# OPTIMUM DISTRIBUTION OF SWITCHING CENTERS IN A COMMUNICATION NETWORK AND SOME RELATED GRAPH THEORETIC PROBLEMS†

**S. L. Hakimi**

*The Technological Institute, Northwestern University, Evanston, Illinois*

(Received August 3, 1964)

The concept of a median in a weighted graph is generalized to a multi-median. Then, it is shown that the optimum distribution of $p$ switching centers in a communication network is at a $p$-median of the corresponding weighted graph. The following related problem in highway networks is also considered: What is a minimum number of policemen that can be distributed in a highway network so that no one is farther away from a policeman than a given distance $d$? This problem is attacked by generating all vertex-coverings (externally stable sets) of a graph by means of a Boolean function defined over the vertices of a graph. Then this idea is extended to Boolean functions that generate all matchings, all factors, and all possible subgraphs of $G$ with given degrees.

IN A previous paper,[1] we presented a method for finding the optimum location of a switching center in a communication network. A natural generalization of that problem is considered here. The problem may be described as follows. In a communication network $N$, such as a telephone interconnection system, there are usually a number of switching centers $S_1, S_2, \cdots, S_p$. All traffic flows (messages) within the network must arrive at one of the switching centers and then be processed and sent to their proper destination. It is assumed that the cost of providing the necessary links between switching centers is negligible. A model of such a network is a graph $G$ with weights (nonnegative numbers) attached to each of its vertices as well as its branches. The weight $w_i$ attached to the branch (element) $b_i$ of $G$ represents the length (or the cost per unit capacity) of that branch. The weight $h_i$ attached to a vertex (node) $v_i$ of $G$ represents the number of wires (lines) that must be connected between vertex $v_i$ and a switching center to handle the information flows (messages) that originate or terminate at $v_i$. We assume that any switching center can be anywhere along any branch of $G$, i.e., the locations of switching centers are not necessarily limited to the vertices of $G$. The problem is to find the

† This work was supported by the Air Force Office of Scientific Research, Grant No. AF-AFOSR-98-63 and the U.S. Army Research Office (Durham).

distribution of switching centers $S_1, S_2, \cdots, S_p$ such that the total length of wires is a minimum. Such a distribution will be referred to as the optimum distribution of switching centers.

It is evident that the concept of median of a graph must be generalized.[1,2] Consider a weighted graph $G$. A point $x$ on $G$ is a point along a branch $b_i$ of $G$ that may or may not be a vertex of $G$. The distance between any two points $x$ and $y$ on $G$, denoted by $d(x, y)$, is the length of the shortest path in $G$ between points $x$ and $y$, where the length of a path is the sum of the weights of the branches of that path.[3,4] Let $X_p$ be a set of $p$ points $x_1, x_2, \cdots, x_p$ on $G$, and let

$$d(v_i, X_p) = \min[d(v_i, x_1), d(v_i, x_2), \cdots, d(v_i, x_p)]. \tag{1}$$

The set of points $X_p^*$ is a $p$-median of $G$, if for every $X_p$ on $G$

$$\textstyle\sum_{i=1}^{i=n} h_i\, d(v_i, X_p^*) \leqq \sum_{i=1}^{i=n} h_i\, d(v_i, X_p), \tag{2}$$

where, as before, $h_i$ is the weight attached to vertex $v_i$ in $G$. Clearly the set of points $X_p^*$ can be identified with the optimum locations of the switching centers in a communication network. A procedure for computing $X_p^*$ is presented.

A somewhat related problem is the problem of policing a highway network. This problem is formulated as a vertex-covering problem in a graph. Consider a graph in which the weights of all branches are equal to one. Let $V$ be the set of vertices of $G$. Let $V_q$ be a subset of $V$. The subset $V_q$ covers $G$, if

$$d(v_j, V_q) \leqq d. \qquad (j=1, 2, \cdots, n) \tag{3}$$

If no subset of $V_q$ has the same property, then $V_q$ is called a vertex-covering of $G$ within the distance $d$. A subset $V_r$ of $V$ is said to be an optimum vertex-covering of $G$, if there exists no vertex-covering of $G$ with fewer vertices than $V_r$. Clearly an optimum vertex-covering of $G$ can be identified with a distribution of minimum number of policemen to police a highway network.

We will assume that the weights of all branches are equal to one. This is not a severe restriction, because otherwise one could introduce sufficiently many additional vertices on the branches of the graph so that the length of each section identified as a separate branch is equal to one. This can be done as long as the weight of each branch is an integer.

To solve the problem, we present a method, originally due to MAGHOUT,[4] for generating all vertex-covering of $G$. Then, we extend his result to the case of connected vertex-coverings and coverings of subset of vertices of graph. All of this is done by constructing Boolean functions[5,6] of variables attached to the vertices of $G$.

Finally, we demonstrate the usefulness of constructing Boolean functions over a graph $G$ by showing that functions may be devised that generate all matchings, all factors, and all subgraphs of $G$ with given degrees.

## MULTIMEDIAN

OUR AIM is to outline a procedure for finding a $p$-median $X_p{}^*$ of a weighted graph. The following theorem is a major step in that direction.

