COMPUTATIONAL PRACTICUM assignment
Differential Equations

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y' = sin(x) + y

Exact solution of IVP(Initial Value Problem)

there x0 = 0 and y0 = 1

Given Differential Equation:

$$y' = \sin(x) + y$$

Solution of DE:

$$y = C_1 e^x - \frac{(\sin(x) + \cos(x))}{2}$$

Solution of IVP:

$$y = \frac{3}{2} e^x - \frac{(\sin(x) + \cos(x))}{2}$$

There isn't any point of discontinuity in solution of given differential equation.

System contains 3 classes: Equation, Numeric methods and supporting class Plotting

UML diagram for Equation and Numeric methods

Equation

+x0 : Float
+y0 : Float
+X : Float
+n : Integer
+h : Integer

+get_derivative(x : Float, y : Float)
+__init__(x0 : Float, y0 : Float, X : Float, n : Integer)
+local_errors(ff : Equation)
+max_error(ff : Equation)
+exact_solution()

Equation For numeric methods

Implementation of numeric methods for equation

Numeric methods

+euler_method(f : Equation)
+euler_method_improved(f : Equation)
+runge_kutta_method(f : Equation)
+total_approximation_errors(start_n : Integer, end_n : Integer, x0 : Float, y0 : Float, X : Float)

Explanation of Equation class attributes and methods

Equation class is used to operate with given y' = sin(x) + y differential equation, change initial values, grid size to solve IVP inside class.

Attributes:

+x0 : Float **Starting point of x-axis of IVP problem segment**
+y0 : Float **Value of function in x0 point y(x0) = y0**
+X : Float **Ending point of x-axis of IVP problem segment**
+n : Integer **Grid size**
+h : Integer **Value of one grid step(depends on x0, X0 and n)**

Methods:

+__init__(x0 : Float, y0 : Float, X : Float, n : Integer)

```
def __init__(self, x0, y0, X, n): #initialization
    self.x0 = x0
    self.y0 = y0
    self.X = X
    self.n = n

    h = (X - x0)/n
    self.h = h

    self.x = numpy.arange(x0, X + 0.000001, h)
```

Method initialize initial values, scope and grid size for solving IVP problem, these values are assigned to Equation instance

+get_derivative(x : Float, y : Float)

```
def get_derivative(self, x, y):  #local derivative
    return math.sin(x) + y
```

Method returns local derivative of function in the point (x, y) using following function y' = sin(x) + y

+local_errors(ff : Equation)

```
def local_errors(self, ff):  #graph of local errors

    y = ff[1]

    y_error = [0] * len(self.x)
    ex = self.exact_solution()

    for i in range(len(self.x)):
        y_error[i] = ex[1][i] - y[i]

    return [self.x, y_error]
```

Method calculates function of local errors of function ff with respect to exact solution of given differential equation

+max_error(ff : Equation)

```
def max_error(self, ff):  #maximum error of function ff

    l = self.local_errors(ff)
    mx = 0.0
    a = l[1]

    for i in range(len(a)):
        mx = max(mx, abs(a[i]))

    return mx
```

Method calculate value of the maximum error of function ff with respect to exact solution of given differential equation

+exact_solution()

```
def exact_solution(self):

    y = [0] * len(self.x)

    for i in range(len(self.x)):
        y[i] = 3.0/2.0 * (math.e**self.x[i]) - (math.sin(self.x[i]) + math.cos(self.x[i]))/2.0

    return [self.x, y]
```

Method calculate values of exact solution for given differential equation with determined scope and grid size in __init__

Explanation of Numeric_methods class attributes and methods

Numeric_methods class is used to operate solve IVP problem using Euler's, Euler' Improved and Runge-Kutta methods for given Equation e that is accepted by the class.

Numeric_methods has no any attribute.

+euler_method(f : Equation)

```
def euler_method(self, f):  #function got by Euler's method

    y = [0] * len(f.x)
    y[0] = f.y0

    for i in range(1, len(f.x)):
        y[i] = y[i - 1] + f.h * f.get_derivative(f.x[i - 1], y[i - 1]) #augmentation

    return [f.x, y]
```

Method calculate values of function of given differential equation ff on scope [x; X] with grid step n. It uses Euler's method for calculations.

