System of eqns.

One requires solution of the system of simultaneous ODE rather than single equation

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \dots y_m)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \dots y_m)$$

$$\vdots$$

$$\frac{dy_m}{dx} = f_m(x, y_1, y_2, \dots y_m)$$

One need m initial condition at initial x

In Euler it is straight forward to solve simultaneously

$$\cancel{y_{n+1} = y_n + h f(x_n, y_n)}$$

$$y_1(x_{n+1}) = y_1(x_n) + h f_1(x_n, y_1(x_n), y_2(x_n), \dots y_m(x_n))$$

$$y_2(x_{n+1}) = y_2(x_n) + h f_2(x_n, y_1(x_n), y_2(x_n), \dots, y_m(x_n))$$

$$\vdots$$

$$y_m(x_{n+1}) = y_m(x_n) + h f_m(x_n, y_1(x_n), y_2(x_n), \dots, y_m(x_n))$$

x_0	$y_1(x_0)$	$y_2(x_0)$	$y_3(x_0)$	$\dots$	$y_m(x_0)$
x_1	$y_1(x_1)$	$y_2(x_1)$	$y_3(x_1)$	$\dots$	$y_m(x_1)$
$\vdots$					
x_n	$y_n(x_n)$	$y_2(x_n)$	$y_3(x_n)$	$\dots$	$y_m(x_n)$

$$\frac{dy_1}{dx} = -0.5 y_1$$

$$\frac{dy_2}{dx} = 4 - 0.3 y_2 - 0.1 y_1$$

$$x = 0 ; \; y_1 = 4, \; y_2 = 6$$

$$y_1(0.5) = 4 + [-0.5 \times 4] 0.5 = 3$$

$$y_2(0.5) = 6 + [4 - 0.3 \times 6] 0.5 = 6.9$$

One can do in same way.

If one has to ~~do~~ use RK method,
Then it is a bit similar way

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2)$$

3^{rd} order (one version)

$$y_{n+1} \simeq y_n + \frac{h}{4}(k_1 + 3k_3)$$

$$k_1 = f(x, y)$$

$$k_2 = f\left(x + \frac{1}{3}h, y + \frac{1}{3}hk_1\right)$$

$$k_3 = f\left(x + \frac{2}{3}h, y + \frac{2}{3}hk_2\right)$$

Therefore

$$y_1(x_{n+1}) = y_1(x_n) + \frac{h}{4}(k_1 + 3k_3)$$

$$k_1 = f_1(x_n, y_1(x_n), y_2(x_n))$$

$$k_2 = f_2\left(x_n + \frac{1}{3}h, y_1(x_n) + \frac{1}{3}hk_1, \; y_2(x_n) + \frac{1}{3}hl_1\right)$$

$$k_3 = f_3\left(x_n + \frac{2}{3}h, y_1(x_n) + \frac{2}{3}hk_2, \; y_2(x_n) + \frac{2}{3}hl_2\right)$$

$$y_2(x_{n+1}) = y_1(x_n) + \frac{h}{4}(l_1 + 3l_3)$$

$$l_1 = f_1(x_n, y_1(x_n), y_2(x_n))$$

$$l_2 = f_2\left(x_n + \frac{1}{3}h, y_1(x_n) + \frac{1}{3}hk_1, \; y_2(x_n) + \frac{1}{3}hl_1\right)$$

$$l_3 = f_2\left(x_n + \frac{2}{3}h, y_1(x_n) + \frac{2}{3}hk_2, \; y_2(x_n) + \frac{2}{3}hl_2\right)$$

Higher order =ns

One can convert them into system of simultaneous ODE.

Let

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \ldots \frac{d^{n-1} y}{dx^{n-1}}\right) \quad \text{—(1)}$$

With initial conditions

$y(x_0)$, $\frac{dy}{dx}(x_0)$, $\frac{d^2y}{dx^2}(x_0)$, ..., $\frac{d^{n-1}y}{dx^{n-1}}(x_0)$ are known.

