Matrix Assignment - Conics

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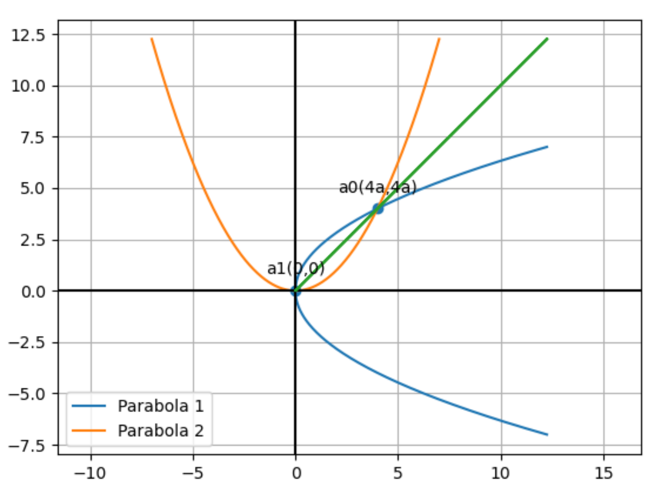
Contents

I. Problem

If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passess through the point of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then
A) $d^2 + (3b - 2c)^2 = 0$
B) $d^2 + (3b + 2c)^2 = 0$
C) $d^2 + (2b - 3c)^2 = 0$
D) $d^2 + (2b + 3c)^2 = 0$

II. Figure



III. Solution

Let us assume a =1 ; **STEP-1**

The given equation of parabola $x^2 = 4y$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \tag{3}$$

$$f_1 = 0 \tag{4}$$

And also for second Parabola,
The given equation of parabola $y^2 = 4x$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{5}$$

where

$$\mathbf{V_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6}$$

$$\mathbf{u_2} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \tag{7}$$

$$f_2 = 0 \tag{8}$$

STEP-2

The intersection of the given conics is obtained as

$$\mathbf{x}^\top \left(\mathbf{V}_1 + \mu \mathbf{V}_2\right) \mathbf{x} + 2 \left(\mathbf{u}_1 + \mu \mathbf{u}_2\right)^\top \mathbf{x} + \left(f_1 + \mu f_2\right) = 0 \tag{9}$$

$$\tag{10}$$

$$\mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \tag{11}$$

$$\mathbf{u}_1 + \mu \mathbf{u}_2 = -\begin{pmatrix} 2\mu \\ 2 \end{pmatrix} \tag{12}$$

$$f_1 + \mu f_2 = 0 \tag{13}$$

This conic is a single straight line if and only if,

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{14}$$

And,

$$\left|\mathbf{V}_1 + \mu\mathbf{V}_2\right| < 0 \tag{15}$$

Substituting (11) (12) (13) in (14)
We get,

$$\implies \begin{vmatrix} 1 & 0 & -2\mu \\ 0 & \mu & -2 \\ -2\mu & -2 & 0 \end{vmatrix} = 0 \tag{16}$$

Solving the above equation we get,

$$\mu = -1, 1 \tag{17}$$

case 1 : when $\mu = 1$
So,

$$\left|\mathbf{V}_1 + \mu\mathbf{V}_2\right| > 0 \tag{18}$$

So, $\mu = 1$ is discarded
case 2 : when $\mu = -1$
So,

$$\left|\mathbf{V}_1 + \mu\mathbf{V}_2\right| < 0 \tag{19}$$

Therefore $\mu = 1$
The eigenvalue decomposition of a symmetric matrix $\mathbf{V}$ is given by

$$\mathbf{P}^\top\mathbf{V}\mathbf{P} = \mathbf{D} \qquad \mathbf{P} = \begin{pmatrix}\mathbf{P_1} & \mathbf{P_2}\end{pmatrix} \tag{20}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{21}$$

On solving (20) with $\mathbf{P_1} = \begin{pmatrix}1\\0\end{pmatrix}$ & $\mathbf{P_2} = \begin{pmatrix}0\\1\end{pmatrix}$, we get

