Hearing Fluid Subspaces (placeholder)

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Abstract

(Placeholder.)

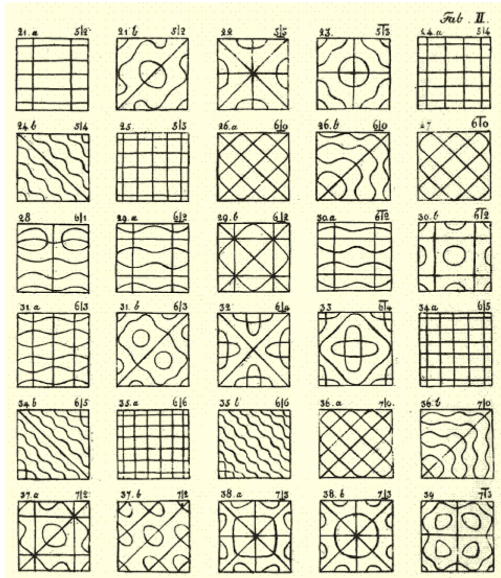
Introduction

In 1787, Ernst Chladni demonstrated a physical experiment to visually reveal the modes of vibration of a rigid surface. By placing sand over a metal plate and running a violin bow along the edge, he excited the plate at different resonant frequencies, forcing the sand to accumulate along the corresponding nodal curves and producing a number of beautiful patterns [1]. Mathematically, these correspond to the eigenfunctions of the biharmonic operator Δ^2 on a square domain with free boundary conditions, as discussed extensively by Gander and Wanner [3]. Inspired by this phenomenon, we consider an analogy in computational fluid dynamics in which particles of smoke are placed in a domain and an underlying velocity field is excited at different resonant frequencies, pushing the smoke around accordingly. We also construct a sonification of these frequencies to produce a mapping between fluid trajectories and sound.

