Axiom of Choice Presentation

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Date: 16 February 2022

1 Warm-up: Infinite Hat Problem (James)

Accordikng to Mathologer Video and Mauro's 2230 Discussion Problem Set:

Infinitely many prisoners stand in a line, each with a black or white hat on their head. Each prisoner can see the hats on the prisoners in front of him, but not his own hat. They take turns guessing the color of their hats from back to front. If a prisoner guesses correctly, he is set free. If a prisoner guesses incorrectly, he is executed. Before guessing, the prisoners can get together and come up with a strategy.

1. Suppose the prisoners don't know whether the previous prisoners guessed correctly or not. Can you ensure that infinitely many prisoners are saved? (This is easy and does not require AoC.)
2. Suppose the prisoners don't know whether the previous prisoners guessed correctly or not. Can you ensure that only finitely many prisoners are executed?
3. Suppose the others can hear what the previous prisoners guessed and are informed about the result (that is, if one guesses incorrectly, he is executed on the spot). Can you ensure that at most one prisoner dies?

Solution: Two sequences are said to be close if they differ only in finitely many places (that is, they have the same tail). One can verify that "closeness" is an equivalence relation (reflexive, symmetric, transitive). Classify the set of all sequences into equivalence classes using the "closeness" relation. (How many equivalence classes are there?) Choose one element from each of these equivalence classes and make it the representative of the class. Each prisoner guess according to that predetermined representative.

As for the case where prisoners are informed about the result, the first prisoner's guess should be the parity of the difference between the actual sequence and the representative. The other prisoners can then use this information to determine whether the color of their hat is different from the representative and thus deduce the color of their own hat.

2 Axiom of Choice (Andres)

2.1 Formal Statement of Axiom

According to wikipedia:
For any set X of nonempty sets (i.e, $\emptyset \notin X$), choice function f $\exists f : X \to \bigcup X$ such that $\forall A \in X$, it is true that $f(A) \in A$ (i.e., existence of a choice function that is defined on X that maps each set of X to an element of the set).

3 Transfinite Induction and Transfinite Recursion (James)

3.1 Total, Partial, and Well Ordering

According to Wikipedia:

A total order (aka linear order) is a binary relation $\leq$ on some set X which satisfies the following for all $a, b, c \in X$:

1. $a \leq a$ (reflexive)
2. $a \leq b \wedge b \leq c \Rightarrow a \leq c$ (transitive)
3. $a \leq b \wedge b \leq a \Rightarrow a = b$ (antisymmetric)
4. $a \leq b \vee b \leq a$ (strongly connected/total)

A non-strict partial order only has reflexivity, antisymmetry, and transitivity.
A strict partial order is a relation $<$ on some set X which satisfies the following for all $a, b, c \in X$:

1. $\neg(a < a)$ (Irreflexivity)
2. $a < b \wedge b < c \Rightarrow a < c$ (Transitivity)
3. $a < b \Rightarrow \neg(b < a)$ (Asymmetry)

A well-order (aka well-ordering) is a total order $\leq$ on S such that every subset T of S has a smallest element under $\leq$. That is, $\forall T \subseteq S, \exists s_0 \in T$ such that $\forall s \in T, s_0 \leq s$.

3.2 Transfinite Induction

According to videos by Antonio Montalban (Set Theory Playlist):
Ordinals go "beyond" the natural/finite numbers.

Definition 3.1. Let $(A, <)$ be a linear ordering. Then for $t \in A$, we define $\operatorname{seg} t = \{x \in A : x < t\}$ to be the initial segment up to t.

Theorem 3.2 (Transfinite Induction)**.** *Let $(A, <)$ be a well-ordering. If a subset $B \subseteq A$ satisfies $\forall t \in A(\operatorname{seg} t \subseteq B \Rightarrow t \subset B)$, then $B = A$.*

Where's the base case? Since $\varnothing \subset B$ and $\operatorname{seg} 0 = \varnothing$, we have $0 \in B$.

