Generative Adversarial Networks (Part 2)

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Optimal Values in GANs

❖ GANs can be framed as a min-max problem where $\hat{x}$ is the output of generator G:

$$\min_G \max_D V(G, D) = \mathbb{E}_{x\sim P_r(x)}[\log D(x)] + \mathbb{E}_{\hat{x}\sim P_r(\hat{x})}[\log(1 - D(\hat{x}))]$$

❖ The optimal discriminator $D^\star(x)$ is:

$$D^\star(x) = \frac{P_r(x)}{P_r(x) + P_r(\hat{x})}$$

❖ For the optimal discriminator, $V(G, D^\star)$ becomes:

$$\begin{aligned}
V(G, D^\star) &= \mathbb{E}_{x\sim P_r(x)}\left[\log \frac{P_r(x)}{P_r(x) + P_r(\hat{x})}\right] + \mathbb{E}_{\hat{x}\sim P_r(\hat{x})}\left[\log \frac{P_r(\hat{x})}{P_r(x) + P_r(\hat{x})}\right] \\
&= \int P_r(x) \log \frac{P_r(x)}{\frac{P_r(x)+P_r(\hat{x})}{2}}\, dx + \int P_r(\hat{x}) \log \frac{P_r(\hat{x})}{\frac{P_r(x)+P_r(\hat{x})}{2}}\, d\hat{x} - \log 4 \\
&= D_{KL}\left[P_r(x) \mid\mid \frac{P_r(x) + P_r(\hat{x})}{2}\right] + D_{KL}\left[P_r(\hat{x}) \mid\mid \frac{P_r(x) + P_r(\hat{x})}{2}\right] - \log 4 \\
&= 2D_{JSD}\left[P_r(x) \mid\mid P_r(\hat{x})\right] - \log 4
\end{aligned}$$

Jensen-Shannon Divergence

JSD is also known as symmetric KL divergence:

$$D_{JSD}\left[p \mid\mid q\right] = \frac{1}{2}\left(D_{KL}\left[p \mid\mid \frac{p+q}{2}\right] + D_{KL}\left[q \mid\mid \frac{p+q}{2}\right]\right)$$

Properties:

1. $D_{JSD}\left[p \mid\mid q\right] \geq 0$
2. $D_{JSD}\left[p \mid\mid q\right] = 0 \quad \text{iff} \quad p = q$
3. $D_{JSD}\left[p \mid\mid q\right] = D_{JSD}\left[q \mid\mid p\right]$

For GANs, when both G and D are at their optimal values:

$$P_{r(x)} = P_{r(\hat{x})}, \quad D^{\star} = \frac{1}{2}, \quad V(G^{\star}, D^{\star}) = -\log 4$$