MAL516: Algebra

Major, May, 2010

Time: **2 hours** Total Marks: 40

All questions are compulsory. No credit will be given unless appropriate argument is provided.

1. Determine whether the following statements are True or False. Justify your answer in each case. $2 \times 5 = 10$

   (a) In an integral domain every prime is an irreducible element.

   (b) The ring $\frac{\mathbb{R}[X]}{\langle X^2 - a \rangle}$ is a field for every $a \in \mathbb{R}$.

   (c) The polynomial $\sum_{i=0}^{6} x^i$ is irreducible over $\mathbb{Z}$.

   (d) Suppose that K is a normal extension over $\mathbb{Q}$. Then every subfield of K is a normal extension of $\mathbb{Q}$.

   (e) Suppose that L is an algebraic extension of a field F. Then every element of L belongs to a finite extension of F.

2. Let H be a p-group. Recall that H acts on itself by conjugation. Use this action to show that the center $Z(H)$ is a nontrivial normal subgroup of H. 4

3. Assume that a cyclic group G of order n has a unique subgroup of order d if d divides n. Determine the number of elements in G of order $m \in \mathbb{N}$. Use it to compute the following sum:

$$\sum_{d|n} \varphi(d)$$

   where $\varphi(d)$ is the number of positive integers less than d which are co-prime to d. $2 + 2 = 4$

4. Determine the splitting field of $X^6 - 1$ over $\mathbb{Q}$ in the form $\mathbb{Q}[\alpha]$. What is its extension degree? Describe its Galois group. $2 + 2 = 4$

5. Let $n \geq 5$. Show that the group S_n is not solvable. 4

6. Let I be an ideal of the ring $M(3, \mathbb{Z})$ of 3×3 matrices over $\mathbb{Z}$. If $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in I$, then show that $I = M(3, \mathbb{Z})$. 2

7. Let F be a field. Suppose $f(X) \in F[X]$. Show that there is an extension of F which contains a zero of $f(X)$. 4

8. Let K be a finite field having m elements. What are all possible values of m? Justify your answer. Is there a field having 64 elements? 4

9. Define 'solvable by radicals'. Suppose F is a field containing all the roots of unity. Use fundamental theorem of Galois theory to prove that $p(X) \in F[X]$ is solvable by radicals then the Galois group of $p(X)$ is a solvable group. You may use (without proof) the following results: If u is a root of $X^n - a$ for $n \in \mathbb{N}$ and $a \in F - \{0\}$, then $F(u)$ is its spitting field and the Galois group is abelian. State results that you use. 4