Proof. Suppose by contradiction that $B \neq A$. Thus $A \setminus B \neq \varnothing$. Since A is a well-ordering, $A \setminus B$ has a least element. Call it t_0. Thus for all $t < t_0$ we have $t \notin (A \setminus B)$. This implies that for all $t < t_0$ we have $t \in B$, so $\operatorname{seg} t_0 \subseteq B$. Thus by our assumption, $t_0 \in B$, which is a contradiction. $\square$

3.3 Transfinite Recursion

According to videos by Antonio Montalban (Set Theory Playlist):

Definition 3.3. Let A, B be sets. Then B^A denotes the functions from A to B, and $B^{<A}$ denotes the functions that have domain $\operatorname{seg} t$ for some $t \in A$ and codomain B.

For example, $B^{<\omega} = B^0 \cup B^1 \cup B^2 \ldots$, which is the set of finite tuples from B.
The idea of the transfinite recursion is to define a function by defining the value of a function at a point using the values of the function at the previous points.

Theorem 3.4 (Transfinite Recursion)**.** *Let $(A, <)$ be a well-ordering. Given a function $G : B^{<A} \to B$, there is a unique function $F : A \to B$ that satisfies*

$$\forall t \in A, F(t) = G(F|_{\operatorname{seg} t}).$$

where $F|_{\operatorname{seg} t}$ is F restricted to the set $\operatorname{seg} t$.

We'll not prove this theorem in our presentation. If you're interested, please check out the Set Theory Playlist by Antonio Montalban (See links at the end of this document).

4 Well-Ordering Theorem (James)

(Note: This theorem is also known as Zermelo's Theorem.)

4.1 Statement of Theorem

According to Wikipedia:

Theorem 4.1. *Every set can be well-ordered. That is, for every set X, there exists a total order R such that every nonempty subset of X has a member which is minimal under R.*

4.2 Well-Ordering Theorem Implies Axiom of Choice

According to Proof Wiki:

Proof. Suppose the Well-Ordering Theorem holds. Let $\mathcal{F}$ be a collection of nonempty sets. Then we can find a well ordering for $\mathcal{F}$. That is, there exists a total order R such that for each set $A \in \mathcal{F}$, there exists a minimal element $a \in A$. The choice function can map the set A to the minimal element of A. Thus, we have constructed a choice function, so the Axiom of Choice holds. □

4.3 Axiom of Choice Implies Well-Ordering Theorem

According to UChicago REU paper:

Proof. Let S be a set, and let F be a choice function for all nonempty subsets of S. Then we can create a well ordering as follows:

Let a_0 be $F(S)$, a_1 be $F(S \setminus \{a_0\})$, and continue this process using transfinite recursion: $a_k = F(S \setminus \{a_i : i < k\})$. Then the set $\{a_0, a_1, \ldots, a_k, \ldots\}$ will eventually contain every element of F, and we can define a well-ordering using the subscripts: $a_i \leq a_j$ iff $i \leq j$. □

5 Zorn's Lemma (Andres)

5.1 Necessary Background

Information in this section from wikipedia.

1. Preordered set: a set equipped with a binary relation that adds relations between elements in the set.
2. Partially ordered set (poset): A type of proset with a binary relation that compares some but not every element with every other element, designating which one precedes which. An example is the set of all subsets of $\{x, y, z\}$ with the binary relation of inclusion. $\emptyset$ and $\{x, y\}$ have a relation (among many others) but, for instance, $\{x\}$ and $\{y\}$ do not.
3. Chain: a subset of a poset that is totally ordered. Usually, chains are subsets of posets with inclusion as their binary relation.
4. Upper bound: An upper bound of a subset of a proset (e.g., a chain) is an element of the proset that is "greater than or equal to" (with respect to how this is defined by the binary relation) every element in the subset.
5. Maximal element: A maximal element of a subset S of a proset is an element that is an upper bound for the subset S ("not smaller than" any element in S) that is in S.

5.2 Statement of Theorem

From Wikipedia:

Suppose a non-empty partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

5.3 WOP Implies Zorn's Lemma

The equivalence to the Axiom of Choice can be shown by equating the Well-Ordering Theorem (which we have proven is equivalent to Axiom of Choice) to Zorn's Lemma. Proof from UChicago REU.

