Zero Theorem

The Governing Bitcoin Pricing Equations

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We suffered...a series of great losses...and we struggled for a long time to understand the reason for it...the reason we were chosen, if chosen we were, to go on living with this misfortune... We came to this place...It seemed to suit us, so we remained silent and isolated. From the ashes...from zero...an theory was conceived...something to cultivate...something to hold on to...something of potential use. We exclusively worked on this masked concept...this concealed truth...seemingly hidden in plain sight...this theorem. One day the phone rang...we answered...and a familiar voice said, "Kristian?", and suddenly we felt a flash of light...and was reminded...that we are still connected to something greater... that we are not judged on our failures rather on the quality of our intent...with the willingness to endure its resultant sacrifices...and with that single realisation...the true purpose behind our loss was revealed...which would give us the courage to publish Zero Theorem... with the faith that it could be used for the greater good; to improve pricing certainty and market efficiency.

- Dr. Kristian Haehndel

Contents

Nomenclature

α_k absorption rate of market k

ω_k substitution of market k

b ???

d ???

h ???

M_k capitalization of market k

M_{BTC} total market capitalization of the asset

P_k price of market k

p_k, q_k Bass diffusion model parameters for market k

P_{BTC} price of the asset

Q_{BTC} output power of the asset

R_k volume of market k

T_j transactions of market j

U_k unit of market k

U_{s_k} ????

V_{BTC} velocity of the asset

Abstract

A theoretical framework is presented to value a new asset type based on its absorption of monetary value from a pre-existing asset class (or weaker substitute).

1 Introduction

2 Critique of Literature

2.1 Economic Literature

[1] introduces Bitcoin as a peer-to-peer means of payment. Peer-to-peer system doesn't require the presence of trusted third parties to process payments which leads to a significant decrease in transaction costs. Moreover, a public ledger allows tracing of historical transactions which prevents double spending and increases security and trust in the system.

[2] and [3] papers examine if Bitcoin satisfies Mises's regression theorem. The theorem states that most of the monies that become an accepted medium of exchange, for example gold, can trace back (regress) their subject value as a medium of exchange to their intrinsic, objective, value in consumption or production. [2] argue that examining if Bitcoin satisfies this theorem, i.e. if it has an intrinsic value in direct use, is "barking up the wrong tree". The theorem refers to the case of transition from barter to monetary economy, and for something to serve as a money in barter economy it must have an intrinsic value. However, Bitcoin appeared directly in the monetary economy and never served in a Barter economy, so the theorem doesn't apply. [3], on the other hand, finds that Bitcoin meets the requirements of the regression theorem because there is a non-monetary demand for Bitcoin: hackers use it as a toy and try to crack its advanced cryptography, libertarians and tech geeks look at it as a membership card and symbol of their community. Moreover, Bitcoin is powered by advanced technology that can increase the productivity of many economic sectors that rely on smart contracts.

In [4] model Bitcoin derives its fundamental value from net transactional benefits. On the one hand, cryptocurrencies provide transactional benefits because they are not subject to Government imposed restrictions, such as sanctions and taxes, nor they can be devalued by central bank policies. On the other hand transacting in cryptocurrencies also endures costs in terms of exchange fees to traditional currencies as well as hacking risk. Adding noise to the model can explain high volatility in Bitcoin pricing.

[5] builds a model of two competing but intrinsically worthless currencies (Dollar and Bitcoin) with a difference that the supply of latter is predetermined while supply of Dollars is stochastic and in the hands of a central bank. Authors find conditions under which Bitcoin price is martingale, and hence there is no speculative demand, hoarding in expectation of price appreciation, for Bitcoin.

[6] relates Bitcoin price to its cost of production, i.e. electricity costs. [7] takes this idea further by conducting an empirical research on the relationship between Bitcoin price and mining costs. The author finds that Bitcoin price drives mining costs which is quite surprising as normally production costs influence the price. The adjustment of mining costs is not instantaneous, but it takes from several months up to a year for production costs to catch up with the price.

[8] takes a skeptical view and argues that the Bitcoin cannot be priced at all. Simply, there is too much ambiguity: "What share of conventional money will Bitcoin replace?", "Will there be competition from other cryptocurrencies?" So the author concludes by saying that Bitcoin pricing should be subject of study in psychology rather than economics.

[9] aim to explain the phenomenon of currency substitution, a situation when due to macroeconomic instability the domestic money is partially replaced by a foreign one. In their model, foreign money plays a role both as a means of payment, because it is expected to appreciate against the domestic currency, and as a store of value because agents are not able to invest in foreign assets due to capital flow restrictions and they decide to keep foreign money instead.

[10] uses portfolio balance model to explain the demand for foreign money. Portfolio balance means that the net change in portfolio value is zero so increase in the share of foreign money in the portfolio needs to be accompanied by the decrease in the share of domestic money.

[11] analyze statistical propertied of Bitcoin prices and find that the Bitcoin is uncorrelated with traditional asset classes both in normal times as well as in the times of financial crisis. By examining transaction data authors find that the significant amount of Bitcoins never changes hands which they interpret as a sign that Bitcoin is mainly used as a speculative investment

and not as a medium of exchange.

[12] conduct an empirical study of Bitcoin price formation. Using time-series analysis they find that the demand for Bitcoin as a medium of exchange, as proxied by the number of transactions and number of addresses exert strong effect on the Bitcoin price. This relationship is expected and holds for traditional currencies too. In addition, the authors find that the arrival of new information, as measured by the number of views on Wikipedia, increases the trust among the users in the new currency and positively impacts the price in the long-term. In the short term, the price is impacted by the speculative activity which leads to high volatility and questions the usefulness of Bitcoin as a medium of exchange. Interestingly, the study does not find any relationship between global macro-financial developments and Bitcoin price. The paper proxies for Bitcoin velocity by looking at the days destroyed, but does not find significant impact of this variable on Bitcoin price.

[13] find that Bitcoin is a strong hedge against down movements in Asian stocks, but not so good hedge against stocks in other geographies and commodities.

However by looking at the [14] review of empirical literature there is a significant number of studies that find strong diversification benefits of Bitcoin, and argue that in terms of return characteristics Bitcoin is somewhere between the gold and the dollar. [15] develop "fundamentals-based dynamic valuation model for cryptocurrencies". In their model cryptocurrency derives value from it's usage as a means of exchange, an ability to settle transactions. Their model exhibits strong network effects, the more users adopt the currency, the more valuable the currency is and the more likely it is to be adopted by other users. In addition, the model allows for the interaction between cryptocurrency and other financial markets through user deciding whether to hold the cryptocurrency, and profit from network effects and lower transaction costs but forego the return on other assets, or not to hold the cryptocurrency and invest in other assets instead. Adoption curve from their model is S-shaped, which is in accordance with the model proposed in this paper.

One of the most frequently used models to model diffusion and adoption of new products is Bass model, [16]. The model rests on the assumption that there are two types of users. The first type are innovators who adopt the new product independently of the decisions of other users. The second type are imitators whose decision to adopt the product depends on the size of the existing user base. The speed of adoption of the new product depends on the share of innovators in the population, parameter p, and share of imitators, parameter q. The higher the share of innovators the faster the product growth. The model has been used widely and tested in different application. For example, [17] use Bass diffusion model to predict the adoption of electric cars in Germany. However, the authors report that the model is very sensitive to the choice of parameter values, and warn against its indiscriminate usage.

2.2 Reinforcement Learning (RL) and Machine Learning Methods

In literature, there exist different survey articles that covers RL methods used in finance. For instance, authors in [18] surveyed 50 studies where RL is applied to financial markets. However, this survey only covers studies up to the year 2017. There are other survey articles such as [19–21]. However, these studies do not provide useful insights for our project. Therefore, we selected 30 of the most relevant studies for this review. After careful selection of the reviewed studies, this section is divided into two subsections. First subsection reviews studies which are of high impact and provides useful insights for our project. Second subsection provides summaries of studies that do not present any notable results.

In [22], a financial model-free RL framework for portfolio management has been proposed. The proposed framework consists of the following: an ensemble of identical independent evaluators topology that allows scalability, a portfolio vector memory that takes transaction cost into account while training policy network, an online stochastic batch learning that allows the agent to continuously learn from incoming market information, and a reward 3 function that aim to maximise accumulated wealth. The proposed framework can be used with a variety of neural networks such as CNN, RNN, and LSTM. To evaluate the performance, accumulative portfolio value, maximum drawdown, and Sharpe ratio are used. It is shown that the proposed method surpasses traditional portfolio selection methods in a cryptocurrency market.

Authors in [23] applied continuous control DRL algorithms to asset allocation. Specifically, DDPG, PPO, and PG algorithms were used. Moreover, an adversarial PG algorithm is proposed to overcome the limitations of the portfolio management problem. Each algorithm used risk-adjusted accumulative portfolio value and different feature combinations as objective function and input respectively. Extensive experimentation on the China Stock market showed that the proposed adversarial PG can greatly improve training efficiency as well as average daily return and Sharpe ratio in back test. In [24], a DDPG-based approach is proposed for the dynamic stock market. The DRL environment is created using the Dow Jones data from Compustat database. The data is divided into three phases: training, validation, and trading. The proposed approach is compared with Dow Jones Industrial Average and min-variance portfolio allocation strategy. Following metrics were used for evaluation: final portfolio value, annualized return, annualised standard error, and the Sharpe ratio. Through experiments, it is shown that the proposed DDPGbased approach outperformed both Dow Jones Industrial Average and min-variance strategy.

Authors in [25] used different DRL algorithms (DQN, PG, and A2C) to design trading strategies. Both discrete and continuous action spaces were considered. For the reward function, volatility scaling is integrated such that trade positions can be scaled based on market volatility. LSTM was used as a neural network and a comparison was made with the following approaches: Long only, Sign (R), and MACD Signal. Following evaluation metrics were considered: annualised expected trade return, annualised standard deviation of trade return, downside deviation, sharpe, sortino, maximum drawdown, calmer ratio, percentage positive return, and the ratio between positive and negative trade returns. All approaches were tested on the 50 most liquid futures contracts from 2011 to 2019. It is shown through experimentation that DRL outperformed most asset classes except for the equity index.

[26] proposed a model-based DRL architecture for dynamic portfolio optimisation. The proposed architecture consists of an infused prediction module, a generative adversarial data augmentation module, and a behaviour cloning module. DDPG is used as a RL algorithm while the proposed architecture is flexible enough that both on-policy and off-policy RL algorithms can be used. A mix of US equities data on an hourly frequency is used whereas 4 the agent produces decisions daily. Multiple performance metrics are used to evaluate the performance such as Sharpe ratio, Sortino ratio, and annual return. As a baseline, modelfree DDPG is considered. The proposed architecture is empirically validated on several independent experiments with real market data and practical constraints. This study should be explored in detail as it can be useful in designing model-based DRL solution for the project. The code of this study is not available.

The studies [27,28] proposed a DRL approach that ensures consistent rewards for the trading agent and mitigation of the noisy nature of profit-and-loss. A price trailing-based reward shaping is proposed that significantly improves the performance of the agent in terms of Sharpe ratio, profit, and maximum drawdown. Moreover, a data prepossessing method is designed that allows for training the agent on different FOREX currency pairs, providing a way for developing market-wide RL agents and allowing, at the same time, to exploit more powerful recurrent deep learning models without the risk of over-fitting. The proposed reward method is tested with double DQN and PPO algorithms where each used LSTM along with FC neural network to enable recurrent properties. To evaluate the performance of the proposed method, a challenging dataset containing 28 instruments (provided by Speedlab AG) is considered. It is shown that with the proposed reward function, DRL algorithms achieve superior performance.

The study [29] proposed an attention network-based RL called AlphaStock whose goal is to implement the buy-winners-and-sell-losers strategy for stock trading. AlphaStock integrated deep attention networks with a Sharpe ratio-oriented RL framework to achieve a risk-return balanced investment strategy. AlphaStock contains the following three components. First, LSTM with history state attention network that extracts stock representation from historical states. Second, a cross-asset attention network that describes interrelationships among the stocks. Third, a portfolio generator that calculates investment proportions of all stocks. These components are optimised using RL. Different performance evaluation metrics are considered such as cumulative wealth, annualised percentage rate, annualised volatility, annualised Sharpe ratio, maximum drawdown, calmar ratio, downside deviation ratio, downside deviation. Performance comparison on US and Chinese market data showed that AlphaStock performed better than other methods. This is an interesting study that can provide some pointers on how to model interrelationships among assets to avoid selection bias.

Authors in [30] proposed an action-specialised expert ensemble trading system using DQN. Instead of using a common ensemble technique, the proposed ensemble method where each action has an expert model. Three different reward functions were used containing profit, 5 Sharpe ratio, and Sortino ratio. The proposed method is trained and tested with S&P500, Hang Seng Index, and Eurostoxx50 data. Through experiments, it is shown that the proposed method performed better than the traditional ensemble and single-agent model. The proposed action-specialised expert ensemble can be used for the project.

The study [31] proposed imitative recurrent deterministic policy gradient (iRDPG) which is an adaptive trading model. To deal with noisy financial data, the problem is formulated as POMDP. The iRDPG model consists of recurrent DPG and imitation learning. GRU network is used as a recurrent neural network and behaviour cloning is used for imitation learning. The proposed method and few simple baseline methods, such as DDPG, are trained and tested with Chinese stock market data. Different performance evaluation metrics are used such as total return rate, Sharpe ratio, and maximum drawdown. Experiment results showed that iRDPG can better learn than baseline methods. From the proposed method, we can further explore the possibility of using an imitation learning algorithm as well as the inclusion of differential Sharpe ratio in the reward function. The code of this study is not available.

