Boundary Value Problem

$$\frac{d^2y}{dx^2} = f(x) \quad \text{with } y(a) = \gamma_1 \text{ and } y(b) = \gamma_2$$

As we saw, one can use shooting method to change it into IVPs and solve.

a) Superposition [only linear]

$$y(b) = \lambda\, y(b;\alpha) + (1-\lambda)\, y(b;\beta)$$

$$y'(a) = \alpha \rightarrow y(b;\alpha)$$
$$y'(a) = \beta \rightarrow y(b;\beta)$$

$$\boxed{\lambda = \frac{\gamma_2 - y(b;\beta)}{y(b;\alpha) - y(b;\beta)}}$$

b) Iterative Eqs

Secant method

$$\alpha_{n+1} = \alpha_n - \frac{(\alpha_n - \alpha_{n-1})}{[\phi(\alpha_n) - \phi(\alpha_{n-1})]} \cdot \phi(\alpha_n)$$

$$\phi(\alpha) = y(b;\alpha) - \gamma_2$$

Need two α guess & then keep modifying the α to get the proper soln.

Simple and effective.

One can also do →

$$\text{Differential eqn} \xrightarrow[\text{into}]{\text{Convert}} \text{Difference eqn}$$

Consider Taylor series

$$y(x+h) = y(x) + h\,y'(x) + \frac{h^2}{2!}\,y''(x) + \cdots \qquad -①$$

$$y'(x) = \frac{y(x+h) - y(x)}{h} - \frac{h}{2}\,y''(\xi)$$

Forward

Approximation for 1st derivative.

$$x < \xi < x+h$$

Now consider

$$y(x-h) = y(x) - h\,y'(x) + \frac{h^2}{2!}\,y''(x) - \cdots \qquad -②$$

$\boxed{\text{Backward}}$

$$y'(x) = \frac{y(x) - y(x-h)}{h} + \frac{h}{2}\,y''(\xi)$$

$$x-h < \xi < x$$

Another approximation

Any grid point, one get

$$y'(x_i) = y_i' \simeq \frac{y(x_i+h) - y(x_i)}{h} = \frac{y_{i+1} - y_i}{h} + O(h)$$

Forward

$$y_i' \simeq \frac{y_i - y_{i-1}}{h} + O(h)$$

Backward

$$y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \dots \quad -(1)$$

$$y(x-h) = y(x) - h y'(x) + \frac{h^2}{2!} y''(x) - \frac{h^3}{3!} y'''(x) + \dots \quad (2)$$

(1) - (2)

$$y(x+h) - y(x-h) = 2 h y'(x) + \frac{h^3}{3} y'''(\xi)$$

$$x-h < \xi < x+h$$

$$\Rightarrow \quad y'(x) \simeq \frac{y(x+h) - y(x-h)}{2h} - \frac{h^2}{6} y'''(\xi)$$

$$y'(x_i) \simeq y_i' \simeq \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2) \quad -(3)$$

This approximation is second order

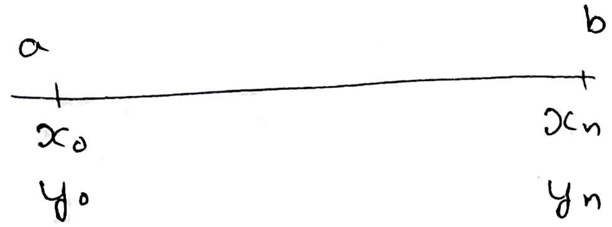
Similarly (1) + (2)

$$y(x+h) + y(x-h) = 2y(x) + h^2 y''(x) + \frac{h^4}{12} y''''(\xi)$$

$$y''(x) = \frac{y(x+h) + y(x-h) - 2y(x)}{h^2} - \frac{h^2}{12} y''''(\xi)$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + O(h^2) \quad -(4)$$

Lets domain for x is $[a, b]$



Lets proceed to solve the simple BVP

Simple two point BVP

$$y''(x) = f(x) \qquad a < x < b$$
$$0 < x < 1 \quad — (1)$$

$$y(0) = \alpha \qquad y(1) = \beta$$

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + O(h^2)$$

$$(1) \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f(x_i) = f_i$$

$$i = 1, 2, \ldots, n$$

$\boxed{a = x_0}$ $\quad x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_{n-1} \quad x_n \quad$ $\boxed{x_{n+1} = b}$

$i = 1$ involves y_0 (which is α)

$$y_1'' = \frac{\overset{\alpha}{y_0} - 2y_1 + y_2}{h^2} = f_1 \Rightarrow \frac{1}{h^2}[-2y_1 + y_2] = f_1 - \frac{\alpha}{h}$$

