How can adolescent height be mathematically modeled for US girls?

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"Wow! You're so tall!" At 5' 10" (177 cm), I tend to hear this phrase a lot. While I stand out amongst many of my friends at school, at home I feel short. In the midst of my towering cousins, I'm destined for the front row in any family pictures. This prompted me to research the average height for American girls my age and how I compare.

This posed a more interesting question as I thought about the rate at which people typically grow across different years. I've always found it so interesting that people grow taller and taller, and then suddenly stop growing and they will remain that height for the rest of their life. Even at 17, I still find it hard to believe that I will be this height for the rest of my life.

While everyone matures at a different time, I wanted to determine if there was an overall growth pattern that could be mathematically modeled. After researching growth statistics, I plotted the data for the average heights of American girls at different ages, as shown in Figure 1 below. Looking at the data, I was surprised that I couldn't immediately think of any common functions that resembled the curve.

Age (years)	Height (cm)
5	112.5669546
6	119.4294477
7	125.6331436
8	131.4797074
9	137.2674035
10	143.280893
11	149.5433668
12	155.0457394
13	158.8160249
14	161.0629124
15	162.2600828
16	162.7590867
17	162.8814282
18	162.9666324

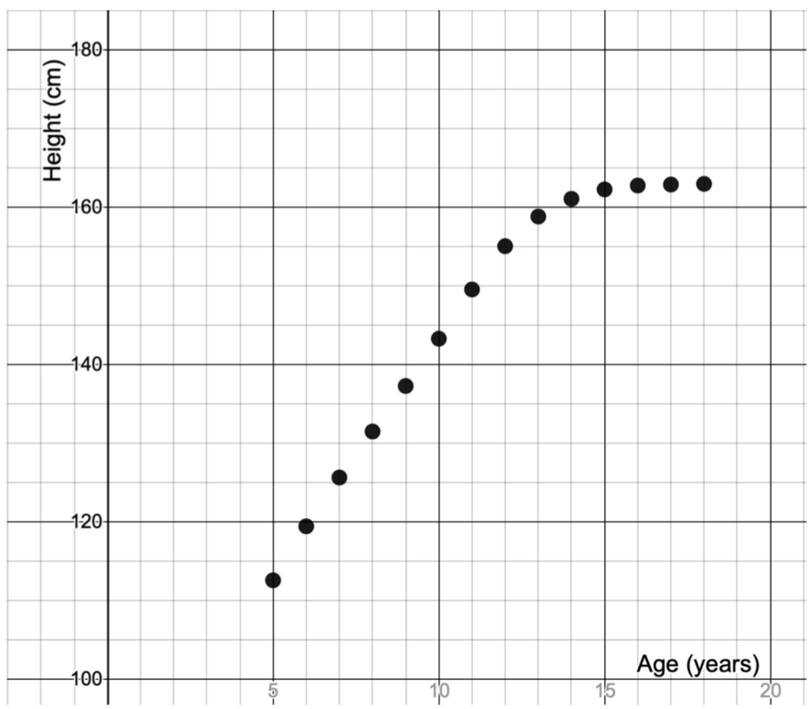

Figure 1: National average heights of adolescent American girls listed by year from ages 5 to 18 years old[1] (see Appendix 1).

[1] "NCD_RisC_Lancet_2020_height_child_adolescent_United States of America." *RisC*, www.ncdrisc.org/data-downloads-height.html.

Even more intriguing, I struggled to find other examples of mathematical functions for modeling this curve after researching online. The goal of my research paper is to determine the best fit curve of average adolescent girls' height in the US over time. From my model, I will be able to analyze how height changes over time and determine how I compare.

After experimenting with various regression models on a calculator, I found that none seemed to best fit all of the data. Often, the function would fit a specific portion of the graph better, rather than the entire data set. This led me to consider that a piecewise model might take this into consideration and result in a better fit. Given that humans grow at different rates throughout their life and level off after their late teens, this appeared to be a better approach.

I decided to break the data up into two sections. The first portion of the data seemed to fit more of a linear model, while the second half was curved. After experimenting with the best cut off point, I found that at year 12 ($x = 12$) was the optimal place where this data could be split. Twelve also made sense as an appropriate dividing point as it is around the age where puberty begins in women.