THEOREM 1. *There exists a subset $V_p{}^*$ of $V$ containing $p$ vertices such that for every set of $p$ points $X$ on $G$*

$$\sum_{i=1}^{i=n} h_i\, d(v_i, X) \geqq \sum_{i=1}^{i=n} h_i\, d(v_i, V_p{}^*). \tag{4}$$

*Proof.* Let the points in $X$ be $x_1, x_2, \cdots, x_p$ on $G$. Let the integers $i_1, i_2, \cdots, i_n$ represent a one to one rearrangement (a permutation) of the integers $1, 2, \cdots, n$ such that

$$\begin{aligned} &d(v_{i_k}, X) = d(v_{i_k}, x_1), && (k=1, 2, \cdots, k_1)\\ &d(v_{i_k}, X) = d(v_{i_k}, x_2), && (k=k_1+1, \cdots, k_2)\\ &\cdots \quad \cdots && \\ &d(v_{i_k}, X) = d(v_{i_k}, x_p). && (k=k_{p-1}+1, \cdots, k_p=n)\end{aligned} \tag{5}$$

Then, we have

$$\begin{aligned}\sum_{i=1}^{i=n} h_i\, d(v_i, X) &= \sum_{k=1}^{k=k_1} h_{i_k}\, d(v_{i_k}, x_1)\\ &\quad + \sum_{k=k_1+1}^{k=k_2} h_{i_k}\, d(v_{i_k}, x_2) + \cdots + \sum_{k=k_{p-1}+1}^{k=n} h_{i_k}\, d(v_{i_k}, x_p).\end{aligned} \tag{6}$$

Consider the first term in the right-hand side of (6)

$$\sum_{k=1}^{k=k1} h_{i_k}\, d(v_{i_k}, x_1) = \sum_{k=1}^{k=k_1} h_{i_k}\, d(v_{i_k}, x_1) + \sum_{k=k_1+1}^{k=n} 0\; d(v_{i_k}, x_1). \tag{7}$$

Therefore, we may write

$$\sum_{k=1}^{k=k_1} h_{i_k}\, d(v_{i_k}, x_1) = \sum_{k=1}^{k=n} h'_{i_k}\, d(v_{i_k}, x_1), \tag{8}$$

where

$$h'_{i_k} = \begin{cases} h_{i_k}, & (k=1, 2, \cdots, k_1)\\ 0. & (k=k_1+1, \cdots, n)\end{cases} \tag{9}$$

However, in the previous paper,[1] we have shown that there exists a vertex $v_1{}^*$ in $G$ such that

$$\sum_{k=1}^{k=n} h'_{i_k}\, d(v_{i_k}, x_1) \geqq \sum_{k=1}^{k=n} h'_{i_k}\, d(v_{i_k}, v_1{}^*). \tag{10}$$

Combining (9) and (10), we obtain

$$\sum_{k=1}^{k=k_1} h_{i_k}\, d(v_{i_k}, x_1) \geqq \sum_{k=1}^{k=k_1} h_{i_k}\, d(v_{i_k}, v_1{}^*). \tag{11-1}$$

With identical reasoning, we can arrive at the following inequalities:

$$\textstyle\sum_{k=k_1+1}^{k=k_2} h_{i_k}\, d(v_{i_k}, x_2) \geqq \sum_{k=k_1}^{k=k_2} h_{i_k}\, d(v_{i_k}, v_2^*) \tag{11-2}$$

$$\cdots \qquad \cdots$$

$$\textstyle\sum_{k=k_{p-1}+1}^{k=n} h_{ik}\, d(v_{i_k}, x_p) \geqq \sum_{k=k_{p-1}+1}^{k=n} h_{i_k}\, d(v_{i_k}, v_p^*). \tag{11-p}$$

Adding the corresponding sides of the inequalities (11-1), (11-2), $\cdots$, and (11-$p$) and using (6), we obtain

$$\begin{aligned}\textstyle\sum_{i=1}^{k=n} h_i\, d(v_i, X) \geqq \sum_{k=1}^{k=k_1} h_{i_k}\, d(v_{i_k}, v_1^*)&\\ +\textstyle\sum_{k=k_1+1}^{k=k_2} h_{i_k}\, d(v_{i_k}, v_2^*) + \cdots + \sum_{k=k_{p-1}+1}^{k=n} h_{i_k}\, d(v_{i_k}, v_p^*).&\end{aligned} \tag{12}$$

Let the set of vertices $v_1^*, v_2^*, \cdots, v_p^*$ be represented by $V_p^*$. It can be seen that

$$\begin{aligned}\textstyle\sum_{k=1}^{k=n} h_{i_k}\, d(v_{i_k}, V_p^*) \leqq \sum_{k=1}^{k=k_1} h_{i_k}\, d(v_{i_k}, v_1^*)&\\ +\textstyle\sum_{k=k_1+1}^{k=k_2} h_{i_k}\, d(v_{i_k}, v_2^*) + \cdots + \sum_{k=k_{p-1}+1}^{k=n} h_{i_k}\, d(v_{i_k}, v_p^*).&\end{aligned} \tag{13}$$

