Elliptic PDEs

Best known are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g \quad \text{Poisson's equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{Laplace's equation}$$

u - dependent variable; x, y are independent variable

The domain of integration of an elliptic equation is

→ Area bounded by a closed curve C (in 2D)

→ Volume bounded by closed surface S (in 3D)

Possible Boundary value problem

1) Solve $\nabla^2 u = 0$ in Ω

known → $u = f$ on $\partial\Omega$ (boundary of Ω) $\boxed{\text{Dirichlet boundary condition}}$

f is specified on the boundary

2) Solve $\nabla^2 u = 0$ in Ω

where $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$ $\boxed{\text{Neumann boundary Condition}}$

→ known

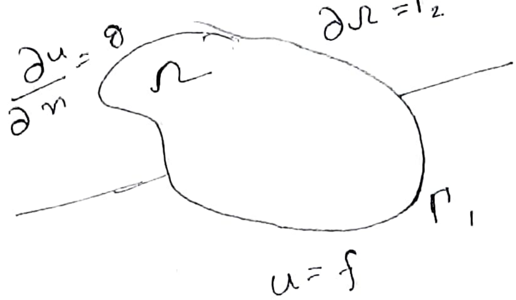
Derivative normal to the boundary is specified

3) Solve $\nabla^2 u = 0$ in Ω

$$\alpha u + \beta \frac{\partial u}{\partial n} = h \text{ on } \partial\Omega$$

$\boxed{\text{Mixed boundary Condition or Robin problem}}$

Combination of function and its normal derivative is specified



→ $u = f$ on Γ_1
and $\frac{\partial u}{\partial n} = g$ on Γ_2

$\Gamma = \Gamma_1 \cup \Gamma_2$

→ Having both u and $\frac{\partial u}{\partial n}$ at same point simultaneously is not possible.

Note :-
For an elliptic PDE, solution domain must be closed, and continuous boundary condition must be specified along the entire physical boundary. The boundary conditions may be of three types:-

1) Dirichlet boundary condition
2) Neumann boundary condition
3) Mixed boundary condition

"well-posed" problem.

Finite Difference Scheme

Laplace equation - 2D

$$\nabla^2 u = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad -①$$

Using central

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\delta y)^2} = 0 \quad -②$$

δx → correspond to x grid length

δy → correspond to y grid length.

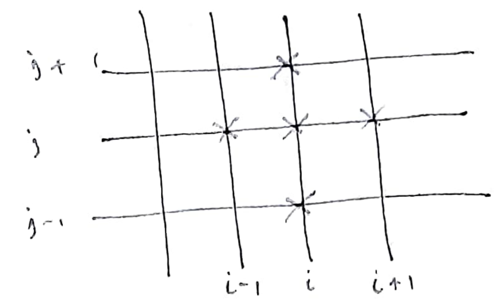
Let $\delta x = \delta y = h$, then (2) reduces to

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = 0$$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

Standard 5 point formulae

Demand four values



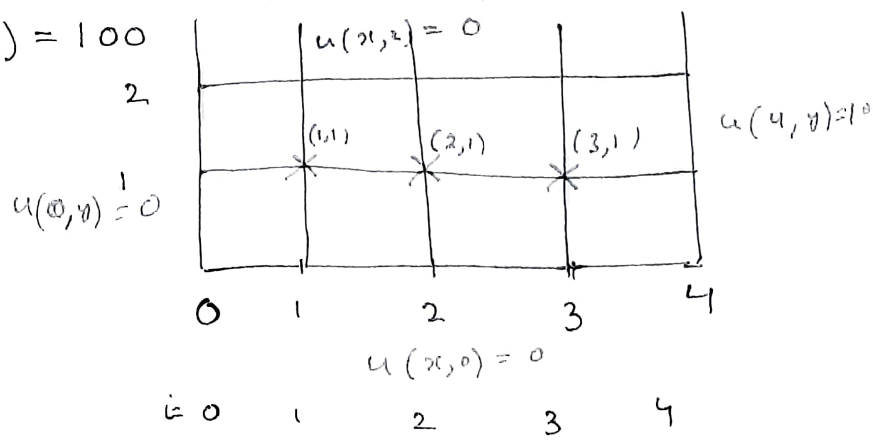
Example

Solve $\nabla^2 u = 0$

BC. $u(x,0) = 0$; $u(0,y) = 0$

$u(x,2) = 0$; $u(4,y) = 100$

$h = 1$



$\overset{i\ j}{(1,1)}$

$$u_{2,1} + \underbrace{\overset{=0}{u_{0,1}}} + \underbrace{\overset{=0}{u_{1,2}}} + \underbrace{\overset{=0}{u_{1,0}}} - 4u_{1,1} = 0$$

