Department of Physics, IIT Delhi

II Semester 2007-2008: PHL742 General Relativity Major

Time: 2 Hours Answer all questions Maximum Marks:50

(For convenience, $c = 1 = G$, so time, length and mass have same dimensions)

1. If $F_{ab} = -F_{ba}$, prove $F^{ab}F_{bc} = F_{cb}F^{ba}$.

2. If $\widetilde{g_{ab}} = \Omega^2 g_{ab}$. where Ω is a scalar field, then the Christoffel symbols of the two metrics are related through

$$\widetilde{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + \left[\delta^a{}_b \frac{\Omega_{,c}}{\Omega} + \delta^a{}_c \frac{\Omega_{,b}}{\Omega} - \frac{\Omega^{,a}}{\Omega} g_{bc}\right]$$

3. For the Schwarzschild metric,

$$(ds)^2 = -\left(1 - \frac{r_s}{r}\right)(dt)^2 + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} + r^2\left[(d\theta^2) + \sin^2\theta\,(d\phi^2)\right].$$

Find the time taken for light to travel from $r = 4r_s$ to $r = 10r_s$ as perceived by an observer situated at $= \infty$. What will be this interval for an observer situated at $r = 4r_s$, who remains at rest there? 6

4. For the Robertson Walker metric, given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

answer the following, giving brief reasons or when appropriate, whether the statement is true or false with reasons.

(a) t is the time as measured by which observer?

(b) Find the ratio of the square of radial length to the area of a spherical surface surrounding the origin, at any fixed time t

(c) If $H_0 = \frac{1}{a(t)}\frac{da(t)}{dt}$ is the Hubble constant at time t then $\frac{1}{H_0}$ is the time in which this universe will change in size by a factor of e.

(d) what is the equation of the surface describing the night sky as seen by you at any instant? 4×3

5. A weak gravitational wave propagating along the z axis is described by the metric $g_{ab} = \eta_{ab} + h_{ab}$, where the only non-zero terms in h_{ab} are $h_{xy} = h_{yx} = h\sin[\omega(x-t)]$. write an equivalent metric in diagonal form. If a small rectangle of sides $\Delta x \times \Delta y$ is placed at the origin in the X-Y plane, find the dimensions of the rectangle as a function of time. What would be the shape of the rectangle (qualitative) in the original non-diagonal metric as a function of time 8

6. The Kerr metric is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\rho}\right)dt^2 \quad - \quad \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt$$
$$+\left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 \quad + \quad \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2,$$

where

$$a = \frac{J}{M}, \quad \rho^2 = r^2 + a2\cos^2\theta \quad \Delta = r^2 - 2Mr + a^2.$$

In the equatorial plane carry out a transformation which makes the metric diagonal for r outside the surfaces at which the metric has a zero determinant. What is the meaning of the new variables T and Φ? can this be carried out for arbitrary but fixed theta? 10

Formulae

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}[g_{bd.c} + g_{dc,b} - g_{bc,d}]$$

$$\frac{\partial^2 x^a}{\partial s^2} + \Gamma^a{}_{bc}\frac{\partial x^b}{\partial s}\frac{\partial x^c}{\partial s} = 0$$

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc.d} + \Gamma^a{}_{cp}\Gamma^p{}_{bd} - \Gamma^a{}_{dp}\Gamma^p{}_{bc}$$

$$R_{abcd} = -R_{abcd} = -R_{abdc} = R_{abcd}.$$