SIR Rumor Spreading

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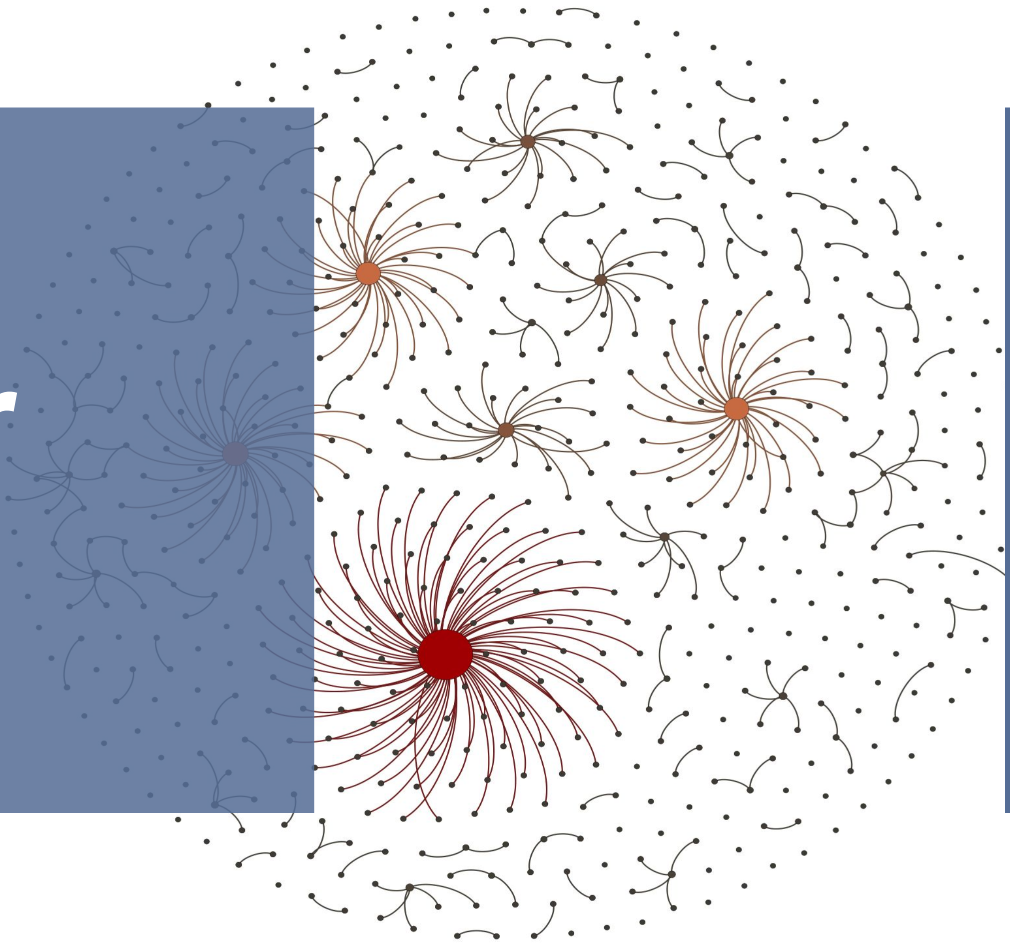
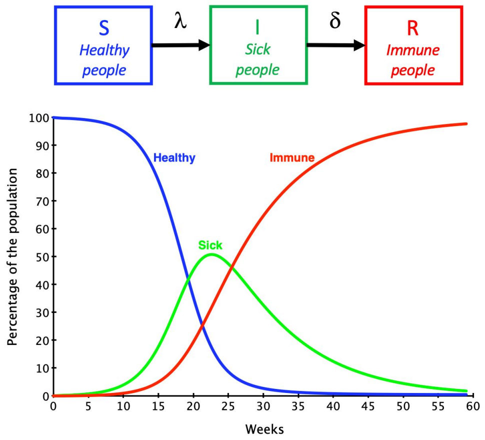
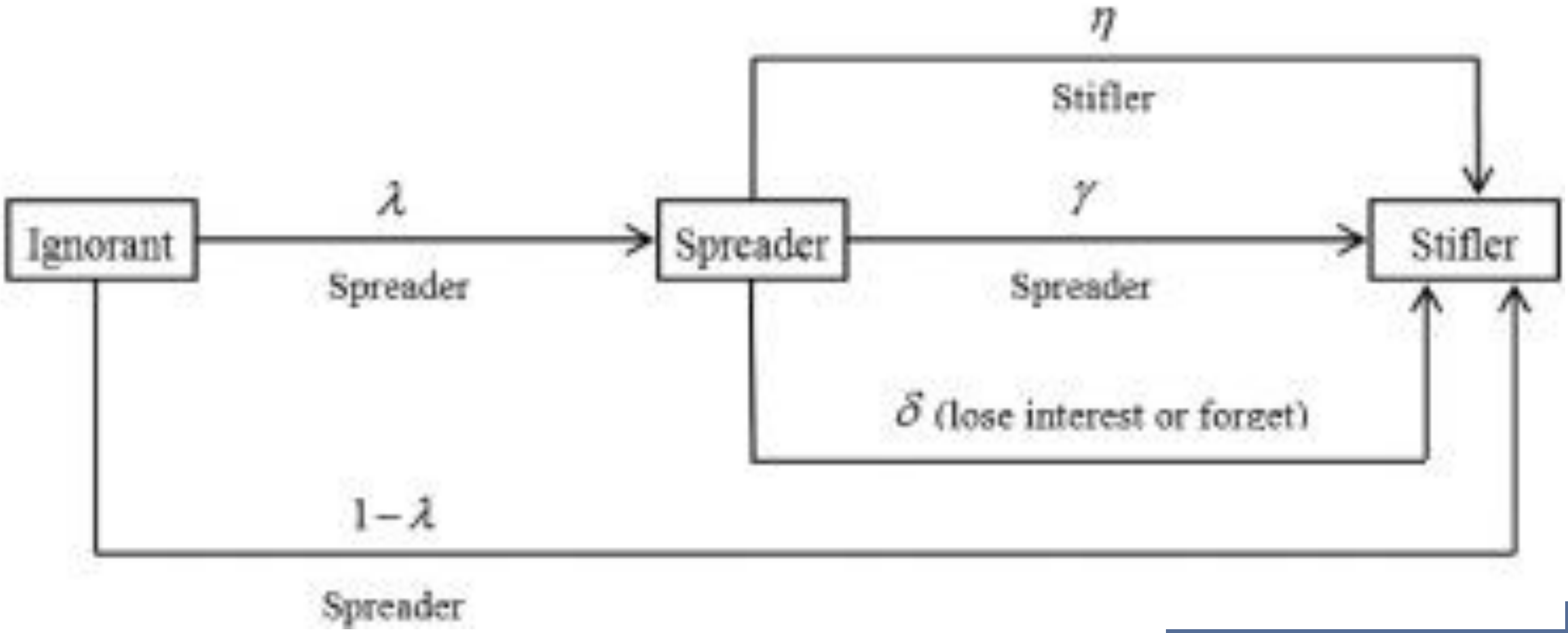