✓ → unknown

= → known

$i = 2, j = 1$

$$u_{3,1} + u_{1,1} + \underbrace{u_{2,2}}_{=0} + \underbrace{u_{2,0}}_{=0} - 4u_{2,1} = 0$$

$i = 3, j = 1$

$$\underbrace{u_{4,1}}_{=100} + u_{2,1} + \underbrace{u_{3,2}}_{=0} + \underbrace{u_{3,0}}_{=0} - 4u_{3,1} = 0$$

$$u_{2,1} - 4u_{1,1} = 0$$
$$u_{3,1} + u_{1,1} - 4u_{2,1} = 0$$
$$u_{2,1} - 4u_{3,1} = -100$$

$u_{1,1}, u_{2,1}, u_{3,1}$

3 unknown

3 equation

1. Discretize
2. Identify points
3. run system at each grid point
4. Solve the system.

→ What if the right-hand system is non-zero

Poisson equation

Solve $\nabla^2 u = f(\bar{x})$; $\bar{x} \in \Omega$

2D →
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y)$$

$\phi \rightarrow \phi(x, y)$

) will see later.

A bit similar to the Laplace.

Let assume δx & δy to be different

(2) become

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + \beta^2 u_{i,j+1} - 2\beta^2 u_{i,j} + \beta^2 u_{i,j-1} = 0$$

Rearranging

$$u_{i+1,j} + \beta^2 u_{i,j+1} + u_{i-1,j} + \beta^2 u_{i,j-1} - 2(1+\beta^2) u_{i,j} = 0$$

$$\beta = \frac{\delta x}{\delta y}$$

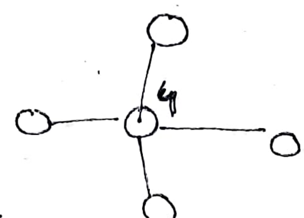
$$u_{i,j} = \frac{u_{i+1,j} + \beta^2 u_{i,j+1} + u_{i-1,j} + \beta^2 u_{i,j-1}}{2(1+\beta^2)}$$

Implicit Nature of the finite difference equation is typical of the finite difference approximation of elliptic PDEs.

Comment

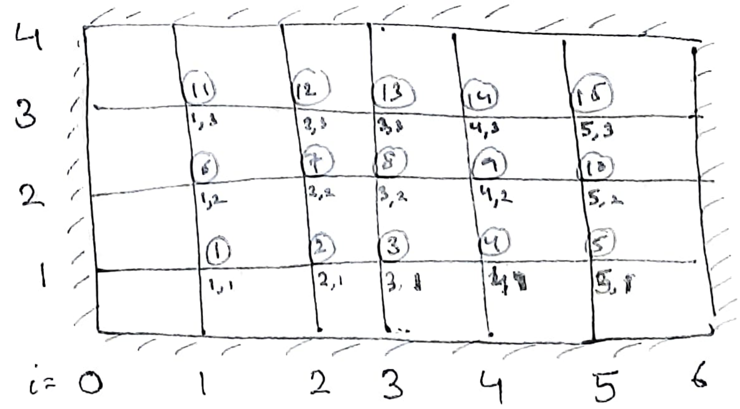
→ Values of β greater than 1 tend to produce less accurate solutions

→ For a grid having β to to 1, solution at every point is the arithmetic average of the solutions at the four neighbouring points.



→ Truncation Error is $O(\delta x^2) + O(\delta y^2)$. Total error decreases quadratically as δx and $\delta y \to 0$. Second order accurate in space!

