Rough Isometries and Hyperbolic Graphs

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1 Rough Isometries

Theorem 1.1. ***Pólya's Theorem***: *Simple random walk on the nearest-neighbour graph $\mathbb{Z}^d$ is recurrent for $d = 1, 2$, and transient for $d \geq 3$.*

Previously we proved Pólya's theorem by calculating the Green function. However, that calculation depends on the precise structure of the graph. Here, we will show that the type doesn't change when we make changes in the lattice graph.

Suppose the diagonal edges are added to the square lattice in the plane. We can still use Nash-Williams criterion to show recurrence. Similar addition of edges in higher dimensions preserves transience by Rayleigh's Monotonicity principle. Suppose that in $\mathbb{Z}^3$, we remove each edge $[(x, y, z), (x, y, z + 1]$ with $x + y$ odd to obtain graph H. Is the resulting graph still transient? How much has the effective resistance to infinity changed?

Definition 1.2. (k-Fuzz). The k-fuzz of a graph G adjoins the edges of G a new edge between every pair of vertices whose distance in G lies between 2 and k.

Then the graph H defined above has a 3-fuzz that includes the original graph on $\mathbb{Z}^3$. We can solve the problem with the following theorem.

Theorem 1.3. *Let G be a connected graph of bounded degree and k a positive integer. Then G and the k-fuzz of G have the same type, that is, both are transient or both are recurrent.*

We can think of the resistance $r(e)$ are the length of the edge e. We will establish a more powerful result.

Definition 1.4. (Rough embedding). Given two networks G and G' with resistances r and r', we say that a map ϕ form the vertices of G to the vertices of G' is a rough embedding if there are constants $\alpha, \beta < \infty$ and a map Φ defined on the edges of G such that

(i) $\forall \langle x, y\rangle \in G,\ \Phi(\langle x, y\rangle)$ is a nonempty simple oriented path of edges in G' from $\phi(x)$ to $\phi(y)$ with

$$\sum_{e' \in \Phi(\langle x,y\rangle)} r'(e') \leq \alpha r(x, y)$$

and $\Phi(\langle y, x\rangle)$ is the reverse of $\Phi(\langle x, y\rangle)$

(ii) $\forall e' \in G'$, there are no more than β edges in G whose image under Φ contains e' We call such map (α, β)-rough. We call two networks roughly equivalent if there are rough embeddings in both directions.

Every two Euclidean lattices of the same dimensions are roughly equivalent. Every graph G of bounded degree and every k, the graph G and its k-fuzz are roughly equivalent.

Theorem 1.5. ***Rough Embeddings and Transience:*** *If G and G' are roughly equivalent connected networks, then G is transient iff G' is transient. In fact, if there is a rough embedding from G to G' and G is transient, then G' is transient.*

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Proof: Suppose G is transient and ϕ is an (α, β)-rough embedding from G to G'. Let θ be a unit flow of finite energy from a to infinity. Define

$$\theta'(e') = \sum_{e' \in \Phi(e)} \theta(e)$$

It is easy to see that θ' is antisymetric and $\forall x' \in G',\ d^*\theta'(x') = \sum_{x \in \phi^{-1}(x')} d^*\theta(x)$. Thus, θ' is a unit flow from $\phi(a)$ to infinity.

By Cauchy-Schwartz inequality and condition (ii), we have

$$\theta'(e')^2 \leq \beta \sum_{e' \in \Phi(e)} \theta(e)^2$$

$$\sum_{e' \in E'} \theta'(e')^2 r'(e') \leq \beta \sum_{e' \in E'} \sum_{e' \in \Phi(e)} \theta(e) = \beta \sum_{e \in E} \sum_{e' \in \Phi(e)} \theta(e) \leq \alpha\beta \sum_{e \in E} \theta(e)^2 r(e) < \infty$$

Problem 1.6. Show that the above graph H, obtained by removing each edge $[(x, y, z), (x, y, z + 1)]$ in $\mathbb{Z}^3$ with $x + y$ odd, is transient with effective resistance to infinity at most 6 times what it was before removal.