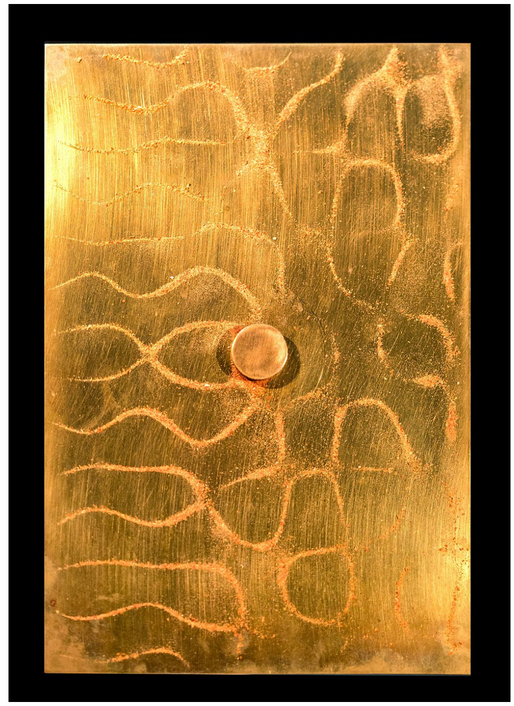
Subspace fluids

Many approaches to fluid simulation in computer graphics numerically solve the incompressible Navier-Stokes equations on a regular spatial grid. Unfortunately, the nonlinearity of these equations and the sensitivity of the resulting flow to small-scale effects means that high-quality results are extremely time-consuming to compute. One strategy for speeding up these computations is the subspace approach, or model reduction. In this method, we consider our velocity field on a regular three-dimensional grid of N cells abstractly as the real vector space $\mathbb{R}^{3N}$. We then discover a collection of linearly independent vectors $v_1, \ldots, v_r \in \mathbb{R}^{3N}$, with $r \ll N$. (For the strategy to work, the subspace $S \subset \mathbb{R}^{3N}$ spanned by these vectors must capture the fluid motions of interest.) Letting $P \colon \mathbb{R}^{3N} \to S$ denote the projection from $\mathbb{R}^{3N}$ to S, we can represent this projection in coordinates with a matrix $\mathbf{U} \in \mathbb{R}^{3N \times r}$. That is, given a vector $\mathbf{u}_t \in \mathbb{R}^{3N}$ at time step t, we can compute its reduced-order counterpart $\mathbf{q}_t \in S$ by computing $\mathbf{U}^T \mathbf{u}_t = \mathbf{q}_t$. At each time step t, we integrate the equations of motion for $\mathbf{q} \in S$ to advance forward to $\mathbf{q}_{t+1} \in S$. Being over a reduced number of variables, this phase is greatly accelerated compared to the full-space integration. Once the integration is complete, we must reverse the projection so that the results can be displayed in the original space of full coordinates. This reconstruction can be achieved simply by computing $\mathbf{u}_{t+1} = \mathbf{U}\mathbf{q}_{t+1}$.

There are several strategies for discovering a useful subspace S, both analytical and statistical in nature. In work by de Witt [2] the subspace S is determined analytically by eigenfunctions of the Laplacian operator on a box up to a desired threshold frequency. In previous work by Kim [4], the subspace S is statistically



(a) A table of hand-drawn Chladni figures. Source: Wikimedia Commons.



(b) The physical experiment with a rectangular plate. Source: Wikimedia Commons.

Figure 1: Chladni figures, both hand-drawn and experimentally realized.

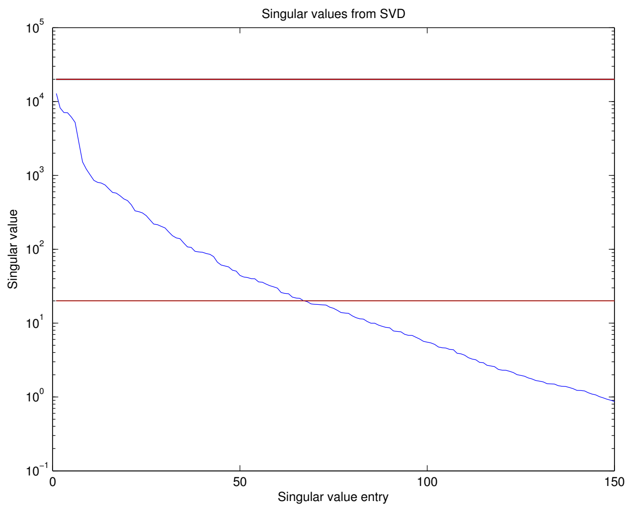
discovered by first performing an appropriate training simulation in full coordinates and performing a singular value decomposition over a matrix A whose columns are velocity fields at each time step. Discarding the singular values less than a small threshold leaves us with r singular values, with $r \ll 3N$, and their corresponding r singular vectors, which are the basis vectors in the full space $\mathbb{R}^{3N}$ that generate S. In either the analytical or statistical case, in analogy to the Chladni plate vibrations, the singular vectors correspond to the eigenfunctions, and the singular values correspond to the frequencies. Because the eigenfunctions of the Laplacian operator on a box are well-known to be separable products of sine and cosine functions, we use the statistical approach to generating the subspace S in this paper in order to discover more unfamiliar modes.

For each time step of the subspace re-simulation, when we perform the reconstruction phase, obtaining a full-space velocity field $\mathbf{v} \in \mathbb{R}^{3N}$ from a reduced-order subspace vector $\mathbf{q} \in \mathbb{R}^r$, we are simply taking a linear combination of the r singular vectors using the corresponding coefficients given by the entries of $\mathbf{q}$. Hence, the subspace vectors q can be intuitively thought of as instructions for how to mix together the corresponding singular vectors with the appropriate weightings. Indeed, we can even identify a path $\varphi\colon [0,1] \to S$, sampled at T timesteps $\varphi_1, \ldots, \varphi_T$, as determining the corresponding T timesteps of the full-space re-simulation, after reconstruction.

