Partial differential equations (PDE) arises in all fields of Science.

PDE is an equation stating relationship b/w a function of 2 or more independent variable & partial derivatives of this function w.r.t. those independent variables.

Example

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{[2D Laplace]}$$

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2} \quad \text{[1D diffusion eqn]}$$

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \quad \text{[1D wave eqn]}$$

Second order PDE.

$$R(x,y)\frac{\partial^2 u}{\partial x^2} + S(x,y)\frac{\partial^2 u}{\partial x \partial y} + T(x,y)\frac{\partial^2 u}{\partial y^2} + g\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0$$

One can also see this as in literature

$$R(x,y)\,u_{xx} + S(x,y)\,u_{xy} + T(x,y)\,u_{yy} + g(x,y,u,u_x,u_y) = 0$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}; \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}; \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}; \quad u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}$$

u is dependent variable; x, y are independent variable

R, S, T are continuous function of (x, y)

Classification of Second order PDE

Consider $Lu + g(x, y, u, u_x, u_y) = 0$

where

$$L = R u_{xx} + S u_{xy} + T u_{yy}$$

One divide into 3 categories

→ $S^2 - 4RT > 0$ Hyperbolic PDEs

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{wave eqn}$$

→ $S^2 - 4RT = 0$ Parabolic PDE.

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad \text{Diffusion eqn}$$

→ $S^2 - 4RT < 0$ Elliptic PDEs

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{Laplace eqn}$$

Finite Difference Approximation to Parabolic PDE

When a function U and its derivative are single-valued finite and continuous function of x, then

Taylor series

$$u(x+h) = u(x) + h u'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \ldots + \underset{\text{remainder}}{R} \quad \text{—— (1)}$$

u → dependent variable

x → independent variable

$$u(x-h) = u(x) - h\,u'(x) + \frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u'''(x) + \ldots + R \quad —(2)$$

(1) and (2) are added

$$u(x+h) + u(x-h) = 2\left[u(x) + \frac{h^2}{2!}u''(x)\right] + O(h^4)$$

as divided by h^2

One obtain approximation

$$u''(x) \simeq \frac{u(x+h) + u(x-h) - 2u(x)}{h^2} + O(h^2)$$

$$\simeq \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} + O(h^2) \quad —(3)$$

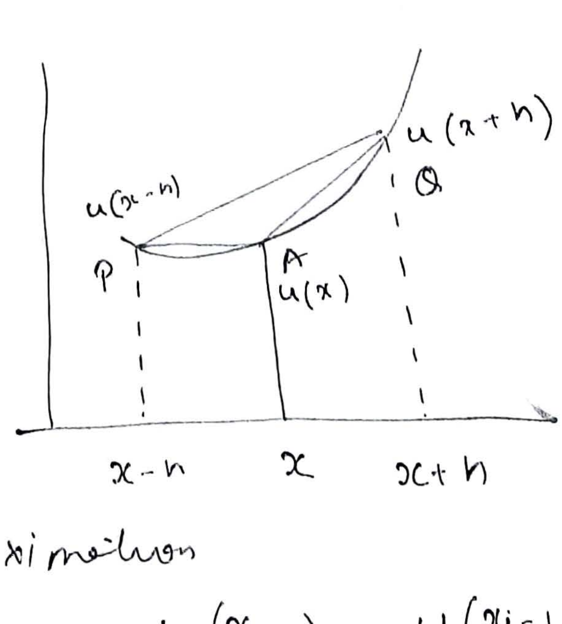
(1) − (2)

$$u(x+h) - u(x-h) = 2h\,u'(x) + \frac{h^3}{3!}u'''(x) + O(h^3)$$

divide by h

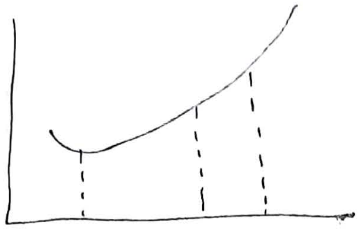
$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2) \quad —(4)$$

This approximates the slope of the tangent at A by slope of the chord PQ.