Definition 1.7. (Rough isometry). Given two graphs $G = (V, E)$ and $G' = (V', E')$, a function $\phi : V \to V'$ is a rough isometry is there are positive constants α, β such that for all $x, y \in V$,

$$\alpha^{-1} d(x, y) - \beta \leq d'(\phi(x), \phi(y)) \leq \alpha d(x, y) + \beta$$

and such that every vertex in G' is within distance β of the image of V.

Here d and d' denote the usual graph distances, and ϕ does not need to be bijective. The same definition applies to metric spaces, with vertices replaced by points. $\mathbb{Z}^d$ is roughly isometric to $\mathbb{R}^d$.

Problem 1.8. Show that being roughly isometric is an equivalence relation.

Problem 1.9. Show that $\mathbb{Z}$ and $\mathbb{Z}^2$ are not roughly isometric graphs.

Proposition 1.10. ***Rough Isometry and Rough Equivalence****: Let G and G' be two infinite roughly isometric graphs with conductances c and c'. If c, c', c^{-1}, c'^{-1} are all bounded and the degrees in G and G' are all bounded, then G is roughly equivalent to G'.*

We can now give a simple proof of Pólya's Theorem. Consider simple random walk in one dimension. The probability to return to the origin after 2n steps s exactly $\binom{2n}{n}2^{-2n}$. By Stirlings formula, this is asymptotic to $1/\sqrt{\pi n}$. If we consider random walk in d dimensions where each coordinate is independent of other coordinates and does simple random walk in one direction, then the return probability after $2n$ steps is $(\binom{2n}{n}2^{-2n})^d \sim (\pi n)^{-d/2}$, this is summable precisely when $d \geq 3$. Moreover, this independent-coordinate walk is simpler random walk on another graph whose vertices are a subset of $\mathbb{Z}^d$, and this other graph is clearly isometric to the usual graph on $\mathbb{Z}^d$. We have deduced Pólya's theorem.

2 Hyperbolic Graphs

Let $\mathbb{H}^d$ denote the standard hyperbolic space of dimension $d \geq 2$, it has scalar -1 everywhere. The figure uses the Poincaré disc model. The corresponding ball model of $\mathbb{H}^d$ uses the unit ball $\{x \in \mathbb{R}^d; |x| < 1\}$ with the arclength metric $\frac{2|dx|}{(1-|x|)^2}$. Here we write $|x|$ for the Euclidean norm.

The Length of a smooth curve $t \to x(t)$ parameterized by $t \in [0, 1]$ is

$$\int_0^1 \frac{2|dx/dt|}{(1 - |x(t)|)^2} dt$$

The minimum of such lengths among curves joining $x_1, x_2 \in \mathbb{H}^d$ is the hyperbolic distance between x_1 and x_2. A curve that achieves the minimum is called a **geodesic**. For example, if x_1 is the origin

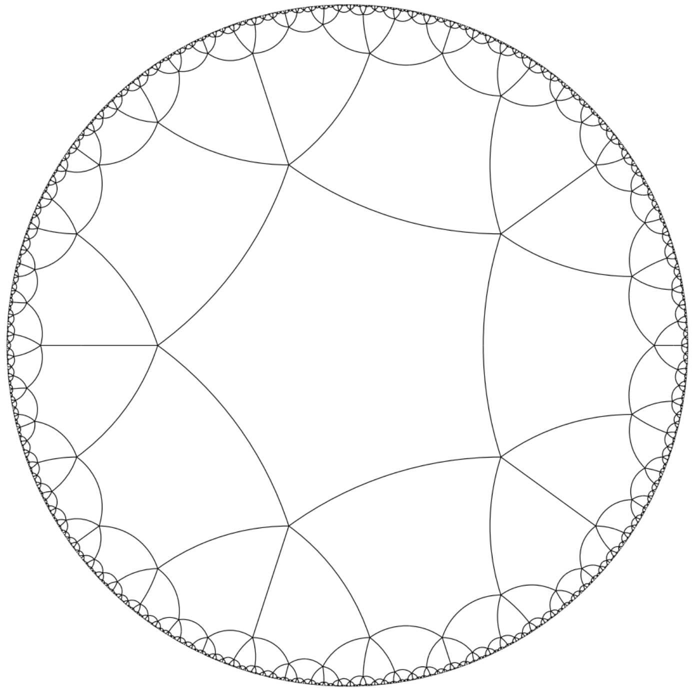

Figure 1: A graph in the hyperbolic disc formed from congruent regular hyperbolic pentagons of interior angle $2\pi/5$.