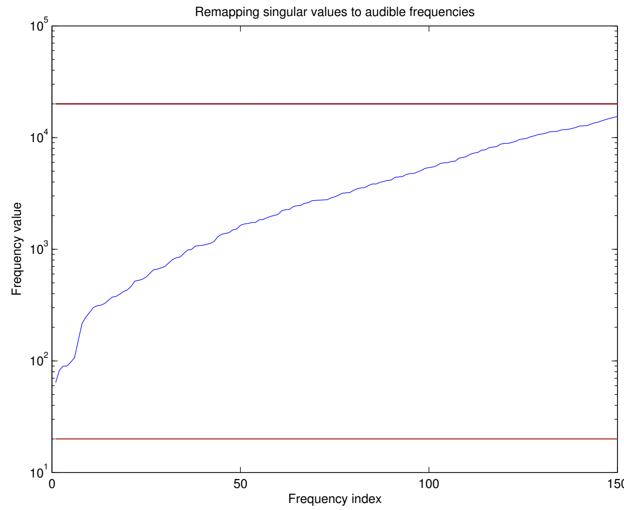
Sonification

We interpret the r singular values $\sigma_1, \ldots, \sigma_r$ from the singular-value decomposition as characteristic frequencies.

There are two immediate concerns about the raw data. Firstly, from Figure 2a, we see that the val-



(a) Singular values: r = 150



(b) Remapped frequencies: f = 64 Hz, s = 1.75

Figure 2: Note the logarithmic scale on both y-axes. The red lines on both plots indicate the bounds of the human audible frequency range.

ues range over approximately 4 orders of magnitude, which is approximately 13 octaves, and decrease to numbers less than 1. However, the human audible dynamic range is generally considered to be from 20 Hz to 20000 Hz, which is only about 10 octaves, and starts greater than 1. Thus, we will have to rescale the octaves into an appropriate range and recenter the data. Secondly, starting from the principal singular value, the singular values are decreasing, whereas an audio spectrum increases from its fundamental frequency. Hence, we will have to invert the singular values. One way to proceed with a sensible mapping is to specify a desired fundamental frequency f and an octave scaling s. Writing our audible frequencies as $f_1, \ldots, f_r$, we can define our mapping from singular values to audible frequencies as follows:

$$\begin{aligned} f_i &= f \cdot \left(\frac{\sigma_i^{-1}}{\sigma_{\text{max}}^{-1}} \right)^{\frac{1}{s}} \\ &= f \cdot \left(\frac{\sigma_{\text{max}}}{\sigma_i} \right)^{\frac{1}{s}}, \; i = 1, \ldots, r. \end{aligned} \tag{1}$$

The effect of this remapping can be seen in Figure 2b, where we have used a fundamental frequency $f = 64$ Hz and an octave scaling of $s = 1.75$. The spectrum now begins at the fundamental, $f = 64$ Hz, and ranges up to a maximum of approximately 15000 Hz, an acceptable spread.

With an audible spectrum in hand, we now turn to the problem of choosing amplitudes for each individual frequency. These can be mapped from a corresponding subspace vector $\mathbf{q} \in S$, as each of its r components, $q_1, \ldots, q_r$ can be thought of as an amplitude for the r corresponding frequencies $f_1, \ldots, f_r$. Care must be taken here too, as the overall sum of all the frequencies can exceed unity gain, which would lead to clipping. Mathematically, this means that we require the L^1 norm of $\mathbf{q}$, $|\mathbf{q}|_1 = \sum_{i=1}^{r} |q_i|$, to be at most 1. A typical subspace vector $\mathbf{q}$ may also contain negative components. However, these can simply be thought of as encoding a positive amplitude and a reversal of phase. The phase reversal can be discarded, as its perceivable effect is typically undesirable clicking artifacts.

