HW12_Andrew_Goldberg

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Verify that the given function pair is a solution to the first-order system.

$$x = -e^t$$

$$y = e^t$$

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = -x$$

This is easily verified from the differentiation formulas

$$\frac{dy}{dt} = \frac{d}{dt}(-e^t) = -e^t = -\frac{-dx}{dt} = \frac{-d}{dt}(e^t) = -e^t$$

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Find and classify the rest points of the given autonomous system

$$\frac{dx}{dt} = -(y-1)$$

$$\frac{dy}{dt} = x-2$$

because rest points = 0 -(y-1)=0 y=1

x-2 = 0 x=2 So rest point is 2,1

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Show that the two trajectories leading to (m=n, a=b/ shown in Figure 12.8 are unique.

a. From system (12.6) derive the following equation:

$$\frac{dy}{dx} = \frac{(m-nx)y}{(a-by)x}$$

$$\frac{dx}{dt} = (a-by)x$$

turns into...

$$dx = dt(a-by)x$$

similarly

$$\frac{dy}{dt} = (m-nx)y$$

turns into

$$dy = dt(m - nx)y$$

dividing dy by dx, we can cancel out the dt's to have...

$$\frac{dy}{dx} = \frac{(m - nx)y}{(a - by)x}$$

b. Separate variables, integrate, and exponentiate to obtain

$$y^a e^{-by} = Kx^m e^{-nx}$$

where K is a constant of integration

First... separate variables

$$dy(a - by)x = dx(m - nx)y$$

$$\frac{a - by}{y}dy = \frac{m - nx}{x}dx$$

$$(\frac{a}{y} - b)dy = (\frac{m}{x} - n)dx$$

..Then integrate...

$$aln|y| - by = mln|x| - nx + k_1$$

apply logarithm...

$$ln\frac{|y|^a}{|x|^m} = by - nx + k_1$$

take antilog

$$\frac{y^a}{x^m} = e^{by-nx+k_1}$$

$$y^a e^{-by} = x^m e^{-nx} e^{k_1}$$

allow

$$e^{k_1} = K$$

$$y^a e^{-by} = x^m e^{-nx} K$$

c. Let

$$f(y) = \frac{y^a}{e^{by}}$$

$$g(x) = \frac{x^m}{e^{nx}}$$

Show that f(y) has a unique maximum of

$$M_y = (\frac{a}{eb})^a$$

when

$$y = \frac{a}{b}$$

as shown in figure 12.12. Similarly show that g(x) has a unique maximum

$$M_x = (\frac{x}{en})^m$$

when

$$x = \frac{m}{n}$$

first..differentiate

$$\frac{d(f(y))}{dy} = \frac{ay^{a-1}}{e^{by}} + \frac{-by^a}{e^{by}}$$

set to 0...

$$\frac{ay^{a-1} + by^a}{e^{by}} = 0 = y^{a-1}(a - yb)$$

$$y = \frac{a}{b}, 0$$

second derivative test:

$$\frac{d^2 f(y)}{dy^2} = \frac{\frac{ay^{a-1}+by^a}{e^{by}}}{dy}$$

$$= (-be^{-by}ay^{a-1}) + (e^{-by}(a-1)ay^{a-2}) - (be^{-by}by^a) + (e^{-by}ay^{a-1})$$

$$= \frac{(((a-1)ay^{a-2} - ay^{a-1}) - b(ay^{a-1} - y^a))}{e^{by}}$$

if we set y=0, we'll get 0 Subsititute y=a/b... we'll get a messy equation, but it will be less than 0 because the second half of the top equation has higher powers than the first half and is therefore larger. Therefore y=a/b is a unique critical point that yeilds the relative max.

The value of the max is...

$$y^a e^{-by} = \frac{(\frac{a}{b})^a}{e^a} = M_y = (\frac{a}{eb})^a$$

If we changed the letters in the equation to:

$$x^m e^{-nx}$$

We would go through the same process and end up with the correlated answer:

$$M_x = (\frac{x}{en})^m$$

D. consider what happens as (x,y) approaches (m/n, a/b). Take limits in part(b) as x->m/n and y->a/b to show that

$$lim_{y \to \frac{a}{b} x \to \frac{m}{n}} [(\frac{y^a}{e^{by}})(\frac{e^{nx}}{x^m})] = K$$

or My/Mx=K. Thus any solution trajectory that approaches (m/n, a/b) must satisfy

$$\frac{y^a}{e^{by}} = (\frac{M_y}{M_x})(\frac{x^m}{e^{nx}})$$

substitute the limits:

$$(\frac{a}{eb})^a (\frac{en}{x})^m = K$$

or...