Age (years)	Height (cm)
5	112.5669546
6	119.4294477
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9	137.2674035
10	143.280893
11	149.5433668
12	155.0457394

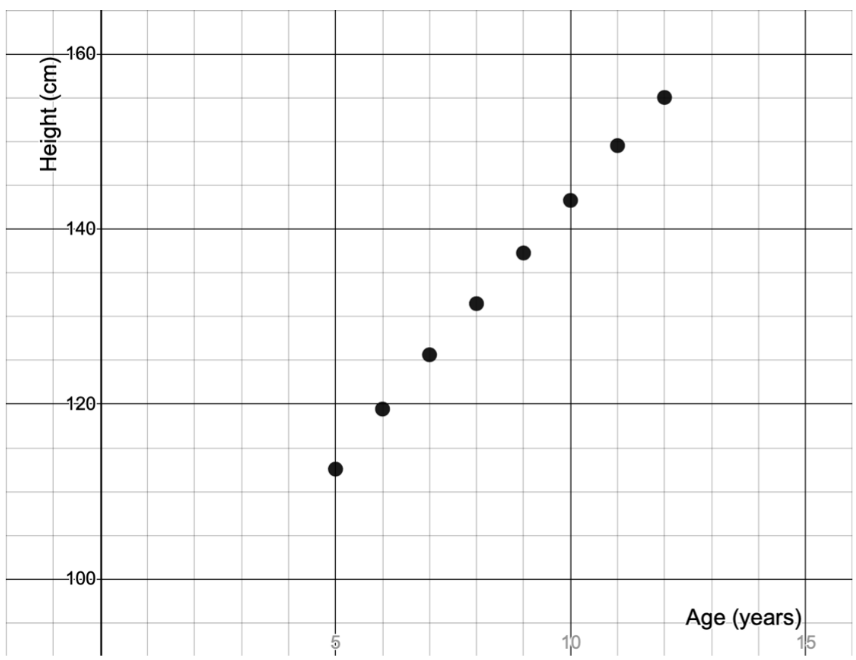


Figure 2: Original data from Figure 1, truncated to show height values from ages 5 to 12.

Visually, Figure 2 appeared to follow a linear model. So I decided to calculate Pearson's correlation coefficient, which will indicate if a linear regression model is appropriate by determining its linearity[2]. Below, Figure 3 gives both the raw x and y values for the data set and the additional terms needed for the calculation of the r value.

x (Age, years)	y (Height, cm)	x^2	y^2	xy
5	112.5669546	25	12671.31927	562.834773
6	119.4294477	36	14263.39298	716.5766862
7	125.6331436	49	15783.68677	879.4320052
8	131.4797074	64	17286.91346	1051.837659
9	137.2674035	81	18842.34006	1235.406632
10	143.280893	100	20529.4143	1432.80893
11	149.5433668	121	22363.21855	1644.977035
12	155.0457394	144	24039.18131	1860.548873
Σx =68	Σy =1074.246656	Σx^2 =620	Σy^2 =145779.4667	Σxy =9384.422593

Figure 3: Data from Figure 2 in preparation for calculating the correlation coefficient, r.

With these values, I then used the following formula to calculate the data's r value.

$$r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{n(\sum x^2) - (\sum x)^2} \times \sqrt{n(\sum y^2) - (\sum y)^2}}$$

$$r = \frac{(8)(9384.422593) - (68)(1074.246656)}{\sqrt{(8)(620) - (68)^2} \times \sqrt{(8)(145779.4667) - (1074.246656)^2}}$$

$$r \approx 0.9997$$

Given that my r value is so close to 1, my data set, therefore, has a very strong, positive linear correlation between age and height for ages 5 to 12. This means that a linear regression model would be a good fit for the data, so I chose to use a least squares regression line. However, I realized that finding the model of best fit for this data would likely result in an equation that did not go exactly through the last point of my linear data, at $(12,\ 155.0457394)$. Because I am creating a piecewise function, I want my entire function to be continuous, so the challenge was to fit a line that specifically went through that particular value at age 12.

[2] Peterson, Richard. "Linear Regression by Hand." *Medium*, Towards Data Science, 11 Apr. 2020, https://towardsdatascience.com/linear-regression-by-hand-ee7fe5a751bf.

h = Age (years)	k = Height (cm)
-7	-42.4787848
-6	-35.6162917
-5	-29.4125958
-4	-23.566032
-3	-17.7783359
-2	-11.7648464
-1	-5.5023726
∑h = -28	∑k = -166.1192592

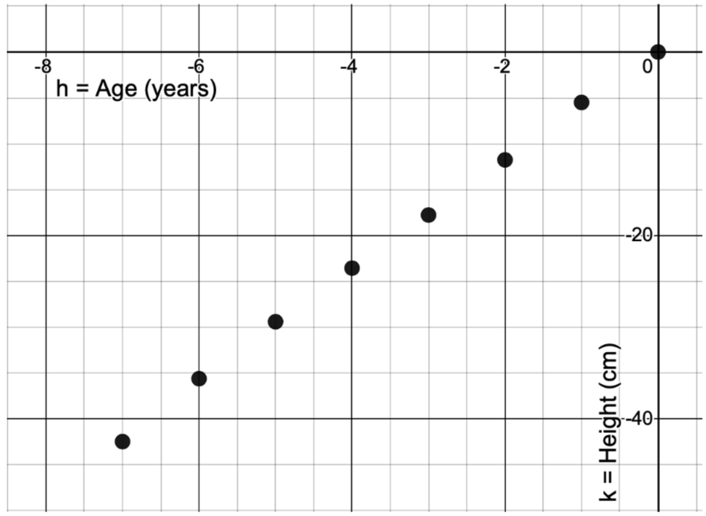


Figure 4: Data from Figure 2 translated to have final data point at origin.

