



Matrix Assignment - Conics

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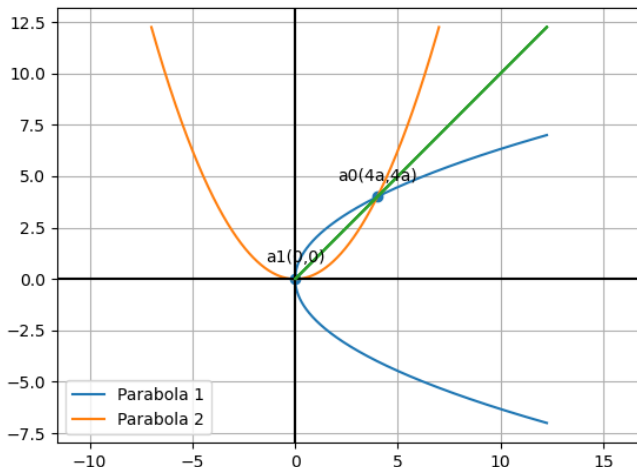
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I. PROBLEM

If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passess through the point of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then

- A) $d^2 + (3b - 2c)^2 = 0$
- B) $d^2 + (3b + 2c)^2 = 0$
- C) $d^2 + (2b - 3c)^2 = 0$
- D) $d^2 + (2b + 3c)^2 = 0$

II. FIGURE



III. SOLUTION

Let us assume $a = 1$; **STEP-1**

- 1 The given equation of parabola $x^2 = 4y$ can be written in the general quadratic form as

$$1 \quad \mathbf{x}^T \mathbf{V}_1 \mathbf{x} + 2\mathbf{u}_1^T \mathbf{x} + f_1 = 0 \quad (1)$$

- 1 where

$$3 \quad \mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (2)$$

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad (3)$$

$$f_1 = 0 \quad (4)$$

And also for second Parabola,

The given equation of parabola $y^2 = 4x$ can be written in the general quadratic form as

$$\mathbf{x}^T \mathbf{V}_2 \mathbf{x} + 2\mathbf{u}_2^T \mathbf{x} + f_2 = 0 \quad (5)$$

where

$$\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (6)$$

$$\mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad (7)$$

$$f_2 = 0 \quad (8)$$

STEP-2

The intersection of the given conics is obtained as

$$\mathbf{x}^T (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 + \mu \mathbf{u}_2)^T \mathbf{x} + (f_1 + \mu f_2) = 0 \quad (9)$$

$$(10)$$

$$\mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \quad (11)$$

$$\mathbf{u}_1 + \mu \mathbf{u}_2 = - \begin{pmatrix} 2\mu \\ 2 \end{pmatrix} \quad (12)$$

$$f_1 + \mu f_2 = 0 \quad (13)$$

The locus of intersection of these two parabola is a pair of straight lines if,

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \quad (14)$$

And,

$$|\mathbf{V}_1 + \mu \mathbf{V}_2| < 0 \quad (15)$$

Substituting (11) (12) (13) in (14)

We get,

$$\Rightarrow \begin{vmatrix} 1 & 0 & -2\mu \\ 0 & \mu & -2 \\ -2\mu & -2 & 0 \end{vmatrix} = 0 \quad (16)$$

Solving the above equation we get,

$$\mu = -1, 1 \quad (17)$$

case 1 : when $\mu = 1$

So,

$$|\mathbf{V}_1 + \mu \mathbf{V}_2| > 0 \quad (18)$$

So, $\mu = 1$ is discarded

case 2 : when $\mu = -1$

So,

$$|\mathbf{V}_1 + \mu \mathbf{V}_2| < 0 \quad (19)$$

Therefore $\mu = 1$

The eigenvalue decomposition of a symmetric matrix \mathbf{V} is given by

$$\mathbf{P}^\top \mathbf{V} \mathbf{P} = \mathbf{D} \quad \mathbf{P} = (\mathbf{P}_1 \quad \mathbf{P}_2) \quad (20)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (21)$$

