Indian Institute of Technology Delhi
Department of Mathematics

MAL 513: Real Analysis

MAJOR TEST (2008-09) Semester I

Maximum Marks: 50 Time: 2 Hours

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1. (a) Let $\mathbb{N}$ be the set of positive integers. Give examples of three different metrics on $\mathbb{N}$, no two of which are multiples of each other.
   (b) Which of the following subsets of $\mathbb{R}^2$ are compact?
   $$E = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1\},\ F = \{(x,y) \in \mathbb{R}^2 : x \geq 1, 0 \leq y \leq \tfrac{1}{x}\}.$$
   (c) Let (X,d) be a metric space. For a fixed $x_0 \in X$, define $f : X \to [0,\infty)$ by $f(x) = d(x,x_0)$. Show that f is continuous. Use this to prove that a connected metric space is uncountable.
   (d) Let (X,d) be a metric space. Let (x_n) and (y_n) be two sequences in X such that (y_n) is a Cauchy sequence and $d(x_n,y_n) \to 0$ as $n \to \infty$. Prove that (x_n) converges to a limit x if and only if (y_n) also converges to x. **[2+2+6+5]**

2. (a) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that $T'(x) = T$ for all $x \in \mathbb{R}^n$.
   (b) Show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \cos x$, is not a strict contraction.
   (c) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be differentiable functions. Using the chain rule for several variables, prove that
   $$\nabla(fg) = g\nabla f + f\nabla g.$$
   (d) State the implicit function theorem.
   (e) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by
   $$f(x_1,x_2,x_3) = x_1x_3^2 + e^{x_3} + x_2.$$
   Show that $f(1,-1,0) = 0$ and $\frac{\partial f}{\partial x_3}(1,-1,0) \neq 0$. So there exists an open subset U of $\mathbb{R}^2$ and a differentiable function $g : U \to \mathbb{R}$ such that $g(1,-1) = 0$ and $f(x_1,x_2,g(x_1,x_2)) = 0$ for all $(x_1,x_2) \in U$. Find $\frac{\partial g}{\partial x_1}(1,-1)$ and $\frac{\partial g}{\partial x_2}(1,-1)$.
   [3+3+4+3+3]

3. (a) Define outer measure of a subset E of $\mathbb{R}^n$. Show that the outer measure of a countable subset of $\mathbb{R}^n$ is zero.
   (b) Let $A \subseteq B \subseteq \mathbb{R}^n$. Show that if B is Lebesgue measurable with measure zero, then A is also Lebesgue measurable with measure zero.
   (c) Define a measurable function. Let Ω be a measurable subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a continuous function. Show that f is measurable.
   (d) Prove that
   $$\int_0^1 (e^x - 1)(\log x + \tfrac{1}{x})\, dx = \sum_{n=1}^{\infty} \frac{n^2+n+1}{(n-1)!\ (n^2+n)^2}.$$
   (e) State Lebesgue dominated convergence theorem. Use this to prove that
   $$\lim_{n\to\infty} \int_a^{\infty} \frac{n^2 x e^{-n^2x^2}}{1+x^2}\, dx = 0 \quad \text{for all } a > 0.$$
   [3+3+3+4+6]