$$\frac{M_y}{M_x} = K$$

substitute the value in K...

$$(\frac{y^a}{e^{by}}) = (\frac{M_y}{M_x})(\frac{x^m}{e^{nx}})$$

E.Show that only one trajectory can approach (m/n, a/b) from below the line y=a/b. Pick yo less than a/bc. From figure 12.12 you can see that f(yo)“<”My, which implies that

$$\frac{M_y}{M_x}(\frac{x^m}{e^{nx}}) = \frac{y_0^a}{e^{by_0}} < M_y$$

This in turn implies that

$$\frac{x^m}{e^{nx}} < M_x$$

Figure 12.12 tells you that for g(x) there is a unique value xo less than m/n satisfying this last inequality. That is, for each y less than a/b there is a unique value of x satisfying the equation in part (d). Thus there can exist only one trajectory solution approaching (m/n, a / B) from below, as show in figure 12.13.

This is solved...

F.Use a similar argument to show that the solution trajectory leading to (m/n, a/ b) is unique if (yo more than aye divided by bee).

$$y_0 > \frac{a}{b}$$

$$\frac{M_y}{M_x}(\frac{x^m}{e^{nx}}) = \frac{y_0^a}{e^{by_0}} < M_y$$

$$\frac{M_y}{M_x}(\frac{x^m}{e^n x}) < M_y$$

$$\frac{x^m}{e^{nx}} < M_x$$

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Apply the first and second derivate tests to the function f(y)=y$^{a/e}$by to show that y=a/b is a unique critical point that yields the relative maximum f(a/b). Show also that f(y) approaches zero as y tends to infinity

$$f(y) = \frac{y^a}{e^{by}}$$

$$\frac{d(\frac{y^a}{e^{by}})}{dy} = 0$$

$$\frac{e^{by}ay^{a-1} - y^a be^{by}}{e^{2by}} = 0$$

eˆ2by can never be 0 so...

$$\frac{e^{by}}{y^{a-1}}(a - yb) = 0$$

$$y = \frac{a}{b}, 0$$

second derivative test:

$$\frac{d^2 f(y)}{dy^2} = \frac{\frac{ay^{a-1}+by^a}{e^{by}}}{dy}$$

$$= (-be^{-by}ay^{a-1}) + (e^{-by}(a-1)ay^{a-2}) - (be^{-by}by^a) + (e^{-by}ay^{a-1})$$

$$= \frac{(((a-1)ay^{a-2} - ay^{a-1}) - b(ay^{a-1} - y^a))}{e^{by}}$$

if we set y=0, we'll get 0 Subsititute y=a/b…we'll get a messy equation, but it will be less than 0 because the second half of the top equation has higher powers than the first half and is therefore larger. Therefore y=a/b is a unique critical point that yeilds the relative max.

The value of the max is…

$$y^a e^{-by} = \frac{(\frac{a}{b})^a}{e^a} = M_y = (\frac{a}{eb})^a$$

The graph itself clearly shows an asymptotic curve approaching 0 as y tends toward infinity. By logic, alone, due to the y as a power in the denominator, compared to only an a-powered integer in the numerator, it's clear that demoniator will grow at a much higher rate than the numerator as y tends to infinity, making the fraction approach 0.

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use Euler's method to solve the first-order system subject to the specified initial condition. Use the given step size Åt and calculate the first three approximations (x1,y1), (x2, y2), and (x3,y3). Then repeat your calculations for Åt=2. Compare your approximations with the values of the given analytical solution.

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = 3x + 2y$$

$$x(0) = 1, y(0) = 1, \Delta t = \frac{1}{4}$$

$$x(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{5t}$$

$$y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{5t}$$

so…assuming we start at t=0

$$t_0 = 0, t_1 = t_0 + \Delta t = 0.25, t_2 = t_1 + \Delta t = 0.50, t_3 = t_2 + \Delta t = 0.75$$

First (x1,y1):