SIR Models

- SIR models are designed for epidemic outbreaks, but their framework can be applied to many other scenarios
- **S** - Susceptible, **I** - Infected, **R** - Recovered
- Assumptions:
    - fixed population
    - each individual interacts with fixed number of people
    - an infected individual is contagious immediately
- Limitations:
    - More simplistic → more limitations
    - Doesn’t account for incubation periods
    - Immunity may expire



Structure for Rumor Spreading SIR Model



Rumor Spreading SIR Model Assumptions

- Fixed Population: $S(t) + R(t) + I(t) = N$
- Each individual interacts with a fixed number of people (k)
- Once someone becomes a Stifler they cannot regain interest
- A Spreader who forgets/loses interest becomes a Stifler
- All individuals start off as either a Spreader or Ignorant
- An Ignorant can only become a Spreader/Stifler through contact with a Spreader & any contact with a Spreader results in an Ignorant becoming a Stifler or Spreader

Framework/Initial Conditions

Variables:

I(t) Ignorants → S(t) Spreaders → R(t) Stiflers

Parameters:

- **λ** : probability I ⟶ S, when ignorant contacts spreader
- **1-λ** : probability I ⟶ R, when ignorant contacts spreader
- **η** : probability S ⟶ R, when spreader contacts stifler
- ***γ*** : probability S ⟶ R, when spreader contacts another spreader
- ***δ*** : probability S ⟶ R, when spreader loses interest/forgets on their own
- **k** : average degree of the network (each individual interacts with k other individuals

Initial Conditions:

$$I(0) = \frac{N-1}{N}, \qquad S(0) = \frac{1}{N}, \qquad R(0) = 0.$$

where $N = 10^6$

Creating the Equations:

Rate of Change of Number of Spreaders → dS(t)/dt

=

Number of Ignorants that become Spreaders when contacting a Spreader → **λ k I(t) S(t)**

-

Number of Spreaders that become Stifler when contacting Spreader → **k *γ* S(t) S(t)**

-

Number of Spreaders that become Stifler when contacting Stifler → **k η S(t) R(t)**

-

Number of Spreaders that forget/lose interest → ***δ*** **S(t)**

$$\Rightarrow \quad \frac{\mathrm{d}S(t)}{\mathrm{d}t} = \lambda\bar{k}I(t)S(t) - \bar{k}S(t)(\gamma S(t) + \eta R(t)) - \delta S(t)$$

Creating the Equations:

Rate of Change of Number of Stiflers→ dR(t)/dt

=

Number of Ignorant that become Stifler when contacting Spreader → **(1-λ) k I(t) S(t)**

+

Number of Spreaders that become Stifler when contacting Spreader → ***γ*** **k S(t) S(t)**

+

Number of Spreaders that become Stifler when contacting Stifler → **η k S(t) R(t)**

+

Number of Spreaders that forget/lose interest → ***δ*** **S(t)**

$$\Rightarrow \quad \frac{dR(t)}{dt} = (1-\lambda)\bar{k}I(t)S(t) + \bar{k}S(t)(\gamma S(t) + \eta R(t)) + \delta S(t)$$

Creating the Equations:

Rate of Change of Number of Ignorants → dI(t)/dt

=

Number of Spreaders/ Stiflers that become Ignorant → **0**

-

Number of Ignorant that become Spreader when Interacting with Spreader → **k λ I(t) S(t)**

-

Number of Ignorant that become Stifler when interacting with Spreader → **(1-λ) k I(t) S(t)**

$$\Rightarrow \quad \frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\bar{k}I(t)S(t)$$