Method of arranging / counting



$u_{i,j} = v_{i+(n-1)(j+1)}$

$n = 6$

$u_{1,1} = v_{1+(6-1)(1-1)} = v_1$

$u_{2,1} = v_{2+(6-1)(1-1)} = v_2$

$\vdots$

$u_{5,1} = v_{5+(6-1)(1-1)} = v_5$

$u_{1,2} = v_{1+(6-1)(2-1)} = v_6$

$\vdots$

$u_{5,2} = v_{5+(6-1)(2-1)} = v_{10}$

$u_{3,3} = v_{3+(6-1)(3-1)} = v_{13}$

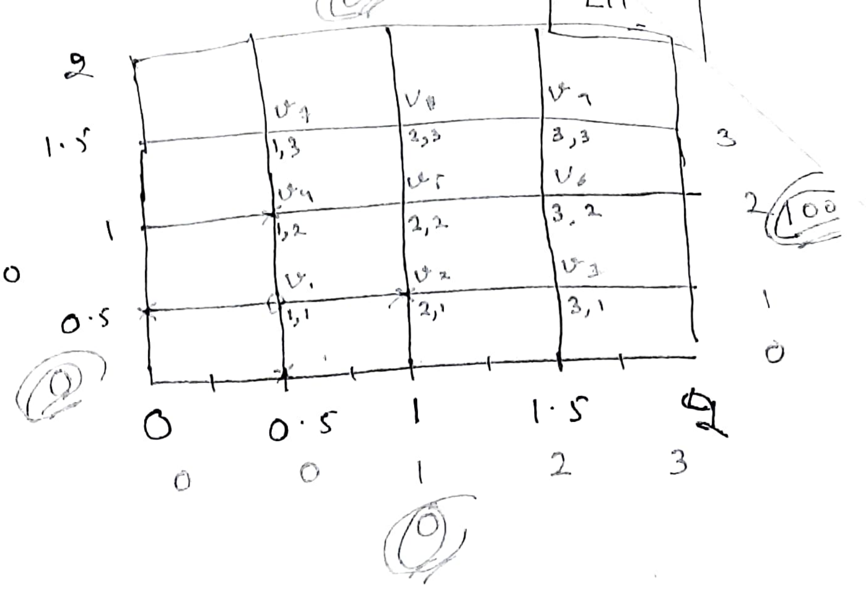
Example

$$\nabla^2 u = 0$$

$$u(x,0) = 0;\quad u(0,y) = 0$$

$$u(x,2) = 0,\quad u(2,y) = 100$$

$$\delta x = \delta y = 0.5$$



$$u_{i,j} = \frac{u_{i+1,j} + \beta^2 u_{i,j+1} + u_{i-1,j} + \beta^2 u_{i,j-1}}{2(1+\beta^2)} \qquad \beta = 1$$

$$u_{i,j} = \frac{u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}}{4}$$

$$v_1 = \frac{v_2 + v_4 + 0 + 0}{4} \qquad -v_2 - v_4 + 4v_1 = 0$$

$$4v_1 - v_2 - v_4 = 0$$

$$v_2 = \frac{v_1 + v_3 + v_5}{4} \qquad -v_1 + 4v_2 - v_3 - v_5 = 0$$

$$v_3 = \frac{v_2 + v_6 + 100}{4} \qquad -v_2 + 4v_3 - v_6 = \cancel{0}\ 100$$

$$v_4 = \frac{v_1 + v_7 + v_5}{4} \qquad -v_1 + 4v_4 - v_5 - v_7 = 0$$

$$v_5 = \frac{v_1 + v_2 + v_4 + v_6}{4} \qquad -v_1 - v_2 - v_4 + 4v_5 - v_6 = 0$$

$$v_6 = \frac{v_3 + v_5 + v_9 + 100}{4} \Rightarrow -v_3 - v_5 + 4v_6 - v_9 = 100$$

$$4v_7 = v_4 + v_8 \Rightarrow -v_4 + 4v_7 + v_8 = 0$$

$$4v_8 = v_7 + v_9 + v_5 \Rightarrow -v_7 + 4v_8 - v_5 - v_9 = 0$$

$$4v_9 = v_8 + v_6 + 100 \Rightarrow -v_6 - v_8 + 4v_9 = 100$$

$$
\underbrace{\begin{pmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
-1 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -4
\end{pmatrix}}_{A}
\underbrace{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{pmatrix}}_{v}
=
\underbrace{\begin{pmatrix} 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 100 \\ 0 \\ 0 \\ 100 \end{pmatrix}}_{G}
$$

l

Spase mt

$$A \cdot v = G$$

Poisson Example

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -20$$

5 point formula

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\delta y)^2} = -20$$

Assume $\delta x = \delta y$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -20(\delta x)^2$$

