Comp 480: Relating Walks on Markov Chains to Circuits

Massimo Vicenzo

January 2023

1 Setup

In order to do this transformation we consider only time-reversible Markov chains. Recall that this means that:

$$\pi(x)p(x,y) = \pi(y)p(y,x)$$

Where π is the stationary measure, in this section we will actually not require that π is a distribution. The way that we will construct the graph allows for us to scale π to make more computations convenient later. So π still has the following two properties, $\pi(x) \geq 0$ and $\pi(x) = \sum_{x\sim y} \pi(y)p(y,x)$ for all states x.

Now, if we have a Markov chain, we construct the graph G by representing the states of the Markov chain as vertices, and every transition with $p(x,y) > 0$ we place an undirected edge between x and y with the weight

$$c(x,y) := \pi(x)p(x,y) = \pi(y)p(y,x)$$

As an example, we can consider gambler's ruin. The the states are represented by numbers 0 to n and with probability p the gambler gains a dollar, and $1-p$ they lose a dollar. The walk normally ends once the gambler reaches 0 or n, whichever happens first. The graph is still a path and on the edge between i and $i+1$ the weight is $(\frac{p}{1-p})^i$.

We will mainly be concerned with hitting times for the beginning of this section. For a random walk on our constructed graph G let τ_A be the time step at which the walk reaches a vertex in the set A. If the walk starts in A, then $\tau_A = 0$. We use τ_A^+ to exclude this case, i.e. its the first time step ≥ 1 which the walk reaches a vertex of A.

For now, we are interested in the following probability:

$$F(x) := \mathbb{P}_x(\tau_A < \tau_Z)$$

In words, given a starting point x, what is the probability that the walk will reach a vertex in A before a vertex in Z. We assume $A \cap Z = \emptyset$. What we know, for $x \in A$, $F(x) = 1$, and for $x \in Z$ we have $F(x) = 0$. However everywhere in between seems like a mystery. We can do the following analysis to notice an interesting property:

$$F(x) = \sum_{x\sim y} \mathbb{P}_x(\text{First step is to } y)\mathbb{P}_x(\tau_A < \tau_Z | \text{First step is to } y) = \sum_{x\sim y} p(x,y)F(y)$$

This can be equivalently stated as,

$$F(x) = \frac{1}{\pi(x)} \sum_{x\sim y} c(x,y)F(y)$$

A special case is when our Markov chain is a simple random walk on a graph (pick an edge to walk down uniformly):

$$F(x) = \frac{1}{\deg(x)} \sum_{x\sim y} F(y)$$

So the value of F is an average of the adjacent vertices' values for F. In the general case it is more of a weighted average.

2 Harmonic Functions

In order to work with the function $F(x)$ from the previous section, we would like to understand this averaging property and its consequences.

A function f is called *harmonic at* x if the following holds:

$$f(x) = \sum_{x \sim y} p(x, y) f(y)$$

Theorem 1 (Maximum Principle). *Let W be a set of states in a Markov chain, and let f be harmonic on W. If the supremum of f is on some element $x_0 \in W$, then f is constant everywhere reachable from x_0 in W and up to one step away from W.*

Proof. $f(x_0)$ is the supremum so all other states y must have $f(y) \leq f(x_0)$. But on the other hand, it is a weighted average of all the adjacent vertices, meaning the only way it could attain the supremum is if for all adjacent y, $f(y) = f(x_0)$. This argument can now be repeated on all of the vertices reachable from x_0, up to vertices adjacent to W. □

Theorem 2 (Uniqueness Principle). *Let W be a connected set of states in a Markov chain, and let f be harmonic on W. Suppose that every state in W is adjacent to $V \setminus W$. Now if f, g are both harmonic on W, and equal off of W ($f(x) = g(x)$ for all $x \notin W$), then $f = g$.*

Proof. Let $h := f - g$, and let x_0 be the state which achieves the supremum. If $x_0 \notin W$ then $h \leq 0$. On the other hand, if $x_0 \in W$ then by the maximum principle h is constant on all of the state space (since all states not in W are adjacent to W). This also gives us $h \leq 0$. Then by symmetry, $h = 0$. □

Theorem 3 (Existence Principle). *Let $W \subset V$ be a set of states in a Markov chain. If f_0 is a real-valued function on the states not in W, then we can create a function $f : V \to \mathbb{R}$ such that $f_0 = f$ for $x \in V \setminus W$ and harmonic on W.*

Proof. Let $Y := f_0(X)$ where X is the first state that the walk reaches in $V \setminus W$. We can now define f as the following on W:

$$f(x) = \mathbb{E}_x\{Y\} = \sum_{x \sim y} p(x, y) \mathbb{E}\{Y | \text{First step is to } y\} = \sum_{x \sim y} p(x, y) f(y)$$

□

From this, we now can see that for any graph, if we fix two vertices with some values, then there is a unique harmonic function that will be able to cover the remaining vertices.

It just so happens that the measure of voltage along nodes in a circuit happens to be a harmonic function. This is our connection to circuits.

3 Electricity and Circuits Recap

In this section we will just remind ourselves of the various parameters and properties of circuits. First we denote $c(x,y)$ as the *conductance*, this is a value that is attached to a wire, and can be thought of as a weight which indicates how much electricity will flow into said wire. If the value is larger compared to other wires connected to the node, it will receive more electricity flowing through it. Inversely we have *resistance* $r(x,y)$, which is just the reciprocal of $c(x,y)$.

