Lecture Notes for Comp 480 - Winter 2023

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Here are lecture summaries for the sections of Probability on Trees and Networks that I will present.

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Course Introduction

If you lose my notes please consult my web page:
`http://arthurayestas.xyz/comp480.html`

Introduction to graphs... mainly notation

Definition 1 (Graph)**.** A **graph** $G = (V, E)$ is an ordered pair containing a set of vertices V and a set of edges $E \subseteq V \times V$.

Remark 2. We may also write the vertex and edge set of G as $V(G)$ and $E(G)$ respectively.

Definition 3 (Edge)**.** An **edge** is a member of E. We write such an edge as: $\langle x, y \rangle$ with $x, y \in V$.

Definition 4 (Oriented Edge)**.** An **(oriented) edge** is an edge that is direction agnostic. We write such an edge as: $[x, y]$.

Theorem 5.

$$[x, y] \iff \langle x, y \rangle \wedge \langle y, x \rangle \iff (x, y), (y, x) \in E$$

Remark 6. In these notes I will add the notation E_s to denote a symmetric edge set.

Definition 7 (Vertex Adjacency)**.** Vertices x and y are **adjacent** if $(x, y) \in E \vee (y, x) \in E$. We write this as $x \sim y$. Vertex adjacency is **symmetric**.

Definition 8 (Edge Adjacency)**.** Edges $e_1 = (x, y)$ and $e_2 = (w, z)$ are **adjacent** if $y = z$. We write this as $e_1 \sim e_2$. Edge adjacency is **symmetric**.

Definition 9 (Incident)**.** Vertex x and edge $e = (y, z)$ are **incident** if $x = z$.

Definition 10 (Degree)**.** The **degree** of x is the number vertecies that are adjacent to x. We write this quantity as: $\deg(x)$.

Definition 11 (Locally Finite)**.** The graph $G = (V, E)$ is locally finite if $deg(x) < \infty$ for all $x \in V$.

Definition 12 (Regularity)**.** The graph $G = (V, E)$ is d-**regular** if $deg(x) = d$ for all $x \in V$.

Definition 13 (Cartesian Product)**.** The **Cartesian product** of graphs G_1 & G_2 is written as: $G_1 \square G_2$, with the following vertex and edge sets:

$$V(G_1 \square G_2) := V(G_1) \times V(G_2)$$

$$E(G_1 \square G_2) := \{((x_1, x_2), (y_1, y_2)) : (x_1 = y_1 \wedge (x_2, y_2) \in E(G_2)) \vee (x_2 = y_2 \wedge (x_1, y_1) \in E(G_2))\} \subseteq V(G_1 \square G_2)$$

Definition 14 (Tensor Product)**.** The **Tensor product** of graphs G_1 & G_2 is written as: $G_1 \times G_2$, with the following vertex and edge sets:

$$V(G_1 \times G_2) := V(G_1) \times V(G_2)$$

$$E(G_1 \times G_2) := \{((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in E(G_1) \wedge (x_2, y_2) \in E(G_2)\} \subseteq V(G_1 \times G_2)$$

Definition 15 (Strong Product)**.** The **Strong product** of graphs G_1 & G_2 is written as: $G_1 \boxtimes G_2$, with the following vertex and edge sets:

$$V(G_1 \boxtimes G_2) := V(G_1) \times V(G_2)$$

$$E(G_1 \boxtimes G_2) := E(G_1 \square G_2) \cup E(G_1 \times G_2)$$

Definition 16 (Vertex Simple)**.** A path that does not visit a vertex more than once is **vertex simple**.

Definition 17 (Edge Simple)**.** A path that does not traverse an edge more than once is **edge simple**.

Definition 18 (Distance)**.** The **distance** between two vertices x & y is the number of edges of the shortest path between x and y, denoted $d(x, y)$.

Definition 19 (Network)**.** A **network** is a graph G, along with weight function $c : E(G) \to \mathbb{R}$.

Definition 20 (Induced Subnetwork)**.** Given $K \subseteq V(G)$, the **induced subnetwork** $G \restriction K$, is the subnetwork with vertex set K, edges $(K \times K) \cap E$, and weight function $c : (K \times K) \cap E \to \mathbb{R}$.