One can do a trick

$$u_1 = y$$

$$u_2 = \frac{dy}{dx}$$

$$u_3 = \frac{d^2y}{dx^2}$$

$$\vdots$$

$$u_n = \frac{d^ny}{dx^n}$$

One modify

$$u_1(x_0) = y(x_0)$$

$$u_2(x_0) = y'(x_0)$$

$$\vdots$$

$$u_n(x_0) = y^{n-1}(x_0$$

Using them one get

$$\frac{du_1}{dx} = f_1(x, u_1, u_2, \ldots u_n)$$

$$\frac{du_2}{dx} = f_2(x, u_1, y_2, \ldots, u_n)$$

$$\vdots$$

$$\frac{du_m}{dx} = f_m(x, u_1, u_2, \ldots u_n)$$

System of high order equations are treated in the same way. One can convert higher order ODE, into a system of first order ODE. Then one can solve these system of simultaneous ODE.

Example

$$\frac{d^2y}{dx^2} + 4y\frac{dy}{dx} + x^2 y + x = 0$$

$$y(0) = 1 \quad ; \quad \frac{dy}{dx}(0) = 2$$

One can do a transform

$u_1 = y$	$u_1(0) = y(0) = 1$
$u_2 = \frac{dy}{dx}$	$u_2(0) = \frac{dy}{dx}(0) = 2$

$$\frac{du_1}{dx} = \frac{dy}{dx} = u_2$$

$$\frac{du_2}{dx} = \frac{d^2y}{dx^2} = -4y\frac{dy}{dx} - x^2 y + x$$

$$= -4u_1u_2 - x^2u_1 - x$$

$\frac{du_1}{dx} = u_2$	I.C.
$\frac{du_2}{dx} = -(4u_1u_2 + x^2u_1 + x)$	$u_1(0) = 1$ $u_2(0) = 2$

One can then simply solve these ~~2~~ system of simultaneous 1^{st} order ODE xample

$$\frac{d^3y}{dx^3} \cancel{+} - 2\frac{d^2y}{dx^2} + x = 0$$

$$\left.\begin{aligned} y(x_0) &= 0 \\ \frac{dy}{dx}(x_0) &= 3 \\ \frac{d^2y}{dx^2}(x_0) &= -2 \end{aligned}\right\} \text{random}$$

$u_1 = y$	$u_1 = 0$
$u_2 = \frac{dy}{dx}$	$u_2 = 3$
$u_3 = \frac{d^2y}{dx^2}$	$u_3 = -2$

$$\frac{du_3}{dx} = \frac{d^3y}{dx^3} = \frac{2d^2y}{dx^2} - x = 2u_3 - x$$

$$\frac{du_3}{dx} = 2u_3 - x \qquad u_3 = -2$$

$$\frac{du_2}{dx} = u_3 \qquad u_2 = 3$$

$$\frac{du_1}{dx} = u_2 \qquad u_1 = 0$$

General nonlinear, ODE exact soln is complicated
Numerical method can be used to estimate the solution.

However, it is not straight forward
Need to use proper method and h size.

<u>Stiffness</u>

Stiffness is a special problem that may arise in the solution of the ODE

A stiff system is one involving rapidly changing component together with slowly changing one.

Concept is easy to understand. However not trivial.
Stiffness is an built a property of the system

Let see an example

$$\frac{dy}{dx} = -cy \quad y(0) = 1 \qquad - (A)$$

$c > 0$

Exact soln is $y(x) = e^{-cx}$

Seen in most of the decays ?

As x goes to ∞, soln should goes to zero.

Lets try to use
Eulers method to solve (A)