Authors in [32] proposed a cost-sensitive portfolio policy network (PPN) for portfolio selection. The PPN consists of three modules: sequential information network, correlation information network, and decision-making module. The sequential information network extracts price sequential patterns whereas the correlation information network extracts asset correlation. DDPG is used to optimise PPN. A cost-sensitive reward function is developed which considers transaction and risk costs. PPN is trained on real-world crypto-currency datasets accessed from Poloneix. PPN is compared with various baselines for different performance evaluation metrics such as accumulated portfolio value, Sharpe ratio, and maximum drawdown. PPN showed better performance than its counterparts. The PPN architecture and cost-sensitive reward function can provide useful insights for the project.

An ensemble trading stock trading strategy is proposed in [33]. PPO, A2C, and DDPG are used as DRL algorithms in the ensemble strategy. A well-defined state space is used consisting of different technical indicators such as moving average, relative strength index, and commodity channel index. Continuous action space is used where each index represents a stock. A simple reward function is used whose goal is to maximise wealth. Each DRL algorithm is trained for a window of n months, then validated on a 3-month window and the best performing agent with the highest Sharpe ratio is selected. The best-selected agent is used to predict and trade for the next quarter. The performance of the proposed strategy is tested against Dow Jones industrial average index and min-variance portfolio allocation on 30 Dow Jones stocks. Results showed that the proposed ensemble strategy performed better than individual algorithms. The proposed strategy is not tested with a larger dataset as well as a simple reward function is used.

A DRL framework consists of two sub-networks is proposed in [34]. The first sub-network takes portfolio returns and standard deviation whereas the second sub-network takes contextual information, i.e., other assets price, other predictive data, and other unstructured data. The second sub-network plays an important role as it captures dependencies with common financial indicators and correlations between assets. Different neural network architectures are used such as CNN and LSTM. The Markovian property is satisfied by stacking previous observations in each state. Two reward functions are used, i.e., the Sharpe ratio and the net value of the final portfolio. Experiment results showed that the proposed method outperformed traditional static and dynamic Markovitx methods and the inclusion of social data can help in anticipating the crisis like the current Covid one. However, these results are generated through a small dataset, and comparison is only made with simple methods.

Authors extended their work in [35] by incorporating walk-forward procedure and multioutput neural networks. The walk-forward analysis allows to use past test data as training data, therefore, it is useful for small datasets. However, the downside of using walk-forward analysis is that it makes neural networks slow in terms of adapting to new information. The feature of multi-input and multi-output is achieved through a modified version of the adversarial policy gradient method. Through experiment results, it is shown that this method performed better than baselines.

Authors further extended their work in [36] where recurrent architecture (LSTM) is used. Rest of the contributions are same as earlier studies. From these studies, the creation of state space, use of CNN, and use of walk-forward analysis can be adopted.

[37] proposed a DRL library called FinRL designed for beginners to learn stock trading. FinRL has a three-layered architecture: application layer, DRL agent layer, and finance market environment layer. FinRL provides various datasets and state-of-the-art DRL algorithms that users can choose and develop their trading strategies. FinRL follows a trainingvalidation-testing flow and provides automated backtesting as well as benchmark tests. This library can be used to set up an environment from the user-given dataset and train a DRL algorithm such that it makes decisions on whether to buy, hold, or sell the stocks at each timestep. However, this

approach might not be very practical.

The study [38] proposed a sentiment and knowledge graph-based DRL algorithm for trading. specifically, using the Reuters Twitter account, historical news headlines are scarped and a knowledge graph is created. Then, sentiment classification is performed. The stock pricing data and sentiments are provided to DRL agent as state space. The action space of the 7 DRL agent is to either buy, sell, or hold. The goal of the agent is to maximise net profit and reward is given on daily decisions. DQN is used as a DRL algorithm. The proposed approach is tested on Microsoft corporation's stock data and a comparison is made with random policy and without sentiment analysis.

[39] proposed state augmented RL (SARL) framework for portfolio management. SARL augments financial prices data with financial news using NLP encoder. The task of the encoder is to transform different types of data into informative representations. The augmented state is fed into DPG algorithm that selects portfolio vector values as output. The reward function consists of the profit of each time step and transaction cost. The proposed framework is tested with Bitcoin and HighTech datasets. Comparison with different baseline methods is also made for different evaluation metrics such as portfolio value and Sharpe ratio.

[40] investigated DRL algorithm (PPO) in the context of high-frequency trading. Three different state spaces are considered: s1 is defined as limit order book (LOB) volumes, s2 is defined as s1 plus mark to market value, and s3 is defined as s2 plus the current bid-ask spread. The DRL algorithm is trained for each state definition and how they affect the outof-sample performance of the algorithm is shown. The results indicated that the knowledge of the mark to market return of the current position is highly beneficial for global PnL and single trade reward. Moreover, the proposed state definitions can potentially increase the net positive profit. This study is related to high-frequency trading and it does not offer any useful insight for the project.

The study [41] proposed a multi-objective DRL approach for intraday financial signal representation and trading. A recurrent neural network specifically LSTM is used to make trading decisions. TO balance risks and profits, two objectives with different weights were used. The effectiveness of the proposed method was shown on stock index future contracts.

The study [42] investigated the application of A3C algorithm for portfolio management problem. CNN is used as a neural network. Using S&P500 index stocks from Yahoo Finance website, a dataset for training is created. It is shown that A3C performs better in the test period. However, the scope of this study is quite limited and cannot be generalised to real-world portfolio management problem.

In this paper [43], authors explored the possibility of using recurrent neural networks, specifically GRUs, for finance portfolio management. A risk-adjusted reward function is crafted to evaluate the expected total rewards for policy. The authors only presented a discussion of the proposed method and did not present any results.

This paper [44] proposed a LSTM-based policy gradient RL algorithm for quantitative trading. The reward function is to maximise ultimate profit. Different combinations of technical indicators are used to train the agent. The performance of the proposed method is compared with a FC-based agent on different stocks. Although the inclusion of technical indicators reduced noise in the stock market, there still remains uncertainty and the proposed method can not be generalised and used for real-world trading.

[45] proposed a LSTM-based DQN to maximise profit in financial trading markets. The proposed method is a simple extension of earlier works with a slight modification. This study does not provide any additional information on the topic and should not be explored further. [46] used DQN and variants of DQN for trading and compared their results. This paper does not provide any useful insights.

The study [47] used DDPG for day trade stocks where the goal of the agent is to find an optimal portfolio vector every business day. This paper has a well-defined state space, action space, and reward function. The proposed method is trained and tested with Brazilian stock exchange data and showed superior performance than baselines. However, they made certain assumptions such as sufficient liquidity, zero slippage, zero market impact, odd lot equivalent to round lot prices, and zero latency impact, which make this approach unreliable for real-world scenarios.

In this study [48], an actor-critic architecture is used to solve asset allocation. Different neural networks are used such as CNN and LSTM. A simple state space, action space, and reward function were used. The DRL algorithm was trained on 24 stocks and compared with traditional methods. Overall, this study does not provide novel results.

[49] investigated the performance of DRL algorithm with different neural network architecture and inputs for learning trading strategies. Specifically, dense networks, CNN with 1d kernel, CNN with 2d kernel, and GRU neural networks are used. The performance of each model for different evaluation metrics is shown. This study does not provide any useful insights.

A significant strand of literature is busy with applying machine-learning algorithms to predict Bitcoin price. [50] consider a set of 26 Bitcoin related features, in addition to the past prices, out of which they select 16 and tests various models: Generalized Linear Model, Random Forest, Support Vector Machine on 10 minutes and 10 seconds interval and get an amazing result. They are able to predict the sign of a daily price change with an astonishing accuracy of 98.7%. Authors do argue, though, that so high accuracy is partially the result of the uninterrupted rise in Bitcoin prices over time. While another paper [51] explains this astonishing performance by the lack of cross-validation.

[52] apply Bayesian neural networks on the features related to Blockchain information, Macro-economic developments, and Currency markets with a goal of predicting daily Bitcoin prices and volatility. They find that Bayesian neural networks perform better than other linear and non-linear models, that predictions are satisfactory, and that price predictions have lower error than volatility predictions.

[51] perform a rigorous test of various models with respect to their ability to predict the Bitcoin price. They find that deep learning models perform better than traditional time series models like ARIMA. However, even the best performing model LSTM, Long Short Term Memory network, achieved sign prediction accuracy of only 52%. Such a low accuracy testifies to the fact that predicting Bitcoin price is not at all an easy task.

[53] first apply different feature selection techniques and then blend several models to achieve an accuracy of 62.91%, which is a 10 p.p. improvement in accuracy compared to state-of-the-art papers.

3 The Zero Theorem

3.1 Purchasing power

3.2 Market Substitution

3.3 Representation of a New Asset in Terms of Existing Assets and Respective Adoptions

First, consider the case of 2 assets. Then, the Wronskian of the functions M_1 and M_2 will be

$$W(M_1, M_2) = \begin{vmatrix} M_1 & M_2 \\ M_1' & M_2' \end{vmatrix} = M_1 \cdot M_2' - M_1' \cdot M_2.$$

On the other hand, we know that

$$\frac{\partial (M_1 + M_2)}{\partial t} = M_1' + M_2' = 0, \tag{1}$$

leading to

$$M_1' = -M_2'.$$

Substituting this into the expression for $W(M_1, M_2)$, we obtain that

$$W(M_1, M_2) = M_1 \cdot M_2' - (-M_2') \cdot M_2 = M_1 \cdot M_2' + M_2' \cdot M_2 = M_2' \cdot (M_1 + M_2).$$

Apparently, $W(M_1, M_2) \neq 0$ at least for some t, providing that M_1 and M_2 are linearly independent functions of time. However, it is important to note that in view of the equality (1), M_1' and M_2' are not linearly independent.

The linear independence of M_1 and M_2 means that any new asset M_3 in the vector space with basis functions M_1 and M_2 can be represented as

$$M_3 = \omega_1 \cdot M_1 + \omega_2 \cdot M_2,$$

where ω_1 and ω_2 are not simultaneously 0.

Evidently, for any $\omega_3 \neq -1$, we can write

$$\omega_1 \cdot M_1 + \omega_2 \cdot M_2 + \omega_3 M_3 \neq 0$$

for the above ω_1 and ω_2. Therefore, M_1, M_2 and M_3 are linearly independent functions forming a basis for a new, higher-dimensional vector space, in which, any new asset M_4 can be represented as

$$M_4 = \omega_1 \cdot M_1 + \omega_2 \cdot M_2 + \omega_3 M_3.$$

In a similar fashion, one can show that in the n-dimensional space with basis functions M_1, M_2, ..., M_n, the new asset M_{BTC} can be represented as

$$M_{\mathrm{BTC}} = \sum_{k=1}^{n} \omega_k \cdot M_k \tag{2}$$

with $\omega_k \neq -1$ which is satisfied in our case since all $\omega_k > 0$.

Hereinafter, M_k is regarded as the total capitalization of Market k, and ω_k– as the substitution of Market k.

4 The Generalised Model

We start with the equation of exchange below:

$$P_{\mathrm{BTC}} \cdot Q_{\mathrm{BTC}} = M_{\mathrm{BTC}} \cdot V_{\mathrm{BTC}},$$

where P_{BTC} is the price of asset, Q_{BTC} is the output power, M_{BTC} is the market capitalization, and V_{BTC} is the velocity of the asset class.

We consider the general case when Q_{BTC}, M_{BTC} and V_{BTC} are functions of time variable t, i.e., in what follows, $Q_{\mathrm{BTC}} = Q_{\mathrm{BTC}}(t)$, $M_{\mathrm{BTC}} = M_{\mathrm{BTC}}(t)$ and $V_{\mathrm{BTC}} = V_{\mathrm{BTC}}(t)$.

Evidently, taking natural logarithm in both sides of the previous equality, we derive

$$\ln P_{\mathrm{BTC}} + \ln Q_{\mathrm{BTC}} = \ln M_{\mathrm{BTC}} + \ln V_{\mathrm{BTC}}.$$

Differentiating both sides of this expression with respect to t, we obtain

$$\frac{\partial}{\partial t} \left[\ln P_{\mathrm{BTC}} + \ln Q_{\mathrm{BTC}} - (\ln M_{\mathrm{BTC}} + \ln V_{\mathrm{BTC}})\right] = 0.$$

or

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial t} = \frac{\partial}{\partial t} \left[\ln M_{\mathrm{BTC}} + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right]. \tag{3}$$

For the sake of simplicity, we denote

$$\frac{\partial \ln P_{\mathrm{BTC}}}{\partial t} = \Pi_{\mathrm{BTC}}.$$

We now need to substitute the representation (2) into (3) to have

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n}\omega_k \cdot M_k\right) + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right].$$

4.1 A Particular Model for Capitalization

We now assume that

$$M_k = P_k \cdot U_k, \tag{4}$$

where P_k is the price and U_k is the unit of the kth asset.

Then,

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n}\omega_k \cdot P_k \cdot U_k\right) + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right]. \tag{5}$$

4.2 A Particular Model for ω_k

In this section, we consider a specific model for the absorption rate α_k as follows

$$\omega_k = \frac{U_{s_k}}{U_k} \tag{6}$$

for $k = 1, 2, \ldots, n$. Substituting it into (5), we derive

$$\begin{aligned}\Pi_{\mathrm{BTC}} &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_k \cdot \frac{U_{s_k}}{U_k}\right) + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right] = \\ &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right].\end{aligned}$$

4.3 The Velocity Consideration

Consider the following model for V_{BTC}:

$$V_{\mathrm{BTC}} = \frac{1}{n}\sum_{j=1}^{n} T'_j, \tag{7}$$

where T_j represents the transactions. Substituting it into the final expression of Π_{BTC} leads us to

$$\Pi_{\text{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T'_j\right) - \ln Q_{\text{BTC}}\right].$$

4.4 The Output Consideration

Now, we assume that

$$Q_{\text{BTC}} = \frac{b \cdot h}{d}, \tag{8}$$

where b, h and d are time-dependent production parameters. Therefore, for Π_{BTC}, we will have

$$\begin{aligned}\Pi_{\text{BTC}} &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T'_j\right) - \ln\left(\frac{b \cdot h}{d}\right)\right] = \\ &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T'_j\right) - (\ln b + \ln h - \ln d)\right] = \\ &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T'_j\right) - \ln b - \ln h + \ln d\right].\end{aligned}$$

4.5 The Absorption Consideration

Assume that

$$U_{s_k} = \alpha_k \cdot R_k,$$

where α_k is the absorption rate of the market k, R_k, $k = 1, 2, \cdot n$, are time-dependent.

