# CSL863: Randomized Algorithms
## II semester, 2007-08

Major Exam
8 AM to 10 AM

Thursday, 1st May, 2008

All questions carry equal weight. Attempt all questions.

1. The diameter of a graph $G = (V, E)$ is defined as $\max_{u,v \in V} d(u, v)$ where $d(u, v)$ is the distance between $u, v$. The straightforward approach to compute the diameter is to compute all-pairs-shortest-paths and output the maximum pairwise distance - this leads an $O(n^3)$ algorithm.

   For an unweighted, undireeted graph, consider the following observation to compute the diameter without computing All Pairs Shortest Paths. Suppose $(u*, v*)$ is a diametral pair and let the shortest path be $\Pi = \{u* = w_0, w_1, \ldots w_k = v*\}$. Then if we run Single Source Shortest Paths algorithm (like Dijkstra's algorithm) from any of the vertices in $\Pi$, we can compute the diametral pair in $O(n^2)$ steps.

   (a) Justify this observation.

   (b) Given that $\Pi$ is not known, how would you use randomization to design an algorithm that will be $o(n^3)$ when $\Pi$ is $\Omega(n^c)$ for $c > 0$.

2. Given a graph $G = (V, E)$, a set $U \subseteq V$ is called a *vertex cover* if each edge in $E$ has at least one endpoint in $U$. Let us consider the following algorithm for finding a vertex cover:

   > Start with an empty cover $U$. While there are uncovered edges left, pick one of them arbitrarily and place one arbitrarily chosen endpoint of this edge in $U$.

   (a) Show that this algorithm could produce a vertex cover that is $\Omega(n)$ times the size of the smallest vertex cover.

   (b) if instead of placing an arbitrarily chosen endpoint of the current edge in the vertex cover, we place a randomly chosen endpoint (one of the two chosen uniformly), show that if the size of the smallest vertex cover is $k$, then $\mathrm{E}(U) \leq 2k$.

3. A line of $n$ airline passengers is waiting to board a plane. they each hold a ticket to one of the $n$ seats on that flight. The $i$th passenger in line has a ticket for the seat number $i$. Unfortunately, the first person in line is crazy, and will ignore the seat number on his ticket, picking a random seat to occupy. All the other passengers are quite normal, and will go to their proper seat unless it is already occupied. If their given seat is occupied, they will then find a free seat to sit in, (uniformly) at random. What is the probability that the last ($n$th) person to board the plane will sit in their proper seat (number $n$)?