Combining inequalities (12) and (13), we obtain

$$\textstyle\sum_{i=1}^{i=n} h_i\, d(v_i, X) \geqq \sum_{k=1}^{k=n} h_{i_k}\, d(v_{i_k}, V_p^*). \tag{14}$$

The inequality (14) is, however, identical with the desired inequality, if we observe that

$$\textstyle\sum_{k=1}^{k=n} h_{i_k}\, d(v_{i_k}, V_p^*) = \sum_{i=1}^{i=n} h_i\, d(v_i, V_p^*). \tag{15}$$

Theorem 1 proves that to find a $p$-median, one must only examine all subsets of $V$ containing $p$ vertices. It does not, however, prove that every $p$-median of $G$ consists of a set of $p$ vertices of $G$. The following numerical example is designed to show how the details of computation for finding a $p$-median of a graph may be carried out.

Consider the weighted graph of Fig. 1. Note that weights are attached to the vertices as well as the branches.

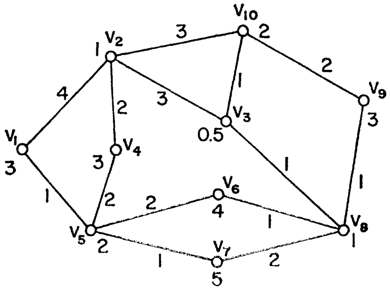


**Fig. 1.** A weighted graph.

We would like to find a 3-median of the graph $G$ of Fig. 1. The first step is to calculate the distance matrix $D$ of $G$.[7] The distance matrix $D = [d_{ij}]$ is an $n \times n$ symmetric matrix in which $d_{ij} = d(v_i, v_j)$. The distance matrix of the graph of Fig. 1 is

$$D = \begin{array}{c} \begin{array}{cccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \end{array} \\ \begin{Vmatrix} 0 & 4 & 5 & 3 & 1 & 3 & 2 & 4 & 5 & 6 \\ 4 & 0 & 3 & 2 & 4 & 5 & 5 & 4 & 5 & 3 \\ 5 & 3 & 0 & 5 & 4 & 2 & 3 & 1 & 2 & 1 \\ 3 & 2 & 5 & 0 & 2 & 4 & 3 & 5 & 6 & 5 \\ 1 & 4 & 4 & 2 & 0 & 2 & 1 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 2 & 0 & 3 & 1 & 2 & 3 \\ 2 & 5 & 3 & 3 & 1 & 3 & 0 & 2 & 3 & 4 \\ 4 & 4 & 1 & 5 & 3 & 1 & 2 & 0 & 1 & 2 \\ 5 & 5 & 2 & 6 & 4 & 2 & 3 & 1 & 0 & 2 \\ 6 & 3 & 1 & 5 & 5 & 3 & 4 & 2 & 2 & 0 \end{Vmatrix} \end{array}. \tag{16}$$

The next step is to write out a matrix $D^*$ which is obtained from $D$ by multiplying each row of $D$ by the weight attached to the corresponding vertex in $G$. Matrix $D^*$ is found to be

$$D^* = \begin{Vmatrix} 0 & 12 & 15 & 9 & 3 & 9 & 6 & 12 & 15 & 18 \\ 4 & 0 & 3 & 2 & 4 & 5 & 5 & 4 & 5 & 3 \\ 2.5 & 1.5 & 0 & 2.5 & 2 & 1 & 1.5 & 0.5 & 1 & 0.5 \\ 9 & 6 & 15 & 0 & 6 & 12 & 9 & 15 & 18 & 15 \\ 2 & 8 & 8 & 4 & 0 & 4 & 2 & 6 & 8 & 10 \\ 12 & 20 & 8 & 16 & 8 & 0 & 12 & 4 & 8 & 12 \\ 10 & 25 & 15 & 15 & 5 & 15 & 0 & 10 & 15 & 20 \\ 4 & 4 & 1 & 5 & 3 & 1 & 2 & 0 & 1 & 2 \\ 15 & 15 & 6 & 18 & 12 & 6 & 9 & 3 & 0 & 6 \\ 12 & 6 & 2 & 10 & 10 & 6 & 8 & 4 & 4 & 0 \end{Vmatrix}. \tag{17}$$

To find a 3-median of $G$, we must examine the following sum for all $i$, $j$, and $k$, $(1 \leqq i, j, k \leqq n)$.

$$\sum_{r=1}^{r=n} \min(d^*_{ir}, d^*_{jr}, d^*_{kr}). \tag{18}$$

It is found that the above sum is a minimum when $i, j, k = 4, 5, 8$, and the value of the sum for these values of $i$, $j$, and $k$ is 21.5. Therefore, a 3-median of $G$ is the set of vertices $\{v_4, v_5, v_8\}$. If graph $G$ of Fig. 1 represents a communication network, then three switching centers should be placed at vertices $v_4$, $v_5$, and $v_8$. Each community, represented by a vertex in $G$, must be connected to its closest switching center. A nonconnected graph representing the interconnection of the vertices to the closest switching centers is shown in Fig. 2.