The solution I came up with was to transform the data so the last point was shifted to the origin, as calculated below and shown graphically in Figure 4.

$$h = x - 12,\ \ k = y - 155.0457394$$

Then, I could find the least squares regression line slope while forcing the y-intercept to be zero. In doing so, I am sacrificing a degree of complete accuracy for the optimal least squares equation of this dataset. But, given my extremely high r value, I know that my data is nearly linear. The difference will be minimal, and I can ensure that the piecewise function is continuous overall.

With the y-intercept established as 0, I can use the equation below to solve for the slope of the linear regression model[3] from the transformed data in Figure 4.

$$k = mh + b \quad m = slope \quad b = y\ intercept$$

$$b = \frac{\sum k - (m)(\sum h)}{n} = 0 \Rightarrow m = \frac{\sum k}{\sum h}$$

$$m = \frac{-166.1192592}{-28} = 5.932830686$$

After solving for the slope, I then transformed my data back from the origin onto the graph, shown in Figure 5, for my final equation of:

$$y - 155.0457394 = 5.932830686(x - 12)$$

[3] Ibid.

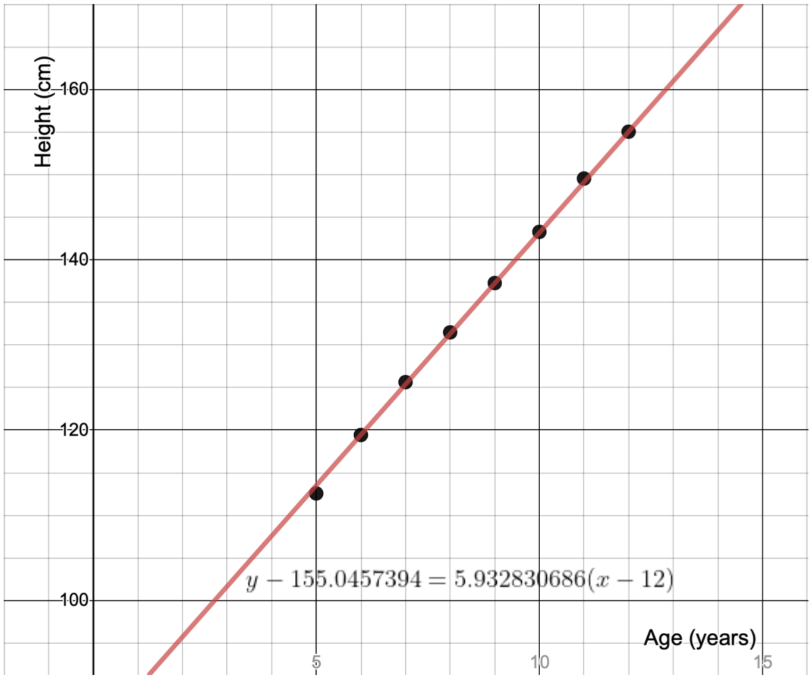


Figure 5: Data from Figure 2 along with the calculated linear regression fit, specifically through the data point at age 12.

My next challenge was determining the best model for the second, curved section of my data. I decided the best approach would be to linearize my data set by altering the x and y axes. Then, I could find a least squares regression line of the linearized data and transform it back.

x	ln(y)
12	5.043720166
13	5.067746456
14	5.081795049
15	5.089200497
16	5.092271112
17	5.093022502
18	5.093545471

