LOCATION-ALLOCATION PROBLEMS

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The calculational aspects of solving certain classes of location-allocation problems are presented. Both exact extremal equations and a heuristic method are presented for solving these problems. Directions for further investigation are also indicated because of the need for methods to shorten the total amount of calculation involved.

THERE frequently arise in practical affairs problems that are concerned with how to serve or supply, in an optimum fashion, a set of destinations that have fixed and known locations. What must be determined, in these problems, is the number, location, and size of the sources that will, most economically, supply the given set of destinations with some commodity, material, or service. Examples of such problems are the determination and location of warehouses, distribution centers, communication centers, or production facilities.

The general problem may be stated as follows:

Given: (1) The location of each destination.
(2) The requirements at each destination.
(3) A set of shipping costs for the region of interest.

To Determine: (1) The number of sources.
(2) The location of each source.
(3) The capacity of each source.

The problem as stated is an exceedingly difficult one, not only from a theoretical point of view, but also from a computational standpoint. The existence and use of digital computers barely makes a dent into some of the combinatorial problems arising when computation is undertaken.

The present paper will attempt to describe some methods that have been devised to deal with a certain class of restricted problems that can and do arise in industrial situations. In many of these cases, the full problem in all its generality, seldom arises. Frequently, say in a warehouse location problem, certain locations are fixed and known by outside considerations, total costs are either not known or are subject to wide fluctuations, revisions and additions to source locations and sources will be necessary, etc.

The assumptions that will be made for the problems considered in the present paper are as follows:

1. There are no restrictions on the permissible source capacities.
2. Unit shipping costs are independent of total source output.

Even with these restrictions the problem is a formidable one. The general outline of this paper will be to mention briefly previous work in the field and then to state the problem in mathematical language, indicate the computational problem and the approaches examined by the author, as well as further studies now under way.

In the purely mathematical literature this problem, in one form or another, is a very old one. CAVALIERI[3] in 1647 considered the problem of finding the point the sum of whose distances from three given points is a minimum. He showed that each side must make an angle of less than 120° with the given minimum point. HEINEN in 1834 noted that in a triangle, which has an angle of 120°, that the vertex of this angle is itself the minimum point. FAGNANO in 1775 showed that the point for which the sum of the distances from the vertices of a quadrilateral (4 points) is a minimum is given by the intersection of the diagonals. TEDENAT in 1810 found for the case of n points the following necessary condition: The sum of the cosines of the angles between any arbitrary line in the plane and the set of lines connecting the given points with the minimum point must equal zero. Finally, STEINER proved in 1837 that the necessary and sufficient conditions are that the sum of the cosines and sines of the above mentioned angles be zero.

In ALFRED WEBER'S[4] classic study of the location of industries, the mathematical appendix by GEORG PICK contains a characterization of the locational triangle from the point of view of an analog model and discusses the concept of the minimum point (minimum sum of distances of a single source to three destinations) as being that point at which three forces, from the variable source point to the destination points, are in physical equilibrium. The argument follows Steiner's geometrical analysis although no explicit reference to Steiner's analysis is made. There is also an extension of the argument to more than three destinations. Only a single source is considered.

In ISARD'S[5] comprehensive study of industrial location, land use, and related problems there is a mathematical formulation of some of these problems. Isard restates the Weberian theory. He also generalizes in the direction of considering the shipment of many products to consuming points (destinations). Isard mentions the problem of a multiplicity of producers (sources), but restricts consideration to the case of several market areas each of which has its own source. This effectively reduces the problem to a single source problem for each market area. This, of course, avoids the problem of a multiplicity of sources to be determined simultaneously.

The present paper seeks to examine the problem of simultaneous source determination, to indicate the difficult computational problems involved, and to suggest approaches to be used in practice.

First the general approach will be indicated. The mathematical problem will be stated and then possible methods of solution as well as some actual computational experience will be presented.

The general approach to the problem is as follows: Suppose that, if m is the number of sources, then the capital depreciation and operating costs of these sources, C, is given by some function of m, i.e.,

$$C=g_1(m). \tag{1}$$

Suppose further, that the *minimum* cost of supplying the given set of n destinations by these m sources (however this is determined) is given by:

$$\varphi_{\min}=g_2(m), \tag{2}$$

where $\varphi_{\min}$ is the *minimum* cost solution, i.e., the cost of using a set of m optimally located sources to supply the destinations.