## VERTEX-COVERINGS OF A GRAPH AND POLICING A HIGHWAY NETWORK

A SUBSET $V_k$ of the set of the vertices $V$ of an $n$-vertex graph $G$ is called a covering (a vertex-covering) of $G$, if

$$d(v_i, V_k) \leqq 1, \qquad (i=1, 2, \cdots, n)$$

*and* no subset of $V_k$ has the same property. Our goal is to find a covering of $G$ containing a minimum number of vertices. To do this, we will present a method for generating all coverings of a graph.

Let $v^i$ be a subset of vertices of a graph $G$ such that $d(v_i, v^i) \leqq 1$. The set $v^i$ is called the adjacent set of vertex $v_i$. Let a Boolean variable $x_i$ be associated to vertex $v_i$, $(1 \leqq i \leqq n)$. Let $X_i$ be the Boolean sum of the

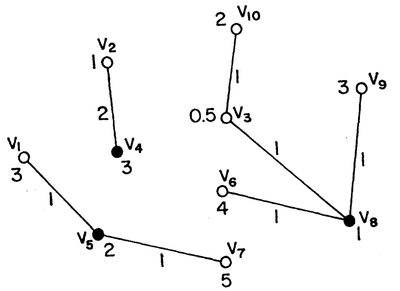


**Fig. 2.** A nonconnected graph connecting each vertex to one of the '3-medians.'

variables associated with vertices in $v^i$. The Boolean function, obtained as a Boolean product of $X_1, X_2, \cdots, X_n$

$$f(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{i=n} X_i$$

is called a covering function of $G$. Since every Boolean variable can assume only two values, (0, 1), every subset of vertices of $G$ may be defined as a union of

$$x_1v_1 \, U \, x_2v_2 \, U \cdots U \, x_nv_n = \bigcup_{i=1}^{i=n} x_iv_i$$

for some choice of the variables $x_1, \cdots, x_n$.

THEOREM 2. *The covering function*

$$f(x_1, x_2, \cdots, x_n) = 1$$

*if, and only if,* $U_{i=1}^{i=n} x_iv_i$ *contains a covering of G.*

*Proof.* (1) If $f(x_1, x_2, \cdots, x_n)=1$, then for the same values of variables $x_1, x_2, \cdots, x_n$

$$X_i = 1. \qquad (i=1, 2, \cdots, n)$$

This however proves that, at least, one variable associated with every adjacent set is equal to one. Which, in turn, implies that $\bigcup_{i=1}^{i=n} x_i v_i = V_q$ contains, at least, one vertex from every adjacent set; therefore, $d(v_i, V_q) \leq 1$ for $i=1, 2, \cdots, n$. Hence $V_q$ contains a covering of $G$.

(2) Suppose for some particular values of the variables $x_1, x_2, \cdots, x_n$, $\bigcup_{i=1}^{i=n} x_i v_i = V_q$ contains a covering of $G$. First, we claim that $V_q$ contains, at least, one vertex of every adjacent set. Suppose not; then let $v^j$ be the adjacent set none of whose vertices are in $V_q$. It is evident that $d(v_j, V_q) > 1$, which means $V_q$ does not contain a covering of $G$. This is a contradiction; therefore $V_q$ contains a vertex of every adjacent set. This means that for the same choice of Boolean variables, $X_i = 1$ for $i=1, 2, \cdots, n$, which proves that $f(x_1, x_2, \cdots, x_n) = 1$; hence the theorem.

Theorem 2 suggests that to generate all coverings of a graph one can construct the covering function of that graph and expand it in form of a 'minimum sum of products.'[5] Each product, corresponds to a covering of $G$ and every covering of $G$ appears as a product in the expansion. The

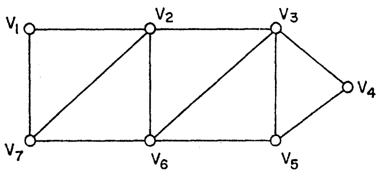


**Fig. 3.** A graph with a minimal covering $\{v_3, v_7\}$.

details of this process can be explained by means of an example. Consider the graph of Fig. 3. The covering function for the graph of Fig. 3 is

$$\begin{aligned} f(x_1, x_2, \cdots, x_7) = &(x_1+x_2+x_7)(x_2+x_1+x_3+x_6+x_7) \\ &\cdot (x_3+x_2+x_4+x_5+x_6)(x_4+x_3+x_5)(x_5+x_3+x_4+x_6) \\ &\cdot (x_6+x_5+x_3+x_2+x_7)(x_7+x_6+x_2+x_1). \end{aligned} \tag{19}$$

Noting that the first term is contained in the second and seventh terms, and that the fourth term is contained in the third and the fifth, we obtain

$$f(x_1, x_2, \cdots, x_7) = (x_1+x_2+x_7)(x_4+x_3+x_5)(x_6+x_5+x_3+x_2+x_7). \tag{20}$$

Using the Boolean distributive law, there results

$$f(x_1, x_2, \cdots, x_7) = x_1x_3+x_1x_5+x_1x_4x_6+x_2x_3+x_2x_5+x_2x_4+x_3x_7+x_5x_7+x_4x_7. \tag{21}$$

This proves that the set of all coverings in $G$ is

$$\{v_1v_3,\ v_1v_5,\ v_1v_4v_6,\ v_2v_3,\ v_2v_5,\ v_2v_4,\ v_3v_7,\ v_5v_7,\ v_4v_7\}, \tag{22}$$

and there are eight minimum coverings

$$\{v_1v_3,\ v_1v_5,\ v_2v_3,\ v_2v_5,\ v_2v_4,\ v_3v_7,\ v_5v_7,\ v_4v_7\}.$$

To generate all coverings of a graph is time consuming. However, since the subject of simplification of Boolean functions has been widely studied and there are efficient digital computer programs for such a purpose,[6] the above formulation is feasible.

