Local truncation Error

Consider $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Let say $Lu = 0$

Approximated by $L_{i,j} u = 0$

Let $\bar{u}$ be the exact solution.

$L_{ij}\bar{u} \simeq 0$

$\Rightarrow \underbrace{\boxed{L\bar{u}}}_{=0} - L_{i,j}\bar{u} \simeq 0 = \boxed{T_{i,j}}$ ← local truncation error

$$L_{i,j} u = \frac{u_{i,j+1} - u_{i,j}}{k} - \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \cong 0$$
$$+ O(k) \qquad + O(h^2)$$

$\boxed{LT}$

One get $T_{i,j} = \frac{\bar{u}_{i,j+1} - \bar{u}_{i,j}}{k} - \frac{\bar{u}_{i-1,j} - 2\bar{u}_{i,j} + \bar{u}_{i+1,j}}{h^2}$

Taylor's expansion

$$\bar{u}_{i+1,j} = \bar{u}(x_{i+1}, t_j) = \bar{u}(x_i + h, t_j) \qquad \text{(A)}$$
$$= \bar{u}(x_i,t_j) + h\frac{\partial \bar{u}}{\partial x_i}\Big|_{(x_i,t_j)} + \frac{h^2}{2!}\frac{\partial^2 \bar{u}}{\partial x_i^2}\Big|_{(x_i,t_j)} + \frac{h^3}{6}\frac{\partial^3 \bar{u}}{\partial x_i^3}\Big|_{(x_i,t_j)} + \cdots$$

$$\bar{u}_{i-1,j} = \bar{u}(x_i - h, t_j) \qquad \text{(B)}$$
$$= \bar{u}(x_i,t_j) - h\frac{\partial \bar{u}}{\partial x_i}\Big|_{(x_i,t_j)} + \frac{h^2}{2!}\frac{\partial^2 \bar{u}}{\partial x_i^2}\Big|_{(x_i,t_j)} - \frac{h^3}{6}\frac{\partial^3 \bar{u}}{\partial x_i^3}\Big|_{(x_i,t_j)} + \cdots$$

$$\bar{u}_{i,j+1} = \bar{u}(x_i, t_j + k) \qquad \text{(C)}$$
$$= \bar{u}(x_i,t_j) + k\frac{\partial \bar{u}}{\partial t}\Big| + \frac{k^2}{2}\frac{\partial^2 \bar{u}}{\partial t^2}\Big| + \frac{k^3}{6}\frac{\partial^3 \bar{u}}{\partial t^3}\Big|_{(x_i,t_j)}$$

Put Ⓐ, Ⓑ, Ⓒ in (LT)

$$T_{ij} = \frac{1}{k}\left[\bar{u} + k\frac{\partial \bar{u}}{\partial t} + \frac{k^2}{2}\frac{\partial^2 \bar{u}}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 \bar{u}}{\partial t^3} + \cdots - \bar{u}\right]_{(x_i, t_j)}$$

$$-\frac{1}{h^2}\left[\bar{u} - h\frac{\partial \bar{u}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 \bar{u}}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 \bar{u}}{\partial x^4} - \cdots\right.$$

$$\left. - 2\bar{u} + \bar{u} + h\frac{\partial \bar{u}}{\partial x} + \frac{h^2}{2}\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 \bar{u}}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 \bar{u}}{\partial x^4} + \cdots\right]_{(x_i,}$$

Cancels

one left with

$$T_{ij} = \frac{1}{k}\left[k\frac{\partial \bar{u}}{\partial t} + \frac{k^2}{2}\frac{\partial^2 \bar{u}}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 \bar{u}}{\partial t^3} + \cdots\right]$$

$$-\frac{1}{h^2}\left[2\frac{h^2}{2}\frac{\partial \bar{u}}{\partial x^2} + 2\frac{h^4}{24}\frac{\partial^4 \bar{u}}{\partial x^4} + \cdots\right]$$

$$T_{ij} = \boxed{\left(\frac{\partial \bar{u}}{\partial t} - \frac{\partial^2 \bar{u}}{\partial x^2}\right)_{(x_i, t_j)}} + \left(\frac{k}{2}\frac{\partial^2 \bar{u}}{\partial t^2} - \frac{h^2}{12}\frac{\partial^4 \bar{u}}{\partial x^4}\right)_{(x_i, t_j)}$$

$$+ \left(\frac{k^2}{6}\frac{\partial^3 \bar{u}}{\partial t^3} - \frac{h^4}{360}\frac{\partial^6 \bar{u}}{\partial x^6}\right) + \cdots$$