$i = n$ involves y_{n+1} (which is β)

$$y_n'' = \frac{y_{n-1} - 2y_n + \overset{\beta}{y_{n+1}}}{h^2} = f_n$$

$$\Rightarrow \frac{1}{h^2}[y_{n-1} - 2y_n] = f_n - \frac{\beta}{h^2}$$

rest $i = 2, \ldots, n-1$

$i = 2$

$$\frac{1}{h^2}[y_1 - 2y_2 + y_3] = f_2$$

$$i = n-1 \quad \frac{1}{h^2}[y_{n-2} - 2y_{n-1} + y_n] = f_{n-2}$$

$$\frac{1}{h^2}\left[-2y_1 + y_2\right] = f_1 - \frac{\alpha}{h^2}$$

$$\frac{1}{h^2}\left[y_1 - 2y_2 + y_3\right] = f_2$$

$$\frac{1}{h^2}\left[y_2 - 2y_3 + y_4\right] = f_3$$

$$\vdots$$

$$\frac{1}{h^2}\left[y_{n-2} - 2y_{n-1} + y_n\right] = f_{n-2}$$

$$\frac{1}{h^2}\left[y_{n-1} - 2y_n\right] = f_n - \frac{\beta}{h^2}$$

One can write in ~~mtx~~ form

$$A\bar{y} = \bar{F}$$

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & 0 & & & \\ 1 & -2 & 1 & 0 & & \Large 0 & \\ 0 & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & \Large 0 & & 1 & -2 & 1 & \\ & & & 0 & 1 & -2 & 1 \\ & & & 0 & 0 & 1 & -2 \end{pmatrix} \rightarrow \text{Tridiagonal system}$$

$$\bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} \qquad \bar{F} = \begin{pmatrix} f_1 - \frac{\alpha}{h^2} \\ f_2 \\ f_3 \\ \vdots \\ f_n - \beta/h^2 \end{pmatrix}$$

Can be solved for a given $f(x)$

How well does $\bar{y} = (y_1, y_2, \cdots y_n)^T$ approximate $y(x)$?

$y'' \sim O(h^2)$. So approx of the h^2 order.

However, the reality is a bit more complicated!

Local truncation error

$$T_i = \frac{1}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] - f(x_i)$$

$$y''(x_i) - \frac{h^2}{12} y^{(4)}(\xi) + O(h^4) = f(x_i)$$

By virtue of $y'' = f(x)$

$$T_i = -\frac{h^2}{12} y^{(4)}(\xi) + O(h^4)$$

We don't know $y^{(4)}(\xi)$; but believe to be independent of h. and fixed

$\therefore$ $T_i \sim O(h^2)$ as $h \to 0$.

Let $\bar{T}$ - vector containing T_i

$$\bar{T} = A\hat{\bar{y}} - \bar{F}$$

$$\Rightarrow A\hat{\bar{y}} = F + \bar{T}$$

Global error: $A\bar{y} = F$ — (a)

global error $\bar{E} = \bar{y} - \hat{\bar{y}}$

$$A(\bar{y} - \hat{\bar{y}}) = -\bar{T} \Rightarrow A\bar{E} = -\bar{T}$$

L + - (4) a

$$\Rightarrow \frac{1}{h^2}\left(E_{i-1} - 2E_i + E_{i+1}\right) = -T(x_i) \quad \text{—— (E)}$$

$i = 1, 2, \ldots$

with b.c.

$E_0 = 0$, $E_{n+1} = 0$

Same as difference eq^n for y_i, except $f(x_i)$ is replaced by T

(E) $\Rightarrow$

$$e''(x) = -\Delta(x)$$

Δ as some function $\quad \Delta(x) = \frac{1}{12} h^2 y^{(4)}(x)$

$e(a) = 0$; $e(b) = 0$

Integrating twice

$$e(x) \simeq -\frac{1}{12} h^2 y''(x) + \frac{h^2}{12}\left[y''(0) + x\left(y''(1) - y''(0)\right)\right]$$

$$\simeq O(h^2)$$

Global error $O(h^2)$

Linear Finite Difference Method

General form

$$y'' = p(x)\,y' + q(x)\,y + r(x) \quad \text{for } a \le x \le b$$

where $y(a) = \gamma_1$ and $y(b) = \gamma_2$

- Divid $[a,b]$ into $N+1$

a $\qquad\qquad$ b

$$h = \frac{(b-a)}{N+1} \qquad i = 0, 1, 2, \ldots \ldots N-1, N, N+1$$

interior mesh point x_i for $i = 1, 2, \ldots, N$

$$y''(x_i) = p(x_i)\,y'(x_i) + q(x_i)\,y(x_i) + r(x_i)$$

Using Central difference Formulae

$$y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{12} y^{(4)}(\xi^a)$$

$$x_{i-1} < \xi^a < x_{i+1}$$

$$y'(x_i) = \frac{1}{2h}\left[y(x_{i+1}) - y(x_{i-1})\right] - \frac{h^2}{6} y'''(\xi^b)$$

$$x_{i-1} < \xi^b < x_{i+1}$$

This leads to

$$\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} = p(x_i)\left[\frac{y(x_{i+1}) - y(x_{i-1})}{2h}\right]$$

$$+ q(x_i)\,y(x_i) + r(x_i) \underbrace{- \frac{h^2}{12}\left[2p(x_i)\,y'''(\xi^b) - y^{(4)}(\xi

LT-(?)

$$\Rightarrow \frac{y_{i+1}-2y_i+y_{i-1}}{h^2} + p(x_i)\left[\frac{y_{i+1}-y_{i-1}}{2h}\right] + q(x_i)y_i = r_i$$

$i = 1, 2, \ldots, N$

Here conveniently choosen $-p(x)$ to be $p(x)$. Absorb the negative in the constant (px)

$\rightarrow y_0 = \gamma_1$ and $y_{N+1} = \gamma_2$

$i = 1, 2, \ldots, N$

$$y_{i+1} - 2y_i + y_{i-1} + \frac{h}{2}p(x_i)y_{i+1} - \frac{h}{2}p(x_i)y_{i-1} + h^2 q(x_i)y_i = h^2 r_i$$

$$\left[1-\frac{h}{2}p(x_i)\right]y_{i-1} + \left[-2+h^2 q(x_i)\right]y_i + \left[1+\frac{h}{2}p(x_i)\right]y_{i+1} = h^2 r_i$$