An interesting question now arises. Does the set of all coverings of a graph uniquely determine the graph? The answer is no. It can be easily shown that the graph Fig. 4 has the same set of coverings as the graph Fig. 3.

Thus, we have a procedure to determine the locations of the minimum number of policemen to be distributed in the network so that everyone is within a unit distance from a policeman. The following corollary presents a solution to a slightly more general problem; that of finding locations of the minimum number policemen to be distributed in the network so that everyone is within the distance of $d$ units from a policeman, where $d$ is an integer.

COROLLARY. *Let $G$ be an $n$-vertex graph. For a vertex $v_i$ in $V$ we define a*

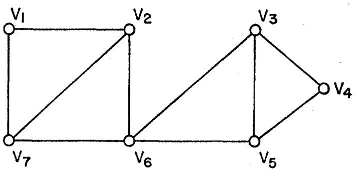


**Fig. 4.** A graph with the same set of covering as the graph of Fig. 3.

*set of vertices*

$$v^i = \{v_j | d(v_j, v_i) \leqq d, v_j \quad \text{in} \quad V\}.$$

*Let $X_i$ be the sum of Boolean variables associated with vertices in $v^i$, then the Boolean function*

$$f(x_1, \cdots, x_n) = \prod_{i=1}^{i=n} X_i = 1$$

*if, and only if, $V_p = \bigcup_{i=1}^{i=n} x_i v_i$ contains a covering of $G$ within the distance $d$.*

On the other hand it might be desirable to have a set of policemen distributed to cover a network within a distance $d_1$ and also such that each policeman is within a distance $d_2$ from another policeman. The following theorem presents a solution to this problem when $d_1 = d_2 = 1$. The general case will be considered later.

THEOREM 3. *Let $G$ be an $n$-vertex graph. For each vertex $v_i$ in $V$, we define a set of vertices $v^i$*

$$v^i = \{v_j | d(v_i, v_j) = 1, v_j \quad \text{in} \quad V\}.$$

*Let $Y_i$ be the sum of Boolean variables associated with $v^i$, then Boolean function*

$$g(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{i=n} Y_i = 1$$

*if, and only if,* $V_p = \bigcup_{i=1}^{i=n} x_i v_i$ *contains a covering of* $G$ *in which each vertex* $v_r$ *in* $V_p$ *is within a distance of one from a vertex* $v_s$ *in* $V_p$, $(v_s \neq v_r)$.

*Proof.* Let us consider the following two Boolean functions

$$f = \prod_{i=1}^{i=n} (x_i + Y_i),$$

and

$$h = \prod_{i=1}^{i=n} (\bar{x}_i + Y_i x_i).$$

From Theorem 2, $f=1$ if, and only if, $\bigcup_{i=1}^{i=n} x_i v_i = V_p$ contains a covering of $G$. The function $h=1$ if, and only if, $\bar{x}_i + Y_i x_i = 1$ for $i=1, 2, \cdots, n$. Therefore $h=1$ if, and only if, either $x_i=0$, or $x_i=1$ and $Y_i=1$. This proves that $h=1$ if, and only if, for every $v_i$ in $V_q$, $(V_q = \bigcup_{i=1}^{i=n} x_i v_i)$, there exists a vertex $v_j$ in $V_q$ such that $d(v_i, v_j)=1$. Now, consider the Boolean function $f.h$.

$$f.h = \prod_{i=1}^{i=n} (x_i + Y_i)(\bar{x}_i + Y_i x_i) = \prod_{i=1}^{i=n} Y_i = g.$$

Hence, we know $g=1$ if, and only if, $f=h=1$ which proves the theorem.

Let $V_a$ and $V_b$ be two nonempty subsets of $V$ which may or may not be disjoint. Let us assume that we would like to find (if possible) a subset $V_q$ of $V_a$ to cover each vertex in $V_b$ within a distance $d_1$ and for every vertex $v_r$ in $V_q$ there exists a vertex $v_s \neq v_r$ in $V_q$ such that $d(v_s, v_r) \leqq d_2$.

