The Canonical Gaussian Field

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1 Hitting, Commute, and Cover Times

Proposition 1.1. *(Hitting-Time Identity) Given a finite network with a vertex a and a disjoint subset of vertices Z, let $v(\cdot)$ be the voltage when a unit current flows from a to Z. We have*

$$E_a[\tau_Z] = \sum_{x \in V} \pi(x) v(x)$$

Proof: $E_a[\tau_Z] = \sum_x \mathcal{G}_Z(a, x) = \sum_{x \in V} \pi(x) v(x)$

The expected time for a random walk started at a to visit z and the return to a, that is, $E_a[\tau_Z] + E_z[\tau_a]$, is called the commute time.

Proposition 1.2. *(Commute-Time Identity) Let G be a finite network and $\gamma := \sum_{e \in E} c(e)$. Let a and z be two vertices of G. The commute time between a and z is $\gamma \mathcal{R}(a \leftrightarrow z)$.*

Proof: The time $E_a[\tau_z]$ is expressed in previous proposition using voltage $v(x)$. Now the voltage at x for a unit-current flow from z to a is $v(a) - v(x)$. Adding these two hitting times, we get that commute time is $v(a) \sum_x \pi(x) = \gamma v(a)$. Finally, we use that $v(a) = \mathcal{R}(a \leftrightarrow z)$.

Recall cover time. For a complete graph, the cover time is studied in the couple-collector problem. The expectation with n vertices is exactly $(n-1) \sum_{k=1}^{n-1} 1/k$. It takes no time to visit the starting state, hence $n - 1$ in place of n.

Theorem 1.3. *(Cover-Time Upper bound) Given an irreducible finite Markov chain whose state space V has size n and start state o, we have*

$$E_o[Cov] \leq \left(\max_{a,b \in V} E_a \tau_b \right) \left(1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right)$$

Theorem 1.4. *(Cover-Time Lower Bound) For any irreducible finite Markov chain on a state space V with starting state o,*

$$E_o[Cov] \geq \max_{A \subseteq V} t_{\min}^A \left(1 + \frac{1}{2} + \ldots + \frac{1}{|A| - 1} \right)$$

2 The Canonical Gaussian Field

Here we consider a model that represents an electrical network via Gaussian random variables. Suppose we want to measure the altitudes at a finite set of locations V. Assume we now the altitude at some location $o \in V$. We can measure the difference in altitudes between pairs of locations, but each measurement Y has a normally distributed error. That is, $x, y \in V, Y(e) \sim N(\alpha(x) - \alpha(y), \sigma_e^2)$, where $\alpha(x)$ is the true altitude and σ_e is known. Let $G = (V, E)$ be a graph associated with the measurements. We can make this into a network by assigning the resistances $r(e) = \sigma_e^2$. The maximum likelihood estimate of the altitudes $\hat{\alpha} : V \to \mathbb{R}$, with $\hat{\alpha}(o) = \alpha(o)$ that maximizes the likelihood

$$\frac{1}{\Pi_e \sqrt{2\pi r(e)}} \exp\left\{ -\frac{1}{2} \sum_e \frac{\left(Y(e) - (d\hat{\alpha})(e)\right)^2}{r(e)} \right\}$$

Maximizing the likelihood is the same as minimizing the sum of squares in the exponent. The random variables $\hat{\alpha}$ form the canonical Gaussian field. Let $Z := \hat{\alpha} - \alpha,\ X(e) := Y(e) - d\alpha(e)$, the above the equivalent to minimizing $\sum_e [X(e) - (dZ)(e)]$. We will work with X and Z in place of Y and $\hat{\alpha}$, equivalently, we take $\alpha \equiv 0$.

Given a network (G, c) and some vertex $o \in V$, let $X(e), e \in E$ be independent normal random variables with mean 0 and variance $r(e)$. Define random variables $Z(x), x \in V$ by the condition that they minimize $\sum_e \frac{[X(e)-(dZ)(e)]^2}{r(e)}$ and that $Z(o) = 0$. The joint distribution of $Z(x), x \in V$ is the canonical Gaussian field.

Example 2.1. Suppose G is the usual nearest-neighbour graph on the integers 0, 1, ..., n and all resistances are 1. Take $o = 0$. Then $X(e) \sim N(0, 1),\ \forall e \in E$, and $dZ = X$, where Z is just n steps of random walk with each step a standard normal random variable.

Example 2.2. If G is a tree rooted at o, then z is a random walk indexed by the tree in the sense that when two paths starting at the root branched off from each other, then the random walks along those paths, which were identical, have independent normal increments thereafter.

One interesting property of the canonical Gaussian field is that $\forall x, y \in V$,

$$\mathbb{V}(Z(x) - Z(y)) = \mathcal{R}(x \leftrightarrow y)$$

To see this, we will first calculate the joint distribution of dZ.

Given a network, define the gradient of a function f on V to be the antisymmetric function

$$(\nabla f)(e) := \frac{(df)(e)}{r(e)}$$

that is, $\nabla f := c\ df$.