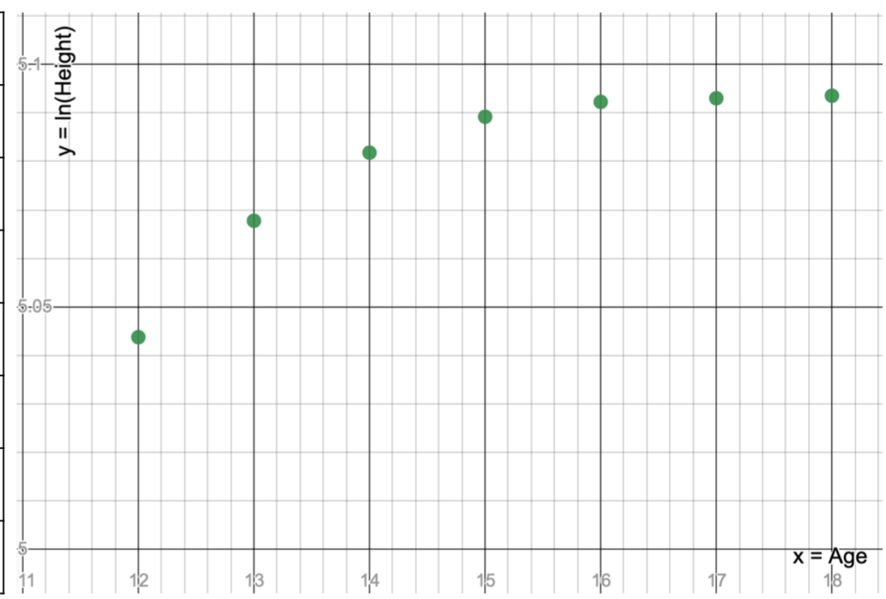


Figure 6: An evaluation of the second half of data from Figure 1 plotted as (x, ln(y)) to see if an exponential model would fit the nonlinear data.

If an exponential function would apply, a graph of plotting x versus the natural log of y would look linear. However, Figure 6's exponential transformation did not linearize the data, which means that the values does not follow an exponential relationship between age and height.

ln(x)	ln(y)
2.48490665	5.043720166
2.564949357	5.067746456
2.63905733	5.081795049
2.708050201	5.089200497
2.772588722	5.092271112
2.833213344	5.093022502
2.890371758	5.093545471

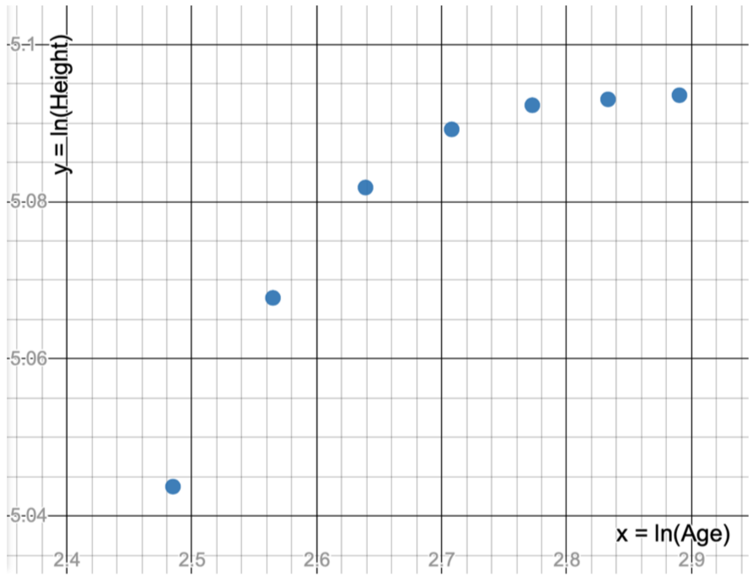


Figure 7: An evaluation of the second half of height data from Figure 1 plotted as (ln(x), ln(y)) to see if an exponential model would fit the nonlinear data.

Next, I considered if it follows a power function model, where transforming both the x and y axis with a natural log would result in a line. However, as seen in Figure 7, my data was still curved. So, it wasn't a power function either. I proceeded to continue attempting to linearize the data through other transformations to no avail, for my data never displayed a linear relationship.

I finally determined that this data was not conforming to any common mathematical models, indicating why I found so little information published online. While I could not find an established mathematical model, I decided that a cubic spline function would have the needed flexibility to best fit the data. A cubic spline function is a piecewise compilation of multiple cubic functions that tie together in specific ways to form a continuous curve through particular points in the data. For example, fitting a spline function, $S(x)$, through 4 fixed points, (x_1, x_2, x_3, x_4), would result in three cubic splines in the following piecewise format:

$$S(x) = \begin{cases} S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 & \text{if } x_1 \leq x \leq x_2 \\ S_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3 & \text{if } x_2 \leq x \leq x_3 \\ S_3(x) = a_3 + b_3(x - x_3) + c_3(x - x_3)^2 + d_3(x - x_3)^3 & \text{if } x_3 \leq x \leq x_4 \end{cases}$$

A spline function ensures that the graph goes through specific fixed points, known as "knots." Although, in order to create an overall smooth curve, the first derivatives of each spline function must match on both sides of the knot. I decided to use every other datapoint, or every two years, for my curved section, because I did not want to overfit my data. Therefore, I can tell if my spline function is a good model for the data if the left-out points follow the overall curve of the spline function. The points highlighted in red in Figure 8 are the designated fixed points that my spline function will be based on.

