Error Analysis

What we have is ODE $\frac{dy}{dx} = f(x,y)$

and ~~IVP~~ IC $y(x_0) = y_0$

Computing the error will suggest, how stable the method is?

In computation, two types of errors

→ Round off error

$$y_{machine} + R = y_{true\ representation}$$

$$R = y_{true} - y_{machine}$$

→ Truncation error

$$y_{true\text{-}representation} + T = y_{exact}$$

This depends on the order of the method

$$\Rightarrow \quad y_{n+1} = y(x_{n+1}) + O(h^{p+1}).$$

In truncation error

→ Local truncation error

→ Total error.

a) Local truncation error :- is the error one gets at the each time step or the iteration.

~~$e_k = |y_{k+1} - x|$~~

$e_k = | y_{k+1} - \hat{y}(x_k + h) |$

↙ approximated by some method

→ exact solution

Lets say one uses Taylor series

$$y(x_1+h) = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \quad + \frac{h^m}{n!} y_1^{(n)} + O(h^{m+1})$$

If one uses m-order then

$$e_k = |y_{k+1} - y(x_k+h)| = \frac{h^{m+1}}{(m+1)!} \left| y^{(m+1)}(x_k, x_{k+1}) \right|$$

$$= \frac{h^{m+1}}{(m+1)!} \left| \frac{d^m f(x, y(x))}{dx^m} \right|$$

Assume $\frac{d^m f}{dx^m}$ is bounded such that $\left| \frac{d^m f}{dx^m} \right| \leq M$

Then one gets

$$e_k \leq \frac{M}{(m+1)!} h^{m+1} = O(h^{m+1})$$

One gets the local error to be of order $(m+1)$ $\boxed{O(h^{m+1})}$

Now, <u>Total error</u>

Find computing time

$x = 0$ to x_{k+1}

Let say T ; we have choosen the step to be of h (grid size)

N - total no. of steps one has to perform

Here $N = \frac{T}{h}$ $\boxed{N \cdot h = T}$

Small h; more steps one has to perform

Total error is $E = |y(T) - y_N|$

Lets assume that $\frac{dy}{dx} = f(y, x)$; IC $y(x_0) = y_0$ is well-posed [i.e. the solution is stable w.r.t any perturbation on the initial condition] If one has 2 initial data (not far away) from each other then the soln. (for both) should not be very far away.

Let $y_0, \bar{y}_0$ be two initial condition

$y(x), \bar{y}(x)$ be the two solution, respectively,

Then $|y(x) - \bar{y}(x)| \le C|y_0 - \bar{y}_0|$

↳ initial error

$0 \le x \le *$

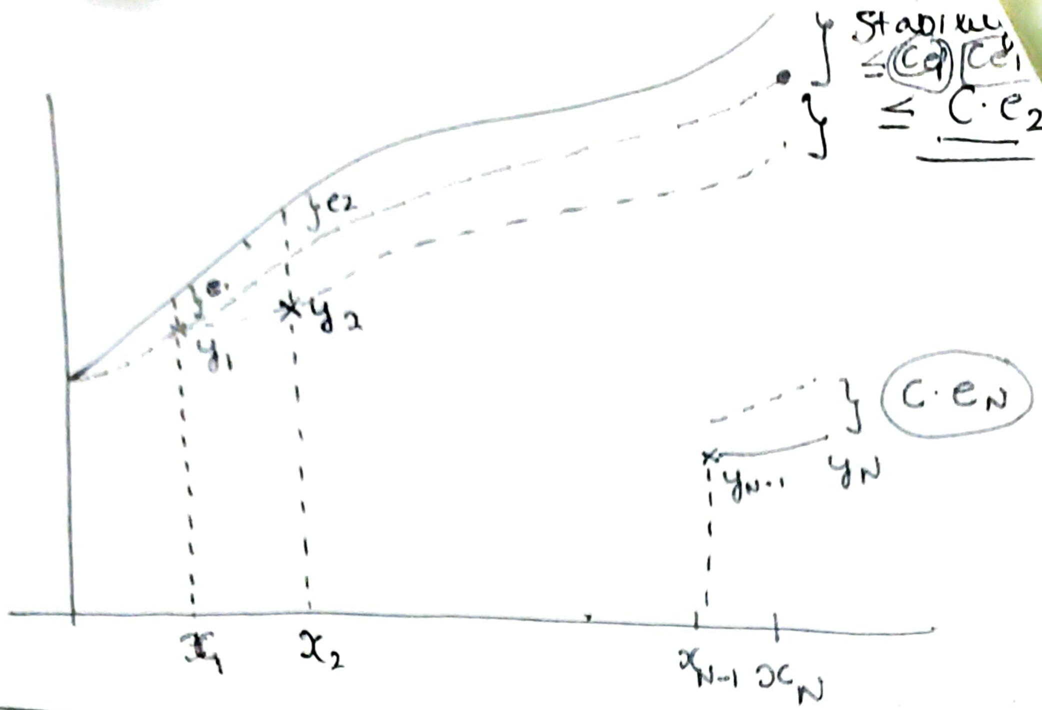
C - constt. (independent on x)

