MAL 728 Category Theory
Major Test (May 2010)

Time: 2 Hours $\qquad$ Max. Marks: 50

1. Let $\underline{\underline{Bn}}(I)$ be the category whose objects are functions $(A, f) = A \xrightarrow{f} I$ and whose morphisms $(A, f) \xrightarrow{h} (B, g)$ are functions $A \xrightarrow{h} B$ such that $gh = f$. Show that $\underline{\underline{Bn}}(I)$ has (i) a terminal object, (ii) pullbacks, and (iii) a subobject-classifier. Construct cartesian products $(A, f) \times (B, g)$ and Hom objects $((A, f) \Longrightarrow (B, g))$ towards showing that $\underline{\underline{Bn}}(I)$ is cartesian-closed (you are not required to prove that it is actually cartesian-closed).

   [15 Marks]

2. What is a topos? Define the natural numbers object in a topos.

   Find the natural numbers object in the category (i) $\underline{\underline{\text{sets}}}$ whose objects are sets and whose morphisms are functions, and (ii) $\underline{\underline{Bn}}(I)$ of Problem 1.

   [15 Marks]

3. Prove that the category of finite sets as objects and morphisms $X \xrightarrow{\alpha} Y$ as $m \times n$ matrices with entries from a field $\mathbb{F}$ $[\![$ we denote this category by $\underline{\underline{\text{Fin Set}}}_{\mathbb{F}}$ $]\!]$ where $|X| = n$, $|Y| = m$, is a fuzzy theory.

   Further show that it is $*$—autonomous. What is $X \otimes Y$ here and what is the dualizer?

   [10 Marks]

4. Let $\underline{\underline{Ab}} \xrightarrow{G} \underline{\underline{Gr}}$ be the functor which forgets that a given abelian groups is abelian remembering only that it is a group $[\![$ $\underline{\underline{\text{Ab}}}$ is the category with objects abelian groups, $\underline{\underline{\text{Gr}}}$ is the categroy with objects as groups; morphisms are homomorphisms in both cases $]\!]$. For a group X let N be the subgroup generate by $\{xy(yx)^{-1} \mid x, y \in X\}$. Show that N is a normal subgroup, that X/N is abelian and that the functor $FX := X/N$ is left adjoint to G.

   [10 Marks]

—Smile—