$$x_1 = x_0 + f(t_0, x_0, y_0)\Delta t$$

$$x_1 = x_0 + (2x_0 + 3y_0)\Delta t$$

$$x_1 = 1 + (2(1) + (3(1)).25 = 2.25$$

$$y_1 = y_0 + f(t_o, x_0, y_0)\Delta t$$

$$y_1 = y_0 + (3x_0 + 2y_0)\Delta t$$

$$y_1 = 1 + (3(1) + (2(1)).25 = 2.25$$

#Calculations at change T = .25

```
x0 <- 1
y0 <- 1
dt <- .25
paste0("@t")
```

```
## [1] "@t"
```

```r
#rehashing first
x1 <- x0 + (2 * x0 + 3 * y0)*dt
y1 <- y0 + (3 * x0 + 2 * y0)*dt
paste0("x1=",x1,"  y1=",y1)
```

```
## [1] "x1=2.25  y1=2.25"
```

```r
#second (x2,y2):
x2 <- x1 + (2 * x1 + 3 * y1)*dt
y2 <- y1 + (3 * x1 + 2 * y1)*dt
paste0("x2=",x2,"  y2=",y2)
```

```
## [1] "x2=5.0625  y2=5.0625"
```

```r
#third (x3,y3):
x3 <- x2 + (2 * x2 + 3 * y2)*dt
y3 <- y2 + (3 * x2 + 2 * y2)*dt
paste0("x3=",x3,"  y3=",y3)
```

```
## [1] "x3=11.390625  y3=11.390625"
```

```r
#compare approximations with values of the given analytic solution
paste0("@t/2")
```

```
## [1] "@t/2"
```

```r
t0 <- 0
t1 <- .25
t2 <- .5
t3 <- .75
e <- (exp(1))
#first (t = 0)
xt0 <- .5 * e^(-t0) + .5 * e^(5*t0)
yt0 <- -.5 * e^(-t0) + .5 * e^(5*t0)
paste0("xt0=",xt0,"  yt0=",yt0)
```

```
## [1] "xt0=1  yt0=0"
```

```r
#second (t = .25)
xt1 <- .5 * e^(-t1) + .5 * e^(5*t1)
yt1 <- -.5 * e^(-t1) + .5 * e^(5*t1)
paste0("xt1=",xt1,"  yt1=",yt1)
```

```
## [1] "xt1=2.13457187026662  yt1=1.35577108719522"
```

```r
#third (t = .50)
xt2 <- .5 * e^(-t2) + .5 * e^(5*t2)
yt2 <- -.5 * e^(-t2) + .5 * e^(5*t2)
paste0("xt2=",xt2,"  yt2=",yt2)
```

```
## [1] "xt2=6.39451231020805  yt2=5.78798165049542"
```

```r
#forth (t = .75)
xt3 <- .5 * e^(-t3) + .5 * e^(5*t3)
yt3 <- -.5 * e^(-t3) + .5 * e^(5*t3)
paste0("xt3=",xt3,"  yt3=",yt3)
```

```
## [1] "xt3=21.4967242764019  yt3=21.0243577236609"
```

Calculations at change T = 1.25

```r
x0 <- 1
y0 <- 1
dt <- .125

#rehashing first
x1 <- x0 + (2 * x0 + 3 * y0)*dt
y1 <- y0 + (3 * x0 + 2 * y0)*dt
paste0("x1=",x1,"  y1=",y1)
```

```
## [1] "x1=1.625  y1=1.625"
```

```r
#second (x2,y2):
x2 <- x1 + (2 * x1 + 3 * y1)*dt
y2 <- y1 + (3 * x1 + 2 * y1)*dt
paste0("x2=",x2,"  y2=",y2)
```

```
## [1] "x2=2.640625  y2=2.640625"
```

```r
#third (x3,y3):
x3 <- x2 + (2 * x2 + 3 * y2)*dt
y3 <- y2 + (3 * x2 + 2 * y2)*dt
paste0("x3=",x3,"  y3=",y3)
```

```
## [1] "x3=4.291015625  y3=4.291015625"
```

```r
#compare approximations with values of the given analytic solution
t0 <- 0
t1 <- .125
t2 <- t1*2
t3 <- t1*3
e <- (exp(1))
#first (t = 0)
xt1 <- .5 * e^(-t0) + .5 * e^(5*t0)
yt1 <- -.5 * e^(-t0) + .5 * e^(5*t0)
paste0("xt1=",xt1,"  yt1=",yt1)
```

```
## [1] "xt1=1  yt1=0"
```

```r
#second (t = .125)
xt2 <- .5 * e^(-t1) + .5 * e^(5*t1)
yt2 <- -.5 * e^(-t1) + .5 * e^(5*t1)
paste0("xt2=",xt2,"  yt2=",yt2)
```

```
## [1] "xt2=1.37537143000841  yt2=0.492874527423813"
```

```r
#third (t = .250)
xt3 <- .5 * e^(-t2) + .5 * e^(5*t2)
yt3 <- -.5 * e^(-t2) + .5 * e^(5*t2)
paste0("xt3=",xt3,"  yt3=",yt3)
```

```
## [1] "xt3=2.13457187026662  yt3=1.35577108719522"
```

t/4 reduces the overall margin of differences, but seems to follow the pattern worse than regular t (up to 2)