Discrete error

Error at each step

Then keep accumulating.

Let look at the fig.



local error e_1

Add up the total error in order to get the accumulate behaviour

$$E = C \sum_{k=1}^{N} |e_k|$$

As we saw, $e_k \sim O(h^{m+1}) \le \frac{M}{(m+1)!} h^{m+1}$

$$E \le C \sum_{k=1}^{N} \frac{M}{(m+1)!} h^{m+1}$$

$$N \frac{CM}{(m+1)!} h^{m+1} \Rightarrow \underbrace{(N \cdot h)}_{\text{this is constant}} \frac{CM}{(m+1)!} h^m$$

constant

$$\boxed{E \le C' \cdot h^m} \sim O(h^m)$$

Taylor series is mth order

local truncation error is order $O(h^{m+1})$.

Then total error is $O(h^m)$.

Total error is one order less than local error

Lets check some method

Most simplest will be to do

Heun's Method is 2nd order RK method

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

Taylor series for two variable fn

$$g(x+r, y+s) = g(x,y) + r\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial y} + \dots$$

$$f(x_i+h, y_i+k_1h) =$$

$$f(x_i, y_i) + h\frac{\partial f}{\partial x}(x_i, y_i) + \underset{\downarrow}{k_1} h \frac{\partial f}{\partial y}(x_i, y_i) + O(h^2, k_1^2)$$

$k_1 = f$

$$k_2 = f(x_i, y_i) + h\frac{\partial f}{\partial x} + \overset{f}{\cancel{k_1}} h \frac{\partial f}{\partial y} + O(h^2)$$

$\rightarrow k_1 = f$

$$y_{i+1} = y_i + \frac{1}{2}\left[hf + hf + h^2\frac{\partial f}{\partial x} + h^2 f\frac{\partial f}{\partial y} + O(h^3)\right]$$

$$= y_i + hf + \frac{1}{2}h^2\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) + O(h^3) \quad \text{— (A)}$$

Compare with Taylor expansion for $y(x_{i+1}) = y(x_i + h)$

$$y(x_i + h) = y(x_i) + h y'(x_i) + \frac{1}{2}h^2 y''(x_i) + O(h^3)$$

$$= y(x_i) + h f(x_i, y_i) + \frac{1}{2}h^2\left[\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) + O(h^3)$$

$$= y_i + hf + \frac{1}{2}h^2\left[\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) + O(h^3) \quad \text{— (B)}$$

Compare (A) & (B)

first three terms cancels

One is left with $O(h^3)$

Which give local truncation error

$$\boxed{e_L = O(h^3)}$$

Hence 2nd order method has local truncation error of $O(h^3)$

In general, R-K method of order m takes the form

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

$$\phi(x_i, y_i, h) = a_1k_1 + a_2k_2 + \dots + a_nk_n$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1h, y_i + pq_{11}k_1h)$$

$$k_3 = f(x_i + p_2h, y_i + q_{21}k_1h + q_{22}k_2h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1}h; y_i + q_{(n-1)1}h + q_{(n-1)2}k_2h + \dots + q_{(n-1)(n-1)}k_{n-1}h$$

Local truncation error is $O(h^{m+1})$

· while total error is $O(h^m)$.

what we can say or have observed

① small value of h leads to smaller error

② Higher the order used for approximation provides better smaller error

③ Uniform grid → Local error different at ~~varying~~ to every step [not same error].