+euler_method_improved(f : Equation)

```
def euler_method_improved(self, f):  #function got by improved Euler's method

    y = [0] * len(f.x)
    y[0] = f.y0

    for i in range(1, len(f.x)):
        d = f.h * f.get_derivative(f.x[i - 1] + f.h/2, y[i - 1] + f.h/2 * f.get_derivative(f.x[i - 1], y[i - 1]))
        y[i] = y[i - 1] + d #augmentation

    return [f.x, y]
```

Method calculate values of function of given differential equation ff on scope [x; X] with grid step n. It uses Improved Euler's method for calculations.

+runge_kutta_method(f : Equation)

```
def runge_kutta_method(self, f):  #function got by improved Runge-Kutta method

    y = [0] * len(f.x)
    y[0] = f.y0

    for i in range(1, len(f.x)):

        d1 = f.get_derivative(f.x[i - 1], y[i - 1])
        d2 = f.get_derivative(f.x[i - 1] + f.h/2, y[i - 1] + f.h * d1/2)
        d3 = f.get_derivative(f.x[i - 1] + f.h/2, y[i - 1] + f.h * d2/2)
        d4 = f.get_derivative(f.x[i - 1] + f.h,   y[i - 1] + f.h * d3)

        d = f.h * (d1 + 2 * d2 + 2 * d3 + d4)/6
        y[i] = y[i - 1] + d #augmentation

    return [f.x, y]
```

Method calculate values of function of given differential equation ff on scope [x; X] with grid step n. It uses Runge-Kutta method for calculations(calculating local derivatives, getting local result and augment it.

+total_approximation_errors(start_n : Integer, end_n : Integer, x0 : Float, y0 : Float, X : Float)

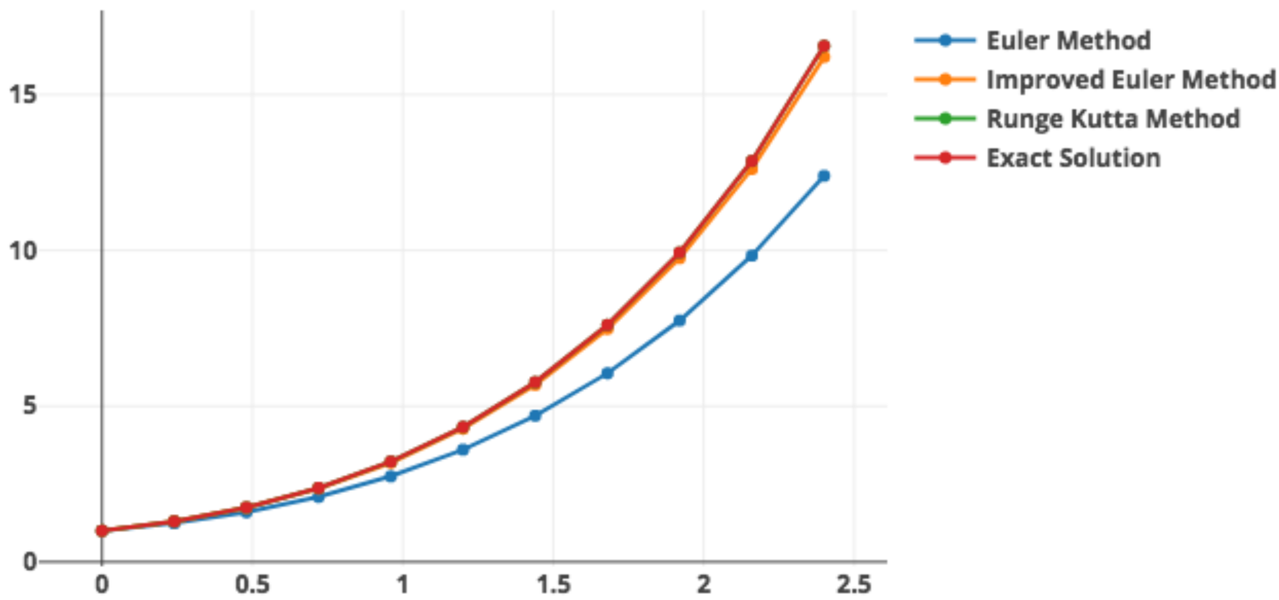
```
def total_approximation_errors(self, start_n, end_n, x0, y0, X):

    euler_total_y = [0] * (end_n - start_n + 1)
    euler_improved_total_y = [0] * (end_n - start_n + 1)
    runge_kutta_total_y = [0] * (end_n - start_n + 1)
    total_x = [0] * (end_n - start_n + 1)

    for i in range(start_n, end_n + 1):

        j = i - start_n
        total_x[j] = i

        e = Equation(x0, y0, X, i)

        euler_result = (self.euler_method(e))
        euler_improved_result = (self.euler_method_improved(e))
        runge_kutta_result = (self.runge_kutta_method(e))

        euler_total_y[j] = e.max_error(euler_result)
        euler_improved_total_y[j] = e.max_error(euler_improved_result)
        runge_kutta_total_y[j] = e.max_error(runge_kutta_result)

    euler_total = [total_x, euler_total_y]
    euler_improved_total = [total_x, euler_improved_total_y]
    runge_kutta_total = [total_x, runge_kutta_total_y]

    return [euler_total, euler_improved_total, runge_kutta_total]
```

