Department of Mathematics
MAL 110: Mathematics I
Major Exam

Maximum marks: 50 **Time: 2 hrs.**
Attempt all questions.

1. Evaluate the integral $\int_0^\infty e^{-x^2} cos\,(\alpha x)\, dx$. **(4 marks)**

2. (a) Find the length of the portion of cardiod $r = a(1 + cos\theta)$ lying outside the circle $r = a$.

   (b) Let $F(x, y, z)$ be a homogenous function of degree n and z is an implicit function of x, y defined by $F(x, y, z) = 0$. Show that
   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$
   (3 + 3 marks)

3. Let $\lambda \neq 0, \mu \neq 0$ be two real numbers. Suppose W_1 is a linear span of the vectors $\begin{pmatrix} 0 \\ 1 \\ \mu \end{pmatrix}$ and $\begin{pmatrix} -\lambda \\ 1 \\ 0 \end{pmatrix}$ and W_2 is a linear span of the vectors $\begin{pmatrix} -1 \\ \lambda \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ \mu \end{pmatrix}$. Find the values of λ and μ for which $W_1 = W_2$. **(3 marks)**

4. Let $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$. Find all critical points of $f(x, y)$ and determine their nature.
   (4 marks)

5. (a) Find the interval of existence of the unique solution of the initial value problem
   $$\frac{dy}{dx} = y^2 + 4, \quad y(0) = 0,$$
   for the domain $\mathcal{R} = \{(x, y) : |x| \leq a, |y| \leq b, a > 1, b > 0\}$, by using the existence and uniqueness theorems of the first order initial value problem.

   (b) Solve the differential equation
   $$(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0.$$
   (3 + 3 marks)

...contd. on page 2

6. Using the method of variation of parameter, find the general solution of the differential equation

$$\frac{d^2y}{d^2x} - 4\frac{dy}{dx} + 4y = (x^3 + x)e^{2x}. \qquad \text{(5 marks)}$$

7. (a) Find the general solution of the differential equation

$$\frac{d^4y}{dx^4} - y = e^x.$$

(b) Determine all real numbers $L > 1$ such that the differential equation

$$x^2\frac{d^2y}{dx^2} + y = 0, \quad 1 \leq x \leq L,$$

satisfying the conditions $y(1) = 0,\ y(L) = 0$, has a nonzero solution. **(3 + 3 marks)**

8. Using the Laplace transform, solve the initial value problem

$$t\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + ty = 0, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1. \qquad \text{(4 marks)}$$

9. (a) Prove that the eigenvalues of skew-Hermitian matrix are pure imaginary or zero.

(b) Let $F(s)$ be the Laplace transform of the function $f(t) = \begin{cases} 1 & 0 \leq t < \pi \\ 0 & \pi \leq t < 2\pi \\ 1 & 2\pi \leq t. \end{cases}$

Find $\mathcal{L}^{-1}\left(\dfrac{F(s)}{s^2 + 2s + 1}\right)$.

(2 + 4 marks)

10. Solve the following homogenous linear system

$$\begin{aligned} \frac{dx_1}{dt} &= 2x_1 + 2x_2 + x_3 \\ \frac{dx_2}{dt} &= x_1 + 3x_2 + x_3 \\ \frac{dx_3}{dt} &= x_1 + 2x_2 + 2x_3. \end{aligned}$$

(6 marks)