Simulation of Mean change

- How the **mean of a time series changes** under structural breaks?
- Two scenarios:
  - **Single break** (one change point)
  - **Multiple breaks** (two change points, three regimes)
- Compare forecasting methods and evaluate accuracy using **RMSE**, **MAE**, and **Bias**.

Data Generating Process (Mean-Break)

We simulate an SARIMA process with time-varying mean:

$$y_t = \mu_t + \phi y_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0, \sigma^2), \quad |\phi| < 1.$$

Single break:

$$\mu_t = \begin{cases} \mu_0, & t \leq T_b, \\ \mu_1, & t > T_b. \end{cases}$$

Multiple breaks:

$$\mu_t = \begin{cases} \mu_0, & t \leq b_1, \\ \mu_1, & b_1 < t \leq b_2, \\ \mu_2, & t > b_2. \end{cases}$$

Monte Carlo Forecast Design

For each replication $i = 1, \ldots, N$:

1. Simulate $\{y_t^{(i)}\}_{t=1}^{T}$ from the DGP.
2. Choose the forecast time after the break(s):

$$t_0 = \begin{cases} T_b + g, & \text{single break,} \\ b_2 + g, & \text{multiple breaks,} \end{cases}$$

   where g is a fixed gap after the (last) break.
3. Estimate each model using $\{y_1, \ldots, y_{t_0-1}\}$ and compute a one-step forecast $\hat{y}_{t_0}$.
4. Forecast error:

$$e_i = y_{t_0}^{(i)} - \hat{y}_{t_0}^{(i)}.$$

Method 1: SARIMA (Global)

Model class:

$$\text{SARIMA}(p, d, q)(P, D, Q)_s$$

implemented in practice using SARIMAX (state-space form).

One-step-ahead forecast (conceptually):

$$\hat{y}_{t_0} = \mathbb{E}(y_{t_0} \mid y_1, \ldots, y_{t_0 - 1}; \hat{\theta})$$

where $\hat{\theta}$ are parameters estimated on the full training sample.

Key point: structural breaks in μ_t are not explicitly modelled.

Method 2: Rolling SARIMA

Rolling estimation: re-estimate the model using only the most recent window of size w:

$$\mathcal{W}_{t_0-1} = \{y_{t_0-w}, \ldots, y_{t_0-1}\}.$$

One-step forecast:

$$\hat{y}_{t_0}^{\text{roll}} = \mathbb{E}(y_{t_0} \mid y_{t_0-w}, \ldots, y_{t_0-1}).$$

Motivation:

- adapts faster after breaks by down-weighting older regimes;
- trade-off: fewer observations may increase estimation noise.

Method 3: SARIMA + Break Dummies (Oracle)

Idea: include break dummies as exogenous regressors (break dates assumed known).

Single break dummy:

$$d_t = \mathbf{1}\{t > T_b\}, \qquad y_t = c + \phi y_{t-1} + \delta d_t + u_t.$$

Multiple breaks dummies:

$$d_{1,t} = \mathbf{1}\{t > b_1\}, \qquad d_{2,t} = \mathbf{1}\{t > b_2\},$$

$$y_t = c + \phi y_{t-1} + \delta_1 d_{1,t} + \delta_2 d_{2,t} + u_t.$$

Key point: dummy variables allow the model intercept (mean) to shift at the break date(s).

Estimated Breaks and Evaluation Metrics

Estimated break(s) (grid search idea):

- Single break: choose $\hat{T}_b$ minimizing pre- and post-break fit error.

$$\hat{T}_b = \arg\min_{T_b} \Big(\mathrm{SSE}_{\text{pre}}(T_b) + \mathrm{SSE}_{\text{post}}(T_b) \Big).$$

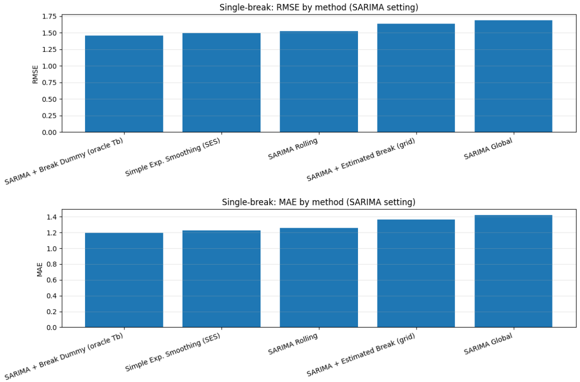
- Two breaks: choose $(\hat{b}_1, \hat{b}_2)$ minimizing total three-segment fit error.