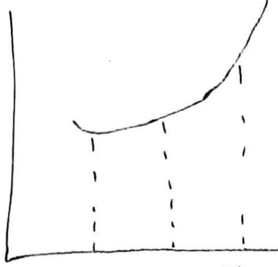


This is known as "Central difference" approximation

One see → $u'(x) = \cancel{u(x+h)} \frac{u(x_{i+1}) - u(x_{i-1})}{(x_{i+1}) - (x_{i-1})}$



unequal spacing



equal spacing

Generally we go by equal spacing but in some cases unequal spaces are required

One might also approximate the slope of tangent at A by AQ

$$u'(x) \simeq \frac{u(x+h) - u(x)}{h}$$ → forward approximation

or by PA

$$u'(x) \simeq \frac{u(x) - u(x-h)}{h}$$ → backward approximation

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Let's consider first order PDE

$$A \frac{\partial u(x,t)}{\partial x} + B \frac{\partial u}{\partial t}(x,t) = 0 \quad \text{——— (5)}$$

Depends both on space and time

$u(x,0) = u_0(x)$ Initial condition (at $t = t_0 = 0$)

$u(0,t) = u_1(t)$ boundary condition. (at $x = x_0 = 0$)

Discretize (5)

x is discretized as $x_{i+1} = x_i + h$; $i = 0, 1, 2, \ldots N$

t is discretized as $t_{j+1} = t_j + k$; $j = 0, 1, 2, \ldots N$

Lets discretize this using explicit method

$t_j = t_0 + jk$

$x_i = x_0 + ih$

Using Forward time and central space

$$\frac{u_{i,j+1} - u_{i,j}}{k} + O(k) = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2)$$

This leads to

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(k+h^2)$$

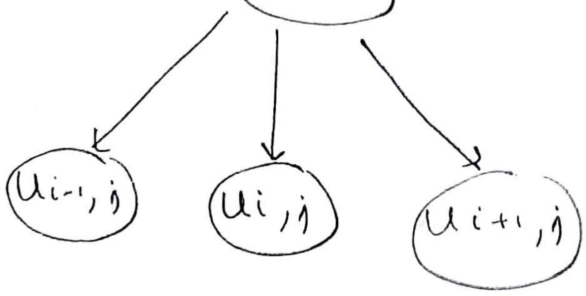
One can further simplify as

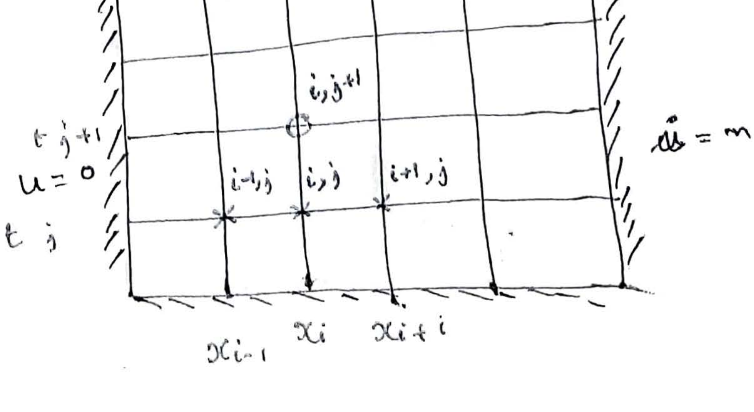
$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i+1,j}$$

$\boxed{\lambda = k/h^2}$ grid parameter

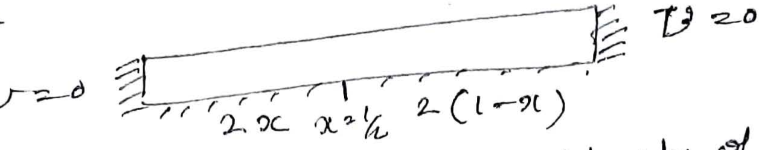
Compute all time solution at various grid points.

Two-level method   Schmidt Method





Knowing the value of previous time step (j^{th} level) one can calculate ($j+1$) level. Explicit in Nature fee Example



Ends of rod are kept in contact with blocks of melting ice and the initial temperature distribution in non dimensional form is

(a) $U = 2x \qquad 0 \le x \le \frac{1}{2}$

b) $U = 2(1-x) \qquad \frac{1}{2} \le x \le 1$

Soln $\quad \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

B.C. → $u = 0$ at $x = 0$ and 1 for all $t > 0$

I.C. → $\left.\begin{array}{l} U = 2x \text{ fo } 0 \le x \le \frac{1}{2} \\ u = 2(1-x) \text{ for } \frac{1}{2} \le x \le 1 \end{array}\right\} t = 0$