$\bar{u}$ is exact soln. $\therefore \dfrac{\partial \bar{u}}{\partial t} - \dfrac{\partial^2 \bar{u}}{\partial x^2} = 0$

$$\therefore T_{i,j} = \underbrace{\left(\frac{k}{2}\frac{\partial^2 \bar{u}}{\partial t^2} - \frac{h^2}{12}\frac{\partial^4 \bar{u}}{\partial x^4}\right)_{(x_i, t_j)}} + \cdots$$

Leading non-zero term: Principal part of the local truncation error

$$T_{ij} \simeq k \boxed{\frac{1}{2}\frac{\partial^2 \bar{u}}{\partial t^2}} + h^2 \boxed{\left(-\frac{1}{12}\right)\frac{\partial^2 \bar{u}}{\partial x^4}} + \ldots$$

$$\qquad\quad c_1 \qquad\qquad\qquad c_2$$

Leading term becomes $c_1 k + c_2 h^2$

which is $O(k + h^2)$.

Can one reduce this error or minimize it further.

which means $c_1 k + c_2 h^2 \rightarrow$ to zero.

$$T_{ij} = k \frac{1}{2}\frac{\partial^2 \bar{u}}{\partial t^2} - h^2 \frac{1}{12}\frac{\partial^4 \bar{u}}{\partial x^4} + O(k^2 + h^4)$$

Consider $\dfrac{\partial}{\partial t} = \dfrac{\partial^2}{\partial x^2}$

$$\frac{\partial}{\partial t}\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2}{\partial x^2}\frac{\partial \bar{u}}{\partial t} = \frac{\partial^2}{\partial x^2}\frac{\partial^2 \bar{u}}{\partial x^2}$$

$$\frac{\partial^2 \bar{u}}{\partial x^4}$$

$$\therefore\ T_{ij} = \underbrace{\left(\frac{k}{2} - \frac{h^2}{12}\right)}_{\text{Force this to be zero}} \frac{\partial^4 \bar{u}}{\partial x^4} + O(k^2 + h^4)$$

$$\frac{k}{\cancel{2}} - \frac{h^2}{\cancel{12}\,6} = 0 \Rightarrow k = \frac{h^2}{6}$$

One can say that choosing $k = h^2/6$. one can minimize the error to the order of $O(k^2 + h^4)$.

Choice of step size $\longrightarrow$ error is minimized.

Connergence

Finite Difference scheme approximating a PDE is a connergent scheme if the solution of the finite difference scheme $u_{i,j}$ connerges to the $\bar{u}(x,t)$ (exact soln) of the PDE as $\Delta x, \Delta t \to 0$.

Stability

Errors caused by a small perturbation in the numerial method remains bounded.

Might happen unconditionally in the entire domain or conditionally within a range.

If given a small peturbation → grows up and blows then the system is not stable.

Consistency

Given a PDE $Lu = f$ approximated by $L_{i,j}\,u = f$

Then finite difference scheme $L_{i,j}\,u = f$ is consistent with the PDE if $L\phi - L_{i,j}\,\phi \to 0$ as $\Delta x, \Delta t \to 0$ for ϕ smooth enough.

A scheme approximating a PDE may be stable but has a solution that connerges to the solution of a different PDE (equation) as the mesh lengths goes to zero. ["in-consistent"]

Example $\quad L = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}$ $\quad$ First order example

$$L\phi = \frac{\partial \phi}{\partial t} + a\frac{\partial \phi}{\partial x}$$

$L_{i,j}$:- forward space and forward time

$$L_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j}}{\Delta t} + a\,\frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} \quad \text{--- (1)}$$

Expand in Taylor series

$$\phi_{i,j+1} = \phi_{i,j} + \frac{\Delta t}{1}\frac{\partial \phi}{\partial t}\Big|_{(i,j)} + \frac{(\Delta t)^2}{2}\frac{\partial^2 \phi}{\partial t^2}\Big|_{(i,j)} + O(\Delta t^3)$$

$$\phi_{i+1,j} = \phi_{i,j} + \Delta x\,\frac{\partial \phi}{\partial x}\Big|_{(i,j)} + \frac{(\Delta x)^2}{2}\frac{\partial^2 \phi}{\partial x^2}\Big|_{(i,j)} + O(\Delta x^3)$$

Substitute them in (1)

$$L_{i,j} = \frac{\partial \phi}{\partial t}\Big|_{(i,j)} + \frac{\Delta t}{2}\frac{\partial^2 \phi}{\partial t^2}\Big|_{(i,j)} + a\left(\frac{\partial \phi}{\partial x}\Big|_{(i,j)} + \frac{\Delta x}{2}\frac{\partial^2 \phi}{\partial x^2}\Big|_{(i,j)}\right) + \cdots$$

$$= \frac{\partial \phi}{\partial t}\Big|_{i,j} + a\frac{\partial \phi}{\partial x}\Big|_{i,j} + \frac{1}{2}\left[\Delta t\frac{\partial^2 \phi}{\partial t^2}\Big|_{i,j} + a\,\Delta x\frac{\partial^2 \phi}{\partial x^2}\Big|_{i,j}\right] + \cdots$$

$$L_{i,j} = L\phi + \frac{1}{2}\left[\qquad\qquad\right]$$

$$\therefore\ L\phi - L_{i,j} = -\frac{1}{2}\Delta t\frac{\partial^2 \phi}{\partial t^2} - \frac{a}{2}\Delta x\frac{\partial^2 \phi}{\partial x^2} + O\left((\Delta x)^2 + (\Delta t)^2\right)$$

goes to zero as $\Delta x, \Delta t \to 0$

$\therefore$ The scheme is consistent.