Evaluation metrics (from errors $\{e_i\}_{i=1}^{N}$):

$$\text{Bias} = \frac{1}{N}\sum_{i=1}^{N} e_i, \quad \text{MAE} = \frac{1}{N}\sum_{i=1}^{N} |e_i|, \quad \text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} e_i^2}.$$

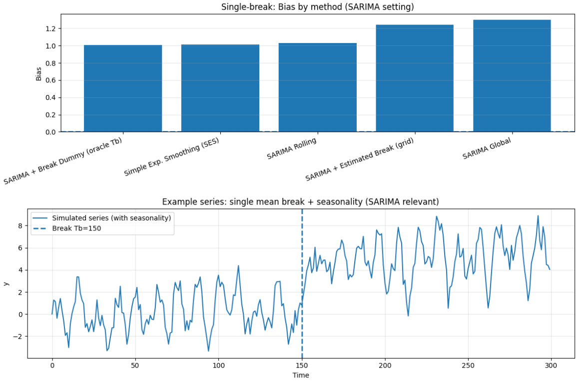
Ranking rule: lower RMSE indicates better forecast accuracy.

Single-break: Forecast Accuracy (RMSE and MAE)



Lower values indicate better forecast accuracy.

Single-break: Bias and Example Simulated Series



Bias closer to 0 is preferred. Dashed line shows $T_b = 150$.

Single-break: Numerical Results (Monte Carlo, $N = 200$)

Method	RMSE	MAE	Bias	N	Fails
SARIMA + Break Dummy (oracle T_b)	1.455137	1.194346	1.005753	200	0
Simple Exp. Smoothing (SES)	1.496262	1.224923	1.015489	200	0
SARIMA Rolling	1.525482	1.257133	1.028853	200	0
SARIMA + Estimated Break (grid)	1.634600	1.367799	1.243084	200	0
SARIMA Global	1.692063	1.423220	1.301679	200	0

Conclusion (RMSE criterion): Best method is **SARIMA + Break Dummy (oracle T_b)**.

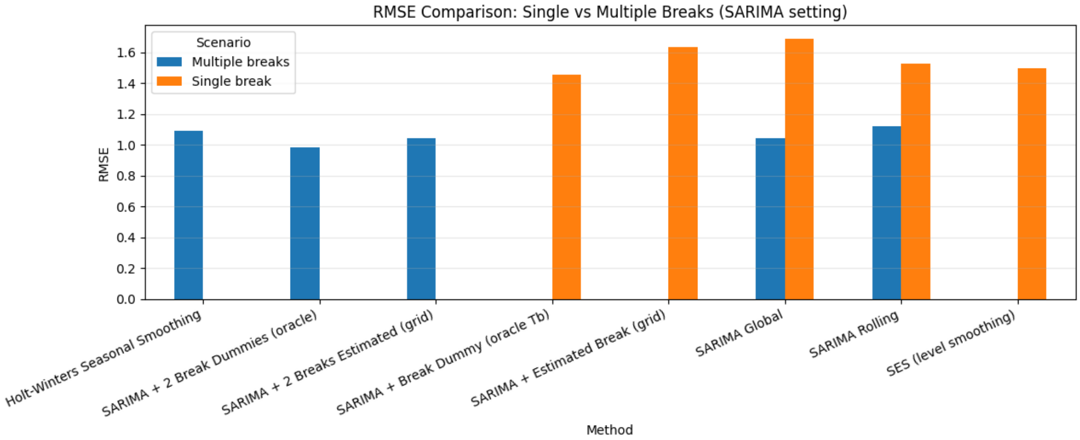
Comparison Table (Single vs Multiple Breaks, $N = 200$)

