IIT Delhi, CSE

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Major Exam (30 marks)

02.12.2006

CSL303 Logic for Computer Science

Exercise 1 *Monadic First Order Logic* (5 marks)

Consider the fragment of first order logic which can be generated by:

$$\varphi = P(x) \mid Q(x) \mid \exists x \varphi \mid \neg\varphi \mid (\varphi \wedge \varphi)$$

where x can be any variable, but P and Q are the only predicate symbols of the logic.

(a) Give a model A for the formula $\neg\exists x(P(x) \wedge Q(x))$ where $|A|$ is the set of natural numbers. (no proof required)

(b) Let $D = \{00, 01, 10, 11\}$ and A be an arbitrary structure. The mapping $f : |A| \to D$ is defined by

$$f(d) = \begin{cases} 00 & \text{if } d \notin P^A, d \notin Q^A, \\ 01 & \text{if } d \notin P^A, d \in Q^A, \\ 10 & \text{if } d \in P^A, d \notin Q^A, \\ 11 & \text{if } d \in P^A, d \in Q^A. \end{cases}$$

Based on f we define structure C_A with domain $|C_A| = D$ by

$$\begin{aligned} P^{C_A} &= \{f(d) \mid d \in P^A\}, \\ Q^{C_A} &= \{f(d) \mid d \in Q^A\}. \end{aligned}$$

i) What are the elements of P^{C_A} and Q^{C_A} if A is the structure you have given under (a)?

ii) Prove that for all formulas ψ the following holds for all structures A and value assignments s:
$\models_A \psi[s]$ if and only if $\models_{C_A} \psi[s']$ where $s'(x) = f(s(x))$.

Exercise 2 *Löwenheim/Skolem/Gödel* (4 marks)

(a) Give a Skolem form for $\alpha = \neg\exists x(\forall y P(y, c) \vee \neg\forall y P(x, y))$.

(b) Give the Herbrand Expansion for the matrix of the Skolem form under (a).

(c) Is α satisfiable? Give reason.

Exercise 3 *Hilbert System* (3 marks)

Give a deduction sequence for $\{\forall x(P(x) \rightarrow Q(y)), \forall z P(z)\} \vdash Q(z)$.

Exercise 4 *Consistency* (3 marks)

Let K be the clausal form of a Skolem form φ. Let K be such that each clause in K contains at least two literals and in each clause either all literals are either positive or all are negative. Is it generally true that $\{\varphi\}$ is consistent? Prove or give a counterexample.

Exercise 5 (4 marks)

Prove or reject by giving a counterexample:
The most general unifier η computed by the unification algorithm is always *idempotent* that is $\eta\eta = \eta$.

Exercise 6 *Separation Property* (5 marks)

(a) Let Σ_1 and Σ_2 be two sets of closed first order formulas such that no structure satisfies both sets that is Σ_1 and Σ_2 do not have a common model. Let $\mathrm{Mod}(\Sigma_i) = \{A \mid \text{ for all } \psi \in \Sigma_i\text{: } \models_A \psi\}$. Show that there is a formula φ such that every structure in $\mathrm{Mod}(\Sigma_1)$ satisfies φ and every structure in $\mathrm{Mod}(\Sigma_2)$ satisfies $\neg\varphi$.

(b) Where in the proof of (a) do you need the fact that Σ_1 and Σ_2 do only contain closed formulas? Give reason.

Exercise 7 *CTL* (5 marks)

Let K and K′ be serial Kripke structures with exhibited intial state s and t, respectively. We say that s in K is equivalent to t in K′, denoted $s \,\underline{\text{in}}\, K \sim t \,\underline{\text{in}}\, K'$ iff

(a) for each finite path $s = s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_n$ in K there is a finite path $t = t_0 \rightarrow t_1 \rightarrow \cdots \rightarrow t_n$ in K′ such that $\kappa(s_i) = \kappa'(t_i)$ for all $i \leq n$ and

(b) for each finite path $t = t_0 \rightarrow t_1 \rightarrow \cdots \rightarrow t_n$ in K′ there is a finite path $s = s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_n$ in K such that $\kappa(s_i) = \kappa'(t_i)$ for all $i \leq n$.

(κ and κ' denote the labelling functions of K and K'.)
Is it true that

$$s \,\underline{\text{in}}\, K \sim t \,\underline{\text{in}}\, K' \text{ iff } \{\varphi \mid \varphi \in \text{CTL}, s \models_K \varphi\} = \{\varphi \mid \varphi \in \text{CTL}, t \models_{K'} \varphi\}? \qquad (*)$$

Prove $(*)$ or show by giving counter examples which of the CTL operators have to be removed to obtain the validity of $(*)$ and then prove the validity of $(*)$ for the remaining fragment of the CTL.

<u>Exercise 8</u> *Applications* (4 marks)

You are given a road map that is a finite number of towns A_1 to A_n and and a finite number of roads E_1 to E_m directly connecting two of these towns. Each road permits traffic in both directions.

(a) Write a logic progam (including a goal) which decides whether one can travel from A_1 to A_n on road. The route itself does not have to be computed.

(b) Define a Kripke structure K which models the road map. Give a CTL formula which is true in a state you have to specify if and only if one can travel from A_1 to A_n on road.

——— *no more exercises* ———

The axioms of Hilbert system:

All generalizations of the following forms:

1. Tautologies ;
2. $\forall x\alpha \to \alpha^x_t$, where t is substitutable for x in α;
3. $\forall x(\alpha \to \beta) \to (\forall x\alpha \to \forall x\beta)$;
4. $\alpha \to \forall x\alpha$, where x does not occur free in α.