Indian Institute of Technology, Delhi
Department of Mathematics
Major Test: Mathematics-1 (MA110P)

Maximum Marks: 50 Time: 2 Hours.

Note: There are ten questions and all questions carry equal marks.

1. Prove that the function $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, \quad if\ x=0,\ y=0, & \end{cases}$

is continuous, possesses first order partial derivatives at (0, 0) but is not differentiable at (0,0).

2. A function $y(x)$ satisfies the differential equation

$$(x^2+1)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=0;\quad y(0)=1,\ y'(0)=0.$$

Using Leibtniz theorem, or otherwise, show that

$$(x^2+1)\frac{d^{n+2}y}{dx^{n+2}}+(2n+3)x\frac{d^{n+1}y}{dx^{n+1}}+(n+1)^2\frac{d^ny}{dx^n}=0,\ n\in N.$$

Hence find the Maclaurin's series for y as far as the term in x^6. Show that the co-efficient of x^{2n} in the series is $(-1)^n\dfrac{(2n-1)(2n-3)....................3.1}{2n(2n-2)(2n-4)..............4.2}$.

3. Using $y(x)=\int_{-\infty}^{\infty}e^{-t^2}\cos(tx)dt$, form the differential equation $2\dfrac{dy}{dx}+xy=0$, and obtain its solution $y=\sqrt{\pi}\,e^{-\frac{x^2}{4}}$.

4a. Making substitution $y=z^{-\frac{1}{3}}$, or otherwise, solve the differential equation

$$y-x\frac{dy}{dx}=2x^3y^4$$

4b. Solve, $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+8y=8x^2$.

5a. Find the Laplace Transform of the function, $f(t)=\dfrac{1}{\sqrt{t}}$ and hence, solve the differential equation $\dfrac{dy}{dt}+y=\dfrac{e^{-t}}{\sqrt{t}},\quad y(0)=0.$

5b. $P_n(x)$ is a Legendre polynomial of degree n, show that

$$\int_{-1}^{1}P_m(x)P_n(x)dx=0,\quad m\neq n.$$

6. Using power series method obtain $J_1(x)$ from the Bessel differential equation

$$x^2y'' + xy' + (x^2 - 1)y = 0$$, and find value of $J_1'(0)$.

7. Show that $\begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2 = \begin{vmatrix} 2yz - x^2 & z^2 & y^2 \\ z^2 & 2xz - y^2 & x^2 \\ y^2 & x^2 & 2xy - z^2 \end{vmatrix} = (x^3 + y^3 + z^3 - 3xyz)^2$.

8. Let AX=B be a system of linear equations given by

$$A=\begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 3 \\ 2 & 4 & 5 \\ -1 & 1 & 3 \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 2 \\ 1 \\ -8 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find the rank of the augmented matrix of the system. Is the system consistent? If so, find its solution using Crammer's Rule.

9a. Show that a skew-symmetric matrix of odd order is singular.

9b. If, $A=\begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -1 \\ 4 & 4 & -1 \end{bmatrix}$, show that $A^2 - 4A + 3I = 0$ and using induction or otherwise prove that $2A^n = (3^n - 1)A + (3 - 3^n)I$.

10. Given that the matrix $A=\begin{bmatrix} 5 & -2 & -2 \\ -2 & 5 & 2 \\ -2 & 2 & 9 \end{bmatrix}$ has an eigenvector $(1,\ -1,\ 1)^T$, and also that one of its eigenvalues is 3. Obtain all its eigenvalues and corresponding eigenvectors. Form a matrix C with columns as the orthonormal eigenvectors of A. Let $f(x_1, x_2, x_3) = 5x_1^2 + 5x_2^2 + 9x_3^2 - 4x_1x_2 + 4x_2x_3 - 4x_3x_1$, show that f can be expressed as $X^T AX$ with $X = (x_1, x_2, x_3)^T$. Now using the transformation X=CY, $Y = (y_1, y_2, y_3)^T$, express f as a function of $y_1, y_2, and\ y_3$,