DEPARTMENT OF MATHEMATICS
MAJOR TEST MAL 733 (Stochastics of Finance)

Max Marks: 55 Time: Two Hours

YOU MAY ATTMPT ALL TEN QUESTIONS. AWARDED SCORE WILL BE RESTRICTED TO 55

1. Let X be a random variable on a probability space (Ω, Γ, P) and G be a sub σ-field of Γ. Define conditional expectation of X given G and prove that it always exists. (6)

2. Assuming the quadratic variation in a Brownian Motion converges a. s., prove that it accumulates at a rate of one per unit time a. s. (6)

3. State Levy's theorem (one dimension) and prove it by clearly stating the main results assumed in the proof. (6)

4. For the standard Brownian motion W(t), show that
$$\int_{t_0}^{t_1} W^n(t)\, dW(t) = \frac{1}{n+1}[W^{n+1}(t_1) - W^{n+1}(t_0)] - \frac{n}{2}\int_{t_0}^{t_1} W^{n-1}(t)\, dt. \quad (5)$$

5. Let Z be a random variable on a probability space (Ω, Γ, P) with filtration $\Gamma(t)$, $0 \le t \le T$, T fixed. Suppose Z>0, a. s., E(Z)=1, $\widetilde{P}$ be a probability measure such that its Radon- Nikodym derivative with respect to P is Z and $Z(t) = E(Z/\Gamma(t))$, $0 \le t \le T$. If Y is a $\Gamma(t)$-measurable random variable then prove that
$$\widetilde{E}(Y/\Gamma(s)) = \frac{1}{Z(s)} E(YZ(t)/\Gamma(s)), 0 \le s \le t \le T, \text{where } \widetilde{E} \text{ is expectation w.r.t. } \widetilde{P}.$$
(6)

6. Given that a process S(t), $0 \le t \le T$, is Geometric Brownian motion with constant drift parameter α, constant volatility parameter σ and driven by P-Brownian motion W(t). Find another probability measure $\widetilde{P}$ by specifying Radon- Nikodym derivative $d\widetilde{P}/dP$ such that under $\widetilde{P}$, the process S(t) follows Geometric Brownian motion with new drift α', the same volatility σ and driven by $\widetilde{P}$-Brownian motion $\widetilde{W}(t)$. (5)

7. Define a multi- stock market model. Define the risk neutral probability measure and derive Market price of risk equations for this model. (7)

8. Define arbitrage in a market. Prove that a market model does not admit an arbitrage if it has a risk neutral probability measure. (6)

P.T.O.

9. Stating clearly all the assumptions made, use risk neutral pricing formula to derive the formula for the price of a European call option at any time on an asset whose price follows a Geometric Brownian motion. Hence state the price formula for European put option. (7)

10. Let $(W_1(t), W_2(t))$, be two dimensional Brownian motion on probability space (Ω, Γ, P), with an associated filtration $\Gamma(t)$, $0 \le t \le T$. In the notations of multi-dimensional Girsanov theorem, take $\theta_1(t) = 0$, $\theta_2(t) = W_1(t)$, so that $(\widetilde{W}_1(t), \widetilde{W}_2(t))$, where $\widetilde{W}_1(t) = W_1(t)$, $\widetilde{W}_2(t) = W_2(t) + \int_0^t W_1(u)\,du$, is two dimensional Brownian motion under new probability measure $\widetilde{P}$.
(i) Show that $\widetilde{E}(W_1(t)) = \widetilde{E}(W_2(t)) = 0$
(ii) Use $d(W_1(t)\,W_2(t))$ to show that $\widetilde{Cov}(W_1(t), W_2(t)) = -T^2/2$, $\widetilde{Cov}$ being covariance under $\widetilde{P}$. (6)