Then, Π_{BTC} will obtain the following form:

$$\Pi_{\text{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot \alpha_k \cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T'_j\right) - \ln b - \ln h + \ln d\right].$$

4.6 A Particular Model for Absorption (Bass Model)

Consider now the following particular model for the absorption rate α_k:

$$\alpha_k = \frac{1 - \exp\left[-\left(p_k + q_k\right) t\right]}{1 + \dfrac{p_k}{q_k} \cdot \exp\left[-\left(p_k + q_k\right) t\right]},$$

where p_k and q_k are time-dependent coefficients to be estimated.

Then,

$$\Pi_{\text{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{1+\dfrac{p_k}{q_k}\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T_j'\right) - \ln b - \ln h + \ln d\right].$$

5 Sensitivity Analysis - General Framework

In this section, we carry out sensitivity analysis for Π_{BTC} with respect to all parameters included in it. We start with the general case and the incorporate some specific models and assumptions.

5.1 The Case of the Generalised Model

First, consider the general case when

$$\Pi_{\text{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln V_{\text{BTC}} - \ln Q_{\text{BTC}}\right].$$

Now, before proceeding to the sensitivity analysis, we assume that there is no correlation between f' and f for none of the functions entering the expression of Π_{BTC}.

5.1.1 Sensitivity with Respect to Velocity

In this section, we study the sensitivity of Π_{BTC} with respect to V_{BTC} and V_{BTC}'. First note that since (see Appendix 1)

$$\frac{\partial \Pi_{\text{BTC}}}{\partial V_{\text{BTC}}} = -\frac{V_{\text{BTC}}'}{V_{\text{BTC}}^2}.$$

Taking into account that the denominator is always positive, we come to the evident conclusion that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial V_{\text{BTC}}} > 0 \quad \text{when} \quad V_{\text{BTC}}' < 0$$

and

$$\frac{\partial \Pi_{\text{BTC}}}{\partial V_{\text{BTC}}} < 0 \quad \text{when} \quad V_{\text{BTC}}' > 0.$$

In other words, when V_{BTC} is a decreasing (increasing) function of t, then Π_{BTC} is an increasing (decreasing) function of V_{BTC}.

On the other hand, since (see Appendix 1)

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial V'_{\mathrm{BTC}}} = \frac{1}{V_{\mathrm{BTC}}}$$

which is apparently always positive, we conclude that Π_{BTC} is an increasing (linear) function of V'_{BTC} for all values of t.

5.1.2 Sensitivity with Respect to Output

According to the derivative expression derived in Appendix 1,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q_{\mathrm{BTC}}} = \frac{Q'_{\mathrm{BTC}}}{Q^2_{\mathrm{BTC}}}.$$

Taking into account that Q_{BTC} is an increasing function of t, implying that $Q'_{\mathrm{BTC}} > 0$, we conclude that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q_{\mathrm{BTC}}} > 0$$

for all t. In other words, Π_{BTC} is always an increasing function of Q_{BTC}.

Similarly, since (see Appendix 1)

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q'_{\mathrm{BTC}}} = -\frac{1}{Q_{\mathrm{BTC}}},$$

we see that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q'_{\mathrm{BTC}}} < 0$$

for all t. Therefore, Π_{BTC} is a decreasing function of Q'_{BTC}.

5.1.3 Sensitivity with Respect to Asset Prices

Considering that (see Appendix 1)

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P'_k} = \frac{U_{s_k}}{S},$$

in which

$$S = \sum_{k=1}^{n} P_k \cdot U_{s_k},$$

it becomes evident that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P'_k} > 0$$

for all t. Thence, Π_{BTC} is an increasing function of P'_k.

On the other hand, since (see Appendix 1)

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} = \frac{U_{s_k}}{S} \cdot \frac{\partial}{\partial t}\left[\ln\left(\frac{U_{s_k}}{S}\right)\right],$$

we observe that when $\dfrac{U_{s_k}}{S}$ is an increasing function of t, then

$$\frac{\partial}{\partial t}\left[\ln\left(\frac{U_{s_k}}{S}\right)\right] > 0$$

implying that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} > 0.$$

Similarly, when $\dfrac{U_{s_k}}{S}$ is a decreasing function of t, then

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} < 0.$$

Thence, when $\dfrac{U_{s_k}}{S}$ is an increasing (decreasing) function of t, Π_{BTC} is an increasing (decreasing) function of P_k.

5.1.4 Sensitivity with Respect to Absorption (U_{s_k})

According to Appendix 1,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial U'_{s_k}} = \frac{P_k}{S} > 0$$

for all t. Therefore, Π_{BTC} is an increasing function of U'_{s_k}.

On the other hand,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial U_{s_k}} = \frac{P_k}{S} \cdot \frac{\partial}{\partial t}\left[\ln\left(\frac{P_k}{S}\right)\right]$$

providing that when $\dfrac{P_k}{S}$ is an increasing (decreasing) function of t, Π_{BTC} is an increasing (decreasing) function of U_{s_k}.

5.2 The Case of Particular Models

In this section, we are going to study the sensitivity of Π_{BTC} with respect to parameters introduced when considering the particular models above. To this end, we are going to make use of the following expression for Π_{BTC}:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{1+\dfrac{p_k}{q_k}\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T_j'\right) - \ln b - \ln h + \ln d\right].$$

5.2.1 Sensitivity with Respect to Bass Model Parameters

It is easy to see that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_k} = \frac{1}{X^2}\cdot\left[X\cdot\frac{\partial X'}{\partial p_k} - X'\cdot\frac{\partial X}{\partial p_k}\right]$$

and

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial q_k} = \frac{1}{X^2}\cdot\left[X\cdot\frac{\partial X'}{\partial q_k} - X'\cdot\frac{\partial X}{\partial q_k}\right],$$

where

$$X = \sum_{k=1}^{n} P_k\cdot R_k\cdot q_k\cdot\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]},$$

and the expressions for X', $\frac{\partial X}{\partial p_k}$, $\frac{\partial X'}{\partial p_k}$, as well as $\frac{\partial X}{\partial q_k}$, $\frac{\partial X'}{\partial q_k}$ can be found in Appendix 1.

Then, we can carry out a sensitivity analysis for Π_{BTC} with respect to p_k by exploring the inequality

$$X\cdot\frac{\partial X'}{\partial p_k} > X'\cdot\frac{\partial X}{\partial p_k},$$

for which Π_{BTC} is an increasing function of p_k, and the inequality

$$X\cdot\frac{\partial X'}{\partial p_k} < X'\cdot\frac{\partial X}{\partial p_k},$$

for which Π_{BTC} is a decreasing function of p_k.

A similar analysis is required for the inequalities

$$X\cdot\frac{\partial X'}{\partial q_k} > X'\cdot\frac{\partial X}{\partial q_k},$$

for which Π_{BTC} is an increasing function of q_k, and the inequality

$$X \cdot \frac{\partial X'}{\partial q_k} < X' \cdot \frac{\partial X}{\partial q_k},$$

for which Π_{BTC} is a decreasing function of q_k.

On the other hand, due to simpler expressions obtained for $\dfrac{\partial \Pi_{\text{BTC}}}{\partial p'_k}$ and $\dfrac{\partial \Pi_{\text{BTC}}}{\partial q'_k}$ in Appendix 1, the corresponding sensitivities are a way much easier to investigate. Indeed,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial p'_k} = \frac{1}{X} \cdot \frac{P_k \cdot R_k \cdot q_k \cdot \exp\left[-\left(p_k + q_k\right) t\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right) t\right]\right)^2} \left[\left(q_k + p_k\right) \cdot t + p_k \cdot t \cdot \exp\left[-\left(p_k + q_k\right) t\right] - 1\right],$$

providing that when

$$\left(q_k + p_k\right) \cdot t + p_k \cdot t \cdot \exp\left[-\left(p_k + q_k\right) t\right] - 1 > 0$$

Π_{BTC} is an increasing function of p'_k. Otherwise, Π_{BTC} is a decreasing function of p'_k.

Moreover, since (see Appendix 1)

$$\frac{\partial X'}{\partial p'_k} = \frac{\partial X'}{\partial q'_k},$$

then, Π_{BTC} is an increasing or decreasing function of p'_k and q'_k simultaneously.

Apparently, for large values of t, such that

$$\left(q_k + p_k\right) \cdot t - 1 > 0$$

or

$$t > \frac{1}{q_k + p_k},$$

the term with exponent decays fast and the above inequality is satisfied. In other words, for $t > \dfrac{1}{q_k + p_k}$, Π_{BTC} is an increasing function of p'_k and q'_k simultaneously.

5.2.2 Sensitivity Analysis with Respect to Volume

Since (see Appendix 1)

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} = \frac{W_k}{R} \cdot \frac{\partial}{\partial t} \left[\ln \frac{W_k}{R}\right],$$

where

$$W_k = P_k \cdot \frac{1 - \exp\left[-\left(p_k + q_k\right) t\right]}{1 + \dfrac{p_k}{q_k} \cdot \exp\left[-\left(p_k + q_k\right) t\right]}, \quad R = \sum_{k=1}^{n} P_k \cdot R_k,$$

then, we straightforwardly conclude that when $\dfrac{W_k}{R}$ is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of R_k.

On the other hand,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R'_k} = \frac{W_k}{R},$$

leading to the conclusion that Π_{BTC} is an increasing function of R'_k.

5.2.3 Sensitivity with Respect to Velocity

Considering that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial T'_j} = -\frac{1}{\left(T'_j\right)^2} \cdot \sum_{m=1}^{n} T''_m,$$

and taking into account that the denominator is positive for all t, then it is obvious that

$$\sum_{m=1}^{n} T''_m < 0$$

implies that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial T'_j} > 0,$$

i.e., Π_{BTC} is an increasing function of T'_j. On the other hand,

$$\sum_{m=1}^{n} T''_m > 0$$

implies that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial T'_j} < 0,$$

i.e., Π_{BTC} is a decreasing function of T'_j.

5.2.4 Sensitivity with Respect to Output Parameters

Taking into account that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial b} = \frac{b'}{b^2},$$

the denominator of which is positive for all t, we conclude that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial b} > 0$$

when $b' > 0$. Similarly,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial b} < 0$$

when $b' < 0$.

Hence, when b is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of b.

Using the expression (see Appendix 1)

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial h} = \frac{h'}{h^2},$$

we conclude that, when h is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of h.

However, since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial d} = -\frac{d'}{d^2},$$

we come to the conclusion that when d is an increasing (decreasing) function of t, then Π_{BTC} is a decreasing (increasing) function of d.

6 Sensitivity Analysis - Single Asset Case

In this section, we consider the single asset case, i.e., we assume that $n = 1$. As in previous section, we split the current one into two subsections for sensitivity analysis in the general case and in the case when all particular models are considered.

6.1 General case

In the general case of single asset market, we have

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(P_1 \cdot U_{s_1}\right) + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right]$$

or

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln P_1 + \ln U_{s_1} + \ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right].$$

Note that since the dependence of Π_{BTC} on V_{BTC} and Q_{BTC} has not been changed by this assumption, the sensitivity of Π_{BTC} with respect to those variables are as it has been established in the previous section. Recall that

1. When V_{BTC} is a decreasing (increasing) function of t, then Π_{BTC} is an increasing (decreasing) function of V_{BTC}.

2. Π_{BTC} is an increasing function of V'_{BTC} for all values of t.

3. Π_{BTC} is an increasing function of Q_{BTC} for all values of t.

4. Π_{BTC} is a decreasing function of Q'_{BTC} for all values of t.

6.1.1 Sensitivity with respect to Asset Price

Since

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_1} = -\frac{P_1'}{P_1^2},$$

and P_1^2 is positive for all values of t, we conclude that when $P_1' > 0$, $\dfrac{\partial \Pi_{\text{BTC}}}{\partial P_1} < 0$ and when $P_1' < 0$, $\dfrac{\partial \Pi_{\text{BTC}}}{\partial P_1} > 0$.

In other words, when P_1 is an increasing (decreasing) function of t, then Π_{BTC} is a decreasing (increasing) function of P_1.

This kind of behavior can be seen also on real data plotted on Figure 1.

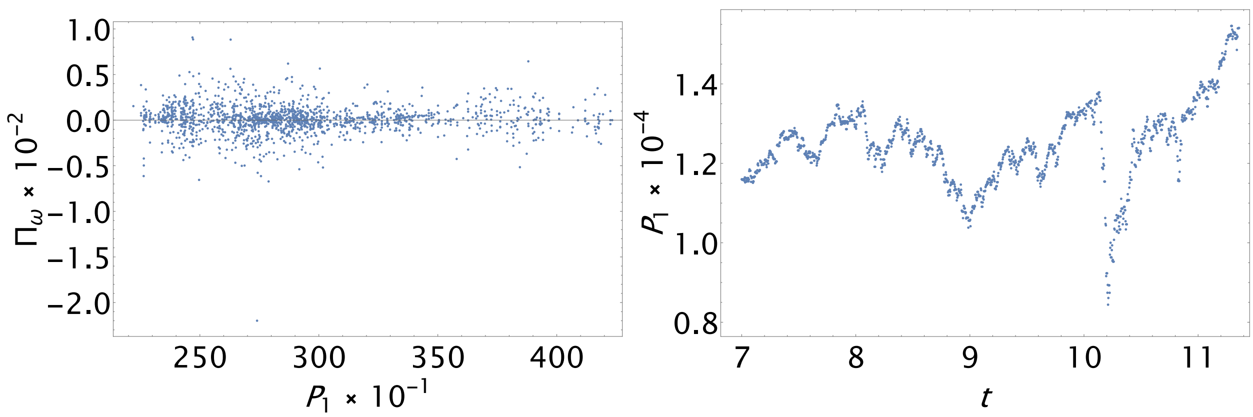

Figure 1: Dependence of Π_{BTC} on P_1 (left) and P_1 on t (right)

On the other hand,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_1'} = \frac{1}{P_1},$$

then, taking into consideration that $P_1 > 0$ for all values of t, we conclude that Π_{BTC} is an increasing function of P_1' for all values of t.