Fixed Points And Stability

Fixed points of the model occur where the number of spreaders is equal to 0.
This results in two possible fixed points:

$$\begin{bmatrix} -kS & -kI & 0 \\ \lambda kS & \lambda kI - 2k\gamma S - k\eta R - \delta & -k\eta S \\ (1-\lambda)kS & (1-\lambda)kI + 2k\gamma S + k\eta R + \delta & k\eta S \end{bmatrix}$$

Jacobian Matrix for the Original Model

1. (0, I*, R*):
- the eigenvalues of the Jacobian are $x = 0, 0, k\lambda I^* - k\eta R^* - \delta$

2. (0, 0 R*):
- The eigenvalues of the Jacobian are $x = 0, 0, k\eta R^* + \delta$

Steady State Analysis:

1. I(t) is always decreasing.

2. S(t) increases only while
$\lambda kI > k(\gamma S + \eta R) + \delta$

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\bar{k}I(t)\,S(t)\,,$$
$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \lambda\bar{k}I(t)\,S(t) - \bar{k}S(t)\,(\gamma S(t) + \eta R(t)) - \delta S(t)\,,$$
$$\frac{\mathrm{d}R(t)}{\mathrm{d}t} = (1-\lambda)\,\bar{k}I(t)\,S(t) + \bar{k}S(t)\,(\gamma S(t) + \eta R(t)) + \delta S(t)\,.$$

⇒ Only stiflers and ignorants left in steady state.

Let R = the density of stiflers in steady state. R ∈ [0,1]

We can solve for R:

$$R = \tfrac{A}{C+1}(1-R) - \left(\tfrac{A}{C+1} - \tfrac{B}{C}\right)(1-R)^{-C} - \tfrac{B}{C} = \tfrac{A}{C+1} - \tfrac{A}{C+1}R$$
$$- \tfrac{A}{C+1}(1-R)^{-C} + \tfrac{B}{C}(1-R)^{-C} - \tfrac{B}{C} = \left(\tfrac{B}{C} - \tfrac{A}{C+1}\right)(1-R)^{-C} - \tfrac{A}{C+1}R$$
$$+ \tfrac{A}{C+1} - \tfrac{B}{C}.$$

Steady State Analysis:

Let z ∈ [0,1). Let

$$f(z) = z - \left(\frac{B}{C} - \frac{A}{C+1}\right)(1-z)^{-C} + \frac{A}{C+1}z - \frac{A}{C+1} + \frac{B}{C} = \left(\frac{A}{C+1} + 1\right)z$$
$$+ \left(\frac{A}{C+1} - \frac{B}{C}\right)(1-z)^{-C} - \frac{A}{C+1} + \frac{B}{C},$$

- Can prove that when $\frac{\lambda}{\delta} > \frac{1}{\bar{k}}$, then ∃ v ∈ (0,1) s.t f(v) = 0.
- I.e: when $\frac{\lambda}{\delta} > \frac{1}{\bar{k}}$, there exists a realistic density v that satisfies the equation for the steady state density of the stiflers (v = R(v)), and thus equilibrium can be reached.

Proposed Models

- Stochastic Model:
  - Introduces individual choice and randomness into the rumor spread
- Model with More Probabilities
  - Introduces new probabilities that affects the rumor spread

Assumptions for Stochastic Model

- We had to assume that an interaction between two individuals is only one-way
- Of course, with every stochastic model, we had to assume Δt to be very small to ensure that everyone interacts with other people at the exact same time
- Instead of treating the population as a network, we had to assume a probability of meeting a certain type (ignorant, spreader, stifler) by calculating the proportion each time step

```
spreaders_prob = S[i] / N / K_BAR
stiflers_prob = R[i] / N / K_BAR
```

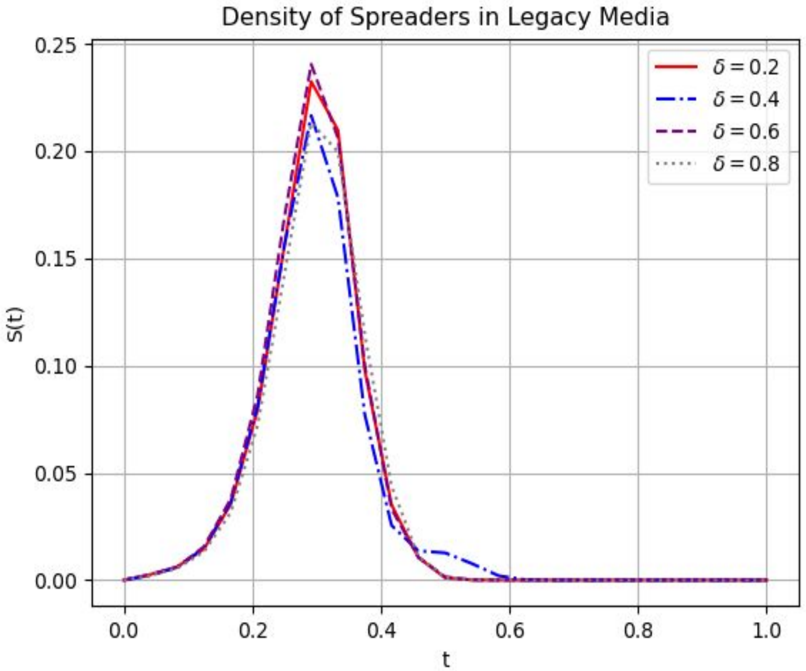
Stochastic Model

- The base model viewed the rumor spreading through a group view rather than individual choice among the individuals
- For our stochastic simulation, we made use of random number generators to provide randomness in individual's choices about the rumor
- Involving randomness makes it more realistic since each individual has their own probabilities in the model, however, it lacks some realism in other aspects

```
for j in range(int(I[i])): # iterate through the ignorants
    r = random.random()
    if r * delta_t < spreaders_prob: # prob of meeting a spreader
        r = random.random()
        if r * delta_t < LAMBDA: # prob of becoming a spreader
            ignorants_to_spreaders += 1
        else: # prob of becoming a stifler
            ignorants_to_stiflers += 1
```