It is clear that the total cost of the supply operation is given by the sum of (1) and (2), i.e.,

$$E=C+\varphi_{\min}=g_1(m)+g_2(m). \tag{3}$$

Further, the minimum cost solution is obtained when:

$$dE/dm=dg_1(m)/dm+[dg_2(m)/dm]=0. \tag{4}$$

Solving equation (4) for m will give the number of sources for the minimum cost solution.

Equation (1) can be obtained by fitting cost data with some appropriate empirical equation and offers no serious problems of execution. The principal problem in the approach outlined above is the determination of the relation implied in equation (2). Therefore, the remainder of this paper is devoted to methods for obtaining $\varphi_{\min}$ for some fixed m. If we can solve this problem, the determination of $\varphi_{\min}=g_2(m)$ is reduced to another problem of fitting data.

Let the location of the set of n known destinations be given by (x_{Di}, y_{Di}) $(i=1, 2, \cdots, n)$, their coordinates in a Cartesian coordinate system. Similarly, let the coordinates of the m sources that are to be determined be given by (x_j, y_j) $(j=1, 2, \cdots, m)$. If it is assumed that there are no source capacity restrictions, and that unit shipping costs are independent of source output, it is clear that no one destination will be supplied by more than one source in the minimum cost solution. However, in addition to not knowing the location of each of the m sources in the minimum cost solution, we also do not know which source is to serve which subset of destinations. In order to avoid this definite association of sources and destinations at the outset, we shall assume, in the definition of the objective function, φ (the cost of supplying n destinations with m sources) that every destination can theoretically be supplied by any source. This is expressed by the use of a multiplier, α_{ij}, which is 0 or 1, depending upon whether the ith destination is not or is supplied by the jth source. Accordingly, we have:

$$\begin{aligned} &\varphi = \textstyle\sum_{j=1}^{j=m} \sum_{i=1}^{i=n} \alpha_{ij}\psi(x_{Di},y_{Di},x_j,y_j), \\ &\alpha_{ij}=0 \quad \text{or} \quad 1, \end{aligned} \tag{5}$$

$\psi(x_{Di},y_{Di},x_j,y_j)=$ cost function for supplying the ith destination by the jth source.

In order to find the set of (x_j,y_j) that will minimize φ we differentiate (5) with respect to x_j and y_j and solve the equations resulting from setting the derivatives equal to zero. Thus:

$$\begin{aligned} \partial\varphi/\partial x_j&=0, \\ \partial\varphi/\partial y_j&=0. \end{aligned} \qquad (j=1,\,2,\,\cdots,m) \tag{6}$$

We therefore have:

$$\begin{aligned} \textstyle\sum_{i=1}^{i=m} \alpha_{ij}\,[\partial\psi(x_{Di},y_{Di},x_j,y_j)/\partial x_j]&=0, \\ \textstyle\sum_{i=1}^{i=m} \alpha_{ij}\,[\partial\psi(x_{Di},y_{Di},x_j,y_j)/\partial y_j]&=0. \end{aligned} \qquad (j=1,2,\,\cdots,m) \tag{7}$$

Solving the $2m$ equations (7) will yield the $2m$ values of (x_j,y_j) which will cause φ to be a minimum, for some particular set of α_{ij}.

Before we proceed to use this technique (as well as explore another one) it is well to consider the amount of computation that may be involved. For m sources and n destinations there are $S(n,m)$ possible assignments of n destinations to m sources, where $S(n,m)$ is the Stirling number of the second kind[1] and is given by

$$S(n,m)=\frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k(m-k)^n, \tag{8}$$

where $\binom{m}{k}$ are the usual bionomial coefficients. For very large n, these Stirling numbers can be formidably large. Equations (7) would have to be solved $S(n,m)$ times to find which particular allocation of sources to destinations, among this *minimum set*, is the absolute minimum we seek.

For small-scale problems this is feasible using a digital computer. However, for large-scale problems of industrial importance the amount of computation is prohibitive. We will return to this matter after first considering an example of a typical minimization problem.