$y_0 = \gamma_1$

$i=1$ $\left[1-\frac{h}{2}p(x_1)\right]y_0 + \left[-2+h^2q(x_1)\right]y_1 + \left[1+\frac{h}{2}p(x_1)\right]y_2 = h^2 r_1$

$i=2$ $\left[1-\frac{h}{2}p(x_2)\right]y_1 + \left[-2+h^2q(x_2)\right]y_2 + \left[1+\frac{h}{2}p(x_3)\right]y_3 = h^2 r_2$

$i=3$ $\left[1-\frac{h}{2}p(x_3)\right]y_2 + \left[-2+h^2q(x_3)\right]y_3 + \left[1+\frac{h}{2}p(x_3)\right]y_4 = h^2 r_3$

$\vdots$

$i=N-1$ $\left[1-\frac{h}{2}p(x_{N-1})\right]y_{N-2} + \left[-2+h^2q(x_p)\right]y_{N-1} + \left[1+\frac{h}{2}p(x_{N-1})\right]y_N = h^2 r_{N-1}$

$i=N$ $\left[1-\frac{h}{2}p(x_N)\right]y_{N-1} + \left[-2+h^2q(x_N)\right]y_N + \left[1+\frac{h}{2}p(x_N)\right]y_{N+} = h^2 r_{N-1}$

$y_{N+1} =$

One can take these B.V. to R.H.S.

One can generalized and write in form of a matrix =n

$$A_i y_{i-1} + B_i y_i + C_i y_{i+1} = h^2 r_i$$
$$i = 1, 2, \dots, N$$

$$A_i = \left(1 - \frac{h}{2} p_i\right)$$

$$B_i = \left(-2 + h^2 q_i\right)$$

$$C_i = \left(1 + \frac{h}{2} p_i\right)$$

From the (A), we see that one get A as a tridiagonal system

$$A \bar{y} = \bar{b}$$

$$A = \begin{pmatrix} B_1 & C_1 & 0 & 0 & 0 & & & \\ A_2 & B_2 & C_2 & 0 & 0 & & & \\ 0 & A_3 & B_3 & C_3 & 0 & & & \\ 0 & 0 & A_4 & B_4 & C_4 & & & \\ & & 0 & \ddots & \ddots & \ddots & 0 & \\ & & & & A_{N-2} & B_{N-2} & C_{N-2} & 0 \\ & & & & 0 & A_{N-1} & B_{N-1} & C_N \\ & & & & 0 & 0 & A_N & B \end{pmatrix}$$

as A_1 and C_N goes to the RHS.

$$\bar{y} = (y_1, y_2, y_3, \ldots, y_{N-1}, y_N)^T$$

$$\bar{b} = (h^2 r_1 - A_1 \gamma_1, h^2 r_2, h^2 r_3, \ldots, h^2 r_{N-1}, h^2 r_N - C_N \gamma_2)^T$$

This system has unique soln. provided that p, q and r are continuous on $[a,b]$, that $q(x) \geq 0$ on $[a,b]$ and that $h < 2/L$ where $L = \max_{a \leq x \leq b} |p(x)|$.

One can use Gauss elimination to solve the eqn.

A brief review

x or y doesn't matter here, something to solve

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Assume a_{21}, a_{31} not zero

Use first row & multiply by $\frac{a_{21}}{a_{11}}$ second row & subtract

— sec. — & multiply by $\frac{a_{31}}{a_{11}}$ third row & subtract

— do — multiply by $\frac{a_{n1}}{a_{11}}$ last row & subtract $\left(\frac{a_{n1}}{a_{11}}\right)$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a'_{22} & \cdots & \cdots & a'_{2n} \\ 0 & a'_{32} & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & a'_{n2} & \cdots & \cdots & a'_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

Now
do same with 2^{nd} row and so on.

One end up with

$$\left[\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots \ldots + a_{1n}x_n = b_1 \\ \quad a'_{22}x_2 + a'_{23}x_3 + \ldots \ldots + a'_{2n}x_n = b'_2 \\ \qquad a'_{33}x_3 + \ldots \ldots + a'_{3n}x_n = b'_3 \\ \qquad\qquad \ddots \\ \qquad\qquad a'_{n-1\,n-1}x_{n-1} + a'_{n-1}x_n = b'_{n-1} \\ \qquad\qquad\qquad a'_{nn}x_n = b'_n \end{array}\right]$$

One get $x_n = \dfrac{b_n'}{a'_{nn}}$

The use it to get $x_{n-1} = \dfrac{1}{a'_{n-1\,n-1}}\left(b'_{n-1} - a_{n-1,n}x_n\right.$

and so on.

Some limitation of Gauss elimination (naive one)
but, not going into ~~pictu~~ the detail here.
Do that yourself as already done.