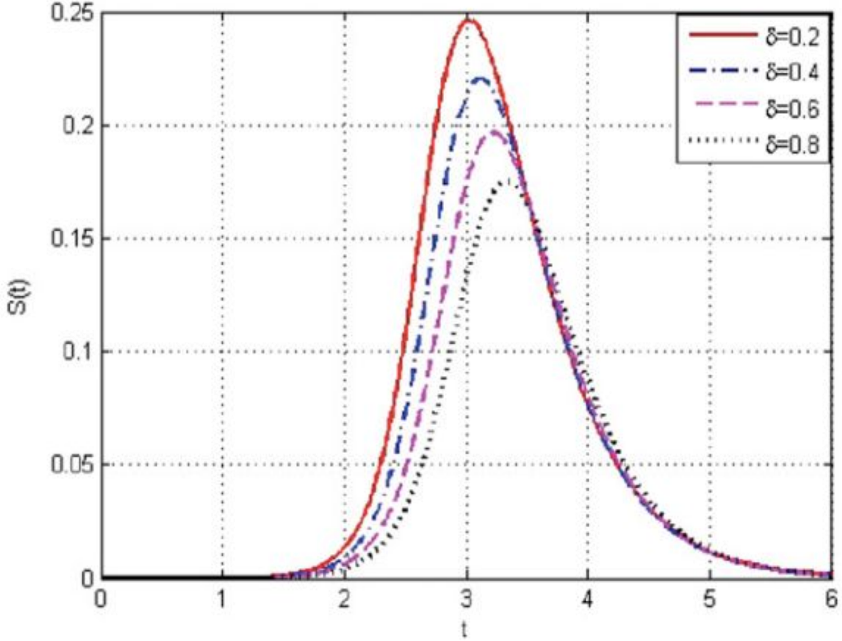
Simulation Results

Stochastic Model



N = 10000, λ=0.5, η=0.2, γ=0.1

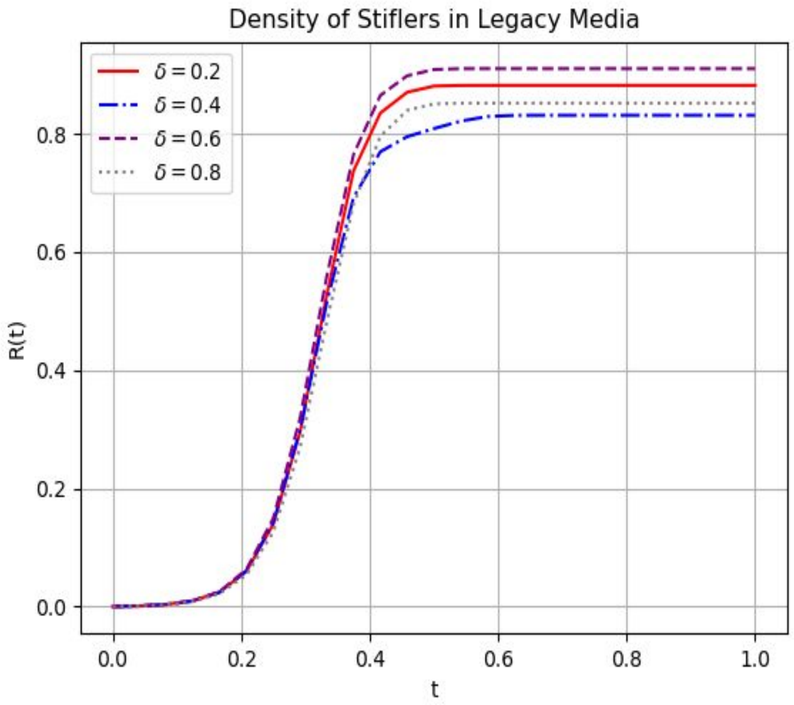
Original Model



N = 10e6, λ=0.5, η=0.2, γ=0.1

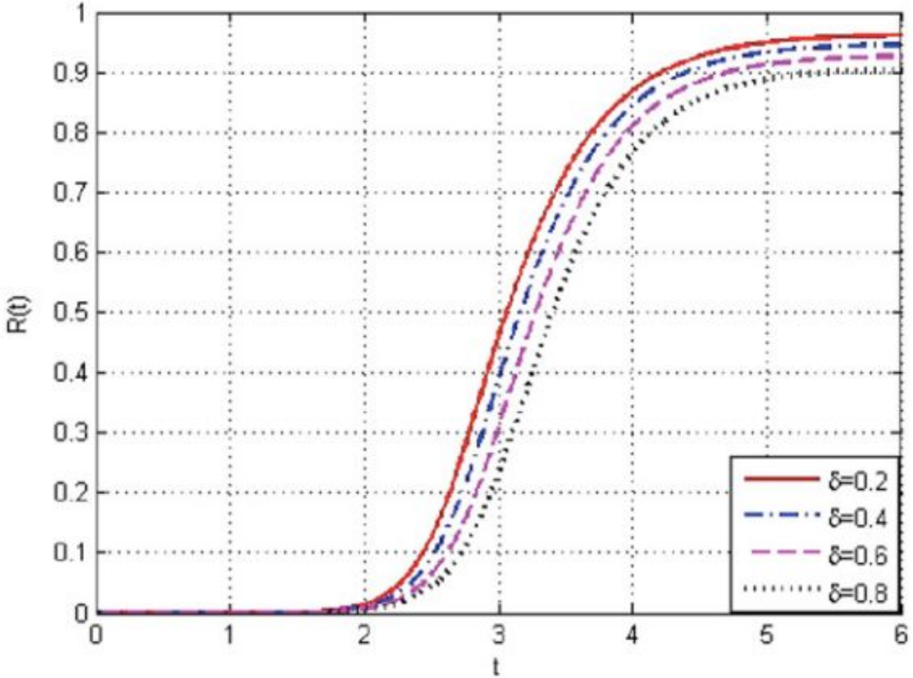
Simulation Results

Stochastic Model



N = 10000, λ=0.5, η=0.2, γ=0.1

Original Model

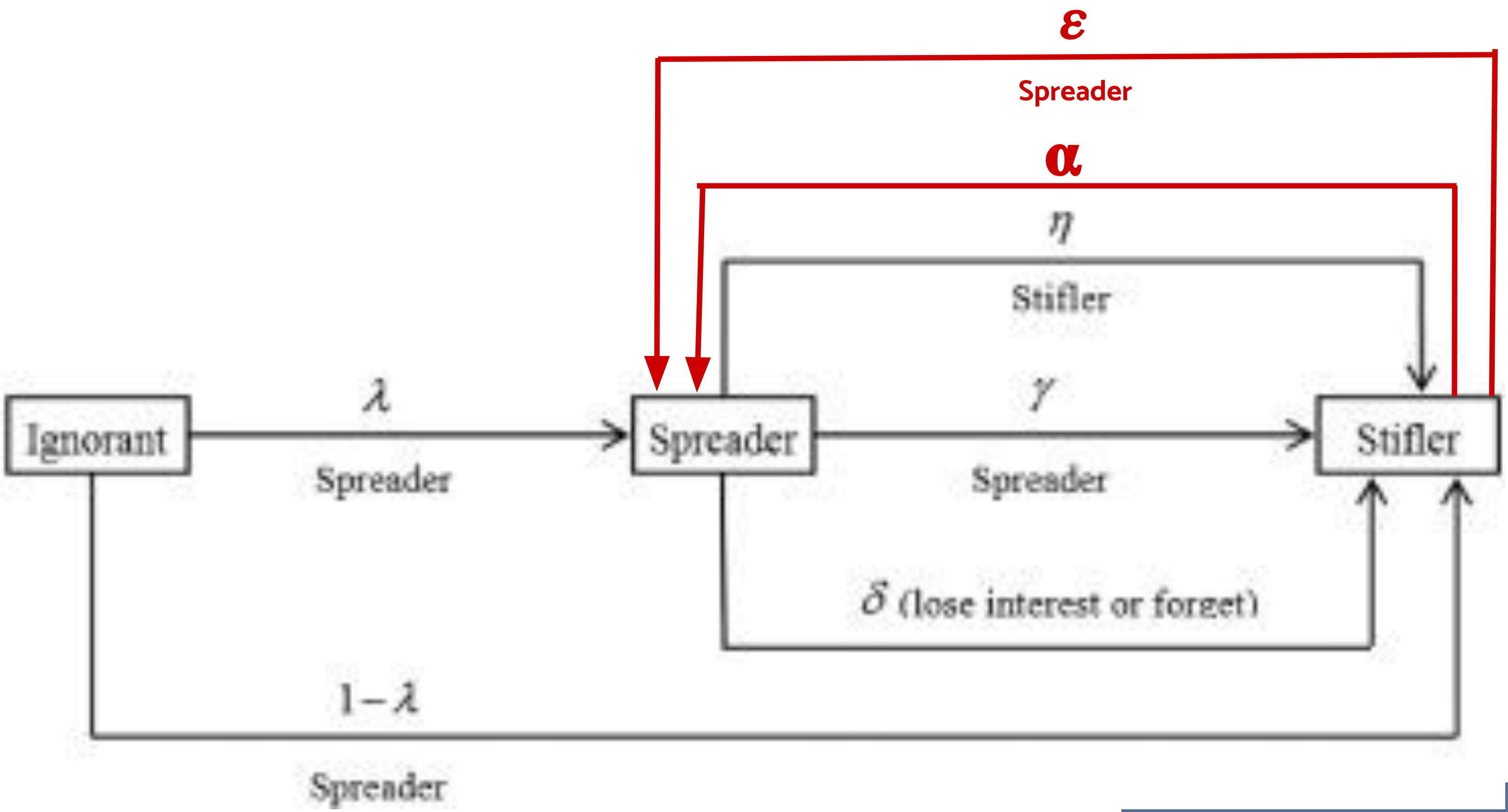


N = 10e6, λ=0.5, η=0.2, γ=0.1

Improvements

- The original model did not consider the possibility that a Stifler can regain interest and return to being a Spreader
  - If an individual could lose interest/forget a rumor, we want to consider the possibilities:
    - an individual regains interest or remembers on their own
    - an individual talks to another spreader, turning the stifler back into a spreader

