INDIAN INSTITUTE OF TECHNOLOGY DELHI
MAL 502: COMPLEX ANALYSIS
MAJOR TEST (2006-2007)

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(1) (a) Let f be analytic in the disc $\{z : |z| < 3\}$. Let $f(\frac{1+2i}{1+i}) = \frac{1+i}{1-i}$ and $|f(z)| \leq 1$ if $|z| < 2$. What is $f'(\frac{1+2i}{1+i})$?

(b) Let $f(z) = \dfrac{i-z}{i+z}$. Show that f maps the upper half plane into the unit disc.

(c) How many zeros does the polynomial $2z^5 + 4z^2 + 1$ have in the disc $|z| < 1$?

(d) Let $f(z) = e^z + \sin z$. Find the angle between the images of $y = 0$ and $y = x$ under f.

[4×2]

(2) (a) Let the points z_1, z_2, z_3 lie on the unit circle of the complex plane and satisfy $z_1 + z_2 + z_3 = 0$. Show that z_1, z_2, z_3 form the vertices of an equilateral triangle.

(b) Evaluate $\displaystyle\int_0^{2\pi} e^{e^{it}}\, dt$.

(c) Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{z}{(1+z^2)^n}$ converges for all complex numbers z satisfying $|1+z^2| > 1$.

(d) Prove that $f \in \mathfrak{F}_a$ for all $a > 0$, where $f(z) = e^{-\pi z^2}$.

(e) Let $\lambda > 1$. Prove that the equation $ze^{\lambda - z} = 1$ has exactly one root in the unit disc and this root is real.

[5×3]

(3) Let $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, be analytic in a domain Ω, where u and v are real valued. Suppose there are real constants a, b and c such that $a^2 + b^2 \neq 0$ and

$$au(x,y) + bv(x,y) = c$$

in Ω. Show that f is constant in Ω.

[5]

(4) Let Γ be a contour whose interior contains 0. Evaluate the integral

$$\int_\Gamma z^n \sin \tfrac{1}{z} dz,$$

where n is a non-negative integer.

[5]

(5) Let f be an entire function and let $a > 0$, $b > 0$ be constants. If $|f(z)| \leq a|z|^{1/2} + b$ for all z, prove that f is a constant.

[5]

(6) Let $a \in \mathbb{R}$ and Γ_R be the rectangle with vertices $0, R, R + ia, ia$ followed in the counterclockwise direction. Integrate the function e^{-z^2} around Γ_R and let $R \to \infty$ to find the value of the integral

$$\int_0^\infty e^{-x^2} \cos 2ax\, dx.$$

You can use $\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$.

[5]

(7) (a) Show that the map

$$\varphi(z) = \frac{1+z}{1-z}$$

maps the unit disc $\mathbb{D} = \{z : |z| < 1\}$ <u>onto</u> the right half plane $H = \{z : \text{Re } z > 0\}$.

(b) Let f be analytic on the right half plane and suppose that $|f(z)| \leq 1$ for $z \in H$. Suppose also that $f(1) = 0$. Show that $|f'(1)| \leq \frac{1}{2}$. Can this upper bound be archived by some such f?

[7]