On solving (20) with $\mathbf{P}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\mathbf{P}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
we get

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (22)$$

where,

$$\lambda_1 = 1 \text{ and } \lambda_2 = 1 \quad (23)$$

Now , Normal vector \mathbf{n}_1 is given by

$$\mathbf{n}_1 = \mathbf{P} \left(\frac{\sqrt{|\lambda_1|}}{\sqrt{|\lambda_2|}} \right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (24)$$

Now, Normal vector \mathbf{n}_2 is given by

$$\mathbf{n}_1 = \mathbf{P} \left(\frac{\sqrt{|\lambda_1|}}{-\sqrt{|\lambda_2|}} \right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (25)$$

Now ,

We know that $C = V^{-1}u$

So,

$$\mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (26)$$

$$\mathbf{u}_1 + \mu \mathbf{u}_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \quad (27)$$

$$\text{So, } \mathbf{C} = V^{-1}u = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

Now ,

$$\mathbf{m}_1 = \text{Omat } \mathbf{n}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (28)$$

And,

$$\mathbf{m}_2 = \text{Omat } \mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (29)$$

Now,

The intersection of line $\mathbf{l}_1 = C + \lambda \mathbf{m}_1$ has no real solution

And,

The intersection of line $\mathbf{l}_2 = C + \lambda \mathbf{m}_2$ is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (30)$$

So, From the above calculations ,

we can conclude that , the general equation of line is $y = x$

The points of intersection of the line

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbf{R} \quad (31)$$

with the conic section are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (32)$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f) (\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \quad (33)$$

From the line $y=x$ the vectors \mathbf{q}, \mathbf{m} are taken,

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (34)$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (35)$$

For Parabola 1,

by substituting (2) (3) (4) (34) (35) in (33)

$$\mu_i = 2 \quad (36)$$

substituting (34) (35) (36) in (32),
the intersection points on the parabola are

$$\mathbf{a}_0 = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (37)$$

$$\mathbf{a}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (38)$$

For parabola 2, by substituting by substituting (6) (7) (8) (34) (35) in (33)

$$\mu_i = 234, 35, 39, 32 \quad (39)$$

substituting (34) (35) (39) in (32),
the intersection points on the parabola are

$$\mathbf{a}_0 = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (40)$$

$$\mathbf{a}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (41)$$

So, Here we can clearly see that the both the Parabolas are intersecting each other at two points i.e

$$\text{at } \mathbf{a}_0 = \begin{pmatrix} 4a \\ 4a \end{pmatrix} \text{ and } \mathbf{a}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now,

Case 1 : As the line $2bx + 3cy + 4d = 0$ passes through the intersection points of both parabola ,

$$\text{So, at } \mathbf{a}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2b(0) + 3c(0) + 4d = 0$$

$$\Rightarrow 4d = 0$$

$$\text{Also , } d = 0$$

Case 2 : As the line $2bx + 3cy + 4d = 0$ passes through the intersection points of both parabola ,

$$\text{So, at } \mathbf{a}_0 = \begin{pmatrix} 4a \\ 4a \end{pmatrix}$$

$$2b(4a) + 3c(4a) + 4d = 0$$

So,

$$\Rightarrow 8ab + 12ac = 0 \text{ (Since } d=0)$$

$$\Rightarrow 4a(2b + 3c) = 0$$

Since $a \neq 0$

we can write ,,

$$2b + 3c = 0$$

$$\text{Also , } (2b + 3c)^2 = 0$$

So,

we can also say that ,

$$d^2 + (2b + 3c)^2 = 0$$

So we can conclude that option D is the correct option

IV. CODE LINK

https://github.com/aadrshptel/Fwc_module1/tree/main/Assignments/Matrix%20assignments/Conics/codes

Execute the code by using the command
python3 conic.py