For every vertex $v_p$ in $V_b$, we define a set $v^p$

$$v^p = \{v_j \mid d(v_j, v_p) \leqq d_1 \quad \text{and} \quad v_j \quad \text{in} \quad V_a\},$$

and similarly for every vertex $v_q$ in $V_a$, we define a set $v^q$

$$v^q = \{v_k \mid 0 < d(v_k, v_q) \leqq d_2 \quad \text{and} \quad v_k \quad \text{in} \quad V_a\}.$$

Let $X_p$ and $Y_q$ be the sum of Boolean variables associated with vertices in $v^p$ and in $v^q$, respectively. We define two Boolean functions as follows

$$F = \prod_{v_p \in V_b} X_p \quad \text{and} \quad G = \prod_{v_q \in V_a} (Y_q + \bar{x}_q).$$

THEOREM 4. *Let the vertices in* $V_a$ *be* $v_{i_1}, v_{i_2}, \cdots, v_{i_a}$, *then*

$$H(x_{i_1}, x_{i_2}, \cdots, x_{i_a}) = F.G = 1$$

*if, and only if, the set of vertices* $\bigcup_{k=1}^{k=a} x_{i_k} v_{i_k} = V_q$ *covers vertices in* $V_b$ *within distance* $d_1$ *and each vertex* $v_r$ *in* $V_q$ *is within distance* $d_2$ *of another vertex* $v_s$ *in* $V_q$.

Proof of this theorem is similar to the proof of Theorems 2 and 3, and is therefore omitted.

Consider the graph of Fig 5. We would like to find a minimal subset $V_0$ of vertices in $\{v_1, v_2, v_3, v_4\}$ that covers the vertices in $\{v_5, v_6, \cdots, v_{10}\}$ within the distance of one and each vertex in $V_0$ is within a distance of two from another vertex in $V_0$. According to Theorem 4, we construct the following two Boolean functions

$$F=x_1(x_1+x_2)(x_1+x_3)(x_1+x_2)(x_3+x_4)(x_2+x_4)=x_1(x_3+x_4)(x_2+x_4),$$

and $$G=(\bar{x}_1+x_2+x_3+x_4)(\bar{x}_2+x_1+x_4)(\bar{x}_3+x_1+x_4)(\bar{x}_4+x_2+x_3+x_1)$$
$$=(x_2+x_3+x_1+x_4)(x_1+x_4)+\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4.$$

Then, the function $H$ is found to be

$$H=F.G=x_1x_2x_3+x_1x_4.$$

Thus, the minimal set of vertices is $\{v_1, v_4\}$.

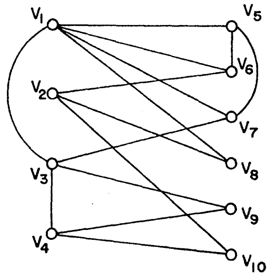


**Fig. 5.** An example of a subset of vertices covering another subset.

## BOOLEAN FUNCTIONS DEFINED ON THE BRANCHES OF A GRAPH

In the previous section, we constructed Boolean functions of variables associated with vertices of a graph $G$. In this section, we will construct Boolean functions of variables associated with branches of $G$. We will show that one can devise Boolean functions that will generate all factors, all matchings, and all subgraphs of $G$ with given degrees.[4] For the sake of completeness we will first define these terms. The *degree* of vertex $v_i$ in $G$, represented by $d_i(G)$, is the number of branches of $G$ incident at $v_i$. Let $g$ be a subgraph of $G$. By $d_i(g)$, we mean the number of branches of subgraph $g$ connected to vertex $v_i$ in $G$. Clearly $d_i(g)=0$ if no branch of $g$ is incident at $v_i$, and $d_i(g)\leqq d_i(G)$. A *matching* of $G$ is a maximal subgraph $g$ of $G$ such that $d_i(g)\leqq 1$ for $i=1, 2, \cdots, n$, where $n$ is the number of vertices of $G$†. Subgraph $g$ is a factor of $G$ if $d_i(g)=2$ for $i=1, 2, \cdots, n$. (These subgraphs have been widely studied by Tutte, Berge, and others[4] and have applications to signal flow graphs, the social sciences, and symbolic logic.)

Let graph $G$ have $m$ branches. Let $x_i$ be a Boolean variable associated with branch $b_i$. Two branches $b_i$ and $b_j$ are adjacent if they have, at least, one vertex in common. For every branch $b_i$, we define a set of branches $B^i$

† If a maximal subgraph $g$ has a property, then no set $g$, which contains $g$ and at least one other element, has the same property.

as follows

$$B^i = \{b_j | b_j \neq b_i \text{ and } b_j \text{ is adjacent to } b_i\}.$$

Let $Z_i$ be the sum of Boolean variables associated with $B^i$. Let the Boolean function $F$ be defined as

$$F = \prod_{i=1}^{i=m} (x_i \bar{Z}_i + \bar{x}_i) = \prod_{i=1}^{i=m} (\bar{Z}_i + \bar{x}_i).$$

THEOREM 5. *$F=1$ if, and only if, $\bigcup_{i=1}^{i=m} x_i b_i$ is contained in a matching of $G$.*

*Proof.* $F=1$ if, and only if, $x_i \bar{Z}_i + \bar{x}_i = 1$ for $i=1, 2, \cdots, n$. Therefore, $F=1$ if, and only if, either $x_i=0$, or $x_i=1$ and $Z_i=0$. A simple interpretation of the above assertion proves the theorem.