Proof. Let $(S, <)$ be a non-empty poset with. By Well-Orering Theorem, S can be well-ordered, resulting in the enumeration $S = \{p_0, .., p_i, ...\}$. Then, using transfinite recursion, we can create a set C with $c_0 = p_0$ and $\forall j > 0$, let $c_j = p_\gamma$ where p_γ is an upper bound of $C_j = \{c_0, ..., c_k | k < j\}$ and $p_\gamma \notin C_j$. C_j is always a chain and p_γ exists unless $c_{\gamma-1}$ is a maximal element. This ensures we will eventually obtain a maximal element of S, resulting in Zorn's Lemma. $\square$

5.4 Zorn's Lemma Implies WOP

Proof from Math Studies Algebra.

Proof. Consider non-empty set X (since empty set is trivially well-ordered). Then consider $(Y, \leq_Y)$ where $Y \subseteq X$ and $\leq_Y$ is a well-ordering on Y. Define partial order on the set of all such $(Y, \leq_Y)$ where $(Y, \leq_Y) \preceq (Y', \leq_{Y'})$ if $Y \subseteq Y'$, Y is an initial segment of Y' in $Y \leq_{Y'}$, and the two orderings $\leq_Y$ and $\leq_{Y'}$ agree on the set Y.

X nonempty means the poset is nonempty. Let C be a chain in this poset, we can define $\bar{Y} = \bigcup_{(Y, \leq_Y) \in C} Y$ and define $x \leq_{\bar{Y}} y$ whenever $x \leq_Y y$ for some $(Y, \leq_Y) \in C$. This makes $\leq_{\bar{Y}}$ a well-ordering on $\bar{Y}$. Here's why. Suppose set $S \in \bar{Y}$ is non-empty and take $(Y, \leq_Y) \in C$ as some pair in the chain such that $S \cap Y \neq \emptyset$. Define $u := min_{\leq_Y}(S \cap Y)$ (i.e., minimum with respect to $\leq_Y$). Then u is a min for S w.r.t $\leq_{\bar{Y}}$, since for some arbitrary $s \in S$, s is either in Y, which means $u \leq_Y s$ means $u \leq_{\bar{y}} s$, or s is not in Y, which means $u \leq_{\bar{Y}}$, since Y is an initial segment from $\bar{Y}$. This means $(\bar{Y}, \leq_{\bar{Y}})$ is an upper bound for each C, and so we can apply Zorn's Lemma. By Zorn's, the poset contains a maximal element $(Y, \leq_Y)$. Retrieving the X from the beginning, we can show that $Y = X$. Suppose $Y \neq X$ and $x \in X \setminus Y$, then we can extend $(Y, \leq_Y)$ to a set $Y \cup \{x\}$ by defining x to be greater than every element in Y, but this would contradict maximality, so $X = Y$. This conclusion that $X = Y$ means X can be well-ordered. $\square$

6 Every vector space has a basis (Andres if time permits)

6.1 Statement of Theorem

Per wikipedia:

Every vector space V has a basis B that is a set of vectors whose linear combinations form every possible $\vec{v} \in V$.

6.2 Zorn's Lemma Implies Every Vector Space has a Basis

We do our equivalence by proving Zorn's Lemma (which we have equated to Axiom of Choice already) implies that every vector space has a basis. Proof from UChicago REU.

Proof. Let V be a vector space, and let X be the set of linearly independent subsets of V. X we know is non-empty, as it at least contains the empty set, since that is an independent subset of V, and X becomes partially ordered by inclusion.

Let Y be a subset of X totally ordered by $\subseteq$ (inclusion) and let $L_Y = \bigcup y, \forall y \in Y$. Because of Y's total ordering, every subset of L_Y is a subset of an element in Y, which is linearly independent, therefore L_Y is linearly independent, which then means $L_Y \in X$. This makes L_Y an upper bound of Y in $(X, \subseteq)$. We then have the conditions for Zorn's Lemma, meaning X has a maximal element L_{max}. We can then show that L_{max} is a basis for V (we already know it is a linearly independent subset of V).