Adaptive methods where we change the h size, helps in setting h such that one get uniform error at each step.

⟶

Stability of a method

A method is said to be stable if the effect of any single fixed round off error is not really growing and is independent of the no. of the mesh points

Lets take $y' = \lambda y$ as reference equation

but why

~~~~ $y' = f(x, y)$; $y(x_0) = y_0$

is the IVP.
~~~~

$y' = f(x,y);\ y(x_0) = y_0$

Let predict the behavior of the IVP in the neighbourhood of a point $(\hat{x}, \bar{y})$

One can linearize $[y' = f(x,y);\ y(x_0) = y_0]$

$$f(x,y) = f(\hat{x},\bar{y}) + \frac{\partial f}{\partial x}\cdot(x-\bar{x}) + \frac{\partial f}{\partial y}\cdot(y-\bar{y}) + 2^{nd}\text{ order} + .$$

$$= y\frac{\partial f}{\partial y}(\hat{x},\bar{y}) + f(\hat{x},\bar{y}) - \bar{y}\frac{\partial f}{\partial y}(\hat{x},\bar{y}) + (x-\bar{x})\frac{\partial f}{\partial y}(\hat{x},\bar{y})$$

One can write it as for $x \in [x_0, b]$

$$f(x,y) = \lambda y + c$$

where $\lambda = \frac{\partial f}{\partial y}(\bar{x}, \bar{y})$

and $c = f(\hat{x},\bar{y}) - \bar{y}\frac{\partial f}{\partial y}(\hat{x},\bar{y}) + (x-\bar{x})\frac{\partial f}{\partial x}(\hat{x},\bar{y})$

One get $y' = \lambda y + c$

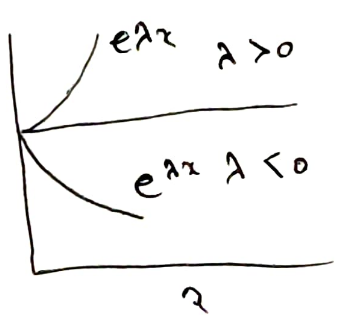
Lets transform y as $\lambda w = \lambda y + c$

$$\Rightarrow w' = y'$$

So, $w' = \lambda w$

whose soln is $w(x) = e^{\lambda x}$

One can linearize $f(x,y)$ and estimate the ~~behavior~~ solution behavior



$y' = \lambda y \quad y(x_0) = y_0$

$x \in [x_0, b]$

One get exact soln at that pow

$$y(x) = c e^{\lambda x}$$

$$y(x) = y_0 e^{\lambda(x - x_0)}$$

In order to compute $y(x)$ at $x = x_0 + kh$

$k = 1, 2, \ldots, N$

$$y(x_1) = y_0 e^{\lambda h}$$

$$y(x_2) = y(x_1) e^{\lambda h}$$

$\vdots$

$$y(x_{n+1}) = y_0(x_n) e^{\lambda h} \quad n = 0, 1, 2, \ldots$$

$e^{\lambda h}$ is difficult to compute

One can approximate $e^{\lambda h}$ suitably

$$y_{n+1} = y_n + h f(x_n, y_n) \quad [\text{Euler}]$$

$$y' = \lambda y = f(x, y)$$

So $y_{n+1} \simeq y_n + \lambda h y_n$

$$= (1 + \lambda h) y_n$$

$$y(x_{n+1}) = y(x_n) e^{\lambda h} \Rightarrow \boxed{e^{\lambda h} = 1 + \lambda h}$$