Structure for Improved Rumor Model



Assumptions for Improved Model

- Most assumptions from original model are the same:
  - Fixed Population: $S(t) + R(t) + I(t) = N$
  - Each individual interacts with a fixed number of people (k)
  - A Spreader who forgets/loses interest becomes a Stifler
  - All individuals start off as either a Spreader or Ignorant
  - An Ignorant can only become a Spreader/Stifler through contact with a Spreader & any contact with a Spreader results in an Ignorant becoming a Stifler or Spreader
- Except:
  - A Stifler can now regain interest and become a Spreader either on their own or by interacting with a Spreader

Improved Framework/Initial Conditions

Variables:

$I(t)$ Ignorants → $S(t)$ Spreaders ↔ $R(t)$ Stiflers

Parameters:

- $\boldsymbol{\lambda}$: probability I → S, when ignorant contacts spreader
- $\boldsymbol{\eta}$: probability S → R, when spreader contacts stifler
- $\boldsymbol{\gamma}$: probability S → R, when spreader contacts another spreader
- $\boldsymbol{\delta}$: probability S → R, when spreader loses interest/forgets independently
- $\boldsymbol{\varepsilon}$: probability R → S, when stifler contacts spreader
- $\boldsymbol{\alpha}$: probability R → S, when stifler regains interest/remembers independently

Initial Conditions:

$$I(0) = \frac{N-1}{N}, \qquad S(0) = \frac{1}{N}, \qquad R(0) = 0.$$

where $N = 10^6$

Creating the Equations for Improved Model:

Rate of Change of Number of Ignorants → dI(t)/dt

=

Number of Spreaders that become Ignorant → **0**

-

Number of Ignorant that become Spreader when Interacting with Spreader → **k λ I(t) S(t)**

-

Number of Ignorant that become Stifler when interacting with Spreader → **(1-λ) k I(t) S(t)**

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} = -\bar{k}I(t)S(t)$$

**Remains the same as original model

Creating the Equations for Improved Model:

Rate of Change of Number of Spreaders → dS(t)/dt

=

Number of Ignorant that become spreaders → **λ k I(t) S(t)**

–

Number of Spreaders that become Stifler when contacting Spreader → **k γ S(t) S(t)**

–

Number of Spreaders that become Stifler when contacting Stifler → **k η S(t) R(t)**

–

Number of Spreaders that forget/lose interest → **δ S(t)**

+

Number of Stiflers that become Spreader when contacting Spreader → **k ε R(t) S(t)**

+

Number of Stiflers that regain interest → **α R(t)**

$$\Rightarrow \quad \frac{dS(t)}{dt} = \lambda k I(t) S(t) - k S(t)[\gamma S(t) + \eta R(t)] - \delta S(t) + \varepsilon k R(t) S(t) + \alpha R(t)$$

Creating the Equations for Improved Model:

Rate of Change of Number of Stiflers→ dR(t)/dt

=

Number of Ignorant that become Stifler when contacting Spreader → **(1-λ) k I(t) S(t)**

+

Number of Spreaders that become Stifler when contacting Spreader → ***γ*** **k S(t) S (t)**

+

Number of Spreaders that become Stifler when contacting Stifler → **η k S(t) R(t)**

+

Number of Spreaders that forget/lose interest → ***δ*** **S(t)**

–

Number of Stiflers that become Spreader when contacting Spreader → **k ε R(t) S(t)**

–

Number of Stiflers that regain interest → ***α*** **R(t)**

$$\Rightarrow \quad \frac{dR(t)}{dt} = (1-\lambda)kI(t)S(t) + kS(t)[\gamma S(t) + \eta R(t)] + \delta S(t) - \varepsilon kR(t)S(t) - \alpha R(t)$$

Fixed Points And Stability

There are two possible fixed points:

$$\begin{bmatrix} -kS & -kI & 0 \\ \lambda kS & \lambda kI - 2k\gamma S - k\eta R - \delta + k\epsilon R & -k\eta S + k\epsilon R + \alpha \\ (1-\lambda)kS & (1-\lambda)kI + 2k\gamma S + k\eta R + \delta - k\epsilon R & k\eta S - k\epsilon R - \alpha \end{bmatrix}$$

Jacobian Matrix for the Improved Model

1. (0, I*, 0):

- the eigenvalues of the Jacobian are

$x = 0$ and $x = \frac{1}{2}\left[\lambda kI - \delta - \alpha \pm \sqrt{(k\lambda I - \delta - \alpha)^2 + 4k\alpha I}\right]$

- There is always 1 positive and 1 negative eigenvalue

Fixed Points And Stability

The second fixed point occurs where

I* = 0 and S*=

$$\frac{1}{2k\gamma}\left[Rk(\epsilon - \eta) - \delta + \sqrt{(Rk(\eta - \epsilon) + \delta)^2 + 4k\gamma\alpha R}\,\right]$$

- This is always real and nonnegative for all realistic values of our parameters.
- At this fixed point, the Jacobian has one zero eigenvalue and two nonzero eigenvalues.

x3 sym

1	2	3
(d - (4*R*a + (d - R*e*k + R*k*n)^2)^(1/2) - R*e*k + R*k*n)/(2*g)	0	0
-(l*(d - (4*R*a + (d - R*e*k + R*k*n)^2)^(1/2) - R*e*k + R*k*n))/(2*g)	-(4*R*a + (d - R*e*k + R*k*n)^2)^(1/2)	a - (e*(d - (4*R*a + (d - R*e*k + R*k*n)^2)^(1/2) - R*e*k + R*k*n))/(2*g) + (n*(d - (4*R*a + (d ...
((l - 1)*(d - (4*R*a + (d - R*e*k + R*k*n)^2)^(1/2) - R*e*k + R*k*n))/(2*g)	(4*R*a + (d - R*e*k + R*k*n)^2)^(1/2)	(e*(d - (4*R*a + (d - R*e*k + R*k*n)^2)^(1/2) - R*e*k + R*k*n))/(2*g) - a - (n*(d - (4*R*a + (d -...