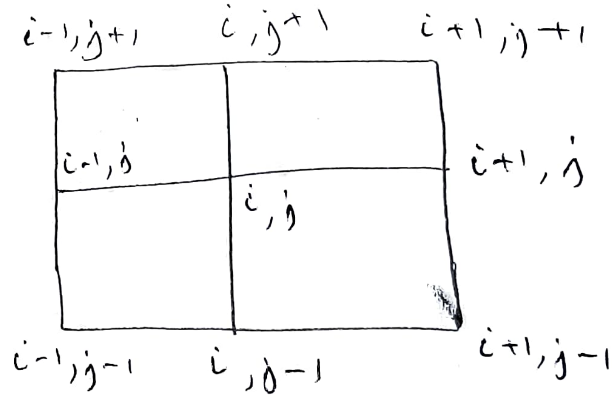
Assume $\delta x = 1/2$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -5$$

One simply end up with system of equation as we get with Laplace eqn

$$A \cdot v = G$$

Nine-point Method



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x, y)$$

$$\frac{\delta x}{\delta y} = \frac{1}{\beta}$$

$$-4\left(\frac{1}{\beta} + \beta\right) u_{i,j} + \left(\frac{2}{\beta} - \beta\right) u_{i+1,j} + \left(2\beta - \frac{1}{\beta}\right) u_{i,j+1}$$

$$+ \left(\frac{2}{\beta} - \beta\right) u_{i-1,j} + \left(2\beta - \frac{1}{\beta}\right) u_{i,j-1} + \frac{1}{2}\left(\frac{1}{\beta} + \beta\right)\big(u_{i+1,j+1} + u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j-1}\big)$$

$$- \frac{\delta x \, \delta y}{4}\left(F_{i+1,j} + F_{i,j+1} + F_{i-1,j} + F_{i,j-1}\right) = 0$$

$\beta = 1$, $\delta x = \delta y = h$ and F = constant

$$-8u_{i,j}+(u_{i+1,j}+u_{i,j+1}+u_{i-1,j}+u_{i,j-1}+u_{i+1,j+1}+u_{i-1,j+1}+u_{i+1,j-1}+u_{i-1,j-1})-h^2F=0$$

got using Galerkin Weighted Residual Approach.

In most of literature nine-point formula is also written as

$$\nabla^2 u = 0$$

$$\frac{1}{6h^2}\Big[4u_{i+1,j}+4u_{i-1,j}+4u_{i,j+1}+4u_{i,j-1}+u_{i+1,j+1}+u_{i-1,j+1}+u_{i+1,j-1}+u_{i-1,j-1}-20u_{i,j}\Big] = \cancel{F(x,y)}\ 0$$

Nine point formula in general problem has the same order error term as five-point formula $(O(h^2))$]. However in special case, u is a harmonic function, nine point formula is of order $O(h^6)$.

General case

$$\nabla^2 u + fu = g$$

$u(x,y) \to$ known on boundary Ω

$$-u_{i+1,j}-u_{i-1,j}-u_{i,j+1}-u_{i,j-1}+(4-h^2 f_{ij})u_{ij} = -h^2 g_i$$

Soln using

Gauss - Seidel Iteration Method

$$u_{ij}^{(k+1)} = \frac{1}{4-h^2 f_{ij}} \left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k+1)} - h^2 g_{ij} \right)$$

Gauss - Jacobi

$$u_{ij}^{(k+1)} = \frac{1}{4-h^2 f_{ij}} \left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)} - h^2 g_{ij} \right)$$

Successive - Over - relaxation Method

$$u_{ij}^{(k+1)} = u_{ij}^{(k)} + \omega \left[\frac{1}{4-h^2 f_{ij}} \left\{ u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)} - h^2 g_{ij} - (4-h^2 f_{ij}) u_{i,j}^{(k)} \right\} \right]$$