Lax-Richtmyer Equivalence Theorem
(Fundamental Theorem of Numerical Analysis)

If a linear finite difference scheme is consistent with a well defined linear IVP then stability guarantees convergence as mesh length goes to 0.

Consistency + Stability $\Longleftrightarrow$ Convergence.

restrict to "linear problem"

Example Let $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0$ is approximated by

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} - \frac{u_{i+1,j} - \left(\frac{3}{2} u_{i,j+1} + \frac{1}{2} u_{i,j-1}\right) + u_{i-1,j}}{h^2} = 0$$

in book $\theta = 3/4$; uses $-2\{\theta u_{i,j+1} + (1-\theta) u_{i,j-1}\}$ Example 2.7

Expand in taylor series

$$u_{i,j+1} - u_{i,j-1} = u + k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{k^4}{24}\frac{\partial^4 u}{\partial t^4} + \dots$$

$$-\left(u - k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{k^3}{6}\frac{\partial^3 u}{\partial t^2} + \frac{k^4}{24}\frac{\partial^4 u}{\partial t^4} + \dots\right)$$

$$= 2k\frac{\partial u}{\partial t} + \frac{k^3}{3}\frac{\partial^3 u}{\partial t^3} + \frac{2k^5}{5!}\frac{\partial^5 u}{\partial t^5} + \dots$$

$$\frac{3}{2} u_{i,j+1} + \frac{1}{2} u_{i,j-1} = 2u + k\frac{\partial u}{\partial t} + 2k^2\frac{\partial^2 u}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 u}{\partial t^3} + \dots$$

$$u_{i+1,j} + u_{i-1,j} = 2u + h^2\frac{\partial^2 u}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u}{\partial x^4} + \dots$$

$$T_{i,j} = \frac{\partial \bar u}{\partial t} + \frac{k^2}{6}\frac{\partial^3 \bar u}{\partial t^3} + \frac{k^5}{5!}\frac{\partial^5 \bar u}{\partial t^5} - \frac{1}{h^2}\Big[2u + h^2\frac{\partial^2 \bar u}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 \bar u}{\partial x^4} + \dots$$

$$- 2u - k\frac{\partial u}{\partial t} - 2k^2\frac{\partial^2 u}{\partial t^2} + \dots\Big]$$

$$= \frac{\partial \bar u}{\partial t} - \frac{\partial^2 \bar u}{\partial x^2} + \frac{k}{h^2}\frac{\partial \bar u}{\partial t} + \frac{2k^2}{h^2}\frac{\partial^2 \bar u}{\partial t^2} + \frac{k^2}{6}\frac{\partial^3 \bar u}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 \bar u}{\partial x^4} + \frac{k^5}{5!}\frac{\partial^5 \bar u}{\partial t^5}$$

Case (I) $\quad k = \lambda h$

$$T_{i,j} = \left(\frac{\partial \bar u}{\partial t} - \frac{\partial^2 \bar u}{\partial x^2}\right) + \frac{\lambda}{h}\frac{\partial \bar u}{\partial t} + 2\lambda^2\frac{\partial^2 \bar u}{\partial t^2} + \frac{\lambda^2 h^2}{6}\frac{\partial^3 \bar u}{\partial t^3} + \dots$$

As $h \to 0$ $\quad \frac{\lambda}{h}\frac{\partial \bar u}{\partial t}$ terms blow up, which result in inconsistency

$\therefore$ Difference eqn is always inconsistent with $\frac{\partial \bar u}{\partial t} - \frac{\partial^2 \bar u}{\partial x^2} = 0$ when $k = \lambda h$

Case (II) $\quad k = \lambda h^2$

$$T_{i,j} = \left(\frac{\partial \bar u}{\partial t} - \frac{\partial^2 \bar u}{\partial x^2}\right) + \lambda\frac{\partial \bar u}{\partial t} + 2k\lambda\frac{\partial^2 \bar u}{\partial t^2} + \frac{k^2}{6}\frac{\partial^3 \bar u}{\partial t^3} + \dots$$

As $h \to 0$; scheme is stable

$\therefore$ the scheme is consistent.