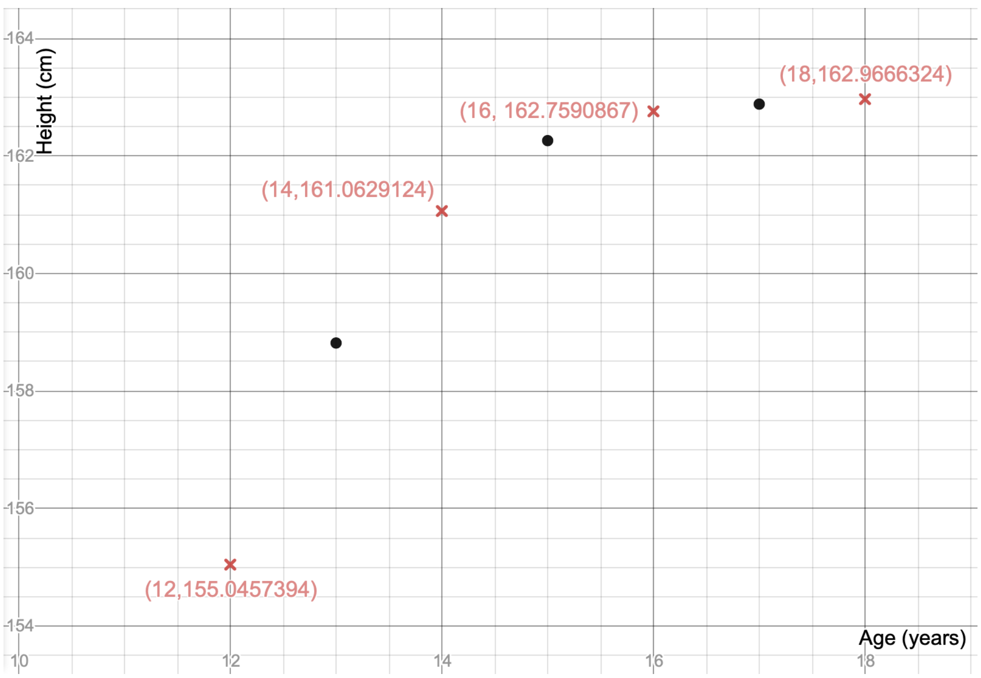


Figure 8: Selected knots for cubic spline function using second half of height data of Figure 1.

From the given formula for cubic equations, I could substitute the data points at ages 12, 14, 16, and 18 to create these equations:

$$S(x) = \begin{cases} S_1(x) = a_1 + b_1(x - 12) + c_1(x - 12)^2 + d_1(x - 12)^3 & \text{if } 12 \leq x \leq 14 \\ S_2(x) = a_2 + b_2(x - 14) + c_2(x - 14)^2 + d_2(x - 14)^3 & \text{if } 14 \leq x \leq 16 \\ S_3(x) = a_3 + b_3(x - 16) + c_3(x - 16)^2 + d_3(x - 16)^3 & \text{if } 16 \leq x \leq 18 \end{cases}$$

The equations above present me with a total of 12 constants, from which I can determine the following 6 conditions based on the specific height data values at those ages:

$$S_1(12) = S(12) \Rightarrow a_1 = 155.0457394$$
$$S_1(14) = S(14) \Rightarrow a_1 + 2b_1 + 4c_1 + 8d_1 = 161.0629124$$
$$S_2(14) = S(14) \Rightarrow a_2 = 161.0629124$$
$$S_2(16) = S(16) \Rightarrow a_2 + 2b_2 + 4c_2 + 8d_2 = 162.7590867$$
$$S_3(16) = S(16) \Rightarrow a_3 = 162.7590867$$
$$S_3(18) = S(18) \Rightarrow a_3 + 2b_3 + 4c_3 + 8d_3 = 162.9666324$$

These equations, in blue above, ensure that the spline function will go through each of my four knots, resulting in the continuous curve, $S(x)$. To make sure the splines are smoothly connected, I also need the first and second derivatives to be equivalent for each internal knot. This stipulation presents me with the following 4 conditions, listed in blue:

$$S_1'(x) = b_1 + 2c_1(x-12) + 3d_1(x-12)^2$$
$$S_2'(x) = b_2 + 2c_2(x-14) + 3d_2(x-14)^2$$
$$S_3'(x) = b_3 + 2c_3(x-16) + 3d_3(x-16)^2$$

$$S_1'(14) = S_2'(14) \Rightarrow b_1 + 4c_1 + 12d_1 = b_2 \Rightarrow b_1 + 4c_1 + 12d_1 - b_2 = 0$$
$$S_2'(16) = S_3'(16) \Rightarrow b_2 + 4c_2 + 12d_2 = b_3 \Rightarrow b_2 + 4c_2 + 12d_2 - b_3 = 0$$

$$S_1''(x) = 2c_1 + 6d_1(x-12)$$
$$S_2''(x) = 2c_2 + 6d_2(x-14)$$
$$S_3''(x) = 2c_3 + 6d_3(x-16)$$

$$S_1''(14) = S_2''(14) \Rightarrow 2c_1 + 12d_1 = 2c_2 \Rightarrow 2c_1 + 12d_1 - 2c_2 = 0$$
$$S_2''(16) = S_3''(16) \Rightarrow 2c_2 + 12d_2 = 2c_3 \Rightarrow 2c_2 + 12d_2 - 2c_3 = 0$$