→ $h = \frac{1}{10}$; $k = \frac{1}{1000}$, $\lambda = k/h^2 = \frac{1}{10}$

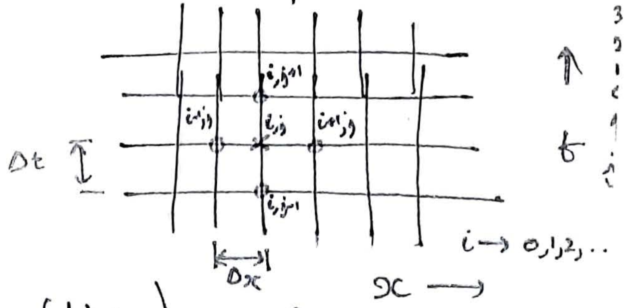
$$u_{i,j+1} = \frac{1}{10}\left(u_{i-1,j} + 8\, u_{i,j} + u_{i+1,j}\right)$$

i =	0	i = 1 →	2	3	4	5	6	7	8	9
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
j = 0	0	0.2	0.4	0.6	0.8	1.0	0.8	0.6	0.4	0.2
j = 1	0	0.2	0.4	0.6	0.8	0.96	0.8	0.6	0.4	0.2
2	0	0.2	0.4	0.6	0.7960	0.928	0.796	0.6	0.4	0.2
3	0	0.2	0.4	0.5991	0.7896	0.9016	0.7896	0.5996	0.4	0.2
4	0	0.2	0.4	0.5986	0.7818	0.8792	0.7818	0.5986	0.4	0.2
5	0	0.2	0.3999	0.5971	0.7732	0.8597	0.7732	0.5971	0.3999	0.2
6	0									
7										

$$u(x,t)\big|_{(x_i,t_j)} = u(x_i, t_j) = u_{i,j}$$

$$\frac{\partial}{\partial t} u(x,t) = \frac{u_{i,j+1} - u_{i,j}}{\Delta t} \quad \text{forward time}$$

$$\frac{\partial}{\partial x} u(x,t) = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \quad \text{Central space}$$



(5) Becomes

$$A\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right) + B\left(\frac{u_{i,j+1} - u_{i,j}}{\Delta t}\right) = 0$$

One can simplify as

$$u_{i,j+1} - u_{i,j} + \frac{A}{B}\,\frac{1}{2}\,\frac{\Delta t}{\Delta x}\left(u_{i+1,j} - u_{i-1,j}\right) = 0$$

Initial condition

$$u(x,0) = u_0(x)$$

$$u(x_i,0) = u_0(x_i)$$

If $u_0(x) = f(x)$, then $u_0(x_i) = f(x_i)$ or f_i for all i at $j=0$

Similarly $u(0,t) = u_1(t) = g(t)$

$$u(0,t_j) = g(t_j) = g_j$$

$u_{0,j} = g_j$ for all j at $i=0$

Lets do Higher order derivative
Start with Parabolic case

Consider Heat conduction $= n$

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

$K \to$ thermal conductivity
Constant,

K has a dimension. To make soln. universally valid

Let us use $x' = x/L$ — L represent length of rod.

$u' = u/u_0$ — $u_0 \to$ some initial temperature at zero time

Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'} \cdot \frac{dx'}{dx} = \frac{\partial u}{\partial x'} \cdot \frac{1}{L}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x'}\left(\frac{1}{L}\frac{\partial u}{\partial x}\right)\frac{d}{d}$$

$$= \frac{1}{L^2}\frac{\partial^2 u}{\partial x'^2}$$

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{L^2}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x'^2}$$

$$\frac{L^2}{K}\frac{\partial (u' u_0)}{\partial t} = \frac{\partial^2 (u' u_0)}{\partial x'^2}$$

$$\Rightarrow \frac{L^2}{K}\frac{\partial u'}{\partial t} = \frac{\partial^2 u'}{\partial x'^2}$$

writing $t' = Kt/L^2$

one get

$$\frac{\partial u'}{\partial t'} = \frac{\partial^2 u'}{\partial x'^2}$$

Dropping prime $\to$ $\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}}$

If one use $h = \frac{1}{10}$, $k = \frac{1}{100}$

$$\lambda = k/h^2 = 1$$

Then

	0	0.1	0.2	0.3	0.4	0.5
t= 0.00	0	0.2	0.4	0.6	0.8	1.0
0.01	0	0.2	0.4	0.6	0.8	0.6
0.02	0	0.2	0.4	0.6	0.4	0.0
0.03	0	0.2	0.4	0.2	1.2	-0.2
0.04	0	0.2	0	1.4	-1.2	2.6

Solutions are not correct. The reason for this is that Explicit methods are stable only for $0 \leq \lambda \leq 1/2$

Implicit method for Parabic PDEs

Consider $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

use $\downarrow$ Backward $\downarrow$ $\downarrow$ Central $\downarrow$

$$\frac{u_{i,j} - u_{i,j-1}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(k + h^2)$$

For convenience lets write j as $j+1$, then $j-1$ will be j.

