Lecture Notes for Comp 480 - Winter 2023

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Here are lecture summaries for the sections of Probability on Trees and Networks that I will present.

Contents

Course Introduction

If you lose my notes please consult my web page:
`http://arthurayestas.xyz/comp480.html`

Introduction to graphs... mainly notation

Definition 1 (Graph)**.** A **graph** $G = (V, E)$ is an ordered pair containing a set of vertices V and a set of edges $E \subseteq V \times V$.

Remark 2. We may also write the vertex and edge set of G as $V(G)$ and $E(G)$ respectively.

Definition 3 (Edge)**.** An **edge** is a member of E. We write such an edge as: $\langle x, y \rangle$ with $x, y \in V$.

Definition 4 (Oriented Edge)**.** An **(oriented) edge** is an edge that is direction agnostic. We write such an edge as: $[x, y]$.

Theorem 5.

$$[x, y] \iff \langle x, y \rangle \wedge \langle y, x \rangle \iff (x, y), (y, x) \in E$$

Remark 6. In these notes I will add the notation E_s to denote a symmetric edge set.

Definition 7 (Vertex Adjacency)**.** Vertices x and y are **adjacent** if $(x, y) \in E \vee (y, x) \in E$. We write this as $x \sim y$. Vertex adjacency is **symmetric**.

Definition 8 (Edge Adjacency)**.** Edges $e_1 = (x, y)$ and $e_2 = (w, z)$ are **adjacent** if $y = z$. We write this as $e_1 \sim e_2$. Edge adjacency is **symmetric**.

Definition 9 (Incident)**.** Vertex x and edge $e = (y, z)$ are **incident** if $x = z$.

Definition 10 (Degree)**.** The **degree** of x is the number vertecies that are adjacent to x. We write this quantity as: $\deg(x)$.

Definition 11 (Locally Finite)**.** The graph $G = (V, E)$ is locally finite if $deg(x) < \infty$ for all $x \in V$.

Definition 12 (Regularity)**.** The graph $G = (V, E)$ is d-**regular** if $deg(x) = d$ for all $x \in V$.

Definition 13 (Graph Product)**.** The **Cartesian product** of graphs G_1 G_2 is written as: $G_1 \square G_2$, with the following vertex and edge sets:

$$V(G_1 \square G_2) := V(G_1) \times V(G_2)$$

$$E(G_1 \square G_2) := \{((x_1, x_2), (y_1, y_2)) : (x_1 = y_1 \wedge (x_2, y_2) \in E(G_2)) \vee (x_2 = y_2 \wedge (x_1, y_1) \in E(G_2))\} \subseteq V(G_1 \square G_2)$$

Branching Number

Definition 14 (nth level)**.** The **nth level** of tree T is the number of vertices of distance n from the root o of T. We write this quantity as: T_n.

Definition 15 (Lower Growth Rate)**.** The **lower growth rate** of a locally finite infinite tree is defined as:

$$\underline{\mathrm{gr}} T := \varliminf_{n \to \infty} \sqrt[n]{|T_n|}$$

Definition 16 (Upper Growth Rate)**.** The **upper growth rate** of a locally finite infinite tree is defined as:

$$\overline{\mathrm{gr}} T := \varlimsup_{n \to \infty} \sqrt[n]{|T_n|}$$

Definition 17 (Growth Rate)**.** The **growth rate** of a locally finite infinite tree, is defined as:

$$\mathrm{gr} T := \lim_{n \to \infty} \sqrt[n]{|T_n|}$$

Definition 18 (Flow)**.** A **flow** function is a nonnegative function θ respecting the property that for all vertices x with parent z and children $y_1, ..., y_d$, $\theta((z, x)) = \sum_{i=1}^{d} \theta((x, y_i))$ holds.

Definition 19 (Branching Number)**.** Let T be a locally finite infinite tree, and let θ be a flow function. We define the **branching number** as follows:

$$\mathrm{br}T := \sup\{\lambda \geq 1 \in \mathbb{R} | \exists\theta\forall n \in \mathbb{N}\forall x \in T_n : \theta((x_{\text{parent}}, x)) \leq \lambda^{-n}\}$$

Theorem 20. *$brT \leq grT$*

Proof. □

Electric Current

Definition 21 (Conductance)**.** A **conductance** function is a function $c : E \to \mathbb{R}_0^+$. We call $c(e) \in c(E)$ **conductances**.

Definition 22 (Energy)**.** The **energy** of flow θ, given conductances $c(E)$ is defined as follows (I have added notation **E**):

$$\mathbf{E} := \sum_e \frac{\theta(e)^2}{c(e)}$$

Theorem 23. *If $\lambda < brT$, then **current flows**. If $\lambda > brT$, then **current does not flow**.*

Random Walks

Percolation

Branching Processes