Lastly, to ensure my overall piecewise approach smoothly captures all ages from 5 to 18, I wanted the first spline of $S(x)$, that begins at age 12, to be continuous with the initial linear regression line. To achieve this, I set the first derivative of the first spline function, S_1 at $x = 12$, equal to the fitted slope of my linear regression function. While the function will already be continuous, adding this condition will provide for a smooth transition from the linear to non-linear portion of the graph. Although, given that $x = 18$ years is the last point of the data set, I left it as a natural spline, where the second derivative is equal to zero. These two requirements led to the following two additional conditions:

$$S_1'(12) = 5.932830686 \Rightarrow b_1 = 5.932830686$$

$$S_2''(18) = 0 \Rightarrow 2c_3 + 12d_3 = 0$$

To solve this multivariable system of equations, I set them up as the following matrix equation $Ax = B$and then solved using the inverse matrix method (see Appendix 2 for detailed calculations).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 12 \end{bmatrix} \times \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} = \begin{bmatrix} 155.0457394 \\ 161.0629124 \\ 161.0629124 \\ 162.7590867 \\ 162.7590867 \\ 162.9666324 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5.932830686 \\ 0 \end{bmatrix}$$

$$x = A^{-1} \times B \;\Rightarrow \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} = \begin{bmatrix} 155.0457 \\ 5.9328 \\ -2.0750 \\ 0.3064 \\ 161.0629 \\ 1.3100 \\ -0.2364 \\ 0.0027 \\ 162.7591 \\ 0.3971 \\ -0.2200 \\ 0.0367 \end{bmatrix}$$

After calculating all of the constants' values in my original equations, I then substituted them in to create my final spline function equation below.

$$S(x) = \begin{cases} 155.0457 + 5.9328(x-12) - 2.0750(x-12)^2 + 0.3064(x-12)^3 & \text{if } 12 \le x \le 14 \\ 161.0629 + 1.3100(x-14) - 0.2364(x-14)^2 + 0.0027(x-14)^3 & \text{if } 14 \le x \le 16 \\ 162.7591 + 0.3971(x-16) - 0.2200(x-16)^2 + 0.0367(x-16)^3 & \text{if } 16 \le x \le 18 \end{cases}$$

Recalling the linear regression line from before, I rearranged the equation, setting it equal to y.

$$y = 5.9328(x-12) + 155.0457$$

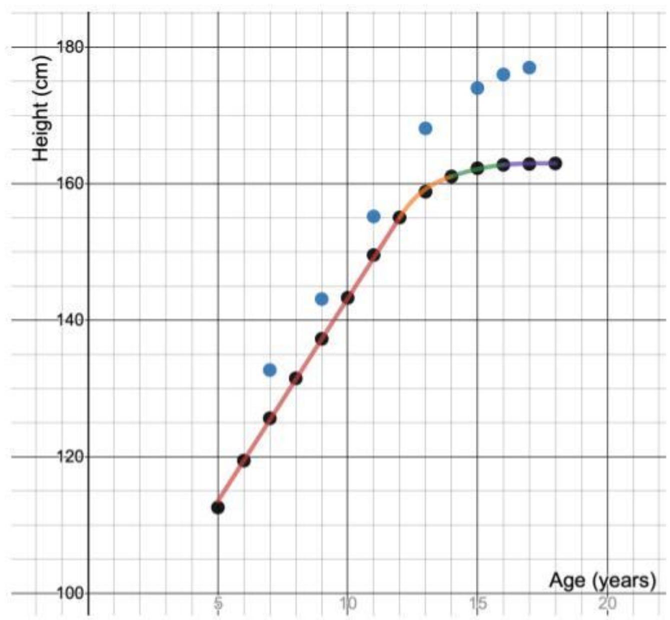
I then combined the spline function with the regression line for the following overall solution to the piecewise curve:

$$h(x) = \begin{cases} 5.9328(x-12)+155.0457, & \text{if } 5 \le x \le 12 \\ 155.0457+5.9328(x-12)-2.0750(x-12)^2+0.3064(x-12)^3, & \text{if } 12 \le x \le 14 \\ 161.0629+1.3100(x-14)-0.2364(x-14)^2+0.0027(x-14)^3, & \text{if } 14 \le x \le 16 \\ 162.7591+0.3971(x-16)-0.2200(x-16)^2+0.0367(x-16)^3, & \text{if } 16 \le x \le 18 \end{cases}$$

Looking at these equations, the coefficient values decreased as age increased on average. This makes sense for as time passes, the rate of growth gradually decreases until adulthood, when it ultimately stops.

Below, Figure 9 shows this same decline in rate of growth graphically. In addition, I also added the data points of my own height at different ages in blue.