Method analyze the total approximation error depending on the number of grid cells, method accepts strat_n – starting point of grid step and end_n – finishing point of grid step and return functions of total errors of Euler's, Improved Euler's and Runge-Kutta methods for given differential equation $y' = \sin(x) + y$. It was implemented by using methods inside Numeric_methods class for all integer grid steps values on [start_n; end_n] and calculating maximum error of taken functions.

Graph of solutions of differential equation y' = sin(x) + y using Euler's method, Improved Euler's method, Runge-Kutta method and graph of exact solution.

Initial values: x0 = 0, y0 = 1, X = 12/5

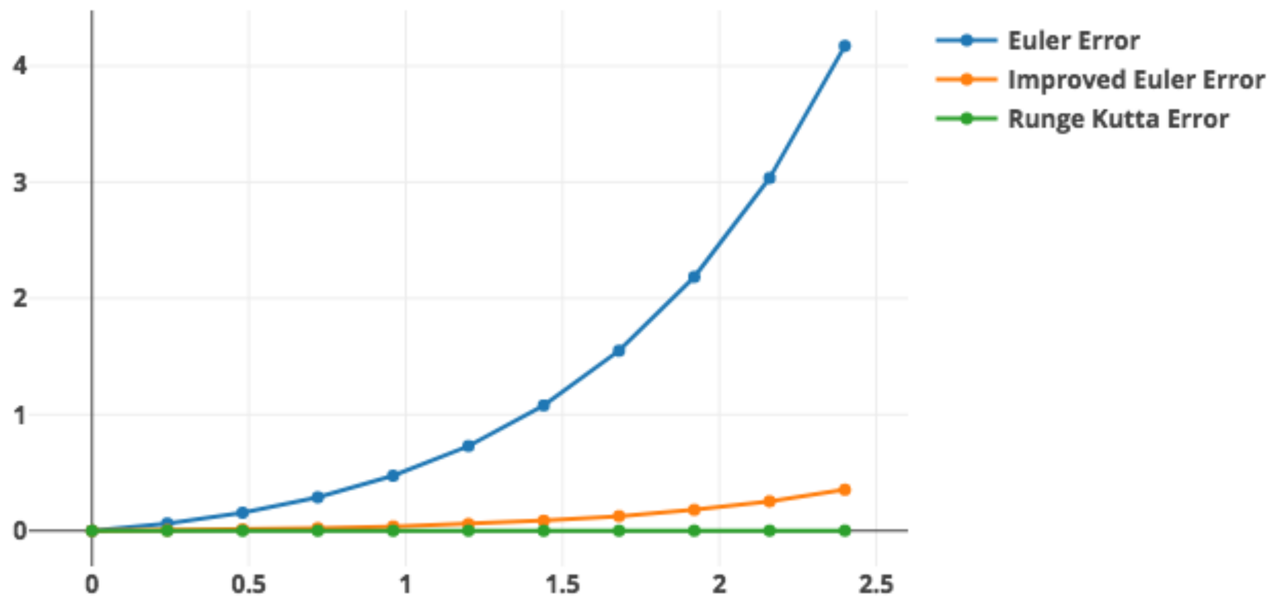
Grid step N = 10



Graphs of local errors, difference between exact solution of differential equation y' = sin(x) + y and solutions calculated using Euler's method, Improved Euler's method, Runge-Kutta method.

Initial values: x0 = 0, y0 = 1, X = 12/5

Number of grid steps N = 10



Graph of total approximation errors of solutions of differential equation y'= sin(x) + y calculated using Euler's method, Improved Euler's method, Runge-Kutta method, depends on number of grid cells (x-axis).

Initial values: x0 = 0, y0 = 1, X = 12/5

Number of grid steps: [400; 600]