6.1.2 Sensitivity with Respect to the Absorption (U_{s_1})

Due to the symmetry of Π_{BTC} with respect to P_1 and U_{s_1}, similar conclusions hold for the sensitivity of Π_{BTC} with respect to U_{s_1}. Indeed, since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial U_{s_1}} = -\frac{U_{s_1}'}{U_{s_1}^2},$$

we conclude that when U_{s_1} is an increasing (decreasing) function of t, then Π_{BTC} is a decreasing (increasing) function of U_{s_1}.

Similarly, since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial U_{s_1}'} = \frac{1}{U_{s_1}} > 0,$$

then Π_{BTC} is an increasing function of U_{s_1}' for all values of t.

6.2 The Case of Particular Models

In the case of a single asset, consideration of particular models leads to the following expression for Π_{BTC}:

$$\Pi_{\mathrm{BTC}} = \frac{\partial}{\partial t}\left[\ln\left(P_1 \cdot \frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{1+\dfrac{p_1}{q_1}\cdot\exp\left[-\left(p_1+q_1\right)t\right]}\cdot R_1\right) + \ln T_1' - \ln b - \ln h + \ln d\right].$$

or

$$
\begin{aligned}
\Pi_{\mathrm{BTC}} &= \frac{\partial}{\partial t}\left[\ln P_1 + \ln\left(\frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{1+\dfrac{p_1}{q_1}\cdot\exp\left[-\left(p_1+q_1\right)t\right]}\right) + \ln R_1 + \ln T_1' - \ln b - \ln h + \ln d\right] = \\
&= \frac{\partial}{\partial t}\left[\ln P_1 + \ln\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right) - \ln\left(1+\frac{p_1}{q_1}\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)\right] + \\
&+ \frac{\partial}{\partial t}\left[\ln R_1 + \ln T_1' - \ln b - \ln h + \ln d\right] = \\
&= \frac{\partial}{\partial t}\left[\ln P_1 + \ln\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right) - \ln\left(\frac{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]}{q_1}\right)\right] + \\
&+ \frac{\partial}{\partial t}\left[\ln R_1 + \ln T_1' - \ln b - \ln h + \ln d\right] = \\
&= \frac{\partial}{\partial t}\left[\ln P_1 + \ln\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right) - \ln\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right) + \ln q_1\right] + \\
&+ \frac{\partial}{\partial t}\left[\ln R_1 + \ln T_1' - \ln b - \ln h + \ln d\right].
\end{aligned}
$$

Again, since the dependence of Π_{BTC} with respect to b, h, d and their derivatives has not been changed, the sensitivity analysis of Π_{BTC} with respect to these variables will not be affected by the single asset assumption. Therefore, we are going to summarize the main conclusions made in the previous section.

1. When b is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of b.

2. Π_{BTC} is a decreasing function of b' for all values of t.

3. When h is an increasing (decreasing) function of t, then Π_{BTC} is an increasing (decreasing) function of h.

4. Π_{BTC} is a decreasing function of h' for all values of t.

5. When d is an increasing (decreasing) function of t, then Π_{BTC} is a decreasing (increasing) function of d.

These conclusions can be directly observed on real data plotted on Figure 2 and 3. In intervals of t where h increases, Π_{BTC} increases as well. On the other hand, in intervals of t where d increases, Π_{BTC} decreases.

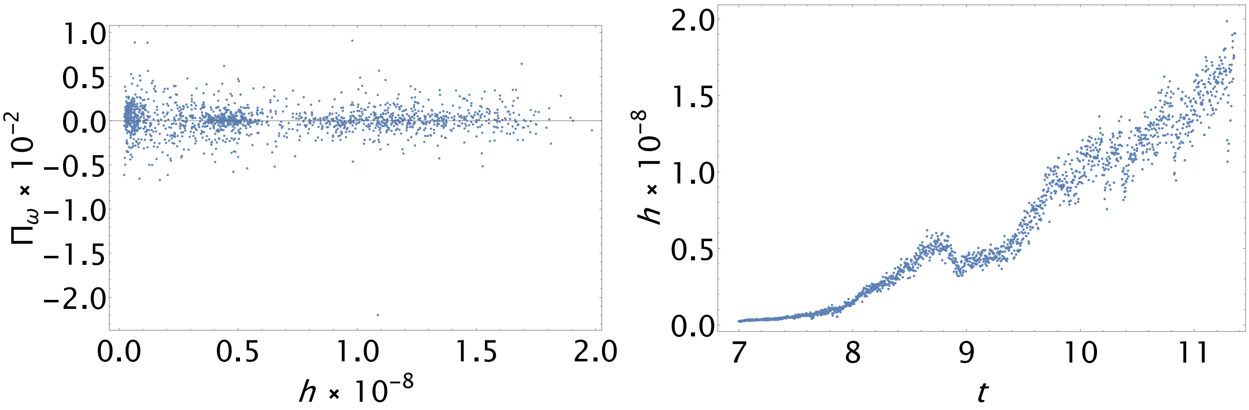

Figure 2: Dependence of Π_{BTC} on h (left) and h on t (right)

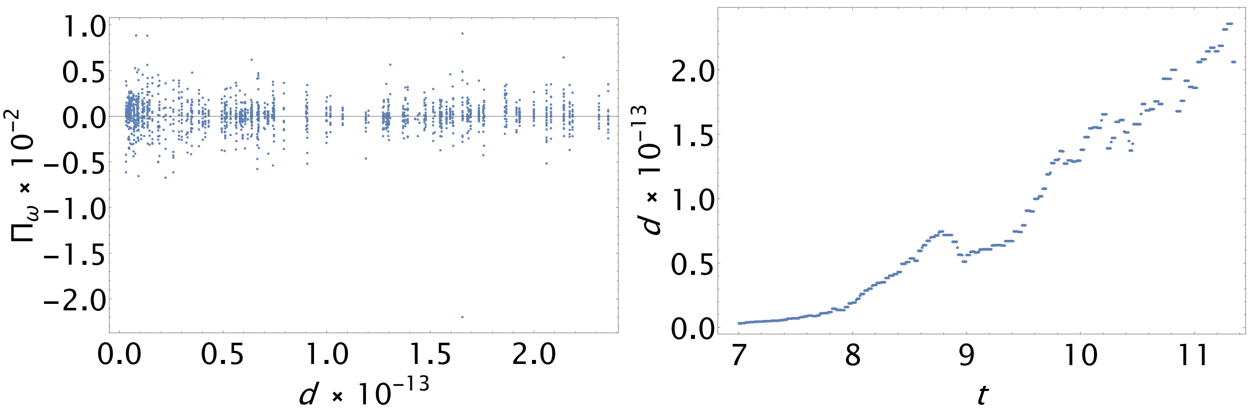

Figure 3: Dependence of Π_{BTC} on d (left) and d on t (right)

6.2.1 Sensitivity with Respect to Asset Volume

In this case, we have

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial R_1} = -\frac{R_1'}{R_1^2}.$$

Since R_1^2 is positive for all values of t, then when $R_1' > 0$, $\dfrac{\partial \Pi_{\mathrm{BTC}}}{\partial R_1} < 0$ and when $R_1' < 0$, $\dfrac{\partial \Pi_{\mathrm{BTC}}}{\partial R_1} > 0$.

Thus, when R_1 is an increasing (decreasing) function of t, then Π_{BTC} is a decreasing (increasing) function of R_1.

This kind of behavior can be seen also on real data plotted on Figure 4.

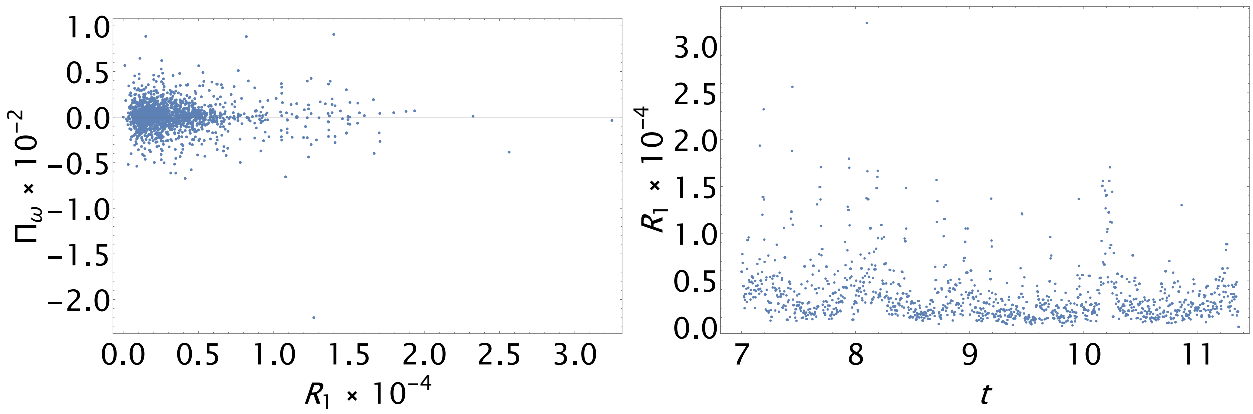

Figure 4: Dependence of Π_{BTC} on R_1 (left) and R_1 on t (right)

Moreover, since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial R_1'} = \frac{1}{R_1}.$$

Since $R_1 > 0$ for all values of t, we conclude that Π_{BTC} is an increasing function of R_1' for all values of t.

6.2.2 Sensitivity with Respect to the Transactions

Since

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial T_1'} = -\frac{T_1''}{(T_1')^2},$$

we conclude that when $T_1'' < 0$, $\dfrac{\partial \Pi_{\mathrm{BTC}}}{\partial T_1'} > 0$ and when $T_1'' > 0$, $\dfrac{\partial \Pi_{\mathrm{BTC}}}{\partial T_1'} < 0$.

In other words, when T_1' is a concave function of t, Π_{BTC} is an increasing function of T_1', and when T_1' is a convex function of t, Π_{BTC} is a decreasing function of T_1'.

Similar pattern can be seen on Figure

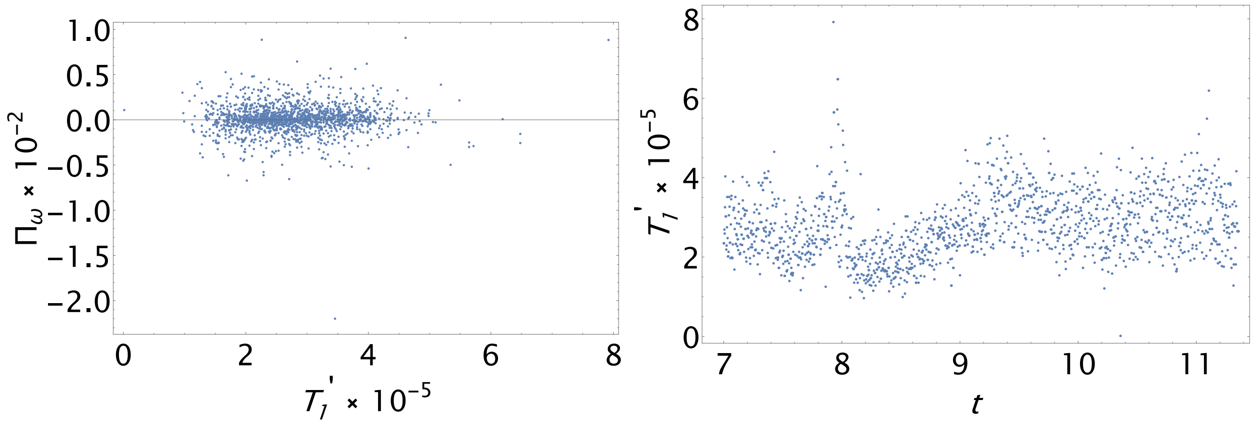

Figure 5: Dependence of Π_{BTC} on d (left) and d on t (right)

6.3 Sensitivity with Respect to Bass Model Parameters

As we obtained in Appendix 1,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_1} = \frac{q_1 \cdot \alpha_1' \cdot \exp\left[-\left(p_1+q_1\right)t\right]}{\alpha_1^2 \cdot \left(q_1 + p_1 \cdot \exp\left[-\left(p_1+q_1\right)t\right]\right)^2} \times$$
$$\times \left[\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right) \cdot \left(1-p_1 \cdot t\right) - t \cdot \left(q_1 + p_1 \cdot \exp\left[-\left(p_1+q_1\right)t\right]\right)\right].$$

Since α_1 is a non-decreasing function of t for any values of p_1 and q_1, then $\alpha_1' \geq 0$, so that the term

$$\frac{q_1 \cdot \alpha_1' \cdot \exp\left[-\left(p_1+q_1\right)t\right]}{\alpha_1^2 \cdot \left(q_1 + p_1 \cdot \exp\left[-\left(p_1+q_1\right)t\right]\right)^2} \geq 0$$

and the sign of $\dfrac{\partial \Pi_{\mathrm{BTC}}}{\partial p_1}$ is equal to the sign of the term

$$\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right) \cdot \left(1-p_1 \cdot t\right) - t \cdot \left(q_1 + p_1 \cdot \exp\left[-\left(p_1+q_1\right)t\right]\right).$$

This expression can be simplified further as below:

$$\begin{aligned}
&\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right) \cdot \left(1-p_1 \cdot t\right) - t \cdot \left(q_1 + p_1 \cdot \exp\left[-\left(p_1+q_1\right)t\right]\right) = \\
&\quad = 1 - p_1 \cdot t - \exp\left[-\left(p_1+q_1\right)t\right] + p_1 \cdot t \cdot \exp\left[-\left(p_1+q_1\right)t\right] - \\
&\quad - t \cdot q_1 - t \cdot p_1 \cdot \exp\left[-\left(p_1+q_1\right)t\right] = \\
&\quad = 1 - p_1 \cdot t - \exp\left[-\left(p_1+q_1\right)t\right] - t \cdot q_1 = \\
&\quad = 1 - \left(p_1+q_1\right) \cdot t - \exp\left[-\left(p_1+q_1\right)t\right].
\end{aligned}$$

Thence, we finally arrive at

$$\operatorname{sign}\left(\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_1}\right) = \operatorname{sign}\left(1 - \left(p_1+q_1\right) \cdot t - \exp\left[-\left(p_1+q_1\right)t\right]\right).$$

Apparently, the maximum of this expression is attained at $t = 0$:

$$1 - (p_1 + q_1) \cdot 0 - \exp\left[-(p_1 + q_1)\, 0\right] = 1 - 0 - 1 = 0,$$

and when $t > 0$,

$$1 - (p_1 + q_1) \cdot t - \exp\left[-(p_1 + q_1)\, t\right] < 0$$

due to negativity of the second and third terms. Therefore, $\dfrac{\partial \Pi_{\mathrm{BTC}}}{\partial p_1} < 0$ for $t > 0$.