Jacobian Matrix Evaluated at the Fixed Point

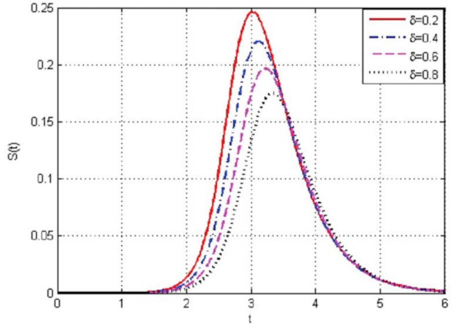
Numerical Simulation - default values

- **N** : size 10^6
- **k** : size 10
- $\boldsymbol{\lambda}$: probability 0.5
- $\boldsymbol{\eta}$: probability 0.2
- $\boldsymbol{\gamma}$: probability 0.1
- $\boldsymbol{\delta}$: probability 0.2
- $\boldsymbol{\epsilon}$: probability 0.1
- $\boldsymbol{\alpha}$: probability 0.1

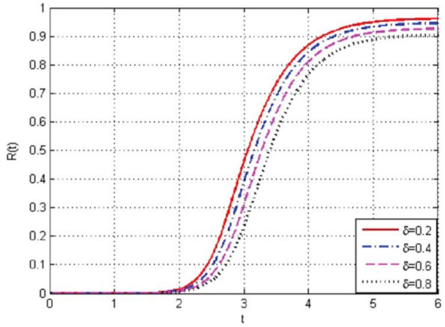
```python
def rumor_func_2(S_0=10e-6,k=10,l=0.5,g=0.1,n=0.2,d=0.2,a=0.1,ep=0.1,
                 t=10,dt=0.0001,I=True,S=True,R=True,
                 lab_i="ignorant",lab_s="spreader",lab_r="stifler"):
    gen=math.floor(t/dt)
    rumors= np.zeros([gen,3])
    rumors[0]=[1-S_0,S_0,0]
    for i in np.arange(gen-1):
        dI=(-1)*k*rumors[i][0]*rumors[i][1]
        #formula for rate of change of ignorants
        dS=l*k*rumors[i][0]*rumors[i][1]
        -k*rumors[i][1]*(g*rumors[i][1]+n*rumors[i][2])
        -d*rumors[i][1]+ep*k*rumors[i][1]*rumors[i][2]+a*rumors[i][2]
        #formula for rate of change of spreaders
        dR=-dI-dS
        #formula for rate of change of stiflers
        change=np.array([dI,dS,dR])
        #derivatives for each subpopulations
        rumors[i+1]=rumors[i]+change*dt
        #subpopulations for next generation using Euler's method
    #loops t/dt times
        x_0=np.arange(gen)*dt
    if I==True:
        sns.lineplot(y=rumors.transpose()[0],x=x_0,label=lab_i)
    if S==True:
        sns.lineplot(y=rumors.transpose()[1],x=x_0,label=lab_s)
    if R==True:
        sns.lineplot(y=rumors.transpose()[2],x=x_0,label=lab_r)
```

**dS is split into 3 lines to fit on slide.*
Code for dS is only one line.

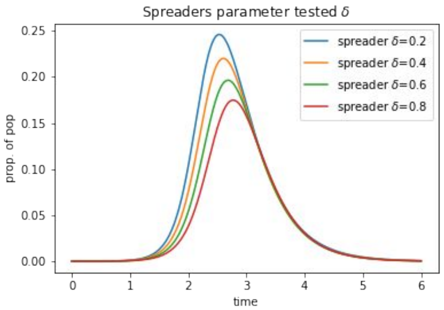
Numerical Simulation - original model

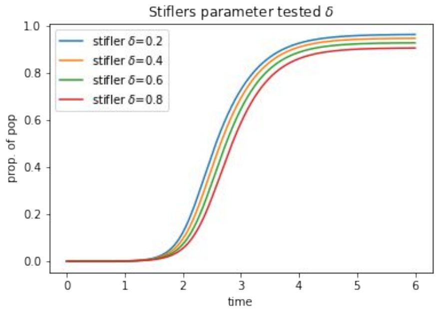


(a) Density of spreaders in legacy media.

