EEL 711 Major Test Semester I 2006-2007

Answer all questions (Marks: Q.1: 30, Q.2: 20, Q.3: 20, Q.4: 10) Full Marks: 80

1. A complex circular Gaussian random process $Z(t)$ is zero-mean and has autocorrelation function

$$R_Z(t_1, t_2) = 16^{-|t_1 - t_2|} e^{j2\pi(t_1 - t_2)} .$$

   (a) Find the p.d.f. of $V = Z(0) + 2Z(\frac{1}{4}) + 3Z(\frac{1}{2})$. [8]

   (b) Using Tchebycheff's Inequality, find an upper bound on the probability $\Pr\{|Z(2) - Z(1)| \geq \epsilon\}$ as a function of ϵ. [6]

   (c) Suppose $Z(t)$ is passed through a whitening filter with transfer function $H(f)$ to produce a white random process $W(t)$ with autocorrelation fnnction

$$R_W(t_1, t_2) = \delta(t_1 - t_2) .$$

   Find $|H(f)|$. [10]

   (d) Define a random vector $\underline{Z}$ as

$$\underline{Z} \triangleq \begin{bmatrix} Z(0) \\ Z(\frac{1}{4}) \end{bmatrix} .$$

   Let $\underline{B}$ be a 2×2 matrix such that $\underline{BZ}$ has the p.d.f.

$$f_{\underline{BZ}}(\underline{v}) = \frac{1}{\pi^2} e^{-\underline{v}^H \underline{v}}, \quad \underline{v} \in \mathcal{C}^2, \tag{1}$$

   where $\mathcal{C}$ is the set of complex numbers. Find a lower triangular matrix $\underline{B}$ satisfying (1). [6]

2. A source emits a weighted sum of N impulses with random time shifts. The source output is a real-valued random process $X(t)$ given by

$$X(t) = \sum_{k=1}^{N} A_k \delta(t - \tau_k) .$$

   $A_1, \ldots, A_N$ are i.i.d. real Gaussian random variables distributed as $\mathcal{N}(m, \sigma^2)$, where $m \in \mathcal{R}$ ($\mathcal{R}$ denotes the set of real numbers) and $\sigma > 0$, and $\tau_1, \ldots, \tau_N$, which are independent of $A_1, \ldots, A_N$, are real-valued i.i.d. exponentially distributed random variables with mean $\frac{1}{\lambda}$.

   (a) Find $\mu_X(t)$ and $R_X(t_1, t_2)$. [6+8]

   (b) What is the condition under which $\mu_X(t)$ is constant for all t? Is $X(t)$ WSS under this condition? Is $X(t)$ white under this condition? [6]

3. Let $X_1, \ldots, X_n$ be i.i.d. one-sided Gaussian random variables with p.d.f.

$$f_{X_i}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}, \quad x \geq 0.$$

   Let $Y = \sum_{i=1}^{n} X_i^2$.

   (a) Find the c.d.f. of X_i. [6]

   (b) Find the p.d.f. of Y. [6]

   (c) For large n, find (i) the c.d.f. of Y, (ii) the c.f. of Y. [8]

4. Packets arrive at a node in a network in two independent simultaneous Poisson streams at times $t \geq 0$. The number of packets arriving in a time interval $[0, t)$ is denoted as $X_1(t)$ for the first stream and by $X_2(t)$ for the second stream, where $X_1(t)$ and $X_2(t)$ are independent Poisson processes. The distribution of

$$N_i(t_0, t_0 + t) = X_i(t_0 + t) - X_i(t_0), \quad i = 1, 2,$$

is given by

$$\Pr[N_i(t_0, t_0 + t) = k] = e^{-i\lambda t}\frac{(i\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots, \qquad \lambda > 0, \qquad i = 1, 2, \qquad \forall t_0 \geq 0.$$

(a) Calculate the probability that a total of two packets arrive at the node in a time interval $\left[\frac{1}{\lambda}, \frac{3}{2\lambda}\right)$. [4]

(b) Let $U = N_1(T, 2T) + 2N_2(2T, 4T)$, and $V = N_1(0, T) + N_1(2T, 4T)$. Find $\Psi_U(\jmath\omega)$ and $\Psi_V(\jmath\omega)$, the c.f.s of U and V, respectively, and the second moment of $U - V$. [6]

Some Formulae

- If $Y \sim \mathcal{N}(0, 1)$, then

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}, \quad -\infty < y < \infty \qquad F_Y(y) = \int_{-\infty}^{y} f_Y(z)dz = 1 - Q(y)$$

- If $\underline{X} \sim \mathcal{N}(\underline{\mu}, \underline{K})$, then

$$\Psi_{\underline{AX}+\underline{b}}(\jmath\underline{\omega}) = \exp\left\{\jmath\underline{\omega}^T\underline{A\mu} + \jmath\underline{\omega}^T\underline{b} - \frac{1}{2}\underline{\omega}^T\underline{AKA}^T\underline{\omega}\right\}$$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\Psi_{X^2}(\jmath\omega) = \frac{e^{\frac{\jmath\omega\mu^2}{1-2\jmath\omega\sigma^2}}}{(1 - 2\jmath\omega\sigma^2)^{\frac{1}{2}}}$$

- Unit step function: $u(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases}$

- Gamma distribution with parameter m and mean Ω: $f_X(x) = \frac{m^m x^{m-1} e^{-\frac{mx}{\Omega}}}{\Gamma(m)\Omega^m} u(x)$

- Exponential p.d.f.: $f_X(x) = ae^{-ax}u(x)$

- $\int_{-\infty}^{\infty} g(x)f(y_1 - x)\delta(y_2 - x)dx = g(y_2)f(y_1 - y_2)$