Coxeter Systems:

A Study of the Intersection of Linear Algebra, Group Theory, and Graph Theory

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Finite Reflection Groups

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A *Finite Reflection Group* is simply a finite group generated by such reflections.

Examples of Finite Reflection Groups

Example

$\mathbf{D_{2n}}$ Generated by two reflections on the $2D$-plane with angle $\frac{\pi}{n}$.

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$\mathbf{D_{2n}}$ Generated by two reflections on the $2D$-plane with angle $\frac{\pi}{n}$.

$\mathbf{S_n}$ Generated by reflections representing the transposition of two components of n-dimensional vectors

Structure of Finite Reflection Groups

Root Systems, Positive Systems and Simple Systems

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Definition

A **Positive System** Π is a subset of ϕ such that for all $\alpha \in \Pi$, $\alpha > 0$ with respect to some lexicographic ordering of V.

Definition

A **Simple System** Δ is a linearly independent subset of ϕ such that each $\alpha \in \phi$, α is either an entirely positive linear combination of the vectors in Delta, or entire negative.

Properties of Finite Reflection Groups

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Theorem
Simple systems exist. Additionally, they define a unique positive system and vice versa.

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The **length function** $l(\omega)$ of $\omega \in W$, a Finite Reflection Group and corresponding simple system Δ is defined as the minimal r such that $s_1 ... s_r = \omega$ where $s_i = s_{a_i}$ for some $a_i \in Delta$.

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$l(\omega) = n(\omega)$

Coxeter Systems

Formal Definition of Coxeter Systems

Theorem

Fix a simple system Δ in some positive system Π, then the finite reflection group generated by Δ is generated with respect to the relations $(s_\alpha s_\beta)^n = 1$ for $n = ord(s_\alpha s_\beta)$ and α, β in W.

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Using this theorem, we can redefine finite reflection groups, and consequently, Coxeter Groups as follows:

Definition

A **Coxeter System** is a pair (W, S) where S is a set of elements and W is a **Coxeter Group** with the following relations:

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$a^2 = 1$ for all $a \in S$

$ord(ab) = n$ for some $n \geq 2$, $\forall a, b \in S$ and $a \neq b$. (The choice of n can be different for distinct pairs a, b)

Definition

Let C be a Coxeter System (W, S). Then define the **Coxeter Graph** G of C as follows.

Let S be the vertex set of the graph G. Then, for any two elements $s_1, s_2 \in S$, if $ord(s_1 s_2) = 2$, do not connect vertices s_1 and s_2. If $ord(s_1 s_2) = 3$, connect the two vertices. If $ord(s_1 s_2) > 3$, connect the two vertices with a labeled edge.

Graph and Matrix Representations of Coxeter Groups

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Definition

Let C be a Coxter System. Then, define the **Coxeter Matrix** A as follows:

$A(s_i, s_j) = \cos \frac{\pi}{ord(s_i s_j)}$

Classification of Positive Definite Coxeter Systems

Positive Definite Coxeter Systems

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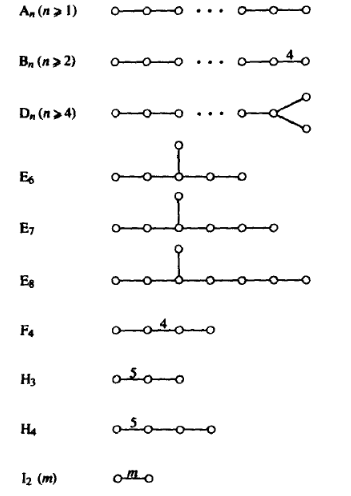
Definition

A Coxeter System is **irreducible** if its Coxeter Graph is connected

Positive Definite Coxeter Systems

Theorem

All the Positive Definite Irreducible Coxeter Graphs:



Positive Semi-Definite Coxeter Systems

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All of the Positive Semi-Definite Irreducible Coxeter Graphs:

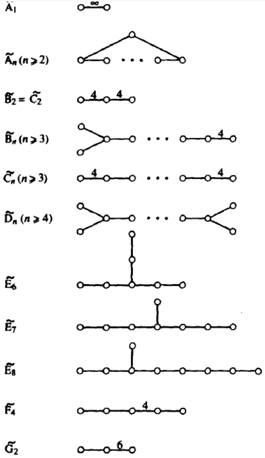

Figure 2: Some positive semidefinite graphs

Why Special Right Triangles are Truly Special

Theorem

This is a really obscure theorem, but it will come to light in just a few minutes.

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \implies \textit{Special Right Triangle}$$