Matrix Assignment - Conics

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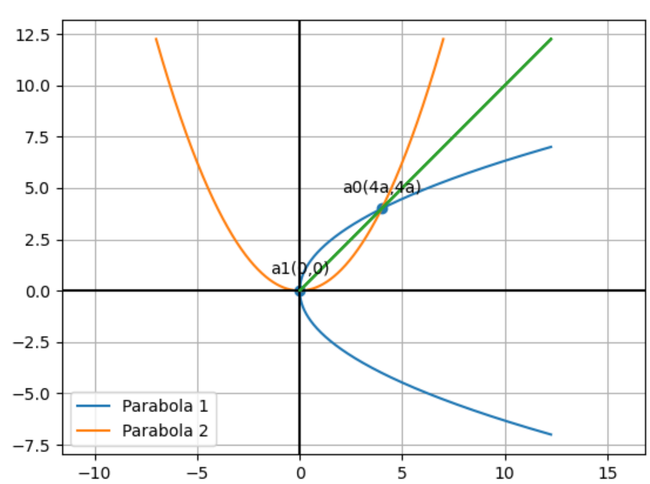
Contents

I. Problem

If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passess through the point of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then
A) $d^2 + (3b - 2c)^2 = 0$
B) $d^2 + (3b + 2c)^2 = 0$
C) $d^2 + (2b - 3c)^2 = 0$
D) $d^2 + (2b + 3c)^2 = 0$

II. Figure



III. Solution

The given equation of parabola $x^2 = 4ay$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2a \end{pmatrix}, \tag{3}$$

$$f = 0 \tag{4}$$

And also for second Parabola,
The given equation of parabola $y^2 = 4ax$ can be written in the general quadratic form as

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{5}$$

where

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{6}$$

$$\mathbf{u} = \begin{pmatrix} -2a \\ 0 \end{pmatrix}, \tag{7}$$

$$f = 0 \tag{8}$$

Since it is given that a line is passing through both the intersection points of these two parabola.
So, the equation of that line is $y = x$.
The points of intersection of the line

$$L: \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbf{R} \tag{9}$$

with the conic section are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{10}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V}\mathbf{m}} \Big(-\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V}\mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V}\mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f)(\mathbf{m}^T \mathbf{V}\mathbf{m})} \Big) \tag{11}$$

From the line y=x the vectors q,m are taken,

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{12}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{13}$$

For Parabola 1,

by substituting eq(2),(3),(4),(12),(13) in eq(11)

$$\mu_i = 2 \tag{14}$$

substituting eq(12),(13),(14) in eq(10) ,
the intersection points on the parabola are

$$\mathbf{a_0} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{15}$$

$$\mathbf{a_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{16}$$

For parabola 2, by substituting eq(6),(7),(8),(12),(13) in eq(11)

$$\mu_i = 2 \tag{17}$$

substituting eq(12),(13),(17) in eq(10) ,
the intersection points on the parabola are

$$\mathbf{a_0} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{18}$$

$$\mathbf{a_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{19}$$

So, Here we can clearly see that the both the Parabolas are intersecting each other at two points i.e at $\mathbf{a_0} = \begin{pmatrix} 4a \\ 4a \end{pmatrix}$ and $\mathbf{a_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Now,
Case 1 : As the line $2bx + 3cy + 4d = 0$ passes through the intersection points of both parabola ,
So, at $\mathbf{a_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$2b(0) + 3c(0) + 4d = 0$
$\Rightarrow 4d = 0$
Also , $d = 0$
Case 2 : As the line $2bx + 3cy + 4d = 0$ passes through the intersection points of both parabola ,
So, at $\mathbf{a_0} = \begin{pmatrix} 4a \\ 4a \end{pmatrix}$
$2b(4a) + 3c(4a) + 4d = 0$
So,
$\Rightarrow 8ab + 12ac = 0$ (Since d=0)
$\Rightarrow 4a(2b + 3c) = 0$
Since $a \neq 0$
we can write ,,
$2b + 3c = 0$
Also , $(2b + 3c)^2 = 0$
So,
we can also say that ,
$d^2 + (2b + 3c)^2 = 0$
So we can conclude that option D is the correct option

IV. Code Link

https://github.com/aadrshptel/Fwc_module1/tree/main/Assignments/Matrix%20assignments/Conics/codes

Execute the code by using the command
python3 conic.py