$\omega \to$ acceleration parameter or relaxation factor

$1 < \omega < 2$ [generally lies in this range] xample

$$\nabla^2 u = 0$$

$u(x,0) = 0$; $u(0,y) = 4$; $u(x,8) = 0$; $u(8,y) = 40$

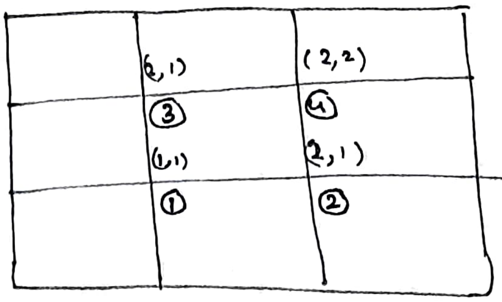
$h = 4$

$u(1,0) = u(2,0) = u(3,0) = 0$

$u(1,3) = u(2,3) = u(3,3) = 0$

$u(0,1) = u(0,2) = u(0,3) = 4$

$u(3,1) = u(3,2) = u(3,3) = 40$



$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

$j = 1$, $i = 1$ (1)

$$u_{2,1} + \underset{4}{u_{0,1}} + u_{1,2} + \underset{0}{u_{1,0}} - 4u_{1,1} = 0$$

$$u_2 + 4 + u_3 - 4u_1 = 0$$

$j = 1$, $i = 2$ (2)

$$40 + u_1 + u_4 + 0 - 4u_2 = 0$$

$j = 2$; $i = 1$ (3)

$$u_4 + 4 + 0 + u_1 - 4u_3 = 0$$

$j = 2$, $i = 2$ (4)

$$50 + u_3 + 0 + u_2 - 4u_4 = 0$$

$$-4u_1 + u_2 + u_3 = -4$$

$$u_1 - 4u_2 + u_4 = -40$$

$$u_1 - 4u_3 + u_4 = -4$$

$$u_2 + u_3 - 4u_4 = -40$$

$$u_1 = \frac{u_2 + u_3}{4} + 1$$

$$u_2 = \frac{u_1 + u_4}{4} + 10$$

$$u_3 = \frac{u_1 + u_4}{4} + 1$$

$$u_4 = \frac{u_2 + u_3}{4} + 10$$

Initial guess $\left(u_1^{(0)}, u_2^{(0)}, u_3^{(0)}, u_4^{(0)}\right) = [0, 0, 0, 0]$

$u_1^{(1)} = 1$; $u_2^{(1)} = 10$; $u_3^{(1)} = 1$; $u_4^{(1)} = 10$

Gauss - Jacobi

$$u_1^{(2)} = \frac{10+1}{4} + 1 = \frac{15}{4}$$

$$u_2^{(2)} = \frac{1+10}{4} + 10 = \frac{51}{4}$$

$$u_3^{(2)} = \frac{1+10}{4} + 1 = \frac{15}{4}$$

$$u_4^{(2)} = \frac{10+1}{4} + 10 = \frac{51}{4}$$

Gauss. Seidel

$$u_1^{(2)} = \frac{15}{4}$$

$$u_2^{(2)} = \frac{\frac{15}{4} + 10}{4} + 10 = \frac{215}{16}$$

$$u_3^{(2)} = \frac{\frac{15}{4} + 10}{4} + 1 = \frac{81}{16}$$

$$u_4^{(2)} = \frac{215 + 81}{16 \times 4} + 10 = \frac{936}{64} \text{ (check)}$$

and so on . . .

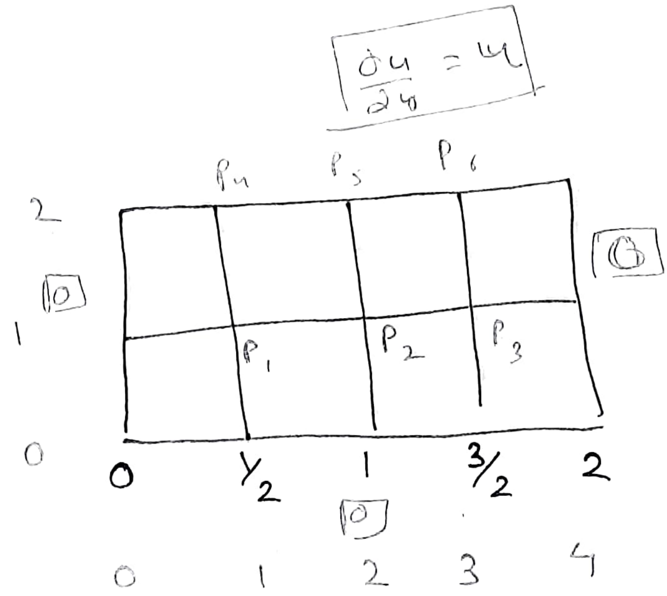
Derivative Boundary condition

Solve $\dfrac{\partial^2 u}{\partial x^2} + 5\dfrac{\partial^2 u}{\partial y^2} = 16$