We define *current* $i(x,y)$ along a wire going from x to y as the following,

$$i(x,y) := c(x,y)(v(x) - v(y))$$

Rearranging this gives us Ohm's law:

$$v(x) - v(y) = i(x,y)r(x,y)$$

We can notice following fact about current passing through a node x which is harmonic:

$$\sum_{x \sim y} i(x,y) = v(x) \sum_{x \sim y} c(x,y) - \sum_{x \sim y} c(x,y)v(y) = v(x)\pi(x) - \pi(x)v(x) = 0$$

This tells us that current acts like a flow on the vertices which are harmonic, this is in fact Kirchoff's Node law.

Now we go through 2 main circuit laws:

Series Law: If two edges are in series, i.e. they form a path with no other internal edges, we can replace the path with a single edge with its resistance being the sum of the two edges resistances.

Parallel Law: If two edges are in parallel, i.e. they have both the same vertices as their ends, we can replace the two edges with a single edge with its conductance being the sum of the two conductances. Both of these do not change the voltage or current of all the unchanged vertices and edges.

With this we can now answer the question posed by the gambler's ruin problem. That being the question, what is $\mathbb{P}_k(\tau_0 < \tau_n)$? Well if we set $v(0) = 1$ and $v(n) = 0$, then the exact probability we want is given by $v(k)$.

Applying the series law to all the edges in between 0 and k leaves us with a single edge with resistance equal to $\sum_{i=0}^{k-1}(\frac{1-p}{p})^i$. Doing the same for the other side we get an edge between k and n with resistance $\sum_{i=k}^{n-1}(\frac{1-p}{p})^i$. Using the fact that $v(k)$ is harmonic, we can solve and simplify giving us a final answer of:

$$\mathbb{P}(\tau_0 < \tau_n) = v(k) = \frac{\sum_{i=k}^{n-1}(\frac{1-p}{p})^i}{\sum_{i=0}^{n-1}(\frac{1-p}{p})^i} = \frac{1 - (\frac{p}{1-p})^{n-k}}{1 - (\frac{p}{1-p})^n}$$

4 Effective Conductance, Green Function and Transience

We would now like to measure the following probability:

$$\mathbb{P}(a \to Z) := \mathbb{P}(\tau_Z < \tau_a^+)$$

In words, we would like to know the probability that an excursion starting from a will reach a vertex in Z before returning to a. Let us set the voltage at a to be $v(a)$, and for any $z \in Z$ set $v(z) = 0$. Let the remaining vertices be harmonic, we can then solve for this probability:

$$\begin{aligned}
\mathbb{P}(a \to Z) &= \sum_x p(a,x)(1 - \mathbb{P}(\tau_a < \tau_Z)) = \sum_x \frac{c(a,x)}{\pi(a)}(1 - \frac{v(x)}{v(a)}) \\
&= \frac{1}{v(a)\pi(a)} \sum_x c(a,x)(v(a) - v(x)) = \frac{1}{v(a)\pi(a)} \sum_x i(a,x)
\end{aligned}$$

This relationship is written as,

$$v(a) = \frac{\sum_x i(a,x)}{\pi(a)\mathbb{P}(a \to Z)}$$

The denominator is something we define as *effective conductance*, written as

$$\mathcal{C}(a,Z) := \pi(a)\mathbb{P}(a \to Z)$$

This looks very similar to our original definition of conductance, in fact, if there is only one edge separating a and Z this is exactly just the conductance of that edge. The interpretation of this quantity is the conductance of a single edge which we could replace the entire circuit with, to maintain the flow between a and Z. Using the series, parallel and other laws gives us ways of reducing a circuit to view it as a single edge and find this value.

Continuing on, what is the expected number of times that we will hit a before we get an excursion which finally hits Z. This is a geometric random variable and can be found to be $\frac{1}{\mathbb{P}(a \to Z)}$. More generally, we let $\mathcal{G}_Z(a,x)$ be the expected number of times that a walk hits the vertex x, which starts at a and dies once it hits Z. This is the *Green function*.

Proposition 4 (Green Function as Voltage). *Let $v(z) = 0$ for all $z \in Z$, and set $v(a)$ so that there is unit flow going through the whole circuit, then $v(x) = \frac{\mathcal{G}_Z(a,x)}{\pi(x)}$.*

Proposition 5 (Edge Crossings as Current). *For any $x \sim y$, let $S_{x,y}$ be the number of transitions from x to y during the random walk, then $\mathbb{E}\{S_{x,y}\} = \mathcal{G}_Z(a,x)p(x,y)$ and $\mathbb{E}\{S_{x,y} - S_{y,x}\} = i(x,y)$.*

For the last section in this chapter, we turn to transience and recurrence in infinite graphs. Given an infinite graph G, we consider growing subgraphs, $G_1 \subset G_2 \subset \cdots \subset G$. Let $Z_n := V(G \setminus G_n)$, then we will define the graph G_n^W as the graph G where Z_n is identified to a single vertex z_n. This way, the probability $\mathbb{P}(a \to Z_n)$ on the graph G can just be seen as $\mathbb{P}(a \to z_n)$ on the finite graph G_n^W.

As the events $[a \to Z_n]$ are a sequence of decreasing events, $\lim_{n\to\infty} \mathbb{P}(a \to Z_n)$ is the probability that a walk starting at a never returns to a. In other words, if this limit is a positive value, the walk on G is transient. By the definition of effective conductance, this actually equivalently means the following:

Theorem 6 (Transience and Effective Conductance). *Let G be an infinite network.*

A random walk on G is transient $\iff \lim_{n\to\infty} \mathcal{C}(a, z_n) > 0$