For euler $e^{\lambda h} = 1 + \lambda h + O(|\lambda h|^2)$

Take taylor Series

$$y_{n+1} = y_n + h f(x_n, y_n) + \frac{h^2}{2!}\left(\frac{\partial f}{\partial x} + f \frac{\partial f}{\partial y}\right)$$

$y' = \lambda y$

$$\Rightarrow y_n + h \lambda y_n + \frac{h^2}{2!}\left(\underline{\quad} + \lambda^2 y_n\right) + O(\lambda^3 h^3)$$

$$\simeq \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) + O(\lambda^3 h^3)$$

$$e^{\lambda h} \simeq \left(1+\lambda h+\frac{\lambda^2 h^2}{2}\right)+O\left(|\lambda h|^3\right)$$
$$\simeq E(\lambda h)$$

$$y_{n+1} \simeq y_n E(\lambda h) \; ; n=0,1,\dots$$

$$y(x_{n+1}) + \epsilon_{n+1} = E(\lambda h)\left[y(x_n)+\epsilon_n\right]$$

$$\epsilon_{n+1} = E(\lambda h)\,y(x_n) + E(\lambda h)\,\epsilon_n - y(x_{n+1})$$

$$y(x_{n+1}) = e^{\lambda h} y(x_n)$$

$$= \left(E(\lambda h) - e^{\lambda h}\right) y(x_n) + E(\lambda h)\,\epsilon_n$$

Absolute stable

$$\epsilon_{n+1} = \underbrace{\left(\overset{\text{approx}}{E(\lambda h)} - \overset{\text{exact}}{e^{\lambda h}}\right) y(x_n)}_{\text{local truncation error}} + \underbrace{E(\lambda h)\,\epsilon_n}_{\text{error at N stage; propogation error}}$$

relative

Minimis by $E(\lambda h)$ more closer to $e^{\lambda h}$.

Propogation err $E(\lambda h) \to$ not magnifying

$E(\lambda h)$ is stability factor.

$$|E(\lambda h)| \leq 1$$

$E(\lambda h)$ should not grow faster the $e^{\lambda h}$

Eulers method

$$y_{n+1} = y_n + h f(x_n, y_n)$$
$$y_n + h\lambda y_n = (1+\lambda h)\, y_n$$

$$\therefore E(\lambda h) = (1+\lambda h)$$

$$|E(\lambda h)| \leq 1 \Rightarrow -2 < \lambda h < 0$$

region of stability

In taylor's method

L4 - (6)

2nd order

$$\left|1+\lambda h+\frac{\lambda^2 h^2}{2}\right| \leq 1 \qquad -2 \leq \lambda h < 0$$

3rd order.

$-2.5 < \lambda h < 0$ (3rd order

$$\left|1+\lambda h+\frac{\lambda^2 h^2}{2}+\frac{\lambda^3 h^3}{6}\right| \leq 1$$

Example $\quad y' = x^2 - y^2 \quad y(0)=1$

Analyze behavior of soln around $(-1,1)$ and $(0,2)$

$f(x,y) = x^2 - y^2 \qquad \frac{\partial f}{\partial y} = -2y$

$\lambda = \frac{\partial f}{\partial y}(\bar{x},\bar{y})$

at $\lambda(1,-1) = 2 \qquad \lambda > 0$

$\lambda(0,-2) = -4 \Rightarrow \lambda < 0$

Stability of Heun Method

$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + h ; y_i + k_1 h)$

$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$

$y_{i+1} = y_i + \frac{h}{2} f(x_i,y_i) + \frac{1}{2} f\left[x_i + h ; y_i + f(x_i,y_i)h\right] h$

$y_i' = \lambda y_i \quad f_i = \lambda y_i$

$f(x_i + h, y_i + h\lambda y_i) = \lambda(y_i + \lambda y_i)$

$$y_{i+1} = y_i + \frac{h\lambda y_i}{2} + \frac{h}{2}(\lambda(y_i + h\lambda y_i))]$$

$$= y_i + \frac{h}{2}\lambda y_i + \frac{h\lambda y_i}{2} + \frac{h^2\lambda y_i}{2}$$

$$= y_i + h\lambda + \frac{h^2\lambda y_i}{2}$$

$$y_{i+1} = \underbrace{\left(y_i + h\lambda + \frac{h^2\lambda}{2}\right)}_{E(\lambda h)} y_i$$