With these considerations in hand, we design a mapping from a subspace vector $\mathbf{q}$ to an amplitude vector $\mathbf{a}$ of unit L^1 norm as follows:

$$a_i = \frac{|q_i|}{|\mathbf{q}|_1},\ i = 1, \ldots, r. \tag{2}$$

Given a sequence $(\mathbf{q}_j)$ of vectors that describe a sampled trajectory $\mathbf{q}_1, \ldots, \mathbf{q}_T$ through the subspace S, we can either normalize each one of the corresponding amplitude vectors on an individual basis, or we can determine the subspace vector of maximum L^1 norm, $\mathbf{q}_{\text{max}}$, and normalize each amplitude vector based on $|\mathbf{q}_{\text{max}}|_1$. The effect of the former is a relatively uniform volume level, while the effect of the latter is usually a more variable volume envelope. Each approach has its own musical merit, depending on the compositional situation.

Synthesis

With our mappings carried out, we can now produce sensible audible sound. The most basic technique to try is additive synthesis—namely, mixing together pure sine tones at the corresponding frequencies and amplitudes, creating an overall sound with a rich spectral content. With an instrument in hand that can produce the corresponding sound given a set of input amplitudes, a particular trajectory $\phi\colon [0,1] \to S$ through the subspace, while retaining the same set of frequencies throughout time, determines a time-varying set of amplitudes, allowing the subspace evolution to dictate the time evolution of the sound. The overall effect is a subtle change in timbre, as the amplitudes of the various overtones fluctuate.

Subtractive synthesis is also an interesting option. We start with a spectrally rich input sound, such as an impulse or broadband noise. We then create a filter bank of resonances at the appropriate set of frequencies, weighted in turn by the appropriate set of amplitudes. Some extra parameters must be addressed, as each resonant mode also is given a ring time for the mode to decay. The nature of the input sound also strongly influences the resulting timbre—broadband noise creates a more atmospheric effect, while impulses can create driving rhythmic textures.

Time evolution

Static sound, while interesting as a new timbre for a few seconds, eventually grows stale. Thus, we would like to capture the time evolution of the subspace trajectory as a dynamic sonic event. This can be achieved by cycling through the the sequence of amplitude vectors $\mathbf{a}_1, \ldots, \mathbf{a}_T$ corresponding to the subspace trajectory $\mathbf{q}_1, \ldots, \mathbf{q}_T$. Indeed, different trajectories through the subspace generated different sequences of amplitude vectors. The unfolding of these trajectories over time occurs on a micro-scale musically, but the choice of different possible abstract trajectories, and the shifting from one trajectory to another, is expressed on more of a meso- or even macro-level scale.

Experimentally, we have explored several different categories of subspace trajectories. The simplest is the original subspace re-simulation trajectory, which is faithful to the original simulation. Another possibility is the 'frequency response' trajectory, which is the result of setting the first subspace vector $\mathbf{q}_1 = \mathbf{1}$, the vector of all $1s$, and allowing the simulation to unfold from there. Both of these possibilities retain the original physics-based dynamics to update each timestep. It is also possible, however, to abandon the physics and simply choose a trajectory by other means. For example, starting from the same initial vector $\mathbf{q}_1$ as in the original re-simulation, we can determine a trajectory simply by walking in a random direction at each time step, or walking along the surface of some abstract manifold $M \subset S$, such as a sphere. These lead both to unique visual and aural results.