Suppose it is desired to minimize Euclidean distance between sources and destinations, under the assumption that cost is proportional to distance. For this case we have:

$$\psi(x_{Di}, y_{Di}, x_j, y_j) = w_{ij}[(x_{Di} - x_j)^2 + (y_{Di} - y_j)^2]^{1/2}, \tag{9}$$

where w_{ij} is a weighting factor relating to multiplicity of supply trips or service calls and effectively increases relative distance. We, therefore have:

$$\varphi = \sum_{j=1}^{j=m} \sum_{i=1}^{i=n} x_{ij}\, w_{ij}[(x_{Di} - x_j)^2 + (y_{Di} - y_j)^2]^{1/2}. \tag{10}$$

Differentiating to find the minimum yields:

$$\begin{gathered} \sum_{i=1}^{i=n} \{\alpha_{ij}\, w_{ij}(x_{Di} - x_j)/[(x_{Di} - x_j)^2 + (y_{Di} - y_j)^2]^{1/2}\} = 0, \\ (j = 1, 2, \cdots, m) \\ \sum_{i=1}^{i=n} \{\alpha_{ij}\, w_{ij}(y_{Di} - y_j)/[(x_{Di} - x_j)^2 + (y_{Di} - y_j)^2]^{1/2}\} = 0. \end{gathered} \tag{11}$$

It must be shown that the solution of equations (11) yields a minimum. The conditions for a minimum are:

$$\begin{gathered} \partial^2\varphi/\partial x_j^2 > 0 \\ (\partial^2\varphi/\partial x_j^2)(\partial^2\varphi/\partial y_j^2) - (\partial^2\varphi/\partial x_j \partial y_j)^2 > 0. \end{gathered} \qquad (j = 1, 2, \cdots, m) \tag{12}$$

Substituting our expressions we have:

$$\begin{gathered} \sum_{i=1}^{n} \frac{\alpha_{ij}\, w_{ij}[(x_{Di} - x_j)^2 + 1]}{[(x_{Di} - x_j)^2 + (y_{Di} - y_j)^2]^{1/2}} > 0, \qquad (j = 1, 2, \cdots, m) \\ \left(\sum_{i=1}^{n} \frac{\alpha_{ij}\, w_{ij}[(x_{Di} - x_j)^2 + 1]}{[(x_{Di} - x_j)^2 + (y_{Di} - y_j)^2]^{1/2}}\right)\left(\sum_{i=1}^{n} \frac{\alpha_{ij}\, w_{ij}[(y_{Di} - y_j)^2 + 1]}{[(x_{Di} - x_j)^2 + (y_{Di} - y_j)^2]^{1/2}}\right) \\ -\left(\sum_{i=1}^{n} \frac{\alpha_{ij}\, w_{ij}(x_{Di} - x_j)(y_{Di} - y_j)}{[(x_{Di} - x_j)^2 + (y_{Di} - y_j)^2]^{1/2}}\right)^2 > 0. \qquad (j = 1, 2, \cdots, m) \end{gathered} \tag{13}$$

It is readily shown that equations (13) are true for all values of x_j and y_j, thus establishing that the solution of equations (11) leads to a minimum.

It becomes necessary to solve the two simultaneous equations (11) m times to find the (x_j, y_j).

The author has found from experience that, of several methods tried, the following 'method of iteration'[2] works best. It has been used in an IBM 704 program to solve problems of this type with excellent results.

Let $D_{ij}=[(x_{Di}-x_j)^2+(y_{Di}-y_j)^2]^{1/2}$; then the extremal equations (11) become:

$$\begin{aligned}&\textstyle\sum_{i=1}^{i=n}[\alpha_{ij}w_{ij}(x_{Di}-x_j)/D_{ij}]=0,\\ &\textstyle\sum_{i=1}^{i=n}[\alpha_{ij}w_{ij}(y_{Di}-y_j)/D_{ij}]=0,\end{aligned}\qquad (j=1,2,\cdots,m)\quad(14)$$

which leads to:

$$\begin{aligned}&\textstyle\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}x_{Di}/D_{ij})-x_j\\ &\qquad\cdot\textstyle\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}/D_{ij})=0,(j=1,2,\cdots,m)\quad(15)\\ &\textstyle\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}y_{Di}/D_{ij}-y_j\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}/D_{ij})=0.\end{aligned}$$