My Personal Height Data	
Age (years)	Height (cm)
7	132.7
9	143.1
11	155.2
13	168.1
15	174.0
16	175.9
17	177.2



Key:
- national average height data
- my height data

- 5.9328(x-12) + 155.0457
- 155.0457 + 5.9328(x-12) - 2.0750(x-12)² + 0.3064(x-12)³
- 161.0629 + 1.3100(x-14) - 0.2364(x-14)² + 0.0027(x-14)³
- 162.7591 + 0.3971(x-16) - 0.2200(x-16)² + 0.0367(x-16)³

Figure 9: Final graph of original averaged American girl height data along with calculated piecewise fit, $h(x)$. Additionally, my personal height values shown in blue, as indicated in the table on the left.

While my personal data is not sampled at exact year intervals, I still found it interesting to compare my own graph with data from America's national average.

In the first half of Figure 9's graph, I am only slightly taller than the average. However, at around the 12 year mark, I seem to have a slight "growth spurt" and grow noticeably taller than the average girl for my age. Because the data that I found was the overall average of the entire United States, individual growth spurts are likely not noticeable after distilling all of the data down.

While the model fits the height data fairly well, it is limited because spline functions do not extrapolate well. As a result, it cannot be accurately applied to predict data outside of the range of ages from 5 to 18. However, given that an individual's height is nearly established by 18, it is unlikely extrapolation would need to be applied to older ages. Additionally, I can see that the omitted data points do appear to follow the curve outlined by the spline function, which demonstrates that this model was an appropriate fit for the data given.

As an American, I chose to only include the data of heights of girls in the United States to see how I compared to others. However, I would love to continue to apply this mathematical approach to model data of other countries and see how I compare globally. My ethnic background is Montenegrin and Norwegian, so maybe my height is not as uncommon somewhere else. It would also be interesting to compare rates of growth between countries and determine if there are genetic differences in the onset age of puberty.

Given that height is a quality shared by every human being, it was surprising to see the lack of mathematical models available, and I wonder if future mathematicians will develop a more sophisticated model to predict height with just one function.

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"Solving Systems of Linear Equations". Matrix Calculator, https://matrixcalc.org/en/slu.html.

Appendix 1: Raw Height Data

Adolescent Child Height Data, United States of America, Girls

Year	Age group	Mean height	Mean height lower 95% uncertainty interval	Mean height upper 95% uncertainty interval	Mean height standard error
2005	5	112.5669546	111.9559474	113.1635095	0.30458444
2006	6	119.4294477	118.9237244	119.9315675	0.254502495
2007	7	125.6331436	125.1540319	126.0972267	0.242430839
2008	8	131.4797074	131.0047782	131.9422484	0.242602313
2009	9	137.2674035	136.7871745	137.7303762	0.242105483
2010	10	143.280893	142.7944945	143.7488246	0.244460975
2011	11	149.5433668	149.0497583	150.0231434	0.249549825
2012	12	155.0457394	154.5345574	155.5463002	0.260150444
2013	13	158.8160249	158.2850456	159.335111	0.27147142
2014	14	161.0629124	160.4981159	161.6021345	0.283112719
2015	15	162.2600828	161.6643902	162.8234633	0.29743943
2016	16	162.7590867	162.1341505	163.3691379	0.319470314
2017	17	162.8814282	162.2195748	163.5356177	0.340470397
2018	18	162.9666324	162.2722351	163.6473952	0.354349532

Appendix 2: Solving Systems of Linear Equations, Matrix Calculator

$$A \cdot X = B$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 \\ 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 12 \end{pmatrix} \quad B = \begin{pmatrix} \frac{775228697}{5000000} \\ \frac{402657281}{2500000} \\ \frac{402657281}{2500000} \\ \frac{1627590867}{10000000} \\ \frac{1627590867}{10000000} \\ \frac{407416581}{2500000} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{2966415343}{500000000} \\ 0 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-57}{104} & \frac{57}{104} & \frac{15}{104} & \frac{-15}{104} & \frac{-3}{104} & \frac{3}{104} & \frac{-3}{13} & \frac{3}{52} & \frac{-7}{52} & \frac{1}{26} & \frac{-45}{52} & \frac{-1}{52} \\ \frac{31}{208} & \frac{-31}{208} & \frac{-15}{208} & \frac{15}{208} & \frac{3}{208} & \frac{-3}{208} & \frac{3}{26} & \frac{-3}{104} & \frac{7}{104} & \frac{-1}{52} & \frac{19}{104} & \frac{1}{104} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-21}{52} & \frac{21}{52} & \frac{-15}{52} & \frac{15}{52} & \frac{3}{52} & \frac{-3}{52} & \frac{-7}{13} & \frac{-3}{26} & \frac{7}{26} & \frac{-1}{13} & \frac{-7}{26} & \frac{1}{26} \\ \frac{9}{26} & \frac{-9}{26} & \frac{-15}{52} & \frac{15}{52} & \frac{3}{52} & \frac{-3}{52} & \frac{6}{13} & \frac{-3}{26} & \frac{-3}{13} & \frac{-1}{13} & \frac{3}{13} & \frac{1}{26} \\ \frac{-15}{208} & \frac{15}{208} & \frac{19}{208} & \frac{-19}{208} & \frac{-9}{208} & \frac{9}{208} & \frac{-5}{52} & \frac{9}{104} & \frac{5}{104} & \frac{3}{52} & \frac{-5}{104} & \frac{-3}{104} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{26} & \frac{-3}{26} & \frac{-9}{26} & \frac{9}{26} & \frac{-3}{13} & \frac{3}{13} & \frac{2}{13} & \frac{-7}{13} & \frac{-1}{13} & \frac{4}{13} & \frac{1}{13} & \frac{-2}{13} \\ \frac{-9}{104} & \frac{9}{104} & \frac{27}{104} & \frac{-27}{104} & \frac{-21}{104} & \frac{21}{104} & \frac{-3}{26} & \frac{21}{52} & \frac{3}{52} & \frac{-3}{13} & \frac{-3}{52} & \frac{-7}{52} \\ \frac{3}{208} & \frac{-3}{208} & \frac{-9}{208} & \frac{9}{208} & \frac{7}{208} & \frac{-7}{208} & \frac{1}{52} & \frac{-7}{104} & \frac{-1}{104} & \frac{1}{26} & \frac{1}{104} & \frac{11}{104} \end{pmatrix}$$