Just a time step change

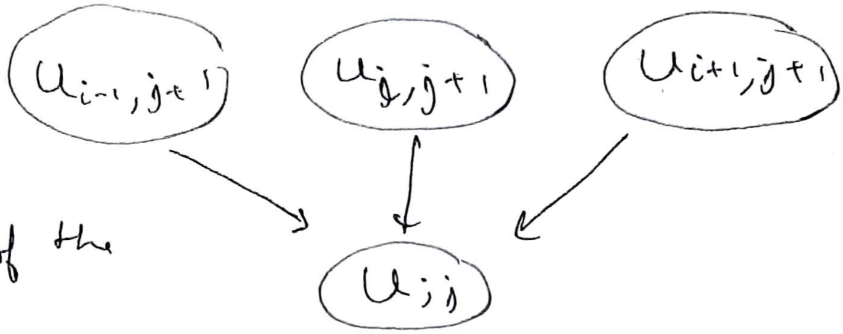
$$\frac{u_{i,j+1}-u_{i,j}}{k} = \frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}$$

Simplify

$$\Rightarrow -\lambda u_{i-1,j+1} + (1+2\lambda)u_{i,j+1} - \lambda u_{i+1,j+1} = u_{i,j}$$

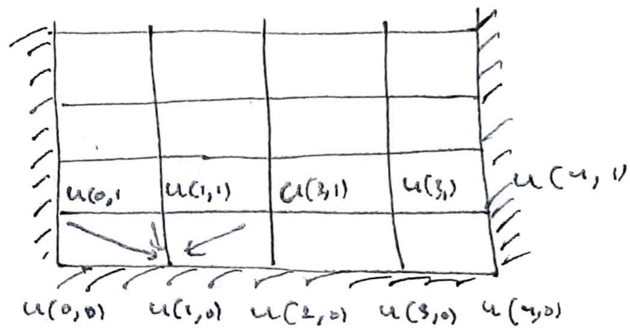
Two-level method.

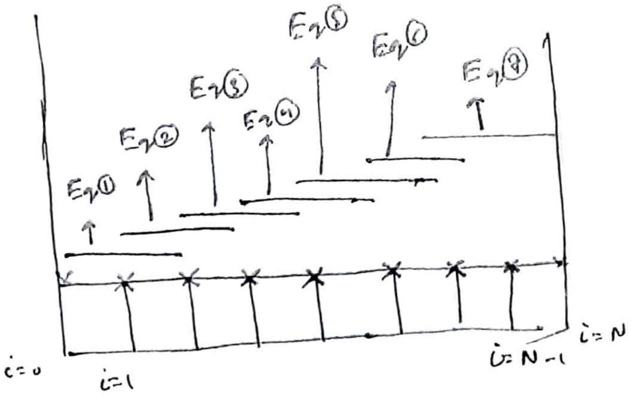
Involve more terms of level above.



Shows ~~explicit~~ implicit nature of the scheme

Implicit → we get ~~a~~ system of eqns, as we have more quantities at level above than below





One simply march past along the points in this method.

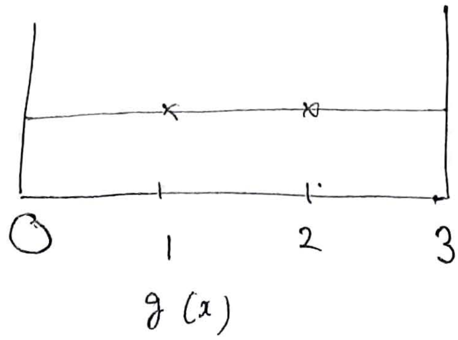
At i=1 and i=N-1,

we get 2 unknown and 1 known for coming from the boundary conditions

Example $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

$u(x,0) = f(x)\,, \quad 0 \le x \le 3$

$u(0,t) = g(x)$



$$-\lambda u_{i-1,j+1} + (1+2\lambda)\, u_{i,j+1} - \lambda u_{i+1,j+1} = u_{i,j}$$

$j = 0$

$$-\lambda u_{i-1,1} + (1+2\lambda)\, u_{i,1} - \lambda u_{i+1,1} = u_{i,0}$$

$i=1$ $\quad -\lambda u_{0,1} + (1+2\lambda)\, \underline{u_{1,1}} - \lambda \underline{u_{2,1}} = u_{1,0}$

$i=2$ $\quad -\lambda \underline{u_{1,1}} + (1+2\lambda)\, \underline{u_{2,1}} - \lambda u_{3,1} = u_{2,0}$

<u>unknowns</u>

One can simply solve for $u_{1,1}$ and $u_{2,1}$

Explicit and Implicit nature of the equation due to the discretization scheme.

Forward with time $\rightarrow$ Explicit

Backward with time $\rightarrow$ Implicit

Approximate the PDE by discretization

We throw away terms containing higher order

Local truncation error

Second order $\rightarrow O(h^2)$

First order $\rightarrow O(k)$

Combining $\quad O(k + h^2)$