To show how Theorem 5 can be used to generate all matchings of a graph, we will consider an example. Consider the graph shown in Fig. 6.

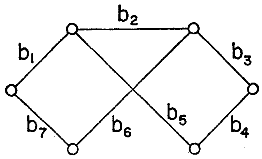


**Fig. 6.** A graph with three perfect matchings.

We first construct the function $F$ in Theorem 5

$$F = (\bar{x}_1 + \bar{x}_2\bar{x}_5\bar{x}_7)(\bar{x}_2 + \bar{x}_3\bar{x}_1\bar{x}_5\bar{x}_6)(\bar{x}_3 + \bar{x}_2\bar{x}_4\bar{x}_6)(\bar{x}_4 + \bar{x}_3\bar{x}_5) \cdot (\bar{x}_5 + \bar{x}_1\bar{x}_2\bar{x}_4)(\bar{x}_6 + \bar{x}_3\bar{x}_2\bar{x}_7)(\bar{x}_7 + \bar{x}_1\bar{x}_6).$$

Using the Boolean distributive law successively, we obtain

$$F = \bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4\bar{x}_7 + \bar{x}_1\bar{x}_2\bar{x}_4\bar{x}_6 + \bar{x}_1\bar{x}_3\bar{x}_5\bar{x}_6 + \bar{x}_2\bar{x}_3\bar{x}_5\bar{x}_7 + \bar{x}_2\bar{x}_4\bar{x}_5\bar{x}_6\bar{x}_7.$$

Looking at the first term on the right-hand side of the equation above, we decide that $F=1$, if $x_1=x_2=x_3=x_4=x_7=0$. This however means that we can pick $x_5=x_6=1$, which proves that the set $\{b_5, b_6\}$ is a matching. Similarly, other matchings are determined to be $\{b_3, b_5, b_7\}$, $\{b_2, b_4, b_7\}$, $\{b_1, b_4, b_6\}$, and $\{b_1, b_3\}$. There are three 'perfect' matchings $\{b_3, b_5, b_7\}$, $\{b_2, b_4, b_7\}$, and $\{b_1, b_4, b_6\}$.

Let $G$ be an $n$-vertex graph. Given nonnegative integers $p_1, p_2, \cdots, p_n$, we would like to find a subgraph $g$ of $G$ such that $d_i(g) = p_i$ for $i=1, 2, \cdots, n$. To begin with it is easy to prove that necessary conditions for existence of such a subgraph are:[8] $\sum_{i=1}^{i=n} p_i$ is even, $p_i \leqq d_i(G)$, for $i=1, 2, \cdots, n$, and $\sum_{i=1}^{i=n} p_i \geqq 2\max(p_1, p_2, \cdots, p_n)$. For an interesting discussion of the above problem the reader is referred to Berge.[4] Here, we are mainly concerned with generating all such subgraphs of $G$.

Let $X$ represent a set of $r$ Boolean variables. Let $\Gamma_s X$ represent a symmetric Boolean function that is equal to one if, and only if, $s$ variables out of $r$ are equal to one.

THEOREM 6. *Let $X_i$ represent the set of Boolean variables associated with branches of $G$ incident at vertex $v_i$. The Boolean function*

$$F=\prod_{i=1}^{i=n} \Gamma_{p_i} X_i=1$$

*if, and only if, $\bigcup_{i=1}^{i=n} x_i b_i$ is a subgraph $g$ in $G$ such that $d_i(g)=p_i$ for $i=1, 2, \cdots, n$.*

*Proof.* $F=1$ if, and only if, $\Gamma_{p_i} X_i=1$ for $i=1, 2, \cdots, n$. However, $\Gamma_{p_i} X_i=1$ if, and only if, any $p_i$ of variables in $X_i$ are equal to one; hence the theorem.

COROLLARY. *In Theorem 6, if $p_i=2$ for $i=1, 2, \cdots, n$, then $F=1$ if, and only if, $\bigcup_{i=1}^{i=n} x_i b_i$ is a factor of $G$.*

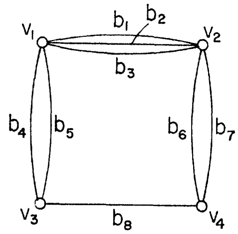


**Fig. 7.** A graph with seven subgraphs with given degrees.

Consider the graph $G$ of Fig. 7. We would like to find all subgraphs $g$ of $G$ with $d_1(g)=d_2(g)=4$ and $d_3(g)=d_4(g)=2$.
To do this, we construct the Boolean function $F$

$$F=\Gamma_4(x_1x_2x_3x_4x_5)\Gamma_4(x_1x_2x_3x_6x_7)\Gamma_2(x_4x_5x_8)\Gamma_2(x_6x_7x_8).$$