Method	RMSE	MAE	Bias	N	Fai
SARIMA + 2 Break Dummies (oracle)	0.985466	0.781121	0.105836	200	0
SARIMA Global	1.041633	0.835788	-0.044973	200	0
SARIMA + 2 Breaks Estimated (grid)	1.046196	0.844797	-0.125415	200	0
Holt–Winters Seasonal Smoothing	1.093613	0.857217	-0.001187	200	0
SARIMA Rolling	1.121625	0.884177	0.298843	200	0
SARIMA + Break Dummy (oracle T_b)	1.455137	1.194346	1.005753	200	0
SES (level smoothing)	1.496262	1.224923	1.015489	200	0
SARIMA Rolling	1.525482	1.257133	1.028853	200	0
SARIMA + Estimated Break (grid)	1.634600	1.367799	1.243084	200	0
SARIMA Global	1.692063	1.423220	1.301679	200	0

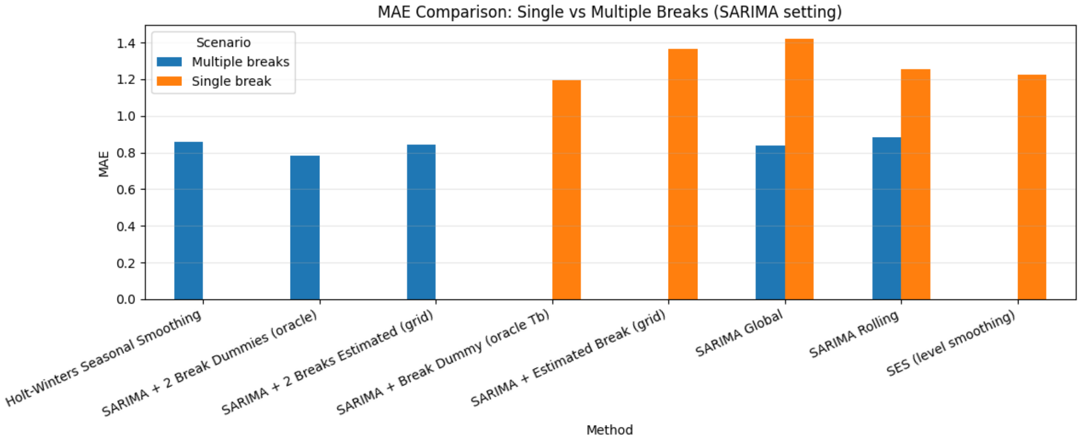
Best method for Single break: SARIMA + Break Dummy (oracle T_b) (RMSE=1.4551, MAE=1.1943)

Best method for Multiple breaks: SARIMA + 2 Break Dummies (oracle) (RMSE=0.9855, MAE=0.7811)

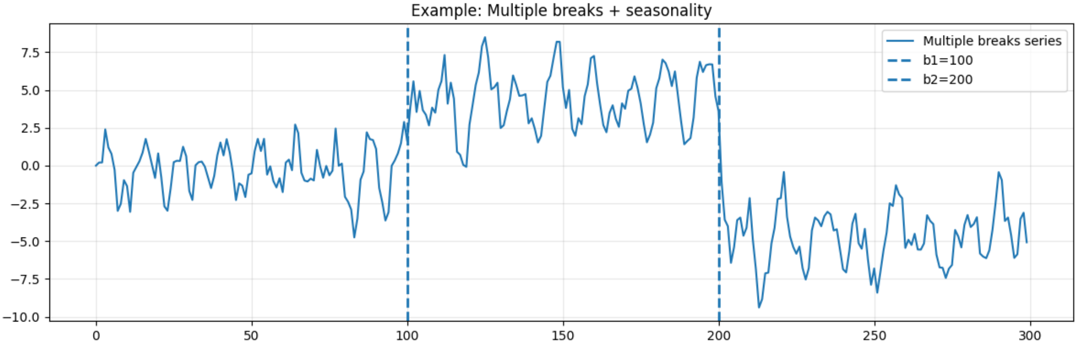
RMSE Comparison: Single vs Multiple Breaks



MAE Comparison: Single vs Multiple Breaks



Example Series: Multiple Breaks with Seasonality



Dashed vertical lines indicate break points $b_1 = 100$ and $b_2 = 200$.