If $\vec{w} \in V$ were not in *span* (L_{max}), then $\vec{w} \notin L_{max}$. Then let $L_w = L_{max} \cup \{\vec{w}\}$. Clearly $L_{max} \subseteq L_w$, but this contradicts the maximality of L_{max}, therefore there is no such $\vec{w}$ that exists. Therefore, L_{max} is linearly independent and spans all of V, and is therefore a basis, meaning every vector space has a basis. $\square$

7 Closing Example: Vitali Set - A Non-measurable Set (James if time is really abundant)

According to PBS Infinite Series video:

A measure μ satisfies two properties:

1. Translation Invariance: $\mu(S) = \mu(S + x)$
2. Countable Additivity: $\mu(S_1) + \mu(S_2) = \mu(S_1 \cup S_2)$ if S_1 and S_2 are disjoint.

Classify real numbers in $[0, 1]$ into equivalence classes. In each equivalence class, any two numbers differ by a rational number. Every number in $[0, 1]$ belongs to exactly one equivalence class. We can think of the equivalence classes as "shifting" the rational numbers by an irrational number. One of the equivalence classes contains all the rational numbers.

Now we pick one element from each equivalence class and call the new set S. We want to show that S is not measurable. We first list all the rational numbers $r_1, r_2, r_3, \ldots$ in $[-1, 1]$. Then we construct sets $S_i = S + r_i$, which means the set formed by adding r_i to each element of S.

Proposition 7.1. If $i \neq j$, then S_i and S_j are disjoint.

Proof. By contradiction. Suppose $\exists x$ such that $x \in S_i$ and $x \in S_j$. Then $x - r_i \in S$ and $x - r_j \in S$. But since r_i, r_j are rational, $x - r_i$ and $x - r_j$ belong to the same equivalence class. Since we only chose one element from each equivalence class to form S, this is a contradiction. $\square$

Proposition 7.2. $\forall x \in [0, 1], \exists i \in \mathbb{N}$ such that $x \in S_i$.

Proof. x must belong to some equivalence class C_i. Thus, $\exists s_i \in C_i$ such that $s_i \in S$. By definition, $\exists i$ such that $x - r_i = s_i$, so $x \in S_i$. $\square$

Now $\mu(\bigcup S_i) = \sum_{i=1}^{\infty} \mu(S_i)$ by countable additivity, and $\mu(S_i) = \mu(S_j) = \mu(S)$ for all i and j by translation invariance. Also, $1 \leq \mu(\bigcup S_i) \leq 3$, because $[0, 1] \subseteq \bigcup S_i$ and $\bigcup S_i \subseteq [-1, 2]$. This shows that $\mu(S)$ doesn't exist.

(In fact, using ZF without AoC, it's impossible to create a set with an undefined measure.)

8 Additional Resources

Here are some articles that we referenced while preparing this presentation:

- The Axiom of Choice and its implications: https://math.uchicago.edu/~may/REU2014/REUPapers/Barnum.pdf
- The Axiom of Choice, the Well Ordering Principle, and Zorn's Lemma: https://www.mn.uio.no/math/tjenester/kunnskap/kompendier/acwozl.pdf
- Axiom of Choice and Zorn's Lemma: http://www.borisbukh.org/MathStudiesAlgebra1718/notes_ac.pdf
- Mathologer video on Infinite Hat Problem: https://www.youtube.com/watch?v=aDOP0XynAzA
- PBS Infinite Series Video on Vitali Set: https://www.youtube.com/watch?v=hcRZadc5KpI
- Videos by Antonio Montalban on Transfinite Induction and Transfinite Recursion (Set Theory Playlist):
  https://www.youtube.com/watch?v=Ue7QsN2lVXc&list=PLjJhPCaCziSQyON7NLc8Ac8ibdm6_iDQf
  https://www.youtube.com/watch?v=VGlK-YPsX1w&list=PLjJhPCaCziSQyON7NLc8Ac8ibdm6_iDQf