consulted solution

Let $y = Ax + b$ be a random vector.

Show that expectation is linear:

$$\mathbb{E}[y] = \mathbb{E}[Ax + b] = A\,\mathbb{E}[x] + b$$

We are given that $y = Ax + b$. Then,

$$\mathbb{E}[y] = \int_S (Ax + b)\,\mathbb{P}(x)\,dx$$
$$= A\,\mathbb{E}[x] + b$$

$$\text{cov}[y] = \text{cov}[Ax + b] = A\,\text{cov}[x]\,A^T = A\Sigma A^T$$

By definition, covariance is $\text{cov}[x] =$

$$\mathbb{E}\left[(x - \mathbb{E}[x])(x - \mathbb{E}[x])^T\right]$$

Given that $\text{cov}[y] = \text{cov}[Ax + b]$, we have that

$$\text{cov}[y] = \mathbb{E}\left[((Ax+b) - \mathbb{E}[Ax + b])(Ax + b - \mathbb{E}[Ax+b])^T\right]$$
$$= \mathbb{E}\left[(Ax - A\mathbb{E}[x])(Ax - A\mathbb{E}[x])^T\right]$$
$$= \mathbb{E}\left[A(x - \mathbb{E}[x])(x - \mathbb{E}[x])^T A^T\right]$$

$$= A\,\mathbb{E}\left[(x-\mathbb{E}[x])(x-\mathbb{E}[x])^T\right] A^T$$

$$= A \operatorname{cov}[x] A^T = A\Sigma A^T.$$

Thus, we can see that

$$\operatorname{cov}[y] = \operatorname{cov}[Ax+b] = A\operatorname{cov}[x]A^T = A\Sigma A^T.$$

2. Given $\mathcal{D} = \{(x,y)\} = \{(0,1),(2,3),(3,6),(4,8)\}$

a) Find least squares estimate $y = \theta^T x$ using Cramer's Rule.

Let

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ and } y = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix}$$

Then, $X^TX = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 9 & 29 \end{bmatrix}$

and

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 \\ 56 \end{bmatrix}$$

$X^T X \theta^* = X^T y$ Cramer's Rule:

$$\theta_0^* = \frac{\begin{vmatrix} 18 & 9 \\ 56 & 29 \end{vmatrix}}{\begin{vmatrix} 4 & 9 \\ 9 & 29 \end{vmatrix}} = \frac{18}{35} \qquad \theta_1^* = \frac{\begin{vmatrix} 4 & 18 \\ 9 & 56 \end{vmatrix}}{\begin{vmatrix} 4 & 9 \\ 9 & 29 \end{vmatrix}} = \frac{62}{35}$$

$y = \theta_0 + \theta_1 x$

b) Using the normal equation, $\theta^* = (X^T X)^{-1} X^T y$

$$= \begin{bmatrix} 4 & 9 \\ 9 & 29 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 29 & -9 \\ -9 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix}$$

$$= \frac{1}{35} \begin{bmatrix} 29 & 11 & 2 & -7 \\ -9 & -1 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 18 \\ 62 \end{bmatrix} = \begin{bmatrix} 18/35 \\ 62/35 \end{bmatrix}$$