IIT Delhi, CSE Dr. Astrid Kiehn Major Exam (30 marks) 02.12.2006

# CSL303 Logic for Computer Science

## Exercise 1 Monadic First Order Logic (5 marks)

Consider the fragment of first order logic which can be generated by:

$$\varphi = P(x) | Q(x) | \exists x \varphi | \neg \varphi | (\varphi \land \varphi)$$

where x can be any variable, but P and Q are the only predicate symbols of the logic.

- (a) Give a model A for the formula  $\neg \exists x (P(x) \land Q(x))$  where |A| is the set of natural numbers. (no proof required)
- (b) Let D =  $\{00, 01, 10, 11\}$  and A be an arbitrary structure. The mapping  $f: |A| \rightarrow D$  is defined by

$$f(d) = \begin{cases} 00 & \text{if } d \not\in P^A, d \not\in Q^A, \\ 01 & \text{if } d \not\in P^A, d \in Q^A, \\ 10 & \text{if } d \in P^A, d \not\in Q^A, \\ 11 & \text{if } d \in P^A, d \in Q^A. \end{cases}$$

Based on f we define structure  $C_A$  with domain  $|C_A| = D$  by

$$P^{C_A} = \{f(d) \mid d \in P^A\},\$$
  
 $Q^{C_A} = \{f(d) \mid d \in Q^A\}.$ 

- i) What are the elements of  $P^{C_A}$  and  $Q^{C_A}$  if A is the structure you have given under (a)?
- ii) Prove that for all formulas  $\psi$  the following holds for all structures A and value assignments s:

$$\models_A \psi[s]$$
 if and only if  $\models_{C_A} \psi[s']$  where  $s'(x) = f(s(x))$ .

# Exercise 2 Löwenheim/Skolem/Gödel (4 marks)

- (a) Give a Skolem form for  $\alpha = \neg \exists x (\forall y P(y, c) \lor \neg \forall y P(x, y))$ .
- (b) Give the Herbrand Expansion for the matrix of the Skolem form under (a).
- (c) Is  $\alpha$  satisfiable? Give reason.

#### Exercise 3 Hilbert System (3 marks)

Give a deduction sequence for  $\{\forall x (P(x) \rightarrow Q(y)), \forall z P(z)\} \vdash Q(z)$ .

### Exercise 4 Consistency (3 marks)

Let K be the clausal form of a Skolem form  $\varphi$ . Let K be such that each clause in K contains at least two literals and in each clause either all literals are either positive or all are negative. Is it generally true that  $\{\varphi\}$  is consistent? Prove or give a counterexample.

#### Exercise 5 (4 marks)

Prove or reject by giving a counterexample:

The most general unifier  $\eta$  computed by the unification algorithm is always idempotent that is  $\eta \eta = \eta$ .

### Exercise 6 Separation Property (5 marks)

- (a) Let  $\Sigma_1$  and  $\Sigma_2$  be two sets of closed first order formulas such that no structure satisfies both sets that is  $\Sigma_1$  and  $\Sigma_2$  do not have a common model. Let  $Mod(\Sigma_i) = \{A \mid \text{ for all } \psi \in \Sigma_i : \models_A \psi\}$ . Show that there is a formula  $\varphi$  such that every structure in  $Mod(\Sigma_1)$  satisfies  $\varphi$  and every structure in  $Mod(\Sigma_2)$  satisfies  $\neg \varphi$ .
- (b) Where in the proof of (a) do you need the fact that  $\Sigma_1$  and  $\Sigma_2$  do only contain closed formulas? Give reason.

## Exercise 7 CTL (5 marks)

Let K and K' be serial Kripke structures with exhibited intial state s and t, respectively. We say that s in K is equivalent to t in K', denoted  $s \underline{in} K \sim t \underline{in} K'$  iff

- (a) for each finite path  $s = s_0 \to s_1 \to \cdots \to s_n$  in K there is a finite path  $t = t_0 \to t_1 \to \cdots \to t_n$  in K' such that  $\kappa(s_i) = \kappa'(t_i)$  for all  $i \le n$  and
- (b) for each finite path  $t = t_0 \to t_1 \to \cdots \to t_n$  in K' there is a finite path  $s = s_0 \to s_1 \to \cdots \to s_n$  in K such that  $\kappa(s_i) = \kappa'(t_i)$  for all  $i \le n$ .

( $\kappa$  and  $\kappa'$  denote the labelling functions of  $\kappa$  and  $\kappa'$ .) Is it true that

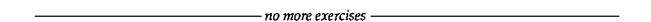
$$s \underline{in} K \sim t \underline{in} K' iff \{ \varphi \mid \varphi \in CTL, s \models_{K} \varphi \} = \{ \varphi \mid \varphi \in CTL, t \models_{K'} \varphi \}?$$
 (\*)

Prove (\*) or show by giving counter examples which of the CTL operators have to be removed to obtain the validity of (\*) and then prove the validity of (\*) for the remaining fragment of the CTL.

### Exercise 8 Applications (4 marks)

You are given a road map that is a finite number of towns  $A_1$  to  $A_n$  and and a finite number of roads  $E_1$  to  $E_m$  directly connecting two of these towns. Each road permits traffic in both directions.

- (a) Write a logic program (including a goal) which decides whether one can travel from  $A_1$  to  $A_n$  on road. The route itself does not have to be computed.
- (b) Define a Kripke structure K which models the road map. Give a CTL formula which is true in a state you have to specify if and only if one can travel from A<sub>1</sub> to A<sub>n</sub> on road.



The axioms of Hilbert system:

All generalizations of the following forms:

- 1. Tautologies;
- 2.  $\forall x \alpha \rightarrow \alpha_t^x$ , where t is substitutable for x in  $\alpha$ ;
- 3.  $\forall x(\alpha \rightarrow \beta) \rightarrow (\forall x\alpha \rightarrow \forall x\beta);$
- 4.  $\alpha \rightarrow \forall x\alpha$ , where x does not occur free in  $\alpha$ .