Department of Mathematics
MAL 002: Preparatory Mathematics
2009-2010: Semester II
Major Exam

5 May 2010

You must attempt *all* ten questions. Explain your answers in as much detail as possible. Calculator may be used ONLY in Question 4.

1. Let $f : \mathbb{R} \to \mathbb{R}$. Show that f can be expressed as a sum of an *even* function and an *odd* function in exactly one way. **[4]**

2. Let $f : [0,1] \to [0,1]$ be given by
$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n}, \text{ with } \gcd(m,n) = 1 \text{ and } n \geq 1; \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$
Discuss continuity of f in $[0,1]$, and justify your answer. **[6]**

3. Suppose that the functions f and g are defined throughout an open interval containing the point x_0. that f is differentiable at x_0, that $f(x_0) = 0$, and that g is continuous at x_0. Show that the product fg is differentiable at x_0. **[4]**

4. Consider points P, Q on a unit circle centred at $O := (0,0)$ such that $\angle\ POQ = 2\theta$. If the area of the triangle POQ equals the area of the region bounded by the line PQ and the arc PQ, use Newton's method to estimate θ to two decimal places. **[5]**

5. Let $a \in \mathbb{R}$. Prove that the function $f(x) = x^3 - 3x + a$ never has two roots in $[0,1]$. **[4]**

6. The period T of a pendulum is proportional to the square root of its length ℓ. If the length is measured with an error of *at most* p %, use differentials to estimate the maximal possible error in calculating the period. **[4]**

7. (a) Let $f : [a,b] \to \mathbb{R}$. Explain the following terms: (i) f is bounded; (ii) $\mathcal{P}$ is a partition of the interval $[a,b]$; (iii) the upper sum $\mathcal{U}(f;\mathcal{P})$; (iv) the lower sum $\mathcal{L}(f;\mathcal{P})$; (v) f is integrable over $[a,b]$.
   (b) For the function f defined in Question 2 and the partition $\mathcal{P} = \left\{0, \frac{1}{m}, \frac{1}{n}, 1\right\}$ of $[0,1]$, compute $\mathcal{U}(f;\mathcal{P})$. You may assume m, n are positive integers with $m > n$.

   [3+4]

8. Suppose f is continuous on $[a,b]$ and that g is integrable and positive on $[a,b]$. Prove that there exists a $c \in [a,b]$ such that
$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$
[5]

9. Find the dimensions of the rectangle of *largest* area that can be inscribed in a circle of radius r. **[5]**

10. Let $ABCD$ denote a parallelogram, with $B := (-3,9)$, $C := (2,4)$ and with AD parallel to BC and tangent to the parabola $y = x^2$. Let A denote the area of the region $\mathcal{R}$ bounded by $y = x^2$ and the line segment BC. Prove that $\mathcal{A}$ equals $\frac{2}{3}$ times the area of the parallelogram $ABCD$. **[6]**