Consulted solution

a) $\sigma'(x) = \frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) = \frac{d}{dx}(1+e^{-x})^{-1}$

$= e^{-x}(1+e^{-x})^{-2}$

$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{e^{-x}}{1+e^{-x}}\right)$

$= \sigma(x)\left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)$

$= \sigma(x)\left(1 - \frac{1}{1+e^{-x}}\right)$

$= \sigma(x)[1-\sigma(x)]$

b) Negative log likelihood for regression is

$nll(\theta) = -\sum_i y_i \sigma(\theta^T x_i)\,\sigma'(\theta^T x_i) + (1-y_i)\frac{1}{1-\sigma(\theta^T x_i)}(-\sigma'(\theta^T x_i))$

$= \sum_i (\mu_i - y_i)x_i = X^T(\mu - y)$

c) Using this, we can find the Hessian matrix.

$H_\theta = \nabla_\theta(\nabla_\theta nll(\theta))^T = \nabla_\theta[X^T(\mu-y)]^T = X^T \operatorname{diag}(\mu(1-\mu))X$

$= X^T S X$

2. $\int_{\mathbb{R}} P(x;\sigma^2)\,dx = \int_{\mathbb{R}} \frac{1}{Z}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx = \frac{1}{Z}\int_{\mathbb{R}}\exp\left(\frac{x^2}{2\sigma^2}\right)dx = 1$

$Z^2 = \int_{\mathbb{R}}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx \int_{\mathbb{R}}\exp\left(-\frac{y^2}{2\sigma^2}\right)dy = \iint_{\mathbb{R}^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)dx\,dy$

$= -2\pi\sigma^2(0-1) = 2\pi\sigma^2$

Hence $Z^2 = 2\pi\sigma^2 \Rightarrow Z = \sqrt{2\pi\sigma^2} = \sqrt{2\pi}\sigma$