Similarly one can do for Tridiagonal system,

$$\begin{bmatrix} d_1 & c_1 & & & & \\ a_2 & d_2 & c_2 & & & \\ & a_3 & d_3 & c_3 & & \\ & & a_4 & d_4 & c_4 & \\ & & & \ddots & \ddots & \ddots \\ & & & a_{n-1} & d_{n-1} & c_{n-1} \\ & & & & a_n & d_n \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} \\ \\ b \\ \\ \\ \end{bmatrix}$$

~~Step 1~~ Let use Same notation

$$\begin{bmatrix} B_1 & C_1 & 0 & 0 & 0 & & & \\ A_2 & B_2 & C_2 & 0 & 0 & & & \\ 0 & A_3 & B_3 & C_3 & 0 & & & \\ 0 & 0 & A_4 & B_4 & C_4 & & & \\ & & & & \ddots & & & \\ & & & & A_{N-2} & B_{N-2} & C_{N-2} & 0 \\ & & & & 0 & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & & 0 & 0 & A_N & B_N \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$

Step ① Eliminate y_1 from row ② by subtracting row ① $\times \frac{A_2}{B_1}$

$$B_2' = B_2 - C_1 \times \frac{A_2}{B_1}$$

$$b_2' = b_2 - b_1 \times \frac{A_2}{B_1}$$

Step ② Eliminate y_2 from row ③ by subtracting row ①' $\times \frac{A_3}{B_2'}$

$$B_3' = B_3 - C_2 \times \frac{A_3}{B_2'}$$

$$b_3' = b_3 - b_2' \times \frac{A_3}{B_2'}$$

Step Ⓝ Eliminate y_{N-1} from row Ⓝ by subtracting row $(N-1)'$ $\times \frac{A_N}{B_{N-1}'}$

$$B_N' = B_N - C_{N-1} \times \frac{A_N}{B_{N-1}'}$$

$$b_N' = b_N - b_{N-1}' \times \frac{A_N}{B_{N-1}'}$$

Eq 1

Eq. 1 gives

$$\because B_N' \, y_n = b_n'$$

$$y_n = \frac{b_n'}{B_N'}$$

and then one can get y for $k = N-1, \ldots$ and 1.

$$y_k = \frac{b_k' - C_k y_{k+1}}{B_k'}$$

Note

Original B an b changes here, while C is same!

One can write the algorithm an solve

Non-linear system

Non linear second order BVP

$$y'' = f(x, y) \; ; \quad a < x < b$$

$$y(a) = \gamma_1 \; , \quad y(b) = \gamma_2$$

If one discretize, one can get

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = f(x_i, y_i)$$

$$\underbrace{y_{i-1} - 2y_i + y_{i+1}}_{\text{LHS linear}} = \underbrace{h^2 f(x_i, y_i)}_{\text{RHS non linear}} \quad \text{— NLa}$$

$$y_0 = \gamma_1 \; , \quad y_{N+1} = \gamma_2$$

Let see an example to understand

$$y'' = x y^2 + x \; ; \quad y(-1) = 2, \quad y(3) = -4 \qquad h = 1$$

Using NLa

$$y_{i-1} - 2y_i + y_{i+1} = (x_i y_i^2 + x_i) \underset{1}{\underset{\shortparallel}{h^2}}$$

$$y_{i-1} - 2y_i + y_{i+1} = x_i y_i^2 + x_i$$

$i = 1$
$$y_0 - 2y_1 + 2y_2 = x_1 y_1^2 + x_1$$
$$= 0$$

$i = 2$
$$y_1 - 2y_2 + 2y_3 = y_2^2 + 1$$

$i = 3$
$$y_2 - 2y_3 + 2y_4 = \cancel{x_3 y_3} \, 2y_3^2 + 2$$

Now, y_0 & y_4 are known

L7

$$\Rightarrow -2y_1 + y_2 + 2 = 0$$
$$y_1 - 2y_2 + y_3 - y_2^2 - 1 = 0$$
$$y_2 - 2y_3 - 2y_3^2 - 3 = 0$$

Not easy to put as $AX = B$ matrix form, due to it non-linear form

$$F(\bar{y}) = 0$$

$$\begin{pmatrix} F_1(\bar{y}) \\ F_2(\bar{y}) \\ F_3(\bar{y}) \end{pmatrix} = \begin{pmatrix} -2y_1 + y_2 + 2 \\ y_1 - 2y_2 + y_3 - y_2^2 - 1 \\ y_2 - 2y_3 - 2y_3^2 - 3 \end{pmatrix} \quad (A)$$

(A) is non-linear system of =ns for y_1, y_2, y_3.

How one can solve the?

Newton Raphson method?

$$\bar{F}(\bar{y}) = 0$$

Suppose we expand in taylor series about $\bar{y}_i$

$$\bar{F}(\bar{y}) = \bar{F}(\bar{y}_i) + (\bar{y} - \bar{y}_i)\frac{\partial \bar{F}}{\partial \bar{y}} + \frac{(\bar{y} - \bar{y}_i)^2}{2!}\frac{\partial^2 \bar{F}}{\partial \bar{y}^2} + \dots$$

$$= \bar{F}(\bar{y}_i) + (\bar{y} - \bar{y}_i)\frac{\partial \bar{F}}{\partial \bar{y}} + O(h^2) \simeq 0$$

$$\cancel{\bar{y} = \bar{y}_i - \bar{F}(\bar{y}_i)\left(\frac{\partial \bar{F}}{\partial \bar{y}}\right)^{-1}}$$

$$\bar{y} = \bar{y}_i - \left(\frac{\partial \bar{F}}{\partial \bar{y}}\right)^{-1} \bar{F}(\bar{y}_i)$$

Jacobian

Hence an ~~iter~~ iterative process

$$\bar{y}^{(k+1)} = \bar{y}^{(k)} - J^{-1}(\bar{y}^{(k)})\,\bar{F}(\bar{y}^{(k)})$$

where $J(y_i, \dots y_n) = \dfrac{\partial \bar{F}}{\partial \bar{y}}$ Jacobian matrix.

