Time 1030 – 1230

Major Test MAL 503 + MAL 255 Linear Algebra

24 Nov 2008 Room WS 204, 209

Max Marks 50

Attempt any FIVE questions.

1. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the eigenvalues of $A \in \mathrm{Mat}_{n \times n}(\mathbb{C})$ with eigenvector x_i associated with λ_i. Prove the following

(i) For $p(\theta) \in \mathbb{C}[\theta]$, the determinant of $p(A)$ is $\prod_{i=1}^{n} p(\lambda_i)$.

(ii) For $a(\theta), b(\theta) \in \mathbb{C}[\theta]$ with $a(\lambda_i) \neq 0$ for all i, the matrix $h(A) = b(A)\,(a(A))^{-1}$ has the eigenvalue $h(\lambda_i) = \dfrac{b(\lambda_i)}{a(\lambda_i)}$ associated with the eigenvector x_i.

(iii) If $g(\theta) \in \mathbb{C}[\theta]$ and $g(A) = B$ is invertible, then

$$B^{-1} = -\frac{1}{\alpha_0}\left[B^{r-1} + \alpha_{r-1} B^{r-2} + \cdots + \alpha_1 I_n\right]$$

where the minimal polynomial of B is given by

$$\mu_B(\theta) = \theta^r + \alpha_{r-1}\theta^{r-1} + \cdots + \alpha_1\theta + \alpha_0$$

$\boxed{2 + 4 + 4 = 10 \text{ marks}}$

2. Suppose A is an $n \times n$ matrix over a field $\mathbb{F}$ with invariant factors of the characteristic polynomial of A being given by $\delta_1 = 1$, $\delta_2 = \theta^2 + 1$, and $\delta_3 = \left[(\theta-1)^2 + 4\right]^2 (\theta^2+1)(\theta-1)$. Then (i) show that $n = 9$, and find (ii) the rational canonical form [[the first one]], (iii) the second rational canonical forms for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, (iv) the standard Jordan form for $\mathbb{F} = \mathbb{C}$, and (v) the real Jordan form for $\mathbb{F} = \mathbb{R}$, of A.

$\boxed{2 \times 5 = 10 \text{ marks}}$

3. If $V \xrightarrow{T} V$ is an $\mathbb{F}$-linear operator on the vector space V over the field $\mathbb{F}$, show that we can think of T as an $\mathbb{F}[\theta]$-module structure on V, denoted by say V_T, with submodules of V_T being precisely the T-invariant vector subspaces of V.

Further, show that if $V \xrightarrow{T} V$, $W \xrightarrow{S} W$ induce the $\mathbb{F}[\theta]$-modules V_T and W_S, an $\mathbb{F}$-linear transformation $V \xrightarrow{A} W$ can be regarded as an $\mathbb{F}[\theta]$-linear transformation $V_T \xrightarrow{A} W_S$ iff $AT = SA$.

According to this, if $\mathbb{F} = \mathbb{R}$, the $\mathbb{R}$-linear $V \xrightarrow{A} W$ is an $\mathbb{R}[\theta]$-linear $V_T \longrightarrow W_S$ iff $AT = SA$. Suppose we now complexify everything. Can we say $(V^{\mathbb{C}})_{T^{\mathbb{C}}} \xrightarrow{A^{\mathbb{C}}} (W^{\mathbb{C}})_{S^{\mathbb{C}}}$ exists and turns out to be $\mathbb{C}[\theta]$-linear?

$\boxed{4 + 4 + 3 = 10 \text{ marks}}$

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4 Use the matrices

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -5 \\ 0 & 1 & -2 \end{bmatrix}$$

to show that:

(i) It is possible for two matrices to have the same characteristic polynomial, one matrix diagonalizable, the other not.

(ii) The concept of diagonalizability depends on the field from which the entries come

$\boxed{5+5 = 10 \text{ marks}}$

5 Show that the Lagrange polynomials

$$L_i := \prod_{j \neq i} \frac{\theta - \lambda_j}{\lambda_i - \lambda_j}, \quad 0 \leq i \leq m-1$$

form a basis for the vector space of polynomials with degree at most $m-1$.

Further, show that if $V \xrightarrow{T} V$ has the spectral form $T = \mu_1 P_1 + \cdots + \mu_k P_k$ then $P_\ell = L_\ell(T)$.

$\boxed{5+5 = 10 \text{ marks}}$

6. Prove that if X is a finite dimensional inner-product space,

(i) every total orthonormal set in X is a Hamel basis for X, and

(ii) an orthonormal Hamel basis for X always exists.

$\boxed{5+5 = 10 \text{ marks}}$

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