and $|x_2| = R \in (0,1)$, then a geodesic between x_1 and x_2 is a Euclidean straight-line segment. To see this, note that $\rho(t) := |x(t)|$, $d|x(t)| \geq |d\rho(t)|$, so the integral is at least

$$\int_0^1 \frac{2|d\rho(t)/dt|}{1-\rho(t)^2} dt \geq \int_0^R \frac{2ds}{1-s^2} = \log\frac{1+R}{1-R}$$

A key difference to Euclidean space is that for each point $o \in \mathbb{H}^d$, the sphere of hyperbolic radius r about o has hyperbolic surface area asymptotic to $\alpha e^{e(d-1)}$ for some positive constant α dependent on d. Therefore, the hyperbolic volume of the ball of hyperbolic radius r is asymptotic to $(d-1)^{-1}\alpha e^{e(d-1)}$. Indeed, if $|x| = R$, then the hyperbolic distance between the origin and x was seen earlier to be

$$r = \log\frac{1+R}{1-R}$$

$$R = \frac{e^r - 1}{e^r + 1}$$

The hyperbolic surface area of this sphere centered at the origin is therefore

$$\int_{|x|=R} \frac{2^{d-1} dS}{(1-|x|^2)^{d-1}}$$

where dS is the element of the Euclidean surface area in $\mathbb{R}^d$. Integrating gives the value

$$C(\frac{R}{1-R^2})^{d-1} = \alpha(e^r - e^{-r})^{d-1}$$

Definition 2.1. (Nets). Graphs that are roughly isometric to $\mathbb{H}^d$ often arises as Cayley graphs of groups, or more generally, as nets. A set is **ϵ-separated** if all nonzero distances between points are at least ϵ. A graph G is called an **ϵ-net** of a metric space M if the vertices of G form a maximal ϵ-separated subset of M and edges join distinct vertices iff their distance in M is at most 3ϵ.

Theorem 2.2. ***Transience of Hyperbolic Space**: If G is roughly isometric to a hyperbolic space $\mathbb{H}^d$ with $d \geq 2$, then simple random walk on G is transient.*

Proof: The proof is quite similar to the first proof of Pólya's theorem.

By Rough Embeddings and Transience, it suffices to show transience for one such G. Let G be a 1-net of $\mathbb{H}^d$. We take the edges of G to be the geodesics segments. Let L be a random uniformly distributed geodesics ray from some point $o \in G$ to ∞. Let $\mathcal{P}(L)$ be a simple path in G from o to ∞ whose vertices stay within distance 1 of L; choose $\mathcal{P}(L)$ measurably. (By choice of G, $\forall p \in L$, there is some vertex $x \in G$ within distance 1 of p.)

Define the flow θ from $\mathcal{P}(L)$

$$\theta(e) := \sum_{n \geq 0} (P[\mathcal{P}(L) = e] - P[\mathcal{P}(L) = -e])$$

Then θ is a unit flow from o to ∞; we claim it has finite energy. There is some constant C such that if e is an edge whose midpoint is at hyperbolic distance r from o, then

$$P[e \in \mathcal{P}(L)] \leq Ce^{-r(d-1)}$$

Given an edge center s, there is a bound on the number of edge centers whose hyperbolic distance from s is at most 1. Therefore, there is also a constant D such that there are at most $De^{n(d-1)}$ edge centers whose hyperbolic distance from the origin is between n and $n+1$. It follows from that the energy of θ is at most $\sum_n C^2 De^{-2n(d-1)} e^{n(d-1)}$, hence is finite, and transience follows.

Theorem 2.3. *If a Riemannian manifold (M, g) has bounded geometry, then the inclusion map ϕ of any $(\epsilon, 3\epsilon)$-net G into M is a rough isometry. Moreoever, M is hyperbolic if and only if G is transient.*