These are solved for x_j and y_j as follows:

$$\begin{aligned}x_j&=\textstyle\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}x_{Di}/D_{ij})/\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}/D_{ij}),(j=1,2,\cdots,m)\\ y_j&=\textstyle\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}y_{Di}/D_{ij})/\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}/D_{ij})\ (j=1,2,\cdots,m)\end{aligned}\quad(16)$$

These equations are solved iteratively. Let the superscript indicate the iteration parameter. The iteration equations for x_j and y_j are simply:

$$\begin{aligned}x_j^{k+1}&=\textstyle\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}x_{Di}/D_{ij}^k)/\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}/D_{ij}^k),\\ y_j^{k+1}&=\textstyle\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}y_{Di}/D_{ij}^k)/\sum_{i=1}^{i=n}(\alpha_{ij}w_{ij}/D_{ij}^k).\end{aligned}\quad(j=1,2,\cdots,m)\quad(17)$$

Results of such calculations will be given later for comparison with another method.

A set of convenient starting values that always yields a convergent algorithm is simply the weighted mean coordinates:

$$\begin{aligned}x_j^0&=\textstyle\sum_{i=1}^{i=n}\alpha_{ij}w_{ij}x_{Di}/\sum_{i=1}^{i=n}\alpha_{ij},\\ y_j^0&=\textstyle\sum_{i=1}^{i=n}\alpha_{ij}w_{ij}y_{Di}/\sum_{i=1}^{i=n}\alpha_{ij}.\end{aligned}\qquad(j=1,2,\cdots,m)\quad(18)$$

As an example of the application of the technique described above, equation (16), for the numerical solution of minimization problems of this type, the following problem with $n=7$, $m=2$ was solved. The destination locations were as follows:

1 (15,15)	4 (16, 8)
2 (5,10)	5 (25,14)
3 (10,27)	6 (31,23)
7 (22,29)	

Figure 1 shows the location of the destinations and the optimal location of two sources that minimize distance from these sources. In this problem all w_{ij} were taken as 1. There were $S(7,2)=63$ possible allocations, which were solved to determine the optimal source locations. As

Fig. 1 shows, one of the sources is located at one of the destinations. The coordinates of the sources and their allocations were found to be:

(15.420, 12.053)—destinations 1, 2, 4, 5.
(22.000, 29.000)—destinations 3, 6, 7.

The above technique is exact and is feasible for small problems on a

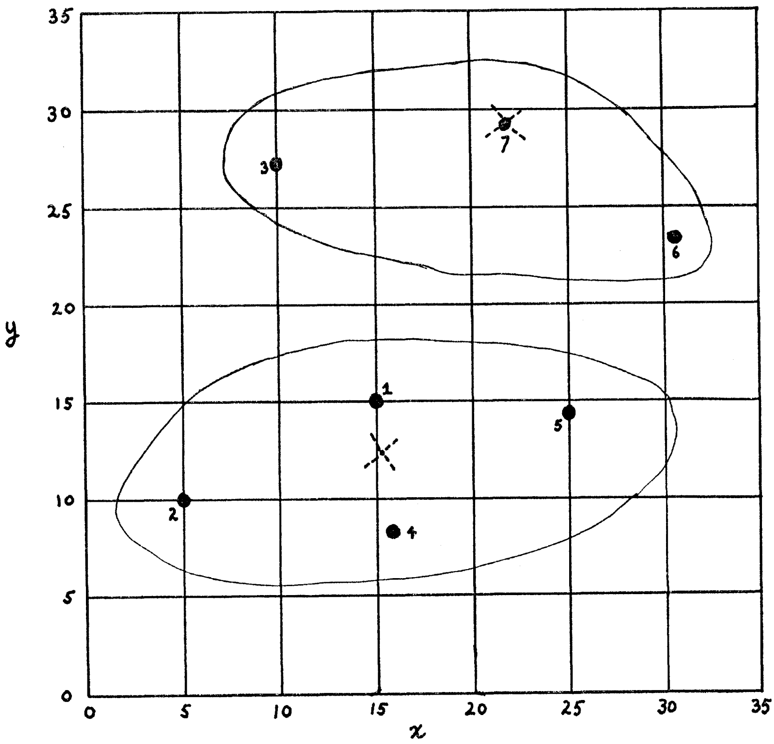

Fig. 1. Solution of distance minimization problem.

digital computer. However, for large numbers of destinations (>10) the method is not computationally attractive. In addition, these equations are valid only if, as stated previously, costs are proportional to distance. In the general case equations (7) must be solved and the existence of a minimum demonstrated.