$$J = \left(\frac{\partial f}{\partial y_1}, \dots \frac{\partial F}{\partial y_n} \right)$$

An example

$$y'' = 9y^2 + \frac{1}{x} \qquad y(0) = 4; \quad y(1) = 1$$

$h = 1/3$

y_0	y_1	y_2	y_3
0	1/3	2/3	1

~~$x_0 = 0$~~

Discretize

$$y_{i+1} - 2y_i + y_{i+1} = h^2\left[9y_i^2 + \frac{1}{x_i}\right]$$

$$y_0 = 4 \;;\; y_3 = 1$$

$i=1$ $\quad y_0 - 2y_1 + y_2 = \left(\frac{1}{3}\right)^2\left[9y_1^2 + \frac{1}{(1/3)^2}\right]$

$$y_0 - 2y_1 + y_2 = y_1^2 + 1$$

$y_0 = 4$

$i=2$ $\quad y_1 - 2y_2 + y_3 = \left(\frac{1}{3}\right)^2\left[9y_2^2 + \frac{1}{(2/3)^2}\right]$ $\qquad y_3 = 1$

$$= y_2^2 + \frac{1}{4}$$

Simplifies to

$$y_1^2 + 2y_1 - y_2 - 3 = 0 \quad (F_1)$$

$$y_2^2 - y_1 + 2y_2 - \frac{3}{4} = 0 \quad (F_2)$$

Compute Jacobian

$$J(y_1, y_2) = \frac{\partial \bar{F}}{\partial \gamma} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix}$$

$$= \begin{pmatrix} 2y_1 + 2 & -1 \\ -1 & 2y_2 + 2 \end{pmatrix}$$

Then one need to compute J^{-1}

$$J^{-1} = \frac{1}{D} \begin{pmatrix} 2y_2 + 2 & 1 \\ 1 & 2y_1 + 2 \end{pmatrix}$$

$$D = (2y_1 + 2)(2y_2 + 2) - 1$$

$$= 4(1 + y_1)(1 + y_2) - 1$$

$\boxed{JD}$

$$\therefore \begin{pmatrix} y_1^{(k+1)} \\ y_2^{(k+1)} \end{pmatrix} = \begin{pmatrix} y_1^{(k)} \\ y_2^{(k)} \end{pmatrix} - \frac{1}{4(1+y_1)(1+y_2) - 1} \begin{pmatrix} 2y_2^{(k)} + 2 & 1 \\ 1 & 2y_2^{(k)} + 2 \end{pmatrix}$$

One need initial guess to start the iteration

Let $y_1^{(0)} = 2$ and $y_2^{(0)} = 1$

one can get $y_1^{(1)}$ and $y_2^{(1)}$ using $\boxed{JD}$

Then $y_1^{(2)}, y_2^{(2)}$

and so on till one gets the desired

accuracy.

BVPs - derivative Boundary condition

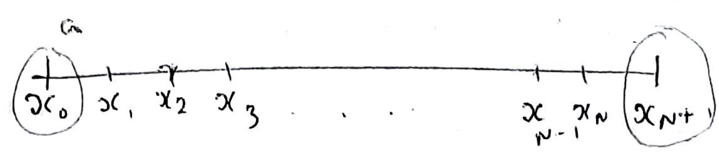
General frame work

Consider $$y'' + p(x)y' + q(x)y = r(x) \quad —(1)$$
$$a < x < b$$

$$\left.\begin{aligned} a_0 y(a) - a_1 y'(a) &= \gamma_1 \\ b_0 y(b) + b_1 y'(b) &= \gamma_2 \end{aligned}\right\} —(2)$$

One can discretize (1) [L7 - 5 b] —(3)
$$A_i y_{i-1} + B_i y_i + C_i y_{i+1} = h^2 r_i \qquad i = 1, 2, \ldots, N$$

where
$$A_i = \left(1 - \frac{h}{2} p_i\right)$$
$$B_i = \left(-2 + h^2 q_i\right)$$
$$C_i = \left(1 + h\frac{p_i}{2}\right)$$



There is total of $(N+2)$ unknown with N equations [y_0, y_{N+1} are not available due to derivative b.c.]

One still has not used the Boundary condition

As $y_i' \simeq \frac{y_{i+1} - y_{i-1}}{2h} + O(h^2)$

at x_0: $\frac{y_1 - y_{-1}}{2h}$

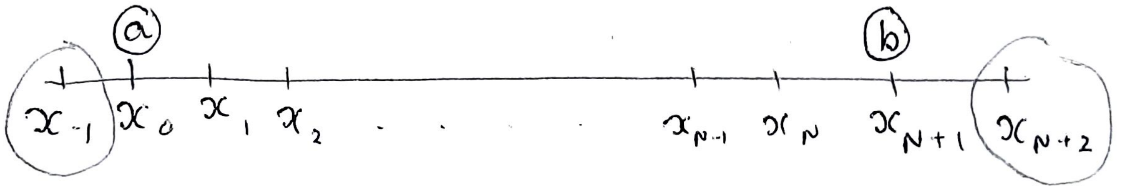
$$\Rightarrow a_0 y(a) - a_1 y'(a) = \gamma_1$$

$$a_0 y_0 - a_1 \left(\frac{y_1 - y_{-1}}{2h}\right) = \gamma_1 \quad \text{— (4)}$$

at $x_\infty = x_{N+1}$ $\quad y'_{N+1} = \frac{y_{N+2} - y_N}{2h} + O(h^2)$

$$b_0 y(b) + b_1 y'(b) = \gamma_2$$

$$b_0 y_{N+1} + b_1 \left(\frac{y_{N+2} - y_N}{2}\right) = \gamma_2 \quad \text{— (5)}$$