Then $F$ may be written as [5]

$$\begin{aligned}F= &(\bar{x}_1x_2x_3x_4x_5+x_1\bar{x}_2x_3x_4x_5+x_1x_2\bar{x}_3x_4x_5+x_1x_2x_3\bar{x}_4x_5+x_1x_2x_3x_4\bar{x}_5)\\ &\cdot(\bar{x}_1x_2x_3x_6x_7+x_1\bar{x}_2x_3x_6x_7+x_1x_2\bar{x}_3x_6x_7+x_1x_2x_3\bar{x}_6x_7+x_1x_2x_3x_6\bar{x}_7)\\ &\cdot(\bar{x}_4x_5x_8+x_4\bar{x}_5x_8+x_4x_5\bar{x}_8)(\bar{x}_6x_7x_8+x_6\bar{x}_7x_8+x_6x_7\bar{x}_8),\end{aligned}$$

which is simplified to be

$$F=x_1x_2x_3x_4\bar{x}_5x_6\bar{x}_7x_8+x_1x_2x_3\bar{x}_4x_5x_6\bar{x}_7x_8+x_1x_2x_3x_4\bar{x}_5\bar{x}_6x_7x_8$$
$$+x_1x_2x_3\bar{x}_4x_5\bar{x}_6x_7x_8+x_1x_2\bar{x}_3x_4x_5x_6x_7\bar{x}_8+x_1\bar{x}_2x_3x_4x_5x_6x_7\bar{x}_8+\bar{x}_1x_2x_3x_4x_5x_6x_7\bar{x}_8.$$

This proves that the sets of all desired subgraphs of $G$ are

$$\{b_1b_2b_3b_4b_6b_8,\ b_1b_2b_3b_5b_6b_8,\ b_1b_2b_3b_4b_7b_8,\ b_1b_2b_3b_5b_7b_8,$$
$$\cdot b_1b_2b_4b_5b_6b_7,\ b_1b_3b_4b_5b_6b_7,\ b_2b_3b_4b_5b_6b_7\}.$$

The above example suggests a possible approach to the problem of structural isomers in organic chemistry.[8] Consider a chemical compound $C_2O_2$, every connected graph in which there are two vertices of degree 4 and two vertices of degree 2 represent a structural isomer of $C_2O_2$. How many such graphs are there? This problem in its general form has not, as yet, been solved. We describe a method of attack, based on Theorem 6. Although this method may be used to generate all isomers, it is time consuming. We shall describe the method using the compound $C_2O_2$. We construct a 4-vertex graph in which two vertices are associated with the carbon atom and two with the oxygen atom. There can be at most three bonds between two carbon atoms and one between 2 oxygen atoms. Two bonds between an oxygen atom and each of the carbon atoms are possible. Therefore, the graph of Fig. 8, contains all possible structural isomers of $C_2O_2$ as subgraphs with degrees $d_1(g) = d_2(g) = 4$, and $d_3(g) = d_4(g) = 2$. By direct use

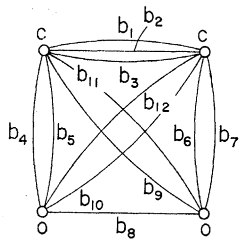


**Fig. 8.** A graph containing all isomers of $c_2O_2$.

of Theorem 6, one can generate all of the structural isomers of $C_2O_2$. Unfortunately, many of the resultant subgraphs are chemically equivalent. For example, the subgraphs of degrees 4, 4, 2, 2 obtained for Fig. 7 all qualify as structural isomers of $C_2O_2$, but only two of them are distinct $b_1b_2b_3b_4b_6b_8$ and $b_1b_2b_4b_5b_6b_7$. The graph of Fig. 8, certainly contains all subgraphs (isomers) of Fig. 7, and hence contains the above two subgraphs but it can be shown that it also contains one more $b_1b_2b_6b_9b_4b_{10}$, which is distinct from the other two. Thus $C_2O_2$ has three isomers. Direct use of Theorem 6 in generation of isomers is prohibitive, but Theorem 6 can easily be modified so that many of these equivalent subgraphs are not generated. The author is, at present, working on a method to eliminate all possible duplications.

## CONCLUSIONS AND FURTHER PROBLEMS

Boolean algebra as a tool for generating classes of subgraphs was introduced. Further exploitation of this tool especially in connection with the isomers problem may lead to fruitful results. In connection with the distribution of policemen on a network, a number of problems remain unsolved. For example if we allow policemen to be placed on the middle of a branch, or if we do not restrict distances to be integers, the problem becomes considerably more challenging. Suppose, on the other hand, instead of specifying the distance $d$ within which the highway network is to be covered, we are given $p$ policemen and we are asked to distribute them in an optimum way. This problem may be interpreted as a generalization of the absolute center[1] to the $p$-center. A set of $p$ points $X_p^*$ on $G$ is called a $p$-center of $G$, if for every set of $p$ points $X_p$ on $G$

$$\max_{1 \leqq i \leqq n} d(v_i, X_p) \geqq \max_{1 \leqq i \leqq n} d(v_i, X_p^*).$$

Unfortunately, we were not able to give a procedure for finding $X_p^*$. Finally, it should be mentioned that most of the results in this paper can be extended to the case of directed graphs by simply making slight modification of the present material.

## ACKNOWLEDGMENT

It is a pleasure for the author to acknowledge the discussions that he had with Mr. Howard Frank who also read the manuscript and suggested improvements.

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