Definition 21 (Multigraph)**.** A **multigraph** is a pair of sets V, E, and a tail function $(\cdot)^- : E \to V$ and a and head function $(\cdot)^+ : E \to V$.

Definition 22 (Identifying)**.** The **multigraph** G/K obtained by **identifying** K to vertex $z \notin V(G)$ has vertex set $(V \backslash K) \cup z$, edge set $E(G)$, and inherit head and tail functions from G altered such that mappings to $x \in K$ now map to z.

Definition 23 (Contraction)**.** Given $e \in E(G)$, the **contraction** G/e is obtained by removing e from $E(G)$, then identifying both e^+ and e^-.

Definition 24 (Multigraph Homomorphism)**.** A **multigraph homomorphism** $\phi : G_1 \to G_2$ preserving incidence and edge orientation.

Definition 25 (Network Homomorphism)**.** A **network homomorphism** is a graph homomorphism satisfying $c_1(e) = c_2(\phi(e))$ for all $e \in E(G_1)$.

Branching Number

Definition 26 (nth level)**.** The **nth level** of tree T is the number of vertices of distance n from the root o of T. We write this quantity as: T_n.

Definition 27 (Lower Growth Rate)**.** The **lower growth rate** of a locally finite infinite tree is defined as:

$$\underline{\mathrm{gr}}T := \varliminf_{n \to \infty} \sqrt[n]{|T_n|}$$

Definition 28 (Upper Growth Rate)**.** The **upper growth rate** of a locally finite infinite tree is defined as:

$$\overline{\mathrm{gr}}T := \varlimsup_{n \to \infty} \sqrt[n]{|T_n|}$$

Definition 29 (Growth Rate)**.** The **growth rate** of a locally finite infinite tree, is defined as:

$$\mathrm{gr}T := \lim_{n\to\infty} \sqrt[n]{|T_n|}$$

Definition 30 (Flow)**.** A **flow** function is a nonnegative function θ respecting the property that for all vertices x with parent z and children $y_1, ..., y_d$, $\theta((z,x)) = \sum_{i=1}^{d} \theta((x,y_i))$ holds.

Definition 31 (Branching Number)**.** Let T be a locally finite infinite tree, and let θ be a flow function. We define the **branching number** as follows:

$$\mathrm{br}T := \sup\{\lambda \geq 1 \in \mathbb{R} | \exists\theta\forall n \in \mathbb{N}\forall x \in T_n : \theta((x_{\text{parent}}, x)) \leq \lambda^{-n}\}$$

Theorem 32. *$brT \leq \underline{gr}T$*

Proof. By induction, it is clear that $\sum_{x\in T_n} \theta(e(x))$, $\forall n \in \mathbb{N}$.

Therefore, given the flow constraint and choosing $\lambda > \underline{\mathrm{gr}}T$, we have:

$$\begin{aligned}
\sum_{x\in T_n} \theta(e(x)) &\leq \sum_{x\in T_n} \lambda^{-n} \\
&= |T_n|\lambda^{-n} \\
\implies \varliminf_{n\to\infty} |T_n|\lambda^{-n} &= 0
\end{aligned}$$

Therefore, there is no flow from the root to infinity. □

Electric Current

Definition 33 (Conductance)**.** A **conductance** function is a function $c : E \to \mathbb{R}^+$. We call $c(e) \in c(E)$ **conductances**.

Definition 34 (Energy)**.** The **energy** of flow θ, given conductances $c(E)$ is defined as follows (I have added notation **E**):

$$\mathbf{E} := \sum_{e} \frac{\theta(e)^2}{c(e)}$$

Theorem 35. *If $\lambda < brT$, then **current flows**. If $\lambda > brT$, then **current does not flow**.*

Random Walks

Definition 36 (Voltage Function)**.** A **voltage function** on vertices a_0 & a_1, given conductance function c, is a function $v : V \to \mathbb{R}_0^+$ satisfying the following conditions:

$$v(a_0) = 0 \tag{1}$$
$$v(a_1) = 1 \tag{2}$$
$$v(x) \sum_{i=1}^{d} c_i = \sum_{i=1}^{d} c_i v(y_i) \tag{3}$$

with $y_1, ..., y_d$ being the children of x.