(4) and (5) involve y_{-1} and y_{N+2} terms which corresponds to nodes outside the interval $[a, b]$

y_{-1} and y_{N+2} are called fictitious values

One can express these fictitious values as

$$y_{-1} = \frac{2h}{a_1}\gamma_1 + y_1 - \frac{2h}{a_1} a_0 y_0$$

$$y_{N+2} = y_N - \frac{2h b_0}{b_1} y_{N+1} + \frac{2h}{b_1}\gamma_2$$

$\rightarrow$ $\boxed{\text{FVs}}$

Unless one don't get rid of y_{-1} and y_{N+2}, one will not end up with (N+2) = ns

Eliminating the fictitious values $\boxed{L7-10a}$

Assume the discretized =n (3) holds for $i = 0, N+1$

i.e. at the boundary points $\boxed{x_0}$ and $\boxed{x_{N+1}}$

$\therefore$ at $i=0$

$$A_0 y_{-1} + B_0 y_0 + C_0 y_1 = h^2 r_0$$

From $\boxed{FVs}$

$$A_0\left[\frac{2h}{a_1}\gamma_1 + y_1 - \frac{2h}{a_1} a_0 y_0\right] + B_0 y_0 + C_0 y_1 = h^2 r_0$$

$$\left(B_0 - \frac{2h a_0 A_0}{a_1}\right) y_0 + (A_0 + C_0) y_1 = h^2 r_0 - \frac{2h}{a_1}\gamma_1 A_0$$

$\boxed{FV-0}$

for $i = N+1$

$$A_{N+1} y_N + B_{N+1} y_{N+1} + C_{N+1} y_{N+2} = h^2 r_{N+1}$$

$\boxed{\text{from FVs}}$

$$A_{N+1} y_N + B_{N+1} y_{N+1} + C_{N+1}\left[y_N - 2h\frac{b_0}{b_1} y_{N+1} + \frac{2h}{b_1}\gamma_2\right] = h^2 r_{N+1}$$

$$(A_{N+1} + C_{N+1}) y_N + \left(B_{N+1} - 2h\frac{b_0}{b_1} C_{N+1}\right) y_{N+1} = h^2 r_{N+1} - \frac{2h}{b_1}\gamma_2 C_{N+1}$$

$\boxed{FV-N+1}$

Using (3) $\boxed{L7-10a}$, $\boxed{FV-0}$ and $\boxed{FV-N+1}$

one get a tridiagonal system

$$
\begin{pmatrix}
B_0 - \frac{2 h a_0}{a_1} A_0 & A_0 + C_0 & 0 & 0 & 0 & & & \\
A_1 & B_1 & C_1 & 0 & 0 & & & \\
0 & A_2 & B_2 & C_2 & 0 & & & \\
0 & 0 & A_3 & B_3 & C_3 & & & \\
 & & & \ddots & \ddots & \ddots & & \\
 & & & & A_{N-1} & B_{N-1} & C_{N-1} & 0 \\
 & & & & 0 & A_N & B_N & C_N \\
 & & & & 0 & 0 & A_{N+1} + C_{N+1} & B_{N+1} - \frac{2 h b_0}{b_1} C_{N+1}
\end{pmatrix}
$$

One get a tri diagonal system

$$\bar{y} = (y_0, y_1, \dots, y_N, y_{N+1})^T$$

$$\left(h^2 r_0 - \frac{2h}{a_1} \gamma_1 A_0,\ h^2 r_1,\ h^2 r_2,\ \dots,\ h^2 r_N,\ h^2 r_{N+1} - \frac{2h}{b_1} \gamma_2 C_{N+1}\right)^T$$

One can solve it

For the derivative Boundary condition

- One get also the end point (y_0, y_{n+2}) fictitious value to evaluate
- One simply eliminate them by discretization
- Give $N+2$ = ns. with $N+2$ unknown

L-7-(12) a

Example Set up FDM for the BVP

$$y''(x) = -4(y-x) \qquad y(0)=1$$
$$y(1)=2$$

Uniform grid

$$x_i = ih \; ; \; h = \frac{1}{m}$$

$$\frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) = -4y_i + 4x_i$$

$i=1$ y_0.

$$y_{i-1} + (4h^2-2)y_i + y_{i+1} = 4h^2 x_i$$

$i=1$

$$y_0 + (4h^2-2)y_1 + y_2 = 4h^2 x_1$$

1

$$(4h^2-2)y_1 + y_2 = 4h^2 x_1 - 1$$

$i=2$

$$y_1 + (4h^2-2)y_2 + y_3 = 4h^2 x_2$$

$i=3$ $$y_2 + (4h^2-2)y_3 + y_4 = 4h^2 x_3$$

$i=N-1$ $$y_{n-2} + (4h^2-2)y_{n-1} + y_n = 4h^2 x_{n-1}$$

2

$i=N$ $$y_{n-1} + (4h^2-2)y_n + y_{n+1} = 4h^2 x_n$$

$$y_{n-1} + (4h^2-2)y_n = 4h^2 x_n - 2$$

$$\begin{pmatrix} 4h^2-2 & 1 & 0 & 0 & 0 & & \\ 1 & 4h^2-2 & 1 & 0 & 0 & & \\ 0 & 1 & 4h^2-2 & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & 1 & 0 & \\ & & & & 1 & 4h^2-2 & 1 \\ & & & & 0 & 1 & 4h^2-2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 4h^2x_1 - 1 \\ 4h^2x_2 \\ 4h^2x_3 \\ \vdots \\ 4h^2x_{n-1} \\ 4h^2 x_n - 2 \end{pmatrix}$$