$u = 0$ on $x = 0$ and 2

$u = 0$ on $y = 0$ and $\dfrac{\partial u}{\partial y} = u$ on $y = 1$

$h = 1/2$



$u_{0,j} = 0$; $u_{4,j} = 0$; $u_{i,0} = 0$

$\dfrac{\partial u}{\partial y}(x,1) = u(x,1)$

__Discretize__

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + 5\left[\frac{\cancel{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}\; u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}\right] = 16$$

$$h^2 = 1/4$$

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + 5u_{i,j+1} - 10u_{i,j} + 5u_{i,j-1} = 4$$

$$5u_{i,j+1} + u_{i+1,j} - 12u_{i,j} + u_{i-1,j} + 5u_{i,j-1} = 4$$

Discretize B.C.

$$\frac{u_{i,j+1} - u_{i,j-1}}{2h} = u_{i,j}$$

$$u_{i,j+1} - u_{i,j-1} = u_{i,j}$$

When $y = 1$; $j = 2$

$$u_{i,3} - u_{i,1} = u_{i,2}$$

$P_1 = (1,1)$

$$5u_{1,2} + u_{2,1} - 12u_{1,1} + \underset{=0}{u_{0,1}} + 5\underset{=0}{u_{1,0}} = 4$$

$$5u_4 + u_2 - 12u_1 = 4$$

$P_2\ (2,1)$

$$5u_{2,2} + u_{3,1} - 12u_{2,1} + u_{1,1} + 5\underset{=0}{u_{2,0}} = 4$$

$$\cancel{5u_3} + 5u_5 + u_3 - 12u_2 + u_1 = 4$$

$P_3\ (3,1)$

$$5u_{3,2} + \underset{=0}{u_{4,1}} - 12u_{3,1} + u_{2,1} + 5\underset{=0}{u_{3,0}} = 4$$

$$5u_6 + u_2 - 12u_3 = 4$$

$P_4\ (1,2)$

$$5u_{1,3} + u_{2,2} - 12u_{1,2} + \underset{=0}{u_{0,2}} + 5u_{1,1} = 4$$

$$\boxed{5u_7} + u_5 - 12u_4 + 5u_1 = 4$$

$\boxed{} \to$ fictitious value.

$P_5\ (2,2)$

$$5u_{2,3} + u_{3,2} - 12u_{2,2} + u_{1,2} + 5u_{2,1} = 4$$

$$\boxed{5u_8} + u_6 - 12u_5 + u_4 + 5u_2 = 4$$

$P_6\ (3,2)$

$$5u_{3,3} + \underset{=0}{u_{4,2}} - 12u_{3,2} + u_{2,3} + 5u_{3,1} = 4$$

$$\boxed{5u_9} - 12u_6 + u_5 + 5u_3 = 4$$

One should eliminate the fictitious using derivative b.c.

$i=1$

$$u_{1,3} - u_{1,1} = u_{1,2}$$

$$u_7 - u_1 = u_4$$

$$u_7 = u_1 + u_4$$

$i=2$

$$u_{2,3} - u_{2,1} = u_{2,2}$$

$$u_8 - u_2 = u_5$$

$$u_8 = u_2 + u_5$$

$i=3$

$$u_{3,3} - u_{3,1} = u_{3,2}$$

$$u_9 - u_3 = u_6$$

$$u_9 = u_6 + u_3$$

$P_1 \rightarrow -12u_1 + u_2 + 5u_4 = 4$

$P_2 \rightarrow u_1 - 12u_2 + u_3 + 5u_5 = 4$

$P_3 \rightarrow u_2 - 12u_3 + 5u_6 = 4$

$P_4 \rightarrow 10u_1 - 7u_4 + u_5 = 4$

$P_5 \rightarrow 10u_2 + u_4 - 7u_5 + u_6 = 4$

$P_6 \rightarrow 10u_3 + u_5 - 7u_6 = 4$

$$\begin{pmatrix} -12 & 1 & 0 & 5 & 0 & 0 \\ 1 & -12 & 1 & 0 & 5 & 0 \\ 0 & 1 & -12 & 0 & 0 & 5 \\ 10 & 0 & 0 & -7 & 1 & 0 \\ 0 & 10 & 0 & 1 & -7 & 1 \\ 0 & 0 & 10 & 0 & 1 & -7 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

One can solve and obtain solution