In other words, Π_{BTC} is a decreasing function of p_1 for all $t > 0$.

Similarly, taking into account that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial q_1} = -\frac{\alpha_1' \cdot \exp\left[-(p_1 + q_1)\, t\right]}{\alpha_1^2 \cdot \left(q_1 + p_1 \cdot \exp\left[-(p_1 + q_1)\, t\right]\right)^2} \cdot \left(p_1 \cdot \left(1 - \exp\left[-(p_1 + q_1)\, t\right]\right) + q_1 \cdot (p_1 + q_1) \cdot t\right).$$

As it was pointed out above, $\alpha_1' > 0$. Taking into account that

$$1 - \exp\left[-(p_1 + q_1)\, t\right] > 0$$

for all $t > 0$, we conclude that

$$p_1 \cdot \left(1 - \exp\left[-(p_1 + q_1)\, t\right]\right) + q_1 \cdot (p_1 + q_1) \cdot t > 0.$$

Therefore,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial q_1} < 0$$

for all $t > 0$. In other words, Π_{BTC} is a decreasing function of q_1 for all $t > 0$.

Figure 6 below shows the evolution of p_1 and q_1 in time corresponding to α_1 plotted on Figure 7.

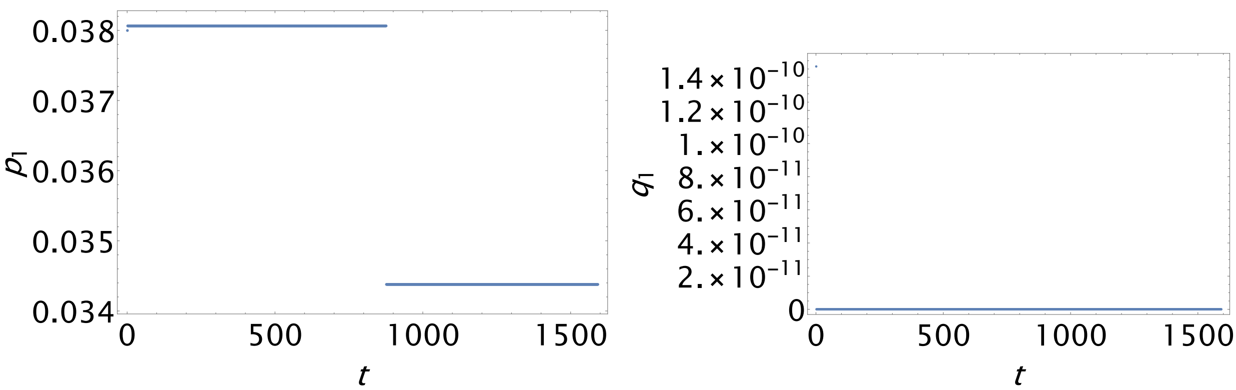

Figure 6: Dependence of p_1 on t (left) and q_1 on t (right)

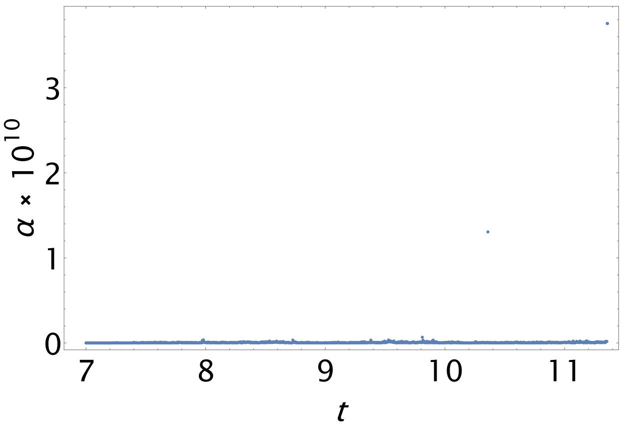

Figure 7: Dependence of α on t

7 An Initial Solution to Zero Theorem

7.1 Simplified Variant

Let us now consider a simplified variant of the assumption regarding the absorption rate considered in Section 4.5. More specifically, we assume that

$$U_{s_k} = \alpha \cdot R_k,$$

where α is a time-dependent function independent of k.

Then, Π_ω will obtain the following form:

$$\begin{aligned}
\Pi_\omega &= \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot \alpha \cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T'_j\right) - \ln b - \ln h + \ln d\right] = \\
&= \frac{\partial}{\partial t}\left[\ln\left(\alpha \cdot \sum_{k=1}^{n} P_k \cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T'_j\right) - \ln b - \ln h + \ln d\right] = \\
&= \frac{\partial}{\partial t}\left[\ln \alpha + \ln\left(\sum_{k=1}^{n} P_k \cdot R_k\right) + \ln\left(\frac{1}{n}\sum_{j=1}^{n} T'_j\right) - \ln b - \ln h + \ln d\right].
\end{aligned}$$

Taking into account that

$$\frac{\partial \Pi_\omega}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\frac{\partial \ln \alpha}{\partial t}\right] = \frac{\partial}{\partial \alpha}\left(\frac{\alpha'}{\alpha}\right) = -\frac{\alpha'}{\alpha^2}$$

and that α^2 is positive for all values of t, we conclude that when $\alpha' > 0$,

$$\frac{\partial \Pi_\omega}{\partial \alpha} < 0,$$

and when $\alpha' < 0$,

$$\frac{\partial \Pi_\omega}{\partial \alpha} > 0.$$

In other words, when α is a decreasing (increasing) function of t, then Π_ω is an increasing (respectively, decreasing) function of α.

It is even simpler in the case of sensitivity analysis with respect to α'. Indeed, since

$$\frac{\partial \Pi_\omega}{\partial \alpha'} = \frac{\partial}{\partial \alpha'}\left(\frac{\alpha'}{\alpha}\right) = \frac{1}{\alpha}.$$

Since $\alpha > 0$ for all values of t, we come to the conclusion that Π_ω is an increasing function of α'.

7.2 Agent and Action Space Design

7.3 Reward Function Design

7.4 Range of Learning Algorithms to Evaluate

7.5 Training and Testing Split

7.6 Results and Discussions

8 On-Going Research and Proof

9 Conclusion

Appendix 1 - Mathematical Derivations

General Framework

With Respect to Velocity

In this appendix, we present derivations of some partial derivatives required for sensitivity analysis carried out in sections above.

In order to study the sensitivity of Π_{BTC} with respect to V_{BTC}, we compute

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial V_{\mathrm{BTC}}} &= \frac{\partial}{\partial V_{\mathrm{BTC}}}\left[\frac{\partial \ln V_{\mathrm{BTC}}}{\partial t} + \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) - \ln Q_{\mathrm{BTC}}\right]\right] = \\ &= \frac{\partial}{\partial V_{\mathrm{BTC}}}\left[\frac{\partial \ln V_{\mathrm{BTC}}}{\partial t}\right] = \frac{\partial}{\partial V_{\mathrm{BTC}}}\left[\frac{V'_{\mathrm{BTC}}}{V_{\mathrm{BTC}}}\right] = -\frac{V'_{\mathrm{BTC}}}{V^2_{\mathrm{BTC}}}.\end{aligned}$$

On the other hand, the sensitivity of Π_{BTC} with respect to V'_{BTC} is carried out on the basis of the derivative

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial V'_{\mathrm{BTC}}} &= \frac{\partial}{\partial V'_{\mathrm{BTC}}}\left[\frac{\partial \ln V_{\mathrm{BTC}}}{\partial t} + \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) - \ln Q_{\mathrm{BTC}}\right]\right] = \\ &= \frac{\partial}{\partial V'_{\mathrm{BTC}}}\left[\frac{\partial \ln V_{\mathrm{BTC}}}{\partial t}\right] = \frac{\partial}{\partial V'_{\mathrm{BTC}}}\left[\frac{V'_{\mathrm{BTC}}}{V_{\mathrm{BTC}}}\right] = \frac{1}{V_{\mathrm{BTC}}}.\end{aligned}$$

With Respect to Output

In this case, we have

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q_{\mathrm{BTC}}} &= \frac{\partial}{\partial Q_{\mathrm{BTC}}}\left[-\frac{\partial \ln Q_{\mathrm{BTC}}}{\partial t} + \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln V_{\mathrm{BTC}}\right]\right] = \\ &= -\frac{\partial}{\partial Q_{\mathrm{BTC}}}\left[\frac{\partial \ln Q_{\mathrm{BTC}}}{\partial t}\right] = -\frac{\partial}{\partial Q_{\mathrm{BTC}}}\left[\frac{Q'_{\mathrm{BTC}}}{Q_{\mathrm{BTC}}}\right] = \frac{Q'_{\mathrm{BTC}}}{Q^2_{\mathrm{BTC}}}.\end{aligned}$$

In the same way, we compute

$$\begin{aligned}\frac{\partial \Pi_{\mathrm{BTC}}}{\partial Q'_{\mathrm{BTC}}} &= \frac{\partial}{\partial Q_{\mathrm{BTC}}}\left[-\frac{\partial \ln Q_{\mathrm{BTC}}}{\partial t} + \frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right) + \ln V_{\mathrm{BTC}}\right]\right] = \\ &= -\frac{\partial}{\partial Q'_{\mathrm{BTC}}}\left[\frac{\partial \ln Q_{\mathrm{BTC}}}{\partial t}\right] = -\frac{\partial}{\partial Q'_{\mathrm{BTC}}}\left[\frac{Q'_{\mathrm{BTC}}}{Q_{\mathrm{BTC}}}\right] = -\frac{1}{Q_{\mathrm{BTC}}}.\end{aligned}$$

With Respect to Asset Price

We start with simplifying the corresponding term in the expression of Π_{BTC},

$$\frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right)\right] = \frac{\sum_{k=1}^{n} (P_k \cdot U_{s_k})'}{\sum_{k=1}^{n} P_k \cdot U_{s_k}} = \frac{1}{S} \cdot \sum_{k=1}^{n} \left(P_k' \cdot U_{s_k} + P_k \cdot U_{s_k}'\right) =$$
$$= \frac{1}{S} \cdot \sum_{k=1}^{n} P_k' \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U_{s_k}',$$

in which

$$S = \sum_{k=1}^{n} P_k \cdot U_{s_k}.$$

As a matter of fact, Π_{BTC} is linear in P_k'. Therefore, the derivative of Π_{BTC} with respect to that variable is easy to compute. Indeed,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_k'} = \frac{\partial}{\partial P_k'}\left[\frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right)\right] + \frac{\partial}{\partial t}\left[\ln V_{\text{BTC}} - \ln Q_{\text{BTC}}\right]\right] =$$
$$= \frac{\partial}{\partial P_k'}\left[\frac{1}{S} \cdot \sum_{k=1}^{n} P_k' \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U_{s_k}'\right] = \frac{U_{s_k}}{S}.$$

On the other hand,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_k} = \frac{\partial}{\partial P_k}\left[\frac{\partial}{\partial t}\left[\ln\left(\sum_{k=1}^{n} P_k \cdot U_{s_k}\right)\right] + \frac{\partial}{\partial t}\left[\ln V_{\text{BTC}} - \ln Q_{\text{BTC}}\right]\right] =$$
$$= \frac{\partial}{\partial P_k}\left[\frac{1}{S} \cdot \sum_{k=1}^{n} P_k' \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U_{s_k}'\right] =$$
$$= \frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P_k' \cdot U_{s_k} + \frac{1}{S} \cdot \frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P_k' \cdot U_{s_k}\right] +$$
$$+ \frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P_k \cdot U_{s_k}' + \frac{1}{S} \cdot \frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P_k \cdot U_{s_k}'\right].$$

Let us compute the derivatives above one by one:

$$\frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] = -\frac{1}{S^2} \cdot \frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P_k \cdot U_{s_k}\right] = -\frac{U_{s_k}}{S^2},$$

$$\frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P_k' \cdot U_{s_k}\right] = 0,$$

$$\frac{\partial}{\partial M_k}\left[\sum_{k=1}^{n} P_k \cdot U'_{s_k}\right] = U'_{s_k}.$$