Higher order BVPs

General case

Consider the third order linear $= n$

$$y''' + A(x)\,y'' + B(x)\,y' + C(x)\,y = D(x) \quad -(1)$$

$$y(a) = \gamma_1 \;;\; y'(a) = \alpha,\; y'(b) = \beta \quad -(2)$$

$$x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_{n-1} \quad x_n \quad x_{N+1}$$

$$x_0 = a, \qquad x_{N+1} = b$$

Approximate by difference

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \;;\; y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$y_i''' = \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3}$$

(1) becomes

$$\frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3} + A(x_i)\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right]$$

$$+ B(x_i)\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + C(x_i)\,y_i = D(x_i)$$

$$y_{i+2} + \left[2hA(x_i) + h^2 B(x_i) - 2\right] y_{i+1} + \left[2h^3 C(x_i) - 4hA(x_i)\right] y_i$$

$$+ \left[2 + 2hA(x_i) - h^2 B(x_i)\right] y_{i-1} - y_{i-2} = 2h^3 D(x_i)$$

HBVP

b.c. $y(a)=\gamma_1$, $y'(a)=\alpha$, $y'(b)=\beta$

$y_0=\gamma_1$, $\dfrac{y_1-y_{-1}}{2h}=\alpha$; $\dfrac{y_{n+1}-y_{n-1}}{2h}=\beta$

When one write (HBVP) at $i=0$

$$y_2+[2hA_0+h^2B_0-2]y_1+[2h^3C_0-4hA_0]y_0+$$
$$[2+2hA_0-h^2B_0]y_{-1}-y_{-2}=2h^3D_0$$

y_{-2}, y_{-1}

2 fictitous values

when $i=N$

$$y_{N+2}+[2hA_N+h^2B_N-2]y_{N+1}+[2h^3C_N-4hA_N]y_N+$$
$$[2+2hA_N-h^2B_N]y_{N-1}-y_{N-2}=2h^3D_N$$

Another 2 fictitous values y_{N+1}, y_{N+2}

We get 4 fictitous values. However we have 3 eqns from b.c.

Let $y'=p$

$$p''+A(x)p'+B(x)p+C(x)y=D(x)$$

$y(a)=\alpha$
$p(a)=\beta$
$p(b)=\gamma$

Discretize it

$\boxed{\text{Average}}$

$y'=p \Rightarrow \dfrac{y_i-y_{i-1}}{} \simeq \dfrac{1}{2}(p_i+p_{i-1})$ —— $\boxed{\text{AVP}}$

$$\frac{p_{i-1}-2p_i+p_{i+1}}{h^2}+A_i\left(\frac{p_{i+1}-p_{i-1}}{2h}\right)+B_i p_i+C_i y_i=D_i \quad — A'$$

Using $\boxed{AVP}$

$$y_{i-1}+\frac{h}{2}p_{i-1}-y_i+\frac{h}{2}p_i=0 \quad — (A)$$

(B)

$$a_i y_i+b_i p_{i-1}+c_i p_i+d_i p_{i+1}=h^2 D(x_i)$$

here $a_i = h^2 C_i$

$b_i = 1-\frac{h}{2}A_i$

$c_i = h^2 B_i - 2$

$d_i = 1+\frac{h}{2}A_i$

$y_0=\gamma_1$
$p_0=\alpha$

Write (A) + (B)

$i=1$

$$y_0+\frac{h}{2}p_0-y_1+\frac{h}{2}p_1=0 \quad — (C)$$

$$a_1 y_1+b_1 p_0+c_1 p_1+d_1 p_2=h^2 D(x_i) \quad — (D)$$

$$\Rightarrow \quad -y_1+\frac{h}{2}p_1=-\gamma_1-\frac{h}{2}\alpha \quad — (E)$$

$$a_1 y_1+c_1 p_1+d_1 p_2=-b_1\alpha+h^2 D(x_1) \quad — (F)$$

$i=2$

$$y_1+\frac{h}{2}p_1-y_2+\frac{h}{2}p_2=0$$

$$a_2 y_2+b_2 p_1+c_2 p_2+d_2 p_3=h^2 D_2$$

Introduce notation $\quad X_i=\begin{pmatrix} y_i \\ p_i \end{pmatrix} \quad ; \quad X_n=\begin{pmatrix} y_n \\ p_n \end{pmatrix}$

Using (A) + (B)

$$A_i X_{i-1} + B_i X_i + C_i X_{i+1} = D_i$$

$$X_i = \begin{pmatrix} y_i \\ p_i \end{pmatrix} \qquad i = 2, 3, \ldots N-1$$

$$A_i = \begin{pmatrix} 1 & h/2 \\ 0 & b_i \end{pmatrix}$$

$$B_i = \begin{pmatrix} -1 & h/2 \\ a_i & c_i \end{pmatrix}$$

$$C_i = \begin{pmatrix} 0 & 0 \\ 0 & d_i \end{pmatrix}$$

$$D_i = \begin{pmatrix} 0 \\ h^2 D(x_i) \end{pmatrix}$$ $D(x_i) \to$ known function