Since the the set of frequencies itself remains static over time, and only the amplitudes are changing, musically, it will also make sense to explore dynamically changing the pitch f which was specified as

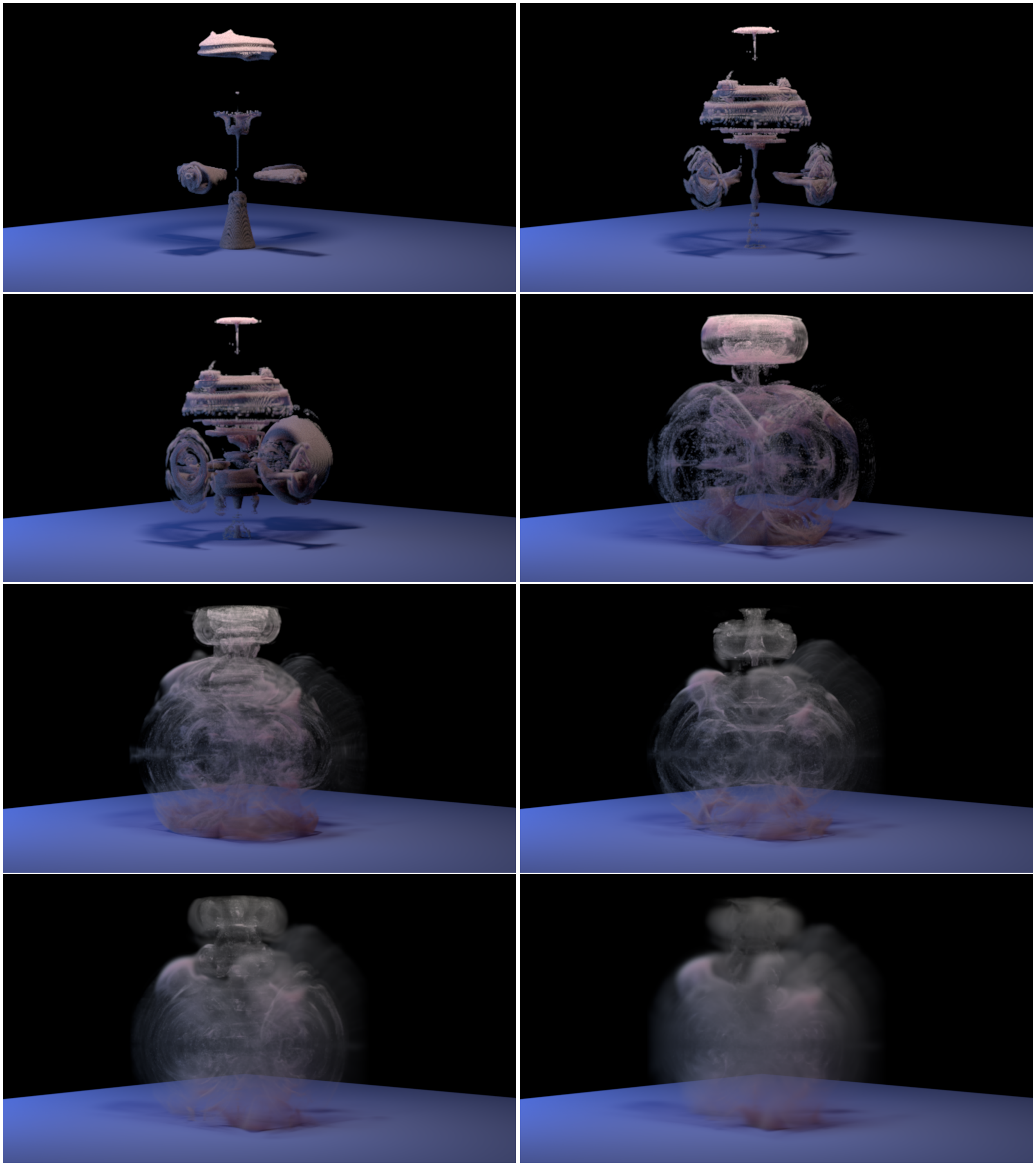


Figure 3: An assortment of subspace fluid modes.

the original fundamental frequency. This can be achieved simply by multiplying the entire spectrum by an interval, which will create a new fundamental without affecting the ratios between the partials. Alternatively, we could consider the original spectrum as a musical scale with which to compose melodies if we desire to compose more at the note scale.

References

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