Substituting these expressions into $\dfrac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k}$, we obtain

$$\begin{aligned}
\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} &= \frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \frac{\partial}{\partial P_k}\left[\sum_{k=1}^{n} P'_k \cdot U_{s_k}\right] + \frac{\partial}{\partial P_k}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k} = \\
&= -\frac{U_{s_k}}{S^2} \cdot \sum_{l=1}^{n} P'_l \cdot U_{s_l} + \frac{U'_{s_k}}{S} - \frac{U_{s_k}}{S^2} \cdot \sum_{l=1}^{n} P_l \cdot U'_{s_l} = \\
&= \frac{U_{s_k}}{S} \cdot \left[\frac{U'_{s_k}}{U_{s_k}} \cdot S - \frac{\partial}{\partial t}\left(\sum_{l=1}^{n} P_l \cdot U_{s_l}\right)\right] = \\
&= \frac{U_{s_k}}{S^2} \cdot S \cdot \left[\frac{U'_{s_k}}{U_{s_k}} - \frac{\partial \ln S}{\partial t}\right] = \\
&= \frac{U_{s_k}}{S} \cdot \frac{\partial}{\partial t}\left[\ln U_{s_k} - \ln S\right]
\end{aligned}$$

or

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial P_k} = \frac{U_{s_k}}{S} \cdot \frac{\partial}{\partial t}\left[\ln\left(\frac{U_{s_k}}{S}\right)\right].$$

With Respect to Absorption

In this case as well, we are going to use the expression

$$\Pi_{\mathrm{BTC}} = \frac{1}{S} \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k} + \frac{\partial}{\partial t}\left[\ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right].$$

Then, obviously,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial U'_{s_k}} = \frac{\partial}{\partial U'_{s_k}}\left[\frac{1}{S} \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k}\right] = \frac{P_k}{S}.$$

On the other hand,

$$\begin{aligned}
\frac{\partial \Pi_{\mathrm{BTC}}}{\partial U_{s_k}} &= \frac{\partial}{\partial U_{s_k}}\left[\frac{1}{S} \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k} + \frac{\partial}{\partial t}\left[\ln V_{\mathrm{BTC}} - \ln Q_{\mathrm{BTC}}\right]\right] = \\
&= \frac{\partial}{\partial U_{s_k}}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P'_k \cdot U_{s_k} + \frac{1}{S} \cdot \frac{\partial}{\partial U_{s_k}}\left[\sum_{k=1}^{n} P'_k \cdot U_{s_k}\right] + \\
&+ \frac{\partial}{\partial U_{s_k}}\left[\frac{1}{S}\right] \cdot \sum_{k=1}^{n} P_k \cdot U'_{s_k} + \frac{1}{S} \cdot \frac{\partial}{\partial U_{s_k}}\left[\sum_{k=1}^{n} P_k \cdot U'_{s_k}\right].
\end{aligned}$$

Apparently,

$$\frac{\partial}{\partial U_{s_k}}\left[\frac{1}{S}\right]=-\frac{P_k}{S^2},$$

$$\frac{\partial}{\partial U_{s_k}}\left[\sum_{k=1}^{n}P_k'\cdot U_{s_k}\right]=P_k',$$

$$\frac{\partial}{\partial U_{s_k}}\left[\sum_{k=1}^{n}P_k\cdot U_{s_k}'\right]=0.$$

Therefore,

$$\begin{aligned}\frac{\partial\Pi_{\mathrm{BTC}}}{\partial U_{s_k}}&=-\frac{P_k}{S^2}\cdot\sum_{k=1}^{n}P_k'\cdot U_{s_k}-\frac{P_k}{S^2}\cdot\sum_{k=1}^{n}P_k\cdot U_{s_k}'+\frac{P_k'}{S}=\\&=\frac{P_k}{S}\cdot\left[-\frac{1}{S}\cdot\sum_{k=1}^{n}P_k'\cdot U_{s_k}-\frac{1}{S}\cdot\sum_{k=1}^{n}P_k\cdot U_{s_k}'+\frac{P_k'}{P_k}\right]=\\&=\frac{P_k}{S}\cdot\left[-\frac{1}{S}\cdot\left(\sum_{k=1}^{n}P_k'\cdot U_{s_k}+\sum_{k=1}^{n}P_k\cdot U_{s_k}'\right)+\frac{P_k'}{P_k}\right]=\\&=\frac{P_k}{S}\cdot\left[\frac{P_k'}{P_k}-\frac{1}{S}\cdot\frac{\partial}{\partial t}\left(\sum_{k=1}^{n}P_k\cdot U_{s_k}\right)\right]=\\&=\frac{P_k}{S}\cdot\left[\frac{P_k'}{P_k}-\frac{P_k'}{S}\right]=\frac{P_k}{S}\cdot\frac{\partial}{\partial t}\left[\ln P_k-\ln S\right]\end{aligned}$$

or

$$\frac{\partial\Pi_{\mathrm{BTC}}}{\partial U_{s_k}}=\frac{P_k}{S}\cdot\frac{\partial}{\partial t}\left[\ln\left(\frac{P_k}{S}\right)\right].$$

With Respect to Bass Model Parameters

It is easy to see that

$$\frac{\partial\Pi_{\mathrm{BTC}}}{\partial p_k}=\frac{\partial}{\partial p_k}\left[\frac{X'}{X}\right]=\frac{1}{X}\cdot\frac{\partial X'}{\partial p_k}+X'\cdot\frac{\partial}{\partial p_k}\left[\frac{1}{X}\right]=\frac{1}{X}\cdot\frac{\partial X'}{\partial p_k}-\frac{X'}{X^2}\cdot\frac{\partial X}{\partial p_k},$$

and, similarly,

$$\frac{\partial\Pi_{\mathrm{BTC}}}{\partial q_k}=\frac{1}{X}\cdot\frac{\partial X'}{\partial q_k}-\frac{X'}{X^2}\cdot\frac{\partial X}{\partial q_k},$$

where

$$X=\sum_{k=1}^{n}P_k\cdot R_k\cdot q_k\cdot\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}.$$

Therefore,

$$X' = \frac{\partial X}{\partial t} = \sum_{k=1}^{n} (P_k \cdot R_k)' \cdot q_k \cdot \frac{1 - \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} +$$
$$+ \sum_{k=1}^{n} P_k \cdot R_k \cdot q_k' \cdot \frac{1 - \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} +$$
$$+ \sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \left(\frac{1 - \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} \right)'.$$

Since p_k and q_k depend on t, we compute

$$\left(\frac{1 - \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} \right)' = \frac{(1 - \exp[-(p_k + q_k)\, t])'}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} -$$
$$- \frac{(1 - \exp[-(p_k + q_k)\, t])\,(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])'}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2}.$$

Now, making use of the chain rule, it is easy to evaluate

$$(1 - \exp[-(p_k + q_k)\, t])' = -\exp[-(p_k + q_k)\, t] \cdot [-(p_k + q_k)\, t]' =$$
$$= [p_k + q_k + (p_k' + q_k') \cdot t] \cdot \exp[-(p_k + q_k)\, t],$$

and

$$(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])' = q_k' + p_k' \cdot \exp[-(p_k + q_k)\, t] +$$
$$+ p_k \cdot \exp[-(p_k + q_k)\, t] \cdot [-(p_k + q_k) \cdot t]' =$$
$$= q_k' + p_k' \cdot \exp[-(p_k + q_k)\, t] +$$
$$- p_k \cdot \exp[-(p_k + q_k)\, t] \cdot [p_k + q_k + (p_k' + q_k') \cdot t] =$$
$$= q_k' + [p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t]) \cdot \exp[-(p_k + q_k)\, t].$$

Hence,

$$\left(\frac{1 - \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} \right)' = \frac{(p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp[-(p_k + q_k)\, t]}{q_k + p_k \cdot \exp[-(p_k + q_k)\, t]} -$$
$$- \frac{q_k' + [p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t]) \cdot \exp[-(p_k + q_k)\, t]}{(q_k + p_k \cdot \exp[-(p_k + q_k)\, t])^2}.$$

Thus,

$$X' = \frac{\partial X}{\partial t} = \sum_{k=1}^{n} (P_k \cdot R_k)' \cdot q_k \cdot \frac{1 - \exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]} +$$

$$+ \sum_{k=1}^{n} P_k \cdot R_k \cdot q_k' \cdot \frac{1 - \exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]} +$$

$$+ \sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{(p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]} -$$

$$- \sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{q_k' + \left[p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)\right] \cdot \exp\left[-(p_k + q_k)\, t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]\right)^2}.$$

Now, in order to continue the sensitivity analysis of Π_{BTC} with respect to p_k, we need to evaluate the derivatives $\dfrac{\partial X}{\partial p_k}$ and $\dfrac{\partial X'}{\partial p_k}$. First,

$$\frac{\partial X}{\partial p_k} = \frac{\partial}{\partial p_k}\left[P_k \cdot R_k \cdot q_k \cdot \frac{1 - \exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]}\right] =$$

$$= P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k}\left[\frac{1 - \exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]}\right],$$

in which

$$\frac{\partial}{\partial p_k}\left[\frac{1 - \exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]}\right] = \frac{1}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]} \frac{\partial}{\partial p_k}\left[1 - \exp\left[-(p_k + q_k)\, t\right]\right] +$$

$$+ \left(1 - \exp\left[-(p_k + q_k)\, t\right]\right) \cdot \frac{\partial}{\partial p_k}\left[\frac{1}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]}\right] =$$

$$= -\frac{\exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]} \cdot \frac{\partial}{\partial p_k}\left[-(p_k + q_k)\, t\right] -$$

$$- \frac{1 - \exp\left[-(p_k + q_k)\, t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]\right)^2} \cdot \frac{\partial}{\partial p_k}\left(q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]\right) =$$

$$= \frac{t \cdot \exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]} -$$

$$- \frac{1 - \exp\left[-(p_k + q_k)\, t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]\right)^2} \cdot \left(\exp\left[-(p_k + q_k)\, t\right] + p_k \cdot \frac{\partial \exp\left[-(p_k + q_k)\, t\right]}{\partial p_k}\right) =$$

$$= \frac{t \cdot \exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]} -$$

$$- \frac{\left(1 - \exp\left[-(p_k + q_k)\, t\right]\right) \cdot \exp\left[-(p_k + q_k)\, t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]\right)^2} \cdot \left(1 + p_k \cdot \frac{\partial}{\partial p_k}\left[-(p_k + q_k)\, t\right]\right) =$$

$$= \frac{t \cdot \exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]} - \frac{\left(1 - \exp\left[-(p_k + q_k)\, t\right]\right) \cdot \exp\left[-(p_k + q_k)\, t\right] \cdot (1 - p_k \cdot t)}{\left(q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]\right)^2} =$$

$$= \frac{\exp\left[-(p_k + q_k)\, t\right]}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]}\left[t - \frac{\left(1 - \exp\left[-(p_k + q_k)\, t\right]\right) \cdot (1 - p_k \cdot t)}{q_k + p_k \cdot \exp\left[-(p_k + q_k)\, t\right]}\right] =$$

$$= \frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$

$$\times\left[t\cdot q_k+t\cdot p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]-\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\left(1-p_k\cdot t\right)\right]=$$

$$= \frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$

$$\times\left[t\cdot q_k+t\cdot p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]-\left(1-p_k\cdot t-\exp\left[-\left(p_k+q_k\right)t\right]+p_k\cdot t\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)\right]=$$

$$= \frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\cdot\left[\left(p_k+q_k\right)\cdot t+\exp\left[-\left(p_k+q_k\right)t\right]-1\right],$$

or

$$\frac{\partial}{\partial p_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right]=\frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$

$$\times\left[\left(p_k+q_k\right)\cdot t+\exp\left[-\left(p_k+q_k\right)t\right]-1\right],$$

leading to

$$\frac{\partial X}{\partial p_k}=P_k\cdot R_k\cdot q_k\cdot\frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\cdot\left[\left(p_k+q_k\right)\cdot t+\exp\left[-\left(p_k+q_k\right)t\right]-1\right],$$

Now, we evaluate the derivative $\dfrac{\partial X'}{\partial p_k}$:

$$\frac{\partial X'}{\partial p_k}=\left(P_k\cdot R_k\cdot q_k\right)'\cdot\frac{\partial}{\partial p_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right]+$$

$$+P_k\cdot R_k\cdot q_k\cdot\frac{\partial}{\partial p_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right]-$$

$$-P_k\cdot R_k\cdot q_k\cdot\frac{\partial}{\partial p_k}\left[\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right]\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right]=$$

$$=\left(P_k\cdot R_k\cdot q_k\right)'\cdot\frac{\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left[\left(p_k+q_k\right)\cdot t+\exp\left[-\left(p_k+q_k\right)t\right]-1\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}+$$

$$+P_k\cdot R_k\cdot q_k\cdot\frac{\partial}{\partial p_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right]-$$

$$-P_k\cdot R_k\cdot q_k\cdot\frac{\partial}{\partial p_k}\left[\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right]\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right].$$

Next, we evaluate

$$\frac{\partial}{\partial p_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}\right]=$$
$$=\frac{\exp\left[-\left(p_k+q_k\right)t\right]+\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot(-t)}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}-$$
$$-\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot\left(1-p_k t\right)=$$
$$=\frac{\exp\left[-\left(p_k+q_k\right)t\right]\cdot\left(1-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot t\right)}{q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]}-$$
$$-\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot \exp\left[-2\left(p_k+q_k\right)t\right]\cdot\left(1-p_k t\right)}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}.$$

and

$$\frac{\partial}{\partial p_k}\left[\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right]=$$
$$=\frac{\left[-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)-p_k\right]\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}+$$
$$+\frac{\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right]\right)\cdot \exp\left[-\left(p_k+q_k\right)t\right]\cdot(-t)}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}-$$
$$-\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^4}\times$$
$$\times\frac{\partial}{\partial p_k}\left[\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2\right]=$$
$$=\frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$
$$\times\cdot\left(\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot t-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)-p_k\right)-$$
$$-\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^4}\times$$
$$\times 2\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\frac{\partial}{\partial p_k}\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)=$$
$$=\frac{\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\times$$
$$\times\cdot\left(\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot t-\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)-p_k\right)-$$
$$-\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot \exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)^4}\times$$
$$\times 2\left(q_k+p_k\cdot \exp\left[-\left(p_k+q_k\right)t\right]\right)\left(1-p_k\cdot t\right).$$

