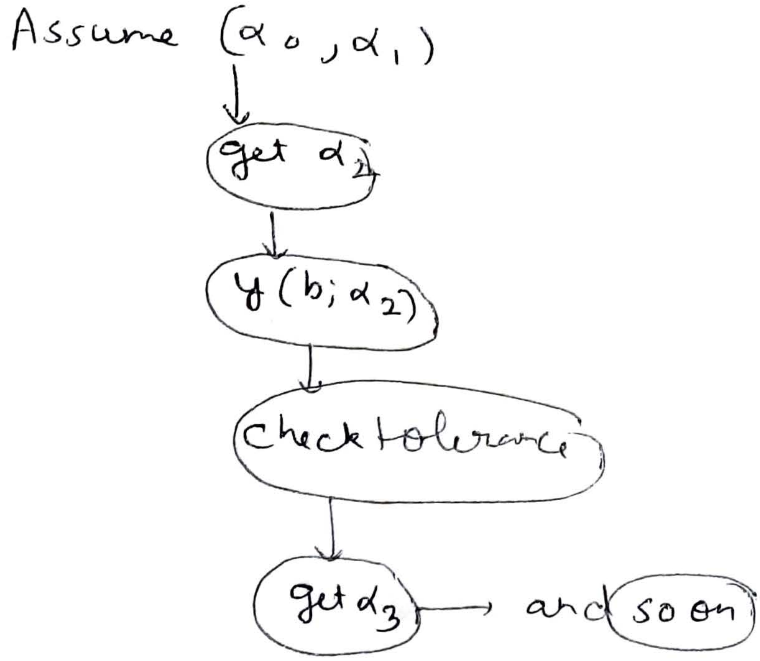
a) Secant method

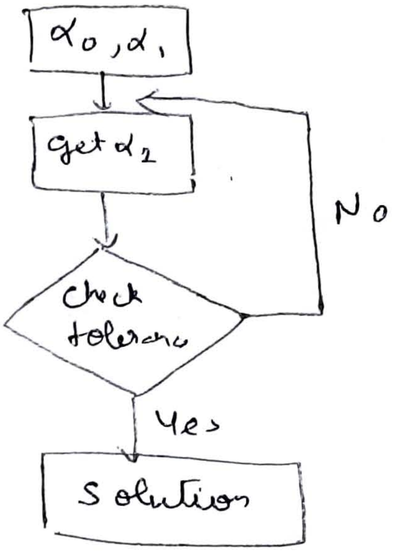
b) Newton Raphson method

Secant method

$$\alpha_{n+1} = \alpha_n - \frac{(\alpha_n - \alpha_{n-1})}{[\phi(\alpha_n) - \phi(\alpha_{n-1})]} \cdot \phi(\alpha_n)$$

One keep doing so



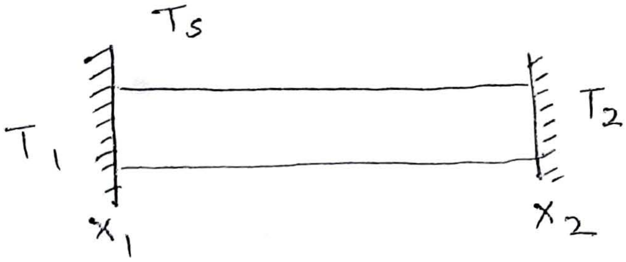


Exampl

Steady 1-dimensional heat transfer problem

T_1 & T_2 different temperature

T_s → surrounding temperature



$$\frac{d^2T}{dx^2} - \alpha^2 T = -\alpha^2 T_s$$

$T(x_1) = T_1$ $\quad$ $T(x_2) = T_2$

$T(0.0) = 0.0°C$ $\quad$ $T(1.0) = 100.0°C$

Let $\quad \alpha = 4.0\, cm^{-1}$ $\quad$ $T_s = 0°C$

$T_i' = \quad u_1 = T \qquad T(0) = 0.03 \Rightarrow u_1(0) = 0.03$

$u_2 = T' \qquad T'(0) = ? \quad u_2(0) = ?$

$$u_1' = u_2$$

$$u_2' = \alpha^2 (T - 0)$$

$$= 16.0\, u_1$$

$$\frac{du_1}{dx} = u_2$$

$$\frac{du_2}{dx} = 16.0\, u_1$$

One can ~~calculate~~ simply assume two choice

$T'(0) = 7.5$ and $T'(0) = 12.5$

If one use RK method one get

$u_1'(0) = 7.5$		$u_1'(0) = 12.5$		New
x	T	x	T	
0.25	2.13	0.25	3.550	2.388
0.5	4.26	0.5	7.100	4.77
0.75	~~26.55~~	0.75	44.26	29.77
1	89.19	1	148.644	100

Should be 100 as $T(1) = 100$

superposition

Now Use (4) $\quad y(x) = \lambda \cdot \tilde{y}(x) + (1-\lambda)\, \tilde{y}(x)$

$$\lambda = \frac{Y_2 - \tilde{y}(b)}{\tilde{y}(b) - \tilde{y}(b)} = \frac{100 - 148.64}{89.19 - 148.64}$$

$$\sim 0.818$$

α_1	α_2
$u_1'(0) = 7.5$	$u_1'(0) = 12.5$
$u_1(1) = 89.19$	$u_2(1) = 148.64$

New α

$$\alpha_3 = \alpha_2 - \frac{(\alpha_2 - \alpha_1)}{[\phi(\alpha_2) - \phi(\alpha_1)]} \cdot \phi(\alpha_2)$$

$$\phi(\alpha_n) = y(b; \alpha) - \gamma_2$$

$$\alpha_3 = 8.409$$

x	α_3: $u_1'(0) = 8.409$	α_4: $u_1'(0) = 8.40937$	α_5: $u_1'(0) = 8.409375$
.25	2.388	2.3884	2.38841
.5	4.776	4.7768	4.77681
.75	29.77	29.776	29.7766
1	99.995	99.9999	100

After few iterations T(b) goes to the expected value!

Shooting Method. One can also use Newton Raphson method. I will not go into the method.