Diffusion Models (Diffusion Kernels)

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Diffusion Kernels

- We can directly sample z_t from $q(z_t \mid x)$ using a closed-form expression, bypassing intermediate variables $z_1, z_2, \ldots, z_{t-1}$.
- This process is referred to as a diffusion kernel.

Consider the variables:

$$z = a \cdot \epsilon_1 + b \cdot \epsilon_2$$

where both ϵ_1 and ϵ_2 are drawn from independent standard normal distributions:

$$\mathbb{E}[z] = 0$$
$$\mathrm{Var}[z] = a^2 + b^2$$

Thus, we can equivalently compute z as:

$$z = \sqrt{a^2 + b^2} \cdot \epsilon$$

where ϵ is drawn from a standard normal distribution.

Consider the first two steps of the forward process:

$$z_1 = \sqrt{1-\beta_1} \cdot x + \sqrt{\beta_1} \cdot \epsilon_1$$
$$z_2 = \sqrt{1-\beta_2} \cdot z_1 + \sqrt{\beta_2} \cdot \epsilon_2$$

Substituting the first equation into the second, we get:

$$z_2 = \sqrt{(1-\beta_2)(1-\beta_1)} \cdot x + \sqrt{1-\beta_2-(1-\beta_2)(1-\beta_1)} \cdot \epsilon_1 + \sqrt{\beta_2} \cdot \epsilon_2$$

We could equivalently rewrite this as:

$$z_2 = \sqrt{(1-\beta_2)(1-\beta_1)} \cdot x + \sqrt{1-(1-\beta_2)(1-\beta_1)} \cdot \epsilon$$

Continuing this process by substituting this equation into the expression for z_3 and so on, we can show that:

$$z_t = \sqrt{\alpha_t} \cdot x + \sqrt{1-\alpha_t} \cdot \epsilon$$

where $\alpha_t = \prod_{s=1}^{t}(1-\beta_s)$. Note that ϵ now denotes the total noise added to the original image, rather than just the incremental noise introduced at this step.
We can equivalently write this in probabilistic form as:

$$q(z_t \mid x) = \mathcal{N}(z_t; \sqrt{\alpha_t} \cdot x, (1-\alpha_t)\mathbf{I})$$