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{22}$$

where,

$$\lambda_1 = 1 \ and \ \lambda_2 = 1 \tag{23}$$

Now , Normal vector $\mathbf{n_1}$ is given by

$$\mathbf{n_1} = \mathbf{P}(\frac{\sqrt{|\lambda_1|}}{\sqrt{|\lambda_2|}}) = \begin{pmatrix}1\\1\end{pmatrix}, \tag{24}$$

Now, Normal vector $\mathbf{n_2}$ is given by

$$\mathbf{n_1} = \mathbf{P}(\frac{\sqrt{|\lambda_1|}}{-\sqrt{|\lambda_2|}}) = \begin{pmatrix}1\\-1\end{pmatrix}, \tag{25}$$

Now ,
We know that $C = V^{-1}u$
So,

$$\mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{26}$$

$$\mathbf{u}_1 + \mu\mathbf{u}_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \tag{27}$$

So, $\mathbf{C} = V^{-1}u = \begin{pmatrix}-2\\-2\end{pmatrix}$
Now ,

$$\mathbf{m_1} = \mathbf{Omat}\ \mathbf{n_1} = \begin{pmatrix}1\\-1\end{pmatrix} \tag{28}$$

And,

$$\mathbf{m_2} = \mathbf{Omat}\ \mathbf{n_2} = \begin{pmatrix}1\\1\end{pmatrix} \tag{29}$$

Now,
The intersection of line $\mathbf{l_1} = C + \lambda\mathbf{m_1}$ has no real solution
And,
The intersection of line $\mathbf{l_2} = C + \lambda\mathbf{m_2}$ is

$$\begin{pmatrix}0\\0\end{pmatrix} and \begin{pmatrix}4\\4\end{pmatrix} \tag{30}$$

So, From the above calculations ,
we can conclude that , the general equation of line is $y = x$

The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbf{R} \tag{31}$$

with the conic section are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{32}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) \pm \sqrt{\left[\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) \right]^2 - \left(\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f \right) \left(\mathbf{m}^T \mathbf{V} \mathbf{m} \right)} \right) \tag{33}$$

From the line y=x the vectors q,m are taken,

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{34}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{35}$$

For Parabola 1,

by substituting (2) (3) (4) (34) (35) in (33)

$$\mu_i = 2 \tag{36}$$

substituting (34) (35) (36) in (32) ,
the intersection points on the parabola are

$$\mathbf{a_0} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{37}$$

$$\mathbf{a_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{38}$$

For parabola 2, by substituting by substituting (6) (7) (8) (34) (35) in (33)

$$\mu_i = 234, 35, 39, 32 \tag{39}$$

substituting (34) (35) (39) in (32),
the intersection points on the parabola are

$$\mathbf{a_0} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{40}$$

$$\mathbf{a_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{41}$$

So, Here we can clearly see that the both the Parabolas are intersecting each other at two points i.e
at $\mathbf{a_0} = \begin{pmatrix} 4a \\ 4a \end{pmatrix}$ and $\mathbf{a_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Now,
Case 1 : As the line $2bx+3cy+4d = 0$ passes through the intersection points of both parabola ,
So, at $\mathbf{a_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$2b(0) + 3c(0) + 4d = 0$
$\Rightarrow 4d = 0$
Also , $d = 0$
Case 2 : As the line $2bx+3cy+4d = 0$ passes through the intersection points of both parabola ,
So, at $\mathbf{a_0} = \begin{pmatrix} 4a \\ 4a \end{pmatrix}$
$2b(4a) + 3c(4a) + 4d = 0$
So,
$\Rightarrow 8ab + 12ac = 0$ (Since d=0)
$\Rightarrow 4a(2b + 3c) = 0$
Since $a \neq 0$
we can write ,,
$2b + 3c = 0$
Also , $(2b + 3c)^2 = 0$
So,
we can also say that ,
$d^2 + (2b + 3c)^2 = 0$
So we can conclude that option D is the correct option

IV. Code Link

https://github.com/aadrshptel/Fwc_module1/tree/main/Assignments/Matrix%20assignments/Conics/codes

Execute the code by using the command
python3 conic.py