$$y_{n+1} = y_n + hf$$

$$y_1 = y_0 + h(-c y_0) = y_0(1 - hc)$$

$$y_2 = y_1 + h(-c y_1)$$

$$= y_0(1-hc) - hc\, y_0 (1-hc)$$

$$\cdots$$

$$= y_0 - hc\, y_0 - hc\, y_0 + (hc)^2 y_0$$

$$= y_0 [1 - 2hc + (hc)^2)$$

$$= y_0 (1-hc)^2$$

$\therefore$ Induction gives

$$y_n = y_0 (1-hc)^n$$

As $y(0) = 1$

$$y_n = (1-hc)^n \quad - \quad \text{B}$$

As $c > 0$, h is always > 0

Exact soln. dictates that $x_n \to 0$ as $n \to \infty$

This statement is fulfilled by the B for the following situation

$$|1-hc| < 1$$

$$-1 < 1-hc < 1$$

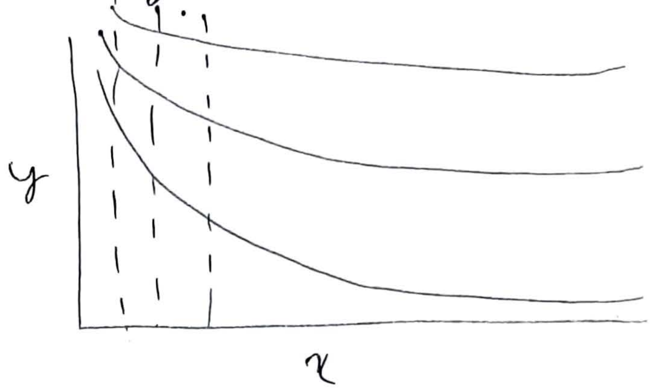
$$-2 < -hc < 0$$

$$\boxed{\frac{2}{c} > h > 0}$$

This gives restriction on the step size

Large C → smaller should be h

small C → one can have large h



One need to use different h based on the C

Therefore $\boxed{h < \frac{2}{C}}$ is stability condition and

tell about the method

If C is very large, one need to be very careful.

⇒ Explicit method leads to some stability condition

How to improve stability condition

What about the implicit method

Simple case is implicit (backward) Euler method

$$y_{n+1} = y_n + h f(y_{n+1}, x_{n+1})$$

need to know !

$$y_{n+1} = y_n + h(-C y_{n+1})$$

$$y_{n+1} = y_n - hC y_{n+1}$$

$$y_{n+1} = \frac{y_n}{1+hC}$$

$$y_1 = \frac{1}{(1+hc)} y_0$$

$$y_2 = \frac{1}{(1+hc)} y_1 = \frac{1}{(1+hc)^2} y_0$$

$$\vdots$$

$$y_n = \frac{1}{(1+hc)^n} y_0$$

As $n \to \infty$ $y_n \to 0$

$$\left(\frac{1}{1+hc}\right)^n < 1$$

C is +ve
h is +ve

$(1+hc)$ is +ve and large than 1

$$\left|\frac{1}{1+hc}\right| < 1 \text{ always hold}$$

This suggest unconditionally stable!

Implicit methods are more stable method.

Advantage of Implicit

↳ stable, any step size

fewer steps

Disadvantage

↳ One should know the expected value in the implicit method (not a good thing).

One simply get a good guess with few iteration it will converge.

If the system is stiff, better to use implicit method..

Explicit → unstable & fast

Implicit → stable & slow

Example

$$\frac{dy}{dx} = -1000y + 3000 - 2000e^{-x}$$

IC $y(0) = 0$

$$y(x) = 3 - 0.998e^{-1000x} - 2.002e^{-x}$$

Analytical soln.

Soln. has different componet, ~~or other solution~~

$\boxed{e^{-1000x}}$ → transient term which dominate is the short and is major stability constraint.

Quickly this transient term die out and soln. depends on the slow exponential (e^{-x})

Another method, which is used to solve the ODEs specially the stiff one are implicit method (based on the work of Gear).

They are called Gear's method

$$\frac{dy}{dx} = f(x,y) \quad y(x_0) = y_0$$

Order

1st

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

2nd

$$y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2}{3} h f(x_{n+1}, y_{n+1})$$

3rd

$$y_{n+1} = \frac{18}{11} y_n - \frac{9}{11} y_{n-1} + \frac{2}{11} y_{n-2} + \frac{6}{11} hf(x_{n+1}, y_{n+1})$$

4th

$$y_{n+1} = \frac{48}{25} y_n - \frac{36}{25} y_{n-1} + \frac{16}{25} y_{n-2} - \frac{3}{25} y_{n-3} + \frac{12}{25} hf(x_{n+1}, y_{n+1})$$

5th

$$y_{n+1} = \frac{300}{137} y_n - \frac{300}{137} y_{n-1} + \frac{200}{137} y_{n-2} - \frac{75}{137} y_{n-3} + \frac{12}{137} y_{n-4}$$
$$+ \frac{60}{137} h f(x_{n+1}, y_{n+1})$$

One can also use them to solve non-linear =ns.

Keep reducing the step size till implicit =ns can be solved.