(E) + (F) can be form $\quad i = 1$

$$B_1 X_1 + C_1 X_2 = D_1$$

$$X_1 = \begin{pmatrix} y_1 \\ p_1 \end{pmatrix} \quad X_2 = \begin{pmatrix} y_2 \\ p_2 \end{pmatrix} \quad X_0 = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} -1 & h/2 \\ a_1 & c_1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & d_1 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} -\alpha - h/2\, \beta \\ -b_1\beta + h^2 D(x_i \end{pmatrix}$$

Now we have $i = 1, 2, \ldots N-1$

$(N-1) = M$ left with (I)

$i=N$

$$y_{N-1} + \frac{h}{2} p_{N-1} - y_N + \frac{h}{2} p_N = 0$$

$$a_N y_N + b_N p_{N-1} + C_N p_N + d_N p_{N+1} = h^2 D(x_N)$$

This can be put as as $p_N = \gamma$

$$A_N X_{N-1} + B_N X_N = D_N$$

$$A_N = \begin{pmatrix} y & \frac{h}{2} p_2 \\ 0 & C_N \end{pmatrix} \qquad B_N = \begin{pmatrix} -1 & 0 \\ a_N & d_N \end{pmatrix}$$

$$D_N = \begin{pmatrix} -\frac{\gamma}{2} h \\ h^2 D(x_N) - C_N \gamma \end{pmatrix} \qquad X_N = \begin{pmatrix} y_N \\ p_{N+1} \end{pmatrix}$$

Now we have total n = n for n unknown. $AX = D$

$$\begin{array}{c} i=1 \\ A_2 \\ \\ i=2, N-1 \\ \\ \\ i=N \end{array} \begin{pmatrix} B_1 & C_1 & 0 & 0 & 0 & & \\ A_2 & B_2 & C_2 & 0 & 0 & & \\ 0 & A_3 & B_3 & C_3 & 0 & & \\ & & \ddots & & & & \\ & & & & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & & 0 & A_N & B_N \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_{N-1} \\ X_N \end{pmatrix}$$

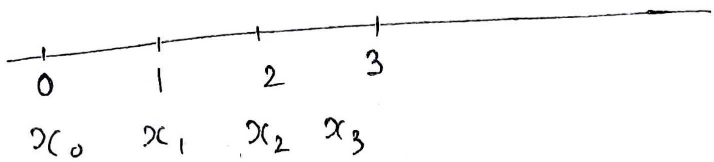
Block-Tridiagonal system

$$X = \begin{bmatrix} \begin{pmatrix} y_1 \\ p_1 \end{pmatrix} \\ \begin{pmatrix} y_2 \\ p_2 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} y_{N-1} \\ p_{N-1} \end{pmatrix} \end{bmatrix} \qquad D = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_{N-1} \\ D_N \end{pmatrix}$$

Example $y''' + y = x$

B.C. : $y(0) = 1$; $y'(0) = 0$ $y'(3) = 1$

$h = 1$



Usual discretization

$$y_{i+2} - 2y_{i+1} + 2y_i + 2y_{i-1} - y_{i-2} = x_i \quad — (1)$$

b.c.

$$\left.\begin{aligned} & y_0 = 1 \\ & y_1 - y_{-1} = 0 \\ & \frac{y_4 - y_2}{2} = 1 \end{aligned}\right) — (2)$$

(1) at $i=1$

$$y_3 - 2y_2 + 2y_1 + 2y_0 - y_{-1} = x_1 = 1$$

$i=2$ $\quad y_4 - 2y_3 + 2y_2 + 2y_1 - y_0 = x_2 = 2$

b.c. $\quad y_0 = 1$; $y_{-1} = y_1$; $y_4 = 2 + y_2$

given

Gives us → y_1, y_2, y_3 as unknowns

$i=0 \quad y_2 - 2y_1 + 2y_0 + 2y_{-1} - \boxed{y_{-2}} = x_0 = 0$. $\quad \square \to$ new fictitious

$i=3 \quad \boxed{y_5} - 2y_4 + 2y_3 + 2y_2 - y_1 = x_3 = 3$

$y_1, y_2, y_3, y_{-2}, y_5 \to$ 5 unknown.

and 4 eqns

$y' = p$, $y(0) = 1$

$p'' + y = x$; $p(0) = 0 \qquad p(1) = 1$

$$2y_{i-1} + p_{i-1} - 2y_i + p_i = 0 \quad \text{— (A)}$$

$$y_i + p_{i-1} - 2p_i + p_{i+1} = x_i \quad \text{— (B)}$$

$i=1$ (A) $\Rightarrow \quad 2y_1 - p_1 = 2$

(B) $\to \quad y_1 - 2p_1 + p_2 = 1$

(A), (B)
for $i = 2$

$$A_i X_{i-1} + B X_i + C_i X_i = D_i$$

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_2 \\ p_2 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

Tridiagonal system to solve

$$\begin{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix} \\ & & \end{pmatrix}\begin{pmatrix} \begin{pmatrix} y_1 \\ p_1 \end{pmatrix} \\ \begin{pmatrix} y_2 \\ p_2 \end{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 5 \end{pmatrix} \\ (\ ) \end{pmatrix}$$