At this point one is led to consider what can be done to treat general nonlinear costs and large destination sets.

A suggestive observation is the following: If, for a set of n destinations and m sources, the location of the sources is known, the determination of the optimal allocation is trivial. It is merely the set of weighted distances (or costs in the general case) that is a minimum. Conversely, if the allocation is fixed, the determination of the optimum location of the sources is merely the exact calculation, with *known* α_{ij}, that has been previously described.

This observation suggests the following as a possible approach: Select a set of m source locations and then determine the optimum allocation. Select another set of source locations and repeat. If a means could be found to consider a reasonable number of the 'important' sets (i.e., those sets that lead to the correct *allocation*) then the final locations of these sources could be determined by solving equations (7) once with the fixed correct allocation.

Let us assume there is some way to determine a number of these sets of source locations and suppose at least one of these sets would yield the same allocation as the exact solution. Then we can examine the entire set of solutions and choose the one with smallest sum of distances or costs. With this fixed allocation we can then solve the extremal equations to locate the sources optimally. There is a further point to consider, however. It may happen that one of the sets that did not have as low a sum of distances or costs as the smallest, would, after relocation of the sources, have a final value that was lower than the final value of the set that had the lowest sum before adjustment of the source locations by the extremal equations. Consequently, several of the 'lowest' values might be considered to be examined. This point will be discussed further at a later point.

There are a variety of possible methods for generating a 'reasonable' set of source location sets. First, let us consider one that was employed by the author for study and to solve several problems. This procedure has been programmed and used for calculations on an IBM 704.

PROCEDURE FOR APPROXIMATE MATRIX SOLUTION

1. Generate an $n\times n$ matrix of distances where d_{kl} = distance from kth destination to the lth destination. (In the general case these can be considered 'costs' of any kind.) We then have:

$$\begin{Vmatrix} d_{11} & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2n} \\ d_{31} & d_{32} & d_{33} & \cdots & d_{3n} \\ \cdots & \cdots & \cdots & & \cdots \\ d_{n1} & d_{n2} & d_{n3} & \cdots & d_{nn} \end{Vmatrix}.$$

In general, $d_{kl}\neq d_{lk}$.

2. First consider the case $m=1$. We solve as follows: Sum each row; the row index of the minimum sum indicates at which destination the single source is to be located.

3. $m>1$

(*a*) Generate all possible combinations of the n destinations taken m at a time:

$$C_n{}^m = n!/m!(n-m)!.$$

(*b*) Each combination tells us that the particular numbers in the combination are to be treated as sources. For example $C_7{}^2 = 7!/2!5! = 21$. A particular combination is 3, 4. This means that destinations 3 and 4 are to be treated as sources and each other destination is assigned to 3 or 4 depending on which source has the smallest distance (or cost matrix entry). Finally all assigned distances are summed.

(*c*) Step (*b*) is repeated for all combinations. The final allocation is the one with minimum sum of distances (costs).

(*d*) With this allocation determine the optimum source locations by solving the extremal equations.

First, it should be noted that there is no assurance at present by arguments from theory that by considering all possible destinations taken m at a time to be the source location sets, we have necessarily included one that has the correct allocation to the m sources. However, there are some intuitive arguments that can be put forth to make it reasonable. Second, even if this set of source location sets does lead to the correct final allocation there is still a great deal of calculation involved. In general, step 3(*b*) of the procedure given must be repeated $\sum_{m=1}^{m=n} [n!/m!(n-m)!]$ times. For large $n(n>40)$ and more significantly, for large m, even this simple summing in a computer becomes far too much computation. However, from a practical point of view one rarely has to perform step 3(*b*) a number of times equal to $\sum_{m=1}^{m=n} [n!/m!(n-m)!]$. The reason for this is that if one has n destinations it is unlikely that we shall want n sources or even $n/2$ sources, even though these cases are theoretically possible. In many industrial problems (although certainly not all) $g_1(m)$ and $g_2(m)$ [equations (1) and (2)] are monotonic functions and so, after calculating a few of the cases for small m, i.e., $m=1, 2, 3$, we can see if the minimum sum of costs is likely to require further calculation. If this is not the case, then the method suggested here, while better than solving nonlinear equations $S(n,m)$ times as the exact method requires is still not applicable to moderately large problems $(10<n<100)$.

