CSL 758

Advanced Algorithms: Final Exam

Q1. Let $G = (V, E)$ be a flow network with source s, sink t and integer capacities. Suppose that we are given a maximum flow in G.

1. Suppose that the capacity of a single edge $(u, v) \in E$ is increased by 1. Give an $O(V+E)$-time algorithm to update the maximum flow. (8)

2. Suppose that the capacity of a single edge $(u, v) \in E$ is decreased by 1. Give an $O(V+E)$-time algorithm to update the maximum flow. (8)

Q2. Let A be an $m \times n$ matrix and b be an m-vector. Then Farkas' lemma states that exactly one of the systems $\{Ax \leq 0, bx > 0\}$ and $\{yA = b, y \geq 0\}$ is solvable, where x is an n-vector and y is an m-vector. Prove Farkas' lemma. (10)

Q3. We are given a graph $G = (V, E)$ and we want to color each node with one of three colors. We say that an edge (u, v) is *satisfied* if the colors assigned to u and v are different.

Consider a coloring that maximizes the number of satisfied edges, and let c^* denote this number. Give a randomized algorithm that produces a coloring in which the expected number of satisfied edges is at least $2c^*/3$. (10)

Q4. Consider a balls and bin experiment with $2n$ balls and 2 bins. Each ball is thrown independently in one of the 2 bins, both bins equally likely. The expected number of balls in each bin is n. We are interested in finding how big the difference is likely to be. Let X_1 and X_2 be random variables denoting the number of balls in the two bins respectively. Prove that for any $\epsilon > 0$ there is a constant $c > 0$ such that the probability $Pr[X_1 - X_2 > c\sqrt{n}] \leq \epsilon$. (10)

Q5. Decide whether the following statements are true or false. If a statement is true, give a short explanation. If it is false give a counterexample.

Let G be an arbitrary flow network with a source s, a sink t and a positive integer capacity c_e on every edge e.

1. If f is a maximum s-t flow in G, then f saturates every edge out of s with flow i.e. for all edges e out of s, we have $f(e) = c_e$. (7)

2. Let (A, B) be a minimum s-t cut with respect to the capacities $\{c_e : e \in E\}$. Now suppose we add 1 to every capacity then (A, B) is still a minimum s-t cut with respect to these new capacities $\{1 + c_e : e \in E\}$. (7)

Q6. Given an undirected graph $G = (V, E)$, the edge colouring problem is an assignment of the smallest number of colours to the edges so that no two edges incident on the same vertex have the same colour. Consider the *online edge colouring* problem where the vertex set V of the graph is fixed and the edges in the graph are presented to you in an online manner, one after another. As each edge e is specified, your algorithm must assign this edge e a colour and this colour of e cannot be changed henceforth. While the offline edge colouring of G knows all the edges of E while deciding on their colours, the online edge colouring algorithm has to decide on the colour of each edge e without any knowledge of the future edges. Design a 2-competitive algorithm for the online edge colouring problem. (10)