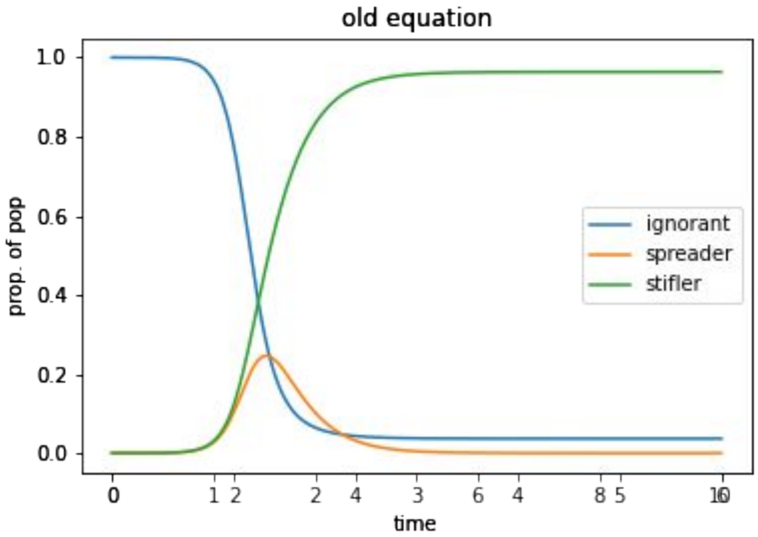


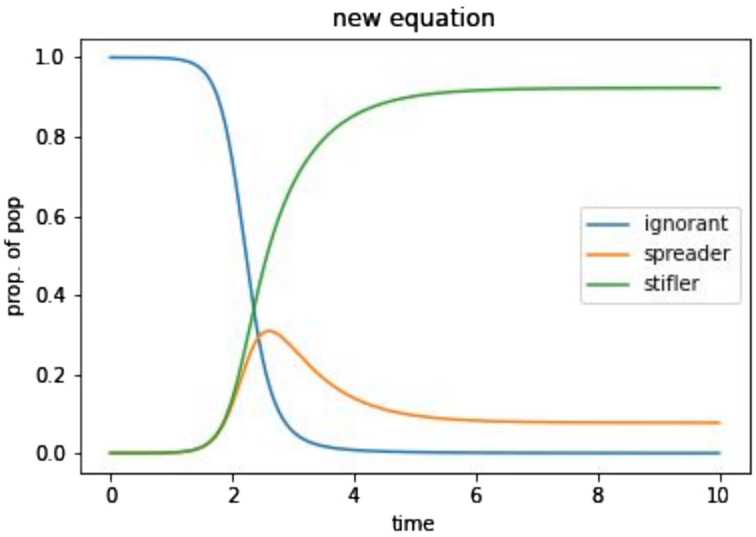
(b) Density of stiflers in legacy media.



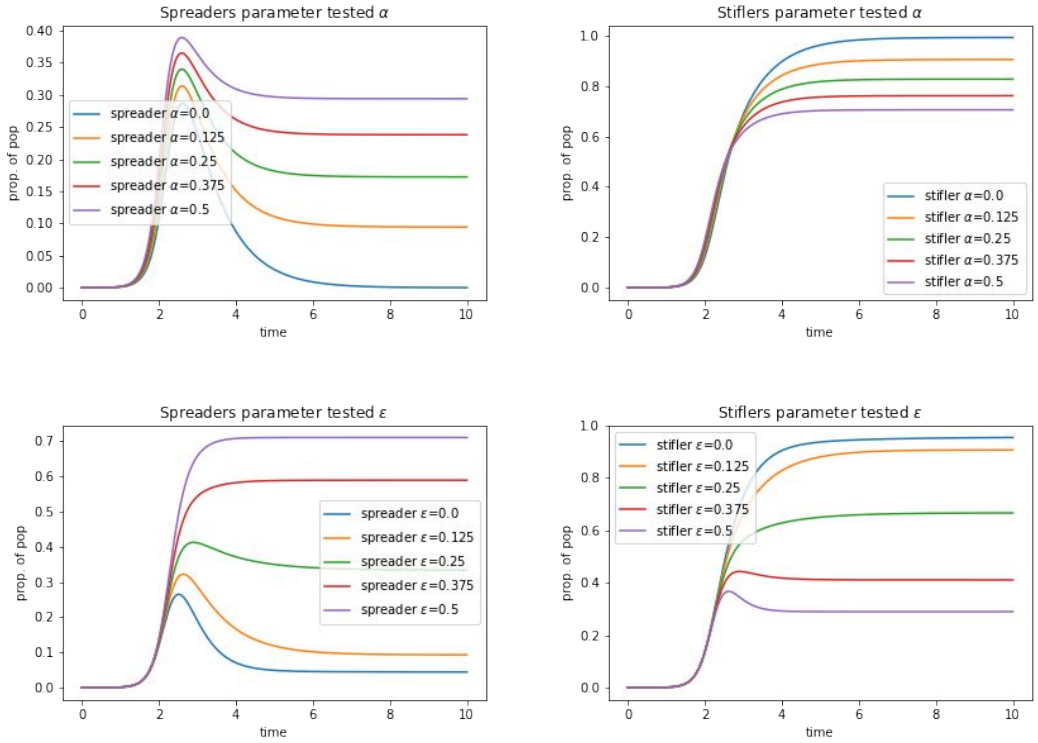


Numerical Simulation - improvements





Numerical Simulation - endemic rumor



Limitations and Future Questions:

Future Questions for the SIR Model:

1. Network connectivity: Our model only considered a homogeneous network, which is unrealistic. It would be interesting to consider heterogeneous networks, and, particularly, the effect that the out-degree of the original spreader has on the spread of the network.
2. Vary initial conditions: It might be unrealistic to initially have only one spreader. Future projects could consider multiple initial spreaders.

Limitations of Our Models:

1. Ignores the individuality of nodes: Nodes may have different levels of susceptibility to becoming a spreader or stifler. Different nodes may also have different levels of activity in the system. Further, whether a node becomes a spreader or stifler after an interaction might depend on the individual relationship between the nodes.
2. Treats the rumor as a static object: Rumors generally change as the rumour is passed along.

Works Cited

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