The approximate method has been tried on a number of test cases and then compared with the exact solution. In every one of these cases, the correct allocation was obtained. These were, however, for $n=7$, $m=2$. With larger destination sets, an examination was made of the procedure of looking at several of the lowest sum allocations rather than the least of the set when determining final source locations. In no case, has this ever yielded any improvement, but it seems possible that such a case may exist.

Table I shows eight different cases that were solved exactly and by the approximate method for $n=7$ and $m=2$.

The approximate method described previously has been used to study optimal location and allocation for several problems ($n=42$, $m=2,3$; $n=14$, $m=2,3,4,5,6$). All of the problems seemed to possess characteristics that argue for the plausibility of the approximate method. First, the region of the global minimum seems always to be very flat rather than peaked. This has been true for widely different patterns of destination distributions. By this we mean that relatively large changes in the values of the coordinates of the sources produced relatively small changes in the value of the objective function. Second, in all problems that have been compared with the exact solution, the allocation was the correct one. Consequently, only one subsequent calculation with the extremal equations was required to produce the exact minimum after the correct allocation had been found. It is quite possible that, in the absence of a rigorous proof, cases will be found for which using the destinations as a base set of sources will not yield the correct allocation. In this case, as previously mentioned, examining several of the lower cost solutions may remedy this.

The major difficulty of the approximate method proposed in this paper is that it is still inadequate for many problems of industrial importance because of the excessive amount of calculation involved. However, the basic notion of using a set of source location sets to find the correct allocation, which is embodied in this method, is probably useful and can perhaps be employed with a method for generating fewer cases to examine than the approximate method reported in this paper.

Three such ideas currently being studied are as follows:

1. Generate sets of m trial source locations randomly.* How many sets that need to be generated to give the correct allocation within a certain probability is presently being studied.

2. Generate sets of m trial source locations randomly from the complete destination set itself and then proceed as in the methods described in this paper. This is also being studied.

3. Use a method of successive approximations as follows: Solve a problem with two sources. With this optimal allocation add a third source at each possible destination and reallocate. This process is continued by adding one source at a time. The total amount of calculation is much less than with the approximate method of this paper. Whether or not this is a convergent process is being studied at present.

* This suggestion was also made by one of the referees reviewing an earlier version of this paper.

TABLE I

Comparison of Results

Case 1	1	2	3	4	5	6	7
Destination co-ordinates:	(15, 15),	(5, 10),	(10, 27),	(16, 8),	(25, 14),	(31, 23),	(22, 29)
Exact method:	$\phi_{min}=50.450\pm0.001$						
Allocations	(1, 2, 4, 5) with source at (15.240, 12.053)						
	(3, 6, 7) with source at (22.000, 28.999)						
Approximate method:	$\phi_{min}=51.283$						
Allocations	(1, 2, 4, 5) with source at (15, 15)—D1						
	(3, 6, 7) with source at (22, 29)—D7						
	Allocation is correct.						
Case 2	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Destination co-ordinates:	(6, 8),	(6, 32),	(20, 8),	(20, 20),	(20, 32),	(36, 8),	(36, 32)
Exact method:	$\phi_{min}=72.002\pm0.001$						
Allocations	(1, 3, 4, 6) with source at (20. , 8.172)						
	(2, 5, 7) with source at (20. , 32.)						
Approximate method:	$\phi_{min}=72.000$						
Allocations	(1, 3, 4, 6) with source at (20, 8)—D3						
	(2, 5, 7) with source at (20, 32)—D5						
	Allocation is correct.						
Case 3	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Destination co-ordinates:	(8, 12),	(5, 19),	(5, 26),	(5, 32),	(35, 20),	(35, 26),	(35, 31)
Exact method:	$\phi_{min}=38.325\pm0.001$						
Allocations	(1, 2, 3, 4) with source at (5.186, 24.837)						
	(5, 6, 7) with source at (35.000, 26.000)						
Approximate method:	$\phi_{min}=38.318$						
Allocations	(1, 2, 3, 4) with source at (5, 26)—D3						
	(5, 6, 7) with source at (35, 26)—D6						
	Allocation is correct.						
Case 4	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Destination co-ordinates:	(5, 23),	(9, 32),	(15, 23),	(21, 32),	(26, 23),	(31, 32),	(16, 12)
Exact method:	$\phi_{min}=48.850\pm0.001$						
Allocations	(1, 2, 3, 7) with source at (12.113, 23.006)						
	(4, 5, 6) with source at (26.000, 29.110)						
Approximate method:	$\phi=52.158$						
Allocations	(1, 2, 3, 7) with source at (15, 23)—D3						
	(4, 5, 6) with source at (21, 32)—D4						
	Allocation is correct. Another equivalent was found because of symmetry.						
Case 5	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Destination co-ordinates:	(8, 10),	(8, 26),	(11, 20),	(17, 15),	(17, 22),	(24, 17),	(31, 19)
Exact method:	$\phi=38.033\pm0.001$						
Allocations	(1, 2, 3, 5) with source at (11, 20)						
	(4, 6, 7) with source at (24, 17)						
Approximate method:	$\phi_{min}=38.033$						
Allocations	(1, 2, 3, 5) with source at (11, 20)—D3						
	(4, 6, 7) with source at (24, 17)—D6						
	Allocation is correct.						
Case 6	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Destination co-ordinates:	(6, 31),	(13, 24),	(13, 31),	(20, 24),	(20, 17),	(27, 17),	(27, 1)
Exact method:	$\phi_{min}=36.175\pm0.001$						
Allocations	(1, 2, 3, 4) with source at (13,27.500)						
	(5, 6, 7) with source at (25.504, 15.504)						
Approximate method:	$\phi_{min}=37.899$						
Allocations	(1, 2, 3, 4) with source at (13, 24)—D2						
	(5, 6, 7) with source at (27, 17)—D6						
	Allocation is correct.						