Absolute stability $|E(\lambda h)| \le 1$

$\boxed{a' \le \lambda h \le b'}$ interval of absolute stability

Multi stepmethod

Two explicit

Implicit

One can write

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + \dots + a_k y_{n-k+1}$$
$$+ h(b_0 y'_{n+1} + b_1 y'_n + \dots + b_k y'_{n-k+1})$$

$b_0 \neq 0$: Implicit

$b_0 = 0$: Explicit

Multistep method

Uses past values of $y(x)$ and/or $y'(x)$ to create a polynomial which can approximates the derivative and extrapolates into the next interval

General method of writing y_{n+1}

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + \dots\dots + a_k y_{n-k+1}$$
$$+ h\,\phi(x_{n+1}, x_n, \dots, x_{n-k+1},$$
$$y'_{n+1}, y'_n, \dots y'_{n-k+1})$$

If one want to estimate the value of y at (x_{n+1}) then it demands values $y(x_n)$, $y(x_{n-1})$ till $y(x_{n-k+1})$ and also derivatives values

One can write

$$y_{n+1} = \sum_{i=1}^{k} a_i y_{n-i+1} + h \sum_{i=0}^{k} b_i y'_{n-i+1}$$

(a) If $b_0 = 0$ then the method is known as Explicit

One only knows the past values.

(b) If $b_0 \neq 0$, then the method is known as implicit

One also need the current value of $(n+1)$

Explicit Method

$$y' = f(x, y) \; ; \quad y(x_0) = y_0$$

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y)\,dx$$

→ how to approximate

As we did in previous lecture

Newtons backward difference formula of degree $(k-1)$
[If the points are k]

$$P_{k-1}(x) = f_n + \frac{(x-x_n)}{h} \nabla f_n + \frac{(x-x_n)(x-x_{n-1})}{2!\, h^2} \nabla^2 f_n + \dots$$

$$+ \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_{n-k+2})}{(k-1)!\, h^{k-1}} \nabla^{k-1} f_n$$

$$+ \frac{(x-x_n)(x-x_{n-1})\cdots(x-x_{n-k+1})}{k!} f^{(k)}(\xi)$$

One ~~step~~

$$(\xi \in [x_{n-k+1}, x_n]$$

$$u = \frac{x - x_n}{h} \; ; \quad u+1 = \frac{x - x_{n-1}}{h}$$

$$P_{k-1}(x) = f_n + u \nabla f_n + \frac{u(u+1)}{2!} \nabla^2 f_n + \dots$$

$$+ \frac{u(u+1)(u+2)\dots(u+k-2)}{(k-1)!} \nabla^{k-1} f_n$$

$$+ \frac{u(u+1)\dots(u+k-1)}{} \, h^k f^{(k)}(\xi)$$

$$y_k = y_{k-1} + h\underbrace{\left(P_{k-1}(x) \right)}$$

Due to which

$$\boxed{h^{k+1}}$$

3rd order Adams - Bashforth method

$$y_{n+1} \simeq y_n + \frac{h}{12}\left[23 f_n - 16 f_{n-1} + 5 f_{n-2}\right] + \underbrace{O(h^4)}_{\text{error}}$$

$$\boxed{\begin{array}{l} (x_n, y_n) \\ (x_{n-1}, y_{n-1}) \\ (x_{n-2}, y_{n-2}) \end{array}}$$

3 previous No.