$$X = A^{-1} \cdot B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-57}{104} & \frac{57}{104} & \frac{15}{104} & \frac{-15}{104} & \frac{-3}{104} & \frac{3}{104} & \frac{-3}{13} & \frac{3}{52} & \frac{-7}{52} & \frac{1}{26} & \frac{-45}{52} & \frac{-1}{52} \\ \frac{31}{208} & \frac{-31}{208} & \frac{-15}{208} & \frac{15}{208} & \frac{3}{208} & \frac{-3}{208} & \frac{3}{26} & \frac{-3}{104} & \frac{7}{104} & \frac{-1}{52} & \frac{19}{104} & \frac{1}{104} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-21}{52} & \frac{21}{52} & \frac{-15}{52} & \frac{15}{52} & \frac{3}{52} & \frac{-3}{52} & \frac{-7}{13} & \frac{-3}{26} & \frac{7}{26} & \frac{-1}{13} & \frac{-7}{26} & \frac{1}{26} \\ \frac{9}{26} & \frac{-9}{26} & \frac{-15}{52} & \frac{15}{52} & \frac{3}{52} & \frac{-3}{52} & \frac{6}{13} & \frac{-3}{26} & \frac{-3}{13} & \frac{-1}{13} & \frac{3}{13} & \frac{1}{26} \\ \frac{-15}{208} & \frac{15}{208} & \frac{19}{208} & \frac{-19}{208} & \frac{-9}{208} & \frac{9}{208} & \frac{-5}{52} & \frac{9}{104} & \frac{5}{104} & \frac{3}{52} & \frac{-5}{104} & \frac{-3}{104} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{26} & \frac{-3}{26} & \frac{-9}{26} & \frac{9}{26} & \frac{-3}{13} & \frac{3}{13} & \frac{2}{13} & \frac{-7}{13} & \frac{-1}{13} & \frac{4}{13} & \frac{1}{13} & \frac{-2}{13} \\ \frac{-9}{104} & \frac{9}{104} & \frac{27}{104} & \frac{-27}{104} & \frac{-21}{104} & \frac{21}{104} & \frac{-3}{26} & \frac{21}{52} & \frac{3}{52} & \frac{-3}{13} & \frac{-3}{52} & \frac{-7}{52} \\ \frac{3}{208} & \frac{-3}{208} & \frac{-9}{208} & \frac{9}{208} & \frac{7}{208} & \frac{-7}{208} & \frac{1}{52} & \frac{-7}{104} & \frac{-1}{104} & \frac{1}{26} & \frac{1}{104} & \frac{11}{104} \end{pmatrix} \cdot \begin{pmatrix} \frac{775228697}{5000000} \\ \frac{402657281}{2500000} \\ \frac{402657281}{2500000} \\ \frac{1627590867}{10000000} \\ \frac{1627590867}{10000000} \\ \frac{407416581}{2500000} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{2966415343}{500000000} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{775228697}{5000000} \\ \frac{2966415343}{500000000} \\ \frac{-10789793907}{5200000000} \\ \frac{15933795117}{52000000000} \\ \frac{402657281}{2500000} \\ \frac{17030245199}{13000000000} \\ \frac{-768448023}{3250000000} \\ \frac{28494387}{10400000000} \\ \frac{1627590867}{10000000} \\ \frac{1290623159}{3250000000} \\ \frac{-5720168379}{26000000000} \\ \frac{1906722793}{52000000000} \end{pmatrix}$$