triangle inequality on boundary

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Let $\alpha, \beta, \zeta \in \partial T$.

$$d(\alpha, \beta) = \frac{1}{e^{\#\text{ common edges between }\alpha,\beta}} \tag{1}$$

$$d(\alpha, \zeta) = \frac{1}{e^{\#\text{ common edges between }\alpha,\zeta}} \tag{2}$$

$$d(\zeta, \beta) = \frac{1}{e^{\#\text{ common edges between }\zeta,\beta}} \tag{3}$$

Case 1: Say ζ diverges from α and β before (or at) the point at which α and β diverge. In this case, ζ diverges from α and β at the same time (before α and β diverge). Therefore, ζ is at least as far away from α as α is from β. Therefore, we have $d(\alpha, \beta) \leq d(\alpha, \zeta) = d(\zeta, \beta)$. Therefore, $d(\alpha, \beta) \leq d(\alpha, \zeta) + d(\zeta, \beta)$.

Case 2: Say ζ follows α (or β, but we will assume α w.o.l.o.g.) after the point at which α and β diverge. In this case, ζ must diverge from β when α diverges from β. Therefore, we have $d(\alpha, \beta) = d(\zeta, \beta)$. And therefore, $d(\alpha, \beta) \leq d(\alpha, \zeta) + d(\zeta, \beta)$.