Thus, eventually,

$$\frac{\partial X'}{\partial p_k} = (P_k \cdot R_k \cdot q_k)' \cdot \frac{\exp\left[-\left(p_k + q_k\right)t\right] \cdot \left[\left(p_k + q_k\right) \cdot t + \exp\left[-\left(p_k + q_k\right)t\right] - 1\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)^2} +$$
$$+ P_k \cdot R_k \cdot q_k \cdot \frac{\exp\left[-\left(p_k + q_k\right)t\right] \cdot \left(1 - \left(p_k + q_k + \left(p_k' + q_k'\right) \cdot t\right) \cdot t\right)}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]} -$$
$$- P_k \cdot R_k \cdot q_k \cdot \frac{\left(p_k + q_k + \left(p_k' + q_k'\right) \cdot t\right) \cdot \exp\left[-2\left(p_k + q_k\right)t\right] \cdot \left(1 - p_k t\right)}{\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)^2} -$$
$$- P_k \cdot R_k \cdot q_k \cdot \frac{\exp\left[-\left(p_k + q_k\right)t\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)^2} \times$$
$$\times \cdot \left(\left[p_k' - p_k \cdot \left(p_k + q_k + \left(p_k' + q_k'\right) \cdot t\right)\right] \cdot t - \left(p_k + q_k + \left(p_k' + q_k'\right) \cdot t\right) - p_k\right) +$$
$$+ P_k \cdot R_k \cdot q_k \cdot \frac{q_k' + \left[p_k' - p_k \cdot \left(p_k + q_k + \left(p_k' + q_k'\right) \cdot t\right)\right] \cdot \exp\left[-\left(p_k + q_k\right)t\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)^4} \times$$
$$\times 2\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)\left(1 - p_k \cdot t\right).$$

Similar steps as above will lead us to the following expressions:

$$\frac{\partial X}{\partial q_k} = \frac{\partial}{\partial q_k}\left[P_k \cdot R_k \cdot q_k \cdot \frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]}\right] =$$
$$= P_k \cdot R_k \cdot \frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]} + P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial p_k}\left[\frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]}\right],$$

in which

$$\frac{\partial}{\partial q_k}\left[\frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]}\right] = \frac{1}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]}\frac{\partial}{\partial q_k}\left[1 - \exp\left[-\left(p_k + q_k\right)t\right]\right] +$$
$$+ \left(1 - \exp\left[-\left(p_k + q_k\right)t\right]\right) \cdot \frac{\partial}{\partial q_k}\left[\frac{1}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]}\right] =$$
$$= -\frac{\exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]} \cdot \frac{\partial}{\partial q_k}\left[-\left(p_k + q_k\right)t\right] -$$
$$- \frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)^2} \cdot \frac{\partial}{\partial q_k}\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right) =$$
$$= \frac{t \cdot \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]} -$$
$$- \frac{1 - \exp\left[-\left(p_k + q_k\right)t\right]}{\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)^2} \cdot \left(1 + p_k \cdot \frac{\partial \exp\left[-\left(p_k + q_k\right)t\right]}{\partial q_k}\right) =$$
$$= \frac{t \cdot \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]} -$$
$$- \frac{\left(1 - \exp\left[-\left(p_k + q_k\right)t\right]\right)}{\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)^2} \cdot \left(1 + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right] \cdot \frac{\partial}{\partial q_k}\left[-\left(p_k + q_k\right)t\right]\right) =$$
$$= \frac{t \cdot \exp\left[-\left(p_k + q_k\right)t\right]}{q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]} - \frac{\left(1 - \exp\left[-\left(p_k + q_k\right)t\right]\right) \cdot \left(1 - p_k \cdot t \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)}{\left(q_k + p_k \cdot \exp\left[-\left(p_k + q_k\right)t\right]\right)^2},$$

leading to

$$\frac{\partial X}{\partial q_k} = P_k \cdot R_k \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} + P_k \cdot R_k \cdot q_k \cdot \frac{t\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} - \\ - P_k \cdot R_k \cdot q_k \cdot \frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\left(1-p_k\cdot t\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}.$$

Now, we evaluate the derivative $\dfrac{\partial X'}{\partial q_k}$:

$$\frac{\partial X'}{\partial q_k} = \left(P_k\cdot R_k\right)'\cdot\frac{\partial}{\partial q_k}\left[q_k\cdot\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] + \\ + P_k\cdot R_k\cdot q_k'\cdot\frac{\partial}{\partial q_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] + \\ + P_k\cdot R_k\cdot\frac{\partial}{\partial q_k}\left[q_k\cdot\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] - \\ - P_k\cdot R_k\cdot\frac{\partial}{\partial q_k}\left[q_k\cdot\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right] = \\ = \left(P_k\cdot R_k\right)'\cdot\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} + \left(P_k\cdot R_k\right)'\cdot q_k\cdot\frac{\partial}{\partial q_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] + \\ + P_k\cdot R_k\cdot q_k'\cdot\frac{\partial}{\partial q_k}\left[\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] + \\ + P_k\cdot R_k\cdot\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} + \\ + P_k\cdot R_k\cdot q_k\cdot\frac{\partial}{\partial q_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] - \\ - P_k\cdot R_k\cdot\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2} - \\ - P_k\cdot R_k\cdot q_k\cdot\frac{\partial}{\partial q_k}\left[\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2}\right]$$

or

$$\frac{\partial X'}{\partial q_k} = \left(P_k\cdot R_k\right)'\cdot\frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} + \\ + \left(P_k\cdot R_k\cdot q_k\right)'\cdot\frac{t\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} - \\ - \left(P_k\cdot R_k\cdot q_k\right)'\cdot\frac{\left(1-\exp\left[-\left(p_k+q_k\right)t\right]\right)\cdot\left(1-p_k\cdot t\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2} + \\ + P_k\cdot R_k\cdot\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]} + \\ + P_k\cdot R_k\cdot q_k\cdot\frac{\partial}{\partial q_k}\left[\frac{\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]}\right] - \\ - P_k\cdot R_k\cdot\frac{q_k'+\left[p_k'-p_k\cdot\left(p_k+q_k+\left(p_k'+q_k'\right)\cdot t\right)\right]\cdot\exp\left[-\left(p_k+q_k\right)t\right]}{\left(q_k+p_k\cdot\exp\left[-\left(p_k+q_k\right)t\right]\right)^2} -$$

$$- P_k \cdot R_k \cdot q_k \cdot \frac{\partial}{\partial q_k}\left[\frac{q_k' + \left[p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)\right] \cdot \exp\left[-(p_k+q_k)\,t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^2}\right].$$

Furthermore,

$$\frac{\partial}{\partial q_k}\left[\frac{(p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp\left[-(p_k+q_k)\,t\right]}{q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]}\right] =$$
$$= \frac{\exp\left[-(p_k+q_k)\,t\right] - (p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp\left[-(p_k+q_k)\,t\right] \cdot t}{q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]} -$$
$$- \frac{(p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp\left[-(p_k+q_k)\,t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^2} \cdot \frac{\partial}{\partial q_k}\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right) =$$
$$= \frac{\exp\left[-(p_k+q_k)\,t\right] - (p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp\left[-(p_k+q_k)\,t\right] \cdot t}{q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]} -$$
$$- \frac{(p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp\left[-(p_k+q_k)\,t\right] \cdot \left(1 - p_k \cdot t \cdot \exp\left[-(p_k+q_k)\,t\right]\right)}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^2},$$

and

$$\frac{\partial}{\partial q_k}\left[\frac{q_k' + \left[p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)\right] \cdot \exp\left[-(p_k+q_k)\,t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^2}\right] =$$
$$= \frac{-p_k \cdot \exp\left[-(p_k+q_k)\,t\right] + \left[p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)\right] \cdot \exp\left[-(p_k+q_k)\,t\right] \cdot (-t)}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^2} -$$
$$- \frac{q_k' + \left[p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)\right] \cdot \exp\left[-(p_k+q_k)\,t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^4} \cdot \frac{\partial}{\partial q_k}\left[\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^2\right] =$$
$$= -\frac{\left(p_k + t \cdot \left[p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)\right]\right) \cdot \exp\left[-(p_k+q_k)\,t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^2} -$$
$$- \frac{q_k' + \left[p_k' - p_k \cdot (p_k + q_k + (p_k' + q_k') \cdot t)\right] \cdot \exp\left[-(p_k+q_k)\,t\right]}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^4} \cdot \left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right) \times$$
$$\times 2 \cdot \left(1 - p_k \cdot t \cdot \exp\left[-(p_k+q_k)\,t\right]\right).$$

Thence,

$$\frac{\partial X'}{\partial q_k} = (P_k \cdot R_k)' \cdot \frac{1 - \exp\left[-(p_k+q_k)\,t\right]}{q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]} +$$
$$+ (P_k \cdot R_k \cdot q_k)' \cdot \frac{t \cdot \exp\left[-(p_k+q_k)\,t\right]}{q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]} -$$
$$- (P_k \cdot R_k \cdot q_k)' \cdot \frac{\left(1 - \exp\left[-(p_k+q_k)\,t\right]\right) \cdot \left(1 - p_k \cdot t \cdot \exp\left[-(p_k+q_k)\,t\right]\right)}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^2} +$$
$$+ P_k \cdot R_k \cdot \frac{(p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp\left[-(p_k+q_k)\,t\right]}{q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]} +$$
$$+ P_k \cdot R_k \cdot q_k \cdot \frac{\exp\left[-(p_k+q_k)\,t\right] - (p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp\left[-(p_k+q_k)\,t\right] \cdot t}{q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]} -$$
$$- P_k \cdot R_k \cdot q_k \cdot \frac{(p_k + q_k + (p_k' + q_k') \cdot t) \cdot \exp\left[-(p_k+q_k)\,t\right] \cdot \left(1 - p_k \cdot t \cdot \exp\left[-(p_k+q_k)\,t\right]\right)}{\left(q_k + p_k \cdot \exp\left[-(p_k+q_k)\,t\right]\right)^2} -$$

$$- P_k \cdot R_k \cdot \frac{q'_k + [p'_k - p_k \cdot (p_k + q_k + (p'_k + q'_k) \cdot t)] \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2} -$$

$$+ P_k \cdot R_k \cdot q_k \cdot \frac{(p_k + t \cdot [p'_k - p_k \cdot (p_k + q_k + (p'_k + q'_k) \cdot t)]) \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2} -$$

$$+ P_k \cdot R_k \cdot q_k \cdot \frac{q'_k + [p'_k - p_k \cdot (p_k + q_k + (p'_k + q'_k) \cdot t)] \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^4} \times$$

$$\times 2\,(q_k + p_k \cdot \exp[-(p_k+q_k)\,t]) \cdot (1 - p_k \cdot t \cdot \exp[-(p_k+q_k)\,t]).$$

Apparently,

$$\frac{\partial X}{\partial p'_k} = \frac{\partial X}{\partial q'_k} = 0,$$

and

$$\frac{\partial X'}{\partial p'_k} = \frac{\partial}{\partial p'_k}\left[\sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{(p_k + q_k + (p'_k + q'_k) \cdot t) \cdot \exp[-(p_k+q_k)\,t]}{q_k + p_k \cdot \exp[-(p_k+q_k)\,t]}\right] -$$

$$- \frac{\partial}{\partial p'_k}\left[\sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{q'_k + [p'_k - p_k \cdot (p_k + q_k + (p'_k + q'_k) \cdot t)] \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2}\right] =$$

$$= P_k \cdot R_k \cdot q_k \cdot \frac{t \cdot \exp[-(p_k+q_k)\,t]}{q_k + p_k \cdot \exp[-(p_k+q_k)\,t]} - P_k \cdot R_k \cdot q_k \cdot \frac{(1 - p_k \cdot t) \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2} =$$

$$= \frac{P_k \cdot R_k \cdot q_k \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2}\,[t \cdot (q_k + p_k \cdot \exp[-(p_k+q_k)\,t]) - 1 + p_k \cdot t] =$$

$$= \frac{P_k \cdot R_k \cdot q_k \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2}\,[(q_k + p_k) \cdot t + p_k \cdot t \cdot \exp[-(p_k+q_k)\,t] - 1].$$

Therefore,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p'_k} = \frac{1}{X} \cdot \frac{\partial X}{\partial p'_k} =$$

$$= \frac{1}{X} \cdot \frac{P_k \cdot R_k \cdot q_k \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2}\,[(q_k + p_k) \cdot t + p_k \cdot t \cdot \exp[-(p_k+q_k)\,t] - 1].$$