Theorem 37 (Voltage as Probability)**.** *Let $x\alpha_1...\alpha_n$ be a random walk starting at x, such that $a_0 = \alpha_i$ & $a_1 = \alpha_j$ have been reached exactly once.*
$v(x) = \mathbb{P}\{j < i\}$

Definition 38 (Transient)**.** A **transient** random walk visits its starting vertex infinitely often.

Definition 39 (Recurrent)**.** A **transient** random walk visits its starting vertex infinitely often.

Remark 40. A random walk is either transient or recurrent.

Theorem 41. *If $\lambda < brT$, then $\mathbb{RW}_\lambda$ is **transient**. If $\lambda > brT$, then $\mathbb{RW}_\lambda$ is **recurrent**.*

Percolation

Theorem 42. *For any tree T, $p_c(T) = (brT)^{-1}$*

Hausdorff Dimension

Definition 43 (Cutset)**.** A **cutset** of a tree T is a set Π of edges whose removal leaves root o in a finite component.

Remark 44. In this section $|x| := d(o, x)$, with o being the root & x being a vertex of some tree.

Theorem 45 (Cutset Flow)**.**

$$\max |\theta| = \inf \left\{ \sum_{e(x_p, x) \in \Pi} \lambda^{-|x|} : \Pi \text{ is a cutset} \right\}$$

Proof. All flow must traverse an edge of the cutset, therefore the cutset with the smallest flow restriction will be the bottleneck that will determine the maximum allowable flow. □

Definition 46 (Ray)**.** An infinite simple path originating at the root is a **ray**.

Definition 47 (Boundary)**.** The **boundary** ∂T of T is the set of all rays of T.

Definition 48 (Distance in the Boundary)**.** Given $\alpha, \beta \in \partial T$:

$$d(\alpha, \beta) := e^{-(\text{ \# common edges in } \alpha,\ \beta)}$$

Definition 49.

$$B_x = \{\beta \in \partial T : \beta_{|}x| = x\}$$

Definition 50.

$$\text{diam } B_x = e^{-|x|}$$

Definition 51 (Cover)**.** A collection C of subsets of ∂T is a **cover** if:

$$\bigcup_{B \in C} B = \partial T$$

Theorem 52 (Kőnig's Lemma)**.** $\partial T \neq \emptyset$

Proof. T is infinite, connected, and locally finite. Therefore, it must contain an infinite simple path containing the root. Therefore, it has an infinite ray. Therefore, $\partial T \neq \emptyset$. ☐

Theorem 53. *∂T is compact.*

Proof. Left as an exercise. Consider the open cover finite subcover definition of compactness. ☐

Definition 54 (Hausdorff Dimension)**.**

$$\dim \partial T := \sup \left\{ \alpha : \inf_{\text{countable } C} \sum_{B \in C} (\text{diam } B)^{\alpha} > 0 \right\}$$

Remark 55. Note that a Π is a cutset if and only if $\{B_x : e(x) \in \Pi\}$ is a cover of ∂T.

Theorem 56.

$$\textit{br } T = \textit{exp } (\textit{dim } \partial T)$$

Proof.

$$\begin{aligned}
\text{br } T &= \sup \left\{ \lambda : \inf \left\{ \sum_{e(x) \in \Pi} \lambda^{-|x|} : \Pi \text{ is a cutset} \right\} \right\} \\
&= \exp \ \sup \left\{ \alpha : \inf \left\{ \sum_{e(x) \in \Pi} e^{-\alpha |x|} : \Pi \text{ is a cutset} \right\} \right\} \\
&= \exp \ \sup \left\{ \alpha : \inf_{\text{cover } C} \sum_{B \in C} (\text{diam } B)^{\alpha} > 0 \right\} \\
&= \exp \text{ dim } \partial T
\end{aligned}$$

□

Remark 57. Haudorff dimension of the boundary of trees (Furstenberg) inspired the branching number (Lyons).