Cardinality Lecture

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Definition of Cardinality

Definition: Let cardinality of a set be denoted $|\cdot|$.

- Notation shorthand: $[n] = \{0, 1, ..., n-1\}$.
- Ex: $[3] = \{0, 1, 2\}$

Definition: *Enumeration* of finite set S with $|S| = n$ is a bijective function $e : S \to [n]$.

Lemma 0.1 — *Suppose X, Y are finite sets such that $|X| \leq |Y|$. Then there exists an injection $f : X \to Y$*

Proof. Let $|X| = n, |Y| = m, n \leq m$. Then let $e_X : X \to [n], e_Y : Y \to [m]$. Define $f : X \to Y$ such that $e_X = e_Y(f(x)), \forall x \in X$.

Every $e_X(x)$ is an integer within the set $[n]$, and since $n \leq m$, it is also within $[m]$, a larger set. Since e_Y is bijection, $\exists$ exactly one $y \in Y$ with $e_Y(y) = e_X(x)$, which we declare to be $f(x)$, so we therefore know $f(x)$ is well-defined since a single x does not map to multiple y.

Next, we want injection. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$. Then, $e_Y(f(x_1)) = e_Y(f(x_2))$, so then $e_X(x_1) = e_X(x_2)$. But we know $e_X(\cdot)$ is a bijection, so $x_1 = x_2$. □

Lemma 0.2 — *If X, Y finite sets such that $|X| > |Y|$, no injective mapping $f : X \to Y$. Proof is due to Pigeonhole Principle.*

Definition: *Given two sets A, B, then $|A| \leq |B| \iff \exists$ injection $A \to B$. $|A| \leq |B|$ and $|B| \leq |A| \Rightarrow |A| = |B|$*

Theorem: *(Schroeder-Bernstein Theorem) Given sets A, B, if there is an injection $f_1 : A \to B$ and surjection $f_2 : A \to B$, then there is a bijection $f : A \to B$.*

Definition: *Set S is countable if it has the same cardinality as a subset of $\mathbb{N}$, countably infinite if $|S| = |\mathbb{N}|$.*

Lemma 0.3 — *Suppose $A \subseteq B$. Then, $|A| \leq |B|$. Proof is there exists an injection.*

Important things about infinite sets:

- *Removing finite number of elements preserves cardinality.*
- *Adding countable set to countable set preserves cardinality.*