Similarly,

$$\frac{\partial X'}{\partial q'_k} = \frac{\partial}{\partial p'_k}\left[\sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{(p_k + q_k + (p'_k + q'_k) \cdot t) \cdot \exp[-(p_k+q_k)\,t]}{q_k + p_k \cdot \exp[-(p_k+q_k)\,t]}\right] -$$

$$- \frac{\partial}{\partial p'_k}\left[\sum_{k=1}^{n} P_k \cdot R_k \cdot q_k \cdot \frac{q'_k + [p'_k - p_k \cdot (p_k + q_k + (p'_k + q'_k) \cdot t)] \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2}\right] =$$

$$= P_k \cdot R_k \cdot q_k \cdot \frac{t \cdot \exp[-(p_k+q_k)\,t]}{q_k + p_k \cdot \exp[-(p_k+q_k)\,t]} - P_k \cdot R_k \cdot q_k \cdot \frac{(1 - p_k \cdot t) \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2} =$$

$$= \frac{P_k \cdot R_k \cdot q_k \cdot \exp[-(p_k+q_k)\,t]}{(q_k + p_k \cdot \exp[-(p_k+q_k)\,t])^2}\,[(q_k + p_k) \cdot t + p_k \cdot t \cdot \exp[-(p_k+q_k)\,t] - 1],$$

i.e.,

$$\frac{\partial X'}{\partial p'_k} = \frac{\partial X'}{\partial q'_k}.$$

With Respect to Volume

In this case,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} = \frac{\partial}{\partial R_k}\left[\frac{\partial}{\partial t}\ln\left(\sum_{k=1}^{n} P_k \cdot R_k\right)\right] = \frac{\partial}{\partial R_k}\left[\frac{\left(\sum_{k=1}^{n} W_k \cdot R_k\right)'}{\sum_{k=1}^{n} W_k \cdot R_k}\right] =$$

$$= \frac{\partial}{\partial R_k}\left[\frac{1}{R}\cdot\sum_{k=1}^{n}\left(W_k' \cdot R_k + W_k \cdot R_k'\right)\right],$$

where

$$W_k = P_k \cdot \frac{1-\exp\left[-\left(p_k+q_k\right)t\right]}{1+\frac{p_k}{q_k}\cdot\exp\left[-\left(p_k+q_k\right)t\right]}, \quad R = \sum_{k=1}^{n} P_k \cdot R_k.$$

Therefore,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} = \frac{\partial}{\partial R_k}\left[\frac{1}{R}\cdot\sum_{k=1}^{n}\left(W_k' \cdot R_k + W_k \cdot R_k'\right)\right] =$$

$$= \frac{\partial}{\partial R_k}\left[\frac{1}{R}\right]\cdot\sum_{k=1}^{n}\left(W_k' \cdot R_k + W_k \cdot R_k'\right) + \frac{1}{R}\cdot\frac{\partial}{\partial R_k}\left[\sum_{k=1}^{n}\left(W_k' \cdot R_k + W_k \cdot R_k'\right)\right].$$

Taking into account the chain rule, we obtain

$$\frac{\partial}{\partial R_k}\left[\frac{1}{R}\right] = -\frac{1}{R^2}\cdot\frac{\partial R}{\partial R_k} = -\frac{W_k}{R^2}.$$

On the other hand,

$$\frac{\partial}{\partial R_k}\left[\sum_{k=1}^{n}\left(W_k' \cdot R_k + W_k \cdot R_k'\right)\right] = W_k',$$

providing us with this final form:

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} = -\frac{W_k}{R^2}\cdot\sum_{k=1}^{n}\left(W_k' \cdot R_k + W_k \cdot R_k'\right) + \frac{P_k'}{R} =$$

$$= \frac{W_k}{R}\cdot\left[-\frac{1}{R}\cdot\sum_{k=1}^{n}\left(W_k' \cdot R_k + W_k \cdot R_k'\right) + \frac{W_k'}{W_k}\right] =$$

$$= \frac{W_k}{R}\cdot\left[-\frac{R'}{R} + \frac{W_k'}{W_k}\right] =$$

$$= \frac{W_k}{R}\cdot\frac{\partial}{\partial t}\left[\ln W_k - \ln R\right]$$

or

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_k} = \frac{W_k}{R} \cdot \frac{\partial}{\partial t} \left[\ln \frac{W_k}{R} \right].$$

It is much easier to establish that

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R'_k} = \frac{\partial}{\partial R'_k} \left[\frac{1}{R} \cdot \sum_{k=1}^{n} \left(W'_k \cdot R_k + W_k \cdot R'_k \right) \right] = \frac{W_k}{R},$$

With Respect to Transactions

In this case,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial T'_j} = \frac{\partial}{\partial T'_j} \left[\frac{\partial}{\partial t} \left[\ln \left(\frac{1}{n} \sum_{j=1}^{n} T'_j \right) \right] \right] = \frac{\partial}{\partial T'_j} \left[\frac{\sum\limits_{j=1}^{n} T''_j}{\sum\limits_{j=1}^{n} T'_j} \right] = -\frac{1}{\left(T'_j \right)^2} \cdot \sum_{m=1}^{n} T''_m.$$

With Respect to Output Parameters

Since

$$\Pi_{\text{BTC}} = \frac{\partial}{\partial t} \left[\ln \left(\sum_{k=1}^{n} P_k \cdot \frac{1 - \exp\left[-\left(p_k + q_k \right) t \right]}{1 + \dfrac{p_k}{q_k} \cdot \exp\left[-\left(p_k + q_k \right) t \right]} \cdot R_k \right) + \ln \left(\frac{1}{n} \sum_{j=1}^{n} T'_j \right) - \ln b - \ln h + \ln d \right],$$

we easily evaluate the following partial derivatives:

$$\frac{\partial \Pi_{\text{BTC}}}{\partial b} = -\frac{\partial}{\partial b} \left[\frac{\partial \ln b}{\partial t} \right] = -\frac{\partial}{\partial b} \left(\frac{b'}{b} \right) = \frac{b'}{b^2},$$

$$\frac{\partial \Pi_{\text{BTC}}}{\partial h} = -\frac{\partial}{\partial h} \left[\frac{\partial \ln h}{\partial t} \right] = \frac{h'}{h^2},$$

and

$$\frac{\partial \Pi_{\text{BTC}}}{\partial d} = \frac{\partial}{\partial d} \left[\frac{\partial \ln d}{\partial t} \right] = -\frac{d'}{d^2}.$$

Single Asset Case

With Respect to Price

Due to the splitting of the logarithm, we have

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_1} = \frac{\partial}{\partial P_1} \left[\frac{P'_1}{P_1} \right] = -\frac{P'_1}{P_1^2}.$$

Moreover, Π_{BTC} is a linear function of P_1' with

$$\frac{\partial \Pi_{\text{BTC}}}{\partial P_1'} = \frac{\partial}{\partial P_1'}\left[\frac{P_1'}{P_1}\right] = \frac{1}{P_1}.$$

With Respect to Absorption

In this case,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial U_{s_1}} = \frac{\partial}{\partial U_{s_1}}\left[\frac{U_{s_1}'}{U_{s_1}}\right] = -\frac{U_{s_1}'}{U_{s_1}^2}$$

and

$$\frac{\partial \Pi_{\text{BTC}}}{\partial U_{s_1}'} = \frac{\partial}{\partial U_{s_1}'}\left[\frac{U_{s_1}'}{U_{s_1}}\right] = \frac{1}{U_{s_1}}.$$

With Respect to Volume

In this case,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_1} = \frac{\partial}{\partial R_1}\left[\frac{R_1'}{R_1}\right] = -\frac{R_1'}{R_1^2}.$$

Moreover, Π_{BTC} is a linear function of R_1' with

$$\frac{\partial \Pi_{\text{BTC}}}{\partial R_1'} = \frac{\partial}{\partial R_1'}\left[\frac{R_1'}{R_1}\right] = \frac{1}{R_1}.$$

With Respect to Transactions

In this case,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial T_1'} = \frac{\partial}{\partial T_1'}\left[\frac{T_1''}{T_1'}\right] = -\frac{T_1''}{(T_1')^2}.$$

With Respect to Bass Model Parameters

When p_1 is the sensitivity parameter, making use of the chain rule, we have

$$\frac{\partial \Pi_{\text{BTC}}}{\partial p_1} = \frac{\partial \Pi_{\text{BTC}}}{\partial \alpha_1} \cdot \frac{\partial \alpha_1}{\partial p_1}.$$

Apparently,

$$\frac{\partial \Pi_{\text{BTC}}}{\partial \alpha_1} = \frac{\partial}{\partial \alpha_1}\left[\frac{\alpha_1'}{\alpha_1}\right] = -\frac{\alpha_1'}{\alpha_1^2},$$

and

$$\frac{\partial \alpha_1}{\partial p_1} = \frac{\partial}{\partial p_1}\left[\frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{1+\dfrac{p_1}{q_1}\cdot\exp\left[-\left(p_1+q_1\right)t\right]}\right] = \frac{\partial}{\partial p_1}\left[q_1\cdot\frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]}\right] =$$
$$= q_1\cdot\frac{\partial}{\partial p_1}\left[\frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]}\right] = q_1\cdot\beta_p,$$

in which

$$\beta_p = \frac{\partial}{\partial p_1}\left[\frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]}\right] =$$
$$= \frac{t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2} -$$
$$- \frac{\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(\exp\left[-\left(p_1+q_1\right)t\right]-p_1\cdot t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2} =$$
$$= \frac{t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2} -$$
$$- \frac{\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(1-p_1\cdot t\right)\cdot\exp\left[-\left(p_1+q_1\right)t\right]}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2} =$$
$$= \frac{\exp\left[-\left(p_1+q_1\right)t\right]}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}\times$$
$$\times\left[t\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)-\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(1-p_1\cdot t\right)\right],$$

Therefore,

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial p_1} = -\frac{\alpha_1'}{\alpha_1^2}\cdot q_1\cdot\frac{\exp\left[-\left(p_1+q_1\right)t\right]}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}\times$$
$$\times\left[t\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)-\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(1-p_1\cdot t\right)\right] =$$
$$= -\frac{q_1\cdot\alpha_1'\cdot\exp\left[-\left(p_1+q_1\right)t\right]}{\alpha_1^2\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}\times$$
$$\times\left[t\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)-\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(1-p_1\cdot t\right)\right] =$$
$$= \frac{q_1\cdot\alpha_1'\cdot\exp\left[-\left(p_1+q_1\right)t\right]}{\alpha_1^2\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}\times$$
$$\times\left[\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(1-p_1\cdot t\right)-t\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)\right].$$

Similarly, taking into account that

$$\frac{\partial \Pi_{\mathrm{BTC}}}{\partial q_1} = \frac{\partial \Pi_{\mathrm{BTC}}}{\partial \alpha_1}\cdot\frac{\partial \alpha_1}{\partial q_1}$$

and

$$\begin{aligned}\frac{\partial \alpha_1}{\partial q_1} &= \frac{\partial}{\partial q_1}\left[q_1 \cdot \frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]}\right] = \\ &= \frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]} + q_1 \cdot \frac{\partial}{\partial q_1}\left[\frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]}\right] = \\ &= \frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]} + q_1 \cdot \beta_q,\end{aligned}$$

in which

$$\begin{aligned}\beta_q &= \frac{\partial}{\partial q_1}\left[\frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]}\right] = \\ &= \frac{t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2} - \\ &- \frac{\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot(-t)\right)}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2} = \\ &= \frac{t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2} - \\ &- \frac{\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(1-p_1\cdot t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}.\end{aligned}$$

Direct simplifications lead to

$$\begin{aligned}& t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right) - \\ & -\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(1-p_1\cdot t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right) = \\ &\qquad = t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot q_1 + t\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right] - \\ &\qquad - 1 + p_1\cdot t\cdot\exp\left[-\left(p_1+q_1\right)t\right] + \exp\left[-\left(p_1+q_1\right)t\right] - p_1\cdot t\cdot\exp\left[-2\left(p_1+q_1\right)t\right] = \\ &\qquad = q_1\cdot t\cdot\exp\left[-\left(p_1+q_1\right)t\right] + p_1\cdot t\cdot\exp\left[-2\left(p_1+q_1\right)t\right] - \\ &\qquad - 1 + p_1\cdot t\cdot\exp\left[-\left(p_1+q_1\right)t\right] + \exp\left[-\left(p_1+q_1\right)t\right] - p_1\cdot t\cdot\exp\left[-2\left(p_1+q_1\right)t\right] = \\ &\qquad = \left(p_1+q_1\right)\cdot t\cdot\exp\left[-\left(p_1+q_1\right)t\right] + \exp\left[-\left(p_1+q_1\right)t\right] - 1 = \\ &\qquad = \left[1+\left(p_1+q_1\right)\cdot t\right]\cdot\exp\left[-\left(p_1+q_1\right)t\right] - 1.\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial \alpha_1}{\partial q_1} &= \frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]}+q_1\cdot\beta_q= \\
&= \frac{1-\exp\left[-\left(p_1+q_1\right)t\right]}{q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]}+q_1\cdot\frac{\left[1+\left(p_1+q_1\right)\cdot t\right]\cdot\exp\left[-\left(p_1+q_1\right)t\right]-1}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}= \\
&= \frac{\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}+ \\
&+ \frac{q_1\cdot\left[1+\left(p_1+q_1\right)\cdot t\right]\cdot\exp\left[-\left(p_1+q_1\right)t\right]-q_1}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}= \\
&= \frac{p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)+q_1\cdot\left(p_1+q_1\right)\cdot t\cdot\exp\left[-\left(p_1+q_1\right)t\right]}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}.
\end{aligned}$$

Thus, eventually,

$$\begin{aligned}
\frac{\partial \Pi_{\mathrm{BTC}}}{\partial q_1} &= -\frac{\alpha_1'}{\alpha_1^2}\cdot\frac{p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\cdot\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)+q_1\cdot\left(p_1+q_1\right)\cdot t\cdot\exp\left[-\left(p_1+q_1\right)t\right]}{\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}= \\
&= -\frac{\alpha_1'\cdot\exp\left[-\left(p_1+q_1\right)t\right]}{\alpha_1^2\cdot\left(q_1+p_1\cdot\exp\left[-\left(p_1+q_1\right)t\right]\right)^2}\cdot\left(p_1\cdot\left(1-\exp\left[-\left(p_1+q_1\right)t\right]\right)+q_1\cdot\left(p_1+q_1\right)\cdot t\right).
\end{aligned}$$

Appendix 2 - Econometric Tables

Appendix 3 - Reinforcement Learning Outcomes

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