Known

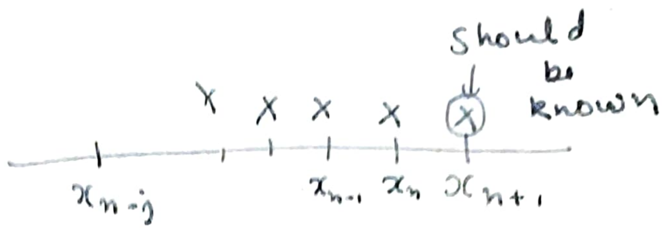
4th order Adams- Bashforth method

$$y_{n+1} \simeq y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right] + \underbrace{O(h^5)}_{\text{error}}$$

One can see that the error depends upon the m^{th} order method. Local error is $O[h^{m+1}]$

Implicit Method

To compute value at $n+1$; one need to have value at $(n+1)$ itself



$$y(x_{n+1}) = y(x_{n-j}) + \int_{x_{n-j}}^{x_{n+1}} f(x,y)\,dx$$

approximate $f(x,y)$ by a polynomial that interpolates at $(k+1)$ points $x_{n+1}, x_n, \ldots x_{n-k+1}$, $k > 0$

Here x_{n+1} is included

$$P_k(x) = f_{n+1} + (u-1)\nabla f_{n+1} + \frac{(u-1)u}{2!}\nabla^2 f_{n+1} + \cdots$$

$$+ \frac{(u-1)u(u+1)\ldots(u-k-2)}{k!}\nabla^k f_{n+1}$$

$$+ \frac{(u-1)u(u+1)\ldots(u+k-1)}{(k+1)!} h^{k+1} f^{(k+1)}(\xi)$$

$$y_{n+1} = y_n + h\, P_k(x)$$

$$y_n + \underline{\qquad\qquad} + O(h^{k+2})$$

Here the error is $(k+2)$

$k=3$ (x_{n+1}, y_{n+1}), (x_n, y_n), (x_{n-1}, y_{n-1}), (x_{n-2}, y_{n-2})

$$y_{n+1} \simeq y_n + \frac{h}{24}\left[9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right]$$

$$+ O(h^5)$$

Adam-Moulton Method

One need $f_{n+1} \approx f(x_{n+1}, y_{n+1})$ That's why Implicit

Local truncation error

$$T_{n+1} \simeq y(x_{n+1}) - \sum_{i=1}^{n} a_i \, y(x_{n-i+1}) - h \sum_{i=0}^{k} b_i \, y'(x_{n-i+1})$$

$O(h^{n+1}) \rightarrow$ Adams-Bashforth Method

$O(h^{n+2}) \rightarrow$ Adams-Moulton Method

Predictor - Corrector

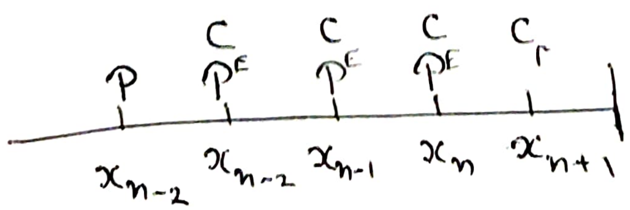
Adam-Bashforth is used as predictor (P) while Adam-Moulton is used as corrector (C)

One can keep doing the method as Estimator (E)

$$PC\cancel{(EP)}^N \rightarrow P(EC)^N$$

N times till the precision of the interest is achieved.

In order to get the initial point one use Runge Kutta 4 method for best precision



Adams Bashforth 5th order

$$y_{n+1} = y_n + \frac{h}{720}\left[1901 y_n' - 2774 y_{n-1}' + 2616 y_{n-2}' - 1274 y_{n-3}' + 251 y_{n-4}'\right)$$

truncation error is $O(h^6)$

Adams Moulton 5th order

$$y_{n+1} = y_n + \frac{h}{251}\left[251 y_{n+1}' + 646 y_n' - 264 y_{n-1}' + 106 y_{n-2}' - 19 y_{n-3}'\right]$$

truncation $O(h^6)$

Method	global error
Euler's	$O(h)$
Heun's	$O(h^2)$
Midpoint	$O(h^2)$
second RK	$O(h^2)$
4th order RK	$O(h^4)$
RK-45	$O(h^5)$