TABLE I—*Continued*

Case 7	1	2	3	4	5	6	7
Destination co-ordinates:	(2, 19),	(35, 11),	(31, 33),	(25, 26),	(18, 23),	(18, 16),	(11, 33)
Exact method:	$\phi_{min}=59.716\pm.001$						
Allocations	(1, 3, 4, 5, 6, 7) with source at (18, 23)						
	(2) with source at (35, 11)						
Approximate method:	$\phi_{min}=59.716$						
Allocations	(1, 3, 4, 5, 6, 7) with source at (18, 23)—D5						
	(2) with source at (35, 11)—D2						
	Allocation is correct.						
Case 8	**1**	**2**	**3**	**4**	**5**	**6**	**7**
Destination co-ordinates:	(28, 6),	(28, 33),	(33, 17),	(33, 23),	(39, 6),	(39, 33),	(6, 10)
Exact method:	$\phi_{min}=62.204\pm.001$						
Allocations	(1, 5, 7) with source at (28.000, 6.000)						
	(2, 3, 4, 6) with source at (33.000, 23.001)						
Approximate method:	$\phi_{min}=62.203$						
Allocations	(1, 5, 7) with source at (28, 6)—D1						
	(2, 3, 4, 6) with source at (33, 23)—D4						
	Allocation is correct.						

The approximate method of this paper and modifications proposed are all predicated upon what has been borne out by limited calculational experience as well as intuitively reasonable arguments, namely, that the value of the objective function after optimum allocation by some heuristic method was usually within a few per cent of the minimum and this, of course, could then be determined by solving the extremal equations.

The appeal of using the destination set (which this paper suggests) as a base from which to generate trial source location sets is that the more 'dense' the set of destinations is (for large problems), the more likely it is that some of the sources should be located at destinations and the less sharply does the value of the objective function vary with less than optimal source locations. Intuitively, it seems that the 'larger' the problem (in the sense of a large and dense destination set), the more likely it is that the approximate method will be correct.

SUMMARY

A DISCUSSION of the calculational aspects of solving certain classes of location-allocation problems has been presented. Exact extremal equations for solving such problems have been derived and discussed. The use of those equations by themselves is limited to relatively small problems because of prohibitively large amounts of calculation time for larger problems. Certain heuristics for solving larger problems have been discussed and results using one of these has also been presented. The approximate method employed, while applicable to larger problems in

certain instances, still is not sufficiently powerful to deal with many problems of industrial importance. Other possible heuristics now under investigation have been briefly listed and discussed.

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