Z is the function β with $\beta(o) = 0$ that minimizes $||X/r - \nabla\beta||_r$. Since $\nabla \mathbb{1}_{\{x\}}$ is the star at $x \in V$, recall $\bigstar$ denote the subspace in $l^2(E)$ spanned by all stars. It follows that the set of all functions of the form $\nabla\beta$ for some function β equals $\bigstar$, we are looking for the element of $\bigstar$ closest to X/r. Such minimization is achieved through orthogonal projection, so that $\nabla Z = P_{\bigstar}(X/r)$. Since $X/r = \sum_e \chi^e X(e)/r(e)$, applying $P_{\bigstar}$ yields

$$\nabla Z = \sum_e i^e X(e)/r(e)$$

The random variables ∇Z are linear combinations of independent normal random variables, so themselves are jointly normal, hence the name Gaussian. Since all $X(e)$ have mean 0, so do all $\nabla Z(e)$.

Another way to look at this orthogonal projection: An orthogonal basis for the space $l^2(E, r)$ is $\langle \chi^e/\sqrt{r(e)};\ e \in E\rangle$. If (Ω, P) is a probability space on which the random variables $X(e)$ are defined, then $\langle X(e)/\sqrt{r(e)};\ e \in E\rangle$ are orthogonal in $L^2(\Omega, P)$. If $\mathcal{H}$ denotes the linear span of the random variables $X(e)$, then $\Phi : \chi^e \to X(e)$ is an isometric isomorphism from $l_2(E, r)$ to $\mathcal{H}$.

Let $Y(e, e') := i^e(e')$, that is, current that flows across e' when a unit current is imposed between the endpoints on e. From previous and reciprocity law, we have

$$dZ(e) = r(e)\nabla Z(e) = \sum_{e'} r(e)Y(e', e)X(e')/r(e')$$

$$= \sum_{e'} Y(e, e')X(e')$$

We have trivially that

$$i^e = \sum_{e'} Y(e, e')\chi^{e'}$$

Comparing the above shows that Φ takes i^e to $dZ(e)$. In particular,

$$Cov\big(dZ(e), dZ(e')\big) = E[dZ(e)dZ(e')] = (i^e, i^{e'})_r = (i^e, \chi^{e'})_r = Y(e, e')r(e')$$

This is the same as the voltage difference across e' when unit current flows from e^- to e^+. Since the meas and covariances determine the distribution uniquely for jointly normal random variables, we could regard this as the definition as dZ; and $Z(o) = 0$, this could also be used as a definition of Z.

The isomorphism Φ takes the subspace $\bigstar$ to a subspace that we will denote $\bigstar(\mathcal{H})$. This latter subspace is simply the linear span of $\sum_{e^-=x} X(e)/r(e)$ over $x \in V$. Furthermore, since $i^e = P_\bigstar \chi^e$, it follows from the isomorphism that $dZ(e) = P_{\bigstar(\mathcal{H})} X(e)$. Since $Z(o) = 0$, we may write $Z(x)$ for $x \in V$ by summing $-dZ$ along a path Ψ from o to x, this gives

$$Z(x) = -\sum_{e\in\Psi} dZ(e) = \Phi(-\sum_{e\in\Psi} i^e) = \Phi(i_{x,o})$$

where $i_{x,o}$ is the unit current flow from x to o. Therefore

$$Z(x) = \sum_e i_{x,o}(e)X(e)$$

Proposition 2.3. *Let Z be the preceding canonical Gaussian field.*

(i) The random variable X are jointly normal with

$$Z(x) - Z(y) \sim N(0, \mathcal{R}(x \leftrightarrow y))$$

for $x \neq y \in V$.

(ii) The covariances of $Z(x) - Z(y)$ and $Z(z) - Z(w)$ equals $v(z) - v(w)$ when v is the voltage associated to a unit current flow from x to y, with $x \neq y$.

(iii)

$$Cov(Z(x), Z(z)) = \frac{\mathcal{G}_o(x, z)}{\pi(z)}$$

where $\mathcal{G}_o(x, z)$ is the expected number of visits to z of the network random walk started x, counting only visits that occur before visiting o.

Proof: (iii) follows from (ii) by $y := w := o$. (ii) extends (i), so we prove only (ii).

Let $v_{x,y}$ be the voltage function corresponding to a unit flow from x to y. Let $\Psi \in l^2(E, r)$ represent a path from w to z, that is, $\Psi = \sum_e \chi^e$, where the sum ranges over the edges in a path from w to z, oriented in the direction of this path. Then as above, we have that $Z(x) - Z(y) = \Phi(i_{i,y})$.

$$\begin{aligned} Cov(Z(x) - Z(y), Z(z) - Z(w)) &= (i_{i,y}, i_{z,w})_r = (\nabla v_{x,y} - P_\bigstar \Psi)_r \\ &= (\nabla v_{x,y} - \Psi)_r